LIFTING OF HOLOMORPHIC MODULAR FORMS

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Abstract. In this article, we are going to discuss some lifting of elliptic cusp forms to Siegel or Hermitian modular forms. The Fourier coefficients of these lifting can be explicitly given and the Fourier coefficient formula is similar to that of Eisenstein series.

§1. Siegel modular forms

Let $\mathfrak{h}_m = \{Z \in \mathrm{M}_m(\mathbb{C}) \,|\, Z = {}^tZ, \mathrm{Im}(Z) > 0\}$ be the Siegel upper half space. The symplectic group

$$\mathrm{Sp}_m(\mathbb{Z}) = \left\{ g \in \mathrm{M}_{2m}(\mathbb{Z}) \middle| g \begin{pmatrix} \mathbf{0}_m & -\mathbf{1}_m \\ \mathbf{1}_m & \mathbf{0}_m \end{pmatrix} {}^tg = \begin{pmatrix} \mathbf{0}_m & -\mathbf{1}_m \\ \mathbf{1}_m & \mathbf{0}_m \end{pmatrix} \right\}$$

acts by $g\langle Z\rangle = (AZ+B)(CZ+D)^{-1}$ for $g = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$. A holomorphic function $F(Z)$ on $\mathfrak{h}_m$ is called a Siegel modular form of weight k if

$$F(g\langle Z\rangle) = F(Z)\det(CZ+D)^k$$

for any $g = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{Sp}_m(\mathbb{Z})$. (If $m = 1$, one needs an addtional condition of holomorphy at the cusp.) Put

$$\mathcal{S}_m(\mathbb{Z}) = \{B = {}^tB = (b_{ij}) \in \tfrac{1}{2}\mathrm{M}_m(\mathbb{Z}), b_{ii} \in \mathbb{Z}, \ (1 \le i \le m)\}$$
$$\mathcal{S}_m(\mathbb{Z})^+ = \{B \in \mathcal{S}_m(\mathbb{Z}) \,|\, B > 0\}.$$

Then a Siegel modular form $F(Z)$ has a Fourier expansion of the form

$$F(Z) = \sum_{\substack{B \in \mathcal{S}_m(\mathbb{Z}) \\ B \ge 0}} A(B)\exp(2\pi\sqrt{-1}\,\mathrm{tr}(BZ)).$$

$A(B) \in \mathbb{C}$ is called the B-th coefficient of $F(Z)$. A Siegel modular form $F(Z)$ is called a cusp form if $A(B) = 0$ unless $B \in \mathcal{S}_m(\mathbb{Z})^+$.

The space of Siegel modular forms and Siegel cusp forms of weight k are denoted by $M_k(\mathrm{Sp}_m(\mathbb{Z}))$ and $S_k(\mathrm{Sp}_m(\mathbb{Z}))$, respectively.

If

$$f(\tau) = \sum_{n=0}^{\infty} a(n)q^n \in M_k(\mathrm{SL}_2(\mathbb{Z})), \quad q = e^{2\pi\sqrt{-1}\tau}$$

is a commen eigenform of Hecke operators, then $a(1) \ne 0$. It is called a normalized Hecke eigenform if $a(1) = 1$. If $f(\tau)$ is a normalized

Hecke eigenform, the L-function $L(s,f)=\sum_{N=1}^{\infty}a(N)N^{-s}$ has an Euler product

$$L(s,f)=\prod_p(1-a(p)p^{-s}+p^{2k-1-2s})^{-1}$$

The Satake parameter $\{\alpha_p,\alpha_p^{-1}\}$ is defined by

$$1-a(p)X+p^{2k-1}X^2=(1-p^{k-(1/2)}\alpha_pX)(1-p^{k-(1/2)}\alpha_p^{-1}X).$$

If a Siegel modular form $F(Z)\in M_k(\mathrm{Sp}_m(\mathbb{Z}))$ is a common eigenform of Hecke operators, the one can also define the "standard" L-function $L(s,F,\mathrm{st})$, which is a Euler procuct of degree $2m+1$. When $m=1$, the standard L-function is given by

$$\prod_p[(1-\alpha_p^2p^{-s})(1-p^{-s})(1-\alpha_p^{-2}p^{-s})]^{-1}.$$

•Fourier coefficients of Eisenstein series

Recall that the Siegel Eisenstein series $E_\kappa(Z)$ on $\mathfrak{h}_m$ is defined by

$$E_{2\kappa}^{(m)}(Z)=\sum_{\left(\begin{smallmatrix}A&B\\C&D\end{smallmatrix}\right)\in\Gamma_0\backslash\mathrm{Sp}_m(\mathbb{Z})}\det(CZ+D)^{-2\kappa},$$

where

$$\Gamma_0^{(m)}=\left\{g=\begin{pmatrix}A&B\\C&D\end{pmatrix}\in\mathrm{Sp}_m(\mathbb{Z})\middle|C=0\right\}.$$

