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Chern's Conjecture in the Dupin case

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Abstract. Chern's conjecture states that a closed minimal hypersurface in the Euclidean sphere is isoparametric if it has constant scalar curvature. When the number g of distinct principal curvatures exceeds three, only limited results have been established. In this work, we examine the conjecture for Dupin hypersurfaces, and establish the following results: a closed proper Dupin hypersurface with constant mean curvature is isoparametric (i) if $g = 3$; (ii) if $g = 4$ and it has constant scalar curvature; (iii) if $g = 4$ and it has constant Lie curvature; and (iv) if $g = 6$ and it has constant Lie curvatures. These cases cover all nontrivial possibilities for closed proper Dupin hypersurfaces. Our proof employs topological and geometric methods, in contrast to earlier algebraic and analytic approaches.

1. Introduction

In the early twentieth century, Italian geometric opticians began studying light wave fronts that propagate at a constant speed. This led to the notion of *isoparametric hypersurfaces*, which consist of parallel hypersurfaces of constant mean curvature (CMC) and, in fact, have constant principal curvatures.

Let M be a hypersurface in S^n, A its shape operator, and $H = \mathrm{Tr}A$ the mean curvature. Using the principal curvatures $\lambda_1, \lambda_2, \ldots, \lambda_{n-1}$ of M, the scalar curvature is expressed as

$$R = (n-1)(n-2) + H^2 - \|A\|^2 = (n-1)(n-2) + \left(\sum \lambda_i\right)^2 - \sum \lambda_i^2. \qquad (1)$$

Chern's conjecture [**14**]: *A closed minimal hypersurface in S^n is isoparametric if it has constant scalar curvature (CSC), where closed means a compact without boundary.*

This problem corresponds to Yau's 105th problem [**43**].

When both $\sum \lambda_i$ and $\sum \lambda_i^2$ are constant, λ_i's themselves are constant if the number g of distinct principal curvatures satisfies $g \leq 2$. However, when $\dim M = 3$, the case $g = 3$ occurs, and the closedness condition is essential for Almeida-Brito's affirmative solution [**16**], completing earlier work of Peng-Terng [**33**]. See Chang [**11**] and Cheng-Wan [**13**] for related results, and Chen-Li [**12**] for the case $\dim M > 3$ and $g \leq 3$. For $g \geq 4$, it becomes difficult to reach a definite conclusion without further assumptions [**39**], [**40**], [**41**]. The latter two works effectively extend the method of [**16**] to arbitrary dimensions.

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Isoparametric hypersurfaces in Euclidean and hyperbolic spaces are only totally umbilic ones and cylinders [**2**]. In contrast, in the sphere S^n, É. Cartan discovered examples with $g = 3$ and $g = 4$ [**3**]. Later, Ozeki-Takeuchi constructed infinitely many homogeneous and non-homogeneous examples with $g = 4$ [**32**], which were extended by Ferus-Karcher-Münzner to those arising from all representations of the Clifford algebra [**18**].

Isoparametric hypersurfaces are algebraic [**28**]: they are level sets of certain homogenous polynomials intersected with S^n. Topological arguments then show that $g \in \{1, 2, 3, 4, 6\}$ [**29**]. All these cases have been classified (Yau's 34th problem [**44**]) [**2**], [**3**], [**6**], [**15**], [**17**],[**24**]-[**26**], and thoroughly studied [**18**], [**28**], [**29**].

To approach the case $\dim M \geq 3$ and $g \geq 3$, we introduce an additional geometric assumption: the *Dupin condition* together with CMC and CSC.

A hypersurface is called *Dupin* if

(i) each principal curvature λ has constant multiplicity, and

(ii) λ is constant along its curvature direction.

Pinkall calls it a proper Dupin in [**34**], but we omit "proper" here. If only (ii) holds, we call it *weak Dupin*. When the multiplicity m of λ exceeds one, (ii) is automatically satisfied. If (ii) also holds for $m = 1$, each leaf of the curvature distribution is an m-dimensional sphere [**30**], [**37**]. Thus a Dupin hypersurface is foliated by spheres and belongs naturally to the broader framework of *Lie sphere geometry*, which extends Riemannian and conformal geometry (see §5, [**4**], [**34**]).

Our main result is:

Theorem 1.1. *A closed CMC Dupin hypersurface M in S^n is isoparametric*

(i) if $g = 3$.

(ii) if $g = 4$ and M has constant scalar curvature.

(iii) if $g = 4$ and M has constant Lie curvature.

(iv) if $g = 6$ and M has constant Lie curvatures.

Remark 1.2 : The *Lie curvature* is the cross ratio of distinct four principal curvatures, discovered by the author as an invarint in Lie sphere geometry [**22**] (see §5).

Corollary 1.3. *Let $\dim M \geq 3$ and $g \geq 3$. A closed CMC hypersurface M in S^n with g principal curvatures, each constant along its curvature direction, is isoparametric*

(i) if $g = 3$.

(ii) if $g = 4$ and M has constant scalar curvature.

(iii) if $g = 4$ and has constant Lie curvature.

(iv) if $g = 6$ and has three independent constant Lie curvatures.

Although the Dupin condition is local, a *closed* embedded Dupin hypersurface has the global topological property *tautness* [**42**] (Fact 2, §2). This is based on the fact that they have the same homology as isoparametric hypersurfaces [**42**]. In particular, $g \in \{1, 2, 3, 4, 6\}$. When $g = 1, 2, 3$, such hypersurfaces are images of isoparametric ones under Lie contact transformations [**9**], [**7**]. The last case extends the author's earlier result in the closed setting [**20**], which motivated Cecil and Ryan to conjecture that every closed embedded Dupin hypersurface is a Lie images of an isoparametric hypersurface

[**9**]. However, *counterexamples* for $g = 4, 6$ were constructed in [**27**] (see also [**36**] to $g = 4$). The *Lie curvature* and the analysis of critical sets of the distance function due to T. Ozawa [**31**] play essential roles. (see the expository article [**5**]).

When $g = 4, 6$, even if all the Lie curvatures are constant, this does not imply Lie equivalence to an isoparametric hypersurface [**22**], [**23**]; see also [**8**]. Thus the Dupin remains substantially weaker, and proving Chern's conjecture in the Dupin case for $g \geq 4$ is still highly nontrivial.

By part (ii) of Theorem 1.1, Chern's conjecture reduces to the following statement: *A closed CMC hypersurface with* $g = 4$ *is Dupin if it has CSC.*

In parts (iii) and (iv), we replace scalar curvature with Lie curvatures, both are quadratic in λ_i's. For $g = 4$, this invariant is unique, while for $g = 6$, three Lie curvatures are essential if they are *independent*, that is, if none of the three can be expressed by the other two (Lemma in the front page of [**23**]).

For further results on Chern's conjecture, see [**39**], [**41**] and the references therein.

In §2, we give basic definitions and the known results. In §3 and in §4, we give proofs of (i) and (ii) of Theorem 1.1, which are relatively elementary. However, the proofs of (iii) and (iv) require Lie sphere geometry, introduced in §5, and are carried out in §§6 –7.

2. Preliminaries

We mainly follow the notation in [**22**]. Let S^n be the n-dimensional unit sphere in $\mathbb{R}^{n+1}$ centered at the origin. Consider an isometrically embedded orientable hypersurface $p : M \to S^n$ with a unit normal vector field n. Let A denote the second fundamental tensor of M and $\lambda_1 \geq \cdots \geq \lambda_{n-1}$ be the principal curvatures. For $\lambda \in \{\lambda_i\}$, the curvature distribution $D(\lambda)$ is defined by

$$D_p(\lambda) = \{X \in T_pM \mid AX = \lambda X\}, \quad p \in M.$$

The following is well-known [**30**], [**37**].

Fact 1. When $\dim D(\lambda)$ is constant (say$= m$) on M, the distribution $D(\lambda)$ is involutive. Moreover if λ is constant along $D(\lambda)$, (which is the case when $m > 1$), the leaf L is a piece of an m-dimensional subsphere of the curvature sphere $C(\lambda)$ at p.

Note that *Dupin* hypersurfaces satisfy the statement of Fact 1.

Example 2.1 : The following are Dupin hypersurfaces:

(1) Isoparametric hypersurfaces.

(2) Conformal or Lie images (§5) of isoparametric hypersurfaces.

(3) If M is a Dupin hypersurface in $\mathbb{R}^m$, then a cylinder or a tube over M in $\mathbb{R}^m \oplus \mathbb{R}^k$ is a new (weak) Dupin hypersurface in $\mathbb{R}^{m+k}$. By a stereographic projection, we obtain a weak Dupin hypersurface in S^{m+k} [**34**].

Remark 2.2 : Item (3) implies that, *locally*, there exist Dupin hypersurfaces with any number of principal curvatures and arbitrary multiplicities.

It is obvious locally that:

(1) *When* $g = 1$, *a Dupin hypersurface is a piece of hypersphere.*

(2) *When $g = 2$, a CMC Dupin hypersurface is a piece of a Clifford hypersurface* (the orthogonal product of two spheres), that is, of an isoparametric hypersurface with $g = 2$.

Indeed, when $g = 2$, CMC condition implies that the principal curvatures λ, μ satisfy $m_1\lambda + m_2\mu = H$ for some constant H, where m_1 (resp. m_2) is the multiplicity of λ (resp. μ). Since λ (resp. μ) is constant along $D(\lambda)$ (resp. $D(\mu)$), so is μ (resp. λ), and hence λ and μ are constant on M.

When $\dim M = 3$, Chern's conjecture has been affirmatively settled [**16**], where analytic methods play a crucial role. In Theorem 1.1 (i), we consider closed CMC hypersurfaces M with $g = 3$ in arbitrary dimensions. Instead of assuming constant scalar curvature, we impose the Dupin condition. Our approach also provides insight into the cases $g = 4, 6$.

The focal points of $p \in M$ are given by

$$f_p^i = \cos\theta_i\, p + \sin\theta_i\, n_p, \quad 0 < \theta_i = \cot^{-1}\lambda_i < \pi$$

and their antipodal points $\bar{f}_p^i$ in S^n, $1 \le i \le n-1$. For $x \in S^n \setminus M$, let $l_x : M \to \mathbb{R}$ be the squared spherical distance function on M, defined by

$$l_x(p) = d(x,p)^2, \quad p \in M. \tag{2}$$

Let $p \in M$ be a critical point of l_x. Then the index of p is given by the sum of the multiplicities of the focal points of p along the oriented open geodesic segment $\widehat{xp}$. For details, see [**9**].

Fact 2. A closed embedded Dupin hypersurface is *taut* [**42**], that is, l_x satisfies the equality in the Morse inequality for generic $x \in S^n \setminus M$, where homology is taken with $\mathbb{Z}_2$ coefficients. In such a case, l_x is called *a perfect Morse function.* Conversely, taut hypersurfaces are *weak Dupin* [**21**], [**35**].

Fact 3. Isoparametric hypersurfaces in S^n consist of parallel CMC hypersurfaces M_θ with constant principal curvatures $\lambda_1 > \cdots > \lambda_g$, $g \in \{1,2,3,4,6\}$ given by [**28**]

$$\lambda_i(\theta) = \cot\theta_i = \cot\Big(\frac{\pi}{2g} + \theta + \frac{(i-1)\pi}{g}\Big),\ i \in \{1,\dots,g\},\ -\frac{\pi}{2g} < \theta < \frac{\pi}{2g}. \tag{3}$$

In particular,

$$0 < \theta_1 = \frac{\pi}{2g} + \theta < \frac{\pi}{g}, \quad \cot\frac{\pi}{g} < \lambda_1(\theta) < \infty. \tag{4}$$

Fact 4. For any normal geodesic γ of M_θ, $M_\theta \cap \gamma$ is a parallel $2g$-gon, that is, a $2g$-gon obtained from a regurlar one by a parallel transformation in the Lie geometric sense (see Figure 3, 6, 11, and Example 5.2).

Fact 5. If M_θ has distinct g principal curvatures, the multiplicity m_i of λ_i is all equal when $g = 1, 3, 6$, and $m_1 = m_{\mathrm{odd}}$ and $m_2 = m_{\mathrm{even}}$ when $g = 2, 4$. In §2 of [**41**], Tang-Yan show that the mean curvature H_θ of M_θ is given by

$$H_\theta = \frac{g}{2}\Big(m_1 t - \frac{m_2}{t}\Big), \quad t = \cot\frac{g\theta_1}{2}, \quad 0 < \theta_1 = \frac{\pi}{2g} + \theta < \frac{\pi}{g}. \tag{5}$$

Thus H_θ is monotone with respect to θ, and θ is uniquely determined by the mean curvature. The minimal case occurs when

$$t^2 = \cot^2\frac{g\theta_1}{2} = \frac{m_2}{m_1}. \tag{6}$$

In particular when $g = 1, 3, 6$, $m_1 t - \dfrac{m_2}{t} = m\dfrac{t^2-1}{t} = 2m\cot g\theta_1$, and hence

$$H_\theta = gm\cot(g\theta_1),$$

which vanishes only when $\theta = 0$, i.e., $\theta_1 = \dfrac{\pi}{2g}$. When $g = 2, 4$, M_0 is not necessarily minimal, since H_θ depends on m_1, m_2 in (5).

3. Closed case: $g = 3$

Hereafter, let M be a *closed embedded* Dupin hypersurface in S^n. Then M is taut [**42**], that is, the function (2) is generically a perfect Morse function. In the following, we make use of this important property in the background.

When $g = 3$, the principal curvatures $\lambda_1 > \lambda_2 > \lambda_3$ have the same multiplicity $m \in \{1, 2, 4, 8\}$ [**3**],[**42**]. In the non-closed case, there is freedom in the choice of multiplicities (Example 2.1 (3)), and hence the following argument does not apply.

For later use, we put $\lambda_1 = \lambda$, $\lambda_2 = \mu$ and $\lambda_3 = \nu$, and denote by D_i the corresponding curvature distributions which decompose the tangent bundle as $D_1 \oplus D_2 \oplus D_3$. Let e_j be any unit vector in D_j. The mean curvature is given by

$$H = m(\lambda + \mu + \nu), \quad m \in \{1, 2, 4, 8\}$$

where we do not take the average.

