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ON HIGHER LIPSCHITZ INVARIANTS

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Abstract. We find new bi-Lipschitz invariants for holomorphic functions of two complex variables. Our new invariants can distinguish function germs that are otherwise indistinguishable by previous methods, as illustrated by the examples given in Section 4.

1. Introduction and preliminaries

While the bi-Lipschitz equivalence of complex analytic set germs does not admit moduli, Henry and Parusiński [HP, Theorem 3.1] firstly showed that, despite the tempting belief that the same holds true for function germs, moduli exist in this case. See also the references to moduli in the related studies [FR] and [CR].

Based on the clustering gradient canyons, we have shown in [PT, Theorem 5.9] that the multi-level "identity card" of a 2-variable holomorphic function germ $f : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ is a discrete bi-Lipschitz invariant. We have pointed out in [MPT, Theorem 4.4] how the modular bi-Lipschitz invariant found by Henry and Parusiński [HP, Theorem 4.3] can be embedded into this picture as the first of the coefficients in the expansion of f along polars.

In this paper we infer a new and different spirit in the research of invariants. Let $f = g \circ \varphi$ for f, g holomorphic function germs of 2 variables, and some bi-Lipschitz homeomorphism φ. Our main result, Theorem 3.4, shows that, surprisingly, one may determine a development of φ to a certain extent. The power of this result is illustrated by our second example in §4.

As a byproduct of our refined study [PT, §5], we find here new bi-Lipschitz invariants within the identity card of the clusters, toward a complete list of bi-Lipschitz invariants enabling us to decide the bi-Lipschitz type of a given two-variables holomorphic function germ. This contributes to a long standing research project, as one can move away from a merely homeomorphism equivalence (too weak) but not too close to a diffeomorphism equivalence (too rigid). Our Corollary 3.5 unveils a second level bi-Lipschitz invariant of Henry-Parusiński type, based on two gradient canyons and their contact order. By Corollary 3.6 we show how to build, in a similar way, a third level bi-Lipschitz invariant of HP-type in case one disposes of 3 or more disjoint gradient canyons, and their contact orders satisfy certain conditions. As expected, this procedure may continue recursively toward still higher level, producing finitely many invariants of HP-type, cf Corollary 3.6.

[1]Sadly Mihai Tibăr passed away after this paper was submitted.

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Our two examples have a key role in understanding two types of applications of Corollary 3.5. Example 4.1 shows how this result decides that any two generic function germs from a certain family are not bi-Lipschitz equivalent. In contrast, Corollary 3.5 is no longer conclusive in Example 4.2, where two generic function germs in a certain family turn out not to be bi-Lipschitz equivalent. To reach this conclusion we need to employ our key Theorem 3.4, which yields the existence of new bi-Lipschitz moduli at this level.
After uploading our manuscript on *arXiv*, N. Nguyen pointed out to us his newly posted arxiv preprint [Ng] in which he observes a version of our Corollary 3.5 by deriving it directly from our constructions in [PT].

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2. Preliminaries

Let $f, g : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ be analytic function germs such that $f = g \circ \varphi$, where $\varphi : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ is a bi-Lipschitz homeomorphism.

Consider a holomorphic map germ

$$\alpha : (\mathbb{C}, 0) \to (\mathbb{C}^2, 0), \quad \alpha(t) = (z(t), w(t)) \not\equiv 0.$$

The image set germ $\alpha_* = \mathrm{Im}(\alpha)$ is a *curve germ* at $0 \in \mathbb{C}^2$, also called a *holomorphic arc* at 0. There is a well-defined tangent line $T(\alpha_*)$ at 0, $T(\alpha_*) \in \mathbb{C}P^1$.

In order to introduce our results, we need to recall here several notations and facts from [PT] and [MPT].

Let $\mathbb{F}$ be the field of convergent fractional power series in an indeterminate y. By the Newton-Puiseux Theorem we have that $\mathbb{F}$ is algebraically closed, cf [BK], [Wa].

A non-zero element of $\mathbb{F}$ has the form

$$\eta(y) = a_0 y^{n_0/N} + a_1 y^{n_1/N} + a_2 y^{n_2/N} + \cdots, \quad n_0 < n_1 < n_2 < \cdots, \tag{1}$$

where $a_i \in \mathbb{C}^*$ and $N, n_i \in \mathbb{N}$ with $\gcd(N, n_0, n_1, \dots) = 1$, $\limsup |a_i|^{\frac{1}{n_i}} < \infty$. The elements of $\mathbb{F}$ are called *Puiseux arcs*. There are $N - 1$ *conjugates* of η, which are the Puiseux arcs of the form

$$\eta^{(k)}_{conj}(y) := \sum a_i \varepsilon^{k n_i} y^{n_i/N}, \quad \varepsilon := e^{\frac{2\pi\sqrt{-1}}{N}},$$

where $k \in \{0, \dots, N - 1\}$.

By the *order of a Puiseux arc* (1) we mean $\operatorname{ord} \eta(y) := \frac{n_0}{N}$, and by the *Puiseux multiplicity* we mean $m_{puiseux}(\eta) = N$, cf [BK], [Wa].

Let $\mathbb{F}_1 := \{\eta \in \mathbb{F} \mid \operatorname{ord}\eta(y) \geq 1\}$. For any $\eta \in \mathbb{F}_1$ with $\operatorname{ord}\eta(y) \geq 1$, the following map germ:

$$\eta_{par} : (\mathbb{C}, 0) \to (\mathbb{C}^2, 0), \quad t \mapsto (\eta(t^N), t^N), \quad N := m_{puiseux}(\eta),$$

is holomorphic, and all the conjugates of η lead to the same irreducible curve $\operatorname{Im}\eta_{par}$, which will be denoted by η_*.

Definition 2.1 (Contact order of holomorphic arcs, cf. [BK], [Wa]).
The *contact order* between two different holomorphic arcs α_* and β_*, where $\alpha, \beta \in \mathbb{F}_1$, is defined as

$$\max_{i,j} \operatorname{ord}\big(\alpha^{(i)}_{\text{conj}}(y) - \beta^{(j)}_{\text{conj}}(y)\big),$$

where $\alpha^{(i)}_{\text{conj}}$, $\beta^{(j)}_{\text{conj}}$ run over all conjugates of α and β, respectively.

Let $f : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ be a holomorphic function germ, and let $m := \operatorname{ord}_0 f$. We say that f is *mini-regular in x of order m*, if the initial form of the Taylor expansion of f is not equal to 0 at the point $(1, 0)$, in other words $f_m(1, 0) \neq 0$ where $f(x, y) = f_m(x, y) + f_{m+1}(x, y) + \text{h.o.t.}$ is the homogeneous Taylor expansion of f.

