INFINITELY MANY KNOTS WITH THE SAME POLYNOMIAL INVARIANT, II

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Abstract. We give infinitely many knots with the same HOMFLYPT and Q polynomials, which are distinguished by the Kauffman polynomial. They are symmetric unions of the trefoil knot.

1. Introduction

The author [8, 9] provided infinite knot families with the same HOMFLYPT polynomial and hence the same specializations of the HOMFLYPT polynomial, Alexander-Conway and Jones polynomials, which are symmetric unions of the figure-eight knot. The notion of symmetric union was introduced by Kinoshita and Terasaka [13] and generalized by Lamm [15]. The knots in these families are classified by the Q polynomial [6]; the Q polynomial is a specialization of the Kauffman polynomial. In this note, we provide another family of knots which share the same HOMFLYPT and Q polynomials; they are distinguished by the Kauffman polynomial. The knots in this family are symmetric unions of the trefoil knot, which are exhibited by Lamm [16].

Theorem 1. *There exist infinitely many knots sharing the same HOMFLYPT and Q polynomials, which are distinguished by the Kauffman polynomial. So, they also share the same Jones and Alexander-Conway polynomials.*

Notice that Miyazawa [18] discovered two 16-crossing knots with trivial Q polynomial, which have nontrivial Jones polynomials. Moreover, he constructed an infinite prime knot family having trivial Q polynomial, which are distinguished by the Jones polynomial. By connected sums we can construct infinitely many knots sharing the same Q polynomial as that of a particular knot.

This note is organized as follows. Section 2 introduces the knot or link family $K(k,l,m)$, k, l, $m \in \mathbb{Z} \cup \{\infty\}$, whose subfamilies satisfy the properties of Theorem 1. In Sects. 3, 4, 5, we calculate the Q polynomial, HOMFLYPT and Conway polynomials, and Jones polynomial of $K(k,l,m)$, respectively. In Sect. 6 we calculate the Kauffman polynomial of $K(k,l,m)$ and prove Theorem 1, and also give some properties of $K(k,l,m)$.

2. The knot/link family $K(k,l,m)$

The knot family for Theorem 1 is contained in the family consisting of $K(k,l,m)$, k, l, $m \in \mathbb{Z} \cup \{\infty\}$, which are presented by any one of the diagrams in Fig. 1; a rectangle labeled n stands for a vertical n-tangle, a 2-string tangle with $|n|$ crossings for $n \in \mathbb{Z}$, or the ∞-tangle for $n = \infty$ as shown in Fig. 2. Note that if k, l, $m \in \mathbb{Z}$, then $K(k,l,m)$ is always a knot.

Date: October 14, 2025.

2020 *Mathematics Subject Classification.* Primary 57K14; Secondary 57K10.

Key words and phrases. Knot, Q polynomial, HOMFLYPT polynomial, Kauffman polynomial, Symmetric union.

These knots are symmetric unions of the trefoil knot. In fact, the diagram Fig. 1(a) is just the second template with determinant 9 for the symmetric union diagram given in Appendix of [16]. The diagram Fig. 1(b) is obtained by deforming the diagram Fig. 1(a), which is also obtained from the diagram Fig. 1(e) by sliding the l-tangle. The diagrams (d), (e), (f) in Fig. 1 are obtained from the diagram (c) by rotating along the x, y, z coordinate axes by the angle $180°$, respectively.

