An algorithm for computing the Υ-invariant and the d-invariants of Dehn surgeries

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Abstract

By using grid homology theory, we give an explicit algorithm for computing Ozsváth-Stipsicz-Szabó's Υ-invariant and the d-invariant of Dehn surgeries along knots in S^3. As its application, we compute the two invariants for all prime knots with up to 11 crossings.

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1 Introduction

The Υ-*invariant* defined by Ozsváth, Stipsicz and Szabó [OSS17] is a group homomorphism

$$\Upsilon\colon \mathcal{C} \to \mathrm{PL}([0,2],\mathbb{R}),$$

where $\mathcal{C}$ denotes the smooth knot concordance group and $\mathrm{PL}([0,2],\mathbb{R})$ the vector space of piecewise-linear functions on $[0,2]$. The Υ-invariant is known as a very effective tool in the study of smooth knot concordance. Indeed, the slope of Υ gives a surjective homomorphism $\mathcal{C} \to \mathbb{Z}^\infty$ whose image is generated by topologically slice knots, and this proves that the subgroup of $\mathcal{C}$ generated by topologically slice knots contains a direct summand isomorphic to $\mathbb{Z}^\infty$. For a given knot K in S^3, we denote the Υ-invariant of K by Υ_K.

The d-*invariant* defined by Ozsváth and Szabó [OS03a] is a group homomorphism

$$d\colon \theta^c \to \mathbb{Q},$$

where θ^c denotes the Spin^c rational homology cobordism group of Spin^c rational homology 3-spheres (see [OS03a, Definition 1.1] for the precise definition of θ^c). The d-invariant is also known as a very powerful invariant so that it enables us

to reprove Donaldson's diagonalization theorem. Moreover, the d-invariants of Dehn surgeries and prime-powered branched covers along knots in S^3 yield an infinite family of smooth knot concordance invariants (see [Hom17] for a survey describing these facts and studies of them). For any knot K in S^3 and coprime integers p and q with $p \neq 0$ and $q > 0$, let $S^3_{p/q}(K)$ denote the p/q-surgery along K. In this paper, we focus on the d-invariant of $S^3_{p/q}(K)$. [1] Note that since $S^3_{p/q}(K)$ has $|p|$ different Spin^c structures, the d-invariant of $S^3_{p/q}(K)$ is obtained as a $|p|$-tuple of rational numbers. We denote the $|p|$-tuple by $d(S^3_{p/q}(K))$.

The invariants Υ_K and $d(S^3_{p/q}(K))$ were originally defined in different packages of Heegaard Floer theory, but later Livingston [Liv17] and Ni-Wu [NW15] have translated them into the words of the doubly filtered chain complex $CFK^\infty(K)$ defined in [OS04]. Moreover, $CFK^\infty(K)$ has a combinatorial description called *grid homology theory* introduced by Manolescu, Ozsváth and Sarkar [MOS09]. Therefore, it is natural to consider computing Υ_K and $d(S^3_{p/q}(K))$ algorithmically by using grid homology theory. However, while $CFK^\infty(K)$ is freely and finitely generated over the Laurent polynomial $\mathbb{F}[U^{\pm 1}]$ (where $\mathbb{F} := \mathbb{Z}/2\mathbb{Z}$), the combinatorial chain complex $C^\infty(G)$ corresponding to $CFK^\infty(K)$ is freely and finitely generated over $\mathbb{F}[U^{\pm 1}, U_2^{\pm 1}, \dots, U_n^{\pm 1}]$ with $n > 1$. In particular, $C^\infty(G)$ has infinite rank over $\mathbb{F}[U^{\pm 1}]$, and hence there is no obvious way to use $C^\infty(G)$ for algorithmic computation of Υ_K and $d(S^3_{p/q}(K))$. In this paper, we provide a method for avoiding the infinite-dimensional problem, and give an explicit algorithm for computing Υ_K and $d(S^3_{p/q}(K))$ of any knot K in S^3 and $p/q \in \mathbb{Q}$. Moreover, as its applications, we complete the lists of

- the Υ-invariants of all prime knots with up to 11 crossings,
- all d-invariants of all Dehn surgeries along all prime knots with up to 11 crossings, and
- all prime knots with up to 12 crossings whose Υ and d-invariants coincide with those of the unknot.

1.1 The Key idea

In order to give an algorithm for computing Υ_K and $d(S^3_{p/q}(K))$, we will translate these invariants several times. The key point is to use the second author's invariant $\mathcal{G}_0(K)$ [Sat23], from which we can easily compute both Υ_K and $d(S^3_{p/q}(K))$. While $\mathcal{G}_0(K)$ was also defined using $CFK^\infty(K)$, we show in this paper that it can be translated into the words of the subcomplex $CFK^-(K)$ over $\mathbb{F}[U]$. The combinatorial chain complex $C^-(G)$ corresponding to $CFK^-(K)$ is finitely and freely generated by $\mathbb{F}[U, U_2, \dots, U_n]$ and so it has infinite rank over $\mathbb{F}[U]$, but as a graded $\mathbb{F}$-vector space, each degree of $C^-(G)$ has finite rank. (It does not hold for $C^\infty(G)$, i.e. each degree of $C^\infty(G)$ has infinite rank over

[1] While 0-surgeries are not rational homology 3-spheres, the d-invariants of them are also defined. For the cases of knots in S^3, they are determined by $d(S^3_{\pm 1}(K))$. See [OS03a, Section 4.2].

$\mathbb{F}$.) This fact enables us to reduce the problem to finite $\mathbb{F}$-linear systems, which makes $\mathcal{G}_0(K)$ algorithmically computable.

1.2 Observations of computation results

Here we show several observations of our computation results. We first state the relationship between the *τ-invariants* [OS03b] and the *ν^+-equivalence classes* [Hom17; KP18] of knots with up to 12 crossings. Here, the *τ-invariant* is a group homomorphism $\mathcal{C} \to \mathbb{Z}$ which is already computed for all knots with up to 11 crossings [BG12]. (Now, the list of the τ-invariants for knots with up to 12 crossings is available in KnotInfo [LM].) The *ν^+-equivalence* is an equivalence relation on $\mathcal{C}$ whose quotient set can be regarded as a quotient group $\mathcal{C}_{\nu^+}$ of $\mathcal{C}$. It is proved in [Hom17] that a knot K is ν^+-equivalent to the unknot if and only if $CFK^\infty(K)$ is filtered chain homotopic to $CFK^\infty(O) \oplus A$, where O denotes the unknot and A is an acyclic complex, i.e. $H_*(A) = 0$. Moreover, all of τ, Υ_K and $d(S^3_{p/q}(K))$ are invariant under ν^+-equivalence. In particular, the triviality of K in $\mathcal{C}_{\nu^+}$ implies $\tau(K) = 0$, while the converse does not hold in general. (For instance, the composite knot $T_{2,7}\#(T_{3,4})^*$ is such an example, where $T_{p,q}$ denotes the positive (p, q) torus knot and K^* the mirror image of K.) On the other hand, it is also known that the converse holds for all quasi-alternating knots [Pet13] and all genus one knots [Sat23]. As an observation of our computation results, we show that the converse also holds for any prime knot with up to 12 crossings.

Theorem 1.1. *For any prime knot K with up to 12 crossings, K is ν^+-equivalent to the unknot if and only if $\tau(K) = 0$.*

Here we also mention that the triviality of a knot in $\mathcal{C}_{\nu^+}$ is determined by the diffeomorphism type of its 0-surgery. (The proof immediately follows from the original definition of ν^+-equivalence in [KP18] with [NW15, Proposition 1.6] and [OS03a, Proposition 4.12].)

Proposition 1.2. *If two knots K and K' have the same 0-surgeries, then K is ν^+-equivalent to the unknot if and only if K' is ν^+-equivalent to the unknot.*

Remark 1.3. The ν^+-equivalence class of a knot is not an invariant of its 0-surgery in general. Indeed, Yasui [Yas26] provides an infinite family of pairs of knots with the same 0-surgeries one of whose τ is greater than the other by 1.

It is meaningful to compare the above propositions with Piccirillo's proof of non-sliceness of the Conway knot $11n34$ [Pic20]. To prove the non-sliceness of the Conway knot, Piccirillo considers the *knot trace*, which is a 4-manifold obtained by attaching a 0-framed 2-handle to B^4 along a given knot. It is known that if two knots have the same knot traces, then the sliceness of them coincides. Piccirillo finds a knot K' whose knot trace is diffeomorphic to that of the Conway knot, and she shows $s(K') \neq 0$, where s denotes Rasmussen's homomorphism $s\colon \mathcal{C} \to 2\mathbb{Z}$ [Ras10]. Here we note that sharing the knot trace implies sharing the 0-surgeries. Moreover, since the Conway knot has 11 crossings and trivial

τ, Theorem 1.1 implies that the Conway knot is ν^+-equivalent to the unknot. Therefore, in contrast to the s-invariant, we have the following:

Theorem 1.4. *For any knot K, if the 0-surgery along K is diffeomorphic to that of the Conway knot, then K is ν^+-equivalent to the unknot. In particular, all of $\tau(K)$, Υ_K and $d(S^3_{p/q}(K))$ coincide with those of the unknot.*

In particular, we cannot replace $s(K')$ in Piccirillo's proof with any of $\tau(K')$, $\Upsilon_{K'}$ and $d(S^3_{p/q}(K'))$.

Organization

In Section 2, we review the invariant $\mathcal{G}_0(K)$. In Section 3, we translate $\mathcal{G}_0(K)$ into the words of $CFK^-(K)$. In Section 4, we review the grid complex $C^-(G)$, and translate $\mathcal{G}_0(K)$ into $C^-(G)$. In Section 5, we give an algorithm for computing $\mathcal{G}_0(K)$ from $C^-(G)$. In Section 6, we show our computation results.

This paper also contains several appendices. In Appendix A, we give the duality theorem for $\mathcal{G}_0(K)$ with respect to the mirroring of K, which proves that $\mathcal{G}_0$ for the mirror K^* is algorithmically determined from $\mathcal{G}_0(K)$. This is used for determining $d(S^3_{p/q}(K))$ where p/q is negative. In Appendix B, we discuss methods for handling sparse linear systems.

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2 A review of the invariant $\mathcal{G}_0$

In this section, we review the invariant $\mathcal{G}_0(K)$ defined in [Sat23], which determines both Υ_K and $d(S^3_{p/q}(K))$.

2.1 Knot complexes, CFK^∞

2.1.1 Poset filtered chain complexes

Let P be a *poset*, i.e. a set P with partial order $\leq$. Then, a *closed region* $R \subset P$ is a subset such that for any $x \in P$, if there exists an element $y \in R$ satisfying $x \leq y$, then $x \in R$. (In other words, a closed region is a *lower set* with respect

[2] `https://academist-cf.com/fanclubs/121`

to the partial order $\leq$.) We denote the set of closed regions of P by $\mathcal{CR}(P)$. In this paper, we mainly consider the partial order $\leq$ on $\mathbb{Z}^2$ given by $(i,j) \leq (k,l)$ if $i \leq k$ and $j \leq l$.

Let $\mathbb{F} := \mathbb{Z}/2\mathbb{Z}$ and Λ be an $\mathbb{F}$-algebra. In this paper, we say that (C, ∂) is a *chain complex C over Λ* if (C, ∂) satisfies the following:

- C is a Λ-module and $\partial \colon C \to C$ is a Λ-linear map with $\partial \circ \partial = 0$.
- As an $\mathbb{F}$-vector space, C is decomposed into $\bigoplus_{n \in \mathbb{Z}} C_n$, where each C_n satisfies $\partial(C_n) \subset C_{n-1}$.

(Note that the Λ-action does not preserve the grading in general.) We often abbreviate (C, ∂) to C. Moreover, we say that C is *P-filtered* if a subcomplex C_R of C over $\mathbb{F}$ is associated to each closed region $R \subset P$ so that if $R \subset R'$ then $C_R \subset C_{R'}$. (Here we remark that C_R is not a Λ-submodule of C in general.) We call the set $\{C_R\}_{R \in \mathcal{CR}(P)}$ a *P-filtration* on C. For instance, a $\mathbb{Z}$-filtration of C can be identified as an ascending filtration in the usual sense, under the identification of an integer i with a closed region $\{m \in \mathbb{Z} \mid m \leq i\}$. Moreover, given two $\mathbb{Z}$-filtrations $\{\mathcal{F}^1_i\}_{i \in \mathbb{Z}}$ and $\{\mathcal{F}^2_j\}_{j \in \mathbb{Z}}$ on C, the set

$$\{C_R\}_{R \in \mathcal{CR}(\mathbb{Z}^2)} := \{ \bigcup_{(i,j) \in R} \mathcal{F}^1_i \cap \mathcal{F}^2_j \}_{R \in \mathcal{CR}(\mathbb{Z}^2)}$$

defines a $\mathbb{Z}^2$-filtration on C. We call it *the $\mathbb{Z}^2$-filtration induced by the ordered pair* $(\{\mathcal{F}^1_i\}_{i \in \mathbb{Z}}, \{\mathcal{F}^2_j\}_{j \in \mathbb{Z}})$. When C is endowed with such an induced $\mathbb{Z}^2$-filtration, we denote by C^r the $\mathbb{Z}^2$-filtered chain complex with the same underlying complex C endowed with the $\mathbb{Z}^2$-filtration induced by $(\{\mathcal{F}^2_i\}, \{\mathcal{F}^1_j\})$. We call *$C^r$ the reflection of C*.

For any two P-filtered chain complexes C and C', a map $f : C \to C'$ is said to be *P-filtered* if $f(C_R) \subset C'_R$ for any closed region R. Two P-filtered chain complexes C and C' are *P-filtered homotopy equivalent over Λ* (and denoted $C \underset{P\text{-filt.}/\Lambda}{\simeq} C'$) if there exists a chain homotopy equivalence map $f : C \to C'$ over Λ, a chain homotopy inverse of f and their chain homotopies such that all of the four maps are P-filtered and graded. Such f is called a *P-filtered homotopy equivalence map over Λ*. Particularly, we call the above f a *P-filtered chain isomorphism over Λ* if f is a chain isomorphism. It is obvious that if $C \underset{P\text{-filt.}/\Lambda}{\simeq} C'$, then $C \underset{P\text{-filt.}/\Lambda'}{\simeq} C'$ for any subalgebra $\Lambda' \subset \Lambda$. Moreover, it is easy to show the following lemma.