Let $f : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ be a mini-regular holomorphic function in x. We have the following Puiseux factorisations of f, and of its derivative f_x with respect to x:

$$f(x, y) = u \cdot \prod_{i=1}^{m}(x - \zeta_i(y)), \quad f_x(x, y) = v \cdot \prod_{i=1}^{m-1}(x - \gamma_i(y)),$$

where u, v are units.
If $f_x = g_1^{q_1} \cdots g_p^{q_p}$ is the decomposition into irreducible factors, then $\Gamma_i = \{g_i = 0\}, i = 1, \ldots, p$, if g_i is not a factor of f as well; there exists at least one Puiseux root γ of f_x sudch that $g_i(\gamma(y), y) \equiv 0$.

2.1. **Gradient degree and gradient canyon.** Let γ be a polar arc of f, thus such that $f(\gamma(y), y) \not\equiv 0$ and $f_x(\gamma(y), y) \equiv 0$. The *gradient degree* $d(\gamma)$ is the smallest number q such that

$$\operatorname{ord}(||\operatorname{grad} f(\gamma(y), y)||) = \operatorname{ord}(||\operatorname{grad} f(\gamma(y) + uy^q, y)||),$$

holds for generic $u \in \mathbb{C}$. The *gradient canyon* $\mathcal{GC}(\gamma_*)$ is the subset of all curve germs α_*, where α is a Puiseux arc of the form

$$\alpha(y) := \gamma + uy^{d(\gamma)} + \text{h.o.t.}$$

for any $u \in \mathbb{C}$.

By [PT, Proposition 3.7(b)] all the polars contained in the same canyon have the same gradient degree, and therefore we may speak about the degree of the canyon (or simply the canyon degree).

2.2. **The order** $\operatorname{ord} f(\gamma(y), y)$**.** The order $h := \operatorname{ord} f(\gamma(y), y)$, where γ is a polar arc of f, is by definition a positive rational number. To each such order h, one associates at least one bar $B(h)$ in the Kuo-Lu tree of f, cf [KL], such that, if γ grows from $B(h)$, then $h = \operatorname{ord} f(\gamma(y), y)$. The order h is constant in the same canyon $\mathcal{C}$, see e.g. [PT, Proposition 3.7(c)], thus we may denote it by $h_{\mathcal{C}}$.

2.3. **Clusters of canyons.** Let $G_\ell(f)$ be the subset of canyons tangent to the line ℓ of the tangent cone $\operatorname{Cone}_0(f)$ at the origin of the curve $Z(f)$. Let then $G_{\ell,d,B(h)}(f) \subset G_\ell(f)$ denote the union of gradient canyons of a fixed degree $d > 1$, the polars of which grow on the same bar $B(h)$.

We write
$$G_{\ell,d,B(h)}(f) = \{\mathcal{GC}_1(f), \dots, \mathcal{GC}_r(f)\}$$
for the finite family of all gradient canyons in this cluster; thus the index i in $\mathcal{GC}_i(f)$ simply labels the distinct canyons.

2.4. **Contact of canyons.** A fixed gradient canyon $\mathcal{GC}_i(f) \in G_{\ell,d,B(h)}(f)$ has an order of contact $k(i,j)$ with any other gradient canyon $\mathcal{GC}_j(f) \in G_{\ell,d,B(h)}(f)$ of the same cluster. We define $k(i,j)$ to be the contact order between some polar arc in $\mathcal{GC}_i(f)$ and some polar arc in $\mathcal{GC}_j(f)$ in the sense of Definition 2.1. By [PT, §5.3], whenever $\mathcal{GC}_i(f)$ and $\mathcal{GC}_j(f)$ are distinct canyons of the same degree $d > 1$, this contact order is strictly smaller than d and hence does not depend on the choice of the polars; in particular $k(i,j)$ is well defined.

The number $k(i,j)$ counts also the multiplicity of each such contact, that is, the number of canyons $\mathcal{GC}_j(f)$ of the cluster $G_{\ell,d,B(h)}(f)$ which have exactly the same contact with $\mathcal{GC}_i(f)$.

2.5. **Sub-clusters of canyons.** Let $K_{\ell,d,B(h),i}(f)$ denote the (un-ordered) set of those contact orders $k(i,j)$ of the fixed canyon $\mathcal{GC}_i(f)$, counted with multiplicity.

Finally, let $G_{\ell,d,B(h),\omega}(f)$ be the union of canyons from $G_{\ell,d,B(h)}$ which have exactly the same set $\omega = K_{\ell,d,B(h),i}(f)$ of orders of contact ≥ 0 with the other canyons from $G_{\ell,d,B(h)}$. We then have a partition:
$$G_{\ell,d,B(h)}(f) = \bigsqcup_{\omega} G_{\ell,d,B(h),\omega}(f).$$

2.6. **Lipschitz invariants and "identity card".** By [PT, Theorem 5.9] and [MPT, Theorem 4.3], for any degree $d > 1$, any bar $B(h)$, and any rational h, the following are bi-Lipschitz invariants:

(a) the cluster of canyons $G_{\ell,d,B(h)}$,
(b) the set of contact orders $K_{\ell,d,B(h),i}(f)$, and for each such set, the sub-cluster of canyons $G_{\ell,d,B(h),K_{\ell,d,B(h),i}}(f)$,
(c) the bi-Lipschitz homeomorphism preserves the contact orders between any two clusters of type $G_{\ell,d,B(h),K_{\ell,d,B(h),i}}(f)$.

On the other hand, let $\mathcal{C} \in G_\ell(f)$ be a canyon, and let γ be some polar arc in $\mathcal{C}$, where ℓ is in the tangent cone of $Z(f)$. If $d_{\mathcal{C}}$ denotes the gradient degree of $\mathcal{C}$, then

$$f(\gamma(y), y) = a_{h_{\mathcal{C}}} y^{h_{\mathcal{C}}} + \text{h.o.t.}$$

by [PT, Proposition 3.7.], where $a_{h_{\mathcal{C}}}$ and $h_{\mathcal{C}}$ depend only on the canyon.

The main theorem of [HP], completed by [PT] (see [MPT, Theorem 4.4]), tells that we also have the following bi-Lipschitz invariant:

(d) The effect of the bi-Lipschitz map φ on each such couple $(h_{\mathcal{C}}, a_{h_{\mathcal{C}}})$ is the identity on $h_{\mathcal{C}}$, and the multiplication of $a_{h_{\mathcal{C}}}$ by $c^{-h_{\mathcal{C}}}$, where c is a certain non-zero constant which is the same for all canyons $\mathcal{C} \in G_\ell(f)$.

We say that the above bunch of data, which yield the described bi-Lipschitz invariants, constitute the "identity card" of f.

3. Construction of bi-Lipschitz invariants

In the following, we will assume that $f, g : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ are analytic function germs such that $f = g \circ \varphi$, where $\varphi = (\varphi_1, \varphi_2) : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ is a bi-Lipschitz homeomorphism and f is mini-regular in x.

