Radical property of the traces of the canonical modules of Cohen-Macaulay rings

Mitsuhiro Miyazaki
Osaka Central Advanced Mathematical Institute,
Osaka Metropolitan University,
3-3-138 Sugimoto, Sumiyoshi-ku Osaka 558-8585, Japan

e-mail:mmyzk7@gmail.com

Dedicated with my deepest gratitude to the memory of Mitsuyasu Hashimoto (1962-2025), my friend and a great mathematician.

Abstract

In this paper, we define a new concept of Noetherian commutative rings which stands between Gorenstein and Cohen-Macaulay properties. We show that this new property keep hold under common operations of commutative rings such as localization, polynomial extension and under mild assumptions, flat extension, tensor product, Segre product and so on. We show that for Schubert cycles, the Ehrhart rings of the stable set polytopes of cycle graphs and perfect graphs, this new concept is close to Gorenstein property.
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1 Introduction

There is a hierarchy of commutative Noetherian local rings.

$$\text{regular} \Rightarrow \text{complete intersection}$$
$$\Rightarrow \text{Gorenstein} \Rightarrow \text{Cohen-Macaulay} \Rightarrow \text{Buchsbaum}.$$

A Cohen-Macaulay rings was originally defined as a Noetherian ring which satisfy the unmixedness theorem. Macaulay showed that the unmixedness theorem holds for polynomial rings over a field and Cohen showed that the unmixedness theorem holds for regular local rings. In old days, Cohen-Macaulay rings were sometimes called semi-regular rings. Further, there are many situations that rings under consideration is Cohen-Macaulay. Therefore Cohen-Macaulay rings are a platform for many theories.

On the other hand, the notion of Gorenstein rings was defined by Bass [Bas]. A Gorenstein local ring is by definition a commutative Noetherian local ring whose self injective dimension is finite. A Gorenstein ring is a Cohen-Macaulay ring whose parameter ideal is irreducible and has very beautiful properties especially concerning symmetry of related objects such as syzygies and dualities.

In pursuing the study of Gorenstein and Cohen-Macaulay rings, many researchers felt that there is a rather large gap between Gorenstein and Cohen-Macaulay properties. Thus, there were attempts to define notions between Gorenstein and Cohen-Macaulay properties and fill this gap. The first one is the level property defined by Stanley [Sta1]. However, the level property can be defined only for semi-standard graded rings over a field.

After that, the almost Gorenstein property [BF, GMP, GTT] and the nearly Gorenstein property [HHS] were defined. However, there are few rings that are non-Gorenstein but almost or nearly Gorenstein.

In this paper, we define a new notion, called *canonical trace radical* (*CTR* for short) property, which lies between Gorenstein and Cohen-Macaulay properties. We define a Cohen-Macaulay local ring to be *CTR* if it admits a canonical module and its trace ideal is a radical ideal. We show that, under mild assumptions, CTR property is preserved or reflected by familiar operations on Noetherian rings, such as localization, flat extension, tensor product, Segre product and some others. We also show that in some combinatorial rings, the combinatorial property corresponding to CTR property is weaker than but is close to the combinatorial property corresponding to Gorenstein property.

Recently, Esentepe [Ese] treated the radical property of the trace ideal of the canonical module of a Cohen-Macaulay ring in relation to Auslander-Reiten conjecture.

This paper is organized as follows. In §2, we establish notation and terminology used in this paper and recall some basic facts, especially the trace of a module. In §3, we define the canonical trace radical (CTR for short) property and show under the following ring operations, with sometimes additional assumptions, CTR property is retained: localization, flat extension, polynomial extension, completion, division by a regular sequence, tensor product and Segre product.

In §§4 and 5, we state criteria of CTR property for certain classes of rings which motivated us to define CTR property. In §4, we deal with Schubert cycles, i.e. the homogeneous coordinate rings of the Schubert subvarieties of Grassmannians: let R be a Schubert cycle. Then by [BV, §8], it is known that R is a Cohen-Macaulay normal domain and there are height 1 prime ideals P_0, P_1, ..., P_t such that the divisor class group $\mathrm{Cl}(R)$ is generated by $\mathrm{cl}(P_0)$, $\mathrm{cl}(P_1)$, ..., $\mathrm{cl}(P_t)$ and $\sum_{i=0}^{t}\mathrm{cl}(P_i)=0$ is the only relation between them. Let $\sum_{i=0}^{t}\kappa_i\mathrm{cl}(P_i)$ be the canonical class of $\mathrm{Cl}(R)$, $\kappa=\max\{\kappa_i : 0\leq i\leq t\}$ and $\kappa'=\min\{\kappa_i : 0\leq i\leq t\}$. Then $\kappa-\kappa'$ is independent of the representation of the canonical class above. It is known that R is Gorenstein if and only if $\kappa-\kappa'=0$. We show that R is CTR if and only if $\kappa-\kappa'\leq 1$. See Theorem 4.3.

In §5, we deal with the Ehrhart ring of the stable set polytope of a cycle graph: let R be such a ring. It is known that R is Gorenstein if and only if the length n of the cycle is even or less than 7. We show that R is CTR if and only if n is even or less than 9. See Theorem 5.2.

Finally in §6, we state a necessary condition that the Ehrhart ring R of the stable set polytope of a perfect graph G is CTR: set $k=\max\{|K| : K$ is a maximal clique in $G\}$ and $k'=\min\{|K| : K$ is a maximal clique in $G\}$. It is known that R is Gorenstein if and only if $k-k'=0$. We show that if R is CTR, then $k-k'\leq 1$. See Proposition 6.2.

2 Preliminaries

In this section, we establish notation and terminology used in this paper. For unexplained term of commutative algebra, we consult [Mat] and [BH].

All rings and algebras are assumed to be commutative with identity element and Noetherian. We denote the set of nonnegative integers, the set of integers, the set of rational numbers and the set of real numbers by $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$ respectively.

For a set X, we denote by $|X|$ the cardinality of X. For sets X and Y, we denote by $X\setminus Y$ the set $\{x\in X : x\notin Y\}$. For nonempty sets X and Y, we denote the set of maps from X to Y by Y^X. If X is a finite set, we identify $\mathbb{R}^X$ with the Euclidean space $\mathbb{R}^{|X|}$. For f, f_1, $f_2\in\mathbb{R}^X$ and $a\in\mathbb{R}$, we define maps $f_1\pm f_2$ and af by $(f_1\pm f_2)(x)=f_1(x)\pm f_2(x)$ and $(af)(x)=a(f(x))$ for $x\in X$. Let A be a subset of X. We define the characteristic function $\chi_A\in\mathbb{R}^X$ of A by $\chi_A(x)=1$ for $x\in A$ and $\chi_A(x)=0$ for $x\in X\setminus A$. For a nonempty subset $\mathscr{X}$ of $\mathbb{R}^X$, we denote by $\mathrm{conv}\mathscr{X}$ the convex hull of $\mathscr{X}$.

Next we fix notation about Ehrhart rings. Let $\mathbb{K}$ be a field, X a finite set and $\mathscr{P}$ a rational convex polytope in $\mathbb{R}^X$, i.e. a convex polytope whose vertices are contained in $\mathbb{Q}^X$. Let $-\infty$ be a new element with $-\infty\notin X$

and set $X^- := X \cup \{-\infty\}$. Also let $\{T_x\}_{x \in X^-}$ be a family of indeterminates indexed by X^-. For $f \in \mathbb{Z}^{X^-}$, we denote the Laurent monomial $\prod_{x \in X^-} T_x^{f(x)}$ by T^f. We set $\deg T_x = 0$ for $x \in X$ and $\deg T_{-\infty} = 1$. Then the Ehrhart ring of $\mathscr{P}$ over a field $\mathbb{K}$ is the $\mathbb{N}$-graded subring

$$\mathbb{K}[T^f : f \in \mathbb{Z}^{X^-}, f(-\infty) > 0, \frac{1}{f(-\infty)} f|_X \in \mathscr{P}]$$

of the Laurent polynomial ring $\mathbb{K}[T_x^{\pm 1} : x \in X^-]$, where $f|_X$ is the restriction of f to X. We denote the Ehrhart ring of $\mathscr{P}$ over $\mathbb{K}$ by $E_{\mathbb{K}}[\mathscr{P}]$. If X is a poset, we define the order on X^- by $-\infty < x$ for any $x \in X$.

Let R be a ring. For $\mathfrak{p} \in \operatorname{Spec}(R)$, we denote by $\kappa(\mathfrak{p})$ the quotient field $R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$ of $R/\mathfrak{p}$. For an ideal I of R, we denote by $\min(I)$ the set of minimal over primes of I. For a ring R and a matrix M with entries in R, we denote by $I_t(M)$ the ideal of R generated by t-minors of M.

An $\mathbb{N}$-graded ring $R = \bigoplus_{n \in \mathbb{N}} R_n$ is said to be an $\mathbb{N}$-graded $\mathbb{K}$-algebra if $R_0 = \mathbb{K}$. An $\mathbb{N}$-graded $\mathbb{K}$-algebra $R = \bigoplus_{n \in \mathbb{N}} R_n$ is said to be standard graded if $R = \mathbb{K}[R_1]$. For an $\mathbb{N}$-graded ring R, a greded R-module M and $m \in \mathbb{Z}$, we denote by $M_{\geq m}$ the graded R-submodule $\bigoplus_{n \geq m} M_n$ of M. When R is a Cohen-Macaulay local ring with a canonical module or an $\mathbb{N}$-graded algebra over a field, we denote by ω_R the (graded) canonical module of R. If R is an $\mathbb{N}$-graded algebra and ω_R is the canonical module of R, $-\min\{m : (\omega_R)_m \neq 0\}$ is called the a-invariant of R and denoted by $a(R)$. See [GW]. For $\mathbb{N}$-graded $\mathbb{K}$-algebras $R^{(1)}, \ldots, R^{(m)}$, we denote by $R^{(1)} \# \cdots \# R^{(m)}$ the Segre product $\bigoplus_{n \in \mathbb{N}} R_n^{(1)} \otimes \cdots \otimes R_n^{(m)}$ of $R^{(1)}, \ldots, R^{(m)}$.

Let $R = \bigoplus_{n \in \mathbb{N}} R_n$ be a standard graded $\mathbb{K}$-algebra, where $\mathbb{K}$ is a field. We say that R is level if the graded canonical module ω_R is generated in one degree. If $\operatorname{Hom}_R(\omega_R, R)$ (whice is a graded module. See [GW].) is generated in one degree, we say that R is anticanonical level. Level and anticanonical level properties are independent, see [Pag, Miy1, Miy2]. We denote $\operatorname{Hom}_R(\omega_R, R)$ by ω_R^{-1}.

Now we recall the following.

Definition 2.1. Let R be a ring and M an R-module. We define the trace of M denoted by $\operatorname{tr}_R(M)$ by

$$\operatorname{tr}_R(M) := \sum_{\varphi \in \operatorname{Hom}_R(M,R)} \varphi(M).$$

If R is clear from context, we omit the subscript R and denote $\operatorname{tr}(M)$.

It follows from the definition, the following.

Lemma 2.2. *Let R be a ring and M an R-module. Then $\mathrm{tr}_R(M)$ is the image of the canonical map*

$$\mathrm{Hom}_R(M,R)\otimes M\to R,\quad f\otimes m\mapsto f(m).$$

In particular, if M is a finitely generated R-module and S is a flat R-algebra, then $\mathrm{tr}_S(M\otimes S)=\mathrm{tr}_R(M)S$ and therefore for $\mathfrak{p}\in\mathrm{Spec}(R)$, $\mathrm{tr}_{R_\mathfrak{p}}(M_\mathfrak{p})=\mathrm{tr}_R(M)_\mathfrak{p}$.