Lemma 3.1. *If M is a closed CMC Dupin hypersurface with $g = 3$, then there exists a point $p \in M$ at which all $d_{ji} = e_j(\lambda_i)$ vanish.*

Proof. Since H is constant, we have, for any $e_j \in D_j$,

$$d_{j1} + d_{j2} + d_{j3} = 0, \quad j = 1, 2, 3.$$

Let $p \in M$ be a critical point of $\lambda = \lambda_1$ on M. Then

$$d_{j1} = 0, \quad j = 1, 2, 3.$$

Since $d_{jj} = 0$ because M is Dupin, we have at p:

$$d_{12} + d_{13} = 0, \quad d_{23} = 0, \quad d_{32} = 0.$$

Thus if we choose a critical point p^1 of λ_2 on $L_1(p)$, the proof is complete. □

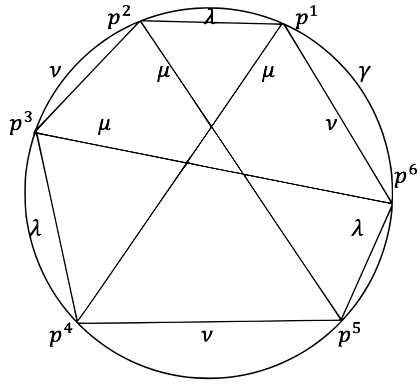

Figure 1: Link of leaves

At $p = p^1$ in the lemma, according to the argument in [**20**], each leaf $L_i(p^1)$ of D_i is totally geodesic in the corresponding curvature sphere C_i, and meets the normal geodesic γ through p^1 orthogonally at $p^{2i} \in \gamma$, $i = 1, 2, 3$, where p^{2i} is the antipodal point of p^1 in each leaf $L_i(p^1)$. The situation is the same as Figure 4.1 $\sim$ 4.3 in [**20**]. By the same argument using tautness, we see that $M \cap \gamma = \{p^1, \ldots, p^6\}$, and that each leaf at p^t intersects some other leaves at points on γ as illustrated in Figure 1. More precisely, denoting $\lambda^t = \lambda(p^t)$ etc., we have

$$\begin{array}{lll} \lambda^1 = \lambda^2, & \lambda^3 = \lambda^4, & \lambda^5 = \lambda^6 \\ \mu^1 = \mu^4, & \mu^2 = \mu^5, & \mu^3 = \mu^6 \\ \nu^1 = \nu^6, & \nu^2 = \nu^3, & \nu^4 = \nu^5. \end{array}$$

Lemma 3.2. *Assume λ attains its maximum at $p^1 \in M$, and $\lambda^1 \geq \lambda^3 \geq \lambda^5$ without loss of generality. Then, from CMC condition, it follows that*

$$\mu^1 \leq \mu^2 = \mu^5 \leq \mu^3, \quad \textit{and} \quad \nu^3 \leq \nu^1 \leq \nu^5.$$

Proof. Since $\lambda^1 \geq \lambda^3 \geq \lambda^5$, we have $\mu^1 + \nu^1 \leq \mu^3 + \nu^3 \leq \mu^5 + \nu^5$. From $\lambda^1 = \lambda^2$, we obtain $\mu^1 + \nu^1 = \mu^2 + \nu^2 \leq \mu^3 + \nu^3$, and $\nu^2 = \nu^3$ implies $\mu^2 \leq \mu^3$. Similarly, from $\lambda^3 = \lambda^4 \geq \lambda^5$, we have $\mu^3 + \nu^3 = \mu^4 + \nu^4 \leq \mu^5 + \nu^5$, and $\nu^4 = \nu^5$ implies $\mu^4 \leq \mu^5$, that is, $\mu^1 \leq \mu^2$. Then $\lambda^1 + \nu^1 \geq \lambda^2 + \nu^2$ follows, hence $\nu^1 \geq \nu^2$. Finally from $\lambda^5 = \lambda^6$, we have $\mu^5 + \nu^5 = \mu^6 + \nu^6$, and $\mu^5 = \mu^2 \leq \mu^3 = \mu^6$ implies $\nu^5 \geq \nu^6 = \nu^1$. □

Proposition 3.3. *In the situation of Lemma 3.2, each principal curvature coincides at all p^t's, and $M \cap \gamma$ is a parallel hexagon (Figure 3).*

Proof. Put

$$\lambda^t = \cot\theta_1^t, \quad \mu^t = \cot\theta_2^t, \quad \nu^t = \cot\theta_3^t, \quad 0 < \theta_i^t < \pi, \quad t = 1, \ldots, 6,$$

and define

$$\bar{\theta}_3^t = \pi - \theta_3^t, \quad \nu^t = \cot\theta_3^t = -\cot\bar{\theta}_3^t,$$

which satisfies $0 < \bar{\theta}_3^t < \pi$. Recall the cotangent is a decreasing function (so $-\cot$ is increasing).

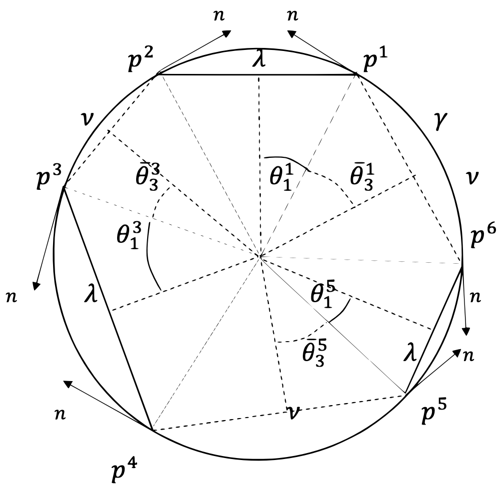

Figure 2: Angles

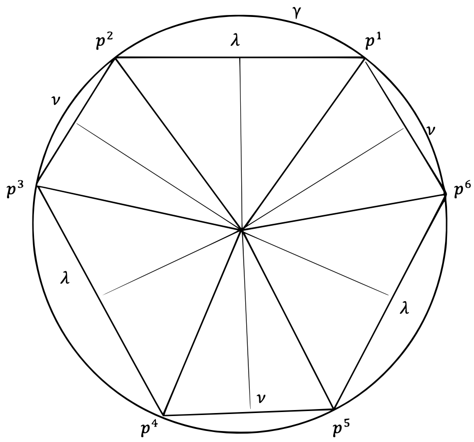

Figure 3: Parallel hexagon

As illustrated in Figure 2, noting the direction of the normal vector, we observe that

$$\begin{aligned} \mu^1 &= \cot\theta_2^1 = \cot\tfrac{1}{2}\angle(p^1Op^4) = \cot(\theta_1^1 + \bar{\theta}_3^3 + \theta_1^3), \\ \mu^2 = \mu^5 &= \cot\theta_2^5 = \cot\tfrac{1}{2}\angle(p^5Op^2) = \cot(\theta_1^5 + \theta_1^1 + \bar{\theta}_3^1), \\ \mu^3 &= \cot\theta_2^3 = \cot\tfrac{1}{2}\angle(p^3Op^6) = \cot(\theta_1^3 + \bar{\theta}_3^5 + \theta_1^5). \end{aligned} \tag{7}$$

Since $\nu^3 \leq \nu^1$ ($\bar{\theta}_3^3 \leq \bar{\theta}_3^1$) and $\lambda^3 \geq \lambda^5$ ($\theta_1^3 \leq \theta_1^5$), we have $\bar{\theta}_3^3 + \theta_1^3 \leq \bar{\theta}_3^1 + \theta_1^5$, and thus $\mu^1 \geq \mu^2$. Then by the previous lemma, $\mu^1 = \mu^2$, and since $\lambda^1 = \lambda^2$, we also have $\nu^1 = \nu^2$. Similary, from $\lambda^1 \geq \lambda^3$ and $\nu^1 \leq \nu^5$, we obtain $\theta_1^1 + \bar{\theta}_3^1 \leq \theta_1^3 + \bar{\theta}_3^5$, which implies $\mu^2 \geq \mu^3$. Hence, by the lemma,

$$\mu^1 = \mu^2 = \mu^3 (= \mu^6).$$

From $\nu^2 = \nu^3$, we deduce $\lambda^2 = \lambda^3$; and from $\nu^1 = \nu^6$, we obtain $\lambda^1 = \lambda^6$. Finally it follows that $\lambda^1 = \lambda^3 = \lambda^5$ and $\nu^1 = \nu^3 = \nu^5$. □

Remark 3.4 : $M \cap \gamma$ is a parallel hexagon when λ^t and ν^t are independent of t.

Next, let q be a minimum point of λ, and let q^5 be a critical point of μ on $L_1(q)$. Then as before, all d_{ji} vanish at q^5, and denoting the normal geodesic through q^5 by γ', we obtain $M \cap \gamma' = \{q^1, \ldots, q^6\}$ replacing p^t by q^t in Figure 1. Setting $\lambda_q^t = \lambda(q^t)$ etc., and assuming $\lambda_q^5 \leq \lambda_q^3 \leq \lambda_q^1$ without loss of generality, we conclude, replacing p^t by q^t in the above argument, $M \cap \gamma'$ is also a parallel hexagon.

Proof of Theorem 1.1 (i): By Proposition 3.3, $\{p^1, \ldots, p^6\} = M \cap \gamma$ is a parallel hexagon, hence isometric to $M_\theta \cap \gamma_\theta$ for some θ. Here, θ is uniquely determined by the mean curvature H of M (Fact 5). Thus at p^1, we have $\lambda = \lambda_\theta = \lambda_1(\theta)$ in (3). Since the same holds at the minimum point q^5 of λ, it follows that λ is constant on M. Then as $m(\mu+\nu)$ and $m(\mu^2+\nu^2)$ become constant, μ, ν are constant, and hence M is isoparametric. □

Remark 3.5 : Tang-Yan [**41**] give the scalar curvature of M_θ

$$R_\theta = 9m(m-1)\big(1+\cot^2(3\theta_1)\big), \quad \theta_1 = \frac{\pi}{6}+\theta.$$

Thus when $m=1$ M_θ is scalar flat, while for $m>1$, it has positive scalar curvature.

Remark 3.6 : There is a simpler proof if we use the result in [**20**] and a conformal invariant $\dfrac{\lambda-\mu}{\mu-\nu}$. However, the above strategy can be applied to the cases $g=4,6$.

4. Closed CMC and CSC Dupin Hypersurfaces with $g = 4$

When $g=4$, let $\lambda_1 > \lambda_2 > \lambda_3 > \lambda_4$ be the principal curvatures with multiplicities $m_1 = m_3, m_2 = m_4$. Let D_i denote the curvature distribution corresponding to λ_i. As before, we write e_j for any unit vector in D_j. We show the following:

Theorem 1.1 (ii). *A closed CMC Dupin hypersurface in S^n with $g=4$ is isoparametric if it has constant scalar curvature (CSC).*

Since the scalar curvature R is given by (1), the assumption implies that both

$$H = m_1\lambda_1 + m_2\lambda_2 + m_3\lambda_3 + m_4\lambda_4, \quad S = m_1\lambda_1^2 + m_2\lambda_2^2 + m_3\lambda_3^2 + m_4\lambda_4^2$$

are constant. Throughout this section, we assume CMC and CSC.

Remark 4.1 : Hence two principal curvatures determine the other two.

For $e_j \in D_j$, we have

$$\sum_{i=1}^{4} m_i d_{ji} = 0, \quad \sum_{i=1}^{4} m_i\lambda_i d_{ji} = 0, \quad j = 1, \dots 4,$$

namely, for $j=1$,

$$m_2d_{12} + m_3d_{13} + m_4d_{14} = 0, \quad m_2\lambda_2d_{12} + m_3\lambda_3d_{13} + m_4\lambda_4d_{14} = 0,$$

because $e_1(\lambda_1) = 0$ as M is Dupin. Thus d_{13} and d_{14} are determined by d_{12}. In particular, they vanish when $d_{12} = 0$, since $m_3m_4(\lambda_4 - \lambda_3) \neq 0$. Just in the same way, we have

Lemma 4.2. *For distinct $i, j, k, l \in \{1,2,3,4\}$, the quantities d_{jk} and d_{jl} are determined by d_{ji}. In particular, $d_{ji} = 0$ implies $d_{jk} = d_{jl} = 0$.*

Lemma 4.3. *There exists a point $p \in M$ at which all d_{ji} vanish.*

Proof. Let $p \in M$ be a critical point of λ_1. Then on $L_1(p)$, $d_{j1} = 0$ holds, and by Lemma 4.2, we have $d_{j2} = d_{j3} = d_{j4} = 0$ for $j = 2,3,4$. Next consider d_{1i}, $i = 2,3,4$. Since λ_2 attains a critical value at some point of $L_1(p)$, take such a point $p^1 \in L_1(p)$. Then $d_{12} = 0$, implying $d_{13} = d_{14} = 0$ by Lemma 4.2. Thus all d_{ji}'s vanish at p^1. □

Let γ be the normal geodesic of M at p^1. Since the leaf L_i of D_i is totally geodesic in the curvature sphere C_i if and only if λ_i takes a critical value on M ([**20**], [**22**]), $L_i(p^1)$ is totally geodesic in C_i. We may put

$$L_1(p^1)\cap\gamma = \{p^1, p^2\},\ L_2(p^1)\cap\gamma = \{p^1, p^4\},\ L_3(p^1)\cap\gamma = \{p^1, p^6\},\ L_4(p^1)\cap\gamma = \{p^1, p^8\},$$

where p^{2i} is the antipodal point of p^1 in $L_i(p^1)$. Thus, at p^1, the situation becomes the same as in Proposition 6.1 of [**22**]. Applying that proposition, in which tautness plays a crucial role, we obtain:

Proposition 4.4. *We have* $M \cap \gamma = \{p^1, p^2, p^3, p^4, p^5, p^6, p^7, p^8\}$, *where* p^1, p^2, p^4, p^6, p^8 *are as above, and* p^3, p^5, p^7 *are as in Figure 4, denoting* $\lambda = \lambda_1$, $\mu = \lambda_2$, $\nu = \lambda_3$ *and* $\tau = \lambda_4$. *The leaves at each* p^t *are all totally geodesic in their respective curvature spheres and meet* γ *orthogonally, intersecting as shown in Figure 4.*

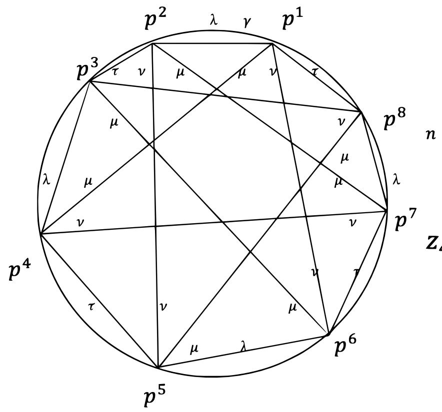

Figure 4: Link of leaves

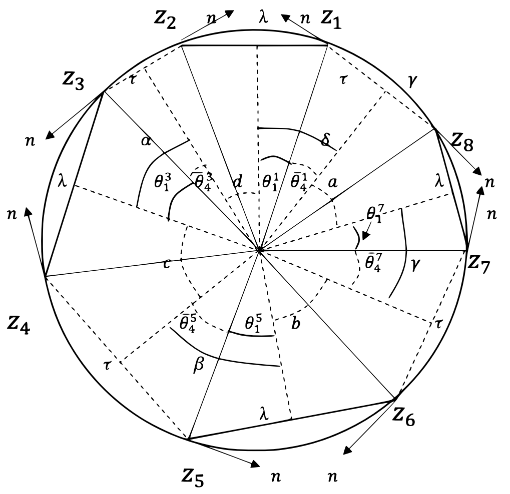

Figure 5: Angles

Proposition 4.5. *Let* $\lambda_i^t = \lambda_i(p^t) = \cot\theta_i^t$, $0 < \theta_i^t < \pi$, *for* $i = 1, 2, 3, 4$, *and* $t = 1, \ldots, 8$. *Putting*

$$\theta_2^1 = \theta_1^1 + \alpha,\ \theta_3^1 = \theta_2^1 + \beta,\ \theta_4^1 = \theta_3^1 + \gamma$$

$$\theta_1^2 = \theta_1^1,\ \theta_2^2 = \theta_1^1 + a,\ \theta_3^2 = \theta_2^2 + b,\ \theta_4^2 = \theta_3^1 + c,$$

and defining $\delta = \pi - (\alpha + \beta + \gamma)$ *and* $d = \pi - a - b - c$, *we obtain the following relations among angles (see Table 1).*

point	θ_1^t		θ_2^t		θ_3^t		θ_4^t		$\pi+\theta_1^t$
p^1	θ_1^1	+	α	+	β	+	γ	+	δ
p^2	θ_1^1	+	a	+	b	+	c	+	d
p^3	θ_1^3	+	β	+	γ	+	δ	+	α
p^4	θ_1^3	+	d	+	a	+	b	+	c
p^5	θ_1^5	+	γ	+	δ	+	α	+	β
p^6	θ_1^5	+	c	+	d	+	a	+	b
p^7	θ_1^7	+	δ	+	α	+	β	+	γ
p^8	θ_1^7	+	b	+	c	+	d	+	a