Proposition 2.1. *Let C and C' be P-filtered chain complexes. If $C \underset{P\text{-filt.}/\Lambda}{\simeq} C'$, then for any closed regions $R \subset R'$, we have an isomorphism between the long exact sequences of Λ'-modules:*

$$\begin{array}{ccccccccc}
\cdots & \xrightarrow{\partial_*} & H_*(C_R) & \xrightarrow{i_*} & H_*(C_{R'}) & \xrightarrow{p_*} & H_*(C_{R'}/C_R) & \xrightarrow{\partial_*} & \cdots \\
 & & \cong\downarrow & & \cong\downarrow & & \cong\downarrow & & \\
\cdots & \xrightarrow{\partial_*} & H_*(C'_R) & \xrightarrow{i_*} & H_*(C'_{R'}) & \xrightarrow{p_*} & H_*(C'_{R'}/C'_R) & \xrightarrow{\partial_*} & \cdots
\end{array}$$

Here, $i\colon C_R \to C_{R'}$ (resp. $p\colon C_{R'} \to C_{R'}/C_R$) denote the inclusion (resp. the projection), and Λ' is the maximal subalgebra of Λ so that all of C_R, $C_{R'}$, C'_R and $C'_{R'}$ are Λ'-submodules of C and C', respectively. Moreover, the above isomorphism induces an isomorphism between the long exact sequences of graded $\mathbb{F}$-vector spaces:

$$\begin{array}{ccccccccc}
\cdots & \xrightarrow{\partial_{*,n+1}} & H_n(C_R) & \xrightarrow{i_{*,n}} & H_n(C_{R'}) & \xrightarrow{p_{*,n}} & H_n(C_{R'}/C_R) & \xrightarrow{\partial_{*,n}} & \cdots \\
 & & \cong\downarrow & & \cong\downarrow & & \cong\downarrow & & \\
\cdots & \xrightarrow{\partial_{*,n+1}} & H_n(C'_R) & \xrightarrow{i_{*,n}} & H_n(C'_{R'}) & \xrightarrow{p_{*,n}} & H_n(C'_{R'}/C'_R) & \xrightarrow{\partial_{*,n}} & \cdots
\end{array}$$

2.1.2 CFK^∞ and CFK^-

To each knot K in S^3, Ozsváth and Szabó [OS04] associate a $\mathbb{Z}^2$-filtered chain complex $CFK^\infty(K)$ over $\Lambda := \mathbb{F}[U, U^{-1}]$ such that if two knots K and J are isotopic, then $CFK^\infty(K) \underset{\mathbb{Z}^2\text{-filt.}/\Lambda}{\simeq} CFK^\infty(J)$. Let $C := CFK^\infty(K)$. Here we summarize the algebraic properties:

1. C is a chain complex over Λ with decomposition $C = \bigoplus_{n\in\mathbb{Z}} C_n$. The grading of a homogeneous element x is denoted $\mathrm{gr}(x)$ and called *the Maslov grading* of x.

2. C has a $\mathbb{Z}$-filtration $\{\mathcal{F}_j^{\mathrm{Alex}}\}_{j\in\mathbb{Z}}$ called *the Alexander filtration*. The filtration level of an element $x \in C$ is denoted $\mathrm{Alex}(x)$ (i.e. $\mathrm{Alex}(x) := \min\{j \in \mathbb{Z} \mid x \in \mathcal{F}_j^{\mathrm{Alex}}\}$).

3. C also has a $\mathbb{Z}$-filtration $\{\mathcal{F}_i^{\mathrm{Alg}}\}_{i\in\mathbb{Z}}$ called *the algebraic filtration*. The filtration level of an element x is denoted $\mathrm{Alg}(x)$. When we regard C as a $\mathbb{Z}^2$-filtered complex, we use the $\mathbb{Z}^2$-filtration induced by the ordered pair $(\{\mathcal{F}_i^{\mathrm{Alg}}\}_{i\in\mathbb{Z}}, \{\mathcal{F}_j^{\mathrm{Alex}}\}_{j\in\mathbb{Z}})$.

4. The action of U lowers the Maslov grading by 2, and lowers the Alexander and algebraic filtration levels by 1.

5. C is a free Λ-module with finite rank, and there exists a basis $\{x_k\}_{1\le k\le r}$ such that

   - each x_k is homogeneous with respect to the homological grading,
   - $\mathcal{F}_0^{\mathrm{Alex}}$ is a free $\mathbb{F}[U]$-module with a basis $\{U^{\mathrm{Alex}(x_k)}x_k\}_{1\le k\le r}$, and
   - $\mathcal{F}_0^{\mathrm{Alg}}$ is a free $\mathbb{F}[U]$-module with a basis $\{U^{\mathrm{Alg}(x_k)}x_k\}_{1\le k\le r}$.

   We call such $\{x_k\}_{1\le k\le r}$ a *filtered basis*.

6. There exists a $\mathbb{Z}^2$-filtered homotopy equivalence map $\iota : C \to C^r$ over Λ.

7. Regard Λ as a chain complex with trivial boundary map, and define the homological grading by

$$\Lambda_n = \begin{cases} \{0, U^{-n/2}\} & (n : \text{ even}) \\ 0 & (n : \text{ odd}) \end{cases}$$

and the Alexander and algebraic filtrations by

$$\mathcal{F}_i^{\mathrm{Alex}}(\Lambda) = \mathcal{F}_i^{\mathrm{Alg}}(\Lambda) = U^{-i} \cdot \mathbb{F}[U].$$

Then there exists a $\mathbb{Z}$-filtered homotopy equivalence map f_{Alex} (resp. f_{Alg}) $: C \to \Lambda$ over Λ with respect to the Alexander (resp. algebraic) filtrations.

Remark 2.2. The above seven conditions are used as the axioms of *formal knot complexes*, to which we can generalize the definitions of the invariants $\tau(K)$, Υ_K and $d(S^3_{p/q}(K))$. For more details, see [Sat23, Section 2].

Particularly, the subcomplex $\mathcal{F}_0^{\mathrm{Alg}}$ is denoted by $CFK^-(K)$.

2.1.3 The dual of CFK^∞

Next, we discuss the following duality theorem for CFK^∞, which will be used to prove the duality theorem for $\mathcal{G}_0$ (Theorem 2.16) and to compute $d(S^3_{p/q}(K))$ with $p/q < 0$.

Theorem 2.3 ([OS04, Proposition 3.7])**.** *For any knot K and its mirror K^*, we have*

$$CFK^\infty(K^*) \underset{\mathbb{Z}^2\text{-}\mathit{filt.}/\Lambda}{\simeq} (CFK^\infty(K))^*,$$

where

$$(CFK^\infty(K))^* := \mathrm{Hom}_\Lambda(CFK^\infty(K), \Lambda)$$

is the dual complex of $CFK^\infty(K)$.

Remark 2.4. Precisely, [OS04, Proposition 3.7] states the theorem for the knot Floer homology $\widehat{HFK}$, while the same proof can be applied to CFK^∞.

Here we explain how to write the Maslov grading and the Alexander and algebraic filtrations on $(CFK^\infty(K))^*$ in the words of $CFK^\infty(K)$. (The following arguments follow from [Sat23, Subsection 2.4].)

Let $C := CFK^\infty(K)$. Define an $\mathbb{F}$-linear map $\varepsilon : \Lambda \to \mathbb{F}$ by $\varepsilon(p(U)) = p(0)$ for each $p(U) \in \Lambda$ (i.e. ε maps a Laurent polynomial to its constant term). Then, the Maslov grading and the Alexander and algebraic filtrations on $C^* = (CFK^\infty(K))^*$ are given by

$$C_n^* = \left\{ \varphi \in C^* \;\middle|\; \varepsilon \circ \varphi(\bigoplus_{m \neq -n} C_m) = \{0\} \right\},$$

$$\mathcal{F}_j^{\mathrm{Alex}}(C^*) = \left\{ \varphi \in C^* \;\middle|\; \varepsilon \circ \varphi(\mathcal{F}_{-j-1}^{\mathrm{Alex}}) = \{0\} \right\},$$

and

$$\mathcal{F}_i^{\mathrm{Alg}}(C^*) = \left\{ \varphi \in C^* \;\middle|\; \varepsilon \circ \varphi(\mathcal{F}_{-i-1}^{\mathrm{Alg}}) = \{0\} \right\}.$$

In addition, we also note that for any filtered basis $\{x_k\}_{1 \le k \le r}$ for C, the dual basis $\{x_k^*\}_{1 \le k \le r}$ becomes a filtered basis for C^*.

Here we also mention the following lemma.

Lemma 2.5 ([Sat23, Lemma 2.17])**.** *Let $C := CFK^\infty(K)$. Then the $\mathbb{F}$-linear map $\varepsilon_n : C^*_{-n} \to \mathrm{Hom}_{\mathbb{F}}(C_n, \mathbb{F})$ defined by $\varphi \mapsto \varepsilon \circ \varphi$ is a cochain isomorphism (where we see $\{C^*_{-n}\}_{n \in \mathbb{Z}}$ as a graded cochain complex over $\mathbb{F}$). In particular, we have $\mathbb{F}$-linear isomorphisms*

$$H_{-n}(C^*) \cong H^n(C_*; \mathbb{F}) \cong \mathrm{Hom}_{\mathbb{F}}(H_n(C_*), \mathbb{F}),$$

where the first isomorphism is induced from ε_n.

2.2 ν^+-equivalence

The *ν^+-equivalence* is an equivalence relation on knots and regarded as a CFK^∞-version of knot concordance.

First, we note that the seventh property of CFK^∞ implies

$$H_*(CFK^\infty(K)) \cong \Lambda$$

and

$$H_n(CFK^\infty(K)) \cong \begin{cases} \mathbb{F} & (n\colon \text{even}) \\ 0 & (n\colon \text{odd}) \end{cases}$$

for any knot K. (This property is called *the global triviality*.) Here we consider morphisms between formal knot complexes in terms of the global triviality. For two knots K and J, a chain map $f\colon CFK^\infty(K) \to CFK^\infty(J)$ over Λ is a *$\mathbb{Z}^2$-filtered quasi-isomorphism* if f is $\mathbb{Z}^2$-filtered, graded, and induces an isomorphism $f_*\colon H_*(CFK^\infty(K)) \to H_*(CFK^\infty(J))$. Now, *the ν^+-equivalence* (or *local equivalence*) is defined as follows.

Definition 2.6. Two knots K and J are *ν^+-equivalent* (and denoted $K \overset{\nu^+}{\sim} J$) if there exist $\mathbb{Z}^2$-filtered quasi-isomorphisms $f\colon CFK^\infty(K) \to CFK^\infty(J)$ and $g\colon CFK^\infty(J) \to CFK^\infty(K)$.

Remark 2.7. Originally, the ν^+-equivalence is defined by using *Hom-Wu's ν^+-invariant* [HW16], which is a $\mathbb{Z}_{\ge 0}$-valued knot concordance invariant. Namely, two knots K and J are ν^+-equivalent if and only if the equalities $\nu^+(K \# -J^*) = \nu^+(-K^* \# J) = 0$ hold. For more details, see [Sat23, Section 2].

It is obvious that $\overset{\nu^+}{\sim}$ is an equivalence relation on knots. We call the equivalence class of a knot K under $\overset{\nu^+}{\sim}$ *the ν^+-equivalence class* or *ν^+-class of K*, and denote it by $[K]_{\nu^+}$. The quotient set of knots under $\overset{\nu^+}{\sim}$ is denoted by $\mathcal{C}_{\nu^+}$. Then, we have the following theorem.

Theorem 2.8 ([Hom17], [Sat23, Theorem 2.37]). *If two knots are concordant, then they are ν^+-equivalent. Moreover, the operation $[K]_{\nu^+}+[J]_{\nu^+} := [K\#J]_{\nu^+}$ endows $\mathcal{C}_{\nu^+}$ with an abelian group structure so that the surjective map*

$$\mathcal{C} \to \mathcal{C}_{\nu^+},\ [K] \mapsto [K]_{\nu^+}$$

is a group homomorphism.

Here we also mention a partial order on $\mathcal{C}_{\nu^+}$. For two elements $[K]_{\nu^+}, [J]_{\nu^+} \in \mathcal{C}_{\nu^+}$, we denote $[K]_{\nu^+} \leq [J]_{\nu^+}$ if there exists a $\mathbb{Z}^2$-filtered quasi-isomorphism $f\colon CFK^\infty(J) \to CFK^\infty(K)$. Then we see that the relation $\leq$ defines a partial order on $\mathcal{C}_{\nu^+}$. For the partial order, we have the following 4-genus bound.

Theorem 2.9 ([Sat23, Theorem 1.5]). *Let g_4 denote the 4-genus of a knot K. Then we have*

$$-g_4[T_{2,3}]_{\nu^+} \leq [K]_{\nu^+} \leq g_4[T_{2,3}]_{\nu^+}.$$

2.3 The invariant $\mathcal{G}_0$

Now we recall the invariant $\mathcal{G}_0$, which is given as an invariant of knots under ν^+-equivalence.

2.3.1 The invariants $\widetilde{\mathcal{G}}_0$ and $\mathcal{G}_0$

For $C := CFK^\infty(K)$, a cycle $x \in C$ is called a *homological generator (of degree 0)* if x is homogeneous with $\mathrm{gr}(x) = 0$ and the homology class $[x] \in H_0(C) \cong \mathbb{F}$ is non-zero. Then, we define

$$\widetilde{\mathcal{G}}_0(K) := \{R \in \mathcal{CR}(\mathbb{Z}^2) \mid C_R \text{ contains a homological generator}\}.$$

Any element $R \in \widetilde{\mathcal{G}}_0(K)$ is called a *realizable region of K*. The set $\widetilde{\mathcal{G}}_0(K)$ behaves naturally with respect to filtered quasi-isomorphism.

Theorem 2.10 ([Sat23, Theorem 5.1]). *If $[K]_{\nu^+} \leq [J]_{\nu^+}$, then $\widetilde{\mathcal{G}}_0(K) \supset \widetilde{\mathcal{G}}_0(J)$.*

As a corollary, we have the invariance of $\widetilde{\mathcal{G}}_0$ under $\overset{\nu^+}{\sim}$. Here $\mathcal{P}(\mathcal{CR}(\mathbb{Z}^2))$ denotes the power set of $\mathcal{CR}(\mathbb{Z}^2)$.

Corollary 2.11. *$\widetilde{\mathcal{G}}_0(K)$ is invariant under ν^+-equivalence. In particular,*

$$\widetilde{\mathcal{G}}_0\colon [K]_{\nu^+} \mapsto \widetilde{\mathcal{G}}_0(K)$$

is a well-defined map $\mathcal{C}_{\nu^+} \to \mathcal{P}(\mathcal{CR}(\mathbb{Z}^2))$.

Here we note that $\widetilde{\mathcal{G}}_0(K)$ is an infinite set for any K. To extract the essential part of $\widetilde{\mathcal{G}}_0$, we consider the minimalization of $\widetilde{\mathcal{G}}_0$.

For a subset $\mathcal{S} \subset \mathcal{CR}(\mathbb{Z}^2)$, an element $R \in \mathcal{S}$ is *minimal in $\mathcal{S}$* if it satisfies

$$\text{if } R' \in \mathcal{S} \text{ and } R' \subset R, \text{ then } R' = R.$$

Define the map

$$\min\colon \mathcal{P}(\mathcal{CR}(\mathbb{Z}^2)) \to \mathcal{P}(\mathcal{CR}(\mathbb{Z}^2))$$

by

$$\mathcal{S} \mapsto \{R \in \mathcal{S} \mid R \text{ is minimal in } \mathcal{S}\}.$$

Now we define $\mathcal{G}_0(K)$ by

$$\mathcal{G}_0(K) := \min \widetilde{\mathcal{G}}_0(K).$$

The invariance of $\mathcal{G}_0$ under $\overset{\nu^+}{\sim}$ immediately follows from Corollary 2.11. Moreover, $\mathcal{G}_0(K)$ has the following nice properties.

Theorem 2.12 ([Sat23, Theorem 5.7])**.** *$\mathcal{G}_0(K)$ is non-empty and finite.*

Theorem 2.13 ([Sat23, Corollary 5.8])**.** *For any closed region R, the following holds:*

$$R \in \widetilde{\mathcal{G}}_0(K) \Leftrightarrow \exists R' \in \mathcal{G}_0(K),\ R' \subset R.$$

Theorem 2.14 ([Sat23, Theorem 5.16])**.** *For any knot K, the following holds:*

$$[K]_{\nu^+} = 0 \Leftrightarrow \mathcal{G}_0(K) = \{R_{(0,0)}\},$$

where $R_{(0,0)} := \{(i,j) \in \mathbb{Z}^2 \mid (i,j) \leq (0,0)\}$.

Here we also mention the relationship of $\mathcal{G}_0$ to filtered quasi-isomorphism.

