EXTENSION-CLOSED SUBCATEGORIES OVER HYPERSURFACES OF FINITE OR COUNTABLE CM-REPRESENTATION TYPE

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Dedicated to Professor Yuji Yoshino on the occasion of his seventieth birthday

Abstract. Let k be an algebraically closed uncountable field of characteristic zero. Let R be a complete local hypersurface over k. Denote by $\mathrm{CM}(R)$ the category of maximal Cohen–Macaulay R-modules and by $\mathrm{D^{sg}}(R)$ the singularity category of R. Denote by $\mathrm{CM}_0(R)$ the full category of $\mathrm{CM}(R)$ consisting of modules that are locally free on the punctured spectrum of R, and by $\mathrm{D}^{\mathrm{sg}}_0(R)$ the full subcategory of $\mathrm{D^{sg}}(R)$ consisting of objects that are locally zero on the punctured spectrum of R. In this paper, under the assumption that R has finite or countable CM-representation type, we completely classify the extension-closed subcategories of $\mathrm{CM}_0(R)$ in dimension at most two, and the extension-closed subcategories of $\mathrm{D}^{\mathrm{sg}}_0(R)$ in arbitrary dimension.

Contents

1. Introduction

Let R be a local hypersurface ring. By virtue of [9, 11] there are one-to-one correspondences between:

- the resolving subcategories of $\operatorname{mod} R$ contained in $\mathrm{CM}(R)$,
- the thick subcategories of $\mathrm{D^{sg}}(R)$,
- the specialization-closed subsets of the singular locus of R,

where $\operatorname{mod} R$, $\mathrm{CM}(R)$, and $\mathrm{D^{sg}}(R)$ respectively stand for the category of finitely generated R-modules, the full subcategory of maximal Cohen–Macaulay R-modules, and the singularity category of R, i.e., the Verdier quotient of the bounded derived category of $\operatorname{mod} R$ by perfect complexes. The bijections are explicitly given, which leads to complete classifications of the resolving subcategories and thick subcategories mentioned above.

An *extension-closed subcategory* of an extriangulated category $\mathcal{C}$ (in the sense of [7]) is defined to be a full subcategory $\mathcal{X}$ of $\mathcal{C}$ which is closed under direct summands and such that for each conflation $X \to Y \to Z$ in $\mathcal{C}$, if X and Z belong to $\mathcal{X}$, then Y also belongs to $\mathcal{X}$. Note by definition that both a resolving subcategory of $\operatorname{mod} R$ and a thick subcategory of $\mathrm{D^{sg}}(R)$ are extension-closed subcategories.

Denote by $\mathrm{CM}_0(R)$ the full subcategory of $\mathrm{CM}(R)$ consisting of maximal Cohen–Macaulay modules which are locally free on the punctured spectrum of R, and by $\mathrm{D}^{\mathrm{sg}}_0(R)$ the full subcategory of $\mathrm{D^{sg}}(R)$ consisting of objects which are locally zero on the punctured spectrum of R. Note that the equalities $\mathrm{CM}_0(R) = \mathrm{CM}(R)$ and $\mathrm{D}^{\mathrm{sg}}_0(R) = \mathrm{D^{sg}}(R)$ hold whenever R has an isolated singularity. The classification theorem stated above implies that, if R is a local hypersurface ring, then there exist only trivial resolving subcategories of $\operatorname{mod} R$ contained in $\mathrm{CM}_0(R)$ and there exist only trivial thick subcategories of $\mathrm{D^{sg}}(R)$ contained in $\mathrm{D}^{\mathrm{sg}}_0(R)$. This fact motivates us to classify the extension-closed subcategories of $\mathrm{CM}_0(R)$ and $\mathrm{D}^{\mathrm{sg}}_0(R)$.

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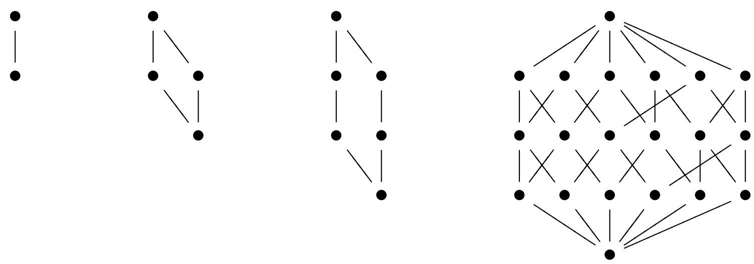
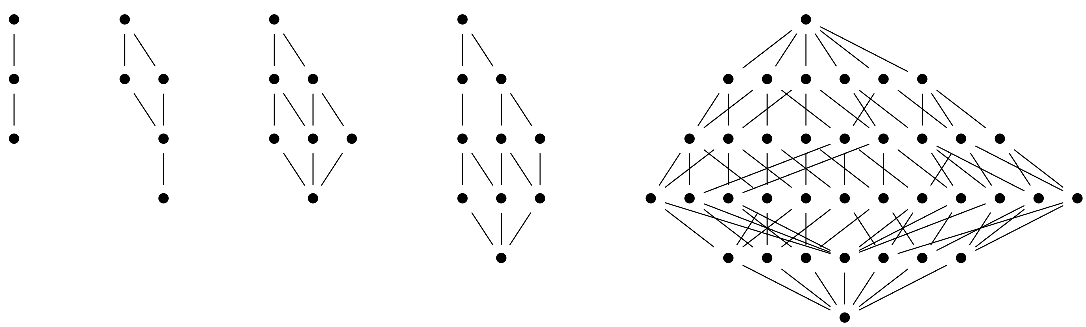
In the present paper, we shall prove the following theorem. The separation of the theorem into two parts is based on the fact that Knörrer's periodicity allows one to describe $\mathrm{D}_0^{\mathrm{sg}}(R)$ in all dimensions, but for $\mathrm{CM}_0(R)$ in higher dimensions it is not clear which extension-closed subcategories contain R.

Theorem 1.1. *Let k be an algebraically closed uncountable field of characteristic 0. Let R be a singular complete local hypersurface ring with residue field k. Suppose that R has either finite or countable CM-representation type.*

(1) *The Hasse diagram of the partially ordered set of extension-closed subcategories of $\mathrm{D}_0^{\mathrm{sg}}(R)$ with respect to the inclusion relation is one of the following four graphs.*



(2) *If R has Krull dimension at most two, then the Hasse diagram of the partially ordered set of extension-closed subcategories of $\mathrm{CM}_0(R)$ with respect to the inclusion relation is one of the following five graphs.*



Note that $0, \operatorname{add} R, \mathrm{CM}_0(R)$ are always extension-closed subcategories of $\mathrm{CM}_0(R)$, and that $0, \mathrm{D}_0^{\mathrm{sg}}(R)$ are always extension-closed subcategories of $\mathrm{D}_0^{\mathrm{sg}}(R)$. Hence the first diagrams in the two assertions of the above theorem mean that there exist only trivial extension-closed subcategories.

The organization of this paper is as follows. In Section 2, we collect definitions and lemmas which are used in later sections. In Sections 3 and 4, we respectively deal with finite CM-representation type and countable CM-representation type in dimension at most two. In the final Section 5, we give proofs of Theorem 1.1.

2. Basic definitions and fundamental lemmas

This section is devoted to stating basic definitions and proving fundamental lemmas for later use. First of all, let us provide our convention.

Convention 2.1. Let k be an algebraically closed field of characteristic zero, so $\sqrt{-1} \in k$. We assume that all rings are commutative and noetherian, all modules are finitely generated, and all subcategories are nonempty and strictly full. Let R be a (commutative noetherian) ring. We denote by $\operatorname{mod} R$ the category of (finitely generated) R-modules, by $\mathrm{CM}(R)$ the (strictly full) subcategory of $\operatorname{mod} R$ consisting of maximal Cohen–Macaulay R-modules (we regard the zero R-module 0 as maximal Cohen–Macaulay, so that 0 belongs to $\mathrm{CM}(R)$), and by $\mathrm{D}^{\mathrm{sg}}(R)$ the singularity category of R, which is defined as the Verdier quotient of the bounded derived category $\mathrm{D}^{\mathrm{b}}(\operatorname{mod} R)$ of $\operatorname{mod} R$ by perfect complexes. For each $\mathrm{P} \in \{\mathrm{A}_n, \mathrm{A}_\infty, \mathrm{D}_n, \mathrm{D}_\infty, \mathrm{E}_6, \mathrm{E}_7, \mathrm{E}_8\}$ we write P^d to indicate that the dimension of the P-singularity considered is d. We may omit a subscript or a superscript when it is clear from the context.

We recall the notion of an exact square which plays an essential role in the proofs of our main results.

Definition 2.2. A commutative diagram of homomorphisms of R-modules

$$\begin{array}{ccc} A & \xrightarrow{a} & B \\ \downarrow{\scriptstyle c} & & \downarrow{\scriptstyle b} \\ C & \xrightarrow{d} & D \end{array}$$

is called an *exact square* if it is a both pushout and pullback diagram, or in other words, if the sequence $0 \to A \xrightarrow{\binom{a}{c}} B \oplus C \xrightarrow{(-b,d)} D \to 0$ is exact.

It can be verified directly that the following lemma holds true; see [12, Lemma 2.2(2)].

Lemma 2.3. *Let* (1) *and* (2) *below be exact squares. Then* (3) *below is an exact square as well.*

$$\begin{array}{ccccc} A & \xrightarrow{a} & B & \xrightarrow{b} & C \\ \downarrow{\scriptstyle d} & (1) & \downarrow{\scriptstyle g} & (2) & \downarrow{\scriptstyle c} \\ D & \xrightarrow{e} & E & \xrightarrow{f} & F \end{array} \qquad\qquad \begin{array}{ccc} A & \xrightarrow{ba} & C \\ \downarrow{\scriptstyle d} & (3) & \downarrow{\scriptstyle c} \\ D & \xrightarrow{fe} & F \end{array}$$

In other words, if the sequences $0 \to A \xrightarrow{\binom{a}{d}} B \oplus D \xrightarrow{(-g,e)} E \to 0$ *and* $0 \to B \xrightarrow{\binom{b}{g}} C \oplus E \xrightarrow{(-c,f)} F \to 0$ *are exact, then so is the sequence* $0 \to A \xrightarrow{\binom{ba}{d}} C \oplus D \xrightarrow{(-c,fe)} F \to 0$.

The following elementary lemma will also be necessary.

Lemma 2.4. (1) *If* $0 \to A \xrightarrow{\binom{f}{g}} B \oplus C \xrightarrow{(h,l)} D \to 0$ *and* $0 \to E \xrightarrow{m} A \xrightarrow{f} B \to 0$ *are exact sequences in* $\operatorname{mod} R$*, then there exists an exact sequence* $0 \to E \xrightarrow{gm} C \xrightarrow{l} D \to 0$ *in* $\operatorname{mod} R$.

(2) *If* $0 \to A \xrightarrow{\binom{h}{l}} B \oplus C \xrightarrow{(f,g)} D \to 0$ *and* $0 \to C \xrightarrow{g} D \xrightarrow{m} E \to 0$ *are exact sequences in* $\operatorname{mod} R$*, then there exists an exact sequence* $0 \to A \xrightarrow{h} B \xrightarrow{mf} E \to 0$ *in* $\operatorname{mod} R$.

Proof. In the situations of (1) and (2), there respectively exist pullback and pushout diagrams

$$\begin{array}{ccccccccc} 0 & \to & E & \xrightarrow{m} & A & \xrightarrow{f} & B & \to & 0 \\ & & \| & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle -h} & & \\ 0 & \to & E & \xrightarrow{gm} & C & \xrightarrow{l} & D & \to & 0 \end{array} \qquad\qquad \begin{array}{ccccccccc} 0 & \to & A & \xrightarrow{h} & B & \xrightarrow{mf} & E & \to & 0 \\ & & \downarrow{\scriptstyle -l} & & \downarrow{\scriptstyle f} & & \| & & \\ 0 & \to & C & \xrightarrow{g} & D & \xrightarrow{m} & E & \to & 0 \end{array}$$

which show the assertions of the lemma. ■

We introduce our two main ambient categories.

Definition 2.5. Let R be a local ring with maximal ideal $\mathfrak{m}$. Denote by $\mathrm{D}_0^{\mathrm{sg}}(R)$ the subcategory of $\mathrm{D}^{\mathrm{sg}}(R)$ consisting of objects that are locally zero on the punctured spectrum of R, and by $\mathrm{CM}_0(R)$ the subcategory of $\mathrm{CM}(R)$ consisting of R-modules that are locally free on the punctured spectrum of R. Thus:

$$\mathrm{D}_0^{\mathrm{sg}}(R) = \{X \in \mathrm{D}^{\mathrm{sg}}(R) \mid X_{\mathfrak{p}} \cong 0 \text{ for all } \mathfrak{m} \neq \mathfrak{p} \in \operatorname{Spec} R\},$$
$$\mathrm{CM}_0(R) = \{M \in \mathrm{CM}(R) \mid M_{\mathfrak{p}} \text{ is } R_{\mathfrak{p}}\text{-free for all } \mathfrak{m} \neq \mathfrak{p} \in \operatorname{Spec} R\}.$$

Remark 2.6. Let R be a local ring. Then R has an isolated singularity if and only if $\mathrm{D}_0^{\mathrm{sg}}(R) = \mathrm{D}^{\mathrm{sg}}(R)$. When R is a Cohen–Macaulay local ring, R has an isolated singularity if and only if $\mathrm{CM}_0(R) = \mathrm{CM}(R)$.

Now we give the precise definition of an extension-closed subcategory. Note that in our sense an extension-closed subcategory is necessarily closed under direct summands.

Definition 2.7. Let $\mathcal{C}$ be an additive category.

(1) We say that a subcategory $\mathcal{X}$ of $\mathcal{C}$ is *closed under finite direct sums* provided that if $X_1, \dots, X_n$ are a finite number of objects of $\mathcal{C}$ that belong to $\mathcal{X}$, then the direct sum $X_1 \oplus \cdots \oplus X_n$ belongs to $\mathcal{X}$ as well.
(2) We say that a subcategory $\mathcal{X}$ of $\mathcal{C}$ is *closed under direct summands* provided that if X is an object of $\mathcal{C}$ that belongs to $\mathcal{X}$ and Y is an object of $\mathcal{C}$ which is a direct summand of X, then Y belongs to $\mathcal{X}$.
(3) An *additively closed* subcategory of $\mathcal{C}$ is defined to be a subcategory of $\mathcal{C}$ which is closed under finite direct sums and direct summands.

(4) Assume that $\mathcal{C}$ is extriangulated in the sense of [7]. Let $\mathcal{X}$ be a subcategory of $\mathcal{C}$ closed under direct summands. We say that $\mathcal{X}$ is *extension-closed* provided that for every conflation $L \to M \to N$ in $\mathcal{C}$, if L and N belong to $\mathcal{X}$, then so does M.

Remark 2.8. (1) Since $\mathrm{D}_0^{\mathrm{sg}}(R)$ is a triangulated category, it is an extriangulated category. A subcategory $\mathcal{X}$ of $\mathrm{D}_0^{\mathrm{sg}}(R)$ closed under direct summands is extension-closed if and only if for each exact triangle $L \to M \to N \rightsquigarrow$ in $\mathrm{D}_0^{\mathrm{sg}}(R)$ with $L, N \in \mathcal{X}$ one has $M \in \mathcal{X}$.

(2) Let R be a Cohen–Macaulay local ring. Then $\mathrm{CM}_0(R)$ is an exact category whose conflations are the short exact sequences of maximal Cohen–Macaulay R-modules that are locally free on the punctured spectrum. Hence $\mathrm{CM}_0(R)$ is an extriangulated category. A subcategory $\mathcal{X}$ of $\mathrm{CM}_0(R)$ closed under direct summands is extension-closed if and only if for an exact sequence $0 \to L \to M \to N \to 0$ of modules in $\mathrm{CM}_0(R)$ with $L, N \in \mathcal{X}$ one has $M \in \mathcal{X}$.

Next we introduce some notations.

Definition 2.9. Let $\mathcal{C}$ be an essentially small additive category, and let $\mathcal{X}$ be a subcategory of $\mathcal{C}$.

(1) We denote by $\operatorname{ind}_{\mathcal{C}} \mathcal{X}$ the set of isomorphism classes of indecomposable objects in $\mathcal{X}$.

(2) Denote by $\operatorname{add}_{\mathcal{C}} \mathcal{X}$ the *additive closure* of $\mathcal{X}$, which is defined as the smallest additively closed subcategory of $\mathcal{C}$ containing $\mathcal{X}$. Hence $\operatorname{add}_{\mathcal{C}} \mathcal{X}$ consists of the direct summands of finite direct sums of objects in $\mathcal{X}$.

(3) Assume that $\mathcal{C}$ is extriangulated. We denote by $\operatorname{ext}_{\mathcal{C}} \mathcal{X}$ the *extension closure* of $\mathcal{X}$, which is by definition the smallest extension-closed subcategory of $\mathcal{C}$ containing $\mathcal{X}$.