Table 1: Angles

Proof. Consider γ as a unit circle[1]. Noting the direction of the normal given in Figure 5 (where we denote p^t by z_t), we have as in (7),

$$\mu^1=\cot(\theta_1^1+\theta_1^3+\bar\theta_4^3),\ \nu^1=\cot(\theta_2^1+\theta_1^5+\bar\theta_4^5),\ \tau^1=\cot(\theta_3^1+\theta_1^7+\bar\theta_4^7),$$
$$\mu^2=\cot(\theta_1^1+\theta_1^8+\bar\theta_4^8),\ \nu^2=\cot(\theta_2^1+\theta_1^6+\bar\theta_4^6),\ \tau^1=\cot(\theta_3^1+\theta_1^4+\bar\theta_4^4),$$

hence

$$\alpha=\theta_1^3+\bar\theta_4^3,\quad \beta=\theta_1^5+\bar\theta_4^5,\quad \gamma=\theta_1^7+\bar\theta_4^7$$
$$a=\theta_1^8+\bar\theta_4^8,\quad b=\theta_1^6+\bar\theta_4^6,\quad c=\theta_1^4+\bar\theta_4^4.$$

Also from Figure 4 and Figure 5, we see,

$$\theta_2^3=\theta_1^3+\theta_1^5+\bar\theta_4^5=\theta_1^3+\beta,\ \theta_3^3=\theta_2^3+\theta_1^7+\bar\theta_4^7=\theta_2^3+\gamma,$$
$$\theta_4^3=\theta_3^3+\theta_1^1+\bar\theta_4^1=\theta_3^3+\delta$$

where $\delta=\theta_1^1+\bar\theta_4^1=\theta_1^1+(\pi-\theta_4^1)=\pi-(\alpha+\beta+\gamma)$. Since $\theta_1^4=\theta_1^3$, we have in the same way,

$$\theta_2^4=\theta_1^4+\theta_1^2+\bar\theta_4^2=\theta_1^3+d,\ \theta_3^4=\theta_2^4+\theta_1^8+\bar\theta_4^8=\theta_2^4+a,$$
$$\theta_4^4=\theta_3^4+\theta_1^6+\bar\theta_4^6=\theta_3^6+b,$$

where $d=\theta_1^2+\bar\theta_4^2=\theta_1^2+(\pi-\theta_4^2)=\pi-a-b-c$. In a similar way, θ_i^5, θ_i^6, θ_i^7 and θ_i^8 are obtained. □

Recall (Figure 4)

$$\lambda^1=\lambda^2,\quad \lambda^3=\lambda^4,\quad \lambda^5=\lambda^6,\quad \lambda^7=\lambda^8$$
$$\mu^1=\mu^4,\quad \mu^2=\mu^7,\quad \mu^3=\mu^6,\quad \mu^5=\mu^8$$
$$\nu^1=\nu^6,\quad \nu^2=\nu^5,\quad \nu^3=\nu^8,\quad \nu^4=\nu^7$$
$$\tau^1=\tau^8,\quad \tau^2=\tau^3,\quad \tau^4=\tau^5,\quad \tau^6=\tau^7.$$

Lemma 4.6. *We have*

$$\theta_1^3=\theta_1^1+\alpha-d,\quad \theta_1^5=\theta_1^1+a+b-\gamma-\delta,\quad \theta_1^7=\theta_1^1+a-\delta. \tag{8}$$

[1]Note that γ is also used to denote an angle; do not confuse the two.

PROOF. Since $\mu^1 = \mu^4$, it follows from Table 1 that $\theta_1^1 + \alpha = \theta_1^3 + d$, hence $\theta_1^3 = \theta_1^1 + \alpha - d$. Similarly, from $\mu^2 = \mu^7$, we obtain $\theta_1^1 + a = \theta_1^7 + \delta$, i.e., $\theta_1^7 = \theta_1^1 + a - \delta$. From $\mu^5 = \mu^8$, we also have $\theta_1^5 + \gamma = \theta_1^7 + b = \theta_1^1 + a - \delta + b$, which gives $\theta_1^5 = \theta_1^1 + a + b - (\gamma + \delta)$. □

LEMMA 4.7. *Assume that at p^1, λ attains its maximum and $\mu^1 \geq \mu^2$ without loss of generality. Then at p^1,*

$$\nu^7 \geq \nu^5 \geq \nu^1 \geq \nu^3, \quad \tau^1 \geq \tau^3, \quad \tau^7 \geq \tau^5. \tag{9}$$

PROOF. Since $m_1 = m_3$ and $m_2 = m_4$, we have at p^1 and p^8,

$$\begin{cases} m_1(\lambda^1 + \nu^1) + m_2(\mu^1 + \tau^1) = H \\ m_1(\lambda^8 + \nu^8) + m_2(\mu^8 + \tau^8) = H \end{cases}$$
$$\begin{cases} m_1((\lambda^1)^2 + (\nu^1)^2) + m_2((\mu^1)^2 + (\tau^1)^2) = S \\ m_1((\lambda^8)^2 + (\nu^8)^2) + m_2((\mu^8)^2 + (\tau^8)^2) = S \end{cases}$$

where $\tau^1 = \tau^8$. Taking the difference of the former two, and the latter two, respectively, we obtain

$$\begin{aligned} &m_1(\lambda^1 - \lambda^8 + \nu^1 - \nu^8) + m_2(\mu^1 - \mu^8) = 0 \\ &m_1((\lambda^1)^2 - (\lambda^8)^2 + (\nu^1)^2 - (\nu^8)^2) + m_2((\mu^1)^2 - (\mu^8)^2) = 0. \end{aligned}$$

Then multiplying the former by $(\mu^1 + \mu^8)$ and subtracting the second, we obtain

$$(\lambda^1 - \lambda^8)(\mu^1 + \mu^8 - \lambda^1 - \lambda^8) + (\nu^1 - \nu^8)(\mu^1 + \mu^8 - \nu^1 - \nu^8) = 0, \tag{10}$$

where $\mu^1 + \mu^8 - \lambda^1 - \lambda^8 < 0$ and $\mu^1 + \mu^8 - \nu^1 - \nu^8 > 0$. Since λ^1 is max, (10) implies

$$\lambda^1 \geq \lambda^8 = \lambda^7, \quad \nu^1 \geq \nu^8 = \nu^3.$$

A similar argument at p^2 and p^7 using $\mu^2 = \mu^7$ implies

$$\begin{aligned} &m_1(\lambda^2 - \lambda^7 + \nu^2 - \nu^7) + m_2(\tau^2 - \tau^7) = 0 \\ &m_1((\lambda^2)^2 - (\lambda^7)^2 + (\nu^2)^2 - (\nu^7)^2) + m_2((\tau^2)^2 - (\tau^7)^2) = 0, \end{aligned}$$

and multiplying the former by $(\tau^2 + \tau^7)$ and subtracting the second, we have

$$(\lambda^2 - \lambda^7)(\tau^2 + \tau^7 - \lambda^2 - \lambda^7) + (\nu^2 - \nu^7)(\tau^2 + \tau^7 - \nu^2 - \nu^7) = 0, \tag{11}$$

where $\tau^2 + \tau^7 - \lambda^2 - \lambda^7 < 0$ and $\tau^2 + \tau^7 - \nu^2 - \nu^7 < 0$. Thus in (11), $\lambda^2 = \lambda^1 \geq \lambda^7$ implies

$$\nu^2 = \nu^5 \leq \nu^7. \tag{12}$$

Next, at p^1 and p^2, using $\lambda^1 = \lambda^2$, we have

$$\begin{aligned} &m_1(\nu^1 - \nu^2) + m_2(\mu^1 - \mu^2 + \tau^1 - \tau^2) = 0 \\ &m_1((\nu^1)^2 - (\nu^2)^2) + m_2((\mu^1)^2 - (\mu^2)^2 + (\tau^1)^2 - (\tau^2)^2) = 0, \end{aligned}$$

and multiplying the former by $(\nu^1 + \nu^2)$ and subtracting the second, we have

$$(\mu^1 - \mu^2)(\nu^1 + \nu^2 - \mu^1 - \mu^2) + (\tau^1 - \tau^2)(\nu^1 + \nu^2 - \tau^1 - \tau^2) = 0, \tag{13}$$

where $\nu^1 + \nu^2 - \mu^1 - \mu^2 < 0$ and $\nu^1 + \nu^2 - \tau^1 - \tau^2 > 0$. Since we have chosen p^1 so that $\mu^1 \geq \mu^2$, we obtain from (13)

$$\mu^1 \geq \mu^2, \quad \tau^1 \geq \tau^2 = \tau^3.$$

This impies $\mu^1 + \tau^1 \geq \mu^2 + \tau^2$ and so $\lambda^1 + \nu^1 \leq \lambda^2 + \nu^2$. Thus by $\lambda^1 = \lambda^2$,

$$\nu^1 \leq \nu^2 = \nu^5$$

holds. Next, at p^4 and p^7, using $\nu^4 = \nu^7$, we have

$$\begin{aligned} &m_1(\lambda^4 - \lambda^7) + m_2(\mu^4 - \mu^7 + \tau^4 - \tau^7) = 0 \\ &m_1((\lambda^4)^2 - (\lambda^7)^2) + m_2((\mu^4)^2 - (\mu^7)^2 + (\tau^4)^2 - (\tau^7)^2) = 0, \end{aligned}$$

and multiplying the former by $(\lambda^4 + \lambda^7)$ and subtracting the second, we obtain

$$(\mu^4 - \mu^7)(\lambda^4 + \lambda^7 - \mu^4 - \mu^7) + (\tau^4 - \tau^7)(\lambda^4 + \lambda^7 - \tau^4 - \tau^7) = 0, \tag{14}$$

where $\lambda^4 + \lambda^7 - \mu^4 - \mu^7 > 0$ and $\lambda^4 + \lambda^7 - \tau^4 - \tau^7 > 0$. Since we are assuming $\mu^1 = \mu^4 \geq \mu^2 = \mu^7$, it holds

$$\tau^4 = \tau^5 \leq \tau^7 = \tau^6,$$

and the lemma is proved. □

PROPOSITION 4.8. *We have*

$$\alpha = a, \quad \beta = b, \quad \gamma = c, \quad \delta = d. \tag{15}$$

PROOF. Since the cotangent function is decreasing, $\mu^1 \geq \mu^2$ implies

$$\alpha \leq a. \tag{16}$$

The second inequality of (9) is written as, using (8),

$$(\theta_1^1 + a - \delta) + \delta + \alpha \leq \theta_1^1 + a + b \leq \theta_1^1 + \alpha + \beta \leq (\theta_1^1 - d + \alpha) + \beta + \gamma$$

and we have immediately

$$\alpha \leq b, \quad a+b \leq \alpha+\beta, \quad d \leq \gamma. \tag{17}$$

Then from $\tau^7 \geq \tau^5$, we have

$$\begin{aligned}(\theta_1^1 + a - \delta) + \delta + \alpha + \beta &= \theta_1^1 + a + \alpha + \beta \\ &\leq (\theta_1^1 + a + b - \gamma - \delta) + (\gamma + \delta + \alpha) = \theta_1^1 + a + b + \alpha,\end{aligned}$$

and so

$$\beta \leq b. \tag{18}$$

Thus from (16) + (18), and the middle innequality of (17), we obtain

$$\alpha + \beta = a + b,$$

and hence

$$\alpha = a, \quad \beta = b. \tag{19}$$

From $\theta_1^1 = \theta_1^2$ and (19), $\mu^1 = \mu^2$ and $\nu^1 = \nu^2$ follow. These imply $\tau^1 = \tau^2$, and we obtain $\gamma = c$ and $\delta = d$. □

Lemma 4.9. *We have*

$$\delta \leq \gamma \leq \alpha \leq \beta. \tag{20}$$

Proof. As we assume $\lambda^1 \geq \lambda^3 = \cot(\theta_1^1 + \alpha - \delta)$, we have

$$\delta \leq \alpha.$$

From $\nu^7 \geq \nu^5 \geq \nu^1 \geq \nu^3$ follows $\theta_3^7 \leq \theta_3^5 \leq \theta_3^1 \leq \theta_3^3$, and using (8) and (15), we have

$$\begin{aligned}\theta_1^7 + \delta + \alpha = \theta_1^1 + 2\alpha& \\ \leq \theta_1^5 + \gamma + \delta &= \theta_1^1 + \alpha + \beta = \theta_3^1 \\ \leq \theta_1^3 + \beta + \gamma &= (\theta_1^1 - \delta + \alpha) + \beta + \gamma,\end{aligned}$$

namely,

$$\delta \leq \alpha \leq \beta, \quad \delta \leq \gamma.$$

Since we have $\lambda^3 = \lambda^4$, $\nu^3 \leq \nu^7 = \nu^4$, and $\mu^3 = \cot(\theta_1^3 + \beta) \leq \cot(\theta_1^3 + \delta) = \mu^4$, we obtain $\tau^3 \geq \tau^4$. Hence $\beta + \gamma + \delta \leq \delta + \alpha + \beta$ implies

$$\gamma \leq \alpha.$$

□

Proposition 4.10. *All the principal curvatures coincide at p^t for $1 \leq t \leq 8$, and*

$M \cap \gamma$ is a parallel octagon.

Proof. Together with $\lambda^3 \leq \lambda^1$, from (15) and (20) we have

$$\mu^3 = \cot(\theta_1^3 + \beta) = \cot(\theta_1^1 + \alpha - \delta + \beta)) \leq \cot(\theta_1^1 + \alpha) = \mu^1$$

and

$$\nu^3 = \cot(\theta_1^3 + \beta + \gamma) = \cot(\theta_1^1 + \alpha - \delta + \beta + \gamma) \leq \nu^1.$$

However, since

$$\theta_4^3 = \theta_1^3 + \beta + \gamma + \delta = (\theta_1^1 - \delta + \alpha) + \beta + \gamma + \delta = \theta_1^1 + \alpha + \beta + \gamma = \theta_4^1,$$

by the CMC condition, we must have $\lambda^1 = \lambda^3$, $\nu^1 = \nu^3$ and $\mu^3 = \mu^1$. Hence

$$\alpha = \beta = \gamma = \delta = \frac{\pi}{4},$$

and $\theta_1^5 = \theta_1^7 = \theta_1^1$ follows. By Proposition 4.8 and by Table 1, all principal curvatures coincide at each p^t, $1 \leq t \leq 8$. Thus, $M \cap \gamma$ forms a parallel octagon. □

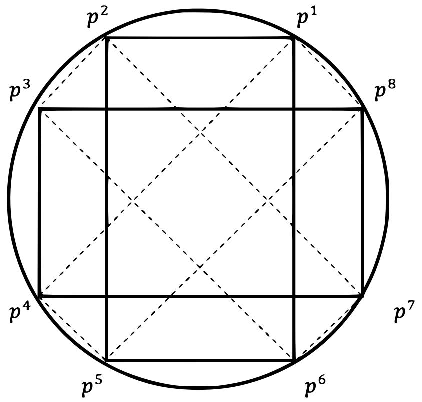

Figure 6: Parallel octagon

Next, we apply the above argument at a minimum point q of λ, instead of the maximum point p. Let q^1 be a minimum point of μ on $L_1(q)$. Then by Lemma 4.2, all d_{ij} vanish at q^1. Let γ' be the normal geodesic through q^1. By tautness again, $M \cap \gamma' = \{q^1, q^2, \ldots, q^8\}$ yields a configuration similar figure to Figure 5 along γ'.