Notations. Let F and G be functions (possible multivaluate) of the variable $x \in \mathbb{R}^n$.

We say that $|F(x)| \lesssim |G(x)|$ or that *"F is $O(G)$"*, if there exists a constant $K > 0$ such that the inequality $|F(x)| \le K|G(x)|$ holds in some neighbourhood of the origin.

We write:
- "$|F| \sim |G|$" if $|F(x)| \lesssim |G(x)|$ and $|G(x)| \lesssim |F(x)|$.
- "$F \ll G$", and we say that *"F is $o(G)$"* if $\frac{|F(x)|}{|G(x)|} \to 0$, whenever $x \to 0$.

3.1. Local variables.

Lemma 3.1. *Let γ be a polar arc of f tangent to the line $\ell \in \mathrm{Cone}_0(f)$, and let $\mathcal{GC}(\gamma_*) \in G_\ell(f)$ be its canyon. There is a polar γ' of g such that its canyon $\mathcal{GC}(\gamma'_*)$ has the same degree d as $\mathcal{GC}(\gamma_*)$, and such that:*

$$\varphi(\gamma(y), y) = \big(\varphi_1(\gamma(y), y), \varphi_2(\gamma(y), y)\big) = \big(\gamma'(Y) + O(Y^d), Y\big),$$

where $Y := \varphi_2(\gamma(y), y)$ is a local variable.

Before giving the proof, we need to recall some more notations from [PT].

Horn domains. For some fixed α_*, where α is a Puiseux series as before, one defines the infinitesimal disks:

$$\mathcal{D}^{(e)}(\alpha_*; \rho) := \{\beta_* \mid \beta(y) = [J^e(\alpha)(y) + cy^e] + \text{h.o.t.}, |c| \le \rho\},$$

where $1 \le e < \infty$, $\rho \ge 0$.

Consider $\mathcal{D}^{(e)}(\alpha_*; \rho)$ of finite order $e \ge 1$ and finite radius $\rho > 0$, and a compact ball $B(0; \eta) := \{(x, y) \in \mathbb{C}^2 \mid \sqrt{|x|^2 + |y|^2} \le \eta\}$ with small enough $\eta > 0$. Let then

$$\mathrm{Horn}^{(e)}(\alpha_*; \rho; \eta) := \big\{(x, y) \in B(0; \eta) \mid x = \beta(y) = J^e(\alpha)(y) + cy^e, |c| \le \rho\big\}$$

be the *horn domain* associated to $\mathcal{D}^{(e)}(\alpha_*; \rho)$. We note that $(x, y) \in \mathrm{Horn}^{(e)}(\alpha_*; \rho; \eta)$ if and only if $|x - J^e(\alpha(y))| \leq \rho|y^e|$, in particular $x = \alpha(y) + O(y^e)$.

Canyon disks within the Milnor fibre. Let $\mathcal{D}^{(e)}_{\gamma_*,\varepsilon}(\lambda; \eta)$ be the union of disks[1] in the Milnor fibre $\{f = \lambda\} \cap B(0; \eta)$ of f defined as follows

$$\mathcal{D}^{(e)}_{\gamma_*,\varepsilon}(\lambda; \eta) := \{f = \lambda\} \cap \mathrm{Horn}^{(e)}(\gamma_*; \varepsilon; \eta),$$

for some rational e, and some small enough $\varepsilon > 0$.

We will only refer to $\mathcal{D}^{(e)}_{\gamma_*,\varepsilon}(\lambda; \eta)$ such that the canyon degree of $\mathcal{GC}(\gamma_*)$ is $d > 1$. If two polars are in the same canyon, then their associated disks coincide by definition.

Let D_f be some disk of the union $\mathcal{D}^{(e)}_{\gamma_*,\varepsilon}(\lambda; \eta)$.

By *canyon disk* (cf. [PT, §5]) we shall mean in the following such a disk D_f.

Proof of Lemma 3.1. The bi-Lipschitz homeomorphism φ maps the Milnor fibre $\{f = \lambda\} \cap B(0; \eta)$ into the Milnor fibre $\{g = \lambda\} \cap B(0; \eta')$, for convenient choices of the Milnor ball radii η and η'. Let D_f be a canyon disk associated with canyon $\mathcal{GC}(\gamma_*)$. Then, by [PT, Theorem 5.8], there exists a polar arc γ' of g such that $\varphi(D_f) = D_g$, where D_g is a canyon disk associated with canyon $\mathcal{GC}(\gamma'_*)$. Making $\lambda \to 0$, we get that $\varphi(J^d(\gamma(y)), y) \in \mathrm{Horn}^{(d)}(\gamma'_*; \varepsilon; \eta)$, for some small enough ε and η. This means that we have:

$$\big|\varphi_1\big(\gamma(y), y\big) - \gamma'(Y)\big| = \big\|\varphi\big(\gamma(y), y\big) - \big(\gamma'(Y), Y\big)\big\| = O(|Y^d|),$$

where $Y := \varphi_2\big(\gamma(y), y\big)$. □

Lemma 3.2. [HP], [PT].
Let $f, g : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ be analytic function germ such that $f = g \circ \varphi$, where $\varphi = (\varphi_1, \varphi_2) : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ is a bi-Lipschitz homeomorphism. Let γ be some polar arc of f tangent to a line $\ell \in \mathrm{Cone}_0(f)$.

Then we have $|\varphi_2(\gamma(y), y)| \sim |y|$. Moreover, if $d_\gamma > 1$, then for all polars $\bar{\gamma}$ of f such that $\mathrm{ord}\big(\gamma(y) - \bar{\gamma}(y)\big) > 1$ we have the equality:

$$\varphi_2\big(\bar{\gamma}(y), y\big) = cy + o(y)$$

with the same constant $c \in \mathbb{C}^*$.

Proof. We may choose the coordinates in $\mathbb{C}^2$ such that both f and g are mini-regular in x, i.e., that the tangent cones of f and g do not contain the direction $[1 : 0]$, see the definition in §2. By our assumptions, the polar γ is tangential, i.e., its tangent is included in $\mathrm{Cone}_0(f)$. This means that γ has contact order $k \geq 1$ with some root ζ of $\{f = 0\}$, in other words:

$$\|(\zeta(y), y) - (\gamma(y), y)\| \sim |y|^k.$$

By the bi-Lipschitz property, there are constants $0 < m < K$ such that in the neighbourhood of the origin we have:

$$m\|(\zeta(y), y) - (\gamma(y), y)\| \leq \|\varphi(\zeta(y), y) - \varphi(\gamma(y), y)\| \leq K\|(\zeta(y), y) - (\gamma(y), y)\|. \tag{2}$$

[1] By *disk* we mean *homeomorphic to an open disk*.