When $\mathcal{X}$ is the subcategory of $\mathcal{C}$ defined by a single object X, we may write $\operatorname{add}_{\mathcal{C}} X$ and $\operatorname{ext}_{\mathcal{C}} X$ instead of $\operatorname{add}_{\mathcal{C}} \mathcal{X}$ and $\operatorname{ext}_{\mathcal{C}} \mathcal{X}$, respectively. Note that $\operatorname{add}_{\mathcal{C}} X$ consists of the direct summands of finite direct sums of copies of X. Let $\mathfrak{E}(R), \mathfrak{E}_0(R)$ be the sets of extension-closed subcategories of $\mathrm{CM}(R), \mathrm{CM}_0(R)$ respectively. Note that $\mathfrak{E}(R), \mathfrak{E}_0(R)$ are posets (partially ordered sets) with respect to the inclusion relation $\subseteq$.

We pose a natural question regarding the set $\mathfrak{E}_0(R)$. Applying our main results, we will give some answers to this question in Remark 4.4. Note that this question is closely related to the problem studied intensively in [10] in the case of artinian local rings. It is also worth pointing out that if the question is affirmative, then for such a ring R as in the question, the extension-closed subcategories of $\mathrm{D}_0^{\mathrm{sg}}(R)$ are the trivial ones: 0 and $\mathrm{D}_0^{\mathrm{sg}}(R)$.

Question 2.10. Let $R = k[\![x_0, x_1, \dots, x_d]\!]/(f)$ be a complete local hypersurface domain over a field k with dimension at most two. Then does it hold that $\mathfrak{E}_0(R) = \{0, \operatorname{add} R, \mathrm{CM}_0(R)\}$?

We recall the definitions of finite and countable CM-representation types.

Definition 2.11. Let R be a Cohen–Macaulay local ring. We say that R has *finite CM-representation type* if $\operatorname{ind} \mathrm{CM}(R)$ is finite. We say that R has *countable CM-representation type* if $\operatorname{ind} \mathrm{CM}(R)$ is countably infinite.

The following lemma states some properties of additive and extension closures which will often be used.

Lemma 2.12. (1) *Let $0 \to L \to M \to N \to 0$ be an exact sequence of R-modules. Let $\mathcal{X}$ be a subcategory of* $\operatorname{mod} R$. *Suppose that L, N belong to* $\operatorname{add} \mathcal{X}$. *Then there exists an exact sequence $0 \to A \to B \to C \to 0$ of R-modules such that A, C are finite direct sums of modules in $\mathcal{X}$ and M is a direct summand of B.*

(2) *Let M and N be R-modules, and let S be a multiplicatively closed subset of R. Suppose that M belongs to the extension closure* $\operatorname{ext}_{\operatorname{mod} R} N$. *Then the localization M_S belongs to the extension closure* $\operatorname{ext}_{\operatorname{mod} R_S} N_S$.

(3) *Let R be a Gorenstein local ring. Let $\mathcal{X}$ and $\mathcal{Y}$ be subcategories of* $\mathrm{CM}(R)$. *If the equality* $\operatorname{ext} \mathcal{X} = \operatorname{add} \mathcal{Y}$ *holds, then the equality* $\operatorname{ext}(\mathcal{X} \cup \{R\}) = \operatorname{add}(\mathcal{Y} \cup \{R\})$ *holds as well.*

Proof. (1) There exist R-modules L', N' and finite direct sums A, C of R-modules in $\mathcal{X}$ such that $L \oplus L' \cong A$ and $N \oplus N' \cong C$. Taking the direct sum with the trivial exact sequences $0 \to L' \to L' \to 0 \to 0$ and $0 \to 0 \to N' \to N' \to 0$, we get an exact sequence $0 \to L \oplus L' \to L' \oplus M \oplus N' \to N \oplus N' \to 0$. Setting $B = L' \oplus M \oplus N'$, we obtain such an exact sequence $0 \to A \to B \to C \to 0$ as in the assertion.

(2) Let $\mathcal{X}$ be the subcategory of $\operatorname{mod} R$ consisting of R-modules X such that $X_S \in \operatorname{ext}_{\operatorname{mod} R_S} N_S$. Then $\mathcal{X}$ contains N. Let Z be an R-module in $\mathcal{X}$ and W is a direct summand of Z. Then Z_S is in $\operatorname{ext}_{\operatorname{mod} R_S} N_S$ and W_S is a direct summand of Z_S. Hence W_S is in $\operatorname{ext}_{\operatorname{mod} R_S} N_S$, so that W is in $\mathcal{X}$. Let $0 \to X \to E \to Y \to 0$ be an exact sequence in $\operatorname{mod} R$ such that $X, Y \in \mathcal{X}$. Then there is an exact sequence $0 \to X_S \to E_S \to Y_S \to 0$ in $\operatorname{mod} R_S$ and X_S, Y_S belong to $\operatorname{ext}_{\operatorname{mod} R_S} N_S$. Hence E_S belongs to $\operatorname{ext}_{\operatorname{mod} R_S} N_S$, and we see that E is in

$\mathcal{X}$. The subcategory $\mathcal{X}$ of $\operatorname{mod} R$ contains N and is extension-closed. It follows that $\operatorname{ext} N$ is contained in $\mathcal{X}$. Since M is in $\operatorname{ext} N$, we observe that M_S is in $\operatorname{ext}_{\operatorname{mod} R_S} N_S$.

(3) We have $R \in \operatorname{add}(\mathcal{Y} \cup \{R\}) \supseteq \operatorname{add} \mathcal{Y} = \operatorname{ext} \mathcal{X} \supseteq \mathcal{X}$. Hence $\operatorname{add}(\mathcal{Y} \cup \{R\})$ contains $\mathcal{X} \cup \{R\}$, and is closed under direct summands by definition. Let $0 \to L \to M \to N \to 0$ be an exact sequence in $\operatorname{mod} R$ with $L, N \in \operatorname{add}(\mathcal{Y} \cup \{R\})$. By assertion (1), we get an exact sequence $0 \to A \oplus R^{\oplus a} \to B \to C \oplus R^{\oplus c} \to 0$ with $a, c \in \mathbb{N}$ and $A, C \in \operatorname{add} \mathcal{Y}$ such that M is a direct summand of B. We have pushout and pullback diagrams:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & R^{\oplus a} & = & R^{\oplus a} & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & A \oplus R^{\oplus a} & \to & B & \to & C \oplus R^{\oplus c} & \to & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \to & A & \to & V & \to & C \oplus R^{\oplus c} & \to & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
\qquad
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & A & \to & W & \to & C & \to & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \to & A & \to & V & \to & C \oplus R^{\oplus c} & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & R^{\oplus c} & = & R^{\oplus c} & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

As R is Gorenstein and V is maximal Cohen–Macaulay, the middle column in the left diagram splits and we get $B \cong R^{\oplus a} \oplus V$. The middle column in the right diagram splits, so $V \cong W \oplus R^{\oplus c}$. Hence $B \cong W \oplus R^{\oplus (a+c)}$. As $\operatorname{add} \mathcal{Y} = \operatorname{ext} \mathcal{X}$ is extension-closed and contains A, C, the first row in the right diagram shows $W \in \operatorname{add} \mathcal{Y}$. Therefore, B is in $\operatorname{add}(\mathcal{Y} \cup \{R\})$, and so is M. We have shown that $\operatorname{add}(\mathcal{Y} \cup \{R\})$ contains $\operatorname{ext}(\mathcal{X} \cup \{R\})$. Let $\mathcal{Z}$ be an extension-closed subcategory of $\operatorname{mod} R$ containing $\mathcal{X} \cup \{R\}$. Since $\mathcal{Z}$ contains $\mathcal{X}$ and is extension-closed, $\mathcal{Z}$ contains $\operatorname{ext} \mathcal{X} = \operatorname{add} \mathcal{Y}$. Hence $\mathcal{Z}$ contains $\operatorname{add}(\mathcal{Y} \cup \{R\})$. It follows that $\operatorname{add}(\mathcal{Y} \cup \{R\}) = \operatorname{ext}(\mathcal{X} \cup \{R\})$. ■

Next we recall some notions from commutative algebra.

Definition 2.13. Let R be a local ring.

(1) We say that R is a *hypersurface* if it satisfies the inequality $\operatorname{edim} R - \operatorname{depth} R \leqslant 1$, where $\operatorname{edim} R$ and $\operatorname{depth} R$ stand for the embedding dimension of R and the depth of R, respectively.
(2) Let M be an R-module. For an integer $n > 0$, we denote by $\Omega^n M$ the *nth syzygy* of M, that is, the image of the nth differential map in a minimal free resolution of M. We put $\Omega M = \Omega^1 M$ and $\Omega^0 M = M$.
(3) An R-module M is called *rigid* if $\operatorname{Ext}^1_R(M, M) = 0$.

Remark 2.14. (1) A local ring $(R, \mathfrak{m})$ is a hypersurface if and only if the $\mathfrak{m}$-adic completion of R is isomorphic to $S/(f)$ for some regular local ring $(S, \mathfrak{n})$ and some element $f \in \mathfrak{n}$; see [1, 5.1].
(2) Let R be a hypersurface. Let M be a nonfree maximal Cohen–Macaulay R-module with no free summand. Then there is an isomorphism $\Omega^2 M \cong M$; see [1, 5.1.1 and 5.1.2].
(3) For an R-module M and an integer $n \geqslant 0$, the nth syzygy $\Omega^n M$ is uniquely determined up to isomorphism, since so is a minimal free resolution of M.

We investigate the relationship between extension closures and syzygies over a local ring.

Lemma 2.15. *Let $(R, \mathfrak{m}, k)$ be a local ring of (Krull) dimension d. Let M be an R-module.*

(1) *Let N be an R-module. If N belongs to $\operatorname{ext} M$, then ΩN belongs to $\operatorname{ext} \Omega M$.*
(2) *Suppose that the local ring R is Gorenstein and singular, and that the R-module M is maximal Cohen–Macaulay. If $\operatorname{ext} M = \operatorname{CM}_0(R)$, then $\operatorname{ext} \Omega^i M = \operatorname{CM}_0(R)$ for all integers $i \geqslant 0$.*

Proof. (1) Let $\mathcal{X}$ be the subcategory of $\operatorname{mod} R$ consisting of R-modules X such that ΩX is in $\operatorname{ext} \Omega M$. Then M is in $\mathcal{X}$. Let X be an R-module in $\mathcal{X}$ and Y a direct summand of X. Then ΩY is a direct summand of ΩX. Since ΩX is in $\operatorname{ext} \Omega M$, so is ΩY. Therefore, Y belongs to $\mathcal{X}$. Let $0 \to A \to B \to C \to 0$ be an exact sequence of R-modules with $A, C \in \mathcal{X}$. Then there is an exact sequence $0 \to \Omega A \to \Omega B \oplus R^{\oplus n} \to \Omega C \to 0$ with $n \in \mathbb{N}$. Since $\Omega A, \Omega C$ are in $\operatorname{ext} \Omega M$, so is ΩB. Hence B is in $\mathcal{X}$. Thus, $\mathcal{X}$ is extension-closed and contains M. It follows that $\mathcal{X}$ contains $\operatorname{ext} M$, and N belongs to $\mathcal{X}$, which means that ΩN is in $\operatorname{ext} \Omega M$.

(2) We may assume $i = 1$. The module M is in $\operatorname{ext} M = \operatorname{CM}_0(R)$, so that ΩM is also in $\operatorname{CM}_0(R)$, whence $\operatorname{ext} \Omega M \subseteq \operatorname{CM}_0(R)$. As R is Gorenstein and M is maximal Cohen–Macaulay, there exists a maximal Cohen–Macaulay R-module N such that $\Omega^d k \cong \Omega N \oplus R^{\oplus n}$ for some $n \in \mathbb{N}$. As R is singular, we have $n = 0$ by [4, Corollary 1.3]. Since $\Omega^d k$ belongs to $\operatorname{CM}_0(R)$, so does ΩN. Since N is a maximal Cohen–Macaulay module

over a Gorenstein ring R, it is observed that N is in $\mathrm{CM}_0(R)$. As $\mathrm{CM}_0(R) = \operatorname{ext} M$, we see from (1) that ΩN belongs to $\operatorname{ext} \Omega M$. Hence $\Omega^d k$ is in $\operatorname{ext} \Omega M$. It follows from [9, Corollary 2.6] that $\operatorname{ext} \Omega M = \mathrm{CM}_0(R)$. ■

We study extension-closedness of additive closures of rigid maximal Cohen–Macaulay modules.

Lemma 2.16. *Let R be a Cohen–Macaulay local ring. Let M be a rigid maximal Cohen–Macaulay R-module.*

(1) *The additive closure $\operatorname{add} M$ is an extension-closed subcategory of $\mathrm{CM}(R)$.*
(2) *If R is Gorenstein, then $\operatorname{add}\{R, M\}$ is an extension-closed subcategory of $\mathrm{CM}(R)$.*
(3) *If M has no nonzero free summand, then $\operatorname{add} M \neq \operatorname{add}\{R, M\}$.*

Proof. (1) By definition, $\operatorname{add} M$ is closed under direct summands. Let $0 \to X \to Y \to Z \to 0$ be an exact sequence of maximal Cohen–Macaulay R-modules, and suppose that X and Z are in $\operatorname{add} M$. By Lemma 2.12(1) there is an exact sequence $\sigma : 0 \to M^{\oplus a} \to W \to M^{\oplus b} \to 0$ in $\operatorname{mod} R$ with $a, b \in \mathbb{N}$ such that Y is a direct summand of W. Since M is rigid, $\operatorname{Ext}^1_R(M^{\oplus b}, M^{\oplus a}) = 0$. Hence the short exact sequence σ splits, and W is isomorphic to $M^{\oplus(a+b)}$. Therefore, Y belongs to $\operatorname{add} M$. It follows that $\operatorname{add} M$ is extension-closed.

(2) If R is Gorenstein, then the maximal Cohen–Macaulay R-module $R \oplus M$ is rigid. It follows from (1) that $\operatorname{add}\{R, M\} = \operatorname{add}(R \oplus M)$ is an extension-closed subcategory of $\mathrm{CM}(R)$.

(3) Assume $\operatorname{add} M = \operatorname{add}\{R, M\}$. Then R is a direct summand of a finite direct sum of copies of M. By [6, Corollaries 1.10 and 1.15], we see that R is a direct summand of M. Thus the assertion follows. ■

The following lemma will be used in the case where S is a regular local ring.

Lemma 2.17. *Let S be a local ring. Let x, y be a regular sequence on S. Take the quotient ring $R = S/(xy)$. Then the R-modules $R/(x)$ and $R/(y)$ are rigid.*

Proof. Using the fact that the elements x and y are regular on S, we observe that the equalities $0 :_R x = yR$ and $0 :_R y = xR$ hold true. This implies that the sequence $\cdots \xrightarrow{y} R \xrightarrow{x} R \xrightarrow{y} R \xrightarrow{x} R \to R/xR \to 0$ is exact. We have $\operatorname{Ext}^1_R(R/xR, R/xR) = \mathrm{H}^1(0 \to R/xR \xrightarrow{0} R/xR \xrightarrow{y} R/xR \xrightarrow{0} R/xR \xrightarrow{y} \cdots) = 0$, since y is regular on $S/xS = R/xR$. Hence R/xR is a rigid R-module. By symmetry, the R-module R/yR is also rigid. ■

Here we recall the definitions of the stable category and cosyzygies of maximal Cohen–Macaulay modules.

Definition 2.18. (1) Let R be a Cohen–Macaulay local ring. Let $\underline{\mathrm{CM}}(R)$ be the *stable category* of $\mathrm{CM}(R)$; the objects of $\underline{\mathrm{CM}}(R)$ are the same as those of $\mathrm{CM}(R)$, and the hom-set from M to N is given by

$$\operatorname{Hom}_{\underline{\mathrm{CM}}(R)}(M, N) = \operatorname{Hom}_R(M, N)/\{f \in \operatorname{Hom}_R(M, N) \mid f \text{ factors through a free } R\text{-module}\}.$$

(2) Let R be a Gorenstein local ring. Let M be a maximal Cohen–Macaulay R-module. We define the *(first) cosyzygy* $\Omega^{-1}M$ of M as $(\Omega(M^*))^*$, where $(-)^* = \operatorname{Hom}_R(-, R)$. Note then that $\Omega^{-1}M$ is maximal Cohen–Macaulay. For $n \geqslant 2$, we define the *nth cosyzygy* $\Omega^{-n}M$ of M inductively by $\Omega^{-n}M = \Omega^{-1}(\Omega^{-(n-1)}M)$. Note that $\Omega^{-n}M = 0$ for every integer $n > 0$ when M is a free R-module.

Remark 2.19. (1) Let R be a Cohen–Macaulay local ring. For each integer $n \geqslant 0$, taking the nth syzygy defines an additive functor $\Omega^n : \underline{\mathrm{CM}}(R) \to \underline{\mathrm{CM}}(R)$.