The following argument is almost parallel as before, but we need to check the change of inequalities. In all the argument of Lemma 4.7, 4.8 and Lemma 4.9 using Table 1, we replace p^t by q^t. Then Lemma 4.6 holds for q^t, and we show instead of Lemma 4.7:

Lemma 4.11. *Take q^1 as a minimum point of λ on M and also minimum of μ on $L_1(q^1)$. Then we have, denoting $\nu^t = \nu(q^t)$, etc.*[2]*,*

[2] should be written as ν_q^t but we omit q for short.

$$\nu^7 \leq \nu^5 \leq \nu^1 \leq \nu^3, \quad \tau^1 \leq \tau^3, \quad \tau^7 \leq \tau^5.$$

PROOF. As before, (10) holds at q^1 and q^8, where $\lambda^1 - \lambda^8$ and $\nu^1 - \nu^8$ should have the same sign. Since λ^1 is the minimum, we have

$$\lambda^1 \leq \lambda^8 = \lambda^7, \quad \nu^1 \leq \nu^8 = \nu^3.$$

At q^2 and q^7 we have (11), and $\lambda^2 = \lambda^1 \leq \lambda^7$ implies

$$\nu^2 = \nu^5 \geq \nu^7.$$

Next, at q^1 and q^2 in (13), since q^1 is chosen so that $\mu^1 \leq \mu^2$, we obtain

$$\mu^1 \leq \mu^2, \quad \tau^1 \leq \tau^2 = \tau^3.$$

This impies $\mu^1 + \tau^1 \leq \mu^2 + \tau^2$ and so $\lambda^1 + \nu^1 \geq \lambda^2 + \nu^2$. As $\lambda^1 = \lambda^2$,

$$\nu^1 \geq \nu^2 = \nu^5$$

follows. Next, at q^4 and q^7, in (14), from $\mu^1 = \mu^4 \leq \mu^2 = \mu^7$, it follows

$$\tau^4 = \tau^5 \geq \tau^7 = \tau^6.$$

□

PROPOSITION 4.12. *Under the situation in Lemma 4.11, denoting the corresponding angles with primes, we have*

$$\alpha' = a', \quad \beta' = b', \quad \gamma' = c', \quad \delta' = d',$$

and all these values are $\dfrac{\pi}{4}$.

PROOF. The argument parallels that of Proposition 4.8, Lemma 4.9 and Proposition 4.10, with reversed inequalities. We check all the processes but omit details. □

Finally, we obtain:

PROPOSITION 4.13. *All principal curvatures coincide at* q^t *for* $1 \leq t \leq 8$*, and* $M \cap \gamma'$ *is a parallel octagon.*

Proof of Theorem 1.1 (ii): Both octagons $p^1 \dots p^8$ and $q^1 \dots q^8$ are isometric to the parallel octagon $M_\theta \cap \gamma_\theta$, where θ is uniquely determined by H (Fact 5). This implies $\lambda(p^1) = \lambda_\theta = \lambda(q^1)$, hence λ is constant all over M. Therefore $d_{j1} = 0$ everywhere on M, and by Lemma 4.2, we have $d_{ji} = 0$ for $j = 2, 3, 4$ and $i \neq 1$. Taking maximum and minimum points of μ where $d_{12} = 0$, and hence $d_{13} = d_{14} = 0$, we see that all d_{ji} vanish at such points. Then by a similar argument, we obtain $\max \mu = \min \mu$ and thus μ is constant on M. By Remark 4.1, ν and τ become constant. This proves Theorem 1.1 (ii).

□

Remark 4.14 : When $g=4$, the scalar curvature R_θ of M_θ is given by ([**41**])

$$R_\theta = 4\Big(m_1(m_1-1)(1+t^2) + m_2(m_2-1)\Big(1+\frac{1}{t^2}\Big)\Big),$$

where $t = \cot 2\theta_1 = \cot 2\big(\frac{\pi}{8}+\theta\big)$, and $n-1 = 2(m_1+m_2)$. Thus M_θ is scalar flat only when $m_1 = m_2 = 1$; otherwise, scalar positive. When M_θ is minimal, i.e. $t^2 = \frac{m_2}{m_1}$ by (6), the scalar curvature is

$$R = 4(m_1+m_2)(m_1+m_2-2).$$

Using the classification of isoparametric hypersurfaces with $g=4$ [**6**],[**15**], we have:

Proposition 4.15. *If a closed* minimal *Dupin hypersurface M in S^n with $g=4$ has constant scalar curvature R, then M is isoparametric, and R lies in the discrete set*

$$\{4(m_1+m_2)(m_1+m_2-2)\},$$

where $(m_1,m_2) = (1,1),(2,2),(4,5)\ldots$ are infinite series given in [**18**]. *The cases other than $(m_1,m_2) = (2,2),(4,5)$ correspond to representation of Clifford algebras.*

5. Review of the Lie sphere geometry

Up to this point, we have used only elementary arguments. However, to prove parts (iii) and (iv) of Theorem1.1, we need to employ *Lie sphere geometry*, in particular the concept of *Lie curvature*. In this section, we briefly review the necessary background. For details, see [**4**].

Let $\mathbb{R}^{n+3}_2 = \mathbb{R}^{n+1} \oplus \mathbb{R}^2_2$ be endowed with the bilinear form $\langle\, ,\, \rangle_2$ of signature $(+\ldots,+,-,-)$. The hyperquadric of $\mathbb{R}P^{n+2}_2$ consisting of null vectors

$$Q^{n+1} = \{[k] \in \mathbb{R}P^{n+2}_2 \mid \langle k,k\rangle_2 = 0\}$$

represents the *space of oriented hyperspheres of S^n*. An oriented hypersphere centered at $p \in S^n$ with oriented radius $-\pi \le \theta \le \pi$ is given by $k = {}^t(p,\cos\theta,\sin\theta) \in \mathbb{R}^{n+3}_2$. We denote k for $[k]$. Then ${}^t(p,1,0) \in \mathbb{R}^{n+3}_2$ represents a *point sphere*, and ${}^t(n,0,1)$ represents an *oriented totally geodesic hypersphere* centered at n. Two elements $k_1 = {}^t(u,\cos\varphi,\sin\varphi), k_2 = {}^t(v,\cos\psi,\sin\psi)$ in Q^{n+1} have *oriented contact* if they meet at a common point with coinciding normal directions. This occurs if and only if

$$\langle k_1,k_2\rangle_2 = \langle u,v\rangle - \cos(\varphi-\psi) = 0.$$

For instance, $k_1 = {}^t(p,1,0)$ and $k_2 = {}^t(n,0,1)$ have oriented contact precisely when p is orthogonal to n. If $\langle k_1,k_2\rangle_2 = 0$, the line

$$l = \{[ak_1+bk_2] \mid a,b \in \mathbb{R}\}$$

lies in Q^{n+1}; it represents a one-parameter family of oriented hyperspheres having oriented contact at p. The space of such lines, denoted by Λ^{2n-1}, is identified with the *unit tangent bundle* T_1S^n, since l is uniquely determined by the contact point $p \in S^n$ and the oriented unit normal $n \in T_pS^n$.

The *Lie contact transformation group* $O(n+1,2)$ is the linear group preserving $\langle\, ,\, \rangle_2$. It preserves Q^{n+1}, the oriented contact between oriented hyperspheres, and the space of lines $\Lambda^{2n-1} \cong T_1S^n$. An element $L = (l_1, l_2, \ldots, l_{n+3}) \in O(n+1,2)$ is characterized by

$$(\langle l_i, l_j \rangle_2) = \begin{pmatrix} I_{n+1} & 0 \\ 0 & -I_2 \end{pmatrix}$$

where I_{n+1} and I_2 are the unit matrices. The set of column vectors l_i (row vectors, respectively) is called *a Lie frame.*

An oriented hypersurface $p : M \to S^n$ with unit normal vector field n can be expressed as a *Lie geometric hypersurface* by the pair (k_1, k_2):

$$k_1 = \begin{pmatrix} p \\ 1 \\ 0 \end{pmatrix} \in Q^{n+1}, \quad k_2 = \begin{pmatrix} n \\ 0 \\ 1 \end{pmatrix} \in Q^{n+1},$$

not both $dk_1(X)$ and $dk_2(X)$ vanish for non-zero vector $X \in TM$. Since $\langle k_1, k_2 \rangle_2 = 0$, (k_1, k_2) defines a line $l \in \Lambda^{2n-1} \cong T_1S^n$. Indeed,

$$(k_1, k_2) : M \to \Lambda^{2n-1} \cong T_1S^n$$

is precisely the *Legendre map* of M into the contact manifold T_1S^n.

A *curvature sphere* of M at p is an oriented hypersphere having oriented contact with M of contact order ≥ 2. For each principal curvature λ, it is given by [**22**]

$$vk_1 + uk_2 \in Q^{n+1}, \quad \lambda = \frac{v}{u} = \cot\theta.$$

Applying $L \in O(n+1,2)$ to k_1, k_2 yields a new Lie geometric hypersurface (Lk_1, Lk_2) : $M \to \Lambda^{2n-1}$:

$$Lk_1 = \begin{pmatrix} q \\ a \\ b \end{pmatrix}, \quad Lk_2 = \begin{pmatrix} m \\ c \\ d \end{pmatrix}.$$

Then the principal curvature $\tilde{\lambda}$ of the image hypersurface is given by (4.2) in [**22**]

$$\tilde{\lambda} = \frac{a\lambda + c}{b\lambda + d}. \tag{21}$$

Remark 5.1 : A suitable projection $\pi : \Lambda^{2n-1} \to S^n$ may be applied to obtain a hypersurface in S^n from (Lk_1, Lk_2). Note that q and m themselves are not, respectively, the position and the normal vector of the resulting hypersurface.

Example 5.2 : (*Parallel transformations*) For

$$L = \begin{pmatrix} L_1 & 0 \\ 0 & L_\alpha \end{pmatrix} \in O(n+1) \oplus O(2), \quad L_\alpha = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix},$$

the transformation L_α deforms a hypersurface M into its *parallel hypersurface* M_α (see Remark 3.8 of [**22**]). If $\lambda = \cot\theta$ is a principal curvature of M, then by (21), M_α has the principal curvature

$$\lambda_\alpha = \frac{a\lambda + c}{b\lambda + d} = \frac{\cos\alpha\cot\theta - \sin\alpha}{\sin\alpha\cot\theta + \cos\alpha} = \cot(\theta + \alpha).$$

Remark 5.3 : The subgroup $O(n+1,1)$ of $O(n+1,2)$ corresponds to the *conformal (Möbius) group.*

Next, for $w_1, w_2, w_3, w_4 \in \mathbb{C}$, define the *cross ratio* by

$$[w_1, w_2; w_3, w_4] = \frac{(w_1 - w_3)(w_2 - w_4)}{(w_1 - w_4)(w_2 - w_3)} \in \mathbb{C},$$

which is real if and only if w_1, w_2, w_3, w_4 are concircular.

Let γ be the normal geodesic of a hypersurface M in S^n at $p \in M$. Suppose M has four principal curvatures $\lambda_i = \cot\theta_i$, each with curvature sphere C_i of radius θ_i. Then C_i intersects $\gamma = S^1$ orthogonally at a point whose spherical distance from p is $2\theta_i$. Labeling these points $z_{2i} \in \gamma = S^1 \subset \mathbb{C}$ and using $\lambda_i = \cot\theta_i$, we have:

Fact 6. (Lemma 6.8 [**22**]) The *Lie curvature* defined by

$$\Phi(p) = \frac{(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_4)}{(\lambda_1 - \lambda_4)(\lambda_2 - \lambda_3)} = [z_2, z_4; z_6, z_8] \in \mathbb{R} \tag{22}$$

is invariant under Lie contact transformations.

This serves as an index for determining whether two hypersurfaces are *Lie equivalent.*

6. Closed CMC and CLC Dupin with $g = 4$ (iii)

In this section, we prove:

Theorem 1.1 (iii) *Let M be a closed CMC Dupin hypersurface with $g = 4$. If M has constant Lie curvature (CLC), then M is isoparametric.*

The following fact plays an essential role in the proof.

Fact 7. (Proposition 8.1 and Corollary 8.3 in [**22**]) *If the Lie curvature of a closed Dupin hypersurface M with $g = 4$ is constant, then at each point of $p \in M$, there exists a Lie contact transformation that maps $M \cap \gamma$ onto a regular octagon, where γ is the normal geodesic of M at p.*

This statement is purely local, and no global consequence follows directly from it. However, since Lie curvatures are invariant under Lie sphere transformations, their values can be computed from the principal curvatures of the (not necessarily minimal, see Fact 5) isoparametric hypersurface $\bar{M}$ satisfying that $\bar{M} \cap \gamma$ is a regular octagon:

$$\lambda = \sqrt{2} + 1 = -\tau, \quad \mu = \sqrt{2} - 1 = -\nu. \tag{23}$$

Nevertheless, we cannot relate this fact directly to the mean curvature of M, because the Lie contact transformation does not preserve the metric structure.

6.1. Critical point of all the principal curvatures

LEMMA 6.1. *If a closed CMC Dupin hypersurface with $g = 4$ has CLC, then there exists a point $p \in M$ at which all d_{ji} vanish.*

PROOF. Since

$$H = m_1\lambda + m_2\mu + m_1\nu + m_2\tau, \quad \Phi = \frac{(\lambda-\mu)(\nu-\tau)}{(\lambda-\tau)(\nu-\mu)} = -1 \tag{24}$$

are constant on M where the value of Φ is computed from (23), we can describe μ, τ by λ, ν. Indeed, putting

$$A = \mu + \tau = \frac{1}{m_2}(H - m_1(\lambda+\nu)),$$

we have from the second equation of (24)

$$0 = (\lambda-\mu)(\nu-\tau) + (\lambda-\tau)(\nu-\mu) = 2\mu\tau - A(\lambda+\nu) + 2\lambda\nu.$$

Hence defining

$$B := \mu\tau = \frac{1}{2}(A(\lambda+\nu) - 2\lambda\nu) = \frac{1}{2}\Big(\frac{1}{m_2}(H - m_1(\lambda+\nu))(\lambda+\nu) - 2\lambda\nu\Big),$$

we know that μ, τ are two solutions of

$$t^2 - At + B = 0. \tag{25}$$

Then on a λ-leaf L_1, μ, τ are functions of only ν, and consequently,

$$d_{12} = f(\nu)d_{13}, \quad d_{14} = g(\nu)d_{13}.$$

Therefore, if λ is critical on M and ν is critical at p on $L_1(p)$, then at p,

$$d_{j1} = 0, \quad d_{1i} = 0, \quad 1 \le j, i \le 4.$$

On the other hand, from (24), we obtain

$$\begin{aligned}
0 &= m_1 d_{j1} + m_2 d_{j2} + m_1 d_{j3} + m_2 d_{j4} = 0\\
0 &= e_j(\log\Phi) = \frac{d_{j1}-d_{j2}}{\lambda-\mu} - \frac{d_{j1}-d_{j4}}{\lambda-\tau} + \frac{d_{j3}-d_{j4}}{\nu-\tau} - \frac{d_{j3}-d_{j2}}{\nu-\mu}\\
&= \Big(\frac{1}{\nu-\mu} - \frac{1}{\lambda-\mu}\Big)d_{j2} + \Big(\frac{1}{\nu-\tau} - \frac{1}{\nu-\mu}\Big)d_{j3} + \Big(\frac{1}{\lambda-\tau} - \frac{1}{\nu-\tau}\Big)d_{j4}.
\end{aligned}$$

At p, the following relations hold:

$$\begin{cases} m_1 d_{23} + m_2 d_{24} = 0 \\ d_{32} + d_{34} = 0 \\ m_2 d_{42} + m_1 d_{43} = 0, \end{cases} \tag{26}$$

$$\begin{cases} \dfrac{\tau-\mu}{\nu-\mu} d_{23} + \dfrac{\nu-\lambda}{\lambda-\tau} d_{24} = 0 \\ \dfrac{1}{(\nu-\mu)(\lambda-\mu)} d_{32} - \dfrac{1}{(\lambda-\tau)(\nu-\tau)} d_{34} = 0 \\ \dfrac{\lambda-\nu}{\lambda-\mu} d_{42} + \dfrac{\tau-\mu}{\nu-\tau} d_{43} = 0. \end{cases} \tag{27}$$

In (26), all coefficients are positive, whereas in (27), two coefficients of the first and third equations each have coefficients of opposite signs. Hence $d_{23} = d_{24} = d_{42} = d_{43} = 0$. Next, rewriting the second equation of (27) and using $\Phi = -1$, we have

$$\begin{aligned} & d_{32} - \frac{(\lambda-\mu)(\nu-\tau)}{(\lambda-\tau)(\nu-\mu)} \Big(\frac{\nu-\mu}{\nu-\tau}\Big)^2 d_{34} \\ & \quad = d_{32} - \Phi \Big(\frac{\nu-\mu}{\nu-\tau}\Big)^2 d_{34} = (1 - \Big(\frac{\nu-\mu}{\nu-\tau}\Big)^2\Big) d_{32} = 0, \end{aligned}$$

Thus from $\big(\frac{\nu-\mu}{\nu-\tau}\big)^2 < 1$, it follows $d_{32} = d_{34} = 0$, namely, all d_{ji}'s vanish at p. □

Finally at $p = p^1$, the situation coincides with that in Proposition 6.1 of [**22**], which is stated as Proposition 4.4 in §4 in the present paper.