Flat extension part of this lemma is shown in [HHS, Lemma 1.5 (iii)]. Further, if I is an ideal of R and contains an R-regular element b, then $\mathrm{Hom}_R(I,R)\cong\{x\in(1/b)R : xI\subset R\}\subset Q(R)$, where $Q(R)$ is the total quotient ring of R, and therefore, $\mathrm{tr}_R(I)=I\{x\in(1/b)R : xI\subset R\}$. See the proof of [HHS, Lemma 1.1].

Let R be a normal domain. For divisorial fractionary ideal I of R and $n\in\mathbb{Z}$, we denote by $I^{(n)}$ the n-th power of I in $\mathrm{Div}(R)$. Note that if I is a height 1 prime ideal of R and $n>0$, then $I^{(n)}$ coincide with the n-the symbolic power of I. Further, by the argument in the previous paragraph, $\mathrm{tr}_R(I)=II^{(-1)}$ for any divisorial fractionary ideal I.

Next we state a tool to compute the trace of a canonical module which is a generalization of [HHS, Corollary 3.2]. A homomorphism $\varphi\colon F\to G$ of finitely generated free modules over a ring can be expressed by a matrix by fixing bases of F and G. Let M be such a matrix and t a positive integer with $t\le\min\{\mathrm{rank}F,\mathrm{rank}G\}$. Then the ideal $I_t(M)$ is independent of the choice of bases of F and G. We denote this ideal by $I_t(\varphi)$.

Lemma 2.3. *Let S be a Gorenstein local ring ($\mathbb{N}$-graded ring over a field), J a (homogeneous) ideal of S such that $R=S/J$ is a Cohen-Macaulay ring. Suppose that there exists a finite (graded) S-free resolution*

$$0\to F_h\overset{\varphi_h}{\to}\cdots\overset{\varphi_1}{\to}F_0\to R\to 0$$

of R with $h=\dim S-\dim R$. Let G be a free R-module and $\psi\colon G\to F_h\otimes R$ a (graded) R-homomorphism with

$$G\overset{\psi}{\to}F_h\otimes R\overset{\varphi_h\otimes 1}{\to}F_{h-1}\otimes R$$

is exact. Then $\mathrm{tr}_R(\omega_R)=I_1(\psi)$.

Proof. Since $\omega_R=\mathrm{Ext}_S^h(R,\omega_S)=\mathrm{Ext}_S^h(R,S)$, we see that $\omega_R\cong\mathrm{Coker}((\varphi_h\otimes 1)^*)$. Therefore, by [HHS, Proposition 3.1], we see the result. □

3 Canonical trace radical rings

In this section, we define the notion of canonical trace radical rings (CTR rings for short) and study basic properties of CTR rings. The reasons that we think CTR property is close to Gorenstein property is shown in the following sections. First we recall the following.

Fact 3.1. *Let R be a Cohen-Macaulay local ring with canonical module ω_R. Then R is Gorenstein if and only if* $\mathrm{tr}_R(\omega_R) = R$.

For the proof, see e.g. [HHS, Lemma 2.1].

Definition 3.2. Let R be a Cohen-Macaulay ring. If for any $\mathfrak{p} \in \mathrm{Spec}(R)$, $R_\mathfrak{p}$ has a canonical module $\omega_{R_\mathfrak{p}}$ and $\mathrm{tr}_{R_\mathfrak{p}}(\omega_{R_\mathfrak{p}})$ is a radical ideal, then we say that R is a canonical trace radical (CTR for short) ring.

Since the unit ideal is a radical ideal, a Gorenstein ring is a CTR ring. Further, it is evident from the definition that CTR property is kept by localization.

By Lemma 2.2, we see the following.

Proposition 3.3. *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring with canonical module. Then R is CTR if and only if $\mathrm{tr}_R(\omega_R)$ is a radical ideal.*

Proof. "Only if" part is a direct consequence of the definition. We prove the "if" part. Let $\mathfrak{p}$ be an arbitrary prime ideal of R. By Lemma 2.2, we see that $\mathrm{tr}_{R_\mathfrak{p}}((\omega_R)_\mathfrak{p}) = \mathrm{tr}_R(\omega_R)_\mathfrak{p}$. Since $\omega_{R_\mathfrak{p}} = (\omega_R)_\mathfrak{p}$ and $\mathrm{tr}_R(\omega_R)_\mathfrak{p}$ is a radical ideal of $R_\mathfrak{p}$, we see the result. □

Next we show a similar fact to the above proposition, which may be regarded as a graded version of the above proposition.

Proposition 3.4. *Let R be an $\mathbb{N}$-graded Cohen-Macaulay ring over a field $\mathbb{K}$ and ω_R the graded canonical module of R. Then R is CTR if and only if $\mathrm{tr}_R(\omega_R)$ is a radical ideal.*

Proof. We first prove the "only if" part. Let $\mathfrak{m}$ be the irrelevant maximal ideal of R. Then $(\omega_R)_\mathfrak{m}$ is the canonical module of $R_\mathfrak{m}$. Therefore, by assumption and Lemma 2.2, $\mathrm{tr}_R(\omega_R)_\mathfrak{m} = \mathrm{tr}_{R_\mathfrak{m}}((\omega_R)_\mathfrak{m})$ is a radical ideal. Since $\mathrm{tr}_R(\omega_R)$ is a graded ideal, every associated prime of $\mathrm{tr}_R(\omega_R)$ is graded and therefore contained in $\mathfrak{m}$. Thus, the radical property of $\mathrm{tr}_R(\omega_R)_\mathfrak{m}$ implies the radical property of $\mathrm{tr}_R(\omega_R)$.

Now we prove the "if" part. Let $\mathfrak{p}$ be an arbitrary prime ideal of R. Take a polynomial ring S with weighted degree over $\mathbb{K}$ and a graded surjective

$\mathbb{K}$-algebra homomorphism $S \to R$. Let P be the preimage of $\mathfrak{p}$. Also take a minimal graded S-free resolution

$$0 \to F_h \stackrel{\varphi_h}{\to} \cdots \stackrel{\varphi_1}{\to} F_0 \to R \to 0$$

of R, a free R-module G and a graded R-homomorphism ψ such that

$$G \stackrel{\psi}{\to} F_h \otimes R \stackrel{\varphi_h \otimes 1}{\to} F_{h-1} \otimes R$$

is exact. Then

$$0 \to (F_h)_P \stackrel{(\varphi_h)_P}{\to} \cdots \stackrel{(\varphi_1)_P}{\to} (F_0)_P \to R_\mathfrak{p} \to 0$$

is a (not necessarily minimal) S_P-free resolution of $R_\mathfrak{p}$ and

$$G_\mathfrak{p} \stackrel{\psi_\mathfrak{p}}{\to} (F_h)_P \otimes_{S_P} R_\mathfrak{p} \stackrel{(\varphi_h)_P \otimes 1}{\to} (F_{h-1})_P \otimes_{S_P} R_\mathfrak{p}$$

is exact.

Since R is Cohen-Macaulay, it follows that $\mathrm{Ass} R = \mathrm{Assh} R$ and therefore $h = \dim S - \dim R = \dim S_P - \dim R_\mathfrak{p}$. Thus, by Lemma 2.3, we see that $\mathrm{tr}_{R_\mathfrak{p}}(\omega_{R_\mathfrak{p}}) = I_1(\psi_\mathfrak{p})$. Since $\mathrm{tr}_R(\omega_R) = I_1(\psi)$ by Lemma 2.3 and $\mathrm{tr}_R(\omega_R)$ is a radical ideal, we see that

$$\mathrm{tr}_{R_\mathfrak{p}}(\omega_{R_\mathfrak{p}}) = I_1(\psi_\mathfrak{p}) = (I_1(\psi))_\mathfrak{p} = \mathrm{tr}_R(\omega_R)_\mathfrak{p}$$

is a radical ideal. $\square$

By Propositions 3.3 and 3.4, we see that nearly Gorenstein rings are CTR rings.

Next we consider the CTR property under the flat extension.

Proposition 3.5. *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring with canonical module and $(R, \mathfrak{m}) \to (S, \mathfrak{n})$ be a flat local homomorphism. Suppose $S/\mathfrak{m}S$ is a Gorenstein ring. Then the followings hold.*

(1) If S is CTR, then so is R.

(2) If R is CTR and for any $\mathfrak{p} \in \min(\mathrm{tr}_R(\omega_R))$, $S_\mathfrak{p}/\mathfrak{p}S_\mathfrak{p} = \kappa(\mathfrak{p}) \otimes_R S$ is a reduced ring, then S is CTR.

Proof. First note that S is Cohen-Macaulay and $\omega_R \otimes S$ is the canonical module of S. See [Mat, Theorem 23.3 Corollary] and [BH, Theorem 3.3.14]. Therefore, by Lemma 2.2 we see that $\mathrm{tr}_S(\omega_S) = \mathrm{tr}_R(\omega_R) S$.

(1) Since the natural map $R/\mathrm{tr}_R(\omega_R) \to S/\mathrm{tr}_R(\omega_R)S = S/\mathrm{tr}_S(\omega_S)$ is faithfully flat, $R/\mathrm{tr}_R(\omega_R)$ is isomorphic to a subring of $S/\mathrm{tr}_S(\omega_S)$. Since $S/\mathrm{tr}_S(\omega_S)$ is reduced by assumption, $R/\mathrm{tr}_R(\omega_R)$ is also reduced.

(2) Let P be an arbitrary associated prime ideal of $\mathrm{tr}_S(\omega_S)$ and set $\mathfrak{p} = P \cap R$. Since $(R/\mathrm{tr}_R(\omega_R))_\mathfrak{p} \to (S/\mathrm{tr}_S(\omega_S))_P$ is a flat local homomorphism, $\mathrm{depth}(R/\mathrm{tr}_R(\omega_R))_\mathfrak{p} \leq \mathrm{depth}(S/\mathrm{tr}_S(\omega_S))_P = 0$ by [Mat, Theorem 23.3]. Therefore, $\mathfrak{p}$ is an associated prime ideal of $\mathrm{tr}_R(\omega_R)$. Since $\mathrm{tr}_R(\omega_R)$ is a radical ideal, we see that $\mathfrak{p} \in \min(\mathrm{tr}_R(\omega_R))$ and $(R/\mathrm{tr}_R(\omega_R))_\mathfrak{p} = \kappa(\mathfrak{p})$. Thus, since $\kappa(\mathfrak{p}) \otimes_R S_P$ is a localization of $\kappa(\mathfrak{p}) \otimes_R S$, we see by assumption that $(S/\mathrm{tr}_S(\omega_S))_P = (S/\mathrm{tr}_R(\omega_R)S)_P = (R/\mathrm{tr}_R(\omega_R))_\mathfrak{p} \otimes_R S_P = \kappa(\mathfrak{p}) \otimes_R S_P$ is reduced. Since P is an arbitrary associated prime ideal of $\mathrm{tr}_S(\omega_S)$, we see that $\mathrm{tr}_S(\omega_S)$ is a radical ideal. □

Corollary 3.6. *Let R be a ring and $S = R[X_1, \ldots, X_n]$ a polynomial ring over R. Then S is CTR if and only if so is R.*

Proof. First assume that S is CTR. Let $\mathfrak{p}$ be an arbitrary prime ideal of R. Set $P = \mathfrak{p}S + (X_1, \ldots, X_n)S$. Then P is a prime ideal of S and $R_\mathfrak{p} = S_P/(X_1, \ldots, X_n)S_P$. Since S_P admits a canonical module, S_P is a homomorphic image of a Gorenstein ring. Therefore, $R_\mathfrak{p}$ is also a homomorphic image of a Gorenstein ring. Thus, $R_\mathfrak{p}$ admits a canonical module.