(2) Let R be a Gorenstein local ring. For each $n > 0$, taking the nth cosyzygy defines an additive functor $\Omega^{-n} : \underline{\mathrm{CM}}(R) \to \underline{\mathrm{CM}}(R)$. The functors $\Omega^n, \Omega^{-n} : \underline{\mathrm{CM}}(R) \to \underline{\mathrm{CM}}(R)$ are mutually quasi-inverse equivalences. The stable category $\underline{\mathrm{CM}}(R)$ is a triangulated category, and the assignment $M \mapsto M$ gives a triangle equivalence $\underline{\mathrm{CM}}(R) \to \mathrm{D}^{\mathrm{sg}}(R)$. For the details, we refer the reader to [2, Theorem 4.4.1].

The lemma below follows from [13, Proposition (3.11)]. For the definition and fundamental properties of the Auslander–Reiten translation functor, we refer the reader to [13, Chapters 3 and 5].

Lemma 2.20. *Let R be a Gorenstein local ring of dimension d. Let $\tau : \underline{\mathrm{CM}}(R) \to \underline{\mathrm{CM}}(R)$ stand for the Auslander–Reiten translation functor. Then $\tau \cong \Omega^{2-d}$.*

Next we introduce the syzygy of a subcategory of modules.

Definition 2.21. Let R be a local ring. For a subcategory $\mathcal{X}$ of $\operatorname{mod} R$ we denote by $\Omega\mathcal{X}$ the subcategory of $\operatorname{mod} R$ consisting of R-modules Y such that there exists an exact sequence $0 \to Y \to P \to X \to 0$ in $\operatorname{mod} R$ with $P \in \operatorname{add} R$ and $X \in \mathcal{X}$. Note that $\Omega\mathcal{X}$ necessarily contains $\operatorname{add} R$.

The lemma below describes commutativity of the syzygy and extension closure of a subcategory of modules.

Lemma 2.22. *Let R be a Gorenstein local ring. Let $\mathcal{X}$ be a subcategory of* $\operatorname{mod} R$ *contained in* $\mathrm{CM}(R)$*. Then there is an equality* $\operatorname{ext}(\Omega\mathcal{X}) = \operatorname{add}\Omega(\operatorname{ext}\mathcal{X})$*.*

Proof. We have that $\operatorname{add}\Omega(\operatorname{ext}\mathcal{X})$ contains $\Omega\mathcal{X}$ and is closed under direct summands. Let $0 \to L \to M \to N \to 0$ be an exact sequence in $\operatorname{mod} R$ with $L, N \in \operatorname{add}\Omega(\operatorname{ext}\mathcal{X})$. Lemma 2.12(1) implies that there exists an exact sequence $0 \to A \to B \to C \to 0$ in $\operatorname{mod} R$ with $A, C \in \Omega(\operatorname{ext}\mathcal{X})$ such that M is a direct summand of B. There exist exact sequences $0 \to A \to F \to V \to 0$ and $0 \to C \to G \to W \to 0$ with $F, G \in \operatorname{add} R$ and $V, W \in \operatorname{ext}\mathcal{X}$. We have a pushout diagram as in the left below. The middle row splits since R is Gorenstein and C is maximal Cohen–Macaulay. Hence we get another pushout diagram as in the right below.

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & A & \to & B & \to & C & \to & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \to & F & \to & D & \to & C & \to & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & V & = & V & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
\qquad\qquad
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & B & \to & F \oplus C & \to & V & \to & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \to & B & \to & F \oplus G & \to & E & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & W & = & W & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

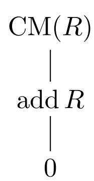
The right column in the right diagram shows that E is in $\operatorname{ext}\mathcal{X}$, and then the middle row in the same diagram shows that B is in $\Omega(\operatorname{ext}\mathcal{X})$. Hence M is in $\operatorname{add}\Omega(\operatorname{ext}\mathcal{X})$, and therefore, $\operatorname{add}\Omega(\operatorname{ext}\mathcal{X})$ is extension-closed.

It remains to prove that if $\mathcal{Y}$ is an extension-closed subcategory of $\operatorname{mod} R$ containing $\Omega\mathcal{X}$, then $\mathcal{Y}$ contains $\operatorname{add}\Omega(\operatorname{ext}\mathcal{X})$. Let $\mathcal{Z}$ be the subcategory of $\operatorname{mod} R$ consisting of R-modules Z with $\Omega Z \in \mathcal{Y}$. Then $\mathcal{Z}$ contains $\mathcal{X}$. If an R-module Z is in $\mathcal{Z}$ and W is a direct summand of Z, then ΩW is a direct summand of ΩZ, which belongs to $\mathcal{Y}$, and so does ΩW since $\mathcal{Y}$ is closed under direct summands. Hence $\mathcal{Z}$ is closed under direct summands. Let $0 \to A \to B \to C \to 0$ be an exact sequence with $A, C \in \mathcal{Z}$. Then there is an exact sequence $0 \to \Omega A \to \Omega B \oplus F \to \Omega C \to 0$ with $F \in \operatorname{add} R$ and $\Omega A, \Omega C \in \mathcal{Y}$. As $\mathcal{Y}$ is extension-closed, ΩB belongs to $\mathcal{Y}$, and hence B is in $\mathcal{Z}$. It follows that $\mathcal{Z}$ is extension-closed. We now observe that $\mathcal{Z}$ contains $\operatorname{ext}\mathcal{X}$, which implies $\Omega(\operatorname{ext}\mathcal{X}) \subseteq \mathcal{Y}$. As $\mathcal{Y}$ is extension-closed, we get the desired inclusion $\operatorname{add}\Omega(\operatorname{ext}\mathcal{X}) \subseteq \mathcal{Y}$. ■

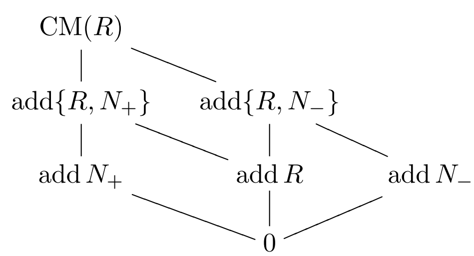
3. On the $\mathrm{A}_n, \mathrm{D}_n, \mathrm{E}_6, \mathrm{E}_7, \mathrm{E}_8$-singularities with dimension at most two

In this section, we give a complete classification of the extension-closed subcategories of $\mathrm{CM}(R)$ in the case where R is a P-singularity with $P \in \{\mathrm{A}_n, \mathrm{D}_n, \mathrm{E}_6, \mathrm{E}_7, \mathrm{E}_8\}$ and has dimension at most two.

Theorem 3.1. *Let $R = k[\![x, y]\!]/(x^2 + y^{n+1})$ be the A_n^1-singularity, where n is a positive integer. When n is odd, put $N_\pm = R/(y^{(n+1)/2} \pm \sqrt{-1}\,x)$. The Hasse diagram of the poset $\mathfrak{E}(R)$ is*



if n is even, and



if n is odd.

Proof. We begin with the case where n is even. Put $l = n/2$, $I_0 = R$ and $I_j = (x, y^j)R$ for each $1 \leqslant j \leqslant l$. By [13, (5.12)] we have $\operatorname{ind}\mathrm{CM}(R) = \{I_0, I_1, \dots, I_l\}$ and there exist exact sequences $0 \to I_j \to I_{j-1} \oplus I_{j+1} \to I_j \to 0$ for each $1 \leqslant j \leqslant l-1$, and $0 \to I_l \to I_{l-1} \oplus I_l \to I_l \to 0$. Hence $R = I_0 \in \operatorname{ext} I_1 = \operatorname{ext} I_2 = \cdots = \operatorname{ext} I_l$. It is observed that $\operatorname{ext} I_j = \mathrm{CM}(R)$ for all $1 \leqslant j \leqslant l$. We obtain the equality $\mathfrak{E}(R) = \{0, \operatorname{add} R, \mathrm{CM}(R)\}$.

Next we deal with the case where n is odd. Put $l = (n-1)/2$, $N_\pm = R/(y^{l+1} \pm \sqrt{-1}\,x) \cong (y^{l+1} \mp \sqrt{-1}\,x)R$, $M_0 = R$ and $M_j = \operatorname{Cok}\begin{pmatrix} x & y^j \\ y^{n+1-j} & -x \end{pmatrix} \cong (x, y^j)R$ for each integer $1 \leqslant j \leqslant l$. Using [13, (9.9)], we have that $\operatorname{ind}\mathrm{CM}(R) = \{M_0, M_1, \dots, M_l, N_+, N_-\}$ and there exist exact sequences

$$
\begin{aligned}
&0 \to M_j \to M_{j-1} \oplus M_{j+1} \to M_j \to 0 \text{ for each } 1 \leqslant j \leqslant l-1, \\
&0 \to M_l \to M_{l-1} \oplus N_+ \oplus N_- \to M_l \to 0 \text{ and } 0 \to N_\pm \to M_l \to N_\mp \to 0.
\end{aligned}
$$

Hence $R = M_0 \in \operatorname{ext} M_1 = \operatorname{ext} M_2 = \cdots = \operatorname{ext} M_l \ni N_\pm$ and $M_l \in \operatorname{ext}\{N_+, N_-\}$. We get $\operatorname{ext} M_1 = \operatorname{ext} M_2 = \cdots = \operatorname{ext} M_l = \operatorname{ext}\{N_+, N_-\} = \operatorname{CM}(R)$. Applying Lemma 2.17, we see that N_+, N_- are rigid R-modules. By Lemma 2.16, we conclude that $\mathfrak{E}(R) = \{0, \operatorname{add} R, \operatorname{add} N_+, \operatorname{add} N_-, \operatorname{add}\{R, N_+\}, \operatorname{add}\{R, N_-\}, \operatorname{CM}(R)\}$. ■

Theorem 3.2. *Let $R = k[\![x, y]\!]/(x^3 + y^4)$ be the E_6^1-singularity. The Hasse diagram of $\mathfrak{E}(R)$ is the following.*

$$\begin{array}{c} \operatorname{CM}(R) \\ | \\ \operatorname{add} R \\ | \\ 0 \end{array}$$

Proof. By virtue of [13, (9.13)], we have that $\operatorname{ind}\operatorname{CM}(R) = \{R, M_1, M_2, N_1, A, B, X\}$ with $N_1 \cong \mathfrak{m} := (x, y)R$, $B \cong \mathfrak{m}^2$, and there exist exact sequences

$$\begin{array}{lll} a : 0 \to M_2 \to X \to M_2 \to 0, & b : 0 \to X \to M_2 \oplus A \oplus B \to X \to 0, & c : 0 \to A \to X \oplus N_1 \to B \to 0, \\ d : 0 \to B \to X \oplus M_1 \to A \to 0, & e : 0 \to M_1 \to A \to N_1 \to 0, & f : 0 \to N_1 \to B \oplus R \to M_1 \to 0 \end{array}$$

such that the maps $C \to D$ with $C, D \in \operatorname{ind}\operatorname{CM}(R)$ appearing in different exact sequences (e.g., the maps $X \to M_2$ in a, b) are the same. It follows from [9, Corollary 2.7] that $\operatorname{ext} N_1 = \operatorname{CM}(R)$. The associated graded ring $\operatorname{gr}_{\mathfrak{m}} R$ is isomorphic to $k[x, y]/(x^3)$, which has depth one. We see from [5, Theorem 5.5] that $\operatorname{ext} B = \operatorname{CM}(R)$. The exact sequences a and b show that $X \in \operatorname{ext} M_2$ and $B \in \operatorname{ext} X$, respectively. Hence the equalities $\operatorname{ext} M_2 = \operatorname{ext} X = \operatorname{ext} B = \operatorname{CM}(R)$ hold. The exact sequences b, c, d, f make exact squares as in the left below, while b, c, e, f make exact squares as in the right below.

$$\begin{array}{ccccc} A & \longrightarrow & X & \longrightarrow & M_2 \oplus A \\ \downarrow & & \downarrow & & \downarrow \\ N_1 & \longrightarrow & B & \longrightarrow & X \\ \downarrow & & \downarrow & & \downarrow \\ R & \longrightarrow & M_1 & \longrightarrow & A \end{array} \qquad \begin{array}{ccccccccccc} M_1 & \longrightarrow & A & \longrightarrow & X & \longrightarrow & M_2 \oplus A & \longrightarrow & M_2 \oplus N_1 & \longrightarrow & M_2 \oplus R \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & N_1 & \longrightarrow & B & \longrightarrow & X & \longrightarrow & B & \longrightarrow & M_1 \end{array}$$

In view of Lemma 2.3, there exist short exact sequences $0 \to A \to M_2 \oplus A \oplus R \to A \to 0$ and $0 \to M_1 \to M_2 \oplus R \to M_1 \to 0$, which show that $M_2 \in \operatorname{ext} A \cap \operatorname{ext} M_1$. It is seen that the equalities $\operatorname{ext} A = \operatorname{ext} M_1 = \operatorname{ext} M_2 = \operatorname{CM}(R)$ hold. Now it is observed that $\mathfrak{E}(R) = \{0, \operatorname{add} R, \operatorname{CM}(R)\}$. ■

Theorem 3.3. *Let $R = k[\![x, y]\!]/(x^3 + y^5)$ be the E_8^1-singularity. The Hasse diagram of $\mathfrak{E}(R)$ is the following.*

$$\begin{array}{c} \operatorname{CM}(R) \\ | \\ \operatorname{add} R \\ | \\ 0 \end{array}$$

Proof. Put $U = \begin{pmatrix} y & -x & 0 \\ 0 & y & -x \\ x & 0 & y^3 \end{pmatrix}$ and $V = \begin{pmatrix} y & -x & 0 \\ 0 & y^2 & -x \\ x & 0 & y^2 \end{pmatrix}$. In view of [13, (9.15)] and Lemma 2.20, we have that

$$\operatorname{ind}\operatorname{CM}(R) = \{R, M_1, M_2, N_1, N_2, A_1, A_2, B_1, B_2, C_1, C_2, D_1, D_2, X_1, X_2, Y_1, Y_2\},$$

where $A_1 = \operatorname{Cok} U$, $A_2 = \operatorname{Cok} V$, $M_1 \cong (x^2, y)R$, $M_2 \cong (x^2, y^2)R$, $N_1 \cong \mathfrak{m} := (x, y)R$, $N_2 \cong (x, y^2)R$, $N_i \cong \Omega M_i$, $B_i \cong \Omega A_i$, $D_i \cong \Omega C_i$, $Y_i \cong \Omega X_i$ for $i = 1, 2$, and there exist exact sequences

$$\begin{array}{lll} 0 \to Y_2 \to D_1 \oplus X_1 \to X_2 \to 0, & 0 \to X_1 \to X_2 \oplus A_2 \oplus C_2 \to Y_1 \to 0, & 0 \to D_1 \to A_1 \oplus X_2 \to C_1 \to 0, \\ 0 \to X_2 \to C_1 \oplus Y_1 \to Y_2 \to 0, & 0 \to C_2 \to Y_1 \oplus N_2 \to D_2 \to 0, & 0 \to Y_1 \to Y_2 \oplus B_2 \oplus D_2 \to X_1 \to 0, \\ 0 \to C_1 \to B_1 \oplus Y_2 \to D_1 \to 0, & 0 \to B_1 \to N_1 \oplus D_1 \to A_1 \to 0, & 0 \to N_2 \to D_2 \to M_2 \to 0, \\ 0 \to D_2 \to X_1 \oplus M_2 \to C_2 \to 0, & & \end{array}$$

where the maps $E \to F$ with $E, F \in \operatorname{ind}\operatorname{CM}(R)$ are all the same. It is verified that $R \xleftarrow{(xy\ x^2\ -y^2)} R^{\oplus 3} \xleftarrow{U} R^{\oplus 3}$ and $R \xleftarrow{(xy^2\ x^2\ -y^3)} R^{\oplus 3} \xleftarrow{V} R^{\oplus 3}$ are exact sequences, so $A_1 \cong \mathfrak{m}^2$ and $A_2 \cong (x^2, xy^2, y^3)R$. For each $I \in \{M_1, M_2, N_1, N_2, A_1, A_2\}$, by [5, Proposition 6.8] and completion $R/\mathfrak{m}$ is in $\operatorname{ext} R/I$. Lemma 2.15(1) (or

[5, Lemma 5.2(1)]) implies $\mathfrak{m} \in \operatorname{ext} I$. This gives $\operatorname{ext} I = \mathrm{CM}(R)$ by [9, Corollary 2.7]. It follows from Lemma 2.15(2) that $\operatorname{ext} B_1 = \operatorname{ext} B_2 = \mathrm{CM}(R)$. The above exact sequences make the following exact squares.

