INTERMEDIATE JACOBIANS AND BURNSIDE INVARIANTS

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Abstract. We propose new invariants in equivariant birational geometry, combining equivariant intermediate Jacobians and the Burnside formalism, for smooth rationally connected threefolds with actions of finite groups.

1. Introduction

This note is inspired by [CKK25], which introduced a version of *atomic* birational invariants of [KKPY25] into equivariant geometry.

We recall the main problem in this area: to determine whether a given generically free regular action of a finite group G on a smooth projective rational variety X of dimension n, over $\mathbb{C}$, is equivariantly birational to a linear, or projectively linear, action on $\mathbb{P}^n$, i.e., to an action arising from a projectivization $\mathbb{P}(V)$ of a $(n+1)$-dimensional representation V of G, respectively, of a central extension of G. We refer to [HKT21] and [HT22] for an introduction to these notions. The linearization problem is settled in dimension 2 [PSY24], but is largely open in dimensions ≥ 3.

Here, we focus on threefolds. We connect the Burnside formalism of [KT22a] with the theory of (equivariant) intermediate Jacobians to recover the most striking applications in [CKK25] in a more classical framework. Concretely, the invariants we offer take into account only the stabilizer stratification and the G-action on the intermediate Jacobian.

Our main contributions in this paper are:

- definition of new birational invariants of G-actions on rationally connected threefolds over $\mathbb{C}$, for arbitrary finite G, in Section 3;
- Proposition 5.2, a classical analog of [CKK25, Theorem 3.6];
- applications to conic bundles, quadric surface bundles, and nodal cubic threefolds, in Section 6.

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2. Generalities

Notation. Throughout, we work over $k = \mathbb{C}$, the complex numbers. We let G be a finite group. By convention, G-actions on varieties are from the right, we emphasize this by writing $X \circlearrowleft G$; correspondingly, the action on the function field is from the left, $G \circlearrowright k(X)$. We write

$$X \sim_G X'$$

to indicate G-equivariant birationality of X and X'.

We briefly recall the framework of equivariant intermediate Jacobians as in [CTT25] and the main ingredients of the Burnside formalism developed in [KT22a] which are relevant for the construction of enhanced birational invariants.

Curves and their Jacobians. Let C be an irreducible smooth projective curve of genus $g(C) \geq 1$ and $(\mathrm{J}(C), \theta_C)$ its Jacobian, with its principal polarization. We have a homomorphism

$$\mathrm{Aut}(C) \to \mathrm{Aut}((\mathrm{J}(C), \theta_C)). \tag{2.1}$$

It is well-known (see, e.g., [Mat58, Section 4]) that by (2.1),

$$\begin{cases} \mathrm{Aut}(C)/C(k) \cong \mathrm{Aut}((\mathrm{J}(C), \theta_C)), & \text{if } g(C) = 1, \\ \mathrm{Aut}(C) \cong \mathrm{Aut}((\mathrm{J}(C), \theta_C)), & \text{if hyperelliptic}, g(C) \geq 2, \\ \mathrm{Aut}(C) \times \{\pm 1\} \cong \mathrm{Aut}((\mathrm{J}(C), \theta_C)), & \text{otherwise.} \end{cases}$$

G-abelian varieties. Let (A, θ_A) be a principally polarized abelian variety. It is called a G-equivariant principally polarized abelian variety if G acts regularly on A preserving both the origin and the class of θ_A in the Néron-Severi group $\mathrm{NS}(A)$; the action is not assumed to be faithful.

We will use the following observation [CTT25, Corollary 3.2]: A G-equivariant principally polarized abelian variety admits a unique, up to permutation of factors, decomposition as a product of indecomposable G-equivariant principally polarized abelian varieties. In combination with [Deb99, Corollary 9.2], we also see that in the non-equivariant decomposition of A as a product of indecomposable principally polarized abelian varieties A_δ, the union $\bigcup_\delta A_\delta$ is G-invariant, hence there is an induced G-action on the disjoint union $\sqcup_\delta A_\delta$.

Intermediate Jacobians. Let X be a smooth projective rationally connected threefold and

$$\mathrm{IJ}(X) := \mathrm{H}^3(X, \mathbb{C})/(\mathrm{H}^1(X, \Omega^2_X) \oplus \mathrm{H}^3(X, \mathbb{Z}))$$

its intermediate Jacobian, with its principal polarization θ_X arising from the cup product

$$\wedge^2 \mathrm{H}^3(X, \mathbb{Z}) \to \mathrm{H}^6(X, \mathbb{Z}) \simeq \mathbb{Z}.$$

When X is rational, $\mathrm{IJ}(X)$ is a product of Jacobians of curves.

