Derived equivalences and Gorenstein algebras

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In this note, we introduce the notion of Gorenstein algebras. Let R be a commutative Gorenstein ring and A a noetherian R-algebra. We call A a Gorenstein R-algebra if A has Gorenstein dimension zero as an R-module (see [2]), $\mathrm{add}(D({}_AA)) = \mathcal{P}_A$, where $D = \mathrm{Hom}_R(-, R)$, and $A_\mathfrak{p}$ is projective as an $R_\mathfrak{p}$-module for all $p \in \mathrm{Spec}\ R$ with $\dim\ R_\mathfrak{p} < \dim\ R$. Note that if $\dim\ R = \infty$ then a Gorenstein R-algebra A is projective as an R-module and that A is a Gorenstein R-algebra if A is projective as an R-module and $\mathrm{add}(D({}_AA)) = \mathcal{P}_A$. Also, in case R is equidimensional and $A_\mathfrak{p} \neq 0$ for all $\mathfrak{p} \in \mathrm{Spec}\ R$, a Gorenstein R-algebra A with $A \simeq DA$ in Mod-A^{e} is a Gorenstein R-order in the sense of [1]. In Section 3, we see that a Gorenstein R-algebra A enjoys properties similar to those of R. Especially, A satisfies the Auslander condition (see [5]) and for any nonzero $P^\bullet \in \mathsf{K}^-(\mathcal{P}_A)$ we have $\mathrm{Hom}_{\mathsf{K}(\mathrm{Mod}\text{-}A)}(P^\bullet, A[i]) \neq 0$ for some $i \in \mathbb{Z}$.

Unfortunately, the class of Gorenstein R-algebras is not closed under derived equivalence in general (see Example 4.9). In Section 4, for a tilting complex $P^\bullet$ over a Gorenstein R-algebra A we show that $B = \mathrm{End}_{\mathsf{K}(\mathrm{Mod}\text{-}A)}(P^\bullet)$ is also a Gorenstein R-algebra if and only if $\mathrm{add}(P^\bullet) = \mathrm{add}(\nu P^\bullet)$, where $\nu = D \circ \mathrm{Hom}_A(-, A)$. In particular, the class of Gorenstein R-algebras A with $A \simeq DA$ in Mod-A^{e} is closed under derived equivalence. More precisely, for any partial tilting complex $P^\bullet$ over a Gorenstein R-algebra A with $A \simeq DA$ in Mod-A^{e}, $B = \mathrm{End}_{\mathsf{K}(\mathrm{Mod}\text{-}A)}(P^\bullet)$ is also a Gorenstein R-algebra with $B \simeq DB$ in Mod-B^{e}. Then, in Section 5, we provide a construction of such tilting complexes. Namely, we show that tilting complexes $P^\bullet$ associated with a certain sequence of idempotents in a Gorenstein R-algebra A satisfy the condition $\mathrm{add}(P^\bullet) = \mathrm{add}(\nu P^\bullet)$.

In Sections 6 and 7, we deal with the case where R is a complete local ring and A is free as an R-module. For a tilting complex $P^\bullet$ constructed in Section 5, we show that $B = \mathrm{End}_{\mathsf{K}(\mathrm{Mod}\text{-}A)}(P^\bullet)$ is also free as an R-module and then provide a way to construct a two-sided tilting complex corresponding to $P^\bullet$.

2000 *Mathematics Subject Classification.* Primary 18E30, 16G30; Secondary 16E05

Key words. derived equivalence, Gorenstein algebra, tilting complex, two-sided tilting complex.

This is not in final form. The detailed version will be submitted for publication elsewhere.

Simultaneously, we provide a sufficient condition for a free R-algebra B containing A as a subalgebra to be derived equivalent to A.

Finally, in Section 8, we ask when a partial tilting complex $P^\bullet$ appears as a direct summand of a tilting complex. This is not the case in general (see [15, Section 8]). We show that the question is affirmative if $P^\bullet$ has length 1 and $P^\bullet \in \mathrm{add}(\nu P^\bullet)$.

Let A be a ring. We denote by Mod-A the category of right A-modules and mod-A the full subcategory of Mod-A consisting of finitely presented modules. We denote by Proj-A (resp., Inj-A) the full subcategory of Mod-A consisting of projective (resp., injective) modules and by $\mathcal{P}_A$ the full subcategory of Proj-A consisting of finitely generated projective modules. We denote by A^{op} the opposite ring of A and consider left A-modules as right A^{op}-modules. Sometimes, we use the notation X_A (resp., ${}_AX$) to stress that the module X considered is a right (resp., left) A-module. For an object X in an additive category $\mathcal{B}$, we denote by $\mathrm{add}(X)$ the full subcategory of $\mathcal{B}$ whose objects are direct summands of finite direct sums of copies of X and by $X^{(n)}$ the direct sum of n copies of X. In case $\mathcal{B}$ has arbitrary direct sums, we denote by $\mathrm{Add}(X)$ the full subcategory of $\mathcal{B}$ whose objects are direct summands of direct sums of copies of X. For a cochain complex $X^\bullet$ over an abelian category $\mathcal{A}$, we denote by $\mathrm{B}^n(X^\bullet)$, $\mathrm{Z}^n(X^\bullet)$, $\mathrm{B}'^n(X^\bullet)$, $\mathrm{Z}'^n(X^\bullet)$ and $\mathrm{H}^n(X^\bullet)$ the n-th boundary, the n-th cycle, the n-th coboundary, the n-th cocycle and the n-th cohomology of $X^\bullet$, respectively. For an additive category $\mathcal{B}$, we denote by $\mathsf{K}(\mathcal{B})$ (resp., $\mathsf{K}^+(\mathcal{B})$, $\mathsf{K}^-(\mathcal{B})$, $\mathsf{K}^{\mathrm{b}}(\mathcal{B})$) the homotopy category of complexes (resp., bounded below complexes, bounded above complexes, bounded complexes) over $\mathcal{B}$. As usual, we consider objects of $\mathcal{B}$ as complexes over $\mathcal{B}$ concentrated in degree zero. For an abelian category $\mathcal{A}$, we denote by $\mathsf{D}(\mathcal{A})$ (resp., $\mathsf{D}^+(\mathcal{A})$, $\mathsf{D}^-(\mathcal{A})$, $\mathsf{D}^{\mathrm{b}}(\mathcal{A})$) the derived category of complexes (resp., bounded below complexes, bounded above complexes, bounded complexes) over $\mathcal{A}$. We always consider $\mathsf{K}^*(\mathcal{B})$ (resp., $\mathsf{D}^*(\mathcal{A})$) as a full triangulated subcategory of $\mathsf{K}(\mathcal{B})$ (resp., $\mathsf{D}(\mathcal{A})$), where $* = +,\ -$ or b. Finally, we use the notation $\mathrm{Hom}^\bullet(-,-)$ (resp., $- \otimes^\bullet -$) to denote the single complex associated with the double hom (resp., tensor) complex (cf. Remark 1.11).

We refer to [6], [9], [17] for basic results in the theory of derived categories, to [15], [16] for definitions and basic properties of derived equivalences, tilting complexes and two-sided tilting complexes and to [12] for standard commutative ring theory.

1 Preliminaries

Throughout this note, R is a commutative ring and A is an R-algebra, i.e., A is a ring endowed with a ring homomorphism $R \to A$ whose image is contained in the center of A. We set $D = \mathrm{Hom}_R(-, R)$. Note that for any $X \in$ Mod-A we have a functorial isomorphism in Mod-A^{op}

$$DX \xrightarrow{\sim} \mathrm{Hom}_A(X, DA), h \mapsto (x \mapsto (a \mapsto h(xa))).$$

For R-algebras A, B we identify an $(A^{\rm op}\otimes_R B)$-module X with an A-B-bimodule X such that $rx = xr$ for all $r \in R$ and $x \in X$. Also, for an R-algebra A we set $A^{\rm e} = A^{\rm op} \otimes_R A$.