Then we have $E_{2\kappa}^{(m)}(Z)\in M_{2\kappa}(\mathrm{Sp}_m(Z))$ if κ is sufficiently large.

Now we consider the case $m=2n$ is even and the weight 2κ is equal to $k+n$. We recall the Fourier coefficient formula for the normalized Eisenstein series

$$\mathcal{E}_{k+n}^{(2n)}(Z)=2^{-n}\zeta(1-2k-2n)\prod_{i=1}^{n}\zeta(1+2i-2k-2n)E_{k+n}^{(2n)}(Z).$$

We assume $k\equiv n \bmod 2$ and $k\gg 0$. For an element $B\in\mathcal{S}_{2n}(\mathbb{Z})^+$, put $D_B=\det(2B)$, $\mathfrak{d}_B=|\mathrm{Disc}(\mathbb{Q}(\sqrt{(-1)^nD_B}))|$, and $\mathfrak{f}_B=\sqrt{D_B\mathfrak{d}_B^{-1}}\in\mathbb{N}$. Let χ_B be the primitive Dirichlet character associated to $\mathbb{Q}(\sqrt{(-1)^nD_B}))$. We denote $\mathbf{e}(x)=\exp(2\pi\sqrt{-1}x)$.

For each prime p, let $\mathbf{e}_p:\mathbb{Q}_p\to\mathbb{C}^\times$ be the additive character of $\mathbb{Q}_p$ such that $\mathbf{e}_p(x)=\mathbf{e}(-x)$ for $x\in\mathbb{Z}[1/p]$. When x is a square matrix, we write $\mathbf{e}(x)$ for $\mathbf{e}(\mathrm{tr}(x))$ and $\mathbf{e}_p(x)$ for $\mathbf{e}_p(\mathrm{tr}(x))$.

Recall that the Siegel series for $B\in\mathcal{S}_{2n}(\mathbb{Z})^+$ is defined by

$$b_p(B,s)=\sum_{R\in\mathrm{S}_{2n}(\mathbb{Q}_p)/\mathrm{S}_{2n}(\mathbb{Z}_p)}\mathbf{e}_p(\mathrm{tr}(BR))p^{-\mathrm{ord}_p(\nu(R))s}.$$

Here $\mathcal{S}_{2n}(\mathbb{Q}_p) = \{R = {}^tR \mid R \in \mathrm{M}_{2n}(\mathbb{Q}_p)\}$, $\mathcal{S}_{2n}(\mathbb{Z}_p) = \{R = {}^tR \mid R \in \mathrm{M}_{2n}(\mathbb{Z}_p)\}$, and $\nu(R) = [R\mathbb{Z}_p^{2n} + \mathbb{Z}_p^{2n} : \mathbb{Z}_p^{2n}]$. Put

$$\gamma_p(B; X) = (1 - X)(1 - p^n \chi_B(p) X)^{-1} \prod_{i=1}^{n} (1 - p^{2i} X^2).$$

Then there exists a polynomial $F_p(B; X) \in \mathbb{Z}[X]$ such that

$$b_p(B, s) = \gamma_p(B; p^{-s}) F_p(B; p^{-s}).$$

Katsurada proved the following functional equation:

$$F_p(B; p^{-2n-1} X^{-1}) = (p^{2n+1} X^2)^{-\mathrm{ord}_p \mathfrak{f}_B} F_p(B; X).$$

In particular, we have $\deg F_p(B; X) = 2\mathrm{ord}_p \mathfrak{f}_B$.