Proposition 2.15 ([Sat23, Proposition 5.9])**.** *If $[K]_{\nu^+} \leq [J]_{\nu^+}$, then for any $R' \in \mathcal{G}_0(J)$, there exists an element $R \in \mathcal{G}_0(K)$ with $R \subset R'$.*

Next, as a new result on the invariant $\mathcal{G}_0$, we state the duality theorem for $\mathcal{G}_0$ which implies that $\mathcal{G}_0(K)$ and $\mathcal{G}_0(K^*)$ can recover each other. Here, for a subset $S \subset \mathbb{Z}^2$, we define the subset $-S \subset \mathbb{Z}^2$ by

$$-S := \{(i,j) \in \mathbb{Z}^2 \mid (-i,-j) \in S\}.$$

Theorem 2.16. *For any knot K, the equalities*

$$\widetilde{\mathcal{G}}_0(K^*) = \{R \in \mathcal{CR}(\mathbb{Z}^2) \mid \forall R' \in \mathcal{G}_0(K),\ R \cap (-R') \neq \varnothing\}$$

and

$$\mathcal{G}_0(K^*) = \min\{R \in \mathcal{CR}(\mathbb{Z}^2) \mid \forall R' \in \mathcal{G}_0(K),\ R \cap (-R') \neq \varnothing\}$$

hold.

Remark 2.17. This equality holds for any formal knot complex. Namely, we can prove the theorem purely algebraically.

Remark 2.18. Combining with Theorem 2.13, we can also prove the equality

$$\widetilde{\mathcal{G}}_0(K^*) = \{R \in \mathcal{CR}(\mathbb{Z}^2) \mid \forall R' \in \widetilde{\mathcal{G}}_0(K),\ R \cap (-R') \neq \varnothing\},$$

which looks more symmetric with respect to $\widetilde{\mathcal{G}}_0$.

Theorem 2.16 is proved in Appendix A, where we also provide a method for computing $\mathcal{G}_0(K^*)$ from $\mathcal{G}_0(K)$ algorithmically.

2.3.2 Relationship to other concordance invariants

Here we state the relationship of $\mathcal{G}_0(K)$ to the invariants $\tau(K)$, $d(S^3_{p/q}(K))$ and Υ_K. We first explain the translation of $\{d(S^3_{p/q}(K))\}_{p/q\in\mathbb{Q}_{>0}}$ into *Ni-Wu's V_k-sequence* $\{V_k(K)\}_{k\in\mathbb{Z}_{\geq 0}}$ [NW15]. Note that there is a canonical identification between the set of Spin^c structures over $S^3_{p/q}(K)$ and $\{i \mid 0 \leq i \leq p-1\}$. This identification can be made explicit by the procedure in [OS11, Section 4, Section 7]. Let $d(S^3_{p/q}(K), i)$ denote the correction term of $S^3_{p/q}(K)$ with the i-th Spin^c structure $(0 \leq i \leq p-1)$.

Proposition 2.19 ([NW15, Proposition 1.6])**.** *For any knot K, $p/q > 0$ and $0 \leq i \leq p-1$, the equality*

$$d(S^3_{p/q}(K), i) = d(S^3_{p/q}(O), i) - 2\max\left\{V_{\lfloor \frac{i}{q} \rfloor}(K), V_{\lfloor \frac{p+q-1-i}{q} \rfloor}(K)\right\}$$

holds, where O denotes the unknot and $\lfloor\cdot\rfloor$ is the floor function.

Since we have the orientation-preserving diffeomorphism

$$S^3_{-p/q}(K) \cong -S^3_{p/q}(K^*),$$

Proposition 2.19 shows that the two sequences $\{V_k(K)\}$ and $\{V_k(K^*)\}$ determine all values of $\{d(S^3_{p/q}(K))\}_{p/q\in\mathbb{Q}}$.

Here we also need to introduce several classes of closed regions. For each coordinate $(k,l) \in \mathbb{Z}^2$, we define the *simple region $R_{(k,l)}$ with the corner (k,l)* by

$$R_{(k,l)} := \{(i,j) \in \mathbb{Z}^2 \mid i \leq k \text{ and } j \leq l\}.$$

In addition, for each $t \in [0,2]$ and $s \in \mathbb{R}$, we define the *linear region $R^t(s)$ with respect to (t,s)* by

$$R^t(s) := \left\{(i,j) \in \mathbb{Z}^2 \;\middle|\; (1-\frac{t}{2})i + \frac{t}{2}j \leq s\right\}.$$

Now, the relationship of $\mathcal{G}_0(K)$ to the invariants $\tau(K)$, $V_k(K)$ and Υ_K is stated as follows.

Theorem 2.20 ([Sat23, Proposition 5.17])**.** *The invariants $\tau(K)$, $V_k(K)$ and Υ_K are determined from $\mathcal{G}_0(K)$ by the formulas:*

$$\begin{aligned}
\tau(K) &= \min\{m \in \mathbb{Z} \mid \exists R \in \mathcal{G}_0(K), R \subset (\{i \leq -1\} \cup R_{(0,m)})\} \\
V_k(K) &= \min\{m \in \mathbb{Z}_{\geq 0} \mid \exists R \in \mathcal{G}_0(K), R \subset R_{(m,k+m)}\} \\
\Upsilon_K(t) &= -2\left(\min\{s \in \mathbb{R} \mid \exists R \in \mathcal{G}_0(K), R \subset R^t(s)\}\right),
\end{aligned}$$

Figure 1 describes the family of closed regions corresponding to each concordance invariant. Lastly, we give a formula for computing $V_k(K^*)$ from $\mathcal{G}_0(K)$, which is obtained via the duality theorem for $\mathcal{G}_0$.

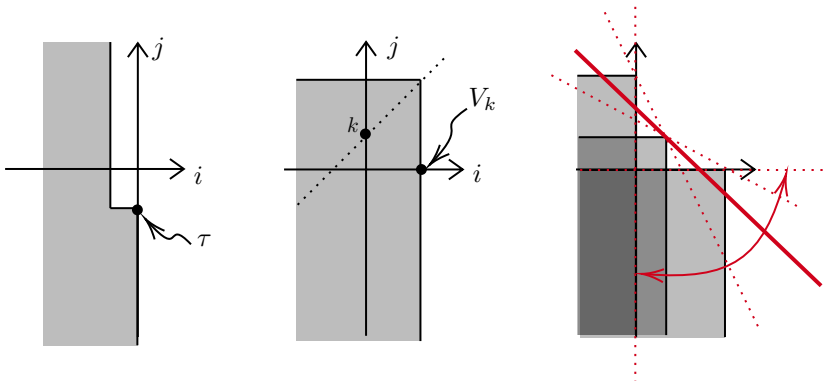


Figure 1: Closed regions corresponding to τ, V_k, and Υ.

Proposition 2.21. *The invariant $V_k(K^*)$ is determined from $\mathcal{G}_0(K)$ by the formula*

$$V_k(K^*) = \min\{m \in \mathbb{Z}_{\geq 0} \mid \forall R \in \mathcal{G}_0(K),\ (-m, -k-m) \in R\}.$$

Proof. For any $R \in \mathcal{CR}(\mathbb{Z}^2)$, we see that

$$\begin{aligned} (-m, -k-m) \in R &\Leftrightarrow R \cap (-R_{(m,k+m)}) \neq \varnothing \\ &\Leftrightarrow R_{(m,k+m)} \cap (-R) \neq \varnothing. \end{aligned}$$

Therefore, it follows from Theorem 2.16 that $R_{(m,k+m)} \in \widetilde{\mathcal{G}}_0(K^*)$ if and only if any $R \in \mathcal{G}_0(K)$ contains $(-m, -k-m)$. Now, by Theorem 2.13 and Theorem 2.20, we have

$$\begin{aligned} V_k(K^*) &= \min\{m \in \mathbb{Z}_{\geq 0} \mid \exists R \in \mathcal{G}_0(K^*), R \subset R_{(m,k+m)}\} \\ &= \min\{m \in \mathbb{Z}_{\geq 0} \mid R_{(m,k+m)} \in \widetilde{\mathcal{G}}_0(K^*)\} \\ &= \min\{m \in \mathbb{Z}_{\geq 0} \mid \forall R \in \mathcal{G}_0(K), (-m, -k-m) \in R\}. \end{aligned}$$

□

2.3.3 On the range of $\mathcal{G}_0$

Here we discuss the range of $\mathcal{G}_0$. We first introduce several notions of closed regions. For any subset $S \subset \mathbb{Z}^2$, define the *closure* of S by

$$cl(S) := \bigcup_{(i,j)\in S} R_{(i,j)}.$$

Then we have $cl(S) \in \mathcal{CR}(\mathbb{Z}^2)$. Moreover, the equality

$$cl(S) = \bigcap_{R\in\mathcal{CR}(\mathbb{Z}^2), S\subset R} R$$

holds. We say that a closed region $R \in \mathcal{CR}(\mathbb{Z}^2)$ is a *semi-simple region* if there exists a non-empty finite subset $S \subset \mathbb{Z}^2$ such that $R = cl(S)$.

As examples of semi-simple regions, we define the *closure* of any chain $x = \sum_{1 \leq k \leq r} p_k(U)x_k \in CFK^\infty(K)$ by

$$cl(x) := cl\left\{ (\mathrm{Alg}(U^{l(p_k)}x_k), \mathrm{Alex}(U^{l(p_k)}x_k)) \;\middle|\; \begin{array}{c} 1 \leq k \leq r \\ p_k(U) \neq 0 \end{array} \right\},$$

where $l(p_k)$ denotes the lowest degree of $p_k(U) \in \Lambda$. Note that the equality

$$cl(x) = \bigcap_{R \in \mathcal{CR}, x \in (CFK^\infty(K))_R} R$$

holds. (The proof is seen in [Sat23, Lemma 5.5].)

Let us denote the set of semi-simple regions by $\mathcal{CR}^{ss}(\mathbb{Z}^2)$. Then, the following proposition immediately follows from [Sat23, Theorem 5.7].

Proposition 2.22. *For any $R \in \mathcal{G}_0(K)$, there exists a homological generator x of $CFK^\infty(K)$ whose closure is equal to R. In particular, for any knot K, we have*

$$\mathcal{G}_0(K) \subset \mathcal{CR}^{ss}(\mathbb{Z}^2).$$

As a corollary, we have the following.

Corollary 2.23. *For any $R \in \mathcal{G}_0(K)$, there exists a non-empty finite subset $\{x_k\}_{k=1}^n \subset CFK^\infty(K)$ of linearly independent 0-chains such that*

- *$\sum_{k=1}^n x_k$ is a homological generator of $CFK^\infty(K)$, and*
- *$R = cl(\{(\mathrm{Alg}(x_k), \mathrm{Alex}(x_k))\}_{k=1}^n)$.*

Next, we introduce the notion of the *corners* of a semi-simple region, which will be used in the coming sections.

For a subset $S \subset \mathbb{Z}^2$, an element $s \in S$ is *maximal in S* if it satisfies

$$\text{if } s' \in S \text{ and } s' \geq s, \text{ then } s' = s.$$

Define the map

$$\max\colon \mathcal{P}(\mathbb{Z}^2) \to \mathcal{P}(\mathbb{Z}^2)$$

by

$$S \mapsto \{s \in S \mid s \text{ is maximal in } S\}.$$

Then, for any $R \in \mathcal{CR}^{ss}(\mathbb{Z}^2)$, the set of *corners* of R is defined by

$$c(R) := \max R.$$

Here we prove that for $R \in \mathcal{CR}^{ss}(\mathbb{Z}^2)$, the set $c(R)$ is non-empty and finite.

Lemma 2.24. *For any $R \in \mathcal{CR}^{ss}(\mathbb{Z}^2)$ and non-empty finite set $S \subset \mathbb{Z}^2$ satisfying $cl(S) = R$, the equality*

$$\max S = c(R)$$

holds. In particular, $c(R)$ is non-empty and finite.

Proof. For proving Lemma 2.24, we use the following lemma, which is obtained as an analogy of [Sat23, Lemma 5.4].

Lemma 2.25. *For any $s \in S$, there exists an element $s' \in \max S$ with $s' \geq s$.*

We first prove $\max S \subset \max R$. Let $s \in \max S$, and suppose that $p \in R$ satisfies $s \leq p$. Then, since $cl(S) = R$, there exists an element $s' \in S$ such that $p \leq s'$. Now, we have

$$s \leq p \leq s',$$

and hence $s = p = s'$. This implies $s \in \max R$.

Next, we prove $\max S \supset \max R$. Let $p \in \max R$, and suppose that $s \in S$ satisfies $p \leq s$. Here, by Lemma 2.25, we may assume that $s \in \max S$. Then, we have

$$s \in S \subset cl(S) = R,$$

and hence the maximality of p gives $p = s \in \max S$. □

Actually, the equality '$\max S = c(R)$' is a necessary and sufficient condition for a non-empty finite subset $S \subset \mathbb{Z}^2$ to generate $R \in \mathcal{CR}^{ss}(\mathbb{Z}^2)$. Namely, we have the following.

Lemma 2.26. *For any semi-simple region R, a non-empty finite set $S \subset \mathbb{Z}^2$ satisfies $cl(S) = R$ if and only if $\max S = c(R)$. In particular, $cl(c(R)) = R$.*

Proof. First, suppose that a non-empty finite set $S \subset \mathbb{Z}^2$ satisfies $cl(S) = R$. Then, by Lemma 2.24, we have

$$\max S = \max R = c(R).$$

Conversely, suppose that a non-empty finite set $S \subset \mathbb{Z}^2$ satisfies $\max S = c(R)$, and take a non-empty finite set $S' \subset \mathbb{Z}^2$ with $cl(S') = R$. Then we have

$$\max S' = \max R = c(R) = \max S.$$

Here, by Lemma 2.25, we see that

$$cl(\max F) = cl(F)$$

for any finite subset $F \subset \mathbb{Z}^2$. This implies

$$cl(S) = cl(\max S) = cl(\max S') = cl(S') = R.$$

□

3 Translating $\mathcal{G}_0$ into CFK^-

In this section, we introduce shifted versions of $\mathcal{G}_0(K)$ which enable us to define $\mathcal{G}_0$-type invariants for $CFK^-(K)$ and $C^-(G)$. Moreover, we prove that after shifting sufficiently large times, such shifted $\mathcal{G}_0$ for $CFK^-(K)$ recovers the original $\mathcal{G}_0(K)$.

3.1 Shifted $\mathcal{G}_0$

Let C be a $\mathbb{Z}^2$-filtered chain complex over $\mathbb{F}$ such that $H_n(C) \cong \mathbb{F}$ for given $n \in \mathbb{Z}$. Then, a homogeneous cycle $x \in C_n$ is called a *homological generator of degree n* if the homology class $[x] \in H_n(C) \cong \mathbb{F}$ is non-zero. Now we define

$$\widetilde{\mathcal{G}}_0^{(n)}(C) := \{R \in \mathcal{CR}(\mathbb{Z}^2) \mid C_R \text{ contains a homological generator of degree } n\}$$

and

$$\mathcal{G}_0^{(n)}(C) := \min \widetilde{\mathcal{G}}_0^{(n)}(C).$$

Let C and C' be two $\mathbb{Z}^2$-filtered chain complexes over $\mathbb{F}$ with $H_n(C) \cong H_n(C') \cong \mathbb{F}$ for $n \in \mathbb{Z}$. Then, a chain map $f\colon C \to C'$ over $\mathbb{F}$ is a *$\mathbb{Z}^2$-filtered quasi-isomorphism at degree n* if f is $\mathbb{Z}^2$-filtered, graded and induces an isomorphism $f_{*,n}\colon H_n(C) \to H_n(C')$.