Examples of X with computable intermediate Jacobians are standard conic bundles $\pi\colon X \to S$, with smooth discriminant curve $C \subset S$. There is an associated étale double cover $\tau\colon \widetilde{C} \to C$, parametrizing lines over C. The intermediate Jacobian $\mathrm{IJ}(X)$ is the Prym variety $\mathrm{P}(\tau)$ associated with τ; it is the identity component of the locus in the Jacobian $\mathrm{J}(\widetilde{C})$ where the involution induced by τ acts as (-1):

$$\mathrm{IJ}(X) = \mathrm{P}(\tau) = \mathrm{im}(1-\tau) = \big(\ker(1+\tau)\big)^0,$$

see, e.g., [Mum74] for more details regarding this construction. Let G act on X. Then $\mathrm{IJ}(X)$ is a G-equivariant principally polarized abelian variety. We consider the non-equivariant decomposition

$$\mathrm{IJ}(X) = \prod_{\delta \in \Delta} \mathrm{IJ}_\delta(X) \tag{2.2}$$

as a product of indecomposable principally polarized abelian varieties. Then there is an induced G-action on Δ, such that the orbits Δ/G index the indecomposable G-equivariant principally polarized abelian varieties

$$\mathrm{IJ}_\omega(X) = \prod_{\delta \in \omega} \mathrm{IJ}_\delta(X), \qquad \omega \in \Delta/G.$$

The G-action on $\mathrm{IJ}_\omega(X)$ induces an action on $\sqcup_{\delta\in\omega} A_\delta$, transitive on components; in particular, the A_δ, for $\delta \in \omega$, are non-equivariantly isomorphic. For given $\delta \in \omega$, we get an action of the stabilizer G_δ on $\mathrm{IJ}_\delta(X)$, which we express as faithful action

$$\mathrm{IJ}_\delta \circlearrowleft G_\delta/H_\delta, \qquad \text{with} \qquad H_\delta \subseteq G_\delta. \tag{2.3}$$

Let C be an irreducible smooth projective curve of positive genus and $\mathrm{J}(C)$ its Jacobian, with its principal polarization. We will say that ω is *C-relevant* if $\mathrm{IJ}_\delta(X) \cong \mathrm{J}(C)$ for $\delta \in \omega$, as principally polarized abelian varieties. Then, with the union of the C-relevant orbits, we have a G-invariant subset

$$\Delta(C) \subseteq \Delta,$$

such that the set of C-relevant orbits may be recovered as

$$\{C\text{-relevant orbits}\} = \Delta(C)/G \subseteq \Delta/G.$$

Burnside formalism. The Burnside formalism takes as input for the analysis of a regular G-action on a smooth projective X the following data:

- the stabilizer stratification,
- representations of the stabilizers in the normal bundles of strata.

A key initial step is passage to a birational model in *divisorial form*. This is a model satisfying the condition called *Assumption 2* in [KT22a, Section 3]. On such a model, the stabilizers are abelian, and their representations in the normal bundles decompose into direct sums of characters. By [KT22a, Proposition 3.6], two birational models in divisorial form can be connected by a sequence of blow-ups and blow-downs with smooth centers, and each intermediate model is in divisorial form.

In particular, this condition is satisfied if the action is in *standard form*, i.e., X is smooth projective, with simple normal crossing boundary divisor

$$D = \cup_\alpha D_\alpha,$$

such that

- we have a free action of G on $X \setminus D$, and
- for all α and $g \in G$, either $g(D_\alpha) = D_\alpha$ or $g(D_\alpha) \cap D_\alpha = \emptyset$.

The class of the action on an n-dimensional X, in divisorial form, is defined as a sum of symbols

$$[X \circlearrowleft G] := \sum_H \sum_F (H, Y \curvearrowright k(F), \beta), \tag{2.4}$$

a sum over representatives H of conjugacy classes of abelian subgroups of G and over strata F of dimension d with abelian stabilizer H. The symbols record

- the residual action of a subgroup $Y \subseteq Z_G(H)/H$, the quotient of the centralizer of H in G by H, on the function field of F, and
- a sequence $\beta = (b_1, \dots, b_{n-d})$ of characters of H, which appear in the normal bundle to F.

The expression takes values in a group

$$\mathrm{Burn}_n(G)$$

defined by symbols as in (2.4), subject to explicit relations [KT22a, Section 4]. This group has an intricate internal structure. In particular, it admits a direct sum decomposition based on the birational class of the MRC quotient of the stratum F, by [KT25, Remark 3.5]. In the following section, we develop this framework in dimension 3, additionally taking into account information about the G-action on the intermediate Jacobian of the threefold.