In this section, we recall several definitions and basic facts which we need in later sections.

Definition 1.1. A module $X \in \text{Mod-}R$ is said to be reflexive if the canonical homomorphism

$$\varepsilon_X : X \to D^2X, x \mapsto (h \mapsto h(x))$$

is an isomorphism, where $D^2X = D(DX)$.

Definition 1.2 (cf. [2]). A module $X \in \text{Mod-}R$ is said to have Gorenstein dimension zero if X is reflexive, $\text{Ext}^i_R(X, R) = 0$ for $i > 0$ and $\text{Ext}^i_R(DX, R) = 0$ for $i > 0$.

Lemma 1.3 ([2, Lemma 3.10]). *Let $0 \to X \to Y \to Z \to 0$ be an exact sequence in* Mod-R. *Then the following hold.*

(1) If Y, Z have Gorenstein dimension zero, so does X.

(2) Assume $\text{Ext}^1_R(Z, R) = 0$. If X, Y have Gorenstein dimension zero, so does Z.

Lemma 1.4. *For any $X^\bullet \in \mathsf{K}(\text{Mod-}R)$ we have a functorial homomorphism*

$$\xi_{X^\bullet} : \mathrm{H}^0(DX^\bullet) \to D\mathrm{H}^0(X^\bullet)$$

and the following hold.

(1) If $\mathrm{B}^0(DX^\bullet) \xrightarrow{\sim} D\mathrm{B}'^0(X^\bullet)$ canonically, then $\xi_{X^\bullet}$ is monic.

(2) If $\mathrm{B}^0(DX^\bullet) \xrightarrow{\sim} D\mathrm{B}'^0(X^\bullet)$ canonically and $\text{Ext}^1_R(\mathrm{B}'^0(X^\bullet), R) = 0$, then $\xi_{X^\bullet}$ is an isomorphism.

Lemma 1.5. *Let A, B be derived equivalent R-algebras. Let $F : \mathsf{K}^{\rm b}(\mathcal{P}_B) \xrightarrow{\sim} \mathsf{K}^{\rm b}(\mathcal{P}_A)$ be an equivalence of triangulated categories and $F^* : \mathsf{K}^{\rm b}(\mathcal{P}_A) \xrightarrow{\sim} \mathsf{K}^{\rm b}(\mathcal{P}_B)$ a quasi-inverse of F. Set $P^\bullet = F(B) \in \mathsf{K}^{\rm b}(\mathcal{P}_A)$ and $Q^\bullet = \text{Hom}^\bullet_B(F^*(A), B) \in \mathsf{K}^{\rm b}(\mathcal{P}_{B^{\rm op}})$. Then for any $i \in \mathbb{Z}$ we have an isomorphism in* Mod-$(B^{\rm op} \otimes_R A)$

$$\text{Hom}_{\mathsf{K}(\text{Mod-}A)}(A, P^\bullet[i]) \simeq \text{Hom}_{\mathsf{K}(\text{Mod-}B^{\rm op})}(B, Q^\bullet[i])$$

and an isomorphism in Mod-$(A^{\rm op} \otimes_R B)$

$$\text{Hom}_{\mathsf{K}(\text{Mod-}A)}(P^\bullet, A[i]) \simeq \text{Hom}_{\mathsf{K}(\text{Mod-}B^{\rm op})}(Q^\bullet, B[i]).$$

Definition 1.6. For any nonzero $P^\bullet \in \mathsf{K}^-(\text{Proj-}A)$ we set

$$a(P^\bullet) = \max\{i \in \mathbb{Z} \mid \mathrm{H}^i(P^\bullet) \neq 0\}$$

and for any nonzero $P^\bullet \in \mathsf{K}^+(\text{Proj-}A)$ we set

$$b(P^\bullet) = \min\{i \in \mathbb{Z} \mid \text{Hom}_{\mathsf{K}(\text{Mod-}A)}(P^\bullet[i], \text{Proj-}A) \neq 0\}.$$

Then for any nonzero $P^\bullet \in \mathsf{K}^{\mathrm{b}}(\text{Proj-}A)$ we set

$$l(P^\bullet) = a(P^\bullet) - b(P^\bullet)$$

which we call the length of $P^\bullet$.

Remark 1.7. For any complex $X^\bullet$ and $n \in \mathbb{Z}$ we define truncations

$$\begin{aligned}\sigma_{\leq n}(X^\bullet) &: \cdots \to X^{n-2} \to X^{n-1} \to \mathrm{Z}^n(X^\bullet) \to 0 \to \cdots ,\\ \sigma'_{\geq n}(X^\bullet) &: \cdots \to 0 \to \mathrm{Z}'^n(X^\bullet) \to X^{n+1} \to X^{n+2} \to \cdots .\end{aligned}$$

Then $P^\bullet \simeq \sigma_{\leq a}(P^\bullet)$ for any nonzero $P^\bullet \in \mathsf{K}^-(\text{Proj-}A)$, where $a = a(P^\bullet)$, and $P^\bullet \simeq \sigma'_{\geq b}(P^\bullet)$ for any nonzero $P^\bullet \in \mathsf{K}^+(\text{Proj-}A)$, where $b = b(P^\bullet)$.

Lemma 1.8. *Assume A is finitely generated projective as an R-module. Then for any $P^\bullet \in \mathsf{K}^+(\mathcal{P}_A)$ and $Q^\bullet \in \mathsf{K}^-(\mathcal{P}_A)$ with* $\mathrm{Hom}_{\mathsf{K}(\text{Mod-}A)}(P^\bullet, Q^\bullet[i]) = 0$ *for* $i > 0$, $\mathrm{Hom}_{\mathsf{K}(\text{Mod-}A)}(P^\bullet, Q^\bullet)$ *is finitely generated as an R-module.*

Definition 1.9 (cf. [3]). An idempotent $e \in A$ is said to be local if eAe is a local ring. A ring A is said to be semiperfect if $1 = e_1 + \cdots + e_n$ in A with the e_i orthogonal local idempotents.

Lemma 1.10. *Assume R is a complete noetherian local ring and A is finitely generated as an R-module. Then A is semiperfect and the Krull-Schmidt theorem holds in* mod-A, *i.e., for any nonzero $X \in$* mod-A *the following hold.*

(1) X decomposes into a direct sum of indecomposable submodules.

(2) X is indecomposable if and only if $\mathrm{End}_A(X)$ *is local.*

Remark 1.11 *([16, Section 4]).* Let A, B and C be projective R-algebras. Then the following hold.

(1) Let $X^\bullet \in \mathsf{K}^-(\text{Mod-}(B^{\mathrm{op}} \otimes_R A))$ and $Y^\bullet \in \mathsf{K}^+(\text{Mod-}(C^{\mathrm{op}} \otimes_R A))$. If either each term of $X^\bullet$ is projective as an A-module or each term of $Y^\bullet$ is injective as an A-module, then the canonical homomorphism in $\mathsf{D}(\text{Mod-}(C^{\mathrm{op}} \otimes_R B))$

$$\mathrm{Hom}^\bullet_A(X^\bullet, Y^\bullet) \to \mathbf{R}\mathrm{Hom}^\bullet_A(X^\bullet, Y^\bullet)$$

is an isomorphism.

(2) Let $X^\bullet \in \mathsf{K}^-(\text{Mod-}(B^{\mathrm{op}} \otimes_R A))$ and $Y^\bullet \in \mathsf{K}^-(\text{Mod-}(A^{\mathrm{op}} \otimes_R C))$. If either each term of $X^\bullet$ is flat as an A-module or each term of $Y^\bullet$ is flat as an A^{op}-module, then the canonical homomorphism in $\mathsf{D}(\text{Mod-}(B^{\mathrm{op}} \otimes_R C))$

$$X^\bullet \otimes^{\mathbf{L}}_A Y^\bullet \to X^\bullet \otimes^\bullet_A Y^\bullet$$

is an isomorphism.