Proposition 3.1. *Let C and C' be two $\mathbb{Z}^2$-filtered chain complexes over $\mathbb{F}$ with $H_n(C) \cong H_n(C') \cong \mathbb{F}$ for given $n \in \mathbb{Z}$. If there exists a $\mathbb{Z}^2$-filtered quasi-isomorphism $f\colon C \to C'$ at degree n, then we have*

$$\widetilde{\mathcal{G}}_0^{(n)}(C) \subset \widetilde{\mathcal{G}}_0^{(n)}(C').$$

In particular, if $C \underset{\mathbb{Z}^2\text{-filt.}/\mathbb{F}}{\simeq} C'$, then we have

$$\widetilde{\mathcal{G}}_0^{(n)}(C) = \widetilde{\mathcal{G}}_0^{(n)}(C')$$

and

$$\mathcal{G}_0^{(n)}(C) = \mathcal{G}_0^{(n)}(C').$$

Proof. If $x \in C_R$ is a homological generator of degree n, then $f(x) \in C'$ is also a homological generator of degree n, and $f(x) \in f(C_R) \subset C'_R$. □

Here let us consider the case of CFK^∞. By the global triviality, we can define $\mathcal{G}_0^{(n)}(CFK^\infty(K))$ for any even integer n. We show that all $\mathcal{G}_0^{(n)}(CFK^\infty(K))$ can recover one another. To show that, we define *shifts* of closed regions.

Let $R \in \mathcal{CR}(\mathbb{Z}^2)$ and $s \in \mathbb{Z}$. Then we define

$$R[s] := \{(i,j) \in \mathbb{Z}^2 \mid (i+s, j+s) \in R\}.$$

For instance, a simple region $R_{(k,l)}$ is shifted to $R_{(k-s,l-s)}$ by $s \in \mathbb{Z}$, i.e. $(R_{(k,l)})[s] = R_{(k-s,l-s)}$. Obviously, for any $s, t \in \mathbb{Z}$, we have

$$(R[s])[t] = R[s+t]$$

and

$$R[0] = R.$$

Next, for $\mathcal{S} \subset \mathcal{CR}(\mathbb{Z}^2)$ and $s \in \mathbb{Z}$, we define

$$\mathcal{S}[s] := \{R[s] \in \mathcal{CR}(\mathbb{Z}^2) \mid R \in \mathcal{S}\},$$

and then we have $(\mathcal{S}[s])[t] = \mathcal{S}[s+t]$ and $\mathcal{S}[0] = \mathcal{S}$.

Proposition 3.2. *For any knot K and $s \in \mathbb{Z}$, we have*

$$\widetilde{\mathcal{G}}_0^{(-2s)}(CFK^\infty(K)) = \widetilde{\mathcal{G}}_0(K)[s]$$

and

$$\mathcal{G}_0^{(-2s)}(CFK^\infty(K)) = \mathcal{G}_0(K)[s].$$

Proof. By the fourth property of CFK^∞, the following hold:

- An element $x \in C$ is a homological generator of degree $-2s$ if and only if $U^{-s}x$ is a homological generator (of degree 0).
- The equality $U^{-s} \cdot C_R = C_{R[-s]}$ holds.

In particular, C_R contains a homological generator of degree $-2s$ if and only if $C_{R[-s]}$ contains a homological generator (of degree 0). This implies

$$\begin{aligned} R \in \widetilde{\mathcal{G}}_0^{(-2s)}(C) &\Leftrightarrow R[-s] \in \widetilde{\mathcal{G}}_0(C) \\ &\Leftrightarrow R \in \widetilde{\mathcal{G}}_0(C)[s]. \end{aligned}$$

This completes the proof. □

As corollaries of Proposition 3.2, we see that $\mathcal{G}_0^{(-2s)}(C)$ shares several nice properties such as Theorem 2.12 and Theorem 2.13 with $\mathcal{G}_0(C)$.

Corollary 3.3. *For any $s \in \mathbb{Z}$, the set $\mathcal{G}_0^{(-2s)}(C)$ is non-empty and finite.*

Corollary 3.4. *For any $s \in \mathbb{Z}$ and closed region R, the following holds:*

$$R \in \widetilde{\mathcal{G}}_0^{(-2s)}(C) \Leftrightarrow \exists R' \in \mathcal{G}_0^{(-2s)}(C),\ R' \subset R.$$

3.2 Recovering $\mathcal{G}_0(K)$ from $\mathcal{G}_0^{(-2s)}(CFK^-(K))$

Let $C := CFK^\infty(K)$. Here we consider $\mathcal{G}_0^{(n)}(CFK^-(K)) = \mathcal{G}_0^{(n)}(\mathcal{F}_0^{\mathrm{Alg}}(C))$. By the seventh property of CFK^∞, we have the isomorphism

$$H_n(\mathcal{F}_0^{\mathrm{Alg}}(C)) \cong H_n(\mathcal{F}_0^{\mathrm{Alg}}(\Lambda)) \cong \begin{cases} \mathbb{F} & (n \leq 0 \text{ and } n\colon \text{ even}) \\ 0 & (\text{otherwise}) \end{cases}.$$

Therefore, we can define $\widetilde{\mathcal{G}}_0^{(-2s)}(\mathcal{F}_0^{\mathrm{Alg}}(C))$ and $\mathcal{G}_0^{(-2s)}(\mathcal{F}_0^{\mathrm{Alg}}(C))$ for any $s \in \mathbb{Z}_{\geq 0}$. Here, $\mathcal{F}_0^{\mathrm{Alg}}(C)$ is regarded as a $\mathbb{Z}^2$-filtered chain complex (over $\mathbb{F}[U]$) by

$$(\mathcal{F}_0^{\mathrm{Alg}}(C))_R := \mathcal{F}_0^{\mathrm{Alg}}(C) \cap C_R$$

for each $R \in \mathcal{CR}(\mathbb{Z}^2)$.

In order to state the relation between $\mathcal{G}_0^{(-2s)}(\mathcal{F}_0^{\mathrm{Alg}}(C))$ and $\mathcal{G}_0(C)$, we first introduce the *shift number* of a closed region. For any $R \in \mathcal{CR}(\mathbb{Z}^2)$, we define the *shift number* of R by

$$\mathrm{shift}(R) := \max\{i \in \mathbb{Z} \mid \exists j \in \mathbb{Z} \text{ s.t. } (i,j) \in R\}.$$

Here we define $\mathrm{shift}(\varnothing) := -\infty$ and $\mathrm{shift}(R) := \infty$ if there exists no integer M such that $R \subset \{i \leq M\}$. Note that $\mathrm{shift}(R)$ is finite if $R \in \mathcal{CR}^{ss}(\mathbb{Z}^2)$. In addition, if $R \subset R'$, then $\mathrm{shift}(R) \leq \mathrm{shift}(R')$.

Now, we can state the relation between $\mathcal{G}_0^{(-2s)}(CFK^-(K))$ and $\mathcal{G}_0(K)$ as follows.

Theorem 3.5. *For any knot K and $s \in \mathbb{Z}_{\geq 0}$, we have*

$$\mathcal{G}_0^{(-2s)}(CFK^-(K)) = \{R[s] \mid R \in \mathcal{G}_0(K),\ \mathrm{shift}(R) \leq s\}.$$

To prove Theorem 3.5, we use the following three lemmas.

Lemma 3.6. *For any $R \in \mathcal{CR}(\mathbb{Z}^2)$, we have*

$$(\mathcal{F}_0^{\mathrm{Alg}}(C))_R = C_{R \cap \{i \leq 0\}} = (\mathcal{F}_0^{\mathrm{Alg}}(C))_{R \cap \{i \leq 0\}}.$$

Proof. By [Sat23, Lemma 2.9], there exists a direct decomposition

$$C = \bigoplus_{(i,j) \in \mathbb{Z}} C_{(i,j)}$$

as an $\mathbb{F}$-vector space such that for any $R \in \mathcal{CR}(\mathbb{Z}^2)$, the equality

$$C_R = \bigoplus_{(i,j) \in R} C_{(i,j)}$$

holds. Now, since $\mathcal{F}_0^{\mathrm{Alg}}(C) = C_{\{i \leq 0\}}$, it is easy to see that

$$\begin{aligned}
(\mathcal{F}_0^{\mathrm{Alg}}(C))_R &= C_{\{i \leq 0\}} \cap C_R \\
&= \Big(\bigoplus_{i \leq 0} C_{(i,j)}\Big) \cap \Big(\bigoplus_{(i,j) \in R} C_{(i,j)}\Big) \\
&= \bigoplus_{(i,j) \in R \cap \{i \leq 0\}} C_{(i,j)} = C_{R \cap \{i \leq 0\}}.
\end{aligned}$$

Now, the second equality follows from $C_{R \cap \{i \leq 0\}} = (\mathcal{F}_0^{\mathrm{Alg}}(C))_R \subset \mathcal{F}_0^{\mathrm{Alg}}(C)$. □

Lemma 3.7. *If $R \in \widetilde{\mathcal{G}}_0^{(-2s)}(\mathcal{F}_0^{\mathrm{Alg}}(C))$, then there exists an element $R' \in \mathcal{G}_0^{(-2s)}(C)$ such that $R' \subset R$ and* $\mathrm{shift}(R') \leq 0$.

Proof. Since $(C, \{\mathcal{F}_i^{\mathrm{Alg}}(C)\}) \underset{\mathbb{Z}\text{-filt.}/\mathbb{F}}{\simeq} (\Lambda, \{\mathcal{F}_i^{\mathrm{Alg}}(\Lambda)\})$, the inclusion $i\colon \mathcal{F}_0^{\mathrm{Alg}}(C) \to C$ induces the isomorphism $H_{-2s}(\mathcal{F}_0^{\mathrm{Alg}}(C)) \cong H_{-2s}(C)$. Moreover, by the definition of the $\mathbb{Z}^2$-filtration on $\mathcal{F}_0^{\mathrm{Alg}}(C)$, it is obvious that the map i is $\mathbb{Z}^2$-filtered. Therefore, the inclusion $i\colon \mathcal{F}_0^{\mathrm{Alg}}(C) \to C$ is a $\mathbb{Z}^2$-filtered quasi-isomorphism at degree $-2s$, and hence Proposition 3.1 gives

$$\widetilde{\mathcal{G}}_0^{(-2s)}(\mathcal{F}_0^{\mathrm{Alg}}(C)) \subset \widetilde{\mathcal{G}}_0^{(-2s)}(C).$$

Now, let $R \in \widetilde{\mathcal{G}}_0^{(-2s)}(\mathcal{F}_0^{\mathrm{Alg}}(C))$. Then, Lemma 3.6 and the above arguments imply that

$$R \cap \{i \leq 0\} \in \widetilde{\mathcal{G}}_0^{(-2s)}(\mathcal{F}_0^{\mathrm{Alg}}(C)) \subset \widetilde{\mathcal{G}}_0^{(-2s)}(C).$$

Therefore, by Corollary 3.4, there exists an element $R' \in \mathcal{G}_0^{(-2s)}(C)$ such that $R' \subset R \cap \{i \leq 0\}$. Obviously, we have $R' \subset R$ and

$$\mathrm{shift}(R') \leq \mathrm{shift}(R \cap \{i \leq 0\}) \leq 0.$$

□

Lemma 3.8. *If $R \in \mathcal{G}_0^{(-2s)}(C)$ and* $\mathrm{shift}(R) \leq 0$, *then $R \in \widetilde{\mathcal{G}}_0^{(-2s)}(\mathcal{F}_0^{\mathrm{Alg}}(C))$.*

Proof. Note that for any $R \in \mathcal{CR}(\mathbb{Z}^2)$, the inequality $\mathrm{shift}(R) \leq 0$ holds if and only if $R \subset \{i \leq 0\}$. Now, let $R \in \mathcal{G}_0^{(-2s)}(C)$ and $\mathrm{shift}(R) \leq 0$. Then there exists a homological generator $x \in C$ of degree $-2s$ which satisfies

$$x \in C_R \subset C_{\{i \leq 0\}} = \mathcal{F}_0^{\mathrm{Alg}}(C).$$

In particular, $x \in \mathcal{F}_0^{\mathrm{Alg}}(C) \cap C_R = (\mathcal{F}_0^{\mathrm{Alg}}(C))_R$, and hence we can regard x as a homological generator of degree $-2s$ for $\mathcal{F}_0^{\mathrm{Alg}}(C)$ which lies in $(\mathcal{F}_0^{\mathrm{Alg}}(C))_R$. This shows $R \in \widetilde{\mathcal{G}}_0^{(-2s)}(\mathcal{F}_0^{\mathrm{Alg}}(C))$. □

Now, let us prove Theorem 3.5.

Proof of Theorem 3.5. For any $R \in \mathcal{CR}^{ss}(\mathbb{Z}^2)$, we see that

$$\mathrm{shift}(R) \leq s \Leftrightarrow \mathrm{shift}(R[s]) \leq 0.$$

Hence, Proposition 3.2 implies

$$\begin{aligned} \{R[s] \mid R \in \mathcal{G}_0(K),\ \mathrm{shift}(R) \leq s\} &= \{R \in \mathcal{G}_0(K)[s] \mid \mathrm{shift}(R) \leq 0\} \\ &= \{R \in \mathcal{G}_0^{(-2s)}(C) \mid \mathrm{shift}(R) \leq 0\}. \end{aligned}$$

Therefore, it suffices to prove that

$$\mathcal{G}_0^{(-2s)}(\mathcal{F}_0^{\mathrm{Alg}}(C)) = \{R \in \mathcal{G}_0^{(-2s)}(C) \mid \mathrm{shift}(R) \leq 0\}.$$

We first suppose that $R \in \mathcal{G}_0^{(-2s)}(\mathcal{F}_0^{\mathrm{Alg}}(C))$. Then $R \in \widetilde{\mathcal{G}}_0^{(-2s)}(\mathcal{F}_0^{\mathrm{Alg}}(C))$, and hence Lemma 3.7 gives an element $R' \in \mathcal{G}_0^{(-2s)}(C)$ such that $R' \subset R$ and $\mathrm{shift}(R') \leq 0$. Moreover, Lemma 3.8 shows that $R' \in \widetilde{\mathcal{G}}_0^{(-2s)}(\mathcal{F}_0^{\mathrm{Alg}}(C))$. By the minimality of R in $\widetilde{\mathcal{G}}_0^{(-2s)}(\mathcal{F}_0^{\mathrm{Alg}}(C))$, we must have $R = R' \in \mathcal{G}_0^{(-2s)}(C)$.

Next, we suppose that $R \in \mathcal{G}_0^{(-2s)}(C)$ and $\mathrm{shift}(R) \leq 0$. Then, Lemma 3.8 shows that $R \in \widetilde{\mathcal{G}}_0^{(-2s)}(\mathcal{F}_0^{\mathrm{Alg}}(C))$. Assume that $R' \in \widetilde{\mathcal{G}}_0^{(-2s)}(\mathcal{F}_0^{\mathrm{Alg}}(C))$ satisfies $R' \subset R$. Then, Lemma 3.7 gives an element $R'' \in \mathcal{G}_0^{(-2s)}(C) \subset \widetilde{\mathcal{G}}_0^{(-2s)}(C)$ such that $R'' \subset R' \subset R$. By the minimality of R in $\widetilde{\mathcal{G}}_0^{(-2s)}(C)$, we must have $R'' = R' = R$. This implies that $R \in \mathcal{G}_0^{(-2s)}(\mathcal{F}_0^{\mathrm{Alg}}(C))$. □

As a corollary of Theorem 3.5, we see that the sequence

$$\{\mathcal{G}_0^{(-2s)}(CFK^-(K))\}_{s=0}^{\infty}$$

converges to shifted $\mathcal{G}_0(K)$.