3. Curve-localized Burnside groups

We proceed with the definition of new invariants for G-actions on rationally connected threefolds combining intermediate Jacobians and Burnside invariants.

Let C be an irreducible smooth projective curve of genus ≥ 1. We define the *C-localized Burnside group*

$$\mathrm{Burn}_3^C(G)$$

by generators and relations. The computation of the class of the G-action on a smooth projective rationally connected threefold X in this group takes into account only those strata in the stabilizer stratification and only those components of the intermediate Jacobian that are "related" to C, i.e., copies of C, ruled surfaces over C, and $\mathrm{J}(C)$.

Generators. The generators are symbols

$$(H, Y \circlearrowright K, (b_1, b_2)),$$
$$(H, Y \circlearrowright L, (b)),$$
$$(H, \mathrm{J} \circlearrowleft Y),$$

where, respectively,

- $H \subseteq G$ is an abelian subgroup, nontrivial characters b_1, b_2 generate the dual $H^\vee$, and $Y \subseteq Z_G(H)/H$ is a subgroup that acts faithfully on $K \cong k(C)$,
- $H \subseteq G$ is nontrivial cyclic, with character group generated by b, and $Y \subseteq Z_G(H)/H$ acts faithfully on $L \cong k(C \times \mathbb{P}^1)$,
- $H \subseteq G$, and $Y \subseteq N_G(H)/H$ is a subgroup, with action on the principally polarized abelian variety $\mathrm{J} \cong \mathrm{J}(C)$ (action and isomorphism compatible with polarization), which
  - is a faithful action, if $g(C) \geq 2$,
  - comes from a faithful action on C, if $g(C) = 1$.

The symbols are subject to permutation of characters and conjugation relations as in [KT22a, Section 4]:

(P) (Permutation)

$$(H, Y \circlearrowright K, (b_1, b_2)) = (H, Y \circlearrowright K, (b_2, b_1)), \quad \forall\, b_1, b_2.$$

(C) (Conjugation)

$$(H, Y \curvearrowright K, (b_1, b_2)) = (H', Y' \curvearrowright K', (b'_1, b'_2)),$$

when there exists a $g \in G$ such that

$$H' = gHg^{-1}, \quad Y' = gYg^{-1},$$

and b'_1, b'_2 are g-conjugates of b_1, b_2; and similarly for the other two kinds of symbols.

The blow-up relations of [KT22a] are modified to reflect the fact that a blow-up of a G-orbit of C contributes one factor $\mathrm{J}(C)$ to the intermediate Jacobian for every component of the G-orbit, with G-action permuting the factors.

(B1): For H, Y, b_1, b_2 as above, with $b_1 + b_2 = 0$, using the action of Y on $\mathrm{J}(C)$ induced by $C \circlearrowleft Y$, we impose

$$(H, Y \curvearrowright k(C), (b_1, -b_1)) + (H, \mathrm{J}(C) \circlearrowleft Y) = 0.$$

(B2): For H, Y, b_1, b_2, and action of Y on $\mathrm{J}(C)$ as above,

$$(H, Y \curvearrowright k(C), (b_1, b_2)) = \Theta_1 + \Theta_2 + (H, \mathrm{J}(C) \circlearrowleft Y),$$

where, with $\beta_1 = (b_1, b_2 - b_1)$, $\beta_2 = (b_2, b_1 - b_2)$,

$$\Theta_1 = \begin{cases} 0, & \text{if } b_1 = b_2, \\ (H, Y \curvearrowright k(C), \beta_1) + (H, Y \curvearrowright k(C), \beta_2), & \text{otherwise,} \end{cases}$$

and

$$\Theta_2 = \begin{cases} 0, & \text{if } \langle b_1 - b_2 \rangle = H^\vee, \\ (\overline{H}, \overline{Y} \curvearrowright k(C \times \mathbb{P}^1), (b_1|_{\overline{H}})), & \text{otherwise.} \end{cases}$$

In the expression for Θ_2 we put $\overline{H} = \ker(b_1 - b_2)$ and apply the action construction, see [KT22a, Section 2] or [KT25, Section 2], to obtain $\overline{Y}$, with action on $k(C \times \mathbb{P}^1)$.

(B3): For any $Y \subseteq G$ and $C \circlearrowleft Y$,

$$(1, \mathrm{J}(C) \circlearrowleft Y) = 0$$

for the corresponding action of Y on $\mathrm{J}(C)$.

(B4): For any C with $g(C) = 1$,

$$(H, \mathrm{J}(C) \circlearrowleft Y) = (H_1, \mathrm{J}(C) \circlearrowleft Y/Y_1),$$

where Y_1 is the subgroup of Y, acting trivially on $\mathrm{J}(C)$, giving rise to H-extension H_1 of Y_1.