2 Nakayama functor

In the following, we set $\nu = D \circ \mathrm{Hom}_A(-, A)$. Note that for any $P \in \mathcal{P}_A$ we have a functorial isomorphism in Mod-A

$$P \otimes_A DA \xrightarrow{\sim} \nu P, x \otimes h \mapsto (g \mapsto h(g(x))).$$

Lemma 2.1. *For any $P^\bullet \in \mathsf{K}^{\mathrm{b}}(\mathcal{P}_A)$ and $Q^\bullet \in \mathsf{K}(\mathrm{Mod}\text{-}A)$ we have a bifunctorial isomorphism of complexes*

$$D\mathrm{Hom}^\bullet_A(P^\bullet, Q^\bullet) \simeq \mathrm{Hom}^\bullet_A(Q^\bullet, \nu P^\bullet).$$

Lemma 2.2. *For any $P^\bullet \in \mathsf{K}^{\mathrm{b}}(\mathcal{P}_A)$ and $Q^\bullet \in \mathsf{K}(\mathrm{Mod}\text{-}A)$ we have a bifunctorial homomorphism*

$$\xi_{P^\bullet,Q^\bullet} : \mathrm{Hom}_{\mathsf{K}(\mathrm{Mod}\text{-}A)}(Q^\bullet, \nu P^\bullet) \to D\mathrm{Hom}_{\mathsf{K}(\mathrm{Mod}\text{-}A)}(P^\bullet, Q^\bullet).$$

Furthermore, in case $Q^\bullet \in \mathsf{K}^-(\mathrm{Proj}\text{-}A)$ and $\mathrm{Hom}_{\mathsf{K}(\mathrm{Mod}\text{-}A)}(P^\bullet, Q^\bullet[i]) = 0$ for $i > 0$, the following hold.

(1) $\xi_{P^\bullet,Q^\bullet}$ is monic if $\mathrm{Ext}^i_R(A, R) = 0$ for $1 \le i < a(Q^\bullet) - b(P^\bullet)$.

(2) $\xi_{P^\bullet,Q^\bullet}$ is an isomorphism if $\mathrm{Ext}^i_R(A, R) = 0$ for $1 \le i \le a(Q^\bullet) - b(P^\bullet)$.

Corollary 2.3. *Assume $\mathrm{Ext}^i_A(A, R) = 0$ for $i > 0$. Then for any $P^\bullet \in \mathsf{K}^{\mathrm{b}}(\mathcal{P}_A)$ with $\mathrm{Hom}_{\mathsf{K}(\mathrm{Mod}\text{-}A)}(P^\bullet, P^\bullet[i]) = 0$ for $i > 0$ we have $\mathrm{Hom}_{\mathsf{K}(\mathrm{Mod}\text{-}A)}(P^\bullet, \nu P^\bullet[i]) = 0$ for $i < 0$.*

Definition 2.4. For any $P^\bullet \in \mathsf{K}^{\mathrm{b}}(\mathcal{P}_A)$, we denote by $\mathcal{C}(P^\bullet)$ the full subcategory of $\mathsf{D}^-(\mathrm{Mod}\text{-}A)$ consisting of $X^\bullet$ with $\mathrm{Hom}_{\mathsf{D}(\mathrm{Mod}\text{-}A)}(P^\bullet, X^\bullet[i]) = 0$ for $i \neq 0$.

Lemma 2.5. *Assume A is reflexive as an R-module and $\mathrm{add}(D({}_AA)) = \mathcal{P}_A$. Then we have an equivalence $\nu : \mathcal{P}_A \xrightarrow{\sim} \mathcal{P}_A$. In particular, for any tilting complex $P^\bullet \in \mathsf{K}^{\mathrm{b}}(\mathcal{P}_A)$, $\nu P^\bullet$ is also a tilting complex and the following are equivalent.*

(1) $\nu P^\bullet \in \mathcal{C}(P^\bullet)$ and $P^\bullet \in \mathcal{C}(\nu P^\bullet)$.

(2) $\mathrm{add}(P^\bullet) = \mathrm{add}(\nu P^\bullet)$.

Lemma 2.6. *Assume $A \simeq DA$ in $\mathrm{Mod}\text{-}A^{\mathrm{e}}$. Then the following hold.*

(1) For any $P^\bullet \in \mathsf{K}(\mathcal{P}_A)$ we have a functorial isomorphism of complexes $\nu P^\bullet \simeq P^\bullet$.

(2) A has Gorenstein dimension zero as an R-module if and only if $\mathrm{Ext}^i_R(A, R) = 0$ for $i > 0$.

Proposition 2.7. *Assume $A \simeq DA$ in $\mathrm{Mod}\text{-}A^{\mathrm{e}}$ and A has Gorenstein dimension zero as an R-module. Let $P^\bullet \in \mathsf{K}^{\mathrm{b}}(\mathcal{P}_A)$ with $\mathrm{Hom}_{\mathsf{K}(\mathrm{Mod}\text{-}A)}(P^\bullet, P^\bullet[i]) = 0$ for $i \neq 0$ and $B = \mathrm{End}_{\mathsf{K}(\mathrm{Mod}\text{-}A)}(P^\bullet)$. Then $B \simeq DB$ in $\mathrm{Mod}\text{-}B^{\mathrm{e}}$.*

3 Gorenstein algebras

In this section, we introduce the notion of Gorenstein R-algebras over a Gorenstein ring R. We refer to [4] for the definition and basic properties of Gorenstein rings.

Lemma 3.1. *For any* $X \in \text{Mod-}R$ *the following hold.*

(1) If X *is injective, so is* $\text{Hom}_R({}_AA, X)$.

(2) Assume A *is finitely generated projective as an* R*-module and* $D({}_AA) \in \mathcal{P}_A$*. If* X *is flat, so is* $\text{Hom}_R({}_AA, X)$.

Definition 3.2. A module $T \in \text{Mod-}A$ is called a tilting module if there exists a tilting complex $P^\bullet \in \mathsf{K}^{\text{b}}(\mathcal{P}_A)$ such that $\text{H}^i(P^\bullet) = 0$ for $i \neq 0$ and $\text{H}^0(P^\bullet) \simeq T$ in $\text{Mod-}A$, i.e., $P^\bullet \simeq T$ in $\mathsf{D}(\text{Mod-}A)$.

Remark 3.3 *(cf. [14]).* A module $T \in \text{Mod-}A$ is a tilting module if and only if the following conditions are satisfied:

(1) $\text{Ext}^i_A(T, T) = 0$ for $i > 0$;

(2) there exists an exact sequence $0 \to P^{-l} \to \cdots \to P^0 \to T \to 0$ in $\text{Mod-}A$ with $P^{-i} \in \mathcal{P}_A$ for all $0 \leq i \leq l$; and

(3) there exists an exact sequence $0 \to A_A \to T^0 \to \cdots \to T^m \to 0$ in $\text{Mod-}A$ with $T^i \in \text{add}(T)$ for all $0 \leq i \leq m$.

Definition 3.4 (cf. [9] and [13]). Assume A is a left and right noetherian ring. Then a complex $V^\bullet \in \mathsf{D}^{\text{b}}(\text{Mod-}A^{\text{e}})$ is called a dualizing complex for A if the following conditions are satisfied:

(1) $\text{H}^i(V^\bullet_A) \in \text{mod-}A$ and $\text{H}^i({}_AV^\bullet) \in \text{mod-}A^{\text{op}}$ for all $i \in \mathbb{Z}$;

(2) $V^\bullet_A \in \mathsf{K}^{\text{b}}(\text{Inj-}A)$ and ${}_AV^\bullet \in \mathsf{K}^{\text{b}}(\text{Inj-}A^{\text{op}})$;

(3) $\text{Hom}_{\mathsf{D}(\text{Mod-}A)}(V^\bullet_A, V^\bullet_A[i]) = 0$ for $i \neq 0$ and $\text{Hom}_{\mathsf{D}(\text{Mod-}A^{\text{op}})}({}_AV^\bullet, {}_AV^\bullet[i]) = 0$ for $i \neq 0$; and

(4) the left multiplication of A on each homogeneous component of $V^\bullet$ gives rise to an R-algebra isomorhism $A \xrightarrow{\sim} \text{End}_{\mathsf{D}(\text{Mod-}A)}(V^\bullet_A)$ and the right multiplication of A on each homogeneous component of $V^\bullet$ gives rise to an R-algebra isomorhism $A \xrightarrow{\sim} \text{End}_{\mathsf{D}(\text{Mod-}A^{\text{op}})}({}_AV^\bullet)^{\text{op}}$.