Since $S_P/\mathfrak{p}S_P$ is a localization of $\kappa(\mathfrak{p})[X_1, \ldots, X_n]$, therefore is Gorenstein and $R_\mathfrak{p} \to S_P$ is a flat local homomorphism, we see by Proposition 3.5 that $R_\mathfrak{p}$ is CTR. Since $\mathfrak{p}$ is an arbitrary prime ideal of R, we see that R is CTR.

Next we assume that that R is CTR. Let P be an arbitrary prime ideal of S and set $\mathfrak{p} = P \cap R$. Then $S_P/\mathfrak{p}S_P = \kappa(\mathfrak{p}) \otimes_R S_P$ is a localization of $\kappa(\mathfrak{p})[X_1, \ldots, X_n]$ and therefore Gorenstein. Further, for any $\mathfrak{q} \in \min(\mathrm{tr}_{R_\mathfrak{p}}(\omega_{R_\mathfrak{p}}))$, $\kappa(\mathfrak{q}) \otimes_{R_\mathfrak{p}} S_P$ is a localization of $\kappa(\mathfrak{q}) \otimes_{R_\mathfrak{p}} S = \kappa(\mathfrak{q}) \otimes_R S = \kappa(\mathfrak{q})[X_1, \ldots, X_n]$ and therefore is reduced. Thus, by Proposition 3.5, we see that S_P is CTR. Since P is an arbitrary prime ideal of S, we see that S is CTR. □

Next consider the CTR property under completion.

Proposition 3.7. *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring with canonical module and $\widehat{R}$ the completion of R with respect to $\mathfrak{m}$. Then the followings hold.*

(1) If $\widehat{R}$ is CTR, then so is R.

(2) If R is a Nagata ring (pseudo-geometric ring in Nagata's terminology) and CTR, then $\widehat{R}$ is also CTR.

Proof. Since $\widehat{R}/\mathfrak{m}\widehat{R} = R/\mathfrak{m}$ is a field, (1) follows from Proposition 3.5. For (2), note that for any $\mathfrak{p} \in \operatorname{Spec}(R)$, $R/\mathfrak{p}$ is analytically unramified, i.e. $\widehat{(R/\mathfrak{p})}$ is reduced. See [Nag, Theorem 36.4]. Therefore, $(R/\operatorname{tr}_R(\omega_R))\widehat{}$ is reduced, since $\operatorname{tr}_R(\omega_R)$ is a radical ideal. Since $\widehat{R}/\operatorname{tr}_{\widehat{R}}(\omega_{\widehat{R}}) = \widehat{R}/\operatorname{tr}_R(\omega_R)\widehat{R} = (R/\operatorname{tr}_R(\omega_R))\widehat{}$, we see that $\operatorname{tr}_{\widehat{R}}(\omega_{\widehat{R}})$ is a radical ideal of $\widehat{R}$. □

Next we consider the CTR property under the quotient of an ideal generated by a regular sequence.

Proposition 3.8. *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring with canonical module or an $\mathbb{N}$-graded algebra over a field with (irrelevant) maximal ideal $\mathfrak{m}$. Suppose that $x_1, \ldots, x_r \in \mathfrak{m}$ is a (homogeneous) regular sequence with $x_1, \ldots, x_r \in \operatorname{tr}_R(\omega_R)$. Set $\overline{R} = R/(x_1, \ldots, x_r)R$. Then $\operatorname{tr}_{\overline{R}}(\omega_{\overline{R}}) = \operatorname{tr}_R(\omega_R)/(x_1, \ldots, x_r)R$. In particular, R is CTR if and only if so is $\overline{R}$.*

Proof. Localizing by $\mathfrak{m}$, we can reduce the $\mathbb{N}$-graded case to the local case.

First note that $\omega_{\overline{R}} = \omega_R/(x_1, \ldots, x_r)\omega_R = \omega_R \otimes_R \overline{R}$. See [BH, Theorem 3.3.5]. By [HHS, Lemma 1.5 (ii)], we see that

$$\operatorname{tr}_R(\omega_R)\overline{R} \subset \operatorname{tr}_{\overline{R}}(\omega_R \otimes \overline{R}) = \operatorname{tr}_{\overline{R}}(\omega_{\overline{R}}).$$

On the other hand, let M be an arbitrary maximal Cohen-Macaulay R-module and x' an R-regular element with $x' \in \operatorname{tr}_R(\omega_R)$. Then by [DKT, Theorem 2.3], we see that $x'\operatorname{Ext}^1_R(M, R) = 0$. By considering the long exact sequence induced by $0 \to M \xrightarrow{x'} M \to M/x'M \to 0$, we get the following exact sequence.

$$\operatorname{Ext}^1_R(M, R) \xrightarrow{x'} \operatorname{Ext}^1_R(M, R) \to \operatorname{Ext}^2_R(M/x'M, R).$$

Since $x'\operatorname{Ext}^1_R(M, R) = 0$ and $\operatorname{Ext}^2_R(M/x'M, R) \cong \operatorname{Ext}^1_{R/x'R}(M/x'M, R/x'R)$ (see, e.g. [BH, Lemma 3.1.16]), we see that $\operatorname{ann}_R(\operatorname{Ext}^1_R(M, R)) \supset \operatorname{ann}_R(\operatorname{Ext}^1_{R/x'R}(M/x'M, R/x'R))$. Therefore, by [DKT, Theorem 2.3], we see that $\operatorname{tr}_R(\omega_R)(R/x'R) \supset \operatorname{tr}_{R/x'R}(\omega_{R/x'R})$. Using this fact repeatedly, we see that

$$\operatorname{tr}_R(\omega_R)\overline{R} \supset \operatorname{tr}_{\overline{R}}(\omega_{\overline{R}}).$$

Therefore,

$$\operatorname{tr}_{\overline{R}}(\omega_{\overline{R}}) = \operatorname{tr}_R(\omega_R)\overline{R} = \operatorname{tr}_R(\omega_R)/(x_1, \ldots, x_r)R.$$

In particular, $\operatorname{tr}_R(\omega_R)$ is a radical ideal if and only if so is $\operatorname{tr}_{\overline{R}}(\omega_{\overline{R}})$. □

Next, we consider the behavior of CTR property under the tensor product.

Proposition 3.9. *Let $R^{(1)}$ and $R^{(2)}$ be $\mathbb{N}$-graded $\mathbb{K}$-algebras, where $\mathbb{K}$ is a field and set $R = R^{(1)} \otimes_{\mathbb{K}} R^{(2)}$.*

(1) If R is CTR and $\mathrm{tr}_{R^{(2)}}(\omega_{R^{(2)}})$ contains an $R^{(2)}$-regular element, then $R^{(1)}$ is CTR.

(2) If $R^{(i)}$ is Cohen-Macaulay and $(R^{(i)}/\mathrm{tr}_{R^{(i)}}(\omega_{R^{(i)}})) \otimes_{\mathbb{K}} R^{(3-i)}$ is reduced for $i = 1, 2$, then R is CTR.

Proof. First note that by [Mat, Theorem 23.3 Corollary], R is Cohen-Macaulay if and only if both $R^{(1)}$ and $R^{(2)}$ are Cohen-Macaulay and by [HHS, Proposition 4.1 and Theorem 4.2], it holds that $\mathrm{tr}_R(\omega_R) = \mathrm{tr}_{R^{(1)}}(\omega_{R^{(1)}})R \cap \mathrm{tr}_{R^{(2)}}(\omega_{R^{(2)}})R$.

We first prove (1). Take an $R^{(2)}$-regular element a from $\mathrm{tr}_{R^{(2)}}(\omega_{R^{(2)}})$. Since

$$0 \to R^{(2)} \xrightarrow{a} R^{(2)}$$

is exact, we see that

$$0 \to R^{(1)}/\mathrm{tr}_{R^{(1)}}(\omega_{R^{(1)}}) \otimes R^{(2)} \xrightarrow{1\otimes a} R^{(1)}/\mathrm{tr}_{R^{(1)}}(\omega_{R^{(1)}}) \otimes R^{(2)}$$

is also exact. Since $R^{(1)}/\mathrm{tr}_{R^{(1)}}(\omega_{R^{(1)}}) \otimes R^{(2)} = R/\mathrm{tr}_{R^{(1)}}(\omega_{R^{(1)}})R$, we see that $1 \otimes a \in R$ is an $R/\mathrm{tr}_{R^{(1)}}(\omega_{R^{(1)}})R$-regular element of R. On the other hand, since $1 \otimes a \in \mathrm{tr}_{R^{(2)}}(\omega_{R^{(2)}})R$, we see that

$$\begin{aligned} & \mathrm{tr}_R(\omega_R)R[(1 \otimes a)^{-1}] \cap R \\ = \; & (\mathrm{tr}_{R^{(1)}}(\omega_{R^{(1)}})R \cap \mathrm{tr}_{R^{(2)}}(\omega_{R^{(2)}})R)R[(1 \otimes a)^{-1}] \cap R \\ = \; & \mathrm{tr}_{R^{(1)}}(\omega_{R^{(1)}})R. \end{aligned}$$

Thus, $\mathrm{tr}_{R^{(1)}}(\omega_{R^{(1)}})R$ is a radical ideal since R is CTR.

Since $R^{(1)} \to R$, $x \mapsto x \otimes 1$ is a faithfully flat homomorphism, we see that $\mathrm{tr}_{R^{(1)}}(\omega_{R^{(1)}}) = \mathrm{tr}_{R^{(1)}}(\omega_{R^{(1)}})R \cap R^{(1)}$ and therefore $\mathrm{tr}_{R^{(1)}}(\omega_{R^{(1)}})$ is a radical ideal of $R^{(1)}$.

Next, we prove (2). Since $\mathrm{tr}_R(\omega_R) = \mathrm{tr}_{R^{(1)}}(\omega_{R^{(1)}})R \cap \mathrm{tr}_{R^{(2)}}(\omega_{R^{(2)}})R$, it is enough to show that $R/\mathrm{tr}_{R^{(i)}}(\omega_{R^{(i)}})R$ is a reduced ring for $i = 1, 2$. However,

$$R/\mathrm{tr}_{R^{(i)}}(\omega_{R^{(i)}})R = R^{(i)}/\mathrm{tr}_{R^{(i)}}(\omega_{R^{(i)}}) \otimes R^{(3-i)}$$

and the right hand side is assumed to be reduced, we see the result. □

Remark 3.10. In the situation of Proposition 3.9 (1), if $R^{(2)}$ is reduced, then for any $\mathfrak{p} \in \mathrm{Ass}(R^{(2)})$, $R^{(2)}_{\mathfrak{p}}$ is a field and therefore Gorenstein. Thus $\mathrm{tr}_{R^{(2)}}(\omega_{R^{(2)}}) \not\subset \mathfrak{p}$. Since $\mathfrak{p}$ is an arbitrary associated prime of $R^{(2)}$, we see that $\mathrm{tr}_{R^{(2)}}(\omega_{R^{(2)}})$ contains an $R^{(2)}$-regular element. On the other hand, if the assumption of Proposition 3.9 (2) is satisfied, then $R^{(1)}$ and $R^{(2)}$ are reduced CTR rings. Conversely, if $R^{(1)}$ and $R^{(2)}$ are reduced CTR and $\mathbb{K}$ is a perfect field, then $R^{(i)}/\mathrm{tr}_{R^{(i)}}(\omega_{R^{(i)}}) \otimes_{\mathbb{K}} R^{(3-i)}$ is reduced for $i = 1, 2$ by Lemma 3.12 below.

By the above remark, we see the following.

Corollary 3.11. *Let $\mathbb{K}$ be a field and let $R^{(1)}$ and $R^{(2)}$ be reduced $\mathbb{N}$-graded $\mathbb{K}$-algebras. Then the followings hold.*

(1) If $R^{(1)} \otimes R^{(2)}$ is CTR, then both $R^{(1)}$ and $R^{(2)}$ are CTR.