The class of the action. Given a smooth projective rationally connected G-threefold in divisorial form, we define the class of the G-action in the C-localized Burnside group

$$[X \circlearrowleft G]^C \in \mathrm{Burn}_3^C(G)$$

as follows:

$$[X \circlearrowleft G]^C := \sum_F (H_F, Y_F \circlearrowright k(F), \beta_F(X)) + \sum_D (H_D, Y_D \circlearrowright k(D), \beta_D(X)) + \sum_\delta (H_\delta, \mathrm{IJ}_\delta(X) \circlearrowleft Y_\delta),$$

where

- the first sum is over orbit representatives F of subvarieties $F \cong C$ of X, where the generic stabilizer is H_F and generic normal bundle representation $\beta_F(X) = (b_1, b_2)$ with nontrivial b_1, b_2,
- the second sum is over orbit representatives D of divisors in X that are ruled surfaces over C, nontrivial generic stabilizer H_D, and generic normal bundle representation $\beta_D(X) = (b)$,
- the third sum is over orbit representatives δ of C-relevant orbits $\omega \in \Delta/G$, with Δ as in (2.2), where H_δ is as in (2.3),
- in each sum there is the residual action of Y_F on $k(F)$, respectively Y_D on $k(D)$, respectively Y_δ on $\mathrm{IJ}_\delta(X)$.

Example 3.1. Let $X = S \times \mathbb{P}^1$, where S is a degree 2 del Pezzo surface. The threefold X has trivial intermediate Jacobian. Let $G = \mathbb{Z}/2\mathbb{Z}$, acting by the standard covering involution of $\pi : S \to \mathbb{P}^2$ and trivially on $\mathbb{P}^1$. Let C be the ramification curve of π, a smooth quadric curve. The G-fixed locus of X is $C \times \mathbb{P}^1$. The class of the G-action in the C-localized Burnside group is

$$[X \circlearrowleft G]^C = (G, 1 \circlearrowright k(C \times \mathbb{P}^1), (1)) \in \mathrm{Burn}_3^C(G).$$

By relations **(B1)**–**(B2)** this is equal to

$$-2(G, \mathrm{J}(C) \circlearrowleft 1);$$

see the further discussion in Section 6.

4. Blow-up relations

Theorem 4.1. *Let X and X' be smooth projective rationally connected threefolds with a regular action of a finite group G, in divisorial form. Let C be a curve of genus ≥ 1. Then*

$$X \sim_G X' \quad \Rightarrow \quad [X \circlearrowleft G]^C = [X' \circlearrowleft G]^C.$$

Proof. By the consequence of functorial weak factorization, recalled in Section 2, the theorem reduces to the equality of classes in the C-localized Burnside group in the case of a G-equivariant blow-up

$$\varrho\colon X' \to X,$$

with smooth center Z. Given the shape of the invariant, it suffices to consider the case that Z is the orbit of a curve, isomorphic to C, which by abuse of notation we denote by C in the analysis of the possibilities:

(1) C has trivial generic stabilizer,
(2) the generic stabilizer H of C is nontrivial and $\beta_C(X)$ is of the form $(0,b)$,
(3) the generic stabilizer H of C is nontrivial and $\beta_C(X)=(b_1,b_2)$ with nontrivial b_1, b_2.

Case (1): The generic stabilizer of the exceptional divisor E of the blow-up is trivial, and there is no curve $C'\subset E$ which is isomorphic to C and has nontrivial generic stabilizer. Thus there is no effect on the first two sums in the expression for $[X \circlearrowleft G]^C$. The third sum gets an extra term, but this vanishes by **(B3)**.

In case $g(C)=1$, it may be necessary to combine with **(B4)** to obtain the claimed vanishing. In the remaining cases there is a similar, implicit use of **(B4)** when $g(C)=1$.

Case (2): In this case, there is a divisor D of the stabilizer stratification such that its generic stabilizer is H, and $C\subset D$. The first sum picks up a term $\beta_F(X)=(b,-b)$, and the third sum gets a contribution from the residual action on C. Their sum vanishes by **(B1)**.