Since φ is bi-Lipschitz, we have:

$$\|\varphi(\zeta(y), y)\| \sim \|(\zeta(y), y)\| \sim |y|. \tag{3}$$

We write:

$$\|(\varphi_1(\zeta(y), y), \varphi_2(\zeta(y), y))\| = |\varphi_2(\zeta(y), y)| \left\| \left(\frac{\varphi_1(\zeta(y), y)}{\varphi_2(\zeta(y), y)}, 1 \right) \right\|.$$

Since $f = g \circ \varphi$, the root ζ is sent by φ to some root $\eta = (\eta_1, \eta_2)$ of g, which means that we have the equality of directions $\left[\frac{\varphi_1(\zeta(y),y)}{\varphi_2(\zeta(y),y)} : 1\right] = \left[\frac{\eta_1}{\eta_2} : 1\right]$. The later tends to the direction of the tangent line η, which is different from $[1 : 0]$ by our assumption. Hence this is of the form $[a, 1]$, where $a \in \mathbb{C}$. Consequently:

$$|\varphi_1(\zeta(y), y)| \le M|\varphi_2(\zeta(y), y)|, \quad \text{for some } M > 0$$

By using (3), we then get:

$$m|y| \le |\varphi_2(\zeta(y), y)| + |\varphi_1(\zeta(y), y)| \le (1 + M)|\varphi_2(\zeta(y), y)|,$$

and further, by using (2), this implies:

$$|\varphi_2(\gamma(y), y)| \sim |y|. \tag{4}$$

Note that property (4) is independent of the choice of the polar γ.

By §2.1 and §2.2, we have the expansion:

$$f(\gamma(y), y) = ay^h + \cdots + \alpha y^{h+d-1} + \text{h.o.t.}, \tag{5}$$

where $d = d_\gamma$ is the degree of the gradient canyon $\mathcal{GC}(\gamma_*)$, and where all the terms before αy^{d+h-1} depend only on the canyon $\mathcal{GC}(\gamma_*)$.

Lemma 3.1 provides a polar γ' of g such that, for $Y := \varphi_2\big(\gamma(y), y\big)$, we also have:

$$(g \circ \varphi)(\gamma(y), y) = g\big(\gamma'(Y) + O(Y^d), Y\big) = a'Y^H + \cdots + O(Y^{H+d-1}),$$

where all the terms before $O(Y^{d+H-1})$ depend only on the canyon $\mathcal{GC}(\gamma'_*)$.

We obtain the equality:

$$ay^h + \cdots + O(y^{h+d-1}) = a'Y^H + \cdots + O(Y^{H+d-1}). \tag{6}$$

In particular, using (4), we see that the order in $|y|$ of the left-hand side of (6) is h, while the order of the right-hand side is H, since $|Y| \sim |y|$. Hence $h = H$.

Since $f(\gamma(y), y) = (g \circ \phi)(\gamma(y), y)$, for $y \neq 0$ we obtain

$$\left(\frac{Y}{y}\right)^h \xrightarrow[y \to 0]{} \frac{a}{a'} \in \mathbb{C}^*.$$

For y small enough the values of the left-hand side lie in a small disc $D \subset \mathbb{C}^*$ around $\frac{a}{a'}$ which does not contain 0, so we can fix a continuous branch of the h-th root on D. This yields a continuous function $\rho(y)$ such that $\left(\frac{Y}{y}\right)^h = \rho(y)^h$, thus the quotient $\frac{Y/y}{\rho(y)}$

takes values in the finite set of h-th roots of unity. Being continuous on a punctured neighbourhood of 0, it must be constant there, in particular

$$\frac{Y}{y} \longrightarrow c \neq 0 \quad \text{when } y \to 0, \tag{7}$$

i.e. $Y = cy + o(y)$.

Suppose now that $\bar{\gamma}$ is another polar arc of f, tangent to the line $\ell \in \mathrm{Cone}_0(f)$, such that $\mathcal{GC}(\gamma_*) \neq \mathcal{GC}(\bar{\gamma}_*)$, and let $\bar{Y} := \varphi_2(\bar{\gamma}(y), y)$. Using the bi-Lipschitz property of φ, we have:

$$|Y - \bar{Y}| = \|\varphi_2(\gamma(y), y) - \varphi_2(\bar{\gamma}(y), y)\| \leq \|(\gamma(y), y) - (\bar{\gamma}(y), y)\|.$$

Since γ and $\bar{\gamma}$ satisfy $\mathrm{ord}\,\|(\gamma(y), y) - (\bar{\gamma}(y), y)\| > 1$, this implies that $Y - \bar{Y} = o(y)$. □

3.2. **First level bi-Lipschitz invariant.** The main theorem of [HP] gives the first level continuous Lipschitz invariant. It is a direct consequence of the above lemma, as an application of (7). Since this will be used in the following, we state it here (in an equivalent version), and give a short proof:

Corollary 3.3 ([HP] main theorem, equivalent version)**.**
Let $f, g : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ be holomorphic functions such that $f = g \circ \varphi$, where $\varphi : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ is a bi-Lipschitz homeomorphism and f is mini-regular in x. Let γ be some polar arc of f tangent to a line $\ell \in \mathrm{Cone}_0(f)$, and let a, h be the coefficient and the order of the leading term of $f(\gamma(y), y)$.

Then the effect of the bi-Lipschitz map φ on (h, a) is the identity on h, and the multiplication of a by c^h, where c is the non-zero constant provided by Lemma 3.2.

Proof. Let $d > 1$ denote the degree of the canyon $\mathcal{GC}(\gamma_*)$. By §2.1 and §2.2, we have the following:

$$f(\gamma(y), y) = ay^h + \cdots + O(y^{h+d-1}), \tag{8}$$

where the terms of orders less than $h + d - 1$ are identical for every arc in $\mathcal{GC}(\gamma_*)$. Similarly, after applying φ, by Lemma 3.2, we obtain:

$$(g \circ \varphi)(\gamma(y), y) = bY^h + \cdots + O(Y^{h+d-1}), \tag{9}$$

where, by Lemma 3.2, Y is a local variable, and the terms of orders less than $h + d - 1$ are the same for every arc in $\mathcal{GC}(\gamma'_*)$.

Since $f = g \circ \varphi$, by comparing (8) to (9), we get:

$$ay^h + \cdots + O(y^{h+d-1}) = bY^h + \cdots + O(Y^{h+d-1}), \tag{10}$$

and by applying the substitution (15) provided by Lemma 3.2, we obtain the equality:

$$b = ac^{-h}, \tag{11}$$

where c is the same constant for all canyons in $G_\ell(f)$, as established in Lemma 3.2. □

Our next result refines in more depth the equality in Lemma 3.2, up to a level depending on the involved canyons. It plays a crucial role in Example 4.2.