For $B \in \mathcal{S}_{2n}(\mathbb{Z})^+$, the B-th Fourier coefficient of $\mathcal{E}_{k+n}^{(2n)}(Z)$ is equal to

$$L(1 - k, \chi_B) \mathfrak{f}_B^{2k-1} \prod_{p \mid \mathfrak{f}_B} F_p(B; p^{-k-n}).$$

Put $\tilde{F}_p(B; X) = X^{-\mathrm{ord}_p \mathfrak{f}_B} F_p(B; p^{-n-(1/2)} X)$. Then the functional equation for $F_p(B; X)$ implies $\tilde{F}_p(B; X^{-1}) = \tilde{F}_p(B; X)$. Then the Fourier coefficient is equal to

$$L(1 - k, \chi_B) \mathfrak{f}_B^{k-(1/2)} \prod_{p \mid \mathfrak{f}_B} \tilde{F}_p(B; p^{k-(1/2)}).$$

• Lifting of cusp forms (Siegel modular case)

Now we consider cusp forms. Let k be arbitrary positive integers such that $k \equiv n \bmod 2$. Recall that the Kohnen plus subspace

$$S^+_{k+(1/2)}(\Gamma_0(4)) \subset S_{k+(1/2)}(\Gamma_0(4))$$

is the space of cusp forms in $S_{k+(1/2)}(\Gamma_0(4))$ whose Fourier coefficient vanishes unless $(-1)^k N \equiv 0, 1 \bmod 4$. Then it is well-known that $S_{2k}(\mathrm{SL}_2(\mathbb{Z})) \simeq S^+_{k+(1/2)}(\Gamma_0(4))$ as Hecke modules.

Now let

$$f(\tau) = \sum_{N=1}^{\infty} a(N) q^N \in S_{2k}(\mathrm{SL}_2(\mathbb{Z}))$$

be a normalized Hecke eigenform and

$$h(\tau) = \sum_{\substack{N>0 \\ (-1)^k N \equiv 0,\, 1\ (4)}} c(N) q^N \in S^+_{k+(1/2)}(\Gamma_0(4))$$

a corresponding Hecke eigenform.

For $B \in \mathcal{S}_{2n}(\mathbb{Z})^+$, we put

$$A(B) := c(\mathfrak{d}_B)\mathfrak{f}_B^{k-(1/2)} \prod_p \tilde{F}_p(B;\alpha_p),$$

$$F(Z) := \sum_{B={}^tB>0} A(B)\mathbf{e}(BZ), \quad Z \in \mathfrak{h}_{2n}$$

Note that $\tilde{F}_p(B;\alpha_p)$ does not depend on the choice of the Satake parameter α_p.

Then our first main theorem is as follows.

Theorem 1. *Assume $k \equiv n \bmod 2$. Then $F \in S_{k+n}(\mathrm{Sp}_{2n}(\mathbb{Z}))$ and $F \not\equiv 0$. Moreover, F is a Hecke eigenform whose standard L-function is equal to*

$$L(s,F,\mathrm{st}) = \zeta(s)\prod_{i=1}^{2n} L(s+k+n-i,f).$$

§2. Hermitian modular case

Now we consider the hermitian modular case. Let $K = \mathbb{Q}(\sqrt{-\mathbf{D}})$ be an imaginary number field with the ring of intergers $\mathcal{O} = \mathcal{O}_K$. We denote the primitive Dirichlet character associated to $K/\mathbb{Q}$ by χ. Put $\mathcal{O}^\sharp = (\sqrt{-\mathbf{D}})^{-1}\mathcal{O}$.

Let $G = \mathrm{SU}(m,m)$ be the special unitary group defined by

$$G(\mathbb{Q}) = \left\{ g \in \mathrm{SL}_{2m}(K) \,\middle|\, g\begin{pmatrix} \mathbf{0}_m & -\mathbf{1}_m \\ \mathbf{1}_m & \mathbf{0}_m \end{pmatrix} {}^t\bar{g} = \begin{pmatrix} \mathbf{0}_m & -\mathbf{1}_m \\ \mathbf{1}_m & \mathbf{0}_m \end{pmatrix} \right\}$$

We put

$$\Gamma_K^{(m)} = G(\mathbb{Q}) \cap \mathrm{GL}_{2m}(\mathcal{O}),$$
$$\Gamma_{K,0}^{(m)} = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_K^{(m)} \,\middle|\, C = 0 \right\}.$$

We define the hermitian upper half space $\mathcal{H}_m$ by

$$\mathcal{H}_m = \{ Z \in M_m(\mathbb{C}) \mid \tfrac{1}{2\sqrt{-1}}(Z - {}^t\bar{Z}) > 0 \}.$$