(2) If $\mathbb{K}$ is a perfect field and $R^{(1)}$ and $R^{(2)}$ are CTR, then $R^{(1)} \otimes R^{(2)}$ is CTR.

Next we consider the behavior of CTR property under the Segre product. First, we state the following fact which is a direct consequence of [Mat, Theorem 26.3].

Lemma 3.12. *Let $\mathbb{K}$ be a perfect field and let $R^{(1)}$ and $R^{(2)}$ be reduced $\mathbb{K}$-algebras. Then $R^{(1)} \otimes_{\mathbb{K}} R^{(2)}$ is a reduced ring.*

Next we state the following.

Lemma 3.13. *Let $\mathbb{K}$ be a perfect field and let $R^{(1)}, \ldots, R^{(n)}$ be $\mathbb{N}$-graded reduced $\mathbb{K}$-algebras and I_i a graded radical ideal of $R^{(i)}$ for $1 \le i \le n$. Then $I_1 \# \cdots \# I_n$ is a radical ideal of $R^{(1)} \# \cdots \# R^{(n)}$.*

Proof. Since $R^{(1)} \# R^{(2)}$is a subring of $R^{(1)} \otimes_{\mathbb{K}} R^{(2)}$, we see that $R^{(1)} \# R^{(2)}$ is a reduced ring by the previous lemma. Therefore, by induction on n, it is enough to prove the case where $n = 2$.

Since

$$0 \to I_1 \to R^{(1)} \to R^{(1)}/I_1 \to 0$$

is exact, we see that

$$0 \to I_1 \# R^{(2)} \to R^{(1)} \# R^{(2)} \to (R^{(1)}/I_1) \# R^{(2)} \to 0$$

is exact. Since $(R^{(1)}/I_1) \# R^{(2)}$ is a subring of $(R^{(1)}/I_1) \otimes R_2$ and $(R^{(1)}/I_1) \otimes R_2$ is reduced by Lemma 3.12, we see that $(R^{(1)}/I_1) \# R^{(2)}$ is a reduced ring. Therefore, $I_1 \# R^{(2)}$ is a radical ideal of $R^{(1)} \# R^{(2)}$. We see by the same way that $R^{(1)} \# I_2$ is a radical ideal of $R^{(1)} \# R^{(2)}$.

Since $I_1 \# I_2 = (I_1 \# R^{(2)}) \cap (R^{(1)} \# I_2)$, we see that $I_1 \# I_2$ is a radical ideal of $R^{(1)} \# R^{(2)}$. □

Next we note a basic fact about non-zero-divizor and Segre product.

Lemma 3.14. *Let $\mathbb{K}$ be a field, $R^{(1)}$ and $R^{(2)}$ $\mathbb{N}$-graded $\mathbb{K}$-algebras, d a positive integer and $x_i \in R^{(i)}_d$ a non-zero-divizor of $R^{(i)}$ for $i = 1, 2$. Then $x_1 \# x_2$ is a non-zero-divizor of $R^{(1)} \# R^{(2)}$.*

Proof. It is enough to show that for any non-zero homogeneous element $\alpha \in R^{(1)} \# R^{(2)}$, it holds that $(x_1 \# x_2)\alpha \neq 0$.

Set $\deg \alpha = d'$, $\alpha = \sum_{i=1}^{\ell} z_i \# w_i$, $z_i \in R^{(1)}_{d'}$, $w_i \in R^{(2)}_{d'}$ $z_1, \ldots, z_\ell$ are linearly independent over $\mathbb{K}$ and $w_i \neq 0$ for $1 \leq i \leq \ell$. Then $\ell \geq 1$ since $\alpha \neq 0$. Further, $(x_1 \# x_2)\alpha = \sum_{i=1}^{\ell} x_1 z_i \# x_2 w_i$, $x_1 z_1, \ldots, x_1 z_\ell$ are linearly independent over $\mathbb{K}$ and $x_2 w_i \neq 0$ for $1 \leq i \leq \ell$. Therefore, $(x_1 \# x_2)\alpha \neq 0$. $\square$

Now we show the following.

Proposition 3.15. *Let $\mathbb{K}$ be a perfect field and let $R^{(1)}, R^{(2)}, \ldots, R^{(n)}$ be standard graded $\mathbb{K}$-algebras. Suppose that $R^{(i)}$ is a reduced CTR ring, $\dim R^{(i)} \geq 2$, $a(R^{(i)}) < 0$, $R^{(i)}$ contains a linear $R^{(i)}$-regular element and $R^{(i)}$ is level and anticanonical level for any i. Set $a_i = a(R^{(i)})$ and $b_i = \min\{m : (\omega^{-1}_{R^{(i)}})_m \neq 0\}$ for $1 \leq i \leq n$. Suppose also that $a_i \geq a_1$ and $b_i \leq b_1$ for $2 \leq i \leq n$. Then the Segre product $R = R^{(1)} \# \cdots \# R^{(n)}$ is also a reduced CTR ring, level and anticanonical level, $a(R) = a_1$, $\min\{m : (\omega_R^{-1})_m \neq 0\} = b_1$ and R contains a linear R-regular element.*

Proof. By induction on n, it is enough to prove the case where $n = 2$, since by [GW, Theorem 4.2.3], $\dim(R^{(1)} \# R^{(2)}) \geq 2$. By [GW, Theorem 4.2.3], $R^{(1)} \# R^{(2)}$ is Cohen-Macaulay and by [GW, Theorem 4.3.1] and [HMP, Theorem 2.4], we see that $\omega_R = \omega_{R^{(1)}} \# \omega_{R^{(2)}}$ and $\omega_R^{-1} = \omega^{-1}_{R^{(1)}} \# \omega^{-1}_{R^{(2)}}$. Note the assumption of [HMP, Theorem 2.4] that $\mathbb{K}$ is an infinite field is used only for the existence of linear non-zero-divizor.

Since $\omega_{R^{(1)}}$ is generated by $(\omega_{R^{(1)}})_{-a_1}$, $\omega_{R^{(2)}}$ is generated by $(\omega_{R^{(2)}})_{-a_2}$ and $-a_2 \leq -a_1$, we see that ω_R is generated by $(\omega_{R^{(1)}})_{-a_1} \otimes (\omega_{R^{(2)}})_{-a_2} R^{(2)}_{a_2 - a_1}$. Similarly, we see that ω_R^{-1} is generated by $(\omega^{-1}_{R^{(1)}})_{b_1} \otimes (\omega^{-1}_{R^{(2)}})_{b_2} R^{(2)}_{b_1 - b_2}$. Therefore, $\mathrm{tr}_R(\omega_R)$ is generated by $(\omega_{R^{(1)}})_{-a_1} (\omega^{-1}_{R^{(1)}})_{b_1} \otimes (\omega_{R^{(2)}})_{-a_2} (\omega^{-1}_{R^{(2)}})_{b_2} R^{(2)}_{a_2 - a_1 + b_1 - b_2}$. Since $b_1 - a_1 \geq b_2 - a_2$, we see that

$$\begin{aligned} \mathrm{tr}_R(\omega_R) &= \mathrm{tr}_{R^{(1)}}(\omega_{R^{(1)}}) \# \mathrm{tr}_{R^{(2)}}(\omega_{R^{(2)}})_{\geq b_1 - a_1} \\ &= \mathrm{tr}_{R^{(1)}}(\omega_{R^{(1)}}) \# \mathrm{tr}_{R^{(2)}}(\omega_{R^{(2)}}). \end{aligned}$$

Therefore, by Lemma 3.13, we see that $\mathrm{tr}_R(\omega_R)$ is a radical ideal, by Lemma 3.14, we see that there is a linear R-regular element and by Lemma 3.12, we see that $R^{(1)} \otimes R^{(2)}$ is a reduced ring and therefore R, a subring of $R^{(1)} \otimes R^{(2)}$, is also a reduced ring. Further, since $\omega_R = \omega_{R^{(1)}} \# \omega_{R^{(2)}}$ is generated in degree $-a_1$ and $\omega_{R^{(1)}}^{-1} \# \omega_{R^{(2)}}^{-1}$ is generated in degree b_1, we see that R is level and anticanoncal level, $a(R) = a_1$ and $\min\{m : (\omega_R^{-1})_m \neq 0\} = b_1$. □

In general, the converse of this proposition does not hold. See Example 3.16.

As a special case of Proposition 3.15, if $R^{(2)}, \dots, R^{(n)}$ are Gorenstein and $a_1 \leq a(R^{(i)}) \leq b_1$ for $2 \leq i \leq n$, then $R^{(1)} \# R^{(2)} \# \cdots \# R^{(n)}$ is a CTR ring. Note that in this case, we do not need to assume that $R^{(1)}$ is a reduced ring, since $\mathrm{tr}_R(\omega_R) = \mathrm{tr}_{R^{(1)}}(\omega_{R^{(1)}}) \# R^{(2)}$ in the equation above.

We state an example of a pair of graded rings one of it is not CTR, but their Segre product is. First we recall the definition of order polytopes. Let P be a poset. The convex polytope

$$\mathrm{conv}\left\{ f \in \mathbb{R}^P : \begin{array}{l} 0 \leq f(x) \leq 1 \text{ for any } x \in P,\ x \leq y \Rightarrow \\ f(x) \geq f(y) \end{array} \right\}$$

is called the order polytope of P and denoted by $\mathscr{O}(P)$. See [Sta2]. Note that we reverse the inequality of $f(x)$ and $f(y)$ above from [Sta2] in order to make the Ehrhart ring of $\mathscr{O}(P)$ over a field $\mathbb{K}$ is identical with the Hibi ring $\mathcal{R}_{\mathbb{K}}[\mathscr{I}(P)]$ defined by Hibi [Hib], where $\mathscr{I}(P)$ is the set of poset ideals of P.

For $n \in \mathbb{Z}$, we set

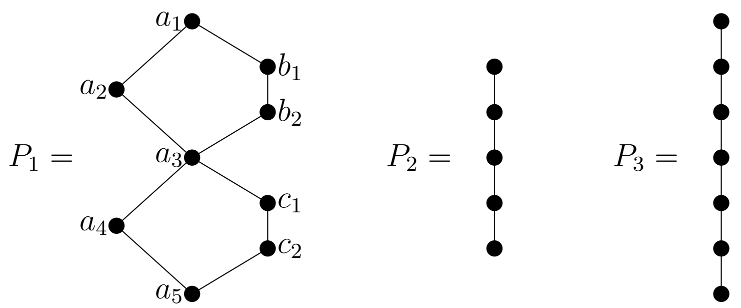
$$\mathcal{T}^{(n)}(P) := \left\{ \nu \in \mathbb{Z}^{P^-} : \begin{array}{r} \nu(x) \geq n \text{ for any maximal element } x \\ \text{of } P \text{ and if } x \lessdot y \text{ in } P^-, \text{ then } \nu(x) \geq \\ \nu(y) + n \end{array} \right\},$$

where $x \lessdot y$ means that y covers x, i.e. $x < y$ and there is no $z \in P^-$ with $x < z < y$. Then by [Miy1, Theorem 2.9], it holds that

$$\omega_{E_{\mathbb{K}}[\mathscr{O}(P)]}^{(n)} = \bigoplus_{\nu \in \mathcal{T}^{(n)}(P)} \mathbb{K} T^{\nu}$$

for any $n \in \mathbb{Z}$.

Example 3.16. Let $\mathbb{K}$ be a perfect field and let P_1, P_2 and P_3 be posets with the following Hasse diagrams.