Theorem 3.4. *Let $d \in \mathbb{Q}_+$ be the degree of the gradient canyon $\mathcal{GC}(\gamma_*)$.*
For any polar arc $\gamma' \in \mathcal{GC}(\gamma_*)$, we have:

$$\varphi_2(\gamma'(y), y) = P(y) + \phi(y)$$

where $P(y) = cy + \cdots$ is a certain finite sum of monomials with rational exponents less than $d-1$, which is well determined by f and g.

Proof. By Lemma 3.2 we have:

$$Y = \varphi_2\big(\gamma(y), y\big) = cy + \xi(y).$$

We claim that $\xi(y) = r_1 y^{\beta_1} + o(y^{\beta_1})$, where $r_1 \in \mathbb{C}$ and $\beta_1 \in \mathbb{Q}_+$, $\beta_1 > 1$, are well determined by the coefficients of the expansions of f and g displayed in (6), which are invariant in the respective canyons.

Let $h+q$ and $h+q'$ denote the second exponent in the expansion at the left hand side, and at the right hand side of (6), respectively, where $q, q' \in \mathbb{Q}_+$ and $q, q' < d-1$. Then (6) reads:

$$ay^h + by^{h+q} + \cdots + O(y^{h+d-1}) = a'Y^h + b'Y^{h+q'} + \cdots + O(Y^{h+d-1}) \tag{12}$$

By Lemma 3.2 we have $a' = a\frac{1}{c^h}$, in particular $Y - cy = \xi(y) = o(y)$. Then, by replacing Y in (12), we get:

$$ay^h + by^{h+q} + o(y^{h+q}) = a\frac{1}{c^h}(cy + \xi(y))^h + b'(cy + \xi(y))^{h+q'} + o(y^{h+q'}), \tag{13}$$

One develops:

$$\big(cy + \xi(y)\big)^h = c^h y^h + c^{h-1} h y^{h-1}\xi(y) + \text{h.o.t.},$$

and thus the right hand side of (13) becomes:

$$ay^h + a\frac{1}{c}hy^{h-1}\xi(y) + b'c^{h+q'}y^{h+q'} + o(y^{h+q'})$$

After reducing the first terms ay^h, the equality (13) becomes:

$$by^{h+q} + o(y^{h+q}) = a\frac{1}{c}hy^{h-1}\xi(y) + o\big(y^{h-1}\xi(y)\big) + b'c^{h+q'}y^{h+q'} + o(y^{h+q'}) \tag{14}$$

The study of this equality falls into 3 cases, as follows.

Case 1: $q < q'$. Then the limit $\lim_{y\to 0} \frac{\xi(y)}{y^{q+1}}$ is well defined, and we deduce:

$$\xi(y) = \frac{bc}{ah}y^{q+1} + o(y^{q+1}).$$

Case 2: $q = q'$. Then the limit $\lim_{y\to 0} \frac{\xi(y)}{y^{q+1}} = \frac{c(b - b'c^{h+q})}{ah}$ is well defined, and therefore:

$$\xi(y) = \frac{c(b - b'c^{h+q})}{ah}y^{q+1} + o(y^{q+1}).$$

Case 3: $q > q'$. Then $a\frac{1}{c}h\lim_{y\to 0}\frac{\xi(y)}{y^{q'+1}} + b'c^{h+q'} = 0$, and therefore

$$\xi(y) = \frac{-b'c^{h+q'+1}}{ah}y^{q'+1} + o(y^{q'+1})$$

This ends the proof of our claim on the function $\xi(y)$. Let us call "Step 1" this determination of its first term $r_1 y^{\beta_1}$ of $\xi(y)$. Next, we set:

$$Y = (cy + r_1 y^{\beta_1}) + \xi'(y)$$

and we claim that we have $\xi'(y) = r_2 y^{\beta_2} + o(y^{\beta_2})$, where $r_2 \in \mathbb{C}$, $\beta_2 \in \mathbb{Q}_+$, and $\beta_2 > \beta_1$.

Like in Step 1, we then do a similar case-by-case study of the equality (12) in which we replace Y. This will involve the first two terms of both sides. The determination of the first two terms will be called "Step 2", and we continue with another step. The last step k is defined by the last time when the highest exponent β_k may influence the terms $< d + h - 1$ of the expansions (12). This happens when, in the term $c^h y^h r_k \beta_k$ of the expansion of Y^h, the exponent of y is less than $h + d - 1$, and therefore we get $\beta_k < d - 1$.

Altogether, we have shown how to determine the polynomial expression $P(y)$ with rational coefficients. This finishes the proof of our theorem. □

Let γ_1 and γ_2 be polar arcs of f such that $\mathcal{GC}(\gamma_{1*}) \neq \mathcal{GC}(\gamma_{2*})$ and that $\mathcal{GC}(\gamma_{1*}), \mathcal{GC}(\gamma_{2*}) \in G_\ell(f)$. By Lemma 3.1, there are polars γ_1', γ_2' of g such that their canyons $\mathcal{GC}(\gamma'_{1*})$, $\mathcal{GC}(\gamma'_{2*})$ have the same degrees d_1, d_2 as $\mathcal{GC}(\gamma_{1*})$, $\mathcal{GC}(\gamma_{2*})$, respectively, and such that:

$$\varphi(\gamma_1(y), y) = (\varphi_1(\gamma_1(y), y), \varphi_2(\gamma_1(y), y)) = (\gamma_1'(Y_1) + O(Y_1^{d_1}), Y_1)$$

and

$$\varphi(\gamma_2(y), y) = (\varphi_1(\gamma_2(y), y), \varphi_2(\gamma_2(y), y)) = (\gamma_2'(Y_2) + O(Y_2^{d_2}), Y_2)$$

where Y_1, Y_2 are local variables.

By Lemma 3.2 we have:

$$Y_i = cy + o(y), \quad i = 1, 2, \tag{15}$$

where c is the same constant for all canyons in $G_\ell(f)$.

We are now prepared to introduce a new class of bi-Lipschitz invariants associated with pairs of gradient canyons.

3.3. **Higher invariants: second level.** We assume that there are at least two distinct gradient canyons contained in $G_\ell(f)$, $\mathcal{GC}(\gamma_{1*})$ and $\mathcal{GC}(\gamma_{2*})$, of canyon degrees d_1 and d_2, respectively. If $\delta \geq 1$ denotes the contact order between them, then this must be lower than both, i.e.:

$$\delta < \min\{d_1, d_2\}. \tag{16}$$

By [PT, Theorem 5.9], the corresponding canyons of g, denoted by $\mathcal{GC}(\gamma'_{1*})$ and $\mathcal{GC}(\gamma'_{2*})$, have the same degrees and the same contact order δ, cf §2.1 and §2.4.