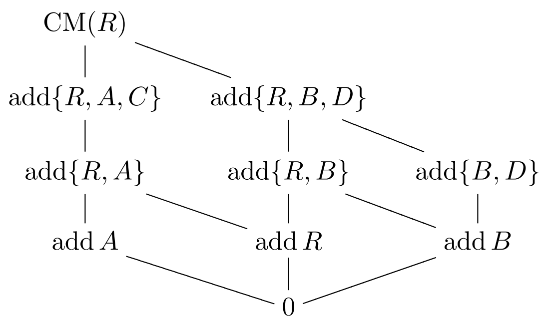
$$\begin{array}{ccccc} Y_2 & \to & D_1 & \to & A_1 \\ \downarrow & & \downarrow & & \downarrow \\ X_1 & \to & X_2 & \to & C_1 \\ \downarrow & & \downarrow & & \downarrow \\ A_2 \oplus C_2 & \to & Y_1 & \to & Y_2 \end{array} \qquad \begin{array}{ccccc} X_1 & \to & X_2 & \to & C_1 \oplus B_2 \\ \downarrow & & \downarrow & & \downarrow \\ A_2 \oplus C_2 & \to & Y_1 & \to & Y_2 \oplus B_2 \\ \downarrow & & \downarrow & & \downarrow \\ A_2 \oplus N_2 & \to & D_2 & \to & X_1 \end{array} \qquad \begin{array}{ccccc} C_1 & \to & B_1 & \to & N_1 \\ \downarrow & & \downarrow & & \downarrow \\ Y_2 & \to & D_1 & \to & A_1 \\ \downarrow & & \downarrow & & \downarrow \\ X_1 & \to & X_2 & \to & C_1 \end{array} \qquad \begin{array}{ccccc} C_2 & \to & Y_1 & \to & Y_2 \oplus B_2 \\ \downarrow & & \downarrow & & \downarrow \\ N_2 & \to & D_2 & \to & X_1 \\ \downarrow & & \downarrow & & \downarrow \\ 0 & \to & M_2 & \to & C_2 \end{array}$$

By Lemma 2.3, these exact squares produce the following four exact sequences:

$$0 \to Y_2 \to A_1 \oplus A_2 \oplus C_2 \to Y_2 \to 0, \quad 0 \to X_1 \to C_1 \oplus B_2 \oplus A_2 \oplus N_2 \to X_1 \to 0,$$
$$0 \to C_1 \to N_1 \oplus X_1 \to C_1 \to 0, \quad 0 \to C_2 \to Y_2 \oplus B_2 \to C_2 \to 0.$$

Hence $A_2 \in \operatorname{ext} Y_2 \cap \operatorname{ext} X_1$, $X_1 \in \operatorname{ext} C_1$ and $B_2 \in \operatorname{ext} C_2$, and we get $\operatorname{ext} I = \mathrm{CM}(R)$ for $I \in \{Y_2, X_1, C_1, C_2\}$. Lemma 2.15(2) implies $\operatorname{ext} I = \mathrm{CM}(R)$ for $I \in \{X_2, Y_1, D_1, D_2\}$. Consequently, $\operatorname{ext} I = \mathrm{CM}(R)$ for all $I \in \{M_1, M_2, N_1, N_2, A_1, A_2, B_1, B_2, Y_2, X_1, C_1, C_2, X_2, Y_1, D_1, D_2\}$. We obtain $\mathfrak{E}(R) = \{0, \operatorname{add} R, \mathrm{CM}(R)\}$. ■

Theorem 3.4. *Let $R = k[\![x, y]\!]/(x^3 + xy^3)$ be the E_7^1-singularity. The Hasse diagram of $\mathfrak{E}(R)$ is the following.*



Here, $A = R/(x)$, $B = \Omega A = R/(x^2 + y^3)$, $C = \operatorname{Cok}\begin{pmatrix} x^2 & xy \\ xy^2 & -x^2 \end{pmatrix}$ and $D = \Omega C = \operatorname{Cok}\begin{pmatrix} x & y \\ y^2 & -x \end{pmatrix}$.

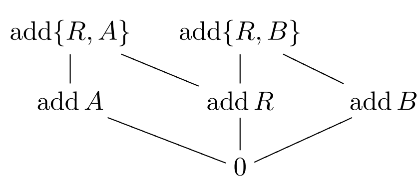
Proof. We observe from [13, (9.14)] and Lemma 2.20 that

$$\operatorname{ind} \mathrm{CM}(R) = \{R, A, B, C, D, M_i, N_i, X_j, Y_j \mid i = 1, 2 \text{ and } j = 1, 2, 3\},$$

where $A = R/(x)$, $\Omega A \cong B = R/(x^2 + y^3)$, $C = \operatorname{Cok}\begin{pmatrix} x^2 & xy \\ xy^2 & -x^2 \end{pmatrix}$, $\Omega C \cong D = \operatorname{Cok} \delta$ with $\delta = \begin{pmatrix} x & y \\ y^2 & -x \end{pmatrix}$, $\Omega M_i \cong N_i$ for $i = 1, 2$, $\Omega X_j \cong Y_j$ for $j = 1, 2, 3$, $N_1 = \operatorname{Cok} \psi_1$ with $\psi_1 = \begin{pmatrix} x^2 & y \\ xy^2 & -x \end{pmatrix}$, $Y_1 = \operatorname{Cok} \eta_1$ with $\eta_1 = \begin{pmatrix} y & 0 & x \\ -x & xy & 0 \\ 0 & -x & y \end{pmatrix}$, and there exist exact sequences

$$\begin{array}{lll} 0 \to X_3 \to X_1 \oplus D \oplus Y_2 \to Y_3 \to 0, & 0 \to Y_2 \to Y_3 \oplus N_2 \to X_2 \to 0, & 0 \to X_1 \to N_1 \oplus Y_3 \to Y_1 \to 0, \\ 0 \to Y_3 \to Y_1 \oplus C \oplus X_2 \to X_3 \to 0, & 0 \to N_1 \to R \oplus Y_1 \to M_1 \to 0, & 0 \to Y_1 \to M_1 \oplus X_3 \to X_1 \to 0, \\ 0 \to M_2 \to Y_2 \oplus B \to N_2 \to 0, & \rho : 0 \to B \to N_2 \to A \to 0, & 0 \to N_2 \to X_2 \oplus A \to M_2 \to 0, \\ 0 \to X_2 \to X_3 \oplus M_2 \to Y_2 \to 0, & \sigma : 0 \to C \to X_3 \to D \to 0, & \tau : 0 \to D \to Y_3 \to C \to 0, \end{array}$$

where the maps $F \to G$ with $F, G \in \operatorname{ind} \mathrm{CM}(R)$ are all the same. Lemmas 2.16 and 2.17 give the subdiagram



of the Hasse diagram of $\mathfrak{E}(R)$. It is easily verified that the sequences $R \xleftarrow{(x\ y)} R^3 \xleftarrow{\psi_1} R^3$ and $R \xleftarrow{(xy\ y^2\ -x^2)} R^3 \xleftarrow{\eta_2} R^3$ are exact. Hence $N_1 \cong \mathfrak{m}$ and $Y_1 \cong \mathfrak{m}^2$, where $\mathfrak{m}$ stands for the maximal ideal of R. Note that

$\operatorname{gr}_{\mathfrak{m}} R \cong k[x,y]/(x^3)$. The extension closures $\operatorname{ext} N_1$ and $\operatorname{ext} Y_1$ coincide with $\operatorname{CM}(R)$ by [9, Corollary 2.7] and [5, Theorem 5.5], respectively. Lemma 2.15(2) yields $\operatorname{ext} M_1 = \operatorname{ext} X_1 = \operatorname{CM}(R)$. There exist exact squares

$$\begin{array}{ccccccc}
X_3 & \to & X_1 \oplus D & \to & N_1 \oplus D & \to & R \oplus D \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
Y_2 & \to & Y_3 & \to & Y_1 & \to & M_1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
N_2 \oplus C & \to & X_2 \oplus C & \to & X_3 & \to & X_1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
D \oplus A \oplus C & \to & D \oplus M_2 \oplus C & \to & D \oplus Y_2 & \to & Y_3
\end{array}
\qquad
\begin{array}{ccccc}
X_2 & \to & M_2 & \to & B \\
\downarrow & & \downarrow & & \downarrow \\
X_3 & \to & Y_2 & \to & N_2 \\
\downarrow & & \downarrow & & \downarrow \\
X_1 \oplus D & \to & Y_3 & \to & X_2
\end{array}
\qquad
\begin{array}{ccccc}
M_2 & \to & Y_2 & \to & Y_3 \\
\downarrow & & \downarrow & & \downarrow \\
B & \to & N_2 & \to & X_2 \\
\downarrow & & \downarrow & & \downarrow \\
0 & \to & A & \to & M_2
\end{array}$$

Applying Lemma 2.3, we obtain the following four exact sequences.

$$\alpha : 0 \to X_3 \to N_1 \oplus D \oplus N_2 \oplus C \to X_3 \to 0, \qquad \beta : 0 \to X_2 \to B \oplus X_1 \oplus D \to X_2 \to 0,$$
$$\gamma : 0 \to M_2 \to Y_3 \to M_2 \to 0, \qquad \zeta : 0 \to X_3 \to R \oplus D \oplus D \oplus A \oplus C \to Y_3 \to 0.$$

The exact sequences α, β, γ respectively show that $N_1 \in \operatorname{ext} X_3$, $X_1 \in \operatorname{ext} X_2$ and $Y_3 \in \operatorname{ext} M_2$. The first two containments give equalities $\operatorname{ext} X_3 = \operatorname{ext} X_2 = \operatorname{CM}(R)$, which give equalities $\operatorname{ext} Y_3 = \operatorname{ext} Y_2 = \operatorname{CM}(R)$ by Lemma 2.15(2). The third containment now shows that $\operatorname{ext} M_2 = \operatorname{CM}(R)$, which implies $\operatorname{ext} N_2 = \operatorname{CM}(R)$ by Lemma 2.15(2) again. In summary, we have

$$\operatorname{ext} E = \operatorname{CM}(R) \qquad \text{for all } E \in \{M_i, N_i, X_j, Y_j \mid i = 1,2 \text{ and } j = 1,2,3\}. \tag{3.4.1}$$

Applying Lemma 2.4 to the exact sequences σ, τ, ζ, we get an exact sequence $0 \to C \to R \oplus A \oplus C \to C \to 0$, which shows that $R, A \in \operatorname{ext} C$. It follows from Lemma 2.15(1) that $B \in \operatorname{ext} D$, and thus $\operatorname{add}\{B, D\} \subseteq \operatorname{ext} D$.

Let $\mathfrak{p}$ be the prime ideal of R generated by x. We claim that B, D are the only nonisomorphic indecomposable maximal Cohen–Macaulay R-modules X with $X_{\mathfrak{p}} = 0$. Indeed, by (3.4.1), for all $E \in \{M_i, N_i, X_j, Y_j \mid i = 1,2 \text{ and } j = 1,2,3\}$ the module R belongs to $\operatorname{ext} E$, so that $R_{\mathfrak{p}}$ belongs to $\operatorname{ext} E_{\mathfrak{p}}$ by Lemma 2.12(2), and in particular, $E_{\mathfrak{p}} \neq 0$. Since $A = R/(x)$ and $B = R/(x^2 + y^3)$, we have $A_{\mathfrak{p}} \neq 0 = B_{\mathfrak{p}}$. There are equivalences

$$\begin{pmatrix} x & y \\ y^2 & -x \end{pmatrix} \cong \begin{pmatrix} x & 1 \\ y^2 & -y^{-1}x \end{pmatrix} \cong \begin{pmatrix} 0 & 1 \\ y^2+y^{-1}x^2 & -y^{-1}x \end{pmatrix} \cong \begin{pmatrix} 0 & 1 \\ y^2+y^{-1}x^2 & 0 \end{pmatrix} \cong \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \cong \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$$

of matrices over $R_{\mathfrak{p}}$. Hence it holds that $D_{\mathfrak{p}} \cong \operatorname{Cok} \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) = 0$. The localized exact sequence $\sigma_{\mathfrak{p}} : 0 \to C_{\mathfrak{p}} \to (X_3)_{\mathfrak{p}} \to D_{\mathfrak{p}} \to 0$ shows that $C_{\mathfrak{p}} \cong (X_3)_{\mathfrak{p}} \neq 0$. Thus the claim follows.

It follows from Lemma 2.12(2) and the equality $D_{\mathfrak{p}} = 0$ that every R-module E that belongs to $\operatorname{ext} D$ is such that $E_{\mathfrak{p}} = 0$. Therefore, the above claim implies that $\operatorname{add}\{B, D\} = \operatorname{ext} D$. Using Lemma 2.12(3), we get $\operatorname{add}\{B, D, R\} = \operatorname{ext}\{D, R\}$. In summary, we have

$$0 \subsetneq (\operatorname{ext} B = \operatorname{add} B) \subsetneq (\operatorname{ext} D = \operatorname{add}\{B, D\}) \subsetneq (\operatorname{ext}\{D, R\} = \operatorname{add}\{B, D, R\}) \subsetneq \operatorname{CM}(R). \tag{3.4.2}$$

Suppose that there exists an extension-closed subcategory $\mathcal{X}$ with $\operatorname{add}\{B, D, R\} \subsetneq \mathcal{X} \subsetneq \operatorname{CM}(R)$. Then $\mathcal{X}$ contains some module

$$E \in \operatorname{ind} \operatorname{CM}(R) \setminus \{B, D, R\} = \{A, C, M_i, N_i, X_j, Y_j \mid i = 1,2 \text{ and } j = 1,2,3\}.$$

In view of (3.4.1), the module E must be either A or C. Hence $\mathcal{X}$ contains either $\operatorname{ext}\{A, B\}$ or $\operatorname{ext}\{C, D\}$. The exact sequences ρ, σ give rise to equalities $\operatorname{ext}\{A, B\} = \operatorname{ext}\{C, D\} = \operatorname{CM}(R)$. Therefore we have $\mathcal{X} = \operatorname{CM}(R)$, which is a contradiction. Thus, the chain (3.4.2) is saturated.

Applying Lemma 2.22 to $\mathcal{X} = \{D\}$, we obtain $\operatorname{ext}(\Omega\{D\}) = \operatorname{add} \Omega(\operatorname{ext} D)$. Hence

$$\operatorname{ext} C = \operatorname{ext}\{C, R\} = \operatorname{ext}(\Omega\{D\}) = \operatorname{add} \Omega(\operatorname{ext} D) = \operatorname{add}\{\Omega B, \Omega D, R\} = \operatorname{add}\{A, C, R\}.$$

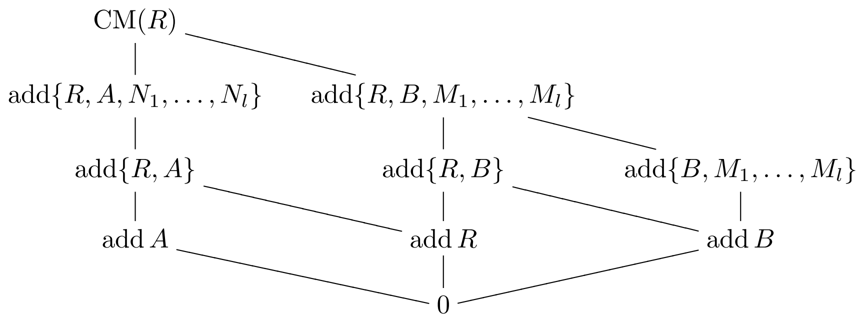
We have a chain of subcategories of $\operatorname{CM}(R)$:

$$0 \subsetneq (\operatorname{ext} A = \operatorname{add} A) \subsetneq (\operatorname{ext}\{A, R\} = \operatorname{add}\{A, R\}) \subsetneq (\operatorname{ext} C = \operatorname{add}\{A, C, R\}) \subsetneq \operatorname{CM}(R). \tag{3.4.3}$$

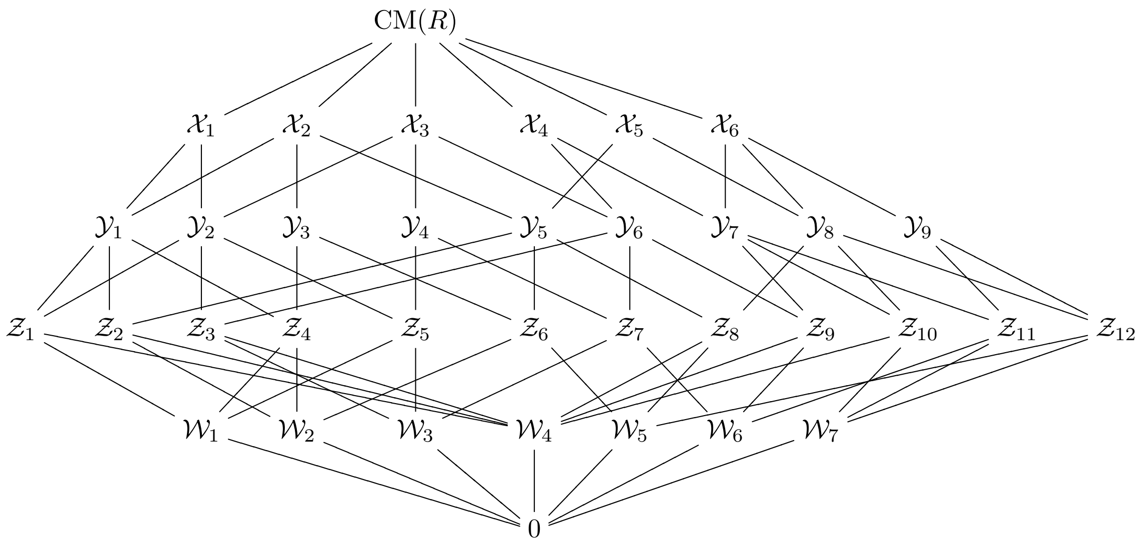
Suppose that there exists an extension-closed subcategory $\mathcal{X}$ with $\operatorname{add}\{A, C, R\} \subsetneq \mathcal{X} \subsetneq \operatorname{CM}(R)$. Then $\mathcal{X}$ contains some module $E \in \operatorname{ind} \operatorname{CM}(R) \setminus \{A, C, R\} = \{B, D, M_i, N_i, X_j, Y_j \mid i = 1,2 \text{ and } j = 1,2,3\}$. By (3.4.1) we must have $E \in \{B, D\}$. Hence $\mathcal{X}$ contains either $\operatorname{ext}\{A, B\}$ or $\operatorname{ext}\{C, D\}$, and get a contradiction as before. Thus, the chain (3.4.3) is saturated. Now we obtain the Hasse diagram in the theorem. ■

Theorem 3.5. *Let $R = k[\![x, y]\!]/(x^2y + y^{n-1})$ be the D_n^1-singularity, where $n \geqslant 4$ is an integer. Set $A = R/(y)$, $B = R/(x^2 + y^{n-2})$, $M_j = \operatorname{Cok} \begin{pmatrix} x & y^j \\ y^{n-j-2} & -x \end{pmatrix}$ and $N_j = \operatorname{Cok} \begin{pmatrix} xy & y^{j+1} \\ y^{n-j-1} & -xy \end{pmatrix}$ for each $0 \leqslant j \leqslant n-3$.*

(1) *Suppose that n is odd and put $l = \frac{n-3}{2}$. Then the Hasse diagram of the poset $\mathfrak{E}(R)$ is the following.*



(2) *Suppose that n is even and put $l = \frac{n-4}{2}$. Then the Hasse diagram of the poset $\mathfrak{E}(R)$ is the following.*



The vertices in the diagram are as follows, where $C_\pm = R/(xy \pm \sqrt{-1}\, y^{l+2})$ and $D_\pm = R/(x \mp \sqrt{-1}\, y^{l+1})$.