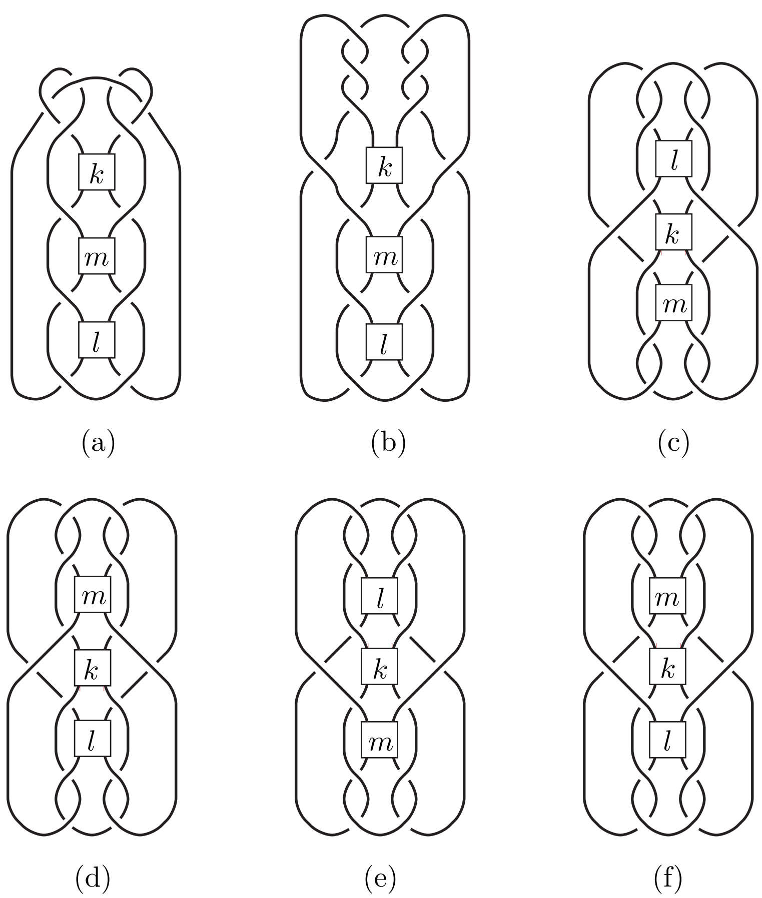


FIGURE 1. Diagrams of the knot/link $K(k, l, m)$.

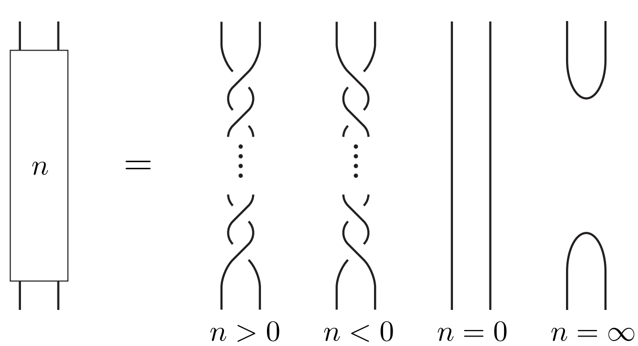


FIGURE 2. The n-tangles and the ∞-tangle.

Let L and L' be links. Then $L \approx L'$ means that L and L' are isotopic. We denote the mirror image of L by $L!$. Comparing the diagrams (c), (d), (e) in Fig. 1 we have:

Proposition 2. *For $k,\ l,\ m \in \mathbb{Z}$, we have $K(k,l,m) \approx K(k,m,l)$ and $K(k,l,m)! \approx K(-k,-m,-l)$.*

We orient $K(k,l,m)$ as shown in Fig. 3, which we denote by $\vec{K}(k,l,m)$. Then $\vec{K}(k,l,m)$ is isotopic to any one of the diagrams illustrated in Fig. 4.

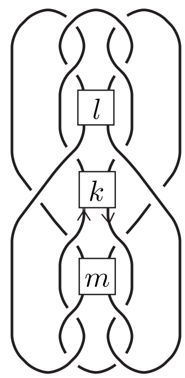
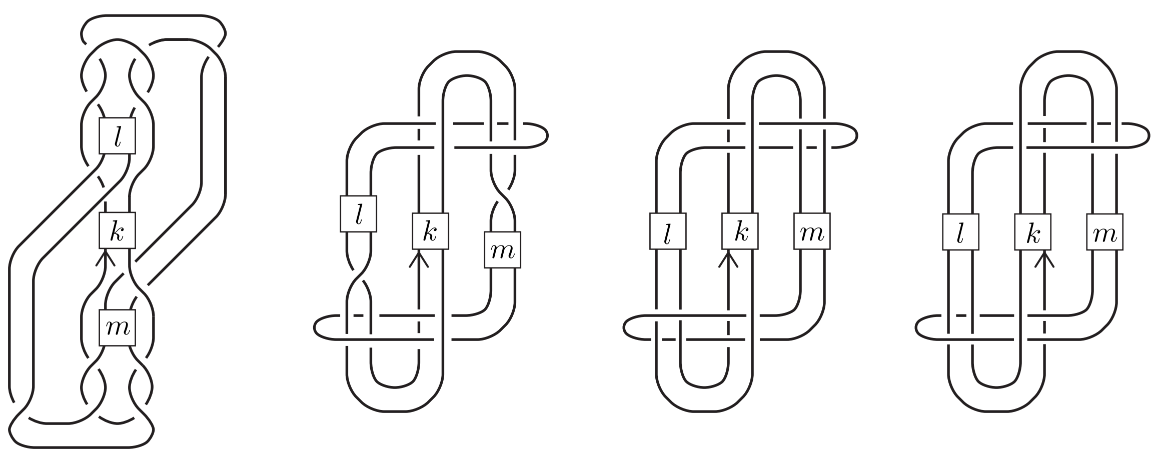


FIGURE 3. Diagram of the oriented knot/link $\vec{K}(k,l,m)$.