In order to state the theorem, we now assume that the orders are equal:

$$\operatorname{ord} f(\gamma_1(y), y) = \operatorname{ord} f(\gamma_2(y), y),$$

and we introduce the following notation:

- a_1 is the coefficient of y^h in the expansion of $f(\gamma_1(y), y)$,
- a_2 is the coefficient of y^h in the expansion of $f(\gamma_2(y), y)$,
- $H = \operatorname{ord}\left[\frac{1}{a_1} f(\gamma_1(y), y) - \frac{1}{a_2} f(\gamma_2(y), y)\right]$,
- $\tilde{a}_1$ is the coefficient of y^H in the expansion of $\frac{1}{a_1} f(\gamma_1(y), y)$,
- $\tilde{a}_2$ is the coefficient of y^H in the expansion of $\frac{1}{a_2} f(\gamma_2(y), y)$.

With these notations, we have[2]:

Corollary 3.5. *Let $f, g : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ be holomorphic functions such that $f = g \circ \varphi$, where $\varphi : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ is a bi-Lipschitz homeomorphism and f is mini-regular in x. Let $h = \operatorname{ord} f(\gamma_1(y), y) = \operatorname{ord} f(\gamma_2(y), y)$. If*

$$H < h + \delta - 1, \tag{17}$$

then the effect of the bi-Lipschitz map φ on the pair $\big(H, (\tilde{a}_1 - \tilde{a}_2)\big)$ is the identity on H, and $(\tilde{a}_1 - \tilde{a}_2)$ multiplies by c^{h-H}.

Proof. By §2.1 and §2.2, we have the following expansions:

$$\begin{aligned} f(\gamma_1(y), y) &= a_1 y^h + \cdots + O(y^{h+d_1-1}), \\ f(\gamma_2(y), y) &= a_2 y^h + \cdots + O(y^{h+d_2-1}), \end{aligned} \tag{18}$$

where the terms of orders less than $h + d_1 - 1$ and $h + d_2 - 1$ are identical for every arc in $\mathcal{GC}(\gamma_{1*})$ and $\mathcal{GC}(\gamma_{2*})$, respectively. Similarly, after applying φ, by Lemma 3.2, we obtain:

$$\begin{aligned} (g \circ \varphi)(\gamma_1(y), y) &= g(\gamma_1'(Y_1), Y_1) + O(Y_1^{h+d_1-1}) &= b_1 Y_1^h + \cdots + O(Y_1^{h+d_1-1}), \\ (g \circ \varphi)(\gamma_2(y), y) &= g(\gamma_2'(Y_2), Y_2) + O(Y_2^{h+d_2-1}) &= b_2 Y_2^h + \cdots + O(Y_2^{h+d_2-1}), \end{aligned} \tag{19}$$

where Y_1, Y_2 are local variables, cf (15), and the terms of orders less than $h + d_1 - 1$ and $h + d_2 - 1$ are the same for every arc in $\mathcal{GC}(\gamma_{1*}')$ and $\mathcal{GC}(\gamma_{2*}')$, respectively. Since $f = g \circ \varphi$, by comparing (18) to (19), we get:

(20)
$$\begin{aligned} a_1 y^h + \cdots + a_1 \tilde{a}_1 y^H + \cdots + O(y^{h+d_1-1}) &= b_1 Y_1^h + \cdots + \bar{b}_1 Y_1^H + \cdots + O(Y_1^{h+d_1-1}), \\ a_2 y^h + \cdots + a_2 \tilde{a}_2 y^H + \cdots + O(y^{h+d_2-1}) &= b_2 Y_2^h + \cdots + \bar{b}_2 Y_2^H + \cdots + O(Y_2^{h+d_2-1}). \end{aligned}$$

As φ is bi-Lipschitz, we have:

$$|Y_1 - Y_2| \le \|\varphi(\gamma_1(y), y) - \varphi(\gamma_2(y), y)\| \sim \|(\gamma_1(y), y) - (\gamma_2(y), y)\|, \tag{21}$$

and since $\delta = \operatorname{ord}(\gamma_1(y) - \gamma_2(y)) \ge 1$, we then obtain:

$$Y_2 = Y_1 + O(Y_1^\delta).$$

[2]After the publication of this manuscript as an arXiv preprint, N. Nguyen pointed up to us his manuscript [Ng] containing a statement which seems equivalent to our Corollary 3.5.

We now use the assumption (17). By using this substitution, together with the inequality (16), and after dividing by the leading coefficients a_i, we may rewrite (20) as follows:

$$\begin{aligned} y^h + \cdots + \tilde{a}_1 y^H + \cdots + O(y^{h+\delta-1}) &= \frac{b_1}{a_1} Y_1^h + \cdots + \frac{\bar{b}_1}{a_1} Y_1^H + \cdots + O(Y_1^{h+\delta-1}), \\ y^h + \cdots + \tilde{a}_2 y^H + \cdots + O(y^{h+\delta-1}) &= \frac{b_2}{a_2} Y_1^h + \cdots + \frac{\bar{b}_2}{a_2} Y_1^H + \cdots + O(Y_1^{h+\delta-1}). \end{aligned} \tag{22}$$

By applying (15), we get $\frac{b_1}{a_1} = \frac{b_2}{a_2} = c^{-h}$, where c is a constant for all canyons in $G_\ell(f)$, as established in Lemma 3.2.

Now, by subtracting the second equation from the first, we obtain:

$$(\tilde{a}_1 - \tilde{a}_2)\, y^H + o(y^H) = \left(\frac{\bar{b}_1}{a_1} - \frac{\bar{b}_2}{a_2} \right) Y_1^H + o(Y_1^H) \tag{23}$$

where $(\tilde{a}_1 - \tilde{a}_2) \neq 0$ due to our assumption that $H > h$.

By using (15), we conclude that we must have the equality:

$$(\tilde{a}_1 - \tilde{a}_2) = c^H \left(\frac{\bar{b}_1}{a_1} - \frac{\bar{b}_2}{a_2} \right) = c^H c^{-h} \left(\frac{\bar{b}_1}{b_1} - \frac{\bar{b}_2}{b_2} \right) = c^{H-h} \left(\tilde{b}_1 - \tilde{b}_2 \right),$$

where

- $\tilde{b}_1$ denotes the coefficient of y^H in the expansion of $\frac{1}{b_1}(g \circ \varphi)(\gamma_1(y), y)$,
- $\tilde{b}_2$ denotes the coefficient of y^H in the expansion of $\frac{1}{b_2}(g \circ \varphi)(\gamma_2(y), y)$.

Conclusion: if the above condition $H < h + \delta - 1$ holds, then the action of the bi-Lipschitz map φ on the pair $\big(H, (\tilde{a}_1 - \tilde{a}_2)\big)$ is the identity on the first component H, and the multiplication of $(\tilde{a}_1 - \tilde{a}_2)$ by c^{h-H}. □

3.4. **Higher invariants: third level.**

Let us give the reccursive receipt for constructing higher level bi-Lipschitz invariants. We only consider here the third step, since the next ones are analogous. Each step hold upon a condition similar to (17), and the length of the chain of such higher invariant is limited by the size of the contact between canyons.