Definition 3.5 (cf. [5]). A left and right noetherian ring A is said to satisfy the Auslander condition if it admits an injective resolution $A_A \to E^\bullet$ in $\text{Mod-}A$ such that flat dim $E^n \leq n$ for all $n \geq 0$.

Throughout the rest of this section, we assume R is noetherian and A is a noetherian R-algebra, i.e., A is finitely generated as an R-module. We denote by dim R the Krull dimension of R, by Spec R the set of prime ideals in R

and by $(-)_\mathfrak{p}$ the localization at $\mathfrak{p} \in \operatorname{Spec} R$. Note that we do not exclude the case where $A_\mathfrak{p} = 0$ for some $\mathfrak{p} \in \operatorname{Spec} R$, i.e., the kernel of the structure ring homomorphism $R \to A$ may not be nilpotent. Also, if R is a Gorenstein ring and $\operatorname{Ext}^i_R(A, R) = 0$ for $i > 0$, then A has Gorenstein dimension zero as an R-module.

Lemma 3.6. *Assume* $\operatorname{Ext}^i_R(A, R) = 0$ *for* $i > 0$. *Then the following hold.*

(1) *For an injective resolution* $R \to I^\bullet$ *in* Mod-R, *we have an injective resolution* $D({}_AA) \to \operatorname{Hom}^\bullet_R({}_AA, I^\bullet)$ *in* Mod-A.

(2) *For any* $X \in$ Mod-A, *we have* $\operatorname{Ext}^i_A(X, DA) \simeq \operatorname{Ext}^i_R(X, R)$ *for all* $i \geq 0$.

(3) *If* R *is an equidimensional Gorenstein ring, then* $\operatorname{inj\,dim} D({}_AA) = \dim R$.

Proposition 3.7. *Assume* R *is a Gorenstein ring with* $\dim R < \infty$ *and* A *has Gorenstein dimension zero as an* R*-module. Then the following hold.*

(1) $\operatorname{proj\,dim} D({}_AA) < \infty$ *if and only if* $\operatorname{inj\,dim} {}_AA < \infty$.

(2) $D({}_AA)$ *is a tilting module if and only if* $\operatorname{inj\,dim} {}_AA = \operatorname{inj\,dim} A_A < \infty$.

(3) *If* $\operatorname{add}(D({}_AA)) = \mathcal{P}_A$, *then* $\operatorname{inj\,dim} {}_AA = \operatorname{inj\,dim} A_A \leq \dim R$.

(4) *For a minimal injective resolution* $R \to I^\bullet$ *in* Mod-R, $\operatorname{Hom}^\bullet_R(A, I^\bullet) \in \mathsf{D}^{\mathrm{b}}(\text{Mod-}A^{\mathrm{e}})$ *is a dualizing complex for* A.

Proposition 3.8. *Assume* R *is a Gorenstein ring,* A *has Gorenstein dimension zero as an* R*-module and* ${}_AA \in \operatorname{add}(D(A_A))$. *Then for any nonzero* $P^\bullet \in \mathsf{K}^-(\mathcal{P}_A)$ *we have* $\operatorname{Hom}_{\mathsf{K}(\text{Mod-}A)}(P^\bullet, A[i]) \neq 0$ *for some* $i \in \mathbb{Z}$.

Proposition 3.9. *Assume* R *is a Gorenstein ring,* A *has Gorenstein dimension zero as an* R*-module,* $\operatorname{add}(D({}_AA)) = \mathcal{P}_A$ *and* $A_\mathfrak{p}$ *is projective as an* $R_\mathfrak{p}$*-module for all* $\mathfrak{p} \in \operatorname{Spec} R$ *with* $\dim R_\mathfrak{p} < \dim R$. *Then* A *satisfies the Auslander condition.*

Now, we propose to define the notion of Gorenstein algebras as follows.

Definition 3.10. Assume R is a Gorenstein ring. A noetherian R-algebra A is called a Gorenstein R-algebra if A has Gorenstein dimension zero as an R-module, $\operatorname{add}(D({}_AA)) = \mathcal{P}_A$ and $A_\mathfrak{p}$ is projective as an $R_\mathfrak{p}$-module for all $\mathfrak{p} \in \operatorname{Spec} R$ with $\dim R_\mathfrak{p} < \dim R$. In particular, if A is projective as an R-module and $\operatorname{add}(D({}_AA)) = \mathcal{P}_A$, then A is a Gorenstein R-algebra.

Remark 3.11. Assume R is a Gorenstein ring and A is a Gorenstein R-algebra. Then the following hold.

(1) If $\dim R = \infty$, then A is projective as an R-module.

(2) For any $\mathfrak{p} \in \operatorname{Spec} R$ with $A_\mathfrak{p} \neq 0$, $A_\mathfrak{p}$ is a Gorenstein $R_\mathfrak{p}$-algebra.

Consider the case where R is an equidimensional Gorenstein ring and $A_\mathfrak{p} \neq 0$ for all $\mathfrak{p} \in \operatorname{Spec} R$. Then a Gorenstein R-algebra A with $A \simeq DA$ in Mod-A^{e} is a Gorenstein R-order in the sense of [1, Chapter III, Section 1].

4 Derived equivalences in Gorenstein algebras

In this section, for a tilting complex $P^\bullet$ over a Gorenstein R-algebra A we ask when $B = \mathrm{End}_{\mathsf{K}(\mathrm{Mod}\text{-}A)}(P^\bullet)$ is also a Gorenstein R-algebra. This question does not seem to depend on the base ring R. So we assume R is an arbitrary commutative ring unless otherwise stated.

We fix a nonzero $P^\bullet \in \mathsf{K}^{\mathrm{b}}(\mathcal{P}_A)$ with $\mathrm{Hom}_{\mathsf{K}(\mathrm{Mod}\text{-}A)}(P^\bullet, P^\bullet[i]) = 0$ for $i \neq 0$. Set $B = \mathrm{End}_{\mathsf{K}(\mathrm{Mod}\text{-}A)}(P^\bullet)$ and $X^\bullet = \mathrm{Hom}^\bullet_A(P^\bullet, P^\bullet) \in \mathsf{K}^{\mathrm{b}}(\mathrm{add}(A_R))$. Since $\mathrm{H}^i(X^\bullet) = 0$ for $i \neq 0$, we have exact sequences of the form

$$(*) \qquad 0 \to \mathrm{Z}^0(X^\bullet) \to X^0 \to \cdots \to X^l \to 0,$$

$$(**) \quad 0 \to X^{-l} \to \cdots \to X^{-1} \to \mathrm{Z}^0(X^\bullet) \to B \to 0.$$

Lemma 4.1. *Assume* $\mathrm{Ext}^i_R(A, R) = 0$ *for* $i > 0$. *Then the following are equivalent.*

(1) $\mathrm{Ext}^i_R(B, R) = 0$ *for* $i > 0$.

(2) $\nu P^\bullet \in \mathcal{C}(P^\bullet)$.

Lemma 4.2. *Assume A has Gorenstein dimension zero as an R-module. Then the following are equivalent.*

(1) B has Gorenstein dimension zero as an R-module.

(2) $\nu P^\bullet \in \mathcal{C}(P^\bullet)$.

Lemma 4.3. *Assume A is finitely generated projective as an R-module. Then the following are equivalent.*

(1) B is finitely generated projective as an R-module.