Next, we denote p^1, p^2 by z_1, z_2 where

$$z_2 = e^{2i\theta_1} z_1, \quad \lambda^1 = \cot\theta_1.$$

Moreover, define

$$z_4 = e^{2i(\theta_1+\alpha)}, \quad z_6 = e^{2i(\theta_1+\alpha+\beta)}, \quad z_8 = e^{2i(\theta_1+\alpha+\beta+\gamma)},$$

where

$$\alpha = \theta_1^3 + \bar\theta_4^3, \quad \beta = \theta_1^5 + \bar\theta_4^5, \quad \gamma = \theta_1^7 + \bar\theta_4^7,$$

which follow from Table 1. From (24), we then have

$$\begin{aligned} -1 = \Phi &= \frac{(\lambda-\mu)(\nu-\tau)}{(\lambda-\tau)(\nu-\mu)} = [z_2, z_6 : z_4, z_8] = \frac{(z_2-z_4)(z_6-z_8)}{(z_2-z_8)(z_6-z_4)} \\ &= \frac{(e^{2i\theta_1} - e^{2i(\theta_1+\alpha)})(e^{2i(\theta_1+\alpha+\beta)} - e^{2i(\theta_1+\alpha+\beta+\gamma)})}{(e^{2i\theta_1} - e^{2i(\theta_1+\alpha+\beta+\gamma)})(e^{2i(\theta_1+\alpha+\beta)} - e^{2i(\theta_1+\alpha)})} \\ &= \frac{(1-e^{2i\alpha})(e^{2i(\alpha+\beta)} - e^{2i(\alpha+\beta+\gamma)})}{(1-e^{2i(\alpha+\beta+\gamma)})(e^{2i(\alpha+\beta)} - e^{2i\alpha})} = \frac{(1-e^{2i\alpha})(e^{2i\beta} - e^{2i(\beta+\gamma)})}{(1-e^{2i(\alpha+\beta+\gamma)}(e^{2i\beta}-1)}, \end{aligned}$$

and hence

$$\begin{aligned} 0 &= (1-e^{2i\alpha})(e^{2i\beta} - e^{2i(\beta+\gamma)}) + (1-e^{2i(\alpha+\beta+\gamma)})(e^{2i\beta}-1) \\ &= 2(e^{2i\beta} + e^{2i(\alpha+\beta+\gamma)}) - e^{2i(\alpha+\beta)} - e^{2i(\beta+\gamma)} - e^{2i(\alpha+2\beta+\gamma)} - 1. \end{aligned}$$

Multiplying $e^{-2i\beta}$ and using $\alpha+\beta+\gamma=\pi-\delta$, we obtain:

LEMMA 6.2. *In this situation,* $\Phi=-1$ *implies*

$$2(1+e^{2i(\alpha+\gamma)})-e^{2i\alpha}-e^{2i\gamma}-e^{-2i\delta}-e^{-2i\beta}=0. \tag{28}$$

6.2. Conformal transformation

It is easy to see that there exists a conformal transformation $\hat{C}$ on S^n which maps $M\to\hat{M}$, so that the leaves $L_\lambda(p^1)$ and $L_\lambda(p^5)$ become antipodally symmetric, and $L_\nu(p^1)$ and $L_\nu(p^2)$ become parallel (see Figure 7). In fact, we may regard $\hat{C}$ as restricted to the plane on which γ lies, keeping its orthogonal complement invariant (see (31) below).

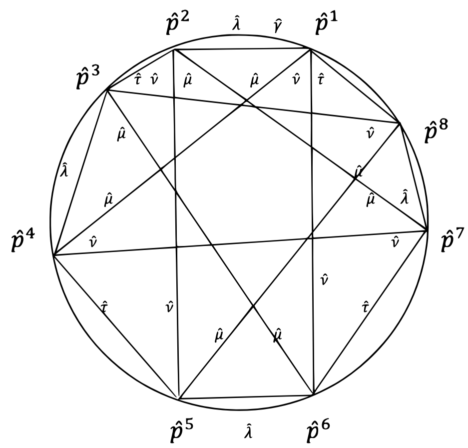

Figure 7: Conformal image

We denote the image objects with hats, such as $\hat{M}$, $\hat{p}^t$, $\hat{\lambda}^t$ etc. Then $\hat{\lambda}^1=\hat{\lambda}^6$ and $\hat{\nu}^1=\hat{\nu}^2=\hat{\nu}^5=\hat{\nu}^6$ hold, and hence, from Table 1 with hats, it follows that

$$\hat{\theta}^1=\hat{\theta}^5, \quad \hat{\alpha}+\hat{\beta}=\hat{a}+\hat{b}=\hat{\gamma}+\hat{\delta}=\hat{c}+\hat{d}=\frac{\pi}{2}. \tag{29}$$

Thus denoting $\hat{\lambda}^t=\cot\hat{\theta}^t_1$, we obtain

$$\begin{aligned}\hat{\lambda}^1\hat{\nu}^1&=\cot\hat{\theta}_1\cot(\hat{\theta}_1+\hat{\alpha}+\hat{\beta})=-1,\\ \hat{\lambda}^2\hat{\nu}^2&=\cot\hat{\theta}^2_1\cot(\hat{\theta}^2_1+\hat{a}+\hat{b})=-1.\end{aligned}$$

Now apply the argument in the previous subsection to $\hat{M}$. This is legitimate because the Lie curvature is invariant under $\hat{C}$. Then we obtain (28) replacing the data with that of $\hat{M}$, namely,

$$\begin{aligned}&2(1+e^{2i(\hat{\alpha}+\hat{\gamma})})-e^{2i\hat{\alpha}}-e^{2i\hat{\gamma}}-e^{-2i\hat{\delta}}-e^{-2i\hat{\beta}}=0,\\ &2(1+e^{2i(\hat{a}+\hat{c})})-e^{2i\hat{a}}-e^{2i\hat{c}}-e^{-2i\hat{d}}-e^{-2i\hat{b}}=0.\end{aligned} \tag{30}$$

From (29), $e^{-2i\hat\beta} = e^{-2i(\pi/2-\hat\alpha)} = -e^{2i\hat\alpha}$ etc. holds, and (30) becomes $1 + e^{2i(\hat\alpha+\hat\gamma)} = 0$ and $1 + e^{2i(\hat a+\hat c)} = 0$. Thus we obtain

$$\hat\alpha = \hat\beta = \hat\gamma = \hat\delta = \hat a = \hat b = \hat c = \hat d = \frac{\pi}{4}.$$

This means that $\hat M \cap \hat\gamma$ is a parallel octagon (see Figure 6 with hat). We can denote

$$\begin{aligned}
\hat p^1 &= \begin{pmatrix}\hat u\\ \hat v\end{pmatrix} = -\hat p^5, \quad \hat n^1 = \begin{pmatrix}-\hat v\\ \hat u\end{pmatrix} = -\hat n^5\\
\hat p^2 &= \begin{pmatrix}-\hat u\\ \hat v\end{pmatrix} = -\hat p^6, \quad \hat n^2 = \begin{pmatrix}\hat v\\ \hat u\end{pmatrix} = -\hat n^6,\\
\hat p^3 &= \begin{pmatrix}-\hat v\\ \hat u\end{pmatrix} = -\hat p^7, \quad \hat n^3 = \begin{pmatrix}-\hat u\\ -\hat v\end{pmatrix} = -\hat n^7\\
\hat p^4 &= \begin{pmatrix}-\hat v\\ -\hat u\end{pmatrix} = -\hat p^8, \quad \hat n^4 = \begin{pmatrix}-\hat u\\ \hat v\end{pmatrix} = -\hat n^8.
\end{aligned}$$

where $\hat n^t$ denotes the oriented unit normal of $\hat M$ at $\hat p^t$, and

$$\hat\lambda^t = \frac{\hat v}{\hat u}, \quad \hat\nu^t = -\frac{\hat u}{\hat v}, \quad \hat u^2 + \hat v^2 = 1.$$

More precisely, omitting t:

LEMMA 6.3. *We have* $\hat\lambda = \cot\hat\theta_1 > 1$ *and*

$$\hat\lambda = \frac{\hat v}{\hat u}, \quad \hat\mu = \frac{\hat v - \hat u}{\hat u + \hat v}, \quad \hat\nu = -\frac{\hat u}{\hat v}, \quad \hat\tau = \frac{\hat u + \hat v}{\hat u - \hat v}.$$

PROOF. From (4), we have $\hat\theta_1 = \frac{\pi}{8} + \theta < \frac{\pi}{4}$, hence $\hat\lambda > 1$. We compute $\hat\mu$, where $\hat\tau = -\frac{1}{\hat\mu}$:

$$\hat\mu = \cot\left(\hat\theta_1 + \frac{\pi}{4}\right) = \frac{-1+\hat\lambda}{\hat\lambda+1} = \frac{-1+\frac{\hat v}{\hat u}}{\frac{\hat v}{\hat u}+1} = \frac{-\hat u+\hat v}{\hat v+\hat u}.$$

□

Next consider the inverse $C : \hat M \to M$ of $\hat C$. Denote the conformal transformation C restricted to $\mathbb{R}^4_2 = \mathbb{R}^2 \oplus \mathbb{R}^2_2$, where $\mathbb{R}^2$ is the plane on which γ lies, by

$$C = \begin{pmatrix} * & * & * & 0\\ * & * & * & 0\\ x & y & r & 0\\ 0 & 0 & 0 & 1\end{pmatrix} \in O(2,1) \subset O(2,2). \tag{31}$$

Apply C to $k_1 = \begin{pmatrix}\hat p^t\\ 1\\ 0\end{pmatrix}$ and $k_2 = \begin{pmatrix}\hat n^t\\ 0\\ 1\end{pmatrix}$, and express

$$Ck_1 = \begin{pmatrix} q^t \\ a_t \\ 0 \end{pmatrix}, \quad Ck_2 = \begin{pmatrix} n^t \\ c_t \\ 1 \end{pmatrix}.$$

The principal curvature λ_i^t at the original point $p^t \in M$ is given by (21):

$$\lambda_i^t = a_t \hat{\lambda}_i + c_t, \quad i = 1, \dots, 4, \quad t = 1, \dots 8, \tag{32}$$

since $\hat{b}_t = 0$ and $\hat{d}_t = 1$. We know

$$\begin{cases} a_1 = \hat{u}x + \hat{v}y + r \\ c_1 = -\hat{v}x + \hat{u}y \end{cases} \qquad \begin{cases} a_2 = -\hat{u}x + \hat{v}y + r \\ c_2 = \hat{v}x + \hat{u}y \end{cases}$$
$$\begin{cases} a_3 = -\hat{v}x + \hat{u}y + r \\ c_3 = -\hat{u}x - \hat{v}y. \end{cases}$$

Note that the mean curvature $\hat{H}^t$ does not depend on t since $\hat{M} \cap \hat{\gamma}$ is a parallel octagon. Denote

$$\hat{H}^t = \hat{H} = m_1(\hat{\lambda} + \hat{\nu}) + m_2(\hat{\mu} + \hat{\tau}).$$

On the other hand, the mean curvature H^t of M is obtained from (32):

$$H^t = a_t \hat{H}^t + c_t K, \quad t = 1, \dots, 8,$$

where $K = 2(m_1 + m_2)$. Since H^t is constant, we have

$$0 = H^1 - H^2 = (a_1 - a_2)\hat{H} + (c_1 - c_2)K = 2x(\hat{u}\hat{H} - \hat{v}K) = 2x\hat{u}(\hat{H} - \hat{\lambda}K).$$

Because $\hat{\lambda}$ is the largest or smallest principal curvature by (32), we have $\hat{H} - \hat{\lambda}K \neq 0$, and it follows

$$x = 0.$$

Next, from

$$\begin{aligned} 0 &= H^1 - H^3 = (a_1 - a_3)\hat{H} + (c_1 - c_3)K = 2y((\hat{v} - \hat{u})\hat{H} + (\hat{u} + \hat{v}))K \\ &= 2y(\hat{v} - \hat{u})(\hat{H} - \hat{\tau}K) \end{aligned}$$

and since $(\hat{v} - \hat{u})(\hat{H} - \hat{\tau}K) \neq 0$ by Lemma 6.3 (as $\hat{\tau}$ is smallest or largest), we have

$$y = 0.$$

Therefore, we obtain:

Proposition 6.4. *C is an isometry $\begin{pmatrix} T & 0 \\ 0 & I' \end{pmatrix} \in O(2,2)$, $I' = \begin{pmatrix} \epsilon & 0 \\ 0 & 1 \end{pmatrix}$, and $M \cap \gamma$ itself is a parallel octagon isometric to $M_\theta \cap \gamma_\theta$, where θ is uniquely determined by the mean curvature H of M (Fact 5).*

Proof of Theorem 1.1 (iii): In the same way as the proof of Theorem 1.1 (ii), $M \cap \gamma$ at a maximum point $p = p^1$ of λ, with ν critical on $L(p^1)$, and $M \cap \gamma'$ at a minimum point $q = q^1$ of λ, with ν critical on $L(q^1)$, are both congruent to a parallel octagon $M_\theta \cap \gamma_\theta$, where θ is uniquely determined by the mean curvature of M. Thus, λ is constant. A similar argument at the maximum and minimum points of ν implies that ν is constant. Then (24) uniquely determines $\mu > \tau$ as solutions of (25), and thus all principal curvatures are constant throughout M. Therefore, M is *isoparametric*. □

7. Closed case: $g = 6$ (iv)

7.1. Strategy for $g = 6$

Now we consider the case $g = 6$, where the multiplicity $m = m_1 = m_2 \in \{1, 2\}$ [**1**]. Let the distinct principal curvatures be $\lambda_1 > \cdots > \lambda_6$, which we also denote by $\lambda = \lambda_1, \mu = \lambda_2, \nu = \lambda_3, \rho = \lambda_4, \sigma = \lambda_5$ and $\tau = \lambda_6$, following the notation in [**23**]. Later, we use upper indices to indicate the corresponding points p^t. The curvature distributions decompose the tangent space into $D_1 \oplus \cdots \oplus D_6$, and we denote by e_j any unit vector in D_j.