$\mathcal{X}_1 = \operatorname{add}\{R, A, C_+, C_-, N_1, \dots, N_l\}$, $\mathcal{X}_2 = \operatorname{add}\{R, A, C_+, D_-\}$, $\mathcal{X}_3 = \operatorname{add}\{R, A, C_-, D_+\}$,
$\mathcal{X}_4 = \operatorname{add}\{R, B, C_-, D_+\}$, $\mathcal{X}_5 = \operatorname{add}\{R, B, C_+, D_-\}$, $\mathcal{X}_6 = \operatorname{add}\{R, B, D_+, D_-, M_1, \dots, M_l\}$,
$\mathcal{Y}_1 = \operatorname{add}\{R, A, C_+\}$, $\mathcal{Y}_2 = \operatorname{add}\{R, A, C_-\}$, $\mathcal{Y}_3 = \operatorname{add}\{A, C_+, D_-\}$, $\mathcal{Y}_4 = \operatorname{add}\{A, C_-, D_+\}$,
$\mathcal{Y}_5 = \operatorname{add}\{R, C_+, D_-\}$, $\mathcal{Y}_6 = \operatorname{add}\{R, C_-, D_+\}$, $\mathcal{Y}_7 = \operatorname{add}\{R, B, D_+\}$, $\mathcal{Y}_8 = \operatorname{add}\{R, B, D_-\}$,
$\mathcal{Y}_9 = \operatorname{add}\{B, D_+, D_-, M_1, \dots, M_l\}$, $\mathcal{Z}_1 = \operatorname{add}\{R, A\}$, $\mathcal{Z}_2 = \operatorname{add}\{R, C_+\}$, $\mathcal{Z}_3 = \operatorname{add}\{R, C_-\}$,
$\mathcal{Z}_4 = \operatorname{add}\{A, C_+\}$, $\mathcal{Z}_5 = \operatorname{add}\{A, C_-\}$, $\mathcal{Z}_6 = \operatorname{add}\{C_+, D_-\}$, $\mathcal{Z}_7 = \operatorname{add}\{C_-, D_+\}$, $\mathcal{Z}_8 = \operatorname{add}\{R, D_-\}$,
$\mathcal{Z}_9 = \operatorname{add}\{R, D_+\}$, $\mathcal{Z}_{10} = \operatorname{add}\{R, B\}$, $\mathcal{Z}_{11} = \operatorname{add}\{B, D_+\}$, $\mathcal{Z}_{12} = \operatorname{add}\{B, D_-\}$, $\mathcal{W}_1 = \operatorname{add} A$,
$\mathcal{W}_2 = \operatorname{add} C_+$, $\mathcal{W}_3 = \operatorname{add} C_-$, $\mathcal{W}_4 = \operatorname{add} R$, $\mathcal{W}_5 = \operatorname{add} D_-$, $\mathcal{W}_6 = \operatorname{add} D_+$, $\mathcal{W}_7 = \operatorname{add} B$.

Proof. Put $X_j = \operatorname{Cok}\begin{pmatrix} x & y^j \\ y^{n-j-1} & -xy \end{pmatrix}$ and $Y_j = \operatorname{Cok}\begin{pmatrix} xy & y^j \\ y^{n-j-1} & -x \end{pmatrix}$ for each $0 \leqslant j \leqslant n-3$. By [13, (9.11) and (9.12)] and Lemma 2.20, we have that

$$\operatorname{ind} \mathrm{CM}(R) \supseteq \{R, A, B, X_{l+1}, Y_{l+1}, X_j, Y_j, M_j, N_j \mid 1 \leqslant j \leqslant l\}, \tag{3.5.1}$$

and that there exist exact sequences

$$\begin{array}{ll} \rho_j : 0 \to Y_{j+1} \to M_{j+1} \oplus N_j \to X_{j+1} \to 0, & 0 \to M_j \to X_j \oplus Y_{j+1} \to N_j \to 0, \\ \zeta_j : 0 \to X_{j+1} \to N_{j+1} \oplus M_j \to Y_{j+1} \to 0, & \sigma_j : 0 \to N_j \to Y_j \oplus X_{j+1} \to M_j \to 0 \end{array} \quad (0 \leqslant j \leqslant l)$$

where the maps $F \to G$ with $F, G \in \operatorname{ind} \mathrm{CM}(R)$ are all the same, and isomorphisms $M_0 \cong B$, $N_0 \cong A \oplus R$, $X_0 \cong Y_0 \cong R$, $\Omega A \cong B$, $\Omega X_j \cong Y_j \cong (x, y^j)R$ and $\Omega N_j \cong M_j \cong (xy, y^{j+1})R$ for all $1 \leqslant j \leqslant l+1$. For each

$1 \leqslant j \leqslant l$ there are exact squares in the left below, which produce exact sequences in the right below.

$$
\begin{array}{ccccccccc}
M_j & \longrightarrow & X_j & \longrightarrow & M_{j-1} & \longrightarrow & X_{j-1} & & \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
Y_{j+1} & \longrightarrow & N_j & \longrightarrow & Y_j & \longrightarrow & N_{j-1} & \longrightarrow & Y_{j-1} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
M_{j+1} & \longrightarrow & X_{j+1} & \longrightarrow & M_j & \longrightarrow & X_j & \longrightarrow & M_{j-1} \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 & & N_{j+1} & \longrightarrow & Y_{j+1} & \longrightarrow & N_j & \longrightarrow & Y_j
\end{array}
\qquad
\begin{array}{c}
0 \to M_j \to M_{j+1} \oplus M_{j-1} \to M_j \to 0, \\
0 \to X_j \to X_{j+1} \oplus X_{j-1} \to X_j \to 0, \\
0 \to N_j \to N_{j+1} \oplus N_{j-1} \to N_j \to 0, \\
0 \to Y_j \to Y_{j+1} \oplus Y_{j-1} \to Y_j \to 0.
\end{array}
$$

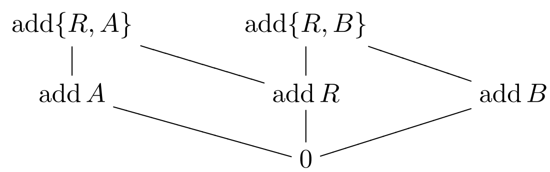
These four exact sequences show that

$$
\begin{array}{ll}
M_{l+1} \in \operatorname{ext} M_l = \operatorname{ext} M_{l-1} = \cdots = \operatorname{ext} M_1 \ni M_0 = B, & \operatorname{ext} X_l = \operatorname{ext} X_{l-1} = \cdots = \operatorname{ext} X_1, \\
N_{l+1} \in \operatorname{ext} N_l = \operatorname{ext} N_{l-1} = \cdots = \operatorname{ext} N_1 \ni N_0 = A \oplus R, & \operatorname{ext} Y_l = \operatorname{ext} Y_{l-1} = \cdots = \operatorname{ext} Y_1.
\end{array}
$$

Since Y_1 is isomorphic to the maximal ideal $(x,y)R$, we have $\operatorname{ext} Y_1 = \operatorname{CM}(R)$ by [9, Corollary 2.7]. Lemma 2.15(2) implies that $\operatorname{ext} X_1 = \operatorname{CM}(R)$. Thus for all $1 \leqslant j \leqslant l$ there are inclusions and equalities

(3.5.2) $\operatorname{ext} M_j \supseteq \operatorname{add}\{B, M_1, \dots, M_{l+1}\}, \quad \operatorname{ext} N_j \supseteq \operatorname{add}\{R, A, N_1, \dots, N_{l+1}\}, \quad \operatorname{ext} X_j = \operatorname{ext} Y_j = \operatorname{CM}(R).$

(1) Let n be odd. By Lemmas 2.16 and 2.17, we get the subdiagram of the Hasse diagram of $\mathfrak{E}(R)$:



By virtue of [13, (9.11)], the inclusion (3.5.1) is an equality. Put $\mathfrak{p} = (y) \in \operatorname{Spec} R$. It is clear that $(M_0)_\mathfrak{p} = B_\mathfrak{p} = 0 \neq A_\mathfrak{p}$. For each $1 \leqslant j \leqslant l+1$, as $M_j \cong (xy, y^{j+1})R$ and $Y_j \cong (x, y^j)R$, we have $(M_j)_\mathfrak{p} \subseteq \mathfrak{p}R_\mathfrak{p} = 0$ and $(Y_j)_\mathfrak{p} \cong R_\mathfrak{p}$. For each $1 \leqslant j \leqslant l+1$, there is an exact sequence $0 \to X_j \to R^{\oplus 2} \to Y_j \to 0$, which shows that $(X_j)_\mathfrak{p} \cong R_\mathfrak{p}$. For each $1 \leqslant j \leqslant l$, the exact sequence σ_j shows $(N_j)_\mathfrak{p} \neq 0$. Hence there is an equality

(3.5.3) $$\{E \in \operatorname{ind} \operatorname{CM}(R) \mid E_\mathfrak{p} = 0\} = \{B, M_1, \dots, M_l\}.$$

Since n is odd, we have isomorphisms $X_{l+1} \cong Y_{l+1}$, $M_l \cong M_{l+1}$ and $N_l \cong N_{l+1}$ by [13, (9.11.5)]. The short exact sequence σ_l gives rise to a short exact sequence $0 \to N_{l+1} \to Y_l \oplus Y_{l+1} \to M_{l+1} \to 0$. Therefore, there are exact squares in the left below, which produce an exact sequence in the right below.

$$
\begin{array}{ccccc}
X_{l+1} & \longrightarrow & M_l & \longrightarrow & X_l \\
\downarrow & & \downarrow & & \downarrow \\
N_{l+1} & \longrightarrow & Y_{l+1} & \longrightarrow & N_l \\
\downarrow & & \downarrow & & \downarrow \\
Y_l & \longrightarrow & M_{l+1} & \longrightarrow & X_{l+1}
\end{array}
\qquad
0 \to X_{l+1} \to X_l \oplus Y_l \to X_{l+1} \to 0.
$$

This exact sequence and (3.5.2) give $\operatorname{CM}(R) = \operatorname{ext} X_l \subseteq \operatorname{ext} X_{l+1}$, which and Lemma 2.15(2) yield $\operatorname{CM}(R) = \operatorname{ext} X_{l+1} = \operatorname{ext} Y_{l+1}$. By (3.5.3), for any $1 \leqslant j \leqslant l$ and any $E \in \operatorname{ext} M_j$ it holds that $E_\mathfrak{p} = 0$. Using (3.5.2) and (3.5.3), we get $\operatorname{ext} M_j = \operatorname{add}\{B, M_1, \dots, M_l\}$ for each $1 \leqslant j \leqslant l$. This equality, (3.5.2), and Lemma 2.15(1) yield $\operatorname{ext} N_j = \operatorname{add}\{R, A, N_1, \dots, N_l\}$ for each $1 \leqslant j \leqslant l$. Applying Lemma 2.12(3), we get $\operatorname{ext}\{R, M_j\} = \operatorname{add}\{R, B, M_1, \dots, M_l\}$ for every $1 \leqslant j \leqslant l$. The exact sequence σ_l shows that $X_{l+1} \in \operatorname{ext}\{M_l, N_l\}$, which implies $\operatorname{ext}\{M_l, N_l\} = \operatorname{CM}(R)$. Therefore, $\operatorname{ext}\{M_j, N_h\} = \operatorname{CM}(R)$ for all $1 \leqslant j, h \leqslant l$. From σ_0 and the proof of Lemma 2.12(3) (or [13, page 78, line 2]) we get an exact sequence $0 \to A \to X_1 \to B \to 0$, which implies $X_1 \in \operatorname{ext}\{A, B\}$, and $\operatorname{ext}\{A, B\} = \operatorname{CM}(R)$ by (3.5.2). Now we obtain the Hasse diagram as in the theorem.

(2) We consider the case where n is even. In view of [13, (9.12)] and Lemma 2.20, we have that

$$\operatorname{ind} \operatorname{CM}(R) = \{R, A, B, C_\pm, D_\pm, X_{l+1}, Y_{l+1}, X_j, Y_j, M_j, N_j \mid 1 \leqslant j \leqslant l\}$$

with $\Omega C_\pm \cong D_\pm$, $M_{l+1} \cong D_+ \oplus D_-$ and $N_{l+1} \cong C_+ \oplus C_-$, and that there are exact sequences

$$
\begin{array}{ll}
\alpha_\pm : 0 \to C_\pm \to Y_{l+1} \to D_\pm \to 0, & \beta : 0 \to B \to Y_1 \to A \to 0, \\
\gamma_\pm : 0 \to D_\pm \to X_{l+1} \to C_\pm \to 0, & \delta : 0 \to A \to X_1 \to B \to 0.
\end{array}
$$

By Lemma 2.17 we see that $R, A, B, C_\pm, D_\pm$ are rigid. As is shown in the proof of [3, Proposition 2.6], the direct sums $C_+ \oplus D_-$, $C_- \oplus D_+$, $C_+ \oplus A$, $D_+ \oplus B$, $A \oplus C_-$, $B \oplus D_-$ are (maximal) rigid. Applying Lemma 2.16, we see that the following subcategories are extension-closed.

$$\mathcal{Y}_1, \mathcal{Y}_2, \mathcal{Y}_5, \mathcal{Y}_6, \mathcal{Y}_7, \mathcal{Y}_8, \mathcal{Z}_1, \mathcal{Z}_2, \mathcal{Z}_3, \mathcal{Z}_4, \mathcal{Z}_5, \mathcal{Z}_6, \mathcal{Z}_7, \mathcal{Z}_8, \mathcal{Z}_9, \mathcal{Z}_{10}, \mathcal{Z}_{11}, \mathcal{Z}_{12}, \mathcal{W}_1, \mathcal{W}_2, \mathcal{W}_3, \mathcal{W}_4, \mathcal{W}_5, \mathcal{W}_6, \mathcal{W}_7.$$

Taking the direct sum of α_+ and α_-, we get an exact sequence $0 \to N_{l+1} \to Y_{l+1} \oplus Y_{l+1} \to M_{l+1} \to 0$. Thus there are exact squares in the left below, which produce an exact sequence in the right below.

$$\begin{array}{ccccc} X_{l+1} & \longrightarrow & M_l & \longrightarrow & X_l \\ \downarrow & & \downarrow & & \downarrow \\ N_{l+1} & \longrightarrow & Y_{l+1} & \longrightarrow & N_l \\ \downarrow & & \downarrow & & \downarrow \\ Y_{l+1} & \longrightarrow & M_{l+1} & \longrightarrow & X_{l+1} \end{array} \qquad\qquad 0 \to X_{l+1} \to X_l \oplus Y_{l+1} \to X_{l+1} \to 0.$$