(2) $\nu P^\bullet \in \mathcal{C}(P^\bullet)$.

Lemma 4.4. *Assume R is noetherian and A is finitely generated as an R-module. Then for any* $\mathfrak{p} \in \mathrm{Spec}\, R$ *with $A_\mathfrak{p}$ projective as an $R_\mathfrak{p}$-module the following are equivalent.*

(1) $B_\mathfrak{p}$ is projective as an $R_\mathfrak{p}$-module.

(2) $\mathrm{Hom}_{\mathsf{K}(\mathrm{Mod}\text{-}A)}(P^\bullet, \nu P^\bullet[i])_\mathfrak{p} = 0$ *for* $i \neq 0$, *this is the case if* $\nu P^\bullet \in \mathcal{C}(P^\bullet)$.

Theorem 4.5. *Assume* $A \simeq DA$ *in* $\mathrm{Mod}\text{-}A^{\mathrm{e}}$ *and A has Gorenstein dimension zero as an R-module. Then the following hold.*

(1) $B \simeq DB$ *in* $\mathrm{Mod}\text{-}B^{\mathrm{e}}$ *and B has Gorenstein dimension zero as an R-module.*

(2) If A is finitely generated projective as an R-module, so is B.

(3) *Assume R is noetherian and A is finitely generated as an R-module. Then for any $\mathfrak{p} \in \operatorname{Spec} R$, if $A_\mathfrak{p}$ is projective as an $R_\mathfrak{p}$-module, so is $B_\mathfrak{p}$.*

Throughout the rest of this section, we assume $P^\bullet$ is a tilting complex.

Proposition 4.6. *Assume A has Gorenstein dimension zero as an R-module and $\operatorname{add}(D({}_AA)) = \mathcal{P}_A$. Then the following are equivalent.*

(1) *B has Gorenstein dimension zero as an R-module and $\operatorname{add}(D({}_BB)) = \mathcal{P}_B$.*

(2) *$\nu P^\bullet \in \mathcal{C}(P^\bullet)$ and $P^\bullet \in \mathcal{C}(\nu P^\bullet)$.*

(3) *$\operatorname{add}(P^\bullet) = \operatorname{add}(\nu P^\bullet)$.*

Proposition 4.7. *Assume A is finitely generated projective as an R-module and $\operatorname{add}(D({}_AA)) = \mathcal{P}_A$. Then the following are equivalent.*

(1) *B is finitely generated projective as an R-module and $\operatorname{add}(D({}_BB)) = \mathcal{P}_B$.*

(2) *$\nu P^\bullet \in \mathcal{C}(P^\bullet)$ and $P^\bullet \in \mathcal{C}(\nu P^\bullet)$.*

(3) *$\operatorname{add}(P^\bullet) = \operatorname{add}(\nu P^\bullet)$.*

Theorem 4.8. *Assume R is a Gorenstein ring and A is a Gorenstein R-algebra. Then the following are equivalent.*

(1) *B is a Gorenstein R-algebra.*

(2) *$\nu P^\bullet \in \mathcal{C}(P^\bullet)$ and $P^\bullet \in \mathcal{C}(\nu P^\bullet)$.*

(3) *$\operatorname{add}(P^\bullet) = \operatorname{add}(\nu P^\bullet)$.*

Example 4.9. Assume R contains a regular element c which is not a unit. Let

$$A = \begin{pmatrix} R & R \\ cR & R \end{pmatrix}$$

be a free R-algebra of rank 4 and set

$$e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad a = \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

It is easy to see that $\nu(e_1A) \simeq e_2A$ and $\nu(e_2A) \simeq e_1A$. In particular, $D({}_AA) \simeq A_A$. Set $P_1^\bullet = e_1A[1]$ and let $P_2^\bullet$ be the mapping cone of $h : e_1A \to e_2A, x \mapsto ax$. Then $\operatorname{Cok} h \simeq R/cR$ in Mod-R and $\operatorname{Hom}_R(\operatorname{Cok} h, e_1A) = 0$. Thus $\operatorname{Hom}_A(\operatorname{Cok} h, e_1A) = 0$ and by [10, Proposition 1.2] $P^\bullet = P_1^\bullet \oplus P_2^\bullet \in \mathsf{K}^{\mathrm{b}}(\mathcal{P}_A)$ is a tilting complex. On the other hand, $\nu P_2^\bullet$ is isomorphic to the mapping cone of the homomorphism $e_2A \to e_1A, x \mapsto bx$, and hence $\operatorname{Hom}_{\mathsf{K}(\text{Mod-}A)}(P_1^\bullet, \nu P_2^\bullet[1]) \neq 0$. Thus $\nu P^\bullet \notin \mathcal{C}(P^\bullet)$ and by Lemma 4.1 $\operatorname{Ext}_R^1(B, R) \neq 0$, where $B = \operatorname{End}_{\mathsf{K}(\text{Mod-}A)}(P^\bullet)$. More precisely, we have an R-algebra isomorphism

$$B \simeq \begin{pmatrix} R & R/cR \\ 0 & R/cR \end{pmatrix}.$$

Note that if R is a Gorenstein ring then A is a Gorenstein R-algebra.

At present, we do not have any example of tilting complexes $P^\bullet$ over a Gorenstein R-algebra A such that $\nu P^\bullet \in \mathcal{C}(P^\bullet)$ and $\mathrm{add}(P^\bullet) \neq \mathrm{add}(\nu P^\bullet)$.

Proposition 4.10. *Assume A, B have Gorenstein dimension zero as R-modules. Then the following hold.*

(1) A is finitely generated projective if and only if so is B.

(2) Assume R is noetherian and A, B are finitely generated as R-modules. Then for any $\mathfrak{p} \in \mathrm{Spec}\ R$, $A_\mathfrak{p}$ is projective if and only if so is $B_\mathfrak{p}$.

(3) If $\mathrm{add}(D({}_AA)) = \mathcal{P}_A$, then $D({}_BB)$ is a tilting module.

5 Suitable tilting complexes

Throughout this section, R is noetherian and A is finitely generated as an R-module. Following [11], we provide a way to construct tilting complexes $T^\bullet \in \mathsf{K}^\mathrm{b}(\mathcal{P}_A)$ such that $\mathrm{add}(T^\bullet) = \mathrm{add}(\nu T^\bullet)$.

Lemma 5.1. *Let $T^\bullet \in \mathsf{K}^\mathrm{b}(\mathcal{P}_A)$ be a tilting complex. Let $P^\bullet \in \mathsf{K}^\mathrm{b}(\mathcal{P}_A)$ be a nonzero complex with $\mathrm{Hom}_{\mathsf{K}(\text{Mod-}A)}(P^\bullet, P^\bullet[i]) = 0$ for $i \neq 0$ and form a distinguished triangle in $\mathsf{K}^\mathrm{b}(\mathcal{P}_A)$*

$$Q^\bullet \to P^{\bullet(n)} \xrightarrow{f} T^\bullet \to$$

such that $\mathrm{Hom}_{\mathsf{K}(\text{Mod-}A)}(P^\bullet, f)$ is epic. Then $Q^\bullet \oplus P^\bullet$ is a tilting complex if the following conditions are satisfied:

(1) $\mathrm{Hom}_{\mathsf{K}(\text{Mod-}A)}(P^\bullet, T^\bullet[i]) = 0$ for $i > 0$ and $i < -1$;

(2) $\mathrm{Hom}_{\mathsf{K}(\text{Mod-}A)}(T^\bullet, P^\bullet[i]) = 0$ for $i > 1$;

(3) $P^\bullet \in \mathrm{add}(\nu P^\bullet)$; and

(4) $\mathrm{Ext}^i_R(A, R) = 0$ for $1 \leq i < a(Q^\bullet) - b(P^\bullet) - 1$.