Case (3): Let H be the generic stabilizer. Let E be the exceptional divisor. If $b_1\neq b_2$, then the exceptional divisor E admits two curves with stabilizer H, and respective weights (b_1, b_2-b_1) and (b_2, b_1-b_2). If $\langle b_1-b_2\rangle$ is a proper subgroup of $H^\vee$, then E has nontrivial generic stabilizer $\ker(b_1-b_2)$. Thus the term $(H, Y \circlearrowright k(C), (b_1,b_2))$ in the first sum gets replaced by Θ_1, with the addition to the second sum of Θ_2, from **(B2)**. The third term gets the required extra term, so that the equality $[X \circlearrowleft G]^C = [X' \circlearrowleft G]^C$ holds by **(B2)**. □

Example 4.2. Let X be a smooth rational threefold with a regular involution ι. Put $G=\langle\iota\rangle$. Let $C\subset X$ be an elliptic curve. We examine the blow-up relations:

- $C\subseteq X^G$, with normal bundle $(1,1)$. Blowing up we obtain an exceptional divisor birational to $C\times\mathbb{P}^1$ with generic stabilizer G. We have the relation

$$-(G, 1 \circlearrowright k(C), (1,1)) + (G, 1 \circlearrowright k(C\times\mathbb{P}^1), (1)) + (G, \mathrm{J}(C) \circlearrowleft 1) = 0.$$

- $C \subseteq X^G$, with normal bundle $(0,1)$, then
$$(G, 1 \circlearrowright k(C), (1,1)) + (G, \mathrm{J}(C) \circlearrowleft 1) = 0.$$
- C has a G-action via translation by $\mathbb{Z}/2$ and no stabilizer. After blowing up, we have the relation
$$(G, \mathrm{J}(C) \circlearrowleft 1) = (1, \mathrm{J}(C) \overset{\mathrm{triv}}{\circlearrowleft} G) = 0.$$
- C has a G-action fixing 4 points. Then
$$(1, \mathrm{J}(C) \circlearrowleft G) = 0.$$
- C has no G-action, no stabilizer. Then
$$(1, \mathrm{J}(C) \circlearrowleft 1) = 0.$$

In conclusion, all symbols involving C vanish.

5. Structure

The paper [KT25] introduced filtrations on the full Burnside group $\mathrm{Burn}_n(G)$, based on combinatorial properties of the subgroup lattice of G; these allow to simplify the analysis of the class
$$[X \circlearrowleft G] \in \mathrm{Burn}_n(G),$$
in some cases.

Briefly, [KT25, Section 3] introduced the notion of a *filter* $\mathbf{H}$, consisting of pairs (H, Y), subject to certain properties, which ensure that the quotient
$$\mathrm{Burn}_n(G) \to \mathrm{Burn}_n^{\mathbf{H}}(G)$$
by symbols with $(H, Y) \notin \mathbf{H}$ is a well-defined homomorphism to a group that is generated by symbols with $(H, Y) \in \mathbf{H}$, with the same relations as in $\mathrm{Burn}_n(G)$, but applied only to these symbols.

Now let C be an irreducible smooth projective curve of genus ≥ 2. Then, by the same reasoning, we have the C-localized Burnside group
$$\mathrm{Burn}_3^C(G) \to \mathrm{Burn}_3^{\mathbf{H},C}(G),$$
generated by symbols $(H, Y) \in \mathbf{H}$ and with relations **(B1)**–**(B3)**, applied only to these symbols. (When $g = 1$ the same reasoning is not applicable on account of the additional relation **(B4)**.)

For G *abelian*, an example of a G-filter is
$$\{(G, 1)\},$$
see [KT25, Example 3.4] and [KT22b, Section 8]. The corresponding Burnside group
$$\mathrm{Burn}_n^G(G)$$

records only the strata with maximal stabilizer G, i.e., G-fixed loci. An analogous formalism applies to the C-localized Burnside group

$$\mathrm{Burn}_3^{G,C}(G).$$

We denote by $[X \circlearrowleft G]^{G,C}$ the image of the class $[X \circlearrowleft G]^G$, by means of the filter, in $\mathrm{Burn}_3^{G,C}(G)$. Concretely, this is given by picking out just the symbols with first argument equal to G, in the formula for $[X \circlearrowleft G]^C$.

In particular, specializing to abelian groups G, we obtain:

Proposition 5.1. *Let G be abelian and $g(C) \geq 2$. Then there is a homomorphism*

$$\varphi^G \colon \mathrm{Burn}_3^{G,C}(G) \to \mathbb{Z},$$

determined by

$$(G, 1 \circlearrowright K, (b_1, b_2)) \mapsto -1,$$
$$(G, 1 \circlearrowright L, (b)) \mapsto -2,$$
$$(G, \mathrm{J} \circlearrowleft 1) \mapsto 1.$$

Proof. The abelian group $\mathrm{Burn}_3^{G,C}(G)$ is generated by symbols with G as first argument, and has relations given by **(B1)**–**(B3)** with $H = G$. It is straightforward to verify that φ^G respects the relations. □

This yields a more classical analog of [CKK25, Theorem 3.6]:

Proposition 5.2. *Let X be a smooth projective rationally connected threefold with a regular action of an abelian group G and*

$$X^G = \sqcup_\alpha F_\alpha$$

the decomposition of the G-fixed locus into a disjoint union of smooth irreducible components. Let C be an irreducible smooth projective curve of genus ≥ 2 and

- *I_1 be the number of F_α isomorphic to C,*
- *I_2 be the number of F_α birational to $C \times \mathbb{P}^1$, and*
- *I_3 be the number of factors of the intermediate Jacobian $\mathrm{IJ}(X)$ isomorphic to $\mathrm{J}(C)$, with trivial G-action.*

Then

$$I := -I_1 - 2I_2 + I_3$$

is a G-equivariant birational invariant, given by

$$\varphi^G([X \circlearrowleft G]^{G,C}).$$

Furthermore, if $I \neq 0$ then the G-action on X is not linearizable, and also not projectively linearizable.

Proof. The proof of the first statement is immediate from the definition of the class $[X \circlearrowleft G]^{G,C}$ and map φ^G.

To show the second statement, we pass to a standard model of the G-action on $\mathbb{P}^3$, as in [KT22b], and observe that the G-action cannot fix higher genus curves or nonrational surfaces. ☐

6. Applications

In this section we provide several applications of the formalism of C-localized Burnside groups. The applications are most interesting when the group G is small – for large G, one can often deploy techniques from birational rigidity. For this reason, we focus on cyclic groups of order 2 and 3. For the treatment of the examples given here, the equivariant Burnside groups without C-localization are insufficient, on account of the relations for symbols in $\mathrm{Burn}_3(G)$.

Involutions. The first steps towards classification of involutions in the Cremona group Cr_3 were undertaken by Prokhorov in [Pro13]. Following the classification of involutions in Cr_2, which is based on the existence of higher-genus curves in the fixed locus of the action, Prokhorov considered involutions ι on rational X with a *nonuniruled* divisor in the fixed locus X^ι. In [CTT25], we constructed nonconjugated involutions in Cr_3 without any divisors in the fixed locus; this was based on the intermediate Jacobian torsor obstruction, which already obstructs linearizability. Here, we offer further examples of involutions ι without nonuniruled divisors in X^ι.

Example 6.1. We return to Example 3.1: $X = S \times \mathbb{P}^1$, where S is a degree 2 del Pezzo surface and $G = \mathbb{Z}/2\mathbb{Z}$, acting via the covering involution on S, which fixes a smooth quartic curve C. We have

$$I_1 = I_3 = 0, \quad I_2 = 1,$$

thus $I = -2$ in Proposition 5.2. In particular, the G-action on X is not linearizable; see [CKK25, Theorem I]. However, by [BP13, Theorem 1.1], we have

$$\mathrm{H}^1(G, \mathrm{Pic}(S)) = (\mathbb{Z}/2\mathbb{Z})^6,$$

so the G-action on S is not even stably linearizable.

Let $G := \langle \iota \rangle$ and C be an irreducible smooth projective curve of genus ≥ 2. We spell out generators and relations in $\mathrm{Burn}_3^C(G)$, taking into account that the only possibilities for a C-relevant components of $\mathrm{IJ}(X)$ are

- $\mathrm{J}(C)$ with trivial or nontrivial G-action,
- $\mathrm{J}(C) \times \mathrm{J}(C)$ with G-action permuting the factors.

For the generators, we have

(1) $\mathfrak{s}_1 := (G, 1 \circlearrowright k(C), (1,1))$,
(2) $\mathfrak{s}_2 := (G, 1 \circlearrowright k(C \times \mathbb{P}^1), (1))$,
(3) $\mathfrak{s}_3 := (G, \mathrm{J}(C) \circlearrowleft 1)$,
(4) $\mathfrak{s}_4 := (1, \mathrm{J}(C) \circlearrowleft G)$,
(5) $\mathfrak{s}_5 := (1, \mathrm{J}(C) \circlearrowleft 1)$.

For C hyperelliptic, every involution on $\mathrm{J}(C)$ can be realized on C; thus, relation **(B3)** implies that the symbols $\mathfrak{s}_4$ and $\mathfrak{s}_5$ vanish in $\mathrm{Burn}_3^C(G)$. Relations **(B1)** and **(B2)** yield

$$\mathfrak{s}_1 + \mathfrak{s}_3 = 0, \quad \mathfrak{s}_1 = \mathfrak{s}_2 + \mathfrak{s}_3.$$

It follows that

$$\mathrm{Burn}_3^C(G) \simeq \mathbb{Z}$$

and the homomorphism φ^G of Proposition 5.1 is an isomorphism; in particular, for C hyperelliptic, the invariant I of Proposition 5.2 is the *only* invariant of involutions accessible via the C-localized Burnside groups.