Hence X_l belongs to $\operatorname{ext} X_{l+1}$. Thanks to (3.5.2) and Lemma 2.15(2), for all integers $1 \leqslant h \leqslant l$, it holds that

$$\operatorname{ext} Y_h = \operatorname{ext} X_h = \operatorname{ext} Y_1 = \operatorname{ext} X_l = \operatorname{ext} X_{l+1} = \operatorname{ext} Y_{l+1} = \operatorname{CM}(R).$$

By $\beta, \gamma_\pm$ we have $\operatorname{ext}\{A, B\} = \operatorname{ext}\{C_+, D_+\} = \operatorname{ext}\{C_-, D_-\} = \operatorname{CM}(R)$. As $M_{l+1} \cong D_+ \oplus D_-$, it holds that

$$\begin{aligned} C_+, D_+ &\in \operatorname{ext}\{C_+, D_+, D_-\} = \operatorname{ext}\{M_{l+1}, C_+\} \subseteq \operatorname{ext}\{M_j, C_+\}, \\ C_-, D_- &\in \operatorname{ext}\{C_-, D_+, D_-\} = \operatorname{ext}\{M_{l+1}, C_-\} \subseteq \operatorname{ext}\{M_j, C_-\} \end{aligned}$$

for all $1 \leqslant j \leqslant l$, where the inclusions follow from (3.5.2). Hence $\operatorname{ext}\{M_j, C_+\} = \operatorname{ext}\{M_j, C_-\} = \operatorname{CM}(R)$ hold for every $1 \leqslant j \leqslant l+1$. Applying Lemma 2.15(2), we get $\operatorname{ext}\{N_j, D_+\} = \operatorname{ext}\{N_j, D_-\} = \operatorname{CM}(R)$ for every $1 \leqslant j \leqslant l+1$. It is observed from (3.5.2) that $A, B \in \operatorname{ext}\{A, M_j\} \cap \operatorname{ext}\{B, N_j\} \cap \operatorname{ext}\{M_j, N_{j'}\}$, which implies that $\operatorname{ext}\{A, M_j\} = \operatorname{ext}\{B, N_j\} = \operatorname{ext}\{M_j, N_{j'}\} = \operatorname{CM}(R)$ for all $1 \leqslant j, j' \leqslant l$. So far, we have shown that

$$\operatorname{ext} \mathcal{C} = \operatorname{CM}(R) \text{ for all } \mathcal{C} \in \left\{ \begin{array}{c} \{X_h\}, \{Y_h\}, \{A, B\}, \{C_+, D_+\}, \{C_-, D_-\}, \\ \{C_+, M_j\}, \{C_-, M_j\}, \{D_+, N_j\}, \{D_-, N_j\}, \\ \{A, M_j\}, \{B, N_j\}, \{M_j, N_{j'}\} \end{array} \middle| \begin{array}{c} 1 \leqslant h \leqslant l+1 \\ 1 \leqslant j, j' \leqslant l \end{array} \right\}. \tag{3.5.4}$$

Since $M_{l+1} \cong D_+ \oplus D_-$ and $N_0 \cong A \oplus R$, the exact sequences $\alpha_\pm, \rho_l, \sigma_l, \rho_{l-1}, \sigma_{l-1}, \dots, \rho_1, \sigma_1, \rho_0, \delta$ and the exact sequences $\gamma_+, \zeta_l, \alpha_-$ respectively give rise to the following two diagrams of exact squares.

$$\begin{array}{ccccccccccccccccc} C_\pm & \rightarrowtail & Y_{l+1} & \rightarrowtail & D_\mp \oplus N_l & \rightarrowtail & D_\mp \oplus Y_l & \rightarrowtail & D_\mp \oplus N_{l-1} & \rightarrowtail \cdots \rightarrowtail & D_\mp \oplus N_1 & \rightarrowtail & D_\mp \oplus Y_1 & \rightarrowtail & D_\mp \oplus A \oplus R & \rightarrowtail & D_\mp \oplus R \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & D_\pm & \longrightarrow & X_{l+1} & \longrightarrow & M_l & \longrightarrow & X_l & \longrightarrow \cdots \longrightarrow & X_2 & \longrightarrow & M_1 & \longrightarrow & X_1 & \longrightarrow & B \end{array}$$

$$\begin{array}{ccccccc} D_+ & \rightarrowtail & X_{l+1} & \rightarrowtail & M_l \oplus C_- & \rightarrowtail & M_l \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & C_+ & \longrightarrow & Y_{l+1} & \longrightarrow & D_- \end{array}$$

We obtain exact sequences $0 \to C_\pm \to D_\mp \oplus R \to B \to 0$ and $\varepsilon : 0 \to D_+ \to M_l \to D_- \to 0$, which show

$$R, D_\pm \in \operatorname{ext}\{B, C_\mp\}, \qquad M_l \in \operatorname{ext}\{D_+, D_-\}. \tag{3.5.5}$$

It follows from Lemma 2.15(1) and (3.5.2) that there are containments

$$C_\pm \in \operatorname{ext}\{A, D_\mp\} \quad \text{and} \quad B, M_j \in \operatorname{ext} M_l \subseteq \operatorname{ext}\{D_+, D_-\} \quad \text{for all } 1 \leqslant j \leqslant l+1. \tag{3.5.6}$$

Put $\mathfrak{p} = (y)$, $\mathfrak{q} = (x + \sqrt{-1}\, y^{l+1})R$ and $\mathfrak{r} = (x - \sqrt{-1}\, y^{l+1})R$; these are the minimal prime ideals of R. For each $E \in \operatorname{ind} \operatorname{CM}(R)$ we denote by $\chi(E) = (\chi_1(E), \chi_2(E), \chi_3(E))$ the triple of the dimensions of the vector spaces $E_\mathfrak{p}, E_\mathfrak{q}, E_\mathfrak{r}$ over the fields $R_\mathfrak{p}, R_\mathfrak{q}, R_\mathfrak{r}$ respectively. Note that

$$A = R/\mathfrak{p}, \qquad B = R/\mathfrak{qr}, \qquad C_+ = R/\mathfrak{pq}, \qquad C_- = R/\mathfrak{pr}, \qquad D_+ = R/\mathfrak{r}, \qquad D_- = R/\mathfrak{q}.$$

We see that $\chi(R) = (1,1,1)$, $\chi(A) = (1,0,0)$, $\chi(B) = (0,1,1)$, $\chi(C_+) = (1,1,0)$, $\chi(C_-) = (1,0,1)$, $\chi(D_+) = (0,0,1)$ and $\chi(D_-) = (0,1,0)$. Fix $1 \leqslant h \leqslant l+1$ and $1 \leqslant j \leqslant l$. As $\Omega X_h \cong Y_h \cong (x, y^h)R$ and X_h is generated by two elements, there is an exact sequence $0 \to (x, y^h)R \to R^{\oplus 2} \to X_h \to 0$. Localizing this at $\mathfrak{p}, \mathfrak{q}, \mathfrak{r}$ shows $\chi(X_h) = \chi(Y_h) = (1,1,1)$. Since $\Omega N_j \cong M_j \cong (xy, y^{j+1})R$, a similar argument shows $\chi(M_j) = (0,1,1)$ and $\chi(N_j) = (2,1,1)$. In summary, for all $1 \leqslant h \leqslant l+1$ and $1 \leqslant j \leqslant l$ we have the following table.

E	R, X_h, Y_h	A	B, M_j	C_+	C_-	D_+	D_-	N_j
$\chi(E)$	$(1,1,1)$	$(1,0,0)$	$(0,1,1)$	$(1,1,0)$	$(1,0,1)$	$(0,0,1)$	$(0,1,0)$	$(2,1,1)$

Since the equalities in the left below hold, so do the equalities in the right below by (3.5.6).

$$\begin{cases} \{E \mid \chi_1(E)=0\}=\{B,M_1,\dots,M_l,D_+,D_-\}, \\ \{E \mid \chi_2(E)=0\}=\{A,C_-,D_+\}, \\ \{E \mid \chi_3(E)=0\}=\{A,C_+,D_-\}. \end{cases} \qquad \begin{cases} \operatorname{ext}\{D_+,D_-\}=\operatorname{add}\{B,M_1,\dots,M_l,D_+,D_-\}=\mathcal{Y}_9, \\ \operatorname{ext}\{A,D_+\}=\operatorname{add}\{A,C_-,D_+\}=\mathcal{Y}_4, \\ \operatorname{ext}\{A,D_-\}=\operatorname{add}\{A,C_+,D_-\}=\mathcal{Y}_3. \end{cases}$$

Hence $\mathcal{Y}_3,\mathcal{Y}_4,\mathcal{Y}_9$ are extension-closed. Using Lemmas 2.12(3) and 2.22, we obtain the following six equalities.

$$\begin{cases} \operatorname{ext}\{R,D_+,D_-\}=\operatorname{add}\{R,D_+,D_-,B,M_1,\dots,M_l\}=\mathcal{X}_6, \\ \operatorname{ext}\{R,A,D_+\}=\operatorname{add}\{R,A,D_+,C_-\}=\mathcal{X}_3, \\ \operatorname{ext}\{R,A,D_-\}=\operatorname{add}\{R,A,D_-,C_+\}=\mathcal{X}_2, \end{cases}$$
$$\begin{cases} \operatorname{ext}\{R,C_+,C_-\}=\operatorname{add}\{R,C_+,C_-,A,N_1,\dots,N_l\}=\mathcal{X}_1, \\ \operatorname{ext}\{R,B,C_+\}=\operatorname{add}\{R,B,C_+,D_-\}=\mathcal{X}_5, \\ \operatorname{ext}\{R,B,C_-\}=\operatorname{add}\{R,B,C_-,D_+\}=\mathcal{X}_4. \end{cases}$$

Consequently, the subcategories $\mathcal{X}_1,\mathcal{X}_2,\mathcal{X}_3,\mathcal{X}_4,\mathcal{X}_5,\mathcal{X}_6$ are extension-closed. It is observed from (3.5.4) that for each integer $1\leqslant p\leqslant 6$ there exists no extension-closed subcategory $\mathcal{C}$ of $\operatorname{CM}(R)$ with $\mathcal{X}_p\subsetneq\mathcal{C}\subsetneq\operatorname{CM}(R)$.

We claim that for each $(p,q)\in\{(1,1),(2,1),(7,6),(8,6)\}$ there exists no extension-closed subcategory $\mathcal{C}$ of $\operatorname{CM}(R)$ with $\mathcal{Y}_p\subsetneq\mathcal{C}\subsetneq\mathcal{X}_q$. Indeed, we have $\operatorname{ext}(\mathcal{Y}_1\cup\{C_-\})=\operatorname{ext}\{R,A,C_+,C_-\}\supseteq\operatorname{ext}\{R,C_+,C_-\}=\mathcal{X}_1$, so that $\operatorname{ext}(\mathcal{Y}_1\cup\{C_-\})=\mathcal{X}_1$. Also, by (3.5.2), for each $1\leqslant j\leqslant l$, it holds that

$$\operatorname{ext}(\mathcal{Y}_1\cup\{N_j\})=\operatorname{ext}\{R,A,C_+,N_j\}\supseteq\operatorname{ext}\{R,C_+,N_{l+1}\}=\operatorname{ext}\{R,C_+,C_+\oplus C_-\}\supseteq\operatorname{ext}\{R,C_+,C_-\}=\mathcal{X}_1.$$

Hence $\operatorname{ext}(\mathcal{Y}_1\cup\{N_j\})=\mathcal{X}_1$, and thus the claim follows for $(p,q)=(1,1)$. A similar argument shows the claim for $(p,q)=(2,1)$. Also, analogously, we can show that $\operatorname{ext}(\mathcal{Y}_7\cup\{D_-\})=\operatorname{ext}(\mathcal{Y}_7\cup\{M_j\})=\mathcal{X}_6$ for $1\leqslant j\leqslant l$, which deduces the claim for $(p,q)=(7,6)$. The claim for $(p,q)=(8,6)$ is shown similarly.

It holds that $\operatorname{ext}(\mathcal{Z}_{11}\cup\{D_-\})=\operatorname{ext}\{B,D_+,D_-\}\supseteq\operatorname{ext}\{D_+,D_-\}=\mathcal{Y}_9$, and therefore $\operatorname{ext}(\mathcal{Z}_{11}\cup\{D_-\})=\mathcal{Y}_9$. By using (3.5.2), for each integer $1\leqslant j\leqslant l$ we have

$$\operatorname{ext}(\mathcal{Z}_{11}\cup\{M_j\})=\operatorname{ext}\{B,D_+,M_j\}\supseteq\operatorname{ext}\{B,D_+,M_{l+1}\}=\operatorname{ext}\{B,D_+,D_+\oplus D_-\}\supseteq\operatorname{ext}\{D_+,D_-\}=\mathcal{Y}_9,$$

whence $\operatorname{ext}(\mathcal{Z}_{11}\cup\{M_j\})=\mathcal{Y}_9$. It is observed that there is no extension-closed subcategory $\mathcal{C}$ of $\operatorname{CM}(R)$ with $\mathcal{Z}_{11}\subsetneq\mathcal{C}\subsetneq\mathcal{Y}_9$. By an analogous argument, there is no extension-closed subcategory $\mathcal{C}$ with $\mathcal{Z}_{12}\subsetneq\mathcal{C}\subsetneq\mathcal{Y}_9$.

Now we have obtained the Hasse diagram of $\mathfrak{E}(R)$ as in the theorem. ■

Remark 3.6. In Theorem 3.5(2), we can prove more. Applying Ω to the exact sequence ε, we get $N_l\in\operatorname{ext}\{C_+,C_-\}$. It follows from (3.5.2) that $R\in\operatorname{ext}N_l\subseteq\operatorname{ext}\{C_+,C_-\}$. Using (3.5.5), we obtain equalities

$$\mathcal{X}_1=\operatorname{ext}\{C_+,C_-\},\qquad \mathcal{X}_5=\operatorname{ext}\{B,C_+\},\qquad \mathcal{X}_4=\operatorname{ext}\{B,C_-\}.$$

Using (3.5.2), the isomorphisms $M_{l+1}\cong D_+\oplus D_-$, $N_{l+1}\cong C_+\oplus C_-$ and Lemma 2.12(3), for all $1\leqslant j\leqslant l$ we have

$$\begin{aligned} \operatorname{ext}M_j&=\operatorname{add}\{B,M_1,\dots,M_{l+1}\}=\operatorname{add}\{B,M_1,\dots,M_l,D_+,D_-\}=\operatorname{ext}\{D_+,D_-\}=\mathcal{Y}_9, \\ \operatorname{ext}N_j&=\operatorname{add}\{R,A,N_1,\dots,N_{l+1}\}=\operatorname{add}\{R,A,N_1,\dots,N_l,C_+,C_-\}=\operatorname{ext}\{C_+,C_-\}=\mathcal{X}_1, \\ \operatorname{ext}\{R,M_j\}&=\operatorname{add}\{R,B,M_1,\dots,M_l,D_+,D_-\}=\operatorname{ext}\{R,D_+,D_-\}=\mathcal{X}_6. \end{aligned}$$

Theorem 3.7. *Let R be one of the following hypersurfaces, where n is an integer.*

$$\begin{cases} \mathrm{A}_n^0: & k[\![x]\!]/(x^{n+1}) & (n\geqslant 1), \\ \mathrm{A}_n^2: & k[\![x,y,z]\!]/(x^2+y^{n+1}+z^2) & (n\geqslant 1), \\ \mathrm{D}_n^2: & k[\![x,y,z]\!]/(x^2y+y^{n-1}+z^2) & (n\geqslant 4), \\ \mathrm{E}_6^2: & k[\![x,y,z]\!]/(x^3+y^4+z^2), \\ \mathrm{E}_7^2: & k[\![x,y,z]\!]/(x^3+xy^3+z^2), \\ \mathrm{E}_8^2: & k[\![x,y,z]\!]/(x^3+y^5+z^2). \end{cases}$$

Then the Hasse diagram of the poset $\mathfrak{E}(R)$ is the following graph.

$$\begin{array}{c} \mathrm{CM}(R) \\ | \\ \operatorname{add} R \\ | \\ 0 \end{array}$$

Proof. The case A_n^0 follows from [9, Proposition 5.6]. In the other cases, since $\dim R = 2$, Lemma 2.20 implies that the Auslander–Reiten translation functor is isomorphic to the identity functor. From the Auslander–Reiten quivers exhibited in [13, (10.15)] it is easy to observe that for any nonfree indecomposable maximal Cohen–Macaulay R-module E the equality $\operatorname{ext} E = \mathrm{CM}(R)$ holds. The assertion follows from this. ■

4. On the $\mathrm{A}_\infty, \mathrm{D}_\infty$-singularities with dimension at most two

In this section, we give a complete classification of the extension-closed subcategories of $\mathrm{CM}_0(R)$ in the case where R is a P-singularity with $P \in \{\mathrm{A}_\infty, \mathrm{D}_\infty\}$ and has dimension at most two.