The action of $G(\mathbb{R})$ on $\mathcal{H}_m$ is given by

$$g\langle Z\rangle = (AZ+B)(CZ+D)^{-1}, \quad Z \in \mathcal{H}_m, g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

We put

$$\Lambda_m(\mathcal{O}) = \{ h = {}^t\bar{h} = (h_{ij}) \in \tfrac{1}{\sqrt{-\mathbf{D}}}\mathrm{M}_m(\mathcal{O}) \mid h_{ii} \in \mathbb{Z} \},$$
$$\Lambda_m(\mathcal{O})^+ = \{ h \in \Lambda_m(\mathcal{O}) \mid h > 0 \}.$$

For $H \in \Lambda_m(\mathcal{O})$, $\det H \neq 0$, we put $\gamma(H) = (-\mathbf{D})^{[m/2]} \det(H)$. Then the Siegel series for H is defined by

$$b_p(H,s) = \sum_{R \in \mathcal{H}_m(K_p)/\mathcal{H}_m(\mathcal{O}_p)} \mathbf{e}_p(\mathrm{tr}(HR)) p^{-\mathrm{ord}_p(\nu(R))s}$$

for $\mathrm{Re}(s) \gg 0$. Here, $\mathcal{H}_m(K_p)$ (resp. $\mathcal{H}_m(\mathcal{O}_p)$) is the additive group of all hermitian matrices with entries in $K_p = K \otimes \mathbb{Q}_p$ (resp. $\mathcal{O}_p = \mathcal{O} \otimes \mathbb{Z}_p$).

The ideal $\nu(R) \subset \mathbb{Z}_p$ is defined as follows: Choose an element $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{SU}(m,m)(\mathbb{Q}_p)$ such that $\det D \neq 0$, $D^{-1}C = R$. Then $\nu(R) = \det(D)\mathbb{Z}_p$.

We define a polynomial $t_p(K/\mathbb{Q}; X) \in \mathbb{Z}[X]$ by

$$t_p(K/\mathbb{Q}; X) = \prod_{i=1}^{[(m+1)/2]} (1 - p^{2i}X) \prod_{i=1}^{[m/2]} (1 - p^{2i-1}\chi(p)X).$$

Then there exists a polynomial $F_p(H; X) \in \mathbb{Z}[X]$ such that

$$b_p(H,s) = t_p(K/\mathbb{Q}; p^{-s}) F_p(H; p^{-s}).$$

$$\mathrm{deg} F_p(H; X) = \mathrm{ord}_p \gamma(H).$$

Moreover, $F_p(H; X)$ satisfies the functional equation

$$F_p(H; p^{-2m}X^{-1}) = \zeta_p(H)(p^m X)^{-\mathrm{ord}_p \gamma(H)} F_p(H; X).$$

Put $\tilde{F}_p(H; X) = X^{\mathrm{ord}_p \gamma(H)} F_p(H; p^{-m}X^{-2})$. Then the following functional equation holds:

$$\begin{cases} \tilde{F}_p(H; X^{-1}) = \tilde{F}_p(H; X), & 2 \nmid m \\ \tilde{F}_p(H; X^{-1}) = \underline{\chi}_p(\gamma(H)) \tilde{F}_p(H; X), & 2 | m \\ \tilde{F}_p(H; \chi(p) X^{-1}) = \tilde{F}_p(H; X), & 2 | m, \text{ and } \chi(p) \neq 0. \end{cases}$$

Assume $k \gg 0$. Put $n = [m/2]$. We define the Eisenstein series

$$E_{2k+2n}^{(m)}(Z) = \sum_{\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma_{K,0}^{(m)} \backslash \Gamma_K^{(m)}} \det(CZ + D)^{-2k-2n}$$

and its normalization

$$\mathcal{E}_{2k+2n}^{(m)}(Z) = 2^{-m} \prod_{i=1}^{m} L(i - 2k - 2n, \chi^{i-1}) E_{2k+2n}^{(m)}(Z).$$

We first consider the case $m = 2n+1$. In this case, the H-th Fourier coefficient of $\mathcal{E}_{2k+2n}^{(2n+1)}(Z)$ is equal to

$$|\gamma(H)|^{k-(1/2)} \prod_{p | \gamma(H)} \tilde{F}_p(H; p^{-k+(1/2)})$$

for any $H \in \Lambda_{2n+1}(\mathcal{O})^+$.