When C is non-hyperelliptic and the G-action on $\mathrm{J}(C)$ does not arise from an automorphism of C, the corresponding symbol of type $\mathfrak{s}_4$ does not participate in blow-up relations. In this case, $\mathrm{Burn}_3^C(G)$ is a free abelian group generated by $\mathfrak{s}_1$ and such symbols of type $\mathfrak{s}_4$.

The following example gives an alternative approach to [CTT25, Example 6.9].

Example 6.2. Consider

$$X \subset \mathbb{P}^1_{(t_1:t_2)} \times \mathbb{P}^3_{(x_1:x_2:x_3:x_4)},$$

given by the vanishing of

$$\sum_{i=0}^{n} t_1^i t_2^{n-i}(f_i(x_1,x_2) + g_i(x_3,x_4)), \quad n \geq 3,$$

for general binary quadratic forms f_i, g_i, so that X is smooth. Projection to $\mathbb{P}^1$ yields a quadric surface bundle, with discriminant cover a smooth hyperelliptic curve C of genus $g(C) = 2n-1$, see [CTT25, Section 6]. The threefold X is rational, with $\mathrm{IJ}(X) = \mathrm{J}(C)$. The involution

$$\iota\colon (x_1 : x_2 : x_3 : x_4) \mapsto (-x_1 : -x_2 : x_3 : x_4)$$

fixes two (nonisomorphic, for general f_i, g_i) hyperelliptic curves C', C'' of genus $n-1$. Applying Proposition 5.2 to either C' or C'' shows that $I \neq 0$ and the action is not linearizable.

Example 6.3. Assume that X is rational with $\mathrm{IJ}(X) = \mathrm{J}(\mathrm{C})$, for a smooth projective non-hyperelliptic curve C of genus ≥ 3. Assume that the G-action on $\mathrm{IJ}(X)$ does not come from any G-action on C, i.e., some element of G acts by an automorphism, not in the image of the homomorphism (2.1). Then the G-action is not linearizable. Examples of this arise from conic bundles $X \subset \mathbb{A}^2 \times \mathbb{P}^2$ given by

$$t_1 t_2 = f(x_1, x_2, x_3),$$

where f is a form of degree 4 defining a smooth non-hyperelliptic curve $C \subset \mathbb{P}^2$, and ι acts by switching t_1 and t_2. This situation arises also from 2-nodal cubic threefolds, see [CTZ25, Theorems 2.3 and 3.3].

We claim that the action of ι on $\mathrm{IJ}(X)$ is by (-1). There are two families $\{\ell_c\}, \{\ell'_c\}$ of vertical lines in the conic bundle, each parametrized by C. The Abel-Jacobi map for such a family of lines defines a non-equivariant isomorphism

$$\mathrm{J}(C) \cong \mathrm{IJ}(X).$$

The two families are swapped by the G-action. The evident rational equivalence

$$\ell_{c_0} + \ell'_{c_0} \sim_{\mathrm{rat}} \ell_{c_1} + \ell'_{c_1},$$

for c_0, $c_1 \in C$, justifies the claim.

Actions of $\mathbb{Z}/3\mathbb{Z}$. Our first application is a down-to-earth version of [CKK25, Example 3.10].

Example 6.4. Let $X \subset \mathbb{A}^4$ be given by

$$x_1 x_2 x_3 = P(x_4),$$

where P is a general polynomial of degree $3d$ with $d \geq 2$, with an action of $G = \mathbb{Z}/3\mathbb{Z}$ permuting the first 3 variables. By embedding X in

$$\mathbb{P}^1_{(s_1:t_1)} \times \mathbb{P}^1_{(s_2:t_2)} \times \mathbb{P}^1_{(s_3:t_3)} \times \mathbb{A}^1_{x_4},$$

with $x_i = s_i/t_i$, for $i = 1, 2, 3$, we obtain the defining equation

$$s_1 s_2 s_3 = P(x_4) t_1 t_2 t_3$$

of a fibration in degree 6 del Pezzo surfaces, with 3 singular points of type A_1 in the fiber over each zero of P for a total of $9d$ singular points. Compactification in the $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$-fibration

$$\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(d) \oplus \mathcal{O}_{\mathbb{P}^1}) \times_{\mathbb{P}^1} \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(d) \oplus \mathcal{O}_{\mathbb{P}^1}) \times_{\mathbb{P}^1} \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(d) \oplus \mathcal{O}_{\mathbb{P}^1})$$

over $\mathbb{P}^1$ does not introduce any further singularities. By blowing up the singular points we get a smooth projective model $\widetilde{X}$ with