Theorem 4.1. *Let R be one of the following hypersurfaces.*

$$\begin{cases} \mathrm{A}_\infty^1 : \ k[\![x, y]\!]/(x^2), \\ \mathrm{D}_\infty^2 : \ k[\![x, y, z]\!]/(x^2y + z^2). \end{cases}$$

Then the Hasse diagram of the poset $\mathfrak{E}_0(R)$ is the following.

$$\begin{array}{c} \mathrm{CM}_0(R) \\ | \\ \operatorname{add} R \\ | \\ 0 \end{array}$$

Proof. Let E be a nonfree indecomposable maximal Cohen–Macaulay R-module that is locally free on the punctured spectrum of R. Then there is an isomorphism $\tau E \cong E$; this follows from [8, (6.1)] in case A_∞^1, and from Lemma 2.20 in case D_∞^2. From the Auslander–Reiten quiver in [8, (6.1)] in case A_∞^1 and in [8, (6.2)] in case D_∞^2, it is easy to observe that $\operatorname{ext} E = \mathrm{CM}_0(R)$. The assertion follows from this. ■

Theorem 4.2. *Let $R = k[\![x, y, z]\!]/(xy)$ be the A_∞^2-singularity. For each integer $i > 0$, let $I_i = (x, z^i)$ and $J_i = (y, z^i)$ be ideals of R. Then the Hasse diagram of the poset $\mathfrak{E}_0(R)$ is the following.*

$$\begin{array}{ccc} \mathrm{CM}_0(R) & & \\ | & \diagdown & \\ \operatorname{add}\{R, I_1, I_2, \dots\} & & \operatorname{add}\{R, J_1, J_2, \dots\} \\ & \diagdown & | \\ & & \operatorname{add} R \\ & & | \\ & & 0 \end{array}$$

Proof. By Lemma 2.20 we have $\tau E \cong E$ for every $E \in \operatorname{ind} \mathrm{CM}_0(R) \setminus \{R\}$. From the Auslander–Reiten quiver given in [8, (6.2)], we see that $\operatorname{ind} \mathrm{CM}_0(R) = \{R, I_i, J_i \mid i \in \mathbb{Z}_{>0}\}$ and that

$$R \in \operatorname{ext} I_1 = \operatorname{ext} I_2 = \operatorname{ext} I_3 = \cdots, \qquad R \in \operatorname{ext} J_1 = \operatorname{ext} J_2 = \operatorname{ext} J_3 = \cdots.$$

Hence $\operatorname{ext}\{I_p, J_q\} = \mathrm{CM}_0(R)$ for all integers $p, q > 0$. To get the Hasse diagram in the assertion, it suffices to show that $J_q \notin \operatorname{ext} I_p$ (and then by symmetry we get $I_q \notin \operatorname{ext} J_p$) for all integers $p, q > 0$. We have only to verify $J_1 \notin \operatorname{ext} I_1$, as if $J_q \in \operatorname{ext} I_p$, then $J_1 \in \operatorname{ext} J_1 = \operatorname{ext} J_q \subseteq \operatorname{ext} I_p = \operatorname{ext} I_1$.

Now, assume that $J_1 \in \operatorname{ext} I_1$. It follows from [10, Propositions 2.2(1) and 2.4] that there exist an R-module M and a filtration $0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_n = M$ of R-submodules of M with $M_i/M_{i-1} \cong I_1$ for each $1 \leqslant i \leqslant n$ such that J_1 is a direct summand of M. We establishes two claims.

Claim 1. Let N be an R-module. Denote by $R^\times$ the set of units of R. Put $A_j = \begin{pmatrix} y & -z^j \\ 0 & x \end{pmatrix}$ for each $j > 0$.

(1) The following are equivalent for all positive integers r and $l_1, \dots, l_r$.
  (a) There exists an exact sequence $0 \to I_{l_1} \oplus \cdots \oplus I_{l_r} \to N \to I_1 \to 0$.
  (b) The module N is isomorphic to the cokernel of the R-linear map given by the matrix $D(u_1, \dots, u_r) = \begin{pmatrix} A_{l_1} & O & \cdots & O & U_1 \\ O & A_{l_2} & \cdots & O & U_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ O & O & \cdots & A_{l_r} & U_r \\ O & O & \cdots & O & A_1 \end{pmatrix}$, where $U_i = \left(\begin{smallmatrix} 0 & u_i \\ 0 & 0 \end{smallmatrix}\right)$ with $u_i \in R^\times \cup \{0\}$ for each $1 \leqslant i \leqslant r$.

(2) Assume that condition (1b) is satisfied.
  (a) If $u_j = 0$ for some $1 \leqslant j \leqslant r$, then there exist an exact sequence $0 \to I_{l_1} \oplus \cdots \oplus I_{l_{j-1}} \oplus I_{l_{j+1}} \oplus \cdots \oplus I_{l_r} \to N' \to I_1 \to 0$ and an isomorphism $N \cong I_{l_j} \oplus N'$.
  (b) If $l_j \leqslant l_h$ and $u_h \in R^\times$ for some $1 \leqslant j \neq h \leqslant r$, then $D(u_1, \dots, u_r) \cong D(u_1, \dots, u_{j-1}, 0, u_{j+1}, \dots, u_r)$.
  (c) If $r = 1$ and $u_1 \in R^\times$, then there exists an isomorphism $N \cong I_{l_1+1} \oplus R$.
  (d) The module N is isomorphic to either $I_{l_1} \oplus \cdots \oplus I_{l_r} \oplus I_1$ or $I_{l_1} \oplus \cdots \oplus I_{l_{s-1}} \oplus I_{l_s+1} \oplus I_{l_{s+1}} \oplus I_{l_r} \oplus R$ for some $1 \leqslant s \leqslant r$.

Proof of Claim. (1) (a) $\Rightarrow$ (b): Applying the horseshoe lemma repeatedly gives rise to a commutative diagram (4.2.1)

$$\begin{array}{ccccccccccc}
 & & 0 & & 0 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longleftarrow & \bigoplus_{i=1}^r I_{l_i} & \xleftarrow{\bigoplus_{i=1}^r (x\ z^{l_i})} & F_0 & \xleftarrow{\bigoplus_{i=1}^r A_{l_i}} & F_1 & \xleftarrow{\bigoplus_{i=1}^r B_{l_i}} & F_2 & \xleftarrow{\bigoplus_{i=1}^r A_{l_i}} & F_3 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longleftarrow & N & \longleftarrow & E_0 & \xleftarrow{H} & E_1 & \longleftarrow & E_2 & \longleftarrow & E_3 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longleftarrow & I_1 & \xleftarrow{(x\ z)} & G_0 & \xleftarrow{A_1} & G_1 & \xleftarrow{B_1} & G_2 & \xleftarrow{A_1} & G_3 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0 & & 0 & & 0
\end{array}$$

with exact rows and columns, where $F_h = R^{\oplus 2r}$, $G_h = R^{\oplus 2}$, $E_h = F_h \oplus G_h = R^{\oplus(2r+2)}$ for $0 \leqslant h \leqslant 3$, $B_j = \left(\begin{smallmatrix} x & z^j \\ 0 & y \end{smallmatrix}\right)$ for $j > 0$, and $H = \begin{pmatrix} A_{l_1} & O & \cdots & O & C_1 \\ O & A_{l_2} & \cdots & O & C_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ O & O & \cdots & A_{l_r} & C_r \\ O & O & \cdots & O & A_1 \end{pmatrix}$ with $C_i = \left(\begin{smallmatrix} a_i & b_i \\ c_i & d_i \end{smallmatrix}\right)$ and $a_i, b_i, c_i, d_i \in R$ for $1 \leqslant i \leqslant r$. We do diagram chasing. Take any vector $v \in G_2$. The vector $B_1 v \in G_1$ comes from $\left(\begin{smallmatrix} \mathbf{0} \\ B_1 v \end{smallmatrix}\right) \in E_1 = F_1 \oplus G_1$. Since $A_1 B_1 v = \mathbf{0}$, the map H sends $\left(\begin{smallmatrix} \mathbf{0} \\ B_1 v \end{smallmatrix}\right)$ to $\begin{pmatrix} C_1 B_1 v \\ \cdots \\ C_r B_1 v \\ \mathbf{0} \end{pmatrix} \in E_0 = F_0 \oplus G_0$, which comes from $\begin{pmatrix} C_1 B_1 v \\ \cdots \\ C_r B_1 v \end{pmatrix} \in F_0$. This goes to $\mathbf{0} \in \bigoplus_{i=1}^r I_{l_i}$ by the map $\bigoplus_{i=1}^r (x\ z^{l_i})$ by the snake lemma, so that it belongs to the image of $\bigoplus_{i=1}^r A_{l_i}$, which coincides with the kernel of $\bigoplus_{i=1}^r B_{l_i}$. Hence $\mathbf{0} = \begin{pmatrix} B_{l_1} & & \\ & \cdots & \\ & & B_{l_r} \end{pmatrix} \begin{pmatrix} C_1 B_1 v \\ \cdots \\ C_r B_1 v \end{pmatrix} = \begin{pmatrix} B_{l_1} C_1 B_1 v \\ \cdots \\ B_{l_r} C_r B_1 v \end{pmatrix}$. Since this holds for any vector $v \in G_2$, it is observed that $B_{l_i} C_i B_1$ is a zero matrix for every $1 \leqslant i \leqslant r$. Thus,

$$(4.2.2) \qquad \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right) = \begin{pmatrix} x & z^{l_i} \\ 0 & y \end{pmatrix} \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix} \begin{pmatrix} x & z \\ 0 & y \end{pmatrix} = \begin{pmatrix} x(xa_i + z^{l_i} c_i) & z(xa_i + z^{l_i} c_i + y z^{l_i - 1} d_i) \\ 0 & y(zc_i + yd_i) \end{pmatrix} \quad \text{for every integer } 1 \leqslant i \leqslant r.$$

Fix $1 \leqslant i \leqslant r$. We see that $xa_i + z^{l_i} c_i + y z^{l_i - 1} d_i = 0$, $xa_i + z^{l_i} c_i = ye_i$ and $zc_i + yd_i = xf_i$ for some $e_i, f_i \in R$. Note that the sequence $R \xleftarrow{\mu = (x\ y\ z)} R^{\oplus 3} \xleftarrow{\nu = \left(\begin{smallmatrix} y & 0 & z & 0 \\ 0 & x & 0 & z \\ 0 & 0 & -x & -y \end{smallmatrix}\right)} R^{\oplus 4}$ is exact; it is part of a minimal free resolution of the residue field k. We have $xf_i + y(-d_i) + z(-c_i) = 0$, so that $\begin{pmatrix} f_i \\ -d_i \\ -c_i \end{pmatrix} \in R^{\oplus 3}$ belongs to the kernel of the above map μ, which is equal to the image of the above map ν. Hence there exist elements $q_i, s_i, t_i, p_i \in R$ such that $\begin{pmatrix} f_i \\ -d_i \\ -c_i \end{pmatrix} = \begin{pmatrix} y & 0 & z & 0 \\ 0 & x & 0 & z \\ 0 & 0 & -x & -y \end{pmatrix} \begin{pmatrix} q_i \\ s_i \\ t_i \\ p_i \end{pmatrix} = \begin{pmatrix} yq_i + zt_i \\ xs_i + zp_i \\ -xt_i - yp_i \end{pmatrix}$. We thus get $c_i = xt_i + yp_i$ and $d_i = -xs_i - zp_i$. As $xa_i + z^{l_i}(xt_i + yp_i) = ye_i$, we have $x(a_i + z^{l_i} t_i) = y(e_i - z^{l_i} p_i) \in (x) \cap (y) = 0$. Hence $a_i + z^{l_i} t_i = yg_i$ for some $g_i \in R$, so that $a_i = yg_i - z^{l_i} t_i$. Write $b_i = u_i + x\alpha_i + y\beta_i + z\gamma_i$ with $u_i \in R^\times \cup \{0\}$ and $\alpha_i, \beta_i, \gamma_i \in R$.

It follows that

$$
\begin{aligned}
H &= \begin{pmatrix} A_{l_1} & & & C_1 \\ & \cdots & & \cdots \\ & & A_{l_r} & C_r \\ & & & A_1 \end{pmatrix}
= \begin{pmatrix} y & -z^{l_1} & & & & a_1 & b_1 \\ 0 & x & & & & c_1 & d_1 \\ & & \cdots & & & \cdots & \cdots \\ & & & y & -z^{l_r} & a_r & b_r \\ & & & 0 & x & c_r & d_r \\ & & & & & y & -z \\ & & & & & 0 & x \end{pmatrix}
= \begin{pmatrix} y & -z^{l_1} & & & & yg_1-z^{l_1}t_1 & b_1 \\ 0 & x & & & & xt_1+yp_1 & -xs_1-zp_1 \\ & & \cdots & & & \cdots & \cdots \\ & & & y & -z^{l_r} & yg_r-z^{l_r}t_r & b_r \\ & & & 0 & x & xt_r+yp_r & -xs_r-zp_r \\ & & & & & y & -z \\ & & & & & 0 & x \end{pmatrix} \\
&\cong \begin{pmatrix} y & -z^{l_1} & & & & -z^{l_1}t_1 & b_1 \\ 0 & x & & & & xt_1+yp_1 & -xs_1-zp_1 \\ & & \cdots & & & \cdots & \cdots \\ & & & y & -z^{l_r} & -z^{l_r}t_r & b_r \\ & & & 0 & x & xt_r+yp_r & -xs_r-zp_r \\ & & & & & y & -z \\ & & & & & 0 & x \end{pmatrix}
\cong \begin{pmatrix} y & -z^{l_1} & & & & -z^{l_1}t_1 & b_1 \\ 0 & x & & & & xt_1+yp_1 & -zp_1 \\ & & \cdots & & & \cdots & \cdots \\ & & & y & -z^{l_r} & -z^{l_r}t_r & b_r \\ & & & 0 & x & xt_r+yp_r & -zp_r \\ & & & & & y & -z \\ & & & & & 0 & x \end{pmatrix} \\
&\cong \begin{pmatrix} y & -z^{l_1} & & & & 0 & b_1 \\ 0 & x & & & & yp_1 & -zp_1 \\ & & \cdots & & & \cdots & \cdots \\ & & & y & -z^{l_r} & 0 & b_r \\ & & & 0 & x & yp_r & -zp_r \\ & & & & & y & -z \\ & & & & & 0 & x \end{pmatrix}
\cong \begin{pmatrix} y & -z^{l_1} & & & & 0 & b_1 \\ 0 & x & & & & 0 & 0 \\ & & \cdots & & & \cdots & \cdots \\ & & & y & -z^{l_r} & 0 & b_r \\ & & & 0 & x & 0 & 0 \\ & & & & & y & -z \\ & & & & & 0 & x \end{pmatrix}
= \begin{pmatrix} y & -z^{l_1} & & & & 0 & u_1+x\alpha_1+y\beta_1+z\gamma_1 \\ 0 & x & & & & 0 & 0 \\ & & \cdots & & & \cdots & \cdots \\ & & & y & -z^{l_r} & 0 & u_r+x\alpha_r+y\beta_r+z\gamma_r \\ & & & 0 & x & 0 & 0 \\ & & & & & y & -z \\ & & & & & 0 & x \end{pmatrix} \\
&\cong \begin{pmatrix} y & -z^{l_1} & & & & 0 & u_1+z\gamma_1 \\ 0 & x & & & & 0 & 0 \\ & & \cdots & & & \cdots & \cdots \\ & & & y & -z^{l_r} & 0 & u_r+z\gamma_r \\ & & & 0 & x & 0 & 0 \\ & & & & & y & -z \\ & & & & & 0 & x \end{pmatrix}
\cong \begin{pmatrix} y & -z^{l_1} & & & & y\gamma_1 & u_1 \\ 0 & x & & & & 0 & 0 \\ & & \cdots & & & \cdots & \cdots \\ & & & y & -z^{l_r} & y\gamma_r & u_r \\ & & & 0 & x & 0 & 0 \\ & & & & & y & -z \\ & & & & & 0 & x \end{pmatrix}
\cong \begin{pmatrix} y & -z^{l_1} & & & & 0 & u_1 \\ 0 & x & & & & 0 & 0 \\ & & \cdots & & & \cdots & \cdots \\ & & & y & -z^{l_r} & 0 & u_r \\ & & & 0 & x & 0 & 0 \\ & & & & & y & -z \\ & & & & & 0 & x \end{pmatrix}.
\end{aligned}
$$

(b) $\Rightarrow$ (a): For each integer $1 \leqslant i \leqslant r$, let $a_i = c_i = d_i = 0$ and $b_i = u_i \in R^\times \cup \{0\}$. We easily verify that the first equality in (4.2.2) holds. This means that (4.2.1) is a commutative diagram with exact rows and columns if we put $C_i = \left(\begin{smallmatrix} a_i & b_i \\ c_i & d_i \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & u_i \\ 0 & 0 \end{smallmatrix}\right) = U_i$ for every integer $1 \leqslant i \leqslant r$.