Then $E_{\mathbb{K}}[\mathscr{O}(P_1)]$ is CTR. In fact, let T^ν be an arbitrary monomial in $\sqrt{\operatorname{tr}(\omega_{E_{\mathbb{K}}[\mathscr{O}(P_1)]})}$, where $\nu \in \mathcal{T}^{(0)}(P_1)$. Then by [MP, Theorem 4.5], we see that $\nu(a_1) < \nu(a_3) < \nu(a_5)$. Set $I_1 = \{b_i : \nu(b_i) > \nu(a_1)\}$, $I_2 = \{c_i : \nu(c_i) > \nu(a_3)\}$ and define ζ and $\eta \in \mathbb{Z}^{P^-}$ by

$$\zeta(x) = \begin{cases} -i, & x = a_i, \\ -i - 1 + \chi_{I_1}(x), & x = b_i, \\ -i - 3 + \chi_{I_2}(x), & x = c_i, \\ -6, & x = -\infty \end{cases}$$

and $\eta = \nu - \zeta$. Then it is verified by hand calculation that $\zeta \in \mathcal{T}^{(-1)}(P_1)$ and $\eta \in \mathcal{T}^{(1)}(P_1)$. Since $\nu = \eta + \zeta$, we see that $T^\nu \in \operatorname{tr}(\omega_{E_{\mathbb{K}}[\mathscr{O}(P_1)]})$. Thus, we see that $E_{\mathbb{K}}[\mathscr{O}(P_1)]$ is a CTR ring. Further, we see by [Miy1, Theorems 3.11 and 3.12] that $E_{\mathbb{K}}[\mathscr{O}(P_1)]$ is level and anticanonical level, $a(E_{\mathbb{K}}[\mathscr{O}(P_1)]) = -8$ and $\min\{m : (\omega^{(-1)}_{E_{\mathbb{K}}[\mathscr{O}(P_1)]})_m \neq 0\} = -6$.

Since $E_{\mathbb{K}}[\mathscr{O}(P_2)]$ and $E_{\mathbb{K}}[\mathscr{O}(P_3)]$ are isomorphic to polynomial rings with 6 and 8 variables respectively, we see that $E_{\mathbb{K}}[\mathscr{O}(P_2)]$ and $E_{\mathbb{K}}[\mathscr{O}(P_3)]$ are Gorenstein rings with $a(E_{\mathbb{K}}[\mathscr{O}(P_2)]) = -6$ and $a(E_{\mathbb{K}}[\mathscr{O}(P_3)]) = -8$. Therefore, we see by Proposition 3.15 that

$$E_{\mathbb{K}}[\mathscr{O}(P_1)] \# E_{\mathbb{K}}[\mathscr{O}(P_2)] \# E_{\mathbb{K}}[\mathscr{O}(P_3)] = E_{\mathbb{K}}[\mathscr{O}(P_1 \cup P_2 \cup P_3)]$$

is a CTR ring. However, by [HMP, Theorem 2.7], the trace of the canonical module of $E_{\mathbb{K}}[\mathscr{O}(P_2)] \# E_{\mathbb{K}}[\mathscr{O}(P_3)]$ is the square of the irrelevant maximal ideal of $E_{\mathbb{K}}[\mathscr{O}(P_2)] \# E_{\mathbb{K}}[\mathscr{O}(P_3)]$. Therefore, $E_{\mathbb{K}}[\mathscr{O}(P_2)] \# E_{\mathbb{K}}[\mathscr{O}(P_3)]$ is not CTR.

4 CTR property of Schubert cycles

In this section and next, we state criteria of CTR property of certain classes of rings which motivated us to define CTR property. First in this section, we study Schubert cycles.

Before going into the details, we first establish notation and recall basic facts. Let $\mathbb{K}$ be a field. For the terms concerning algebras with straightening law (ASL for short) we consult [BV]. In particular, if R is a graded ASL on a poset Π over $\mathbb{K}$, Ω a poset ideal of Π and $I = \Omega R$, we say that Ω or I is straightening closed if for any incomparable elements υ, $\xi \in \Omega$, every standard monomial μ_i appearing in the standard representation

$$\upsilon\xi = \sum_i c_i \mu_i, \quad c_i \in \mathbb{K} \setminus \{0\}$$

has at least 2 factors in Ω. By [DEP, Proposition 1], we see the following.

Lemma 4.1. *Let R be a graded ASL over $\mathbb{K}$ on a poset Π, Ω a straightening closed poset ideal of Π and $I = \Omega R$. Then for any positive integer n, I^n is an ideal of R generated by $\{\xi_1 \cdots \xi_n : \xi_i \in \Omega$ for $1 \leq i \leq n$, $\xi_1 \leq \cdots \leq \xi_n\}$. Also, I^n is a $\mathbb{K}$-vector subspace of R with basis $\{\xi_1 \cdots \xi_\ell : \xi_i \in \Pi$ for $1 \leq i \leq \ell$, $\xi_1 \leq \cdots \leq \xi_\ell$, $\ell \geq n$, $\xi_1, \ldots, \xi_n \in \Omega\}$.*

Since Ω is a poset ideal, $\xi_n \in \Omega$ implies $\xi_1, \ldots, \xi_n \in \Omega$. However, we expressed the above lemma by the above form for convenience of later use.

For integers m and n with $1 \leq m \leq n$ we set $\Gamma(m \times n) := \{[a_1, \ldots, a_m] : a_i \in \mathbb{Z}$ for $1 \leq i \leq m$, $1 \leq a_1 < \cdots < a_m \leq n\}$ and define the order on $\Gamma(m \times n)$ by

$$[a_1, \ldots, a_m] \leq [b_1, \ldots, b_m] \overset{\text{def}}{\iff} a_i \leq b_i \text{ for } 1 \leq i \leq m.$$

Then $\Gamma(m \times n)$ is a distributive lattice whose join, denoted by $\sqcup$, and meet, denoted by $\sqcap$, are

$$[a_1, \ldots, a_m] \sqcup [b_1, \ldots, b_m] = [\max\{a_1, b_1\}, \ldots, \max\{a_m, b_m\}]$$

and

$$[a_1, \ldots, a_m] \sqcap [b_1, \ldots, b_m] = [\min\{a_1, b_1\}, \ldots, \min\{a_m, b_m\}].$$

Further, for $\gamma \in \Gamma(m \times n)$, we set $\Gamma(m \times n; \gamma) := \{\delta \in \Gamma(m \times n) : \delta \geq \gamma\}$. Then $\Gamma(m \times n; \gamma)$ is a sublattice of $\Gamma(m \times n)$.

For an $m \times n$ matrix M and $\gamma = [a_1, \ldots, a_m] \in \Gamma(m \times n)$, we denote by γ_M or $[a_1, \ldots, a_m]_M$ the m-minor of M consisting of columns $a_1, \ldots, a_m$.

Let m and n be integers with $1 \leq m < n$, V an n-dimensional $\mathbb{K}$-vector space and $X = (X_{ij})$ an $m \times n$ matrix of indeterminates. It is known that $\mathbb{K}$-subalgebra $G(X)$ of the polynomial ring $\mathbb{K}[X_{ij} : 1 \leq i \leq m, 1 \leq j \leq n]$ generated by the maximal minors of X is the homogeneous coordinate ring of the Grassmannian of the m-dimensional subspaces $\mathrm{Gr}_m(V)$ of V. Further $G(X)$ is an ASL on $\Gamma(m \times n)$ over $\mathbb{K}$ by the identification $\Gamma(m \times n) \ni \gamma \leftrightarrow \gamma_X \in G(X)$. See [BV, §4] for details.

We introduce the column degree, denoted by cdeg_j by setting

$$\mathrm{cdeg}_j X_{k\ell} = \begin{cases} 1 & \ell = j, \\ 0 & \ell \neq j \end{cases}$$

for $1 \leq j \leq n$. Further, we define grading of $G(X)$ by $\deg \gamma = 1$ for any $\gamma \in \Gamma(m \times n)$. Note that $\deg a = (1/m) \sum_{j=1}^{n} \mathrm{cdeg}_j a$ for any homogeneous element $a \in G(X)$ in the $\mathbb{N}^n$-grading defined by column degree. Note also for any incomparable elements $\upsilon, \xi \in \Gamma(m \times n)$ the standard representation of $\upsilon\xi$ is of the following form.

$$\upsilon\xi = \sum_i c_i \gamma_i \delta_i, \quad c_i \in \mathbb{K} \setminus \{0\}, \gamma_i \leq \delta_i$$

$$\mathrm{cdeg}_j \upsilon + \mathrm{cdeg}_j \xi = \mathrm{cdeg}_j \gamma_i + \mathrm{cdeg}_j \delta_i$$

for $1 \le j \le n$ and for any i. See [BV, §4] for details.

Now let $0 = V_0 \subset V_1 \subset \cdots \subset V_n = V$ be a complete flag of V. For integers $b_1, \ldots, b_m$ with $1 \le b_1 < \cdots < b_m \le n$, set $\Omega(b_1, \ldots, b_m) := \{W \in \mathrm{Gr}_m(V) : \dim(W \cap V_{b_i}) \ge i \text{ for } 1 \le i \le m\}$. This is a subvariety of $\mathrm{Gr}_m(V)$ called the Schubert subvariety of $\mathrm{Gr}_m(V)$. Set $a_i := n - b_{m-i+1} + 1$ for $1 \le i \le m$ and $\gamma := [a_1, \ldots, a_m]$. Then the homogeneous coordinate ring of $\Omega(b_1, \ldots, b_m)$ is $G(X)/(\delta \in \Gamma(m \times n) : \delta \not\ge \gamma)$ and called the Schubert cycle. By [DEP, Proposition 1.2], $G(X)/(\delta \in \Gamma(m \times n) : \delta \not\ge \gamma)$ is an ASL on $\Gamma(m \times n) \setminus \{\delta \in \Gamma(m \times n) : \delta \not\ge \gamma\} = \Gamma(m \times n; \gamma)$. Therefore the homogeneous coordinate ring of $\Omega(b_1, \ldots, b_m)$ is an ASL over $\mathbb{K}$ on $\Gamma(m \times n; \gamma)$. We denote this ring by $G(X; \gamma)$. See [BV, §1.D] for details. Further $G(X; \gamma)$ is a normal domain by [BV, Theorem 6.3].

Now we fix $\gamma = [a_1, \ldots, a_m] \in \Gamma(m \times n)$ and consider the canonical class of $G(X; \gamma)$. If $\gamma = [n - m + 1, \ldots, n]$, then $G(X; \gamma)$ is isomorphic to a polynomial ring with 1 variable over $\mathbb{K}$. Therefore, we assume that $\gamma \ne [n - m + 1, \ldots, n]$ in the following. We first decompose γ into blocks and gaps as in [BV, §6]: $[a_1, \ldots, a_m] = [\beta_0, \beta_1, \ldots, \beta_{t+1}]$, $\beta_0 = a_1, a_2, \ldots, a_{k(1)}$, $\beta_1 = a_{k(1)+1}, a_{k(1)+2}, \ldots, a_{k(2)}$, $\ldots$, $\beta_t = a_{k(t)+1}, a_{k(t)+2}, \ldots, a_{k(t+1)}$, $\beta_{t+1} = a_{k(t+1)+1}, a_{k(t+1)+2}, \ldots, n$. $a_{j+1} - a_j = 1$ if $k(i) < j < k(i+1)$ for some i with $0 \le i \le t + 1$, where $k(0) := 0$, $a_{k(i)+1} - a_{k(i)} \ge 2$ for $1 \le i \le t + 1$, where $a_{m+1} := n + 1$. Note that β_{t+1} may be an empty block: $\beta_{t+1} = \emptyset$ if and only if $a_m < n$. This part is different from [BV] but this makes the case dividing simpler. We also define symbols of gaps between blocks by setting $\chi_i := \{j \in \mathbb{Z} : a_{k(i+1)} < j < a_{k(i+1)+1}\}$ for $0 \le i \le t$.