Throughout the rest of this section, we fix a sequence of idempotents $e_0, e_1, \cdots$ in A such that $\mathrm{add}(e_0A_A) = \mathcal{P}_A$ and $e_{i+1} \in e_iAe_i$ for all $i \geq 0$. We will construct inductively a sequence of complexes $T_0^\bullet$, $T_1^\bullet$, $\cdots$ in $\mathsf{K}^\mathrm{b}(\mathcal{P}_A)$ as follows. Set $T_0^\bullet = e_0A$. Let $k \geq 1$ and assume $T_0^\bullet$, $T_1^\bullet$, $\cdots$, $T_{k-1}^\bullet$ have been constructed. Then we form a distinguished triangle in $\mathsf{K}^\mathrm{b}(\mathcal{P}_A)$

$$Q_k^\bullet \to e_kA^{(n_k)} \xrightarrow{f_k} T_{k-1}^\bullet \to$$

such that $\mathrm{Hom}_{\mathsf{K}(\text{Mod-}A)}(e_kA, f_k)$ is epic and set $T_k^\bullet = Q_k^\bullet \oplus e_kA$.

Lemma 5.2. *For any $l \geq 0$ the following hold.*

(1) $T_l^i = 0$ for $i > l$ and $i < 0$.

(2) $T_l^i \in \mathrm{add}(e_{l-i}A_A)$ *for* $0 \le i \le l$.

(3) $\mathrm{Hom}_{\mathsf{K}(\text{Mod-}A)}(e_l A, T_l^\bullet[i]) = 0$ *for* $i > 0$.

(4) $\mathrm{add}(T_l^\bullet)$ *generates* $\mathsf{K}^{\mathrm{b}}(\mathcal{P}_A)$ *as a triangulated category.*

Lemma 5.3. *For any* $l \ge 1$ *the following hold.*

(1) $\mathrm{H}^j(T_l^\bullet) \in \text{Mod-}(A/Ae_{l-i}A)$ *for* $0 \le i < j \le l$.

(2) *If* $D(e_i A_A) \in \mathrm{add}({}_A Ae_i)$ *for* $1 \le i \le l$, *then* $\mathrm{H}^j(\nu T_l^\bullet) \in \text{Mod-}(A/Ae_{l-i}A)$ *for* $0 \le i < j \le l$.

Lemma 5.4 ([11, Lemma 1.11(1)]). *Let* $l \ge 1$. *Let* $T^\bullet \in \mathsf{K}^{\mathrm{b}}(\mathcal{P}_A)$ *with* $T^i = 0$ *for* $i > l$ *and* $i < 0$ *and with* $T^i \in \mathrm{add}(e_{l-i}A_A)$ *for* $0 \le i \le l$. *Then for any* $S^\bullet \in \mathsf{K}^{\mathrm{b}}(\mathcal{P}_A)$ *with* $S^i = 0$ *for* $i > l$ *and* $i < 0$ *and with* $\mathrm{H}^j(S^\bullet) \in \text{Mod-}(A/Ae_{l-i}A)$ *for* $0 \le i < j \le l$, *we have* $\mathrm{Hom}_{\mathsf{K}(\text{Mod-}A)}(T^\bullet, S^\bullet[i]) = 0$ *for* $i > 0$.

Lemma 5.5 ([11, Remark 2.3]). *Let* $l \ge 0$. *For any* $T^\bullet \in \mathsf{K}^{\mathrm{b}}(\mathcal{P}_A)$, $\mathrm{add}(T^\bullet)$ *is uniquely determined if the following conditions are satisfied:*

(1) $T^i = 0$ *for* $i > l$ *and* $i < 0$;

(2) $T^i \in \mathrm{add}(e_{l-i}A_A)$ *for* $0 \le i \le l$;

(3) $\mathrm{H}^j(T^\bullet) \in \text{Mod-}(A/Ae_{l-i}A)$ *for* $0 \le i < j \le l$; *and*

(4) $\mathrm{add}(T^\bullet)$ *generates* $\mathsf{K}^{\mathrm{b}}(\mathcal{P}_A)$ *as a triangulated category.*

Theorem 5.6. *Let* $l \ge 1$ *and assume* $\mathrm{Ext}_R^i(A, R) = 0$ *for* $1 \le i < l-1$. *Then the following hold.*

(1) *If* $e_i A_A \in \mathrm{add}(D({}_A Ae_i))$ *for* $1 \le i \le l$, *then* $T_l^\bullet$ *is a tilting complex.*

(2) *If* $\mathrm{add}(e_i A_A) = \mathrm{add}(D({}_A Ae_i))$ *for* $1 \le i \le l$ *and* $D(e_i A_A) \in \mathrm{add}({}_A Ae_i)$ *for* $1 \le i \le l$, *then* $\nu T_l^\bullet \in \mathcal{C}(T_l^\bullet)$.

(3) *If* $\mathrm{add}(e_i A_A) = \mathrm{add}(D({}_A Ae_i))$ *for* $0 \le i \le l$ *and* $D(e_i A_A) \in \mathrm{add}({}_A Ae_i)$ *for* $1 \le i \le l$, *then* $\nu T_l^\bullet \in \mathrm{add}(T_l^\bullet)$.

(4) *If* A *is reflexive as an* R*-module and* $\mathrm{add}(e_i A_A) = \mathrm{add}(D({}_A Ae_i))$ *for* $0 \le i \le l$, *then* $\mathrm{add}(T_l^\bullet) = \mathrm{add}(\nu T_l^\bullet)$.

The next lemma enables us to make use of induction in calculating the endomorphism algebra of $T_l^\bullet$.

Lemma 5.7. *Let* $T^\bullet \in \mathsf{K}^{\mathrm{b}}(\mathcal{P}_A)$ *be a tilting complex and* $B = \mathrm{End}_{\mathsf{K}(\text{Mod-}A)}(T^\bullet)$. *Let* $P^\bullet \in \mathsf{K}^{\mathrm{b}}(\mathcal{P}_A)$ *be a direct summand of* $T^\bullet$ *and* $e \in B$ *an idempotent corresponding to* $P^\bullet$. *Form a distinguished triangle in* $\mathsf{K}^{\mathrm{b}}(\mathcal{P}_A)$

$$Q^\bullet \to P^{\bullet(n)} \xrightarrow{f} T^\bullet \to$$

such that $\mathrm{Hom}_{\mathsf{K}(\text{Mod-}A)}(P^\bullet, f)$ *is epic and a distinguished triangle in* $\mathsf{K}^{\mathrm{b}}(\mathcal{P}_B)$

$$S^\bullet \to eB^{(m)} \xrightarrow{g} B \to$$

such that $\mathrm{Hom}_B(eB, g)$ *is epic. Then the following hold.*

(1) $\mathrm{End}_{\mathsf{K}(\text{Mod-}A)}(Q^\bullet \oplus P^\bullet)$ *is Morita equivalent to* $\mathrm{End}_{\mathsf{K}(\text{Mod-}B)}(S^\bullet \oplus eB)$.

(2) Assume $\mathrm{Ext}^i_R(A, R) = 0$ *for* $1 \le i \le l(T^\bullet)$. *If* $\mathrm{add}(P^\bullet) = \mathrm{add}(\nu P^\bullet)$, *then* $\mathrm{add}(eB_B) = \mathrm{add}(D({}_B Be))$.

Remark 5.8. In case A is finitely generated projective as an R-module, according to Lemma 1.8, we do not need to assume R is noetherian.

6 Two-sided tilting complexes

Throughout this and the next sections, R is a complete noetherian local ring with the maximal ideal $\mathfrak{m}$ and A is finitely generated free as an R-module. For a tilting complex $P^\bullet \in \mathsf{K}^{\mathrm{b}}(\mathcal{P}_A)$ as in Theorem 5.6(4), we show that $B = \mathrm{End}_{\mathsf{K}(\text{Mod-A})}(P^\bullet)$ is free as an R-module and then construct a two-sided tilting complex corresponding to $P^\bullet$. To do so, according to Lemma 5.7, we have only to deal with tilting complexes of length 1. Namely, we will show that the construction of two-sided tilting complexes in [11, Sections 4 and 5] remains valid; but, of course, we have to modify the argument in several places. Note that all the R-algebras to be considered are semiperfect (see Lemma 1.10).