In the third step we need at least 3 pairwise disjoint gradient canyons $\mathcal{GC}(\gamma_{1*})$, $\mathcal{GC}(\gamma_{2*})$, $\mathcal{GC}(\gamma_{3*})$.

Let us assume the existence of three polar arcs $\gamma_i \in \mathcal{GC}(\gamma_{i*})$, such that we have the following coincidences of orders:

$$h := \operatorname{ord} f(\gamma_1(y), y) = \operatorname{ord} f(\gamma_2(y), y) = \operatorname{ord} f(\gamma_3(y), y), \tag{24}$$

and

$$H := \operatorname{ord} \left[\frac{1}{a_1} f(\gamma_1(y), y) - \frac{1}{a_2} f(\gamma_2(y), y) \right] = \operatorname{ord} \left[\frac{1}{a_3} f(\gamma_3(y), y) - \frac{1}{a_1} f(\gamma_1(y), y) \right]. \tag{25}$$

Let δ denote the contact order between $\mathcal{GC}(\gamma_{1*})$ and $\mathcal{GC}(\gamma_{2*})$, and let δ' denote the contact order between $\mathcal{GC}(\gamma_{1*})$ and $\mathcal{GC}(\gamma_{3*})$.

We divide each of the two above differences in (25) by $(\tilde{a}_1 - \tilde{a}_2)$, and $(\tilde{a}_3 - \tilde{a}_1)$, respectively, where $\tilde{a}_i$ denotes the coefficient of y^H in the expansion of $\frac{1}{a_i} f(\gamma_1(y), y)$.

The resulting expansions in y will then coincide before the term with a certain exponent $H' > H$. We also need that H' satisfies the condition:

$$H' < \min\{h + \delta - 1, h + \delta' - 1\}.$$

From this moment on we do exactly like in the proof of Thm 2.5, i.e., by subtracting one from the other. Denoting by A_{ij} the coefficient of $y^{H'}$ in the expansion of

$$(\tilde{a}_i - \tilde{a}_j)^{-1} \left[\frac{1}{a_i} f(\gamma_i(y), y) - \frac{1}{a_j} f(\gamma_j(y), y) \right],$$

for $(i, j) \in \{(1, 2), (3, 1)\}$, we have thus proved the following statement:

Corollary 3.6. *Let $f, g : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ be holomorphic functions such that $f = g \circ \varphi$, where $\varphi : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ is a bi-Lipschitz homeomorphism and f is mini-regular in x. Assume that the conditions (24) and (25) hold.*

If $H' < \min\{h + \delta - 1, h + \delta' - 1\}$, then the effect of the bi-Lipschitz map φ on $\big(H', (A_{12} - A_{31})\big)$ is the identity on H', and the multiplication of $(A_{12} - A_{31})$ by $c^{H-H'}$, where c is the non-zero constant provided by (15). □

4. Examples

Computing the invariants of Corollary 3.5 in the family of functions F_t of Example 4.1 displayed below shows that, for small enough generic parameters t and s, F_t and F_s are not bi-Lipschitz equivalent. In contrast, the invariants of Corollary 3.5 turn out to be not sufficient to settle the same question for the family G_t in Example 4.2. It is by the key application of Theorem 3.4 that we can decide that G_t and G_s are generically not bi-Lipschitz equivalent.

Example 4.1. Let us consider the family of function germs $F_t(x, y) = \frac{1}{3}x^3 - t^2 x y^{10} + y^{12}$, for t in a small neighbourhood of $0 \in \mathbb{C}$.

It appears that F_t has two polar arcs: $\gamma_1(y) = ty^5$ and $\gamma_2(y) = -ty^5$. By direct computations we get the canyon degrees $d_{\gamma_1} = d_{\gamma_2} = 6$, and their contact order $\delta := \operatorname{ord}(\gamma_1(y) - \gamma_2(y)) = 5$. This also shows that the two polars belong to different canyons, i.e. $\mathcal{GC}(\gamma_{1*}) \neq \mathcal{GC}(\gamma_{2*})$, with $\mathcal{GC}(\gamma_{1*}), \mathcal{GC}(\gamma_{2*}) \subset G_\ell(F_t)$, where $\ell = \{x = 0\}$. We also get:

$$F_t(\gamma_1(y), y) = y^{12} - \frac{2}{3}t^3 y^{15}, \quad F_t(\gamma_2(y), y) = y^{12} + \frac{2}{3}t^3 y^{15}, \tag{26}$$

thus $h = 12$, and the leading coefficients of $F_t(\gamma_1(y), y)$, $F_t(\gamma_2(y), y)$ are $a_1 = 1$, $a_2 = 1$.

Comparing to (26), we then get:

$$H := \operatorname{ord}\left[a_1^{-1} F_t(\gamma_1(y), y) - a_2^{-1} F_t(\gamma_2(y), y)\right] = 15.$$

We will follow the proof of Corollary 3.5 in order to contradict the assumption that there exists a bi-Lipschitz homeomorphism φ such that $F_t = F_s \circ \varphi$ for some parameters $s, t \in \mathbb{C}$ close enough to 0.

By Lemma 3.2 we have:

$$\varphi(\gamma_1(y), y) = (\gamma_1'(Y_1) + O(Y_1)^d, Y_1) \text{ and } \varphi(\gamma_2(y), y) = (\gamma_2'(Y_2) + O(Y_2^d), Y_2), \tag{27}$$

where $d := d_{\gamma_1} = d_{\gamma_2}$, and where Y_1, Y_2 are local variables, and γ_1', γ_2' are corresponding polar arcs of F_s. Since we have two options for the correspondence by φ between the pair of canyons of F_s and of F_t, the computation of the invariants of Corollary 3.5 falls into two cases.

Case 1. $\gamma_1'(Y_1) = sY_1^5$ and $\gamma_2'(Y_2) = -sY_2^5$.

Then by our assumption we have: $F_t(\gamma_i(y), y) = F_s(\varphi(\gamma_i(y), y)) = F_s(\gamma_i'(Y_i), Y_i)$, $i = 1, 2$, and by applying (27) and (20), we obtain the following equalities:

$$\begin{aligned} y^{12} - \frac{2}{3}t^3 y^{15} &= Y_1^{12} - \frac{2}{3}s^3 Y_1^{15}, \\ y^{12} + \frac{2}{3}t^3 y^{15} &= Y_2^{12} + \frac{2}{3}s^3 Y_2^{15}. \end{aligned} \tag{28}$$

In particular, the leading coefficients b_1 and b_2 of $F_s(\gamma_1'(Y_1), Y_1)$ and $F_s(\gamma_2'(Y_2), Y_2)$, respectively, are both equal to 1. At this moment, since $\frac{1}{1} = \frac{b_1}{a_1}$, we deduce that $c^{-12} = 1$.