Next, we set $\zeta_i := [\beta_0, \beta_1, \ldots, \beta_{i-1}, a_{k(i)+1}, a_{k(i)+2}, \ldots, a_{k(i+1)-1}, a_{k(i+1)} + 1, \beta_{i+1}, \ldots, \beta_{t+1}]$, $\Omega_i := \{\delta \in \Gamma(m \times n; \gamma) : \delta \not\ge \zeta_i\}$ $(= \Gamma(m \times n; \gamma) \setminus \Gamma(m \times n; \zeta_i))$ and $J(x; \zeta_i) := \Omega_i G(X; \gamma)$ for $0 \le i \le t$. Then $G(X; \gamma)/J(x; \zeta_i) \cong G(X; \zeta_i)$ is an integral domain with $\dim(G(X; \gamma)/J(x; \zeta_i)) = \dim G(X; \gamma) - 1$. In particular, $J(x; \zeta_i)$ is a height 1 prime ideal of $G(X; \gamma)$ for $0 \le i \le t$. Further, $\gamma G(X; \gamma) = \bigcap_{i=0}^{t} J(x; \zeta_i)$. See [BV, §§5 and 6]. Note that $\Omega_i = \{[b_1, \ldots, b_m] \in \Gamma(m \times n; \gamma) : b_{k(i+1)} = a_{k(i+1)}\}$ for $0 \le i \le t$.

Set

$$\kappa_i := \sum_{j=0}^{i} |\beta_j| + \sum_{j=i}^{t} |\chi_j|$$

for $0 \le i \le t$, $\kappa := \max\{\kappa_i : 0 \le i \le t\}$ and $\kappa' := \min\{\kappa_i : 0 \le i \le t\}$. Then by [BV, Theorem 8.12 and Corollary 8.13], we see the following.

Fact 4.2. *The class*
$$\sum_{i=0}^{t} \kappa_i \mathrm{cl}(J(x;\zeta_i))$$
in the divisor class group $\mathrm{Cl}(G(X;\gamma))$ *is the canonical class of* $G(X;\gamma)$ *and* $G(X;\gamma)$ *is Gorenstein if and only if* $\kappa - \kappa' = 0$.

Now we state the characterization of CTR property of $G(X;\gamma)$.

Theorem 4.3. *With the above notation,* $G(X;\gamma)$ *is CTR if and only if* $\kappa - \kappa' \leq 1$.

Proof. By Fact 4.2, $G(X;\gamma)$ is Gorenstein if and only if $\kappa-\kappa' = 0$. Therefore, we may assume that $\kappa - \kappa' \geq 1$.

Set $J_i = J(x;\zeta_i)$ for $0 \leq i \leq t$. Then by [BV, Corollary 9.18], we see that $J_i^{(\ell)} = J_i^{\ell}$ for any positive integer ℓ and $0 \leq i \leq t$. Set also $\mathfrak{a} = \bigcap_{i=0}^{t} J_i^{\kappa_i}$. Then by Fact 4.2, we see that $\mathfrak{a}$ is a canonical module of $G(X;\gamma)$ up to shift of degree and $\mathrm{tr}(\mathfrak{a}) = \mathfrak{a}\mathfrak{a}^{(-1)}$ is the trace of the graded canonical module of $G(X;\gamma)$. (In fact, $\mathfrak{a}$ is the graded canonical module of $G(X;\gamma)$ by [FM, Proposition 3.7], but we do not use this fact.)

First we consider the case where $\kappa - \kappa' = 1$. Set
$$\begin{aligned} I_1 &= \{i : 0 \leq i \leq t, \kappa_i = \kappa\} \quad \text{and} \\ I_2 &= \{i : 0 \leq i \leq t, \kappa_i = \kappa'\}. \end{aligned}$$
Then $I_1 \cup I_2 = \{0, 1, \ldots, t\}$. Since $\mathfrak{a} = \bigcap_{i=0}^{t} J_i^{\kappa_i}$ and Ω_i is straightening closed for any i by [BV, Lemma 9.1], $\mathfrak{a}$ is generated by $\{\xi_1 \cdots \xi_\ell : \ell \geq \kappa,$ $\xi_1, \ldots, \xi_\ell \in \Gamma(m \times n;\gamma)$, $\xi_1 \leq \cdots \leq \xi_\ell$, $\xi_1, \ldots, \xi_\kappa \in \Omega_i$ for $i \in I_1$ and $\xi_1, \ldots, \xi_{\kappa-1} \in \Omega_i$ for $i \in I_2\}$ as a $\mathbb{K}$-vector subspace of $G(X;\gamma)$ by Lemma 4.1. Since $\bigcap_{i=0}^{t} \Omega_i = \{\gamma\}$, we see that $\mathfrak{a} = \gamma^{\kappa-1}(\bigcap_{i\in I_1} J_i)$. Note that $\bigcap_{i\in I_1} J_i$ is an ideal of $G(X;\gamma)$ generated by $\bigcap_{i\in I_1} \Omega_i$.

On the other hand, since $\mathfrak{a}^{(-1)} = \bigcap_{i=0}^{t} J_i^{(-\kappa_i)}$ and $\gamma G(X;\gamma) = \bigcap_{i=0}^{t} J_i$, we see that $\gamma^\kappa \mathfrak{a}^{(-1)} = \bigcap_{i=0}^{t} J_i^{(\kappa-\kappa_i)} = \bigcap_{i\in I_2} J_i$. Therefore, $\gamma^\kappa \mathfrak{a}^{(-1)}$ is generated by $\bigcap_{i\in I_2} \Omega_i$ as an ideal of $G(X;\gamma)$. Therefore,
$$\gamma^\kappa \mathrm{tr}(\mathfrak{a}) = \mathfrak{a}(\gamma^\kappa \mathfrak{a}^{(-1)}) = (\gamma^{\kappa-1}(\bigcap_{i\in I_1} J_i))(\bigcap_{i\in I_2} J_i)$$
and we see that
$$\gamma \mathrm{tr}(\mathfrak{a}) = (\bigcap_{i\in I_1} J_i)(\bigcap_{i\in I_2} J_i).$$
Thus, $\gamma \mathrm{tr}(\mathfrak{a})$ is generated by $\{\xi\xi' : \xi \in \bigcap_{i\in I_1} \Omega_i, \xi' \in \bigcap_{i\in I_2} \Omega_i\}$.

Consider the standard representation of $\xi\xi'$ for arbitrary $\xi \in \bigcap_{i\in I_1} \Omega_i$ and $\xi' \in \bigcap_{i\in I_2} \Omega_i$. First note that $\xi \sqcap \xi' \in \bigcap_{i=0}^{t} \Omega_i = \{\gamma\}$ since Ω_i is a poset ideal for any i. Thus, $\xi \sqcap \xi' = \gamma$. Suppose that

$$\xi\xi' = \sum_{\ell} c_\ell \alpha_\ell \beta_\ell$$

is the standard representation of $\xi\xi'$ in $G(X;\gamma)$. Then, since $\xi \sqcap \xi' = \gamma \leq \alpha_\ell \leq \xi$, ξ', $\alpha_\ell = \gamma$ for any ℓ. Moreover, since

$$\operatorname{cdeg}_j \gamma + \operatorname{cdeg}_j \beta_\ell = \operatorname{cdeg}_j \xi + \operatorname{cdeg}_j \xi' = \operatorname{cdeg}_j \xi \sqcap \xi' + \operatorname{cdeg}_j \xi \sqcup \xi'$$

for any j, we see that $\beta_\ell = \xi \sqcup \xi'$ for any ℓ. Therefore, the standard representation of $\xi\xi'$ is of the following form.

$$\xi\xi' = c\gamma(\xi \sqcup \xi'), \quad c \in \mathbb{K}, c \neq 0.$$

(In fact, $c = 1$, but we do not use this fact.)

Now consider $\xi \sqcup \xi'$. Set $k(0) = 0$ and

$$\sigma_i := [\beta_0, \ldots, \beta_{i-2}, a_{k(i-1)+1}, \ldots, a_{k(i)-1}, a_{k(i)+1}, \ldots, a_{k(i+1)}, a_{k(i+1)}+1, \beta_{i+1}, \ldots, \beta_{t+1}]$$

for $1 \leq i \leq t$ as in [BV, (6.8)],

$$\Theta_i := \Gamma(m \times n; \gamma) \setminus \Gamma(m \times n; \sigma_i)$$

and $J(x;\sigma_i) = \Theta_i G(X;\gamma)$ for $1 \leq i \leq t$. Then $G(X;\gamma)/J(x;\sigma_i) \cong G(X;\sigma_i)$ is an integral domain and therefore $J(x;\sigma_i)$ is a prime ideal of $G(X;\gamma)$ for $1 \leq i \leq t$. Note that

$$\Theta_i = \{[b_1, \ldots, b_m] \in \Gamma(m \times n; \gamma) : b_{k(i)} < a_{k(i)+1}\}$$

for $1 \leq i \leq t$. Further, it is easily verified that ζ_{i-1}, $\zeta_i \leq \sigma_i$ and therefore Ω_{i-1}, $\Omega_i \subset \Theta_i$ for $1 \leq i \leq t$.

Set

$$I' := \{i : i \in I_1, i-1 \in I_2\} \quad \text{and} \quad I'' := \{i : i \in I_2, i-1 \in I_1\}.$$

Since $\xi \in \bigcap_{i\in I_1} \Omega_i$ and $\xi' \in \bigcap_{i\in I_2} \Omega_i$, we see that $\xi \in \bigcap_{i\in I'\cup I''} \Theta_i$ and $\xi' \in \bigcap_{i\in I'\cup I''} \Theta_i$. On the other hand, since

$$\Theta_i = \{[b_1, \ldots, b_m] \in \Gamma(m \times n; \gamma) : b_{k(i)} < a_{k(i)+1}\},$$

we see that $\xi \sqcup \xi' \in \bigcap_{i\in I'\cup I''} \Theta_i$. Thus, $\xi \sqcup \xi' \in \bigcap_{i\in I'\cup I''} J(x;\sigma_i)$. Since ξ (resp. ξ') is an arbitrary element of $\bigcap_{i\in I_1} \Omega_i$ (resp. $\bigcap_{i\in I_2} \Omega_i$), we see that

$$\gamma \operatorname{tr}(\mathfrak{a}) \subset \gamma(\bigcap_{i\in I'\cup I''} J(x;\sigma_i)).$$

Thus, $\mathrm{tr}(\mathfrak{a}) \subset \bigcap_{i\in I'\cup I''} J(x;\sigma_i)$.

Now consider the reverse inclusion. It is enough to show that for any $\beta \in \bigcap_{i\in I'\cup I''} \Theta_i$, it holds that $\gamma\beta \in \gamma\mathrm{tr}(\mathfrak{a})$. Set $\beta = [b_1, \ldots, b_m]$. Set also $k(0) = 0$,

$$\begin{aligned} H_1 &= \{j \in \mathbb{Z} : \exists i \in I_1; k(i) < j \le k(i+1)\}, \\ H_2 &= \{j \in \mathbb{Z} : \exists i \in I_2; k(i) < j \le k(i+1)\}, \text{ and} \\ H_3 &= \{k(t+1)+1, \ldots, m\}. \end{aligned}$$

Note that $H_3 = \emptyset$ if and only if $a_m < n$ and $a_j = b_j$ for $j \in H_3$. We define integers $c_1, \ldots, c_m$ and $c'_1, \ldots, c'_m$ by

$$c_j = \begin{cases} a_j, & \text{if } j \in H_1 \cup H_3, \\ b_j, & \text{if } j \in H_2, \end{cases} \qquad c'_j = \begin{cases} a_j, & \text{if } j \in H_2 \cup H_3, \\ b_j, & \text{if } j \in H_1. \end{cases}$$

Here we show the following key fact.