Let $\{e_1, \cdots, e_n\}$ be a basic set of orthogonal local idempotents in A. We fix a nonempty subset I_0 of $I = \{1, \cdots, n\}$ and define $S^\bullet \in \mathsf{K}^{\mathrm{b}}(\text{Mod-}A^{\mathrm{e}})$ as the mapping cone of the multiplication map

$$\rho : \bigoplus_{i \in I_0} Ae_i \otimes_R e_i A \to A.$$

Set $e = \sum_{i \in I_0} e_i$, $B = \mathrm{End}_{\mathsf{K}(\text{Mod-A})}(S^\bullet)$ and $d_{ij} = \mathrm{rank}_R\ e_i A e_j$, the rank of $e_i A e_j$ as a free R-module, for $i, j \in I_0$. We assume the following conditions are satisfied:

(a_1) there exists a permutation σ of I_0 such that $e_i A_A \simeq D({}_A A e_{\sigma(i)})$ for all $i \in I_0$;

(a_2) $e_i A e_i \neq e_i R$ for any $i \in I_0$ with $i = \sigma(i)$; and

(a_3) $e_i A e_i / e_i J e_i \simeq R/\mathfrak{m}$ for all $i \in I_0$, where J is the Jacobson radical of A.

Remark 6.1. For any $i, j \in I_0$ the following hold.

(1) $e_i A e_j \simeq D(e_j A e_{\sigma(i)}) \simeq e_{\sigma(i)} A e_{\sigma(j)}$.

(2) ${}_A\mathrm{Hom}_A(Ae_{\sigma(j)} \otimes_R e_{\sigma(i)} A_A, A_A)_A \simeq {}_A A e_{\sigma(i)} \otimes_R e_j A_A \simeq D({}_A A e_{\sigma(j)} \otimes_R e_i A_A)$.

(3) $e_i \otimes e_j \in A^{\mathrm{e}}$ is a local idempotent.

Remark 6.2. For any $i, j \in I_0$ the following hold.

(1) $d_{ij} = d_{j,\sigma(i)} = d_{\sigma(i),\sigma(j)}$.

(2) $d_{ij} \geq 1$ if either $j = i$ or $j = \sigma(i)$.

(3) $d_{ij} \geq 2$ if $j = i = \sigma(i)$.

Remark 6.3. For any $i \in I_0$ we have $\sum_{j \in I_0} d_{ij} = \sum_{j \in I_0} d_{ji} \geq 2$.

Proposition 6.4. *The following hold.*

(1) $S^\bullet \in \mathsf{K}^{\mathrm{b}}(\mathcal{P}_A)$ *is a tilting complex.*

(2) The left multiplication of A *on each homogeneous component of* $S^\bullet$ *gives rise to an injective* R*-algebra homomorphism* $\varphi : A \to B$.

(3) ${}_A(B/A)_A \simeq \bigoplus_{i,j \in I_0} ({}_A Ae_i \otimes_R e_j A_A)^{(\alpha_{ij})}$, *where*

$$\alpha_{ij} = \begin{cases} d_{ji} - 2 & \text{if } i = j = \sigma(j), \\ d_{ji} - 1 & \text{if } j \neq \sigma(j) \text{ and } i \in \{j, \sigma(j)\}, \\ d_{ji} & \text{otherwise.} \end{cases}$$

(4) For any $i \in I_0$, $e_i B_B \simeq \bigoplus_{j \in I_0} \mathrm{Hom}_{\mathsf{K}(\mathrm{Mod}\text{-}A)}(S^\bullet, e_{\sigma(j)} A[1])^{(\mu_{ij})}$, *where*

$$\mu_{ij} = \begin{cases} d_{ji} - 1 & \text{if } i = \sigma(j), \\ d_{ji} & \text{otherwise.} \end{cases}$$

Proposition 6.5. *For any* $i \in I_0$ *there exists a local idempotent* $f_i \in e_i B e_i$ *such that* $f_i B_B \simeq \mathrm{Hom}_{\mathsf{K}(\mathrm{Mod}\text{-}A)}(S^\bullet, e_{\sigma(i)} A[1])$. *Furthermore, the following hold.*

(1) $f_i B_B \not\simeq f_j B_B$ *unless* $i = j$.

(2) $f_i B_B \simeq D({}_B B f_{\sigma(i)})$ *for all* $i \in I_0$.

(3) $f_i B f_j \simeq e_i A e_j$ *for all* $i, j \in I_0$.

(4) $e_i B_B \simeq \bigoplus_{j \in I_0} f_j {B_B}^{(\mu_{ij})}$ *for all* $i \in I_0$.

(5) $f_i B_A \simeq \bigoplus_{j \in I_0} e_j {A_A}^{(\mu_{ji})}$ *for all* $i \in I_0$.

Theorem 6.6. *The mapping cone* $T^\bullet$ *of the multiplication map*

$$\bigoplus_{i \in I_0} {}_B B f_i \otimes_R e_i A_A \to {}_B B_A$$

is a two-sided tilting complex with $T^\bullet \simeq S^\bullet$ *in* $\mathsf{K}(\mathrm{Mod}\text{-}A)$.

We will prove this in the next section (see Theorem 7.3).

Corollary 6.7. *The following are equivalent.*

(1) $\mathrm{add}(D({}_A A)) = \mathcal{P}_A$.

(2) $\mathrm{add}(D({}_B B)) = \mathcal{P}_B$.

7 Derived equivalent extension algebras

Let R and A be the same as in the preceding section. We will show that an R-algebra B containing A as a subalgebra satisfying (3) of Proposition 6.4 and (1)–(5) of Proposition 6.5 is derived equivalent to A.

More precisely, let B be an R-algebra which is finitely generated free as an R-module and contains A as a subalgebra. We fix a local idempotent $f_i \in e_iBe_i$ for each $i \in I_0$ and assume the following conditions are satisfied:

(b_1) ${}_A(B/A)_A \simeq \bigoplus_{i,j \in I_0} ({}_AAe_i \otimes_R e_jA_A)^{(\alpha_{ij})}$;

(b_2) $f_iB_B \not\simeq f_jB_B$ unless $i = j$ and $f_iB_B \simeq D({}_BBf_{\sigma(i)})$ for all $i \in I_0$;

(b_3) $f_iBf_j \simeq e_iAe_j$ for all $i, j \in I_0$;

(b_4) $e_iB_B \simeq \bigoplus_{j \in I_0} f_jB_B^{(\mu_{ij})}$ for all $i \in I_0$; and

(b_5) $f_iB_A \simeq \bigoplus_{j \in I_0} e_jA_A^{(\lambda_{ij})}$ for all $i \in I_0$.

Remark 7.1. The following hold.

(1) ${}_BBe_i \simeq \bigoplus_{j \in I_0} {}_BBf_j^{(\mu_{ij})}$ for all $i \in I_0$.

(2) ${}_ABf_i \simeq \bigoplus_{j \in I_0} {}_AAe_j^{(\lambda_{\sigma^{-1}(i),\sigma^{-1}(j)})}$ for all $i \in I_0$.

Remark 7.2. For any $i, j \in I_0$, $f_i \otimes e_j \in B^{\mathrm{op}} \otimes_R A$ and $e_i \otimes f_j \in A^{\mathrm{op}} \otimes_R B$ are local idempotents.