(a) (b) (c) (d)

FIGURE 4. The diagrams isotopic to $\vec{K}(k,l,m)$.

For $n \in \mathbb{Z}$, let $J(n)$ be the knot and $L_0(n)$ and $L_1(n)$ the 2-component links as shown in Fig 5. Note that $J(n)$ is the pretzel knot of type $(3,-3,n)$, which is also regarded as a symmetric union of the trefoil knot. Using these diagrams we obtain the following.

Proposition 3. *For $k,\ l,\ m \in \mathbb{Z}$, we have the following.*

(i) $K(k,\infty,m) \approx K(k,l,\infty) \approx U^2$, *where U^2 is the trivial 2-component link.*

(ii) $K(0,l,m) \approx J(l+m)$.

(iii) $K(\infty,l,m) \approx \begin{cases} L_0(l+m) & \textit{if } (l,m) \equiv (0,0),\ (0,1),\ (1,0) \pmod 2, \\ L_1(l+m) & \textit{if } (l,m) \equiv (1,1) \pmod 2. \end{cases}$

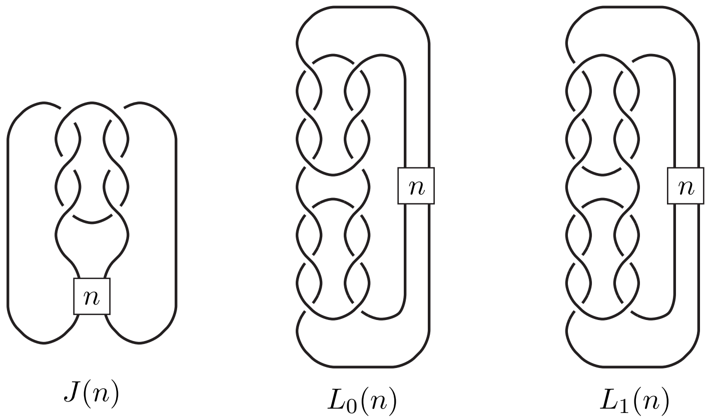


FIGURE 5. The knot $J(n)$ and the links $L_0(n)$ and $L_1(n)$, $n \in \mathbb{Z}$.

For small k, l, m, we can identify the knots $K(k,l,m)$ as follows:

$$\begin{aligned}
&K(0,0,0) \approx K(0,-1,1) \approx J(0) \approx 3_1!\#3_1 \ \text{(the square knot)}, \\
&K(0,1,0) \approx K(0,0,1) \approx J(1) \approx 6_1!, \\
&K(1,0,0) \approx 12\text{n_}605, && K(1,-1,1) \approx K(1,1,-1) \approx 12\text{n_}268, \\
&K(1,0,1) \approx K(1,1,0) \approx 13\text{n_}1849!, && K(1,0,-1) \approx K(1,-1,0) \approx 13\text{n_}1357!, \\
&K(1,-2,1) \approx 13\text{n_}1645!, && K(1,-1,2) \approx 13\text{n_}1805, \\
&K(2,0,0) \approx 13\text{n_}2209, && K(2,-1,1) \approx 13\text{n_}3272,
\end{aligned} \tag{1}$$

where we use the knot names in [17].

3. Q POLYNOMIAL

The Q polynomial $Q(L;x) \in \mathbb{Z}[x^{\pm 1}]$ [2, 5] is an invariant of the isotopy type of an unoriented link L defined by

$$\begin{aligned}
&Q(U) = 1, \\
&Q(L_+) + Q(L_-) = x\left(Q(L_0) + Q(L_\infty)\right),
\end{aligned} \tag{2}$$

where U is the unknot and $(L_+, L_-, L_0, L_\infty)$ is an *unoriented skein quadruple*, which is an ordered set of four unoriented links that are identical except near one point where they are