Corollary 3.9. *For any $s \geq \max\{\mathrm{shift}(R) \mid R \in \mathcal{G}_0(K)\}$, we have*

$$\mathcal{G}_0^{(-2s)}(CFK^-(K)) = \mathcal{G}_0(K)[s].$$

4 Translating $\mathcal{G}_0$ into $C^-(G)$

In this section, we recall the grid complex $C^-(G)$ and its relationship to CFK^-. Combining such arguments with Theorem 3.5, we prove that $\mathcal{G}_0(K)$ is determined from $\mathcal{G}_0^{(-2s)}(C^-(G))$ for sufficiently large $s \in \mathbb{Z}_{\geq 0}$.

4.1 A review of grid complexes

Grid homology theory is a combinatorial description of knot Floer homology theory, introduced by Manolescu, Ozsváth, Sarkar, Szabó and Thurston in [MOS09; Man+07]. For any knot $K \subset S^3$, by choosing a *grid diagram* G of K (also called an *arc representation*), one associates the *grid complex* $C^-(G)$ which is filtered homotopy equivalent to $CFK^-(K)$. Here we briefly review the construction.

A *grid diagram* G lies on an $n \times n$ grid of squares, where some squares are decorated either with an O or an X so that

- every row contains exactly one O and one X;
- every column contains exactly one O and one X.

The number n is called the *grid number* of G. Given such data, one obtains a planar link diagram by drawing horizontal segments from the O's to the X's in each row, and vertical segments from the X's to the O's in each column, while letting the horizontal segment underpass the vertical segment at every intersection point. It is also true that every link in S^3 possesses such representation. We usually place the diagram in the standard plane so that bottom left corner

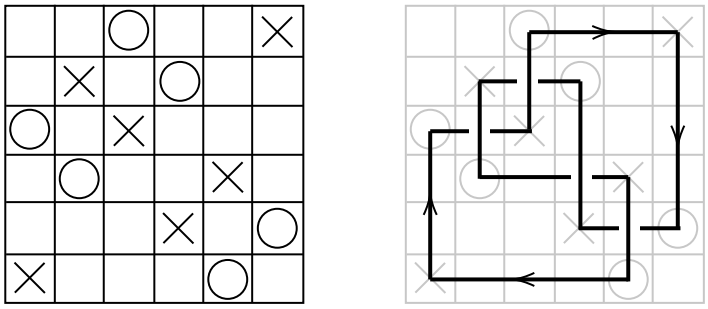


Figure 2: Grid diagram and the corresponding knot

is at the origin, each square has length one, and each O and each X is centered at a half-integer point. We set $\mathbb{O} = \{O_i\}_{1 \le i \le n}$, $\mathbb{X} = \{X_i\}_{1 \le i \le n}$ so that each O_i and X_i has its center in $x = i - \frac{1}{2}$.

Given a grid diagram G, the chain complex $C^-(G)$ is constructed as follows. First we regard G as a diagram on the torus by gluing the two opposite sides. The generating set S is given by n-tuples of intersection points between the horizontal and vertical circles, with the property that each horizontal (or vertical) circle contains exactly one intersection point. By fixing the bottom left corner of the grid, each $\mathbf{x} \in S$ can be identified with the set of permutations of length n under the correspondence

$$\sigma \mapsto \mathbf{x} = \{(i, \sigma(i))\}_{0 \le i < n}.$$

$C^-(G)$ is generated by S over the multivariate polynomial ring $\mathbb{F}[U_1, \cdots, U_n]$. The differential ∂ is defined as

$$\partial(\mathbf{x}) = \sum_{\mathbf{y} \in S} \sum_{r \in \mathrm{Rect}^\circ(\mathbf{x}, \mathbf{y})} U_1^{\epsilon_1} \cdots U_n^{\epsilon_n} \mathbf{y},$$

where $\mathrm{Rect}^\circ(\mathbf{x}, \mathbf{y})$ denotes the set of *empty rectangles* connecting $\mathbf{x}$ to $\mathbf{y}$ (which exist only when $\mathbf{x}$ and $\mathbf{y}$ are related by a single transposition), and for each empty rectangle r, the exponent $\epsilon_i \in \{0, 1\}$ is given by the number of intersections of r and O_i. Figure 3 depicts an empty rectangle r connecting $\mathbf{x}$ to $\mathbf{y}$, and for this r we have $\epsilon_4 = 1$. See [Man+07] for the precise definition.

$C^-(G)$ is endowed the *Maslov grading* and the *Alexander grading* as follows. If P and Q are sets of finitely many points in $\mathbb{R}^2$, we define $I(P, Q)$ by the number of pairs $p \in P$ and $q \in Q$ with $p < q$. We symmetrize this function as

$$J(P, Q) := \frac{I(P, Q) + I(Q, P)}{2}.$$

Here J is extended bilinearly over formal sums of finite subsets of $\mathbb{R}^2$. For any $\mathbf{x} \in S$, we define

$$M(\mathbf{x}) := J(\mathbf{x} - \mathbb{O}, \mathbf{x} - \mathbb{O}) + 1,$$
$$A(\mathbf{x}) := J(\mathbf{x} - \frac{1}{2}(\mathbb{X} + \mathbb{O}), \mathbb{X} - \mathbb{O}) - \frac{n-1}{2}.$$

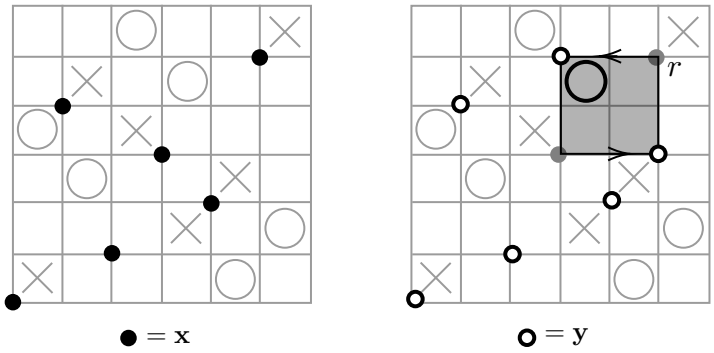


Figure 3: An empty rectangle connecting $\mathbf{x}$ to $\mathbf{y}$

We also declare that each factor U_i decreases M by -2 and A by -1. Then it follows that ∂ decreases M by -1, while A is non-increasing under ∂. Thus M gives a homological grading of $C^-(G)$ and A gives a $\mathbb{Z}$-filtration $\{\mathcal{F}_j^{\mathrm{Alex}}\}$ on $C^-(G)$. We regard $C^-(G)$ as a $\mathbb{Z}$-filtered complex over $\mathbb{F}[U]$, where the action by U is defined to be multiplication by U_1.

Theorem 4.1 ([MOS09, Theorem 3.3])**.** *If G is a grid diagram of a knot K, then we have*

$$(C^-(G), \mathcal{F}_j^{\mathrm{Alex}}) \underset{\mathbb{Z}\text{-}filt./\mathbb{F}[U]}{\simeq} (CFK^-(K), \mathcal{F}_j^{\mathrm{Alex}}).$$

4.2 Computing $\mathcal{G}_0(K)$ from $C^-(G)$

Here we show that the $\mathbb{Z}$-filtered homotopy equivalence stated in Theorem 4.1 induces the $\mathbb{Z}^2$-filtered homotopy equivalence between $C^-(G)$ and $CFK^-(K)$. To show that, we introduce the *algebraic filtration* $\{\mathcal{F}_i^{\mathrm{Alg}}\}$ on $C^-(G)$ as follows: For any $i \in \mathbb{Z}$, the i-th subcomplex $\mathcal{F}_i^{\mathrm{Alg}}$ is defined to be the $\mathbb{F}[U]$-submodule generated by elements of the form $U_1^{a_1} \cdots U_n^{a_n} \mathbf{x} \in C^-(G)$ with $a_1 \geq -i$. By the definition of the differential on $C^-(G)$, it is obvious that $\{\mathcal{F}_i^{\mathrm{Alg}}\}$ is a $\mathbb{Z}$-filtration on $C^-(G)$. Moreover, we see that

$$U \cdot \mathcal{F}_i^{\mathrm{Alg}} = \mathcal{F}_{i-1}^{\mathrm{Alg}}$$

for any $i \leq 0$, and

$$\mathcal{F}_i^{\mathrm{Alg}} = C^-(G)$$

for any $i \geq 0$. (Note that $CFK^-(K)$ also has the same property.) We use the $\mathbb{Z}^2$-filtration induced by $(\{\mathcal{F}_i^{\mathrm{Alg}}\}, \{\mathcal{F}_j^{\mathrm{Alex}}\})$ to regard $C^-(G)$ as a $\mathbb{Z}^2$-filtered complex over $\mathbb{F}[U]$. Then, we have the following.

Corollary 4.2. *If G is a grid diagram of a knot K, then we have*

$$C^-(G) \underset{\mathbb{Z}^2\text{-}filt./\mathbb{F}[U]}{\simeq} CFK^-(K).$$

Proof. Let C and C' be either one of $C^-(G)$ and $CFK^-(K)$, and suppose that $f\colon C \to C'$ is a map over $\mathbb{F}[U]$. Then, obviously we have

$$f(\mathcal{F}_i^{\mathrm{Alg}}(C)) \subset C' = \mathcal{F}_i^{\mathrm{Alg}}(C')$$

if $i \geq 0$. Moreover, if $i < 0$, then

$$\begin{aligned} f(\mathcal{F}_i^{\mathrm{Alg}}(C)) &= f(U^{-i} \cdot \mathcal{F}_0^{\mathrm{Alg}}(C)) \\ &= U^{-i} \cdot f(\mathcal{F}_0^{\mathrm{Alg}}(C)) \subset U^{-i} \cdot \mathcal{F}_0^{\mathrm{Alg}}(C') = \mathcal{F}_i^{\mathrm{Alg}}(C'). \end{aligned}$$

Now, it is easy to see that a $\mathbb{Z}$-filtered homotopy equivalence map given by Theorem 4.1 induces $C^-(G) \underset{\mathbb{Z}^2\text{-filt.}/\mathbb{F}[U]}{\simeq} CFK^-(K)$. □

Now, combining Corollary 4.2 with Proposition 3.1 and Theorem 3.5, we have the following.

Theorem 4.3. *For any knot K, grid diagram G for K and $s \in \mathbb{Z}_{\geq 0}$, we have*

$$\mathcal{G}_0^{(-2s)}(C^-(G)) = \{R[s] \mid R \in \mathcal{G}_0(K),\ \mathrm{shift}(R) \leq s\}.$$

We also have the following convergence theorem.

Corollary 4.4. *For any $s \geq \max\{\mathrm{shift}(R) \mid R \in \mathcal{G}_0(K)\}$, we have*

$$\mathcal{G}_0^{(-2s)}(C^-(G)) = \mathcal{G}_0(K)[s].$$

Moreover, we have the following corollary, which enables us to check the realizability of each closed region with finite shift number via $C^-(G)$.

Corollary 4.5. *For any closed region R with $\mathrm{shift}(R) \leq s$, we have*

$$R \in \widetilde{\mathcal{G}}_0(K) \Leftrightarrow R[s] \in \widetilde{\mathcal{G}}_0^{(-2s)}(C^-(G)).$$

Proof. First, suppose that $R \in \widetilde{\mathcal{G}}_0(K)$. Then, by Theorem 2.13, we have $R' \in \mathcal{G}_0(K)$ with $R' \subset R$. In particular, the inequalities $\mathrm{shift}(R') \leq \mathrm{shift}(R) \leq s$ hold, and hence Theorem 4.3 gives $R'[s] \in \mathcal{G}_0^{(-2s)}(C^-(G))$. This implies that $R[s] \in \widetilde{\mathcal{G}}_0^{(-2s)}(C^-(G))$.

Next, suppose that $R[s] \in \widetilde{\mathcal{G}}_0^{(-2s)}(C^-(G))$. Then Corollary 4.2 gives $R[s] \in \widetilde{\mathcal{G}}_0^{(-2s)}(CFK^-(K))$. Moreover, Lemma 3.7 and Proposition 3.2 implies

$$R[s] \in \widetilde{\mathcal{G}}_0^{(-2s)}(CFK^\infty(K)) = \widetilde{\mathcal{G}}_0(K)[s].$$

Therefore, we have $R \in \widetilde{\mathcal{G}}_0(K)$. □

5 Algorithm

Now we describe the algorithm to compute $\mathcal{G}_0(K)$. First we describe the four main procedures which assures that the algorithm exists, and then describe how these procedures are integrated so that the computations are done effectively. Throughout this section we assume that we are given a grid diagram $G = (\mathbb{O}, \mathbb{X})$ of K, and denote $C = C^-(G)$.

5.1 Enumerating the candidate regions

First we show that there exists a finite set of closed regions that is assured to include $\mathcal{G}_0(K)$.

Proposition 5.1. *For any $g \geq 0$,*

$$\begin{aligned}\mathcal{G}_0(T_{2,2g+1}) &= \{R_{(i,g-i)} \mid i = 0, \cdots, g\},\\ \mathcal{G}_0((T_{2,2g+1})^*) &= \{R_{(-g,0)} \cup R_{(-g+1,-1)} \cup \cdots \cup R_{(0,-g)}\}.\end{aligned}$$

Proof. It is known that $CFK^\infty((T_{2,2r+1})^*)$ possesses a unique homological generator $a := a_0 + \cdots a_g$ such that $(\mathrm{Alg}(a_i), \mathrm{Alex}(a_i)) = (-g+i, -i)$. This implies the latter equality. Similarly, we can easily compute $\mathcal{G}_0(T_{2,2g+1})$. (We can also determine $\mathcal{G}_0(T_{2,2g+1})$ from $\mathcal{G}_0((T_{2,2g+1})^*)$ using Corollary A.7.) □

Proposition 5.2. *Let g_3, g_4 denote the 3-, 4- genus of a knot K respectively. Any $R \in \mathcal{G}_0(K)$ satisfies the following three conditions:*

1. *Each corner (i, j) of R satisfies $|i - j| \leq g_3$.*
2. *R includes the region $R_{(-g_4,0)} \cup R_{(-g_4+1,-1)} \cup \cdots \cup R_{(0,-g_4)}$.*
3. *If R contains a point $(i, g_4 - i)$ for some $i \in \{0, \ldots, g_4\}$, then $R = R_{(i,g_4-i)}$.*

Proof. By [Sat23, Theorem 4.5], up to $\mathbb{Z}^2$-filtered homotopy equivalence, we may assume that CFK^∞ has a filtered basis $\{x_k\}_{1 \leq k \leq r}$ such that

$$|\,\mathrm{Alex}(U^n x_k) - \mathrm{Alg}(U^n x_k)| \leq g_3$$

for any $1 \leq k \leq r$ and $n \in \mathbb{Z}$. Hence the first statement follows from Proposition 2.22. Next, from [Pet13, Section 3.1], we have $\pm g[T_{2,3}]_{\nu^+} = \pm[T_{2,2g+1}]_{\nu^+}$. Now the remaining two statements follow from Theorem 2.9, Proposition 2.15 and Proposition 5.1. □

In particular, any corner (i, j) of $R \in \mathcal{G}_0(K)$ lies in the bounded area

$$\begin{aligned}|i + j| &\leq g_4 \quad (ij \geq 0),\\ |i - j| &\leq g_3 \quad (ij < 0).\end{aligned}$$

Thus the set of all semi-simple regions satisfying the conditions of Proposition 5.2 is finite. Furthermore, another strong condition can be imposed from the bigraded module structure of $\widehat{HFK}(K)$. As mentioned in [KP18, Section 4], by [Ras03, Lemma 4.5], up to $\mathbb{Z}^2$-filtered homotopy equivalence, $CFK^\infty(K)$ may be regarded as a chain complex generated by $\widehat{HFK}(K)$ over $\mathbb{F}[U, U^{-1}]$. Define

$$\widehat{S}_k := \{\ U^a x \mid M(x) \equiv k \bmod 2,\ a = (M(x) - k)/2\ \},$$

where x runs over the homogeneous generators of $\widehat{HFK}(K)$. For a closed region R, we say two corners (i, j) and (k, l) of R are *adjacent* if there are no corners in between them.