Assuming M has *constant mean curvature (CMC)*, we put

$$H = m \sum_{i=1}^{6} \lambda_i, \quad m = 1, 2. \tag{33}$$

Note that when $g = 6$, there exist essentially *three independent Lie curvatures* [**23**]. We now prove the following:

Theorem 1.1 (iv) *Let M be a closed CMC Dupin hypersurface with $g = 6$. If M has three independent constant Lie curvatures, then M is isoparametric.*

Remark 7.1 : Even if all the Lie curvatures are constant, a Dupin hypersurface with $g = 6$ is not necessarily Lie equivalent to an isoparametric hypersurface [**23**].

Recall that for any normal geodesic $\bar{\gamma}$ of the minimal isoparametric hypersurface M_0, the intersection $M_0 \cap \bar{\gamma}$ forms a regular dodecagon. In [**23**] we prove:

Fact 6. (Lemma 4 in [23]) *When all the Lie curvatures of a closed Dupin hypersurface M with $g = 6$ are constant, there exists a Lie transformation at each point of $p \in M$ which maps $M \cap \gamma$ onto a regular dodecagon, where γ denotes the normal geodesic of M at p.*

As before, this is a local statement, and does not yield a global result. However, since the Lie curvatures are *invariant under Lie transformations*, their values can be computed from the principal curvatures of M_0. By setting $g = 6$ and $\theta = 0$ in (3), we obtain:

$$\lambda = 2 + \sqrt{3} = -\tau, \quad \mu = 1 = -\sigma, \quad \nu = 2 - \sqrt{3} = -\rho. \tag{34}$$

Now, using the CMC condition together with these explicit values of the Lie curvatures, we find a point $p \in M$ at which all d_{ji} vanish (Proposition 7.4). Our goal is to show $M \cap \gamma$ at this point is *itself* a *parallel dodecagon* (Proposition 7.10). Then as in the case $g = 4$, the mean curvature uniquely determines θ so that $M \cap \gamma$ is isometric to the parallel dodecagon $M_\theta \cap \gamma_\theta$, and the maximum and minimum values of λ coincide. Hence λ is constant on all of M; with further arguments, we conclude that M is *isoparametric*.

7.2. Critical point of principal curvatures

We use indices $1 \leq i, j \leq 6$. From (33), it follows that for $e_j \in D_j$,

$$d_{j1} + d_{j2} + d_{j3} + d_{j4} + d_{j5} + d_{j6} = 0, \quad j = 1, \dots, 6. \tag{35}$$

Since the multiplicities of λ_i's are common ($m = 1, 2$), we may omit them. As before, it is essential to find a point where all $d_{ji} = 0$ under the conditions CMC and constant Lie curvatures (Proposition 7.4). This is the delicate part of the proof.

For instance, in the case $m = 1$, there are 30 unknowns $d_{ji} = e_j(\lambda_i)$, $1 \leq i, j \leq 6$, since $d_{jj} = 0$ by the Dupin condition. Equation (35) provides six relations, reducing the number of independent unknowns to 24. Let $p \in M$ be a critical point of λ, and assume further that μ is critical at p along $L_1(p)$. Then at p,

$$d_{j1} = 0, \quad d_{12} = 0, \quad 1 \leq j \leq 6.$$

These yield six equations (since $d_{11} = 0$ already), leaving 18 unknowns. Now consider three Lie curvatures

$$\Phi_h = \frac{(\lambda - \mu)(\lambda_h - \sigma)}{(\lambda - \sigma)(\lambda_h - \mu)}, \quad h = 3, 4, 6. \tag{36}$$

Assume that each Φ_h is constant on M for $h = 3, 4, 6$. Then

$$e_j(\Phi_h) = 0, \quad j = 1, \dots, 6, \quad h = 3, 4, 6,$$

provides 18 further equations among the d_{ji}'s for the 18 unknowns, allowing us to examine whether $d_{ji}(p) = 0$ holds.

In the following, we investigate this process for $m = 1, 2$.

LEMMA 7.2. *At a point $p \in M$ where λ is critical on M and μ is critical on $L_1(p)$, all d_{1i} vanish for any $e_1 \in D_1(p)$ and $i \in \{1, \dots, 6\}$.*

PROOF. From (36), for $h \in \{3, 4, 6\}$,

$$\begin{aligned} e_j(\log \Phi_h) &= \frac{d_{j1} - d_{j2}}{\lambda - \mu} - \frac{d_{j1} - d_{j5}}{\lambda - \sigma} + \frac{d_{jh} - d_{j5}}{\lambda_h - \sigma} - \frac{d_{jh} - d_{j2}}{\lambda_h - \mu} \\ &= u_h d_{j2} + v_h d_{j5} + w_h d_{jh} = 0, \quad j = 1, \dots, 6, \ h = 3, 4, 6, \end{aligned} \tag{37}$$

for $j = 1, \dots, 6$, since $d_{j1} = 0$ holds at p, where

$$
\begin{aligned}
u_h &= \frac{1}{\lambda_h-\mu} - \frac{1}{\lambda-\mu} = \frac{(\lambda-\lambda_h)}{(\lambda_h-\mu)(\lambda-\mu)}\\
v_h &= \frac{1}{\lambda-\sigma} - \frac{1}{\lambda_h-\sigma} = \frac{(\lambda_h-\lambda)}{(\lambda-\sigma)(\lambda_h-\sigma)}\\
w_h &= \frac{1}{\lambda_h-\sigma} - \frac{1}{\lambda_h-\mu} = \frac{(\sigma-\mu)}{(\lambda_h-\sigma)(\lambda_h-\mu)}.
\end{aligned}
$$

Putting $j=1$ in (37) and using $d_{12}=0$ at p, we obtain

$$
\begin{aligned}
v_3 d_{15} + w_3 d_{13} = 0, &\quad \text{i.e.,} \quad d_{13} = -\frac{v_3}{w_3} d_{15},\\
v_4 d_{15} + w_4 d_{14} = 0, &\quad \text{i.e.,} \quad d_{14} = -\frac{v_4}{w_4} d_{15},\\
v_6 d_{15} + w_6 d_{16} = 0, &\quad \text{i.e.,} \quad d_{16} = -\frac{v_6}{w_6} d_{15}.
\end{aligned} \tag{38}
$$

Substituting these into (35), we obtain

$$
\Big(1 - \frac{v_3}{w_3} - \frac{v_4}{w_4} - \frac{v_6}{w_6}\Big) d_{15} = 0.
$$

Since $v_3, v_4 < 0$, $v_6 > 0$, $w_3, w_4 > 0$ and $w_6 < 0$, we have

$$
1 - \frac{v_3}{w_3} - \frac{v_4}{w_4} - \frac{v_6}{w_6} > 0,
$$

which implies

$$
d_{1i} = 0.
$$

□

LEMMA 7.3. *On $L_1(p)$ where λ is critical, we have $d_{2i} = d_{3i} = d_{4i} = d_{6i} = 0$.*

PROOF. We do not use $d_{12}=0$ in the proof below. Hence the argument holds all over $L_1(p)$.

1. First, to show $d_{2i}=0$ on $L_1(p)$, put $j=2$ in (37), and we have

$$
\begin{aligned}
v_3 d_{25} + w_3 d_{23} = 0, &\quad \text{i.e.,} \quad d_{23} = -\frac{v_3}{w_3} d_{25}\\
v_4 d_{25} + w_4 d_{24} = 0, &\quad \text{i.e.,} \quad d_{24} = -\frac{v_4}{w_4} d_{25}\\
v_6 d_{25} + w_6 d_{26} = 0, &\quad \text{i.e.,} \quad d_{26} = -\frac{v_6}{w_6} d_{25}.
\end{aligned}
$$

which are identical to (38) with d_{1j} replaced by d_{2j}. Thus

$$
d_{2i} = 0.
$$

2. For $d_{3i}=0$, consider

$$
\check{\Psi}_h = \frac{(\nu-\rho)(\lambda_h-\lambda)}{(\nu-\lambda)(\lambda_h-\rho)}, \quad h = 2,5,6.
$$

A calculation using $d_{31} = 0$ and $d_{33} = 0$ gives

$$\begin{aligned} 0 = e_3(\log \check{\Psi}_h) &= \frac{-d_{34}}{\nu - \rho} + \frac{d_{3h}}{\lambda_h - \lambda} - \frac{d_{3h} - d_{34}}{\lambda_h - \rho} \\ &= \Big(\frac{1}{\lambda_h - \lambda} - \frac{1}{\lambda_h - \rho}\Big) d_{3h} + \Big(\frac{1}{\lambda_h - \rho} - \frac{1}{\nu - \rho}\Big) d_{34} \\ &= \frac{1}{\lambda_h - \rho}\Big(\frac{\lambda - \rho}{\lambda_h - \lambda} d_{3h} + \frac{\nu - \lambda_h}{\nu - \rho} d_{34}\Big). \end{aligned}$$

Putting $h = 2, 5, 6$, we obtain

$$d_{32} = \frac{(\lambda - \mu)(\nu - \mu)}{(\lambda - \rho)(\nu - \rho)} d_{34}, \quad d_{35} = \frac{(\lambda - \sigma)(\nu - \sigma)}{(\lambda - \rho)(\nu - \rho)} d_{34}, \quad d_{36} = \frac{(\lambda - \tau)(\nu - \tau)}{(\lambda - \rho)(\nu - \rho)} d_{34}.$$

Here, we use the Lie curvature. Using (34), we compute

$$\begin{aligned} \frac{(\lambda - \mu)(\nu - \mu)}{(\lambda - \rho)(\nu - \rho)} &= \frac{(\lambda - \rho)(\nu - \mu)}{(\lambda - \mu)(\nu - \rho)} \Big(\frac{\lambda - \mu}{\lambda - \rho}\Big)^2 = -2\Big(\frac{\lambda - \mu}{\lambda - \rho}\Big)^2, \\ \frac{(\lambda - \sigma)(\nu - \sigma)}{(\lambda - \rho)(\nu - \rho)} &= \frac{(\lambda - \rho)(\nu - \sigma)}{(\lambda - \sigma)(\nu - \rho)} \Big(\frac{\lambda - \sigma}{\lambda - \rho}\Big)^2 = 2\Big(\frac{\lambda - \sigma}{\lambda - \rho}\Big)^2, \\ \frac{(\lambda - \tau)(\nu - \tau)}{(\lambda - \rho)(\nu - \rho)} &= \frac{(\lambda - \rho)(\nu - \tau)}{(\lambda - \tau)(\nu - \rho)} \Big(\frac{\lambda - \tau}{\lambda - \rho}\Big)^2 = 4\Big(\frac{\lambda - \tau}{\lambda - \rho}\Big)^2. \end{aligned}$$

Therefore, (35) becomes

$$\begin{aligned} 0 &= d_{32} + d_{34} + d_{35} + d_{36} \\ &= d_{34}\Big(-2\Big(\frac{\lambda - \mu}{\lambda - \rho}\Big)^2 + 1 + 2\Big(\frac{\lambda - \sigma}{\lambda - \rho}\Big)^2 + 4\Big(\frac{\lambda - \tau}{\lambda - \rho}\Big)^2\Big), \end{aligned}$$

but as the coefficient satisfies

$$-2\Big(\frac{\lambda - \mu}{\lambda - \rho}\Big)^2 + 1 + 2\Big(\frac{\lambda - \sigma}{\lambda - \rho}\Big)^2 + 4\Big(\frac{\lambda - \tau}{\lambda - \rho}\Big)^2 > -2 + 1 + 2 + 4 = 5 > 0,$$

we obtain

$$d_{3i} = 0.$$

3. Similarly, for $d_{4i} = 0$, consider

$$\bar{\Psi}_h = \frac{(\rho - \nu)(\lambda_h - \lambda)}{(\rho - \lambda)(\lambda_h - \nu)}, \quad h = 2, 5, 6.$$

Then we have, using $d_{41} = 0$ and $d_{44} = 0$,

$$\begin{aligned} 0 = e_4(\log \bar{\Psi}_h) &= \frac{-d_{43}}{\rho - \nu} + \frac{d_{4h}}{\lambda_h - \lambda} - \frac{d_{4h} - d_{43}}{\lambda_h - \nu} \\ &= \Big(\frac{1}{\lambda_h - \lambda} - \frac{1}{\lambda_h - \nu}\Big) d_{4h} + \Big(\frac{1}{\lambda_h - \nu} - \frac{1}{\rho - \nu}\Big) d_{43} \\ &= \frac{1}{\lambda_h - \nu}\Big(\frac{\lambda - \nu}{\lambda_h - \lambda} d_{4h} + \frac{\rho - \lambda_h}{\rho - \nu} d_{43}\Big). \end{aligned}$$

Putting $h = 2, 5, 6$, we obtain

$$d_{42} = \frac{(\lambda-\mu)(\rho-\mu)}{(\lambda-\nu)(\rho-\nu)} d_{43}, \quad d_{45} = \frac{(\lambda-\sigma)(\rho-\sigma)}{(\lambda-\nu)(\rho-\nu)} d_{43}, \quad d_{46} = \frac{(\lambda-\tau)(\rho-\tau)}{(\lambda-\nu)(\rho-\nu)} d_{43}$$

Using the Lie curvature which we compute from (34), we have

$$\begin{aligned}
\frac{(\lambda-\mu)(\rho-\mu)}{(\lambda-\nu)(\rho-\nu)} &= \frac{(\lambda-\nu)(\rho-\mu)}{(\lambda-\mu)(\rho-\nu)} \Big(\frac{\lambda-\mu}{\lambda-\nu}\Big)^2 = 3\Big(\frac{\lambda-\mu}{\lambda-\rho}\Big)^2, \\
\frac{(\lambda-\sigma)(\rho-\sigma)}{(\lambda-\nu)(\rho-\nu)} &= \frac{(\lambda-\nu)(\rho-\sigma)}{(\lambda-\sigma)(\rho-\nu)} \Big(\frac{\lambda-\sigma}{\lambda-\nu}\Big)^2 = -\Big(\frac{\lambda-\sigma}{\lambda-\rho}\Big)^2 \\
\frac{(\lambda-\tau)(\rho-\tau)}{(\lambda-\nu)(\rho-\nu)} &= \frac{(\lambda-\nu)(\rho-\tau)}{(\lambda-\tau)(\rho-\nu)} \Big(\frac{\lambda-\tau}{\lambda-\nu}\Big)^2 = -3\Big(\frac{\lambda-\tau}{\lambda-\nu}\Big)^2.
\end{aligned}$$

Then (35) becomes

$$\begin{aligned}
0 &= d_{42} + d_{43} + d_{45} + d_{46} \\
&= d_{43}\Big(3\Big(\frac{\lambda-\mu}{\lambda-\rho}\Big)^2 + 1 - \Big(\frac{\lambda-\sigma}{\lambda-\rho}\Big)^2 - 3\Big(\frac{\lambda-\tau}{\lambda-\nu}\Big)^2\Big),
\end{aligned}$$

where the coefficient satisfies

$$3\Big(\frac{\lambda-\mu}{\lambda-\rho}\Big)^2 + 1 - \Big(\frac{\lambda-\sigma}{\lambda-\rho}\Big)^2 - 3\Big(\frac{\lambda-\tau}{\lambda-\nu}\Big)^2 < 3 + 1 - 1 - 3 = 0.$$

Thus we have

$$d_{4i} = 0.$$

4. Finally, for d_{6i}, take

$$\tilde{\Psi}_h = \frac{(\tau-\nu)(\lambda_h-\lambda)}{(\tau-\lambda)(\lambda_h-\nu)}, \quad h = 2, 4, 5.$$