Claim 4.3.1. *It holds that $c_1 < \cdots < c_m$ and $c'_1 < \cdots < c'_m$.*

We prove the claim $c_1 < \cdots < c_m$. Claim $c'_1 < \cdots < c'_m$ is proved similarly.

If j, $j+1 \in H_1 \cup H_3$, then $c_j = a_j < a_{j+1} = c_{j+1}$ and if j, $j+1 \in H_2$, then $c_j = b_j < b_{j+1} = c_{j+1}$. Assume that $j \in H_1 \cup H_3$ and $j+1 \in H_2$. Then $c_j = a_j < a_{j+1} \le b_{j+1} = c_{j+1}$. Finally, assume that $j \in H_2$ and $j+1 \in H_1 \cup H_3$. Then $j = k(i+1)$ for some $i \in I_2$. If $i = t$, then $j+1 \in H_3$ and we see that $c_j = b_j < b_{j+1} = c_{j+1}$. If $i < t$, then $i \in I_2$ and $i+1 \in I_1$. Therefore, $i+1 \in I'$. Since $\beta \in \Theta_{i+1}$, we see that $c_j = b_{k(i+1)} < a_{k(i+1)+1} = c_{j+1}$ and the claim is proved.

By Claim 4.3.1, we see that $[c_1, \ldots, c_m]$, $[c'_1, \ldots, c'_m] \in \Gamma(m \times n)$. Set $\xi := [c_1, \ldots, c_m]$ and $\xi' := [c'_1, \ldots, c'_m]$. Then, since ξ, $\xi' \ge \gamma$, we see that ξ, $\xi' \in \Gamma(m \times n; \gamma)$. Moreover, since $c_{k(i+1)} = a_{k(i+1)}$ for $i \in I_1$ (resp. $c'_{k(i+1)} = a_{k(i+1)}$ for $i \in I_2$), we see that $\xi \in \bigcap_{i\in I_1} \Omega_i$ (resp. $\xi' \in \bigcap_{i\in I_2} \Omega_i$). Since $\xi \sqcap \xi' = \gamma$ and $\xi \sqcup \xi' = \beta$, we see that the standard representation of $\xi\xi'$ is the following form.

$$\xi\xi' = c\gamma\beta, \quad c \in \mathbb{K}, c \neq 0.$$

Since $\xi \in \bigcap_{i\in I_1} J_i$ and $\xi' \in \bigcap_{i\in I_2} J_i$, we see that

$$\gamma\beta = c^{-1}\xi\xi' \in (\bigcap_{i\in I_1} J_i)(\bigcap_{i\in I_2} J_i) = \gamma\mathrm{tr}(\mathfrak{a}).$$

This is what we wanted to show and we see that $\mathrm{tr}(\mathfrak{a}) = \bigcap_{i\in I'\cup I''} J(x;\sigma_i)$. Since $J(x;\sigma_i)$ is a prime ideal for any i, we see that $\mathrm{tr}(\mathfrak{a})$ is a radical ideal.

Next consider the case where $\kappa - \kappa' \geq 2$. Since $\mathfrak{a} = \bigcap_{i=0}^{t} J_i^{\kappa_i}$, we see that $\mathfrak{a}$ is generated by $\{\xi_1 \cdots \xi_\kappa : \xi_1, \ldots, \xi_{\kappa_i} \in \Omega_i \text{ for } 0 \leq i \leq t\}$. On the other hand, since $\gamma^\kappa \mathfrak{a}^{(-1)} = \bigcap_{i=0}^{t} J_i^{\kappa-\kappa_i}$, $\gamma^\kappa \mathfrak{a}^{(-1)}$ is generated by $\{\xi_1 \cdots \xi_{\kappa-\kappa'} : \xi_1, \ldots, \xi_{\kappa-\kappa_i} \in \Omega_i \text{ for } 0 \leq i \leq t\}$. Therefore $\gamma^\kappa \mathrm{tr}(\mathfrak{a}) = \mathfrak{a}(\gamma^\kappa \mathfrak{a}^{(-1)}) = (\bigcap_{i=0}^{t} J_i^{\kappa_i})(\bigcap_{i=0}^{t} J_i^{\kappa-\kappa_i})$ is generated by homogeneous elements of degree $2\kappa - \kappa'$. Thus $\mathrm{tr}(\mathfrak{a})$ is generated by homogeneous elements of degree $\kappa - \kappa'$. In particular, $\gamma \notin \mathrm{tr}(\mathfrak{a})$, since $\kappa - \kappa' \geq 2$.

On the other hand, by the above description, we see that $\gamma^\kappa \in \bigcap_{i=0}^{t} J_i^{\kappa_i}$ and $\gamma^{\kappa-\kappa'} \in \bigcap_{i=0}^{t} J_i^{\kappa-\kappa'}$. Thus, we see that $\gamma^{2\kappa-\kappa'} \in (\bigcap_{i=0}^{t} J_i^{\kappa_i})(\bigcap_{i=0}^{t} J_i^{\kappa-\kappa_i}) = \gamma^\kappa \mathrm{tr}(\mathfrak{a})$, and therefore $\gamma^{\kappa-\kappa'} \in \mathrm{tr}(\mathfrak{a})$. Thus, we see that $\mathrm{tr}(\mathfrak{a})$ is not a radical ideal. $\square$

Remark 4.4. Ficarra et al. [FHST, Theorem 1.1] showed that if $\mathbb{K}$ is a field, m and n are integers with $2 \leq m \leq n$, $X = (X_{ij})$ is an $m \times n$ matrix of indeterminates and t is an integer with $2 \leq t \leq m$ then

$$\mathrm{tr}_R(\omega_R) = I_{t-1}(X)^{n-m} R,$$

where $R = \mathbb{K}[X_{ij} : 1 \leq i \leq m, 1 \leq j \leq n]/I_t(X)$. Since $I_{t-1}(X)R$ is a prime ideal of R, we see that R is CTR if and only if $n - m \leq 1$. While by [BV, Corollary 8.9] R is Gorenstein if and only if $n - m = 0$.

5 CTR property of the Ehrhart rings of the stable set polytopes of cycle graphs

In this section, we establish a criterion of the CTR propety of the Ehrhart ring of the stable set polytope of a cycle graph. For basic terminology and facts of graph theory, we consult [Die].

Let $G = (V, E)$ be a graph. A stable set S of G is a subset of V with no pair of elements in S are adjacent. $\emptyset$ and $\{v\}$ for any $v \in V$ are trivially stable. We define the stable set polytope, denoted by $\mathrm{STAB}(G)$ of G by

$$\mathrm{STAB}(G) := \mathrm{conv}\{\chi_S \in \mathbb{R}^V : S \text{ is a stable set of } G\}.$$

Next we state the following.

Definition 5.1. Let X be a finite set and $\xi \in \mathbb{R}^X$. For $B \subset X$, we set $\xi^+(B) := \sum_{b \in B} \xi(b)$.

We call a graph G a cycle graph if G consists of one cycle only, i.e. $V = \{v_0, v_1, \ldots, v_{n-1}\}$, $E = \{\{v_i, v_j\} : i - j \equiv 1 \pmod{n}\}$ for some n with

$n \geq 3$. A graph $G = (V, E)$ is called a t-perfect graph if

$$\mathrm{STAB}(G) = \left\{ f \in \mathbb{R}^V : \begin{array}{l} 0 \leq f(v) \leq 1 \text{ for any } v \in V,\ f^+(e) \leq 1 \\ \text{for any } e \in E \text{ and } f^+(C) \leq \frac{|C|-1}{2} \text{ for} \\ \text{any odd cycle } C \text{ without chord} \end{array} \right\}.$$

It is known that a cycle graph is t-perfect. See [Mah].

Let $G = (V, E)$ be a t-perfect graph. For $n \in \mathbb{Z}$, set

$$t\mathcal{U}^{(n)} := \left\{ \mu \in \mathbb{Z}^{V^-} : \begin{array}{l} \mu(v) \geq n \text{ for any } v \in V,\ \mu^+(K) + n \leq \\ \mu(-\infty) \text{ for any maximal clique } K \text{ in } G \\ \text{and } \mu^+(C) + n \leq \frac{|C|-1}{2}\mu(-\infty) \text{ for any} \\ \text{odd cycle } C \text{ without chord and length} \\ \text{at least } 5 \end{array} \right\}.$$

Since G is t-perfect, there are no cliques with size greater than 3, we see by [Miy3, Remark 3.10] that

$$\omega^{(n)}_{E_{\mathbb{K}}[\mathrm{STAB}(G)]} = \bigoplus_{\mu \in t\mathcal{U}^{(n)}} \mathbb{K}T^\mu$$

for any $n \in \mathbb{Z}$.

Let $G = (V, E)$ be a cycle graph. If the length of the cycle is even, then G is a bipartite graph and therefore is a perfect graph whose maximal cliques have size 2. Thus, by [OH, Theorem 2.1 (b)], we see that $E_{\mathbb{K}}[\mathrm{STAB}(G)]$ is a Gorenstein ring. Further, if the length n of the cycle is odd, then by [HT, Theorem 1], $E_{\mathbb{K}}[\mathrm{STAB}(G)]$ is Gorenstein if and only if $n \leq 5$.

Therefore, we assume in the following of this section that $G = (V, E)$ is a cycle graph of odd length n with $n \geq 7$. Set $n = 2\ell + 1$, $V = \{v_0, v_1, \ldots, v_{2\ell}\}$ and we consider indices modulo $2\ell+1$, $E = \{\{v_i, v_j\} : i - j \equiv 1 \pmod{2\ell+1}\}$, $e_i = \{v_i, v_{i+1}\}$ for $0 \leq i \leq 2\ell$. Further, we denote by ω the canonical ideal of $E_{\mathbb{K}}[\mathrm{STAB}(G)]$.

Set

$$\mathfrak{p}_i := \bigoplus_{\substack{\mu \in t\mathcal{U}^{(0)}, \\ \mu(v_i) > 0 \text{ or } \mu^+(V) < \ell\mu(-\infty)}} \mathbb{K}T^\mu$$

for $0 \leq i \leq 2\ell$. Then $\mathfrak{p}_i$ is a prime ideal of $E_{\mathbb{K}}[\mathrm{STAB}(G)]$ and

$$\sqrt{\mathrm{tr}(\omega)} = \bigcap_{i=0}^{2\ell} \mathfrak{p}_i$$

by [Miy4, Theorem 3.1].

Now we state the following criterion of CTR property of $E_{\mathbb{K}}[\mathrm{STAB}(G)]$.

Theorem 5.2. *Let $G = (V, E)$ be a cycle graph of odd length n. Then $E_{\mathbb{K}}[\mathrm{STAB}(G)]$ is a CTR ring if and only if $n \le 7$.*

Proof. Set $n = 2\ell + 1$ as above. First we prove the "if" part. Suppose $\ell = 3$, $\mu \in t\mathcal{U}^{(0)}$ and $T^\mu \in \bigcap_{i=0}^{2\ell} \mathfrak{p}_i$. Then by the proof of [Miy4, Lemmas 3.4 and 3.5], we see that $T^\mu \in \mathrm{tr}(\omega)$. Therefore, we see that $\bigcap_{i=0}^{2\ell} \mathfrak{p}_i \subset \mathrm{tr}(\omega)$. Since $\sqrt{\mathrm{tr}(\omega)} = \bigcap_{i=0}^{2\ell} \mathfrak{p}_i$, we see that

$$\sqrt{\mathrm{tr}(\omega)} = \mathrm{tr}(\omega)$$

and $E_{\mathbb{K}}[\mathrm{STAB}(G)]$ is a CTR ring.