Proposition 5.3. *Any $R \in \mathcal{G}_0(K)$ satisfies the following two conditions:*

1. *For each corner (i,j) of R, there exists an element $x \in \widehat{S}_0$ such that $(\mathrm{Alg}(x), \mathrm{Alex}(x)) = (i,j)$.*
2. *For each pair of adjacent corners (i,j) and (k,l) of R with $i < k$, there exists an element $x \in \widehat{S}_{-1}$ such that $(\mathrm{Alg}(x), \mathrm{Alex}(x)) \leq (i,l)$.*

Proof. The first statement is obvious from Corollary 2.23 and the above arguments. For the second statement, take a homological generator z of degree 0 whose closure gives R. Decompose z as $x + y$ so that x consists of terms of z having $\mathrm{Alg} \leq i$. Obviously $x, y \neq 0$ and $\partial x + \partial y = 0$. Also from the minimality of R, we have $\partial x \neq 0$. Thus $\partial x = \partial y$ belongs to $cl(x) \cap cl(y) = R_{(i,l)}$. □

If $g_3(K), g_4(K)$ and $\widehat{HFK}(K)$ are known beforehand, we take them as additional inputs. If not, then we can compute $\widehat{HFK}(K)$ (which also gives $g_3(K)$) from the simplified grid complex

$$\widetilde{C}(G) := \frac{C^-(G)}{U_1 = \cdots = U_n = 0}.$$

From [Man+07, Theorem 3.6] we have

$$H(\widetilde{C}(G)) \cong \widehat{HFK}(K) \otimes W^{\otimes(n-1)},$$

where W is the two-dimensional bigraded vector space spanned by two generators, one generator in bigrading $(0,0)$ and another in bigrading $(-1,-1)$. An effective algorithm for computing $\widehat{HFK}$ is described in [BG12].

We call the set of all semi-simple regions satisfying Proposition 5.2 and 5.3 the set of *candidate regions*. Having enumerated the candidate regions, the remaining task is to determine the realizability for each candidate region. In fact, from the symmetry $CFK^\infty(K) \underset{\mathbb{Z}^2\text{-filt.}/\Lambda}{\simeq} (CFK^\infty(K))^r$, it follows that

$$R \in \mathcal{G}_0(K) \Leftrightarrow R^r \in \mathcal{G}_0(K),$$

where R^r is the *reflection* of R

$$R^r := \{(i,j) \mid (j,i) \in R\}.$$

Thus it suffices to consider only one of the reflection pairs. For reasons to be explained in the coming sections, we check the one that has the smaller shift number.

5.2 Inflating the generators

Recall that the grid complex $C = C^-(G)$ is generated by the set S over $\mathbb{F}[U_1, \cdots, U_n]$, where each $\mathbf{x} \in S$ corresponds one-to-one to a permutation of

length n. *Heap's algorithm* [Hea63] is well known in the area of computer science, which enumerates all permutations of any fixed length by a sequence of transpositions. As for the two gradings M and A, if $\mathbf{x}, \mathbf{y} \in S$ are related by a transposition and r is a rectangle that connects $\mathbf{x}$ to $\mathbf{y}$, it is known that the following relations hold:

$$\begin{aligned} M(\mathbf{y}) - M(\mathbf{x}) &= 2\#(r \cap \mathbb{O}) - 2\#(\mathbf{x} \cap \mathrm{Int}(r)) - 1, \\ A(\mathbf{y}) - A(\mathbf{x}) &= \#(r \cap \mathbb{O}) - \#(r \cap \mathbb{X}). \end{aligned}$$

Note that computing the two gradings sequentially from the above formulas is more effective than computing them directly using the formulas given in Section 4.1. Thus we obtain the generating set S together with the gradings $(M(\mathbf{x}), A(\mathbf{x}))$ for each $\mathbf{x} \in S$. Next we inflate these generators by multiplying monomials in $U_1, \cdots U_n$ and regard C as an (infinitely generated) chain complex over $\mathbb{F}$. Each factor U_i decreases the homological degree by 2, so for any $k \in \mathbb{Z}$, the k-th chain group C_k can be regarded as a finite dimensional vector space over $\mathbb{F}$ with generators of the form

$$U_1^{a_1} \cdots U_n^{a_n} \mathbf{x}, \quad \deg \mathbf{x} - 2\textstyle\sum_i a_i = k.$$

We call generators of this form *inflated generators*. Recall from Section 4.1 that the $\mathbb{Z}^2$-filtration is given by

$$\begin{aligned} \mathrm{Alg}(U_1^{a_1} \cdots U_n^{a_n} \mathbf{x}) &= -a_1, \\ \mathrm{Alex}(U_1^{a_1} \cdots U_n^{a_n} \mathbf{x}) &= A(\mathbf{x}) - \sum_i a_i. \end{aligned}$$

Note that the number of inflated generators increases infinitely as the homological degree k decreases. Nonetheless, from Proposition 5.2 (and reasons to be stated in the coming sections), we only need to consider chain groups of degree between $-2g_3$ and 1, hence the enumeration is finite.

5.3 Computing a homological generator

Next we describe the algorithm for computing a homological generator of $C(G)$, i.e. a representative cycle of the unique generator of $H_0^-(G) \cong \mathbb{F}$. For efficiency, the algorithm avoids direct computation of the homology group. We call the set of points $\{(i, n-i-1) \mid i \in \frac{1}{2}\mathbb{Z}\}$ the *anti-diagonal* of G.

Proposition 5.4. *Suppose all O's of G lie in the anti-diagonal of G. The element $\mathbf{x}_0 := \{(i, n-i-1)\}_{0 \le i < n}$ represents the unique generator of $H_0^-(G) \cong \mathbb{F}$.*

Proof. $M(\mathbf{x}_0) = 0$ is proved in [OSS15, Lemma 4.3.5]. For any $\mathbf{y}$ that is related to $\mathbf{x}_0$ by a transposition, there are exactly two empty rectangles that connect $\mathbf{x}_0$ to $\mathbf{y}$, and neither of them contains O. Thus $\partial \mathbf{x}_0 = 0$. Conversely, rectangles that connect $\mathbf{y}$ to $\mathbf{x}_0$ contains either a point of $\mathbf{x}_0$ or one of O, so there is no chain c such that ∂c contains $\mathbf{x}_0$ with coefficient 1 in its terms. Thus $\mathbf{x}_0$ is non-boundary. □

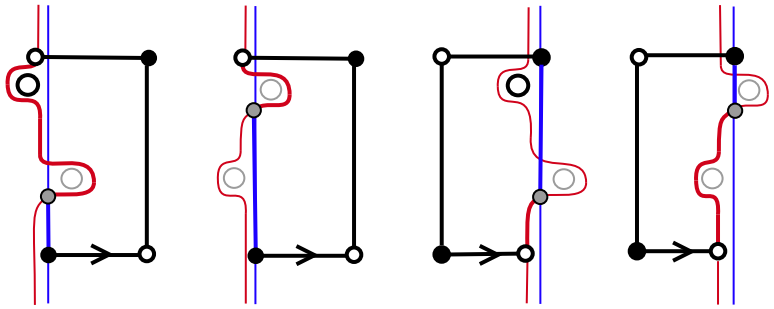


Figure 4: Relative positions of O_i, O_{i+1} and the pentagon Π

Obviously G can be transformed into a grid diagram G_0, whose O's lie in its anti-diagonal, by applying some sequence of *commutation moves* (moves that swap two adjacent columns). Now reverse the sequence. Following the arguments given in [Man+07, Section 3.1], each commutation move $G_i \to G_{i+1}$ is assigned a chain homotopy equivalence map $\Phi : C(G_i) \to C(G_{i+1})$. Starting from $\mathbf{x}_0 \in C_0(G_0)$, we obtain one desired cycle in $C_0(G)$ by sequentially applying the corresponding chain maps. Here we explain the explicit computation of Φ.

Given two diagrams that are related by a single commutation move, we depict the move by drawing the two diagrams on the same grid, where the intermediate circle β is to be replaced with a different circle γ. By perturbing γ so that it intersects β transversally at two points that do not lie in the horizontal circles, the chain map Φ is given by

$$\Phi(\mathbf{x}) = \sum_{\mathbf{y} \in S(G')} \sum_{\Pi \in \mathrm{Pent}^\circ(\mathbf{x},\mathbf{y})} U_1^{\epsilon_1} \cdots U_n^{\epsilon_n} \mathbf{y},$$

where $\mathrm{Pent}^\circ(\mathbf{x}, \mathbf{y})$ denotes the set of empty pentagons connecting $\mathbf{x}$ to $\mathbf{y}$, and for each such pentagon Π, the exponent $\epsilon_i \in \{0, 1\}$ is given by the number of intersections of Π and O_i. (Note that the labels of the O-markings are corresponding to G_{i+1}.) See [Man+07] for precise definition.

The actual computation is easy. Suppose we are swapping the i-th and the $(i+1)$-th columns, so that O_i and O_{i+1} interchanges. Then ϵ_i is always 0, and $\epsilon_{i+1} = 1$ if and only if, before the commutation move, O_i is located at the left-upper of O_{i+1} and its vertical coordinate is between the lower and upper sides of Π. This can be seen from Figure 4, where in each of the four cases β is drawn as a straight vertical blue line, γ is drawn as a curve close to β. The intersection of the two curves is drawn as a small gray disk, and the other four small disks form the vertices of the pentagon Π. Note that we can ignore the X's since they are not involved in both ∂ and Φ. For the remaining exponents ϵ_j, it is easy to check for each O_j whether it is contained in Π.

5.4 Determining the realizability

Having prepared the inflated generators and one homological generator z, we are ready to determine the realizability of the candidate regions. From Corollary 4.5, a closed region R of shift number s is realizable (i.e. $R \in \widetilde{\mathcal{G}}_0(K)$) if and only if there is a cycle that is homologous to $(U_1)^s z$ and is contained in $C_{R[s]}$. This is equivalent to the condition that (the projection of) $(U_1)^s z$ is null homologous in the quotient complex $Q_R := C/C_{R[s]}$. For any $k \in \mathbb{Z}$, we can take a finite basis for $(Q_R)_k$ from the inflated generators contained in C_k and modding out those contained in $(C_{R[s]})_k$. Having fixed such bases in degree $-2s$ and $-2s+1$, the differential $\partial : (Q_R)_{-2s+1} \to (Q_R)_{-2s}$ is represented by a matrix A. Thus the realizability of R is equivalent to the existence of a solution x of the linear system

$$Ax = b,$$

where b is the vector corresponding to $(U_1)^s z \in (Q_R)_{-2s}$. Now that we have reduced the problem to linear systems, the remaining task is purely computational. Here we only mention that the matrix A is generally sparse, and becomes extremely large as n increases. More details on sparse linear systems is given in Appendix B.

Remark 5.5. This procedure of reducing a filtration-level type problem to linear systems can be applied to computing some other filtration-level type invariants. For example, Rasmussen's s-invariant ([Ras10]), which is an integer valued knot invariant defined by a $\mathbb{Z}$-filtration on (a deformed version of) Khovanov homology over $\mathbb{Q}$, can be computed by the same method.

5.5 Integration

The four procedures are integrated into a program that goes through the following six steps:

1. Enumerate the candidate regions.
2. Setup the inflated generators.
3. Compute one homological generator.
4. Compute $\tau(K)$.
5. Determine the simple regions in $\mathcal{G}_0(K)$.
6. Check the remaining candidate regions.

Assuming that we have gone through the first three steps, we explain how remaining three work, together with some techniques for reducing the computational cost. Before moving on, we remark that determining the realizability of a candidate region may contribute to discarding some other candidates. Namely, if we find a realizable region R, then we may discard candidates that include R, since they are obviously realizable and non-minimal. Conversely, if we find

a non-realizable R, then we discard candidates that are included in R, since they are obviously non-realizable. The program terminates at any step if the candidates become empty.

5.5.1 Computing $\tau(K)$

From Theorem 2.20, we know that $\tau(K)$ is given by the minimum integer j such that $R_{(-1,\infty)} \cup R_{(0,j)}$ is realizable and $R_{(-1,\infty)} \cup R_{(0,j-1)}$ is not. Here if $\tau(K) = 0$, there is a chance of an early exit: if $R_{(0,0)}$ is realizable for both K and K^*, then from Theorem 2.16 we immediately have $\mathcal{G}_0(K) = \{R_{(0,0)}\}$. In the following we assume that $\tau(K) \geq 0$, by replacing G with its mirror if necessary.

5.5.2 Determining the simple regions in $\mathcal{G}_0(K)$

Next we narrow down the candidates by determining the simple regions in $\mathcal{G}_0(K)$. Starting from $R_{(0,g_4)}$, we check all simple candidate regions. Note that a simple region $R = R_{(i,j)}$ belongs to $\mathcal{G}_0(K)$ if and only if R is realizable and $R \setminus \{(i,j)\}$ is not. This strategy actually works, as we see from the results that for prime knots K with up to 11 crossings and $\tau(K) \geq 0$, $\mathcal{G}_0(K)$ consist only of regions whose corners lie in the first quadrant, and in many cases they are all simple.

5.5.3 Checking the remaining candidates

Having determined the simple regions in $\mathcal{G}_0(K)$, we expect that the remaining candidates are all non-realizable (we still assume $\tau(K) \geq 0$). Suppose R is the maximal remaining candidate. If we succeed to show that R is non-realizable, then we are done. However such computation tends to be highly expensive, since R has relatively high shift number. To avoid such computation whenever possible, it is useful to consider the mirror simultaneously. For any subset $R \subset \mathbb{Z}^2$, recall that $-R \subset \mathbb{Z}^2$ is defined by

$$-R := \{(i,j) \in \mathbb{Z}^2 \mid (-i,-j) \in R\}.$$

Then the following lemmas are obtained from Theorem 2.16.

Lemma 5.6. *For any pair of realizable regions R of K and R' of K^*, we have $R \cap (-R') \neq \varnothing$.*

Lemma 5.7. *Suppose $\mathcal{S}$ is a set of closed regions that includes $\mathcal{G}_0(K)$. If a closed region R' intersects $-R$ for any $R \in \mathcal{S}$, then R' is a realizable region of K^*.*

Thus the intermediate results obtained for K can be reflected to those of K^* and vice versa. After having determined the simple regions in $\mathcal{G}_0(K)$, we expect that the maximal remaining region of K is non-realizable, and the minimal remaining region of K^* is realizable. We select the one with smaller computational cost (which is simply estimated by the size of the corresponding matrix). This final process is continued until the candidates are empty.

5.5.4 Rejection by a smaller system

As we have seen, there are cases where we expect that a specific region R is non-realizable. Suppose there is a smaller chain complex C' and a $\mathbb{Z}^2$-filtered chain map $f : C \to C'$. The existence of $c \in C$ such that

$$z - \partial c \in C_R$$

implies the existence of $c' \in C'$ such that

$$f(z) - \partial c' \in C'_R.$$

Hence by contraposition, the non-realizability of R can possibly be detected using a smaller system. Our program uses the chain map induced by the following projection:

$$\pi : \mathbb{F}[U_1, \cdots, U_n] \to \mathbb{F}[U_1, U_2], \quad U_1 \mapsto U_1,\ U_{\geq 2} \mapsto U_2.$$

Further reduction techniques on the level of linear systems are described in Appendix B.