Let $f(\tau) = \sum_{N=1}^{\infty} a(N)q^N \in S_{2k}(\mathrm{SL}_2(\mathbb{Z}))$ be a normalized Hecke eigenform, whose L-function is given by

$$\begin{aligned} L(f,s) &= \sum_{N=1}^{\infty} a(N)N^{-s} \\ &= \prod_p \left[(1-p^{k-(1/2)}\alpha_p p^{-s})(1-p^{k-(1/2)}\alpha_p^{-1}p^{-s})\right]^{-1} \end{aligned}$$

For each $H \in \Lambda_{2n+1}(\mathcal{O})^+$, we put

$$A(H) = |\gamma(H)|^{k-(1/2)} \prod_{p|\gamma(H)} \tilde{F}_p(H, \alpha_p).$$

We define

$$F(Z) = \sum_{H \in \Lambda_{2n+1}(\mathcal{O})^+} A(H)\mathbf{e}(HZ), \quad Z \in \mathcal{H}_{2n+1}.$$

Then we have

Theorem 2. *Assume that $m = 2n+1$ is odd. Then $F \in S_{2k+2n}(\Gamma_K^{(2n+1)})$ and $F \not\equiv 0$. Moreover, F is a Hecke eigenform.*

Now we consider the case when $m = 2n$. In this case, the H-th Fourier coefficient of $\mathcal{E}_{2k+2n}^{(2n)}(Z)$ is equal to

$$|\gamma(H)|^k \prod_{p|\gamma(H)} \tilde{F}_p(H; p^{-k})$$

for any $H \in \Lambda_{2n}(\mathcal{O})^+$.

Now let $f(\tau) = \sum_{N=1}^{\infty} a(N)q^N \in S_{2k+1}(\Gamma_0(\mathbf{D}), \chi)$ be a primitive form, whose L-function is given by

$$\begin{aligned} L(f,s) &= \sum_{N=1}^{\infty} a(N)N^{-s} \\ &= \prod_{p\nmid \mathbf{D}} (1-a(p)p^{-s}+\chi(p)p^{2k-2s})^{-1} \prod_{q|\mathbf{D}} (1-a(q)q^{-s})^{-1}. \end{aligned}$$

For each prime $p \nmid \mathbf{D}$, we define the Satake parameter $\{\alpha_p, \beta_p\} = \{\alpha_p, \chi(p)\alpha_p^{-1}\}$ by

$$(1-a(p)X+\chi(p)p^{2k}X^2) = (1-p^k\alpha_p X)(1-p^k\beta_p X).$$

For $p \mid \mathbf{D}$, we put $\alpha_p = p^{-k}a(p)$.

For each $H \in \Lambda_{2n}(\mathcal{O})^+$, we put

$$A(H) = |\gamma(H)|^k \prod_{p|\gamma(H)} \tilde{F}_p(H, \alpha_p).$$

We define

$$F(Z) = \sum_{H \in \Lambda_{2n}(\mathcal{O})^+} A(H)\mathbf{e}(HZ), \quad Z \in \mathcal{H}_{2n}.$$

The we have

Theorem 3. *Assume that $m = 2n$ is even. Then $F \in S_{2k+2n}(\Gamma_K^{(2n)})$. Moreover, F is a Hecke eigenform. $F \equiv 0$ if and only if n is odd and $f(\tau)$ comes from a Hecke character of some imaginary quadratic field.*

Under some normalization, the L-function of F is as follows. For simplicity, we assume the class number of K is one.

$$L(s, F, \rho) = \prod_{i=1}^{2n+1} L(s + k + n - i + (1/2), f) \\ \times \prod_{i=1}^{2n+1} L(s + k + n - i + (1/2), f, \chi)$$

for $m = 2n + 1$, and

$$L(s, F, \rho) = \prod_{i=1}^{2n} L(s + k + n - i + (1/2), f) \\ \times \prod_{i=1}^{2n} L(s + k + n - i + (1/2), f, \chi)$$

for $m = 2n$ and $F \not\equiv 0$. Here, ρ is a $2m$-dimensional representation of the L-group of $U(m, m)$. Note that an extenstion of F to an adelic automoraphic form on $U(m, m)(\mathbb{A})$ is not canonical unless the class number of K is one.

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