(2)(a) We may assume $j = 1$. Then $D(u_1, \dots, u_r) = D(0, u_2, \dots, u_r) = A_{l_1} \oplus D(u_2, \dots, u_r)$. Set $N' = \operatorname{Cok} D(u_2, \dots, u_r)$. We have $N \cong I_{l_1} \oplus N'$. By (1) there is an exact sequence $0 \to I_{l_2} \oplus \cdots \oplus I_{l_r} \to N' \to I_1 \to 0$.

(b) Replacing $A_{l_1}, \dots, A_{l_r}$ if necessary, we may assume $j = 1$ and $h = 2$, so $l_2 - l_1 \geqslant 0$. We have

$$
\begin{aligned}
D(u_1, u_2, \dots, u_r) &= \begin{pmatrix} y & -z^{l_1} & 0 & 0 & \cdots & 0 & u_1 \\ 0 & x & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & y & -z^{l_2} & \cdots & 0 & u_2 \\ 0 & 0 & 0 & x & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}
\cong \begin{pmatrix} y & -z^{l_1} & -u_2^{-1}u_1y & u_2^{-1}u_1z^{l_2} & \cdots & 0 & 0 \\ 0 & x & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & y & -z^{l_2} & \cdots & 0 & u_2 \\ 0 & 0 & 0 & x & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix} \\
&\cong \begin{pmatrix} y & -z^{l_1} & 0 & u_2^{-1}u_1z^{l_2} & \cdots & 0 & 0 \\ 0 & x & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & y & -z^{l_2} & \cdots & 0 & u_2 \\ 0 & 0 & 0 & x & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}
\cong \begin{pmatrix} y & -z^{l_1} & 0 & 0 & \cdots & 0 & 0 \\ 0 & x & 0 & u_2^{-1}u_1z^{l_2-l_1}x & \cdots & 0 & 0 \\ 0 & 0 & y & -z^{l_2} & \cdots & 0 & u_2 \\ 0 & 0 & 0 & x & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix} \\
&\cong \begin{pmatrix} y & -z^{l_1} & 0 & 0 & \cdots & 0 & 0 \\ 0 & x & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & y & -z^{l_2} & \cdots & 0 & u_2 \\ 0 & 0 & 0 & x & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix} = D(0, u_2, \dots, u_r).
\end{aligned}
$$

(c) Put $p = l_1$ and $h = u_1$. Since h is a unit of R, we have

$$
\begin{aligned}
D(u_1) = D(h) &= \begin{pmatrix} y & -z^p & 0 & h \\ 0 & x & 0 & 0 \\ 0 & 0 & y & -z \\ 0 & 0 & 0 & x \end{pmatrix}
\cong \begin{pmatrix} 0 & 0 & 0 & h \\ 0 & x & 0 & 0 \\ h^{-1}yz & -h^{-1}z^{p+1} & y & -z \\ 0 & h^{-1}xz^p & 0 & x \end{pmatrix}
\cong \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & x & 0 & 0 \\ h^{-1}yz & -h^{-1}z^{p+1} & y & 0 \\ 0 & h^{-1}xz^p & 0 & 0 \end{pmatrix} \\
&\cong \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & x & 0 & 0 \\ 0 & -h^{-1}z^{p+1} & y & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}
\cong \begin{pmatrix} y & -z^{p+1} & 0 & 0 \\ 0 & x & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.
\end{aligned}
$$

Therefore, we obtain a desired isomorphism $N \cong I_{p+1} \oplus R$.

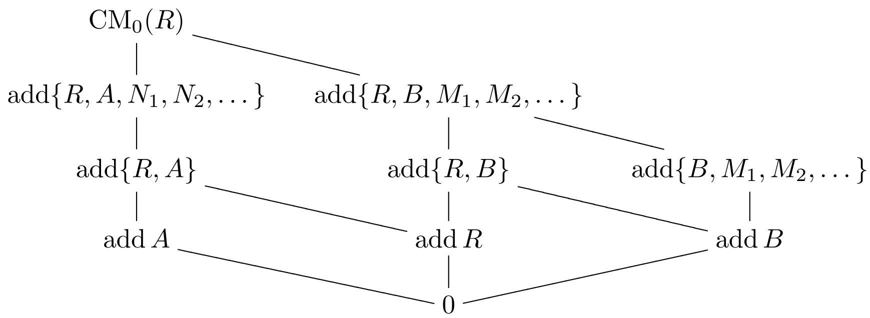
(d) The assertion follows from iterated application of (a), (b), (c) and (1). □

Claim 2. For every integer $1 \leqslant q \leqslant n$ one has $M_q \cong I_{l_1} \oplus \cdots \oplus I_{l_q}$, where $0 \leqslant l_1, \dots, l_q \leqslant q$ and $I_0 := R$.

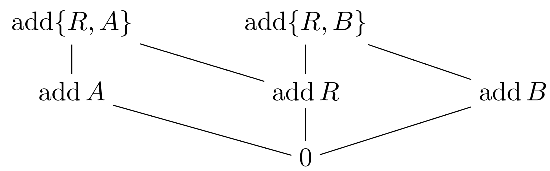
Proof of Claim. We use induction on q. There is an isomorphism $M_1 \cong I_1$, so we are done by putting $l_1 = 1$ when $q = 1$. We consider the case $q \geqslant 2$. There exists an exact sequence $0 \to M_{q-1} \to M_q \to I_1 \to 0$. The induction hypothesis implies $M_{q-1} \cong R^{\oplus e} \oplus I_{l_1} \oplus \cdots \oplus I_{l_r}$ with $e \geqslant 0$, $r = q-1-e$ and $1 \leqslant l_1, \dots, l_r \leqslant q-1$. The same argument as in the proof of Lemma 2.12(3) shows that there is an exact sequence $0 \to L \to K \to I_1 \to 0$ such that $L = I_{l_1} \oplus \cdots \oplus I_{l_r}$ and $R^{\oplus e} \oplus K \cong M_q$. It is observed from Claim 2(2d) that $K \cong I_{h_1} \oplus \cdots \oplus I_{h_{r+1}}$ for some $0 \leqslant h_1, \dots, h_{r+1} \leqslant q$. Now the assertion follows. □

Claim 2 yields an isomorphism $M = M_n \cong I_{l_1} \oplus \cdots \oplus I_{l_n}$ with $0 \leqslant l_1, \dots, l_n \leqslant n$. This is a contrariction, since J_1 is not a direct summand of this direct sum; note that $J_1, R, I_{l_1}, \dots, I_{l_n}$ are all indecomposable R-modules. This contradiction shows that J_1 is not in $\operatorname{ext} I_1$, and now the proof of the theorem is completed. ■

Theorem 4.3. *Let $R = k[\![x,y]\!]/(x^2y)$ be the D_∞^1-singularity. Put $A = R/(y)$ and $B = R/(x^2)$. Moreover, set $M_j = \operatorname{Cok}\left(\begin{smallmatrix} x & y^j \\ 0 & -x \end{smallmatrix}\right)$ and $N_j = \operatorname{Cok}\left(\begin{smallmatrix} xy & y^{j+1} \\ 0 & -xy \end{smallmatrix}\right)$ for each integer $j \geqslant 0$. Then the Hasse diagram of $\mathfrak{E}_0(R)$ is:*



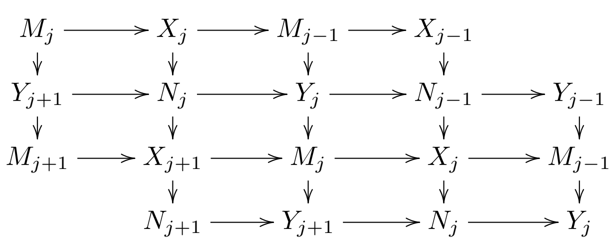
Proof. The proof goes along the same lines as in the proof of Theorem 3.5(1). By Lemmas 2.16 and 2.17, we get the subdiagram of the Hasse diagram of $\mathfrak{E}_0(R)$:



Put $X_j = \operatorname{Cok}\left(\begin{smallmatrix} x & y^j \\ 0 & -xy \end{smallmatrix}\right)$ and $Y_j = \operatorname{Cok}\left(\begin{smallmatrix} xy & y^j \\ 0 & -x \end{smallmatrix}\right)$ for each integer $j \geqslant 0$. By [8, (6.1)] and Lemma 2.20, we have that $\operatorname{ind} \mathrm{CM}_0(R) = \{R, A, B, X_j, Y_j, M_j, N_j \mid j \in \mathbb{Z}_{>0}\}$ and there exist exact sequences

$$\begin{array}{ll} 0 \to Y_{j+1} \to M_{j+1} \oplus N_j \to X_{j+1} \to 0, & 0 \to M_j \to X_j \oplus Y_{j+1} \to N_j \to 0, \\ 0 \to X_{j+1} \to N_{j+1} \oplus M_j \to Y_{j+1} \to 0, & \sigma_j : 0 \to N_j \to Y_j \oplus X_{j+1} \to M_j \to 0 \end{array} \quad (j \in \mathbb{N}),$$

where the maps $F \to G$ with $F, G \in \operatorname{ind} \mathrm{CM}_0(R)$ are all the same, and isomorphisms $M_0 \cong B$, $N_0 \cong A \oplus R$, $X_0 \cong Y_0 \cong R$, $\Omega A \cong B$, $\Omega X_j \cong Y_j \cong (x, y^j)R$ and $\Omega N_j \cong M_j \cong (xy, y^{j+1})R$ for all $j \in \mathbb{Z}_{>0}$. For each $j \in \mathbb{Z}_{>0}$ there are exact squares in the left below, which produce exact sequences in the right below.



$$0 \to M_j \to M_{j+1} \oplus M_{j-1} \to M_j \to 0,$$
$$0 \to X_j \to X_{j+1} \oplus X_{j-1} \to X_j \to 0,$$
$$0 \to N_j \to N_{j+1} \oplus N_{j-1} \to N_j \to 0,$$
$$0 \to Y_j \to Y_{j+1} \oplus Y_{j-1} \to Y_j \to 0.$$

These four exact sequences show that

$$\begin{array}{ll} B = M_0 \in \operatorname{ext} M_1 = \operatorname{ext} M_2 = \operatorname{ext} M_3 = \cdots, & \operatorname{ext} X_1 = \operatorname{ext} X_2 = \operatorname{ext} X_3 = \cdots, \\ A \oplus R = N_0 \in \operatorname{ext} N_1 = \operatorname{ext} N_2 = \operatorname{ext} N_3 = \cdots, & \operatorname{ext} Y_1 = \operatorname{ext} Y_2 = \operatorname{ext} Y_3 = \cdots. \end{array}$$

Since Y_1 is isomorphic to the maximal ideal $(x, y)R$, we have $\operatorname{ext} Y_1 = \mathrm{CM}_0(R)$ by [9, Corollary 2.6]. Lemma 2.15(2) implies that $\operatorname{ext} X_1 = \mathrm{CM}_0(R)$. Thus for all $j \in \mathbb{Z}_{>0}$ there are inclusions and equalities

$$\operatorname{ext} M_j \supseteq \operatorname{add}\{B, M_1, M_2, \dots\}, \quad \operatorname{ext} N_j \supseteq \operatorname{add}\{R, A, N_1, N_2, \dots\}, \quad \operatorname{ext} X_j = \operatorname{ext} Y_j = \mathrm{CM}_0(R).$$

Put $\mathfrak{p} = (y) \in \operatorname{Spec} R$. It is clear that $(M_0)_\mathfrak{p} = B_\mathfrak{p} = 0 \neq A_\mathfrak{p}$. For each $j \in \mathbb{Z}_{>0}$, as $M_j \cong (xy, y^{j+1})R$ and $Y_j \cong (x, y^j)R$, we have $(M_j)_\mathfrak{p} \subseteq \mathfrak{p}R_\mathfrak{p} = 0$ and $(Y_j)_\mathfrak{p} \cong R_\mathfrak{p}$. There is an exact sequence $0 \to X_j \to R^{\oplus 2} \to Y_j \to 0$, which shows that $(X_j)_\mathfrak{p} \cong R_\mathfrak{p}$. The exact sequence σ_j shows $(N_j)_\mathfrak{p} \neq 0$. Hence there is an equality

$$\{E \in \operatorname{ind} \mathrm{CM}_0(R) \mid E_\mathfrak{p} = 0\} = \{B, M_j \mid j \in \mathbb{Z}_{>0}\}.$$

Since every $E \in \operatorname{ext} M_j$ is such that $E_\mathfrak{p} = 0$, we see that $\operatorname{ext} M_j = \operatorname{add}\{B, M_1, M_2, \dots\}$. Lemma 2.15(1) yields that $\operatorname{ext} N_j = \operatorname{add}\{R, A, N_1, N_2, \dots\}$ for every $j \in \mathbb{Z}_{>0}$. Lemma 2.12(3) gives rise to an equality $\operatorname{ext}\{R, M_j\} = \operatorname{add}\{R, B, M_1, M_2, \dots\}$ for every $j \in \mathbb{Z}_{>0}$. The exact sequence σ_1 shows that $X_2 \in \operatorname{ext}\{M_1, N_1\}$, which implies that $\operatorname{ext}\{M_1, N_1\} = \mathrm{CM}_0(R)$. Therefore, $\operatorname{ext}\{M_j, N_h\} = \mathrm{CM}_0(R)$ for all $j, h \in \mathbb{Z}_{>0}$. From σ_0 and the proof of Lemma 2.12(3) we get an exact sequence $0 \to A \to X_1 \to B \to 0$, which implies $X_1 \in \operatorname{ext}\{A, B\}$, and $\operatorname{ext}\{A, B\} = \mathrm{CM}_0(R)$. Now we obtain the Hasse diagram as in the theorem. ■

We close the section by stating a remark about Question 2.10.

Remark 4.4. As a consequence of Theorems 3.1, 3.2, 3.3, 3.7 and 4.1, Question 2.10 has an affirmative answer provided that R is a P-singularity over an algebraically closed uncountable field of characteristic zero, where $P \in \{\mathrm{A}_n, \mathrm{A}_\infty, \mathrm{D}_n, \mathrm{D}_\infty, \mathrm{E}_6, \mathrm{E}_7, \mathrm{E}_8\}$. We should also mention that there are many examples of a hypersurface domain R of dimension bigger than two where $\mathfrak{E}_0(R)$ is not trivial, e.g., an E_7-singularity of dimension 3. This is a direct consequence of Theorem 1.1, whose proof will be given in the next section.

5. Proofs of the main theorems

Now we have reached the stage to prove Theorem 1.1, which contains the main results of this paper.

Proofs of Theorem 1.1. Since R has finite or countable CM-representation type, it is isomorphic to one of the following hypersurfaces; see [6, Theorems 9.8 and 14.16].

$$\begin{cases} \mathrm{A}_n^d: \ k[\![x_0, x_1, \dots, x_d]\!]/(x_0^{n+1} + x_1^2 + x_2^2 + \cdots + x_d^2) & (n \geqslant 1), \\ \mathrm{A}_\infty^d: \ k[\![x_0, x_1, \dots, x_d]\!]/(x_1^2 + x_2^2 + \cdots + x_d^2), \\ \mathrm{D}_n^d: \ k[\![x_0, x_1, \dots, x_d]\!]/(x_0^{n-1} + x_0 x_1^2 + x_2^2 + \cdots + x_d^2) & (n \geqslant 4), \\ \mathrm{D}_\infty^d: \ k[\![x_0, x_1, \dots, x_d]\!]/(x_0 x_1^2 + x_2^2 + \cdots + x_d^2), \\ \mathrm{E}_6^d: \ k[\![x_0, x_1, \dots, x_d]\!]/(x_0^4 + x_1^3 + x_2^2 + \cdots + x_d^2), \\ \mathrm{E}_7^d: \ k[\![x_0, x_1, \dots, x_d]\!]/(x_0^3 x_1 + x_1^3 + x_2^2 + \cdots + x_d^2), \\ \mathrm{E}_8^d: \ k[\![x_0, x_1, \dots, x_d]\!]/(x_0^5 + x_1^3 + x_2^2 + \cdots + x_d^2). \end{cases}$$

Thus Theorem 1.1(2) is a direct consequence of Theorems 3.1, 3.2, 3.3, 3.4, 3.5, 3.7, 4.1, 4.2 and 4.3. To prove Theorem 1.1(1), Knörrer's periodicity theorem [13, Theorem (12.10)] reduces to the case where $\dim R \leqslant 2$. The triangle equivalence $\underline{\mathrm{CM}}(R) \cong \mathrm{D}^{\mathrm{sg}}(R)$ induces a triangle equivalence $\underline{\mathrm{CM}}_0(R) \cong \mathrm{D}_0^{\mathrm{sg}}(R)$. Hence the Hasse diagram of the extension-closed subcategories of $\mathrm{D}_0^{\mathrm{sg}}(R)$ are the same as the Hasse diagram of the extension-closed subcategories of $\underline{\mathrm{CM}}_0(R)$, which is obtained from the Hasse diagrams given in Theorems 3.1, 3.2, 3.3, 3.4, 3.5, 3.7, 4.1, 4.2 and 4.3 by removing the vertices containing R and the edges from/to them. ■

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