Then we have, using $d_{61} = 0$ and $d_{66} = 0$,

$$\begin{aligned}
0 = e_6(\log \tilde{\Psi}_h) &= \frac{-d_{63}}{\tau-\nu} + \frac{d_{6h}}{\lambda_h-\lambda} - \frac{d_{6h}-d_{63}}{\lambda_h-\nu} \\
&= \Big(\frac{1}{\lambda_h-\lambda} - \frac{1}{\lambda_h-\nu}\Big) d_{6h} + \Big(\frac{1}{\lambda_h-\nu} - \frac{1}{\tau-\nu}\Big) d_{63} \\
&= \frac{1}{\lambda_h-\nu}\Big(\frac{\lambda-\nu}{\lambda_h-\lambda} d_{6h} + \frac{\tau-\lambda_h}{\tau-\nu} d_{63}\Big),
\end{aligned}$$

which implies

$$d_{62} = \frac{(\lambda-\mu)(\tau-\mu)}{(\lambda-\nu)(\tau-\nu)} d_{63}, \quad d_{64} = \frac{(\lambda-\rho)(\tau-\rho)}{(\lambda-\nu)(\tau-\nu)} d_{63}, \quad d_{65} = \frac{(\lambda-\sigma)(\tau-\sigma)}{(\lambda-\nu)(\tau-\nu)} d_{63}.$$

The coefficient of d_{63} in $d_{61} + \cdots + d_{65}$ is positive, hence

$$d_{6i} = 0.$$

□

To determine whether $d_{5i} = 0$, consider

$$\tilde{\Psi}_h = \frac{(\sigma-\mu)(\lambda_h-\lambda)}{(\sigma-\lambda)(\lambda_h-\mu)}, \quad h = 3, 4, 6.$$

Then using $d_{51} = 0$ and $d_{55} = 0$, we have

$$\begin{aligned} 0 = e_5(\log \tilde{\Psi}_h) &= \frac{-d_{52}}{\sigma-\mu} + \frac{d_{5h}}{\lambda_h-\lambda} - \frac{d_{5h}-d_{52}}{\lambda_h-\mu} \\ &= \Big(\frac{1}{\lambda_h-\lambda} - \frac{1}{\lambda_h-\mu}\Big)d_{5h} + \Big(\frac{1}{\lambda_h-\mu} - \frac{1}{\sigma-\mu}\Big)d_{52} \\ &= \frac{1}{\lambda_h-\mu}\Big(\frac{\lambda-\mu}{\lambda_h-\lambda}d_{5h} + \frac{\sigma-\lambda_h}{\sigma-\mu}d_{52}\Big), \end{aligned}$$

which implies

$$d_{53} = \frac{(\lambda-\nu)(\sigma-\nu)}{(\lambda-\mu)(\sigma-\mu)}d_{52}, \quad d_{54} = \frac{(\lambda-\rho)(\sigma-\rho)}{(\lambda-\mu)(\sigma-\mu)}d_{52}, \quad d_{56} = \frac{(\lambda-\tau)(\sigma-\tau)}{(\lambda-\mu)(\sigma-\mu)}d_{52}. \quad (39)$$

Thus we have

$$0 = d_{52}\Big(1 + \frac{(\lambda-\nu)(\sigma-\nu)}{(\lambda-\mu)(\sigma-\mu)} + \frac{(\lambda-\rho)(\sigma-\rho)}{(\lambda-\mu)(\sigma-\mu)} + \frac{(\lambda-\tau)(\sigma-\tau)}{(\lambda-\mu)(\sigma-\mu)}\Big).$$

However, even with the values of the Lie curvatures, the sign of the coefficient cannot be determined a priori. Assuming $d_{52} \neq 0$ on an open neighborhood $U \subset L_1$, we have

$$(\lambda-\mu)(\sigma-\mu) + (\lambda-\nu)(\sigma-\nu) + (\lambda-\rho)(\sigma-\rho) + (\lambda-\tau)(\sigma-\tau) = 0, \quad (40)$$

i.e.,

$$\mu^2 + \nu^2 + \rho^2 + \tau^2 - (\lambda+\sigma)(\mu+\nu+\rho+\tau) + 4\lambda\sigma = 0$$

on U. Since $d_{51} = d_{55} = 0$, and $e_5(H) = 0 = d_{52} + d_{53} + d_{54} + d_{56}$, we have

$$\begin{aligned} 0 &= e_5\big(\mu^2 + \nu^2 + \rho^2 + \tau^2 - (\lambda+\sigma)(\mu+\nu+\rho+\tau) + 4\lambda\sigma\big) \\ &= 2(\mu d_{52} + \nu d_{53} + \rho d_{54} + \tau d_{56}) - (\lambda+\sigma)(d_{52} + d_{53} + d_{54} + d_{56}) \\ &= 2(\mu d_{52} + \nu d_{53} + \rho d_{54} + \tau d_{56}), \end{aligned}$$

and using (39) again, we have

$$0 = d_{52}\Big(\mu + \nu\frac{(\lambda-\nu)(\sigma-\nu)}{(\lambda-\mu)(\sigma-\mu)} + \rho\frac{(\lambda-\rho)(\sigma-\rho)}{(\lambda-\mu)(\sigma-\mu)} + \tau\frac{(\lambda-\tau)(\sigma-\tau)}{(\lambda-\mu)(\sigma-\mu)}\Big).$$

This implies, since $d_{52} \neq 0$ on U,

$$\mu(\lambda-\mu)(\sigma-\mu) + \nu(\lambda-\nu)(\sigma-\nu) + \rho(\lambda-\rho)(\sigma-\rho) + \tau(\lambda-\tau)(\sigma-\tau) = 0. \quad (41)$$

On the other hand, from(40), it follows

$$\tau(\lambda-\tau)(\sigma-\tau) = -\tau\big((\lambda-\mu)(\sigma-\mu)+(\lambda-\nu)(\sigma-\nu)+(\lambda-\rho)(\sigma-\rho)\big).$$

Substituting this into (41), we obtain

$$(\mu-\tau)(\lambda-\mu)(\sigma-\mu)+(\nu-\tau)(\lambda-\nu)(\sigma-\nu)+(\rho-\tau)(\lambda-\rho)(\sigma-\rho)=0.$$

However, each term of the LHS is negative, a contradiction. Therefore, we have

$$d_{5i}=0.$$

As a conclusion:

Proposition 7.4. *All d_{ji} vanish at p.*

7.3. Angle relation and the Lie curvature

In [**23**],it is shown that if all d_{ji} vanish at p, then all the leaves L_i through $p^1 = p$ are totally geodesic in their curvature spheres C_i, and that $L_i \cap \gamma = \{p^1, p^{2i}\}$, where γ is the normal geodesic through p^1, and p^{2i} is the antipodal point of p^1 in L_i. From Lemma 3 in [**23**], where tautness is used, we have:

Lemma 7.5. *$M \cap \gamma = \{p = p^1, \ldots, p^{12}\}$, where p^1, p^{2i}, $i = 2,3,4,5,6$ are as above, and p^{odd} are as in Figure 8. At each p^t, all the leaves are totally geodesic in the curvature spheres and intersect γ orthogonally at some p^s. Their mutual intersections are as shown in Figure 8.*

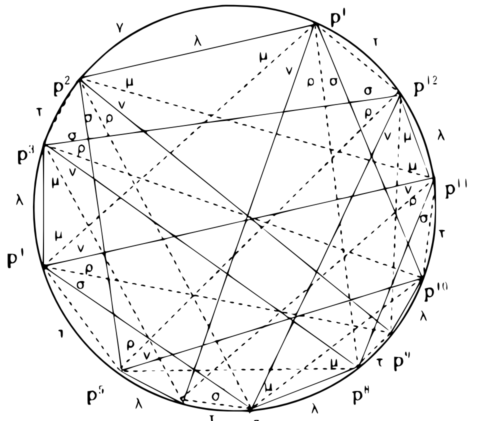

Figure 8: Link of leaves

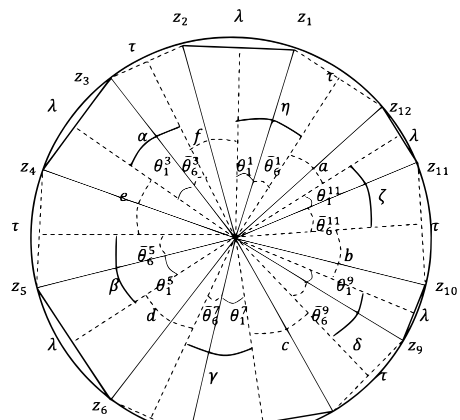

Figure 9: Angle relation

For an isoparametric hypersurface M_θ, the intersection $M_\theta \cap \gamma_\theta$ is a parallel dodecagon (Figure 11) for any normal geodesic γ_θ of M_θ. Our goal is to show that $M \cap \gamma = \{p^1, \ldots, p^{12}\}$ is itself a parallel dodecagon. We prove this in several steps.

Proposition 7.6. *Denote $\lambda_i^t = \lambda_i(p^t) = \cot\theta_i^t$, $0 < \theta_i^t < \pi$, for $i = 1, \ldots, 6$, and $t = 1, \ldots, 12$. Setting*

$$\theta_2^1 = \theta_1^1 + \alpha,\, \theta_3^1 = \theta_2^1 + \beta,\, \theta_4^1 = \theta_3^1 + \gamma, \theta_5^1 = \theta_4^1 + \delta,\, \theta_6^1 = \theta_5^1 + \zeta$$
$$\theta_2^2 = \theta_1^2 + a,\, \theta_3^2 = \theta_2^2 + b,\, \theta_4^2 = \theta_3^2 + c, \theta_5^2 = \theta_4^2 + d,\, \theta_6^2 = \theta_5^2 + e,$$

and putting $\eta = \pi - (\alpha + \beta + \gamma + \delta + \zeta)$ *and* $f = \pi - a - b - c - d - e$*, we can express the angles between* θ_i^t *and* θ_{i+1}^t *at* p^t $(t = 1, \ldots, 12)$ *as follows:*

p^t	θ_1^t		θ_2^t		θ_3^t		θ_4^t		θ_5^t		θ_6^t		$\pi + \theta_1^t$
p^1		α		β		γ		δ		ζ		η	
p^2		a		b		c		d		e		f	
p^3		β		γ		δ		ζ		η		α	
p^4		f		a		b		c		d		e	
p^5		γ		δ		ζ		η		α		β	
p^6		e		f		a		b		c		d	
p^7		δ		ζ		η		α		β		γ	
p^8		d		e		f		a		b		c	
p^9		ζ		η		α		β		γ		δ	
p^{10}		c		d		e		f		a		b	
p^{11}		η		α		β		γ		δ		ζ	
p^{12}		b		c		d		e		f		a	

Table 2: Angles

PROOF. We apply an argument similar to that of Proposition 4.5 and summarize it briefly. Denote p^t by the complex number z_t, and put $\bar{\theta}_i^t = \pi - \theta_i^t$, $0 < \bar{\theta}_i^t < \pi$. From Figure 9, taking the orientation into account, we have for instance:

$$\theta_2^1 = \theta_1^1 + \theta_1^3 + \bar{\theta}_6^3 = \theta_1^1 + \alpha, \quad \theta_3^1 = \theta_2^1 + \theta_1^5 + \bar{\theta}_6^5 = \theta_1^1 + \beta, \ldots$$

Proceeding in this manner, we obtain the table for p^{odd}; starting from $p^2 = z_2$ clockwise, we obtain the table for p^{even}. □

In the following, all the angles stay in $(0, \pi)$ modulo π, and we may use (22), taking care of the order of the principal curvatures,:

$$\Psi_\nu(z_1) = \frac{(\lambda - \mu)(\nu - \sigma)}{(\lambda - \sigma)(\nu - \mu)} = [z_2, z_6; z_4, z_{10}] = -1,$$

where, the value of Ψ is computed from (34).

For the moment, we advance our argument under the assumption $\lambda^1\rho^1 = -1$ at $p^1 \in M$. In §7.4, we apply the argument to a conformal image of M satisfying this condition.

LEMMA 7.7. *Assume* $\lambda^1\rho^1 = -1$ *at* $p^1 \in M$*. Then putting* $\Psi_\nu^t = \Psi_\nu(p^t)$ *and*

$$w_1 = e^{2i\alpha},\, w_2 = e^{2i\beta},\, w_3 = e^{2i\gamma},\, w_4 = e^{2i\delta},\, w_5 = e^{2i\zeta},\, w_6 = e^{2i\eta},$$

we have

$$
\begin{aligned}
\Psi_\nu^1 &= [z_2, z_6; z_4, z_{10}] = \frac{(1-w_1)(1-w_3w_4)}{(1+w_4)(1+w_1w_3)},\\
\Psi_\nu^5 &= [z_6, z_{10}; z_8, z_2] = \frac{(1-w_3)(1-w_5w_6)}{(1+w_3)(1+w_5w_6)},\\
\Psi_\nu^{11} &= [z_{12}, z_4; z_2, z_8] = \frac{(1-w_6)(1-w_2w_3)}{(1+w_6)(1+w_2w_3)}.
\end{aligned}
\tag{42}
$$

Proof. By our assumption $2(\alpha+\beta+\gamma) = \pi = 2(\delta+\zeta+\eta)$ and $e^{i(\pi+\varphi)} = -e^{i\varphi}$, the first relation follows from

$$
\begin{aligned}
&z_4 = e^{2i\alpha} z_2, \quad z_{10} = e^{2i(\alpha+\beta+\gamma+\delta)} z_2 = -z^{2i\delta} z_2,\\
&z_{10} = e^{2i(\gamma+\delta)} z_6, \quad z_4 = e^{2i(\gamma+\delta+\zeta+\eta+\alpha)} z_6 = -e^{2i(\alpha+\gamma)} z_6,
\end{aligned}
$$

as

$$
\begin{aligned}
\Psi_\nu^1 = [z_2, z_6; z_4, z_{10}] &= \frac{(z_2-z_4)(z_6-z_{10})}{(z_2-z_{10})(z_6-z_4)}\\
&= \frac{(1-e^{2i\alpha})(1-e^{2i(\gamma+\delta)})}{(1+e^{2i\delta})(1+e^{2i(\alpha+\gamma)})} = \frac{(1-w_1)(1-w_3w_4)}{(1+w_4)(1+w_1w_3)}.
\end{aligned}
\tag{43}
$$

Similarly, we have

$$
\begin{aligned}
&z_8 = e^{2i\gamma} z_6, \quad z_2 = e^{2i(\gamma+\delta+\zeta+\eta)} z_6 = -e^{2i\gamma} z_6\\
&z_2 = e^{2i(\zeta+\eta)} z_{10}, \quad z_8 = e^{2i(\zeta+\eta+\alpha+\beta+\gamma)} z_{10} = -e^{2i(\zeta+\eta)} z_{10},
\end{aligned}
$$

and we obtain

$$
\begin{aligned}
\Psi_\nu^5 = [z_6, z_{10}; z_8, z_2] &= \frac{(z_6-z_8)(z_{10}-z_2)}{(z_6-z_2)(z_{10}-z_8)}\\
&= \frac{(1-e^{2i\gamma})(1-e^{2i(\zeta+\eta)})}{(1+e^{2i\gamma})(1+e^{2i(\zeta+\eta)})} = \frac{(1-w_3)(1-w_5w_6)}{(1+w_3)(1+w_5w_6)}.
\end{aligned}
$$