Next we prove the contraposition of "only if" part. Suppose that $\ell \ge 4$ and define $\mu \in \mathbb{Z}^{V^-}$ by

$$\mu(x) = \begin{cases} 1 & \text{if } x \in \{v_2, v_4, \ldots, v_{2\ell-2}, -\infty\}, \\ 0 & \text{otherwise.} \end{cases}$$

Then, $\mu \in t\mathcal{U}^{(0)}$ and $\mu^+(V) = \ell - 1 < \ell = \ell\mu(-\infty)$. Therefore, we see that $T^\mu \in \bigcap_{i=0}^{2\ell} \mathfrak{p}_i = \sqrt{\mathrm{tr}(\omega)}$. We assume that μ can be expressed as a sum of elements in $t\mathcal{U}^{(1)}$ and $t\mathcal{U}^{(-1)}$ and deduce a contradiction.

Suppose $\mu = \eta + \zeta$, $\eta \in t\mathcal{U}^{(1)}$ and $\zeta \in t\mathcal{U}^{(-1)}$. Since $\eta(x) \ge 1$ for any $x \in V$ and $\eta^+(V) + 1 \le \ell\eta(-\infty)$, we see that

$$\eta(-\infty) \ge \left\lceil \frac{2\ell + 2}{\ell} \right\rceil = 3.$$

Similarly, since $\zeta(x) \ge -1$ for any $x \in V$ and $\zeta^+(V) - 1 \le \ell\zeta(-\infty)$, we see that

$$\zeta(-\infty) \ge \left\lceil \frac{-2\ell - 2}{\ell} \right\rceil = -2.$$

On the other hand, since $\eta(-\infty) + \zeta(-\infty) = \mu(-\infty) = 1$, we see that

$$\eta(-\infty) = 3 \quad \text{and} \quad \zeta(-\infty) = -2.$$

Moreover, since $\eta(v_i) + \eta(v_{i+1}) + 1 = \eta^+(e_i) + 1 \le \eta(-\infty) = 3$ and $\eta(v_j) \ge 1$ for any i and j, we see that $\eta(v_j) = 1$ for any j. Thus,

$$\zeta(x) = \begin{cases} 0 & \text{if } x \in \{v_2, v_4, \cdots, v_{2\ell-2}\}, \\ -2 & \text{if } x = -\infty, \\ -1 & \text{otherwise.} \end{cases}$$

Therefore, $\zeta^+(V) = -\ell - 2$ and we see that

$$\zeta^+(V) - 1 = -\ell - 3 > -2\ell = \ell\zeta(-\infty),$$

since $\ell \ge 4$. This contradicts to the assumption that $\zeta \in t\mathcal{U}^{(-1)}$. Therefore, we see that $T^\mu \notin \mathrm{tr}(\omega)$ and $\mathrm{tr}(\omega)$ is not a radical ideal. □

6 A necessary condition for the Ehrhart ring of the stable set polytope of a perfect graph to be CTR

In this section, we state a necessary condition of an Ehrhart ring of the stable set polytope of a perfect graph to be CTR, which shows that CTR property is close to Gorenstein property. Let $G = (V, E)$ be a perfect graph and set $k = \max\{|K| : K \text{ is a maximal clique of } G\}$ and $k' = \min\{|K| : K \text{ is a maximal clique of } G\}$.

By Chvátal [Chv, Theorem 3.1], we see that

$$\mathrm{STAB}(G) = \left\{ f \in \mathbb{R}^V : \begin{array}{l} f(x) \geq 0 \text{ for any } x \in V \text{ and } f^+(K) \leq \\ 1 \text{ for any maximal clique } K \text{ of } G \end{array} \right\}.$$

Further, by Ohsugi and Hibi [OH, Theorem 2.1 (b)], $E_{\mathbb{K}}[\mathrm{STAB}(G)]$ is Gorenstein if and only if $k - k' = 0$. We recall notation and basic facts from [Miy3, Remark 3.10]. For $n \in \mathbb{Z}$, we set

$$q\mathcal{U}^{(n)} := \left\{ \mu \in \mathbb{Z}^{V^-} : \begin{array}{l} \mu(x) \geq n \text{ for any } x \in V \text{ and } \mu^+(K) + \\ n \leq \mu(-\infty) \text{ for any maximal clique } K \\ \text{of } G \end{array} \right\}.$$

Then

$$\omega^{(n)} = \bigoplus_{\mu \in q\mathcal{U}^{(n)}} \mathbb{K}T^\mu$$

for $n \in \mathbb{Z}$, where ω is the canonical ideal of $E_{\mathbb{K}}[\mathrm{STAB}(G)]$.

Here, we state a very easily proved but very useful fact.

Lemma 6.1. *If $x \in V$, $\eta \in q\mathcal{U}^{(1)}$, $\zeta \in q\mathcal{U}^{(-1)}$ and $(\eta + \zeta)(x) = 0$, then $\eta(x) = 1$ and $\zeta(x) = -1$.*

Now we state the following necessary condition for $E_{\mathbb{K}}[\mathrm{STAB}(G)]$ to be a CTR ring.

Proposition 6.2. *Let $G = (V, E)$ be a perfect graph. If $E_{\mathbb{K}}[\mathrm{STAB}(G)]$ is CTR, then $k - k' \leq 1$.*

Proof. We prove the contraposition of the proposition, so we assume that $k - k' \geq 2$. Let μ be an element of $\mathbb{Z}^{V^-}$ defined by

$$\mu(x) = \begin{cases} 0 & \text{if } x \in V, \\ 1 & \text{if } x = -\infty. \end{cases}$$

Then $\mu \in q\mathcal{U}^{(0)}$ and therefore $T^\mu \in E_{\mathbb{K}}[\mathrm{STAB}(G)]$. Further, if we define η, $\zeta \in \mathbb{Z}^{V^-}$ by

$$\eta(x) = \begin{cases} 1 & \text{if } x \in V, \\ k+1 & \text{if } x = -\infty, \end{cases} \qquad \text{and} \qquad \zeta(x) = \begin{cases} -1 & \text{if } x \in V, \\ -k'-1 & \text{if } x = -\infty, \end{cases}$$

then $\eta \in q\mathcal{U}^{(1)}$, $\zeta \in q\mathcal{U}^{(-1)}$ and $(k-k')\mu = \eta + \zeta$. Therefore,

$$(T^\mu)^{k-k'} \in \mathrm{tr}(\omega).$$

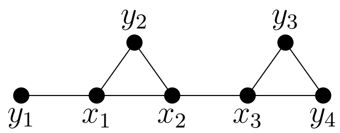
On the other hand, if there are $\eta' \in q\mathcal{U}^{(1)}$ and $\zeta' \in q\mathcal{U}^{(-1)}$ with $\mu = \eta' + \zeta'$, then $\eta'(x) = 1$ and $\zeta'(x) = -1$ for any $x \in V$ by Lemma 6.1. Therefore, $\eta'(-\infty) \geq k+1$ and $\zeta'(-\infty) \geq -k'-1$. Thus,

$$1 = \mu(-\infty) = \eta'(-\infty) + \zeta'(-\infty) \geq k - k' \geq 2.$$

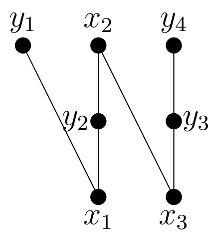
This is a contradiction. Therefore, $T^\mu \notin \mathrm{tr}(\omega)$ and we see that $\mathrm{tr}(\omega)$ is not a radical ideal. □

As the following example shows, the condition in the above proposition is not sufficient.

Example 6.3. Let $G = (V, E)$ be the following graph.



This is a comparability graph of a poset P whose Hasse diagram is



i.e. $G = (V, E)$, $V = P$, $E = \{\{z, w\} : z, w \in P, z < w\}$. In particular, G is perfect. Further, $\max\{|K| : K \text{ is a maximal clique in } G\} = 3$ and $\min\{|K| : K \text{ is a maximal clique in } G\} = 2$.

Define $\mu \in \mathbb{Z}^{V^-}$ by

$$\mu(z) = \begin{cases} 1 & \text{if } z \in \{x_1, x_2, x_3\}, \\ 0 & \text{if } z \in \{y_1, y_2, y_3, y_4\}, \\ 2 & \text{if } z = -\infty. \end{cases}$$

Then $\mu \in q\mathcal{U}^{(0)}$.

We first show that $T^\mu \not\in \mathrm{tr}(\omega)$. Assume the contrary. Then there are $\eta \in q\mathcal{U}^{(1)}$ and $\zeta \in q\mathcal{U}^{(-1)}$ with $\mu = \eta + \zeta$. Then $\eta(y_i) = 1$ and $\zeta(y_i) = -1$ for $1 \le i \le 4$ by Lemma 6.1. Further,

$$\begin{aligned} \eta(-\infty) + \zeta(-\infty) &= \mu(-\infty) = 2, \\ \eta(x_i) + \zeta(x_i) &= \mu(x_i) = 1 \quad \text{for } 1 \le i \le 3, \\ \zeta(x_1) + \zeta(y_1) - 1 &\le \zeta(-\infty), \\ \eta(x_1) + \eta(x_2) + \eta(y_2) + 1 &\le \eta(-\infty), \\ \zeta(x_2) + \zeta(x_3) - 1 &\le \zeta(-\infty) \end{aligned}$$

and

$$\eta(x_3) + \eta(y_3) + \eta(y_4) + 1 \le \eta(-\infty).$$

Therefore

$$\begin{aligned} 4 &= 2\mu(-\infty) \\ &= 2\eta(-\infty) + 2\zeta(-\infty) \\ &\ge (\zeta(x_1) - 2) + (\eta(x_1) + \eta(x_2) + 2) + (\zeta(x_2) + \zeta(x_3) - 1) + (\eta(x_3) + 3) \\ &= \eta(x_1) + \eta(x_2) + \eta(x_3) + \zeta(x_1) + \zeta(x_2) + \zeta(x_3) + 2 \\ &= \mu(x_1) + \mu(x_2) + \mu(x_3) + 2 \\ &= 5. \end{aligned}$$

This is a contradiction. Thus, we see that $T^\mu \not\in \mathrm{tr}(\omega)$.

Next, we show that $T^\mu \in \sqrt{\mathrm{tr}(\omega)}$. Since G is a comparability graph of P, $\mathrm{STAB}(G) = \mathscr{C}(P)$ by [Chv, Theorem 3.1], where $\mathscr{C}(P)$ is the chain polytope of P. See [Sta2] for the definition of the chain polytope. Therefore, by [MP, Theorem 3.7], we see that $T^\mu \in \sqrt{\mathrm{tr}(\omega)}$. In fact, by using the idea of the proof of [MP, Theorem 3.7], we can construct $\eta \in q\mathcal{U}^{(1)}$ and $\zeta \in q\mathcal{U}^{(-1)}$ as follows.

$$\eta(z) = \begin{cases} 1 & \text{if } z \in \{y_1, y_2, y_3, y_4\}, \\ 2 & \text{if } z \in \{x_1, x_2\}, \\ 3 & \text{if } z = x_3, \\ 6 & \text{if } z = -\infty, \end{cases}$$

$$\zeta(z) = \begin{cases} 0 & \text{if } z \in \{x_1, x_2\}, \\ -1 & \text{if } z \in \{x_3, y_1, y_2, y_3, y_4\}, \\ -2 & \text{if } z = -\infty \end{cases}$$

Then $\eta + \zeta = 2\mu$ and therefore $(T^\mu)^2 \in \mathrm{tr}(\omega)$.

We end with the following.

Problem 6.4. Give a criterion of the Ehrhart ring of the chain polytope of a poset to be CTR.

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