On the other hand, since $\operatorname{ord}\big(\gamma_1(y) - \gamma_2(y)\big) = 5$, by (21) we get:

$$Y_2 = Y_1 + O(Y_1^5) = Y_1 + pY_1^5 + o(Y_1^5), \tag{29}$$

for some $p \in \mathbb{C}$. Applying the change of variables (29) in the equations of (28), with signs "$-$" and "$+$", respectively, yields:

$$\begin{aligned} y^{12} \pm \frac{2}{3}t^3 y^{15} &= \big(Y_1 + pY_1^5 + o(Y_1^5)\big)^{12} \pm \frac{2}{3}s^3\big(Y_1 + pY_1^5 + o(Y_1^5)\big)^{15} \\ &= Y_1^{12} \pm \frac{2}{3}s^3 Y_1^{15} + 12pY_1^{16} + o(Y_1^{16}). \end{aligned} \tag{30}$$

By subtracting one equation from the other, we get:

$$-\frac{4}{3}t^3 y^{15} = -\frac{4}{3}s^3 Y_1^{15} + o(Y_1^{15}). \tag{31}$$

Referring to (23), we have in our case, since $H = 15$:

$$\tilde{a}_1 - \tilde{a}_2 = -\frac{4}{3}t^3 \quad \tilde{b}_1 - \tilde{b}_2 = -\frac{4}{3}s^3.$$

where $\tilde{a}_1 = -\frac{2}{3}t^3$ and $\tilde{a}_2 = \frac{2}{3}t^3$ are the coefficients of $y^H = y^{15}$ in the expansions of $a_1^{-1}F_t(\gamma_1(y), y)$ and $a_2^{-1}F_t(\gamma_2(y), y)$, respectively, while $\tilde{b}_1 = -\frac{2}{3}s^3$ and $\tilde{b}_2 = \frac{2}{3}s^3$ are the corresponding coefficients of Y_1^{15} at the right hand side of (30).

By Corollary 3.5, we must have $\frac{\tilde{b}_1 - \tilde{b}_2}{\tilde{a}_1 - \tilde{a}_2} = c^{h-H}$, which in our case becomes $\frac{s^3}{t^3} = c^{-3}$, while we have seen before that $c^{-12} = 1$. The two equalities do not have a common solution c . This contradiction finishes our proof that, in Case 1, F_t is not bi-Lipschitz equivalent to F_s for a generic choice of values t and s of the parameter in our family.

Case 2. $\gamma_1'(Y_1) = -sY_1^5$ and $\gamma_2'(Y_2) = sY_2^5$.

The computations are faithfully similar to Case 1, with a difference of sign, for instance instead of (31) we get:

$$-\frac{4}{3}t^3y^{15} = \frac{4}{3}s^3Y_1^{15} + o(Y_1^{15}).$$

At the end we see that the constant c must satisfy two equalities, namely $-\frac{s^3}{t^3} = c^{-3}$ and $c^{-12} = 1$, which can happen only for t and s satisfying a certain relation. The same conclusion as in Case 1 then follows, i.e. that F_t is not bi-Lipschitz equivalent to F_s for a generic choice of the values t and s.

EXAMPLE 4.2. We consider now the family of function germs $G_t = x^3+y^{12}+xy^9+ty^{13}$, for t in a small neighbourhood of $0 \in \mathbb{C}$.

There are two polars, independent of t, namely:

$$\gamma_a(y) = ay^{\frac{9}{2}} \text{ and } \gamma_{-a}(y) = -ay^{\frac{9}{2}},$$

where $3a^2+1 = 0$, of contact order $\delta = \frac{9}{2}$, in different canyons $\mathcal{GC}(\gamma_a) \neq \mathcal{GC}(\gamma_{-a})$, both tangent to the line $\ell = \{x = 0\}$, and of canyon degrees $d(\gamma_a) = d(\gamma_{-a}) = \frac{13}{2}$. We also find easily $h = 12$.

We follow the same pattern as in the proof and the computations of the preceding example, namely we start by assuming that there is a bi-Lipschitz homeomorphism φ such that $G_t = G_s \circ \varphi$. Here too, there are two options for corresponding the two canyons of G_t and G_s by the bi-Lipschitz homeomorphism φ.

By Theorem 3.4, we have:

$$Y = y + \frac{t-s}{12}y^2 + \text{h.o.t.} \tag{32}$$

In one of the cases, call it Case 1, we get $H = \frac{27}{2}$, $\quad \tilde{a}_1 - \tilde{a}_2 = \frac{4a}{3}$, and the condition $H < h + \delta - 1 = \frac{31}{2}$ holds.

After some more steps in the algorithm displayed in Example 4.1, we get the following equality similar to (31):

$$\frac{4a}{3}y^{\frac{27}{2}} = \frac{4a}{3}Y^{\frac{27}{2}} - 12pY^{\frac{31}{2}} + \text{h.o.t.} \tag{33}$$

where $p \in \mathbb{C}$ comes from the relation $Y_1 = Y + pY^{\frac{9}{2}} + \text{h.o.t.}$ similar to (29).

Pursuing the algorithm, we find (cf Corollary 3.5) that the effect of the bi-Lipschitz map on the pair $\big(H, (\tilde{a}_1 - \tilde{a}_2)\big)$ is the identity on H, and the multiplication of $(\tilde{a}_1 - \tilde{a}_2)$ by $c^{h-H} = c^{-\frac{3}{2}} = 1$. Note that (32) tells that $c = 1$. Up to this point, we do not get any contradiction, thus no proof or disproof of our assumption.

To solve the dilemma, we need to make more efficient use of Theorem 3.4 and thus of the relation (32), as follows. Replacing (32) into the right-hand side of (33), yields the equality:

$$\frac{4a}{3}y^{\frac{27}{2}} = \frac{4a}{3}y^{\frac{27}{2}} + \frac{4a}{3}\cdot\frac{27}{2}\cdot\frac{(t-s)}{12}y^{\frac{29}{2}} - 12py^{\frac{31}{2}} + \text{h.o.t.}$$

which is impossible if $t \neq s$. Therefore the conclusion of Case 1 is that G_t is not bi-Lipschitz equivalent to G_s for any choice $s \neq t$.

The Case 2 is faithfully similar, along the same algorithm. It leads to the condition $c^{h-H} = c^{-\frac{3}{2}} = -1$, which has to be compared to $c = 1$ provided by (32), thus no solution for c. We may draw the conclusion in this case, and actually the final conclusion of the example, as follows: G_t is not Lipschitz equivalent to G_s for any choice of $s \neq t$.

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