Also from

$$
\begin{aligned}
&z_2 = e^{2i\eta} z_{12}, \quad z_8 = e^{2i(\eta+\alpha+\beta+\gamma)} z_{12} = -e^{2i\eta} z_{12}\\
&z_8 = e^{2i(\beta+\gamma)} z_4, \quad z_2 = e^{2i(\beta+\gamma+\delta+\zeta+\eta)} z_4 = -e^{2i(\beta+\gamma)} z_4,
\end{aligned}
$$

we have

$$
\begin{aligned}
\Psi_\nu^{11} = [z_{12}, z_4; z_2, z_8] &= \frac{(z_{12}-z_2)(z_4-z_8)}{(z_{12}-z_8)(z_4-z_2)}\\
&= \frac{(1-e^{2i\eta})(1-e^{2i(\beta+\gamma)})}{(1+e^{2i\eta})(1+e^{2i(\beta+\gamma)})} = \frac{(1-w_6)(1-w_2w_3)}{(1+w_6)(1+w_2w_3)}.
\end{aligned}
$$

□

Proposition 7.8. *Under the assumption $\lambda^1\rho^1 = -1$, we have*

$$
\alpha = \beta = \gamma = \delta = \zeta = \eta = \frac{\pi}{6}.
$$

Proof. From (42), using $w_4w_5w_6 = -1$ and $\Psi^5_\nu = -1$,

$$\frac{(1-w_3)(1-w_5w_6)}{(1+w_3)(1+w_5w_6)} = \frac{(1-w_3)(w_4+1)}{(1+w_3)(w_4-1)} = -1.$$

This gives

$$(1-w_3)(w_4+1) + (1+w_3)(w_4-1) = 0,$$

hence

$$w_3 = w_4 \quad \text{i.e.,} \quad \gamma = \delta.$$

Similarly from $\Psi^{11}_\nu = -1$,

$$\frac{(1-w_6)(1-w_2w_3)}{(1+w_6)(1+w_2w_3)} = \frac{(1-w_6)(w_1+1)}{(1+w_6)(w_1-1)} = -1,$$

and we obtain

$$w_1 = w_6 \quad \text{i.e.,} \quad \alpha = \eta.$$

Then from $2(\alpha+\beta+\gamma) = \pi = 2(\delta+\zeta+\eta)$, it follows that

$$\beta = \zeta \quad \text{i.e.,} \quad w_2 = w_5.$$

Next, from (43),

$$\frac{(1-w_1)(1-w_3w_4)}{(1+w_4)(1+w_1w_3)} = \frac{(1-w_1)(1-w_3^2)}{(1+w_3)(1+w_1w_3)} = \frac{(1-w_1)(1-w_3)}{(1+w_1w_3)} = -1,$$

hence we obtain

$$2(w_1w_3+1) = w_1 + w_3, \tag{44}$$

which provides two real equations allowing us to solve for α and γ.

Since $0 < 2\alpha, 2\gamma < \pi$, we have $\sin 2\alpha, \sin 2\gamma > 0$, and can put $w_1 = e^{2i\alpha} = x + i\sqrt{1-x^2}$, and $w_3 = e^{2i\gamma} = y + i\sqrt{1-y^2}$, $x, y \in \mathbb{R}$. Then (44) becomes

$$2(x+i\sqrt{1-x^2})(y+i\sqrt{1-y^2}) + 1) = x + i\sqrt{1-x^2} + y + i\sqrt{1-y^2},$$

which implies

$$\begin{cases} 2(xy - \sqrt{1-x^2}\sqrt{1-y^2}) + 1) = x + y \\ 2(x\sqrt{1-y^2}) + y\sqrt{1-x^2}) = \sqrt{1-x^2} + \sqrt{1-y^2}, \end{cases}$$

namely

$$\begin{cases}(2xy+2-x-y)^2 = 4(1-x^2)(1-y^2)\\(2x-1)^2(1-y^2) = (2y-1)^2(1-x^2).\end{cases}$$

Thus we have

$$\begin{cases}(x+y)(5(x+y)-4(xy+1)) = 0\\(x-y)(5(x+y)-4(xy+1)) = 0.\end{cases}$$

Here, $x \pm y = 0$ implies $x = y = 0$, and $\alpha = \gamma = \frac{\pi}{4}$, contradicts $0 < \beta = \frac{\pi}{2} - (\alpha + \gamma)$. If $x + y = 0$ and $x - y \neq 0$, we have

$$0 = 5(x+y) - 4(xy+1) = 4(x^2-1)$$

which implies $x = \cot 2\alpha = \pm 1$ and $\alpha = 0$ or $\frac{\pi}{2}$, again impossible. If $x + y \neq 0$ and $x - y = 0$, we have

$$0 = 5(x+y) - 4(xy+1) = -4x^2 + 10x - 4 = -2(2x-1)(x-2),$$

which implies

$$x = \cos 2\alpha = \frac{1}{2} = y = \cos 2\gamma$$

namely,

$$\alpha = \gamma = \frac{\pi}{6},$$

and consequently $\beta = \dfrac{\pi}{6}$, proving the proposition. □

When $\lambda^2\rho^2 = -1$, replacing z_t and w_t suitably, the same argument applies to to a, b, c, d, e, f, and we obtain:

Proposition 7.9. *If $\lambda^1\rho^1 = -1 = \lambda^2\rho^2$, then*

$$\alpha = \beta = \gamma = \delta = \zeta = \eta = a = b = c = d = e = f = \frac{\pi}{6},$$

and hence $M \cap \gamma$ is a parallel dodecagon.

7.4. Conformal transformation

We have obtained Proposition 7.9 under the assumption $\lambda^1\rho^1 = -1 = \lambda^2\rho^2$. Since there exists a conformal transformation $\hat{C} : M \to \hat{M}$ such that the leaves $L_1(\hat{p}^1)$ and $L_1(\hat{p}^7)$ are antipodally symmetric, we denote the conformal image with hats, such as $\hat{M}$, $\hat{p}^t$, $\hat{\lambda}^t$ etc. Then we have $\hat{\lambda}^1\hat{\rho}^1 = -1 = \hat{\lambda}^2\hat{\rho}^2$ (see Figure 10), and we can apply the preceding argument to $\hat{M}$, since $\hat{C}$ preserves the Lie curvatures. Thus $\hat{M} \cap \hat{\gamma}$ is a parallel dodecagon. Let $C : \hat{M} \to M$ be the inverse conformal transformation of $\hat{C}$.

Proposition 7.10. *The conformal transformation $C : \hat{M} \to M$ is an isometry, and $M \cap \gamma$ is itself a parallel dodecagon.*

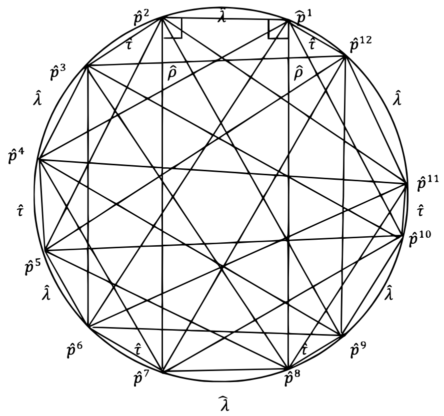

Figure 10: Conformal deformation

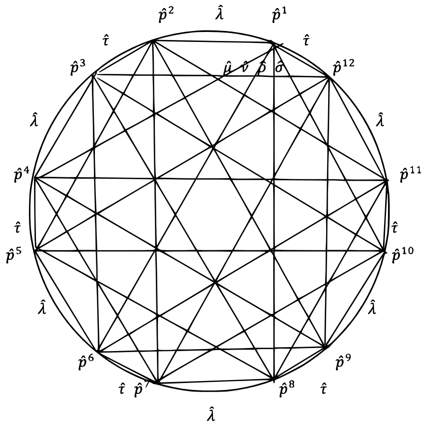

Figure 11: Parallel dodecagon

Proof. Restrict $C : \hat{M} \to M$ to $\mathbb{R}^4_2 = \mathbb{R}^2 \oplus \mathbb{R}^2_2$ where $\gamma \subset \mathbb{R}^2$, and write

$$C = \begin{pmatrix} * & * & * & 0 \\ * & * & * & 0 \\ x & y & r & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in O(2,1) \subset O(2,2).$$

Applying C to $k_1 = \begin{pmatrix} \hat{p}^t \\ 1 \\ 0 \end{pmatrix}$ and $k_2 = \begin{pmatrix} \hat{n}^t \\ 0 \\ 1 \end{pmatrix}$, we express

$$Ck_1 = \begin{pmatrix} q^t \\ \hat{a}_t \\ 0 \end{pmatrix}, \quad Ck_2 = \begin{pmatrix} m^t \\ \hat{c}_t \\ 1 \end{pmatrix}.$$

Then by (21), the principal curvatures λ^t_i at the original point $p^t \in M \cap \gamma$ are:

$$\lambda^t_i = \hat{a}_t \hat{\lambda}_i + \hat{c}_t, \quad i = 1, \dots, \quad t = 1, \dots 8.$$

The mean curvature $\hat{H}^t$ along $\hat{M} \cap \hat{\gamma}$ is constant, since it is a parallel dodecagon. Hence, denoting $\hat{H} = \hat{H}^t$, the mean curvature H^t of M at p^t is

$$H^t = \hat{a}_t \hat{H} + 6m\hat{c}_t, \quad \hat{H} = m(\hat{\lambda} + \hat{\mu} + \hat{\nu} + \hat{\rho} + \hat{\sigma} + \hat{\tau}), \quad m = 1, 2,$$

which is also independent of t as M is CMC.

We express some vertices of the parallel dodecagon $\hat{M} \cap \hat{\gamma} = \{\hat{p}^1, \dots, \hat{p}^{12}\}$ (Figure 11) using positive numbers $\hat{u}, \hat{v}, \hat{k}, \hat{l}$ such that

$$\hat{\lambda}^1 = \frac{\hat{v}}{\hat{u}}, \quad \hat{\tau}^4 = -\frac{\hat{k}}{\hat{l}},$$

as

$$\begin{cases} \hat{p}^1 = \begin{pmatrix} \hat{u} \\ \hat{v} \end{pmatrix} = -\hat{p}^7 \\ \hat{n}^1 = \begin{pmatrix} -\hat{v} \\ \hat{u} \end{pmatrix} = -\hat{n}^7 \end{cases} \qquad \begin{cases} \hat{p}^2 = \begin{pmatrix} -\hat{u} \\ \hat{v} \end{pmatrix} = \hat{p}^8 \\ \hat{n}^2 = \begin{pmatrix} \hat{v} \\ \hat{u} \end{pmatrix} = -\hat{n}^8, \end{cases}$$

$$\begin{cases} \hat{p}^4 = \begin{pmatrix} -\hat{k} \\ \hat{l} \end{pmatrix} = -\hat{p}^{10} \\ \hat{n}^4 = \begin{pmatrix} \hat{l} \\ \hat{k} \end{pmatrix} = -\hat{n}^{10} \end{cases} \qquad \begin{cases} \hat{p}^5 = \begin{pmatrix} -\hat{k} \\ -\hat{l} \end{pmatrix} = -\hat{p}^{11} \\ \hat{n}^5 = \begin{pmatrix} \hat{l} \\ -\hat{k} \end{pmatrix} = -\hat{n}^{11}. \end{cases}$$

Thus we have

$$\begin{cases} \hat{a}_1 = \hat{u}x + \hat{v}y + r \\ \hat{c}_1 = -\hat{v}x + \hat{u}y \end{cases} \qquad \begin{cases} \hat{a}_2 = -\hat{u}x + \hat{v}y + r \\ \hat{c}_2 = \hat{v}x + \hat{u}y \end{cases}$$

$$\begin{cases} \hat{a}_4 = -\hat{k}x + \hat{l}y + r \\ \hat{c}_4 = \hat{l}x + \hat{k}y \end{cases} \qquad \begin{cases} \hat{a}_5 = -\hat{k}x - \hat{l}y + r \\ \hat{c}_5 = \hat{l}x - \hat{k}y. \end{cases}$$

Since M has constant mean curvature (CMC), at points p^1 and p^2, we have

$$0 = H^1 - H^2 = (\hat{a}_1 - \hat{a}_2)\hat{H} + 6m(\hat{c}_1 - \hat{c}_2) = 2\hat{u}x(\hat{H} - 6m\hat{\lambda}),$$

where $\hat{u}(\hat{H} - 6m\hat{\lambda}) < 0$ because $\hat{\lambda}$ is the largest principal curvature. Hence

$$x = 0$$

follows. Similarly, at p^4 and p^5, we have

$$\begin{aligned} 0 = H^4 - H^5 &= (\hat{a}_4 - \hat{a}_5)\hat{H} + 6m(\hat{c}_4 - \hat{c}_5) \\ &= 2\hat{l}y\hat{H} + 6m \cdot 2\hat{k}y = 2y\hat{l}(\hat{H} - 6m\hat{\tau}), \end{aligned}$$

and since $\hat{l}(\hat{H} - 6m\hat{\tau}) > 0$ (because $\hat{\tau}$ is the smallest principal curvature), we obtain

$$y = 0.$$

Therefore, C is an isometry $\begin{pmatrix} T & 0 \\ 0 & I' \end{pmatrix} \in O(2,2)$, $I' = \begin{pmatrix} \epsilon & 0 \\ 0 & 1 \end{pmatrix}$, and $M \cap \gamma$ itself is a parallel dodecagon $M_\theta \cap \gamma_\theta$, where θ is uniquely determined by the mean curvature H of M (Fact 5). □

Proof of Theorem 1.1 (iv): Let $p \in M$ be a point where λ attains its maximum or minimum and suppose $d_{12} = 0$ at p. By the previous argument, $M \cap \gamma$ is isometric to $M_\theta \cap \gamma_\theta$ where θ is uniquely determined by the mean curvature H of M. Thus $\max \lambda = \lambda_\theta = \min \lambda$, so λ is constant on M. Next, let p be a point where μ attains its tmaximum or minimum. Since $d_{j1} = 0$ and $d_{12} = 0$ hold at p, the same argument shows that $M \cap \gamma$ is again isometric to $M_\theta \cap \gamma_\theta$. Hence $\max \mu = \mu_\theta = \min \mu$, and μ is constant on M. The remaining principal curvatures ν, ρ, σ, τ are constant on M, because the CMC and CLC conditions provide four relations among the six principal curvatures. This completes the proof of Theorem 1.1 (iv). □

Remark 7.11 : For $\lambda_1 = \cot\theta_1$ where $\theta_1 = \frac{\pi}{12} + \theta$, $-\frac{\pi}{12} < \theta < \frac{\pi}{12}$, the scalar curvature is given in [**41**] p.147,

$$R_\theta = 36m(m-1)(1 + \cot^2(6\theta_1)).$$

In the minimal case this becomes

$$R = 36m(m-1).$$

Thus the minimal isoparametric hypersurface is scalar flat if $m = 1$, and scalar positive if $m = 2$.

8. Problems

Finally, we propose some problems based on the above results :

Problem 1. *If a closed minimal (or CMC) hypersurface M with $g = 4$ has constant scalar curvature (CSC), is M Dupin?*

This conclusion is weaker than isoparametric, but if the answer is affirmative, it implies Chern's conjecture true for $g = 4$ via Theorem 1.1 (ii).

Problem 2. *If a closed minimal (or CMC) hypersurface M with $g = 4$ has constant Lie curvature (CLC), is M Dupin?*

If so, M is isoparametric via Theorem 1.1 (iii).

Problem 3. *If a closed minimal (or CMC) hypersurface M with $g = 6$ has CSC, is M Dupin with constant Lie curvatures?*

It seems unlikely, but if it is true, it implies Chern's conjecture holds for $g = 6$ via Theorem 1.1 (iv).

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