$$\pi\colon \widetilde{X} \to \mathbb{P}^1.$$

By Proposition 5.2 the G-action is not linearizable, since G fixes a curve of genus $3d-2$, and $\mathrm{IJ}(\widetilde{X})$ is trivial; the last fact makes no reference to the G action and may be explained either directly or via monodromy. Directly, $\widetilde{X}$ can be obtained (non-equivariantly) by repeatedly blowing up rational curves in a $\mathbb{P}^2$-bundle over $\mathbb{P}^1$; in doing so the intermediate Jacobian starts out and remains trivial. But also there is the trivial monodromy of π, from which it is possible to conclude that $\mathrm{IJ}(\widetilde{X})$ is trivial using [Kan89, Corollary 4.3].

Example 6.5. Let

$$X \subset \mathbb{P}^4_{(x_1:x_2:x_3:x_4:x_5)}$$

be the 3-nodal cubic threefold given by

$$x_1x_2x_3 + (x_1+x_2+x_3)x_4x_5 + f_3(x_4,x_5) = 0,$$

where $f_3(x_4,x_5)$ is

(1) $\lambda(x_4+x_5)^3$, or
(2) $\lambda(x_4+x_5)(x_4-x_5)^2$, or
(3) $(x_4+x_5)(\lambda x_4+\mu x_5)(\mu x_4+\lambda x_5)$,

with $\lambda,\mu\in k^\times$ ($\lambda^2\neq -1/16$ in (1), $\lambda^2\neq -27/64$ in (2), $(\lambda+\mu)^4\neq -1$ and $(\lambda-\mu)^6\neq 27\lambda\mu$ in (3)). The intermediate Jacobian of a minimal resolution $\widetilde{X}$ of X is the Jacobian of a genus 2 curve. Indeed, projection from two of the nodes expresses X as a conic bundle over $\mathbb{P}^2$, with a quartic curve as degeneracy locus and equation in split form as in Example 6.3, see also [CTZ25, Section 3]. The quartic curve has exactly one node and thus geometric genus 2.

Let $G=\mathbb{Z}/3\mathbb{Z}$ act by permuting the first three variables. As explained in [CTZ25, Section 4], an equivariant birational model of $\widetilde{X}$ is a fibration

$$\pi\colon Y\to \mathbb{P}^1_{(x_4:x_5)},$$

with generic fiber a del Pezzo surface of degree 6; it is given by

$$(x_1:x_2:x_3:x_4:x_5)\mapsto (x_4:x_5).$$

The model Y is obtained from $\widetilde{X}$ by performing flops. Thus

$$\mathrm{IJ}(Y)\cong \mathrm{IJ}(\widetilde{X}).$$

Since G does not act on x_4,x_5, the G-action is trivial on the base $\mathbb{P}^1$. The G-fixed locus on the model Y is an elliptic curve. The generic fiber of π has Picard rank 3; it admits three conic fibrations. The monodromy factors through the $\mathbb{Z}/2\mathbb{Z}$, exchanging opposite pairs of lines, and determines a double cover C of $\mathbb{P}^1$, branched over 6 points. The relative Fano variety of lines is a union of three copies of C, permuted by G. The variety of vertical conics is a union of three $\mathbb{P}^1$-bundles over $\mathbb{P}^1$, permuted by G. The variety of vertical rational cubics is a $\mathbb{P}^2$-bundle over C, where G acts trivially on C.

Since the class of vertical rational cubics is G-invariant, it defines the equivariant Abel-Jacobi map

$$\mathrm{J}(C) \to \mathrm{IJ}(Y).$$

We claim that this is surjective, so that the triviality of the G-action on the intermediate Jacobian is a consequence of the trivial G-action on C. To see this, we follow the proof of [Kan89, Corollary 4.3], but instead of using the relative Fano variety of lines we use a family of rational cubics parametrized by C, plus conics parametrized by three copies of $\mathbb{P}^1$. Such families may be obtained by choosing a section of the $\mathbb{P}^2$-bundle over C, respectively, the $\mathbb{P}^1$-bundles over $\mathbb{P}^1$.

We have observed that $\mathrm{IJ}(Y)$ is the Jacobian of a genus 2 curve; to this we apply Proposition 5.2 and obtain

$$I_1 = I_2 = 0, \quad I_3 = 1.$$

We conclude that the G-action on X is not linearizable. (Though not needed here, we record the fact $\mathrm{IJ}(Y) \cong \mathrm{J}(C)$, obtained by the analysis of [Kan89, Section 5], with $q = 3$ in the notation of loc. cit.)

Example 6.5 strengthens [CTZ25, Proposition 4.3], which showed nonlinearizability of a *very general* member of the third family, via specialization.

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