6 Computational results

Here we list the computational results performed for prime knots of crossing number up to 11, that are both non-slice and homologically thick with respect to $\widehat{HFK}$. The reason for this selection is that (i) if a knot K is slice then $\mathcal{G}_0(K) = \{R_{(0,0)}\}$, and (ii) if K is homologically thin then $\mathcal{G}_0(K) = \mathcal{G}_0(\tau(K)[T_{2,3}])$, so the non-obvious ones are those complements.

We have developed a program `g-zero` [3] based on the algorithm explained in Section 5. For the input data for actual computations, we used Culler's program `Gridlink` [Cul] to produce the input grid diagrams. This program contains the preset data for prime knots with up to 12 crossings, which are extracted from Livingston's Knotinfo database [LM]. It also has the `simplify` function that randomly modifies a given diagram to reduce its grid number. In addition to the input diagram data, we used values g_3, g_4 and τ from Knotinfo, and also $\widehat{HFK}$ from [BG12].

Table 1 presents the $\mathcal{G}_0$-*types* of the target knots. We say that K has the $\mathcal{G}_0$-*type* of a typical knot T, when $\mathcal{G}_0(K) = \mathcal{G}_0(T)$. We use torus knots and those cables as typical knots, for example we read that 8_{19} has the $\mathcal{G}_0$-type of the $(3, 4)$-torus knot. The listed types are the 'positive side' of the knot, that is, either one of K and K^* having $\tau \geq 0$. In this list, we see that there are only six types that appear as those $\mathcal{G}_0$-types. The next Table 2 presents the values of $\mathcal{G}_0(K)$, $\tau(K)$, $V_k(K)$ and $V_k(K^*)$ for such typical knots K. Note that $T_{2,3;2,1}$ (the $(2, 1)$ cable of $T_{2,3}$) is the only type in this list that possesses a non-simple

[3] The software `g-zero` has since been superseded by `yui-hfk` [San]. We have verified that all computational results reported in this paper can also be reproduced using `yui-hfk`.

minimal region $R_{(0,1)} \cup R_{(1,0)}$. The algorithm for computing $\mathcal{G}_0(K^*)$ from $\mathcal{G}_0(K)$ is described in Corollary A.7. Finally Table 3 presents Υ_K for the typical knots K, where $PL[(x_0, y_0), \cdots, (x_n, y_n)]$ denotes the PL-function from $[x_0, x_n]$ to $\mathbb{R}$ whose graph is given by linearly connecting the adjacent coordinates.

Computations are performed for another set of knots, that is, prime knots of crossing number 12 that are non-slice, homologically thick, and $\tau = 0$. As mentioned in Section 1, results have shown that all such knots have $\mathcal{G}_0$-type of the unknot. Hence we obtain Theorem 1.1.

For the remaining knots with 12 crossings, the computation time of our algorithm on a usual laptop computer becomes too long to complete the list. Here, we remark that if we have some special value of Υ_K, say $\upsilon(K) = \Upsilon_K(1)$, then the value would reduce remaining candidates of $\mathcal{G}_0(K)$ so effectively that we can complete our list for prime knots with up to 12 crossings. Földvári [Föl21] gives a combinatorial description of Υ_K using grid homology, which possibly provides an effective computation of some special value of Υ_K.

K	$\mathcal{G}_0$-type
8_{19}	$T_{3,4}$
9_{42}	O
10_{124}	$T_{3,5}$
10_{128}	$T_{3,4}$
10_{132}	$T_{2,3}$
10_{136}	O
10_{139}	$T_{3,5}$
10_{145}	$T_{2,5}$
10_{152}	$T_{3,5}$
10_{154}	$T_{3,4}$
10_{161}	$T_{3,4}$
$11n6$	O
$11n9$	$T_{3,4}$
$11n12$	$T_{2,3}$
$11n19$	$T_{2,3}$

K	$\mathcal{G}_0$-type
$11n20$	O
$11n24$	O
$11n27$	$T_{3,4}$
$11n31$	$T_{2,5}$
$11n34$	O
$11n38$	O
$11n45$	O
$11n57$	$T_{3,4}$
$11n61$	$T_{2,3;2,1}$
$11n70$	$T_{2,3}$
$11n77$	$T_{3,5}$
$11n79$	O
$11n80$	$T_{2,3}$
$11n81$	$T_{3,4}$
$11n88$	$T_{3,4}$

K	$\mathcal{G}_0$-type
$11n92$	O
$11n96$	O
$11n102$	$T_{2,3}$
$11n104$	$T_{3,4}$
$11n111$	$T_{2,3}$
$11n126$	$T_{3,4}$
$11n133$	$T_{2,3;2,1}$
$11n135$	$T_{2,5}$
$11n138$	O
$11n143$	O
$11n145$	O
$11n151$	$T_{2,3}$
$11n152$	$T_{2,3}$
$11n183$	$T_{3,4}$

Table 1: List of $\mathcal{G}_0$-types

K	$\mathcal{G}_0(K)$	$\tau(K)$	$V_k(K)$	$V_k(K^*)$
O	$R_{(0,0)}$	0	{0}	{0}
$T_{2,3}$	$R_{(0,1)}, R_{(1,0)}$	1	{1, 0}	{0}
$T_{2,5}$	$R_{(0,2)}, R_{(1,1)}, R_{(2,0)}$	2	{1, 1, 0}	{0}
$T_{2,3;2,1}$	$R_{(0,2)}, R_{(0,1)} \cup R_{(1,0)}, R_{(2,0)}$	2	{1, 1, 0}	{0}
$T_{3,4}$	$R_{(0,3)}, R_{(1,1)}, R_{(3,0)}$	3	{1, 1, 1, 0}	{0}
$T_{3,5}$	$R_{(0,4)}, R_{(1,2)}, R_{(2,1)}, R_{(4,0)}$	4	{2, 1, 1, 1, 0}	{0}

Table 2: $\mathcal{G}_0(K)$, $\tau(K)$, $V_k(K)$ and $V_k(K^*)$

K	Υ_K
O	0
$T_{2,3}$	$PL[(0,0),(1,-1),(2,0)]$
$T_{2,5}$	$PL[(0,0),(1,-2),(2,0)]$
$T_{2,3;2,1}$	$PL[(0,0),(2/3,-4/3),(1,-1),(4/3,-4/3),(2,0)]$
$T_{3,4}$	$PL[(0,0),(2/3,-2),(4/3,-2),(2,0)]$
$T_{3,5}$	$PL[(0,0),(2/3,-8/3),(1,-3),(4/3,-8/3),(2,0)]$

Table 3: Υ_K

A The duality theorem for $\mathcal{G}_0$

In this appendix, we prove the duality theorem for $\mathcal{G}_0$, stated as Theorem 2.16. To prove the theorem, we first introduce the notion of *the dual* of a closed region R (denoted R^*). Then, for any $C = CFK^\infty(K)$, it is proved that the subcomplex C^*_R consists of chains which map C_{R^*} to zero. In Appendix A.2, we give a proof of Theorem 2.16. In Appendix A.3, we provide a method for computing $\mathcal{G}_0(K^*)$ from $\mathcal{G}_0(K)$ algorithmically.

A.1 The dual of a closed region

For any given $R \in \mathcal{CR}(\mathbb{Z}^2)$, *the dual region* R^* *of* R is defined by

$$R^* := \mathbb{Z}^2 \setminus (-R) = \{(i,j) \in \mathbb{Z}^2 \mid (-i,-j) \notin R\}.$$

Here we show several basic properties of dual regions.

Lemma A.1. $R^* \in \mathcal{CR}(\mathbb{Z}^2)$.

Proof. Suppose that $(k,l) \in \mathbb{Z}^2$, $(i,j) \in R^*$ and $(k,l) \leq (i,j)$. Under these assumptions, if $(k,l) \notin R^*$, then $(-k,-l) \in R$. On the other hand, since $(-i,-j) \leq (-k,-l)$, we have $(-i,-j) \in R$. This contradicts to $(i,j) \in R^*$, and hence $(k,l) \in R^*$. □

Lemma A.2. $R^{**} = R$.

Proof. $(i,j) \in R^{**} \Leftrightarrow (-i,-j) \neq R^* \Leftrightarrow (i,j) \in R$. □

Lemma A.3. $R^* = \bigcap_{(k,l)\in R}\{i \leq -k-1 \text{ or } j \leq -l-1\}$.

Proof. We first prove $R^* \subset \bigcap_{(k,l)\in R}\{i \leq -k-1 \text{ or } j \leq -l-1\}$. Let $(i,j) \in R^*$. Then, for any $(k,l) \in R$, either $-i > k$ or $-j > l$ holds. Equivalently, we have $i \leq -k-1$ or $j \leq -l-1$.

Next, we prove $R^* \supset \bigcap_{(k,l)\in R}\{i \leq -k-1 \text{ or } j \leq -l-1\}$. Let $(i,j) \in \cap_{(k,l)\in R}\{i \leq -k-1 \text{ or } j \leq -l-1\}$. Then, for any $(k,l) \in R$, either $-i > k$ or $-j > l$ holds. In particular, we have $(-i,-j) \neq (k,l)$, and hence $(-i,-j) \notin R$. □

Using the above lemmas, we have the following proposition, which plays an essential role in the proof of Theorem 2.16.

Proposition A.4. *Let* $C := CFK^\infty(K)$. *Then, for any* $R \in \mathcal{CR}(\mathbb{Z}^2)$, *we have*

$$C^*_R = \{\psi \in C^* \mid \varepsilon \circ \psi(C_{R^*}) = 0\}.$$

Proof. We first prove $C_R^* \subset \{\psi \in C^* \mid \varepsilon \circ \psi(C_{R^*}) = 0\}$. Indeed, this follows from the inequalities

$$\begin{aligned} C_R^* &= \sum_{(k,l)\in R} \mathcal{F}_k^{\mathrm{Alg}}(C^*) \cap \mathcal{F}_l^{\mathrm{Alex}}(C^*) \\ &= \sum_{(k,l)\in R} \left\{ \psi \in C^* \;\middle|\; \varepsilon \circ \psi(\mathcal{F}_{-k-1}^{\mathrm{Alg}} + \mathcal{F}_{-l-1}^{\mathrm{Alex}}) = 0 \right\} \\ &= \sum_{(k,l)\in R} \left\{ \psi \in C^* \;\middle|\; \varepsilon \circ \psi(C_{\{i\le -k-1 \text{ or } j\le -l-1\}}) = 0 \right\} \\ &\subset \left\{ \psi \in C^* \;\middle|\; \varepsilon \circ \psi(C_{\bigcap_{(k,l)\in R}\{i\le -k-1 \text{ or } j\le -l-1\}}) = 0 \right\} \\ &= \{\psi \in C^* \mid \varepsilon \circ \psi(C_{R^*}) = 0\}. \end{aligned}$$

Next, we prove $C_R^* \supset \{\psi \in C^* \mid \varepsilon \circ \psi(C_{R^*}) = 0\}$. Take a filtered basis $\{x_s\}_{1\le s\le r}$ for C, and then any $\psi \in C^*$ is written as $\psi = \sum_{1\le s\le r,\ t\in\mathbb{Z}} a_{s,t} U^t x_s^*$, where $a_{s,t} = 0$ for all but finitely many (s,t). Now, suppose that $\varepsilon \circ \psi(C_{R^*}) = \varepsilon \circ \psi(C_{\bigcap_{(k,l)\in R}\{i\le -k-1 \text{ or } j\le -l-1\}}) = 0$. Then, we see that $U^{-t}x_s \notin C_{R^*}$ for any (s,t) with $a_{s,t} \neq 0$. This implies that $R_{U^{-t}x_s} = R_{(\mathrm{Alg}(U^{-t}x_s),\mathrm{Alex}(U^{-t}x_s))} \notin R^*$, and hence $(\mathrm{Alg}(U^{-t}x_s), \mathrm{Alex}(U^{-t}x_s)) \geq (-k,-l)$ for some $(k,l) \in R$. In particular, we have

$$U^t x_s^*(\mathcal{F}_{-k-1}^{\mathrm{Alg}} + \mathcal{F}_{-l-1}^{\mathrm{Alex}}) = 0$$

for such (k,l). Consequently, we have

$$\psi = \sum_{a_{s,t}\neq 0} a_{s,t} U^t x_s^* \in \sum_{(k,l)\in R} \left\{ \psi \in C^* \;\middle|\; \varepsilon \circ \psi(\mathcal{F}_{-k-1}^{\mathrm{Alg}} + \mathcal{F}_{-l-1}^{\mathrm{Alex}}) = 0 \right\} = C_R^*.$$

□

A.2 Proof of Theorem 2.16

Now we prove Theorem 2.16.

Theorem 2.16. *For any knot K, the equalities*

$$\widetilde{\mathcal{G}}_0(K^*) = \{R \in \mathcal{CR}(\mathbb{Z}^2) \mid \forall R' \in \mathcal{G}_0(K),\ R \cap (-R') \neq \varnothing\}$$

and

$$\mathcal{G}_0(K^*) = \min\{R \in \mathcal{CR}(\mathbb{Z}^2) \mid \forall R' \in \mathcal{G}_0(K),\ R \cap (-R') \neq \varnothing\}$$

hold.

The proof is an analogy of [Sat23, Lemma 2.28].

Proof. By the definition of $\mathcal{G}_0(K)$, it is sufficient to prove the first equality. We denote by $\mathcal{G}_0(K)^\perp$ the right hand side of the desired equality; namely, we set

$$\mathcal{G}_0(K)^\perp := \{R \in \mathcal{CR}(\mathbb{Z}^2) \mid \forall R' \in \mathcal{G}_0(K),\ R \cap (-R') \neq \varnothing\}.$$

Then, by the definition of dual regions, it is obvious that

$$\mathcal{G}_0(K)^\perp = \{R \in \mathcal{CR}(\mathbb{Z}^2) \mid \forall R' \in \mathcal{G}_0(K),\ R' \not\subset R^*\}.$$

We first prove $\widetilde{\mathcal{G}}_0(K^*) \subset \mathcal{G}_0(K)^\perp$. Let $R \in \widetilde{\mathcal{G}}_0(K^*)$ and $C := CFK^\infty(K)$. Then, by Proposition A.4, there exists a homological generator $\varphi \in C_0^*$ lying in

$$C_R^* = \{\psi \in C^* \mid \varepsilon \circ \psi(C_{R^*}) = 0\}.$$

In particular, we have $\varepsilon \circ \varphi(C_{R^*}) = 0$, and $\varepsilon \circ \varphi$ is decomposed as $\varepsilon \circ \varphi = \widetilde{\varphi} \circ p$ where $\widetilde{\varphi} \in \mathrm{Hom}_{\mathbb{F}}(C/C_{R^*}, \mathbb{F})$ is a cocycle and $p : C \to C/C_{R^*}$ is the projection. Now, take $R' \in \mathcal{G}_0(K)$ and a homological generator $x \in C_{R'}$. Then we have $\widetilde{\varphi}(p(x)) = (\varepsilon \circ \varphi)(x) = 1$. This implies that $p(x) \neq 0$, and hence $R^* \not\supset R'$. Therefore, we have $R \in \mathcal{G}_0(K)^\perp$.

Next, we prove $\widetilde{\mathcal{G}}_0(K^*) \supset \mathcal{G}_0(K)^\perp$. Suppose that $R \in \mathcal{G}_0(K)^\perp$. Then, by Theorem 2.13, we see that $R^* \notin \widetilde{\mathcal{G}}_0(K)$, and hence $p_{*,0} \colon H_0(C) \to H_0(C/C_{R^*})$ is injective. Let $x \in C_0$ be a homological generator, and then we have $p_{*,0}([x]) \neq 0$. In addition, $\dim_{\mathbb{F}}(C/C_{R^*})_0$ is finite, and hence we can take a finite $\mathbb{F}$-basis for $H_0(C/C_{R^*})$ containing $p_{*,0}([x])$. Thus, by using the identification (given by Lemma 2.5)

$$\mathrm{Hom}_{\mathbb{F}}(H_0(C/C_{R^*}), \mathbb{F}) \cong H^0(C/C_{R^*}; \mathbb{F}),$$

we can take a cocycle $\psi \in \mathrm{Hom}_{\mathbb{F}}((C/C_{R^*})_0, \mathbb{F})$ whose cohomology class is the dual $(p_{*,0}([x]))^*$. Moreover, the map ε_0 in Lemma 2.5 is bijective, and hence we can take the inverse $\varphi := \varepsilon_0^{-1}(\psi \circ p) \in C_0^*$. Note that since $\varepsilon \circ \varphi(x) = \psi(p(x)) = 1$, the element $\varphi \in C_0^*$ is a homological generator. Moreover, the equalities

$$\varepsilon \circ \varphi(C_{R^*}) = \psi \circ p(C_{R^*}) = 0$$

hold, and hence φ lies in C_R^*. This proves $R \in \widetilde{\mathcal{G}}_0(K^*)$. □

A.3 Computing $\mathcal{G}_0(K^*)$ from $\mathcal{G}_0(K)$

For any non-empty finite set $\mathcal{S} = \{R_i\}_{i=1}^N$ of semi-simple regions, let

$$S^\perp := \{R \in \mathcal{CR}(\mathbb{Z}^2) \mid \forall R_i \in \mathcal{S},\ R \cap (-R_i) \neq \varnothing\}.$$

Then we have the following duality theorem between $\mathcal{S}$ and $\min \mathcal{S}^\perp$.

Theorem A.5. *The equality*

$$\min(\mathcal{S}^\perp) = \min\left\{ cl(\{-p_i\}_{i=1}^N) \;\middle|\; (p_1, \ldots, p_N) \in \prod_{i=1}^N c(R_i) \right\}$$

holds.

Remark A.6. It can happen that $p_i \leq p_j$ for different i and j, and hence the order of $\{p_i\}_{i=1}^N$ can be less than N.

Recall that $\mathcal{G}_0(K) \subset \mathcal{CR}^{ss}(\mathbb{Z}^2)$ for any knot K, and hence, combining with Theorem 2.16, we have the following formula.

Corollary A.7. *Let $\mathcal{G}_0(K) = \{R_i\}_{i=1}^N$. Then, the equality*

$$\mathcal{G}_0(K^*) = \min\left\{ cl(\{-p_i\}_{i=1}^N) \;\middle|\; (p_1, \dots, p_N) \in \prod_{i=1}^N c(R_i) \right\}$$

holds. In particular, $\mathcal{G}_0(K^*)$ is algorithmically determined from $\mathcal{G}_0(K)$.

Now we prove Theorem A.5.

Proof of Theorem A.5. Set $\mathcal{S}' := \{cl(\{-p_i\}_{i=1}^N) \mid (p_1, \dots, p_N) \in \prod_{i=1}^N c(R_i)\}$. We first note that for any $(p_1, \dots, p_N) \in \prod_{i=1}^N c(R_i)$, we have

$$-p_i \in cl(\{-p_i\}_{i=1}^N) \cap (-R_i) \neq \varnothing.$$

This implies $\mathcal{S}' \subset \mathcal{S}^\perp$.

Now we prove $\min(\mathcal{S}^\perp) \subset \min \mathcal{S}'$. Let $R \in \min(\mathcal{S}^\perp)$. Then $R \cap (-R_i) \neq \varnothing$ for any $1 \leq i \leq N$, which is equivalent to $R_i \not\subset R^*$. Here, Lemma 2.26 gives $R_i = cl(c(R_i))$, and hence there exists an element $p_i \in c(R_i)$ such that $p_i \notin R^*$. In particular, we have $-p_i \in R$, and hence $R \supset cl(\{-p_i\}_{i=1}^N)$. Since $cl(\{-p_i\}_{i=1}^N) \in \mathcal{S}' \subset \mathcal{S}^\perp$, it follows from the minimality of R in $\mathcal{S}^\perp$ that

$$R = cl(\{-p_i\}_{i=1}^N) \in \mathcal{S}'.$$

The minimality of R in $\mathcal{S}'$ directly follows from the minimality in $\mathcal{S}^\perp$. Therefore, we have $R \in \min S'$.

Next, we prove $\min(\mathcal{S}^\perp) \supset \min \mathcal{S}'$. Let $cl(\{-p_i\}_{i=1}^N) \in \min \mathcal{S}'$. Then we have $cl(\{-p_i\}_{i=1}^N) \in \mathcal{S}^\perp$. Suppose that $R \in \mathcal{S}^\perp$ satisfies $R \subset cl(\{-p_i\}_{i=1}^N)$. Then, in a similar way to the converse case, we can find an element $cl(\{-q_i\}_{i=1}^N) \in \mathcal{S}'$ such that $cl(\{-q_i\}_{i=1}^N) \subset R$. In particular, we have

$$cl(\{-q_i\}_{i=1}^N) \subset R \subset cl(\{-p_i\}_{i=1}^N),$$

and hence the minimality of $cl(\{-p_i\}_{i=1}^N)$ in $\mathcal{S}'$ gives

$$cl(\{-q_i\}_{i=1}^N) = R = cl(\{-p_i\}_{i=1}^N).$$

This implies $cl(\{-p_i\}_{i=1}^N) \in \min(\mathcal{S}^\perp)$. □

B On sparse linear systems

Here we briefly discuss computational methods for handling large sparse linear system over any computational field, such as $\mathbb{Q}$ or $\mathbb{F}_p$. There are essentially two families of algorithms for handling linear systems: direct methods and iterative methods. Given a matrix A, direct methods (such as Gaussian elimination and LU factorization) perform elementary operations on A so that information can

be read directly from the modified matrix A', whereas iterative methods (such as the Wiedemann algorithm) multiply A repeatedly on a randomly generated vector seeking for a solution. The implementation of our program is based on the former.

LU factorization is one of the standard methods for handling (dense) linear systems. Suppose A is an arbitrary n-by-m matrix of rank r. It is known that, after some permutations of rows and columns, A can be factored into a product of two matrices L and U, where L is a lower triangular n-by-r matrix and U is an upper triangular r-by-m matrix. To be precise, there exists some permutation matrices P and Q such that

$$PAQ = LU$$

holds (in some literature it is expressed in the form $A = PLUQ$ and called the $PLUQ$ factorization of A). This can be seen as factoring a linear map represented by A into a composition of a surjection (represented by U) and an injection (represented by L). Having computed $PAQ = LU$, solving a linear system $Ax = b$ breaks up into solving $Ly = Pb$ and $U(Q^{-1}x) = y$. The former equation may or may not have a solution, and in case it does the solution is unique. The latter equation always has a solution with degree of freedom $m-r$. Both equations can be solved easily from the structure of L and U. In the following, for simplicity, we omit the permutations P, Q and write $A \sim LU$.

LU factorization is obtained from Gaussian elimination. Suppose $A = (a_{ij})_{1\le i\le n,\ 1\le j\le m}$ is non-zero. Then there exists some non-zero component of A, which we may assume that it is a_{11}. We may write

$$A \sim \begin{pmatrix} 1 \\ a_{21}/a_{11} \\ \vdots \\ a_{n1}/a_{11} \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \end{pmatrix} + \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & A' & \\ 0 & & & \end{pmatrix}.$$

If $A' = (a'_{ij})_{2\le i\le n,\ 2\le j\le m}$ is non-zero, then we repeat the same process and obtain

$$A \sim \begin{pmatrix} 1 & 0 \\ a_{21}/a_{11} & 1 \\ \vdots & \vdots \\ a_{n1}/a_{11} & a'_{n2}/a'_{22} \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ 0 & a'_{22} & \cdots & a'_{2m} \end{pmatrix} + \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & A'' & \\ 0 & 0 & & \end{pmatrix}.$$

Repeating this process until the remaining block becomes zero yields the desired factorization. Note that at each step, permutations of the complementary submatrix preserves the structure of the partially computed L and U. This process can also be seen as iteratively computing the *Schur complement* of the upper left non-zero component. In general, for a block matrix

$$X = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

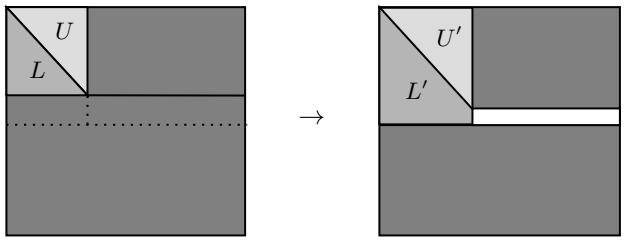


Figure 5: Incremental LU factorization

with A non-singular, the *Schur complement* of A is defined by

$$S := D - CA^{-1}B.$$

Now we assume that the matrix A is large and sparse, say $n, m > 500,000$ and its density is below 0.01%. In such a case, it is impractical to calculate its LU factorization directly. For this problem, Bouillaguet, Delaplace and Voge [BDV17] propose a method of finding a large amount of *structural pivots*, and the algorithm is implemented in their program `SpaSM` [BBD]. Structural pivots are non-zero entries of A such that after some permutations A is transformed into a matrix of the form

$$\begin{pmatrix} U & B \\ C & D \end{pmatrix},$$

where U is a regular upper triangular matrix with pivots lying in its diagonal. Given a set of structural pivots, we immediately obtain a partial LU factorization of A as

$$A \sim \begin{pmatrix} I \\ C' \end{pmatrix} \begin{pmatrix} U & B \end{pmatrix} + \begin{pmatrix} O & O \\ O & S \end{pmatrix},$$

where $C' = CU^{-1}$, and $S = D - CU^{-1}B$ is the Schur complement of U. To complete the LU factorization of A, we either iterate the same process on S, or switch to dense LU factorization, depending on the size and density of S.

The heavy part of the above described method is in the computation of the Schur complement. For our purpose, recall from Section 5.4 that we only need to tell whether the system has a solution or not. Assuming that we have obtained pivots whose number is close to the rank of A, only a few more independent columns might suffice to find one solution. Likewise, only a few more independent rows might suffice to detect that the system has no solution. Thus the approach we take is the following: after we have obtained the pivots and permuted the matrix correspondingly, we divide the remaining columns (resp. rows) into small chunks and incrementally expand the factorization by performing the factorization on the submatrices (see Figure 5). At each step we check whether the subsystem has a solution or not. This enables us to exit the computation in an earlier stage whenever possible.

References

[BG12] John A. Baldwin and William D. Gillam. "Computations of Heegaard-Floer knot homology". In: *J. Knot Theory Ramifications* 21.8 (2012), pp. 1250075, 65.

[BBD] Charles Bouillaguet, Alexis Breust, and Claire Delaplace. *SpaSM: a Sparse Direct Solver Modulo p*. `https://github.com/cbouilla/spasm/tree/v1.2`. Version 1.2; accessed November 5, 2026.

[BDV17] Charles Bouillaguet, Claire Delaplace, and Marie-Emilie Voge. "Parallel Sparse PLUQ Factorization modulo p". In: *Proceedings of the International Workshop on Parallel Symbolic Computation, PASCO@ISSAC 2017, Kaiserslautern, Germany, July 23-24, 2017*. Ed. by Jean-Charles Faugère, Michael B. Monagan, and Hans-Wolfgang Loidl. ACM, 2017, 8:1–8:10.

[Cul] Marc Culler. *Gridlink: A tool for knot theorists*. `http://homepages.math.uic.edu/~culler/gridlink/`. Accessed November 5, 2026.

[Föl21] Viktória Földvári. "The knot invariant Υ using grid homologies". In: *J. Knot Theory Ramifications* 30.7 (2021), Paper No. 2150051, 26.

[Hea63] B. R. Heap. "Permutations by Interchanges". In: *The Computer Journal.* 6.3 (1963), pp. 293–298.

[Hom17] Jennifer Hom. "A survey on Heegaard Floer homology and concordance". In: *J. Knot Theory Ramifications* 26.2 (2017), pp. 1740015, 24.

[HW16] Jennifer Hom and Zhongtao Wu. "Four-ball genus bounds and a refinement of the Ozváth-Szabó tau invariant". In: *J. Symplectic Geom.* 14.1 (2016), pp. 305–323.

[KP18] Min Hoon Kim and Kyungbae Park. "An infinite-rank summand of knots with trivial Alexander polynomial". In: *J. Symplectic Geom.* 16.6 (2018), pp. 1749–1771.

[Liv17] Charles Livingston. "Notes on the knot concordance invariant upsilon". In: *Algebr. Geom. Topol.* 17.1 (2017), pp. 111–130.

[LM] Charles Livingston and Allison H. Moore. *KnotInfo: Table of Knot Invariants*. `https://knotinfo.org/`. Accessed November 5, 2026.

[MOS09] Ciprian Manolescu, Peter Ozsváth, and Sucharit Sarkar. "A combinatorial description of knot Floer homology". In: *Ann. of Math. (2)* 169.2 (2009), pp. 633–660.

[Man+07] Ciprian Manolescu et al. "On combinatorial link Floer homology". In: *Geom. Topol.* 11 (2007), pp. 2339–2412.

[NW15] Yi Ni and Zhongtao Wu. "Cosmetic surgeries on knots in S^3". In: *J. Reine Angew. Math.* 706 (2015), pp. 1–17.

[OS03a] Peter Ozsváth and Zoltán Szabó. "Absolutely graded Floer homologies and intersection forms for four-manifolds with boundary". In: *Adv. Math.* 173.2 (2003), pp. 179–261.

[OS03b] Peter Ozsváth and Zoltán Szabó. "Knot Floer homology and the four-ball genus". In: *Geom. Topol.* 7 (2003), pp. 615–639.

[OS04] Peter Ozsváth and Zoltán Szabó. "Holomorphic disks and knot invariants". In: *Adv. Math.* 186.1 (2004), pp. 58–116.

[OSS15] Peter S Ozsváth, András I Stipsicz, and Zoltán Szabó. *Grid homology for knots and links*. Vol. 208. American Mathematical Soc., 2015.

[OSS17] Peter S. Ozsváth, András I. Stipsicz, and Zoltán Szabó. "Concordance homomorphisms from knot Floer homology". In: *Adv. Math.* 315 (2017), pp. 366–426.

[OS11] Peter S. Ozsváth and Zoltán Szabó. "Knot Floer homology and rational surgeries". In: *Algebr. Geom. Topol.* 11.1 (2011), pp. 1–68.

[Pet13] Ina Petkova. "Cables of thin knots and bordered Heegaard Floer homology". In: *Quantum Topol.* 4.4 (2013), pp. 377–409.

[Pic20] Lisa Piccirillo. "The Conway knot is not slice". In: *Ann. of Math. (2)* 191.2 (2020), pp. 581–591.

[Ras10] Jacob Rasmussen. "Khovanov homology and the slice genus". In: *Invent. Math.* 182.2 (2010), pp. 419–447.

[Ras03] Jacob Andrew Rasmussen. *Floer homology and knot complements*. Thesis (Ph.D.)–Harvard University. ProQuest LLC, Ann Arbor, MI, 2003, p. 126.

[San] Taketo Sano. *yui-hfk: A tool for computing Heegaard Floer knot homology and related invariants*. `https://github.com/taketo1024/yui-hfk/`. Commit 2af9275; accessed November 5, 2026.

[Sat23] Kouki Sato. "The ν^+-equivalence classes of genus one knots". English. In: *Sel. Math., New Ser.* 29.4 (2023). Id/No 63, pp. 1–57.

[Yas26] Kouichi Yasui. "Corks, exotic 4-manifolds and knot concordance". In: *J. Differential Geom.* 132.3 (2026), pp. 547–576.