Theorem 7.3. *Denote by $T^\bullet$ the mapping cone of the multiplication map*

$$\delta : \bigoplus_{i \in I_0} {}_BBf_i \otimes_R e_iA_A \to {}_BB_A.$$

Then $T^\bullet$ is a two-sided tilting complex with $T^\bullet \simeq S^\bullet$ in $\mathsf{K}(\mathrm{Mod}\text{-}A)$ if

$$\alpha_{ij} = \begin{cases} d_{ji} - 2 & \text{if } i = j = \sigma(j), \\ d_{ji} - 1 & \text{if } j \neq \sigma(j) \text{ and } i \in \{j, \sigma(j)\}, \\ d_{ji} & \text{otherwise,} \end{cases}$$

$$\mu_{ij} = \lambda_{ji} = \begin{cases} d_{ji} - 1 & \text{if } i = \sigma(j), \\ d_{ji} & \text{otherwise.} \end{cases}$$

8 Partial tilting complexes

Throughout this section, R is noetherian and A is finitely generated as an R-module. We fix a nonzero $P^\bullet \in \mathsf{K}^{\mathrm{b}}(\mathcal{P}_A)$ with $\mathrm{Hom}_{\mathsf{K}(\mathrm{Mod}\text{-}A)}(P^\bullet, P^\bullet[i]) = 0$ for $i \neq 0$ and ask when $P^\bullet$ appears as a direct summand of a tilting complex. Set $l = l(P^\bullet)$. We may assume $a(P^\bullet) = l$ and $b(P^\bullet) = 0$. In case $l = 0$, by Remark 1.7 $P^\bullet \simeq \mathrm{H}^0(P^\bullet)$ in $\mathsf{K}^{\mathrm{b}}(\mathcal{P}_A)$ and the question is trivial. So we assume $l \geq 1$.

Following [15, Section 4], we will construct inductively a sequence of complexes $Q_0^\bullet$, $Q_1^\bullet$, $\cdots$ in $\mathsf{K}^{\mathrm{b}}(\mathcal{P}_A)$ as follows. Set $Q_0^\bullet = A$. Let $k \geq 0$ and assume $Q_0^\bullet$, $\cdots$, $Q_k^\bullet$ have been constructed. Then we form a distinguished triangle in $\mathsf{K}^{\mathrm{b}}(\mathcal{P}_A)$

$$Q_{k+1}^\bullet \to P^{\bullet(n_k)} \xrightarrow{f_k} Q_k^\bullet \to$$

such that $\mathrm{Hom}_{\mathsf{K}(\mathrm{Mod}\text{-}A)}(P^\bullet, f_k)$ is epic.

Lemma 8.1. *For any $k \geq 1$ the following hold.*

(1) $a(Q_k^\bullet) \leq k + l - 1$ *and* $b(Q_k^\bullet) \geq 0$.

(2) $\mathrm{Hom}_{\mathsf{K}(\mathrm{Mod}\text{-}A)}(P^\bullet, Q_k^\bullet[i]) = 0$ *for* $i > 0$.

(3) $\mathrm{Hom}_{\mathsf{K}(\mathrm{Mod}\text{-}A)}(Q_k^\bullet, Q_k^\bullet[i]) = 0$ *for* $i \geq l$.

(4) If $\mathrm{Ext}_R^i(A, R) = 0$ *for* $1 \leq i < k+l-2$, *then* $\mathrm{Hom}_{\mathsf{K}(\mathrm{Mod}\text{-}A)}(Q_k^\bullet, \nu P^\bullet[i]) = 0$ *for* $i < 0$.

Lemma 8.2. *For any $k \geq l$ the following hold.*

(1) $\mathrm{Hom}_{\mathsf{K}(\mathrm{Mod}\text{-}A)}(P^\bullet, Q_k^\bullet[i]) = 0$ *for* $i < 0$.

(2) $\mathrm{Hom}_{\mathsf{K}(\mathrm{Mod}\text{-}A)}(Q_k^\bullet, P^\bullet[i]) = 0$ *for* $i > 0$.

Lemma 8.3. *Assume $l \geq 2$. Then for any $k \geq l$ the following are equivalent.*

(1) $\mathrm{Hom}_{\mathsf{K}(\mathrm{Mod}\text{-}A)}(Q_k^\bullet, Q_k^\bullet[i]) = 0$ *for* $1 \leq i < l$.

(2) $\mathrm{H}^l(f_i)$ *is epic for* $1 \leq i < l$.

(3) $a(Q_l^\bullet) \leq l$.

(4) $a(Q_k^\bullet) \leq k$.

Theorem 8.4. *Let $k \geq l$ and assume $\mathrm{Ext}_R^i(A, R) = 0$ for $1 \leq i < k + l - 2$. If $P^\bullet \in \mathrm{add}(\nu P^\bullet)$, then the following are equivalent.*

(1) $Q_k^\bullet \oplus P^\bullet$ *is a tilting complex.*

(2) $a(Q_l^\bullet) \leq l$, *this is the case if* $l = 1$.

Proposition 8.5 (cf. [7, Lemma of 1.2]). *Assume $\mathrm{H}^i(P^\bullet) = 0$ for $i \neq l$. Then the following are equivalent.*

(1) $Q_l^\bullet \oplus P^\bullet$ *is a tilting complex with* $\mathrm{H}^i(Q_l^\bullet \oplus P^\bullet) = 0$ *for* $i \neq l$.

(2) $a(Q_l^\bullet) \leq l$, *this is the case if* $l = 1$.

Remark 8.6. In case A is finitely generated projective as an R-module, according to Lemma 1.8, we do not need to assume R is noetherian.

References

[1] M. Auslander, Functors and morphisms determined by objects, in: *Representation theory of algebras (Proc. Conf., Temple Univ., Philadelphia, Pa., 1976)*, 1–244, Lecture Notes in Pure Appl. Math., 37, Dekker, New York, 1978.

[2] M. Auslander and M. Bridger, *Stable module theory*, Mem. Amer. Math. Soc., 94, Amer. Math. Soc., Providence, R.I., 1969.

[3] H. Bass, Finitistic dimension and a homological generalization of semi-primary rings, Trans. Amer. Math. Soc. **95** (1960), 466–488.

[4] H. Bass, On the ubiquity of Gorenstein rings, Math. Z. **82** (1963), 8–28.

[5] J.-E. Björk and E. K. Ekström, Filtered Auslander-Gorenstein rings, in: *Operator algebras, unitary representations, enveloping algebras, and invariant theory (Paris, 1989)*, 425–448, Progr. Math., 92, Birkhäuser, Boston, MA., 1990.

[6] M. Bökstedt and A. Neeman, Homotopy limits in triangulated categories, Compositio Math. **86** (1993), no. 2, 209–234.

[7] K. Bongartz, Tilted algebras, in: *Representations of algebras (Puebla, 1980)*, 26–38, Lecture Notes in Math., 903, Springer, Berlin, 1982.

[8] H. Cartan and S. Eilenberg, *Homological algebra*, Princeton Univ. Press, Princeton, N. J., 1956.

[9] R. Hartshorne, *Residues and duality*, Lecture Notes in Math., 20, Springer, Berlin, 1966.

[10] M. Hoshino and Y. Kato, Tilting complexes defined by idempotents, Comm. Algebra **30** (2002), no. 1, 83–100.

[11] M. Hoshino and Y. Kato, Tilting complexes associated with a sequence of idempotents, J. Pure Appl. Algebra **183** (2003), no. 1-3, 105–124.

[12] H. Matsumura, *Commutative ring theory*, Translated from the Japanese by M. Reid, Cambridge Univ. Press, Cambridge, 1986.

[13] J. Miyachi, Derived categories and Morita duality theory, J. Pure Appl. Algebra **128** (1998), no. 2, 153–170.

[14] Y. Miyashita, Tilting modules of finite projective dimension, Math. Z. **193** (1986), no. 1, 113–146.

[15] J. Rickard, Morita theory for derived categories, J. London Math. Soc. (2) **39** (1989), no. 3, 436–456.

[16] J. Rickard, Derived equivalences as derived functors, J. London Math. Soc. (2) **43** (1991), no. 1, 37–48.

[17] J. L. Verdier, *Catégories dérivées, état 0*, in: *Cohomologie étale*, 262–311, Lecture Notes in Math., 569, Springer, Berlin, 1977.

[18] A. Zaks, Injective dimension of semi-primary rings, J. Algebra **13** (1969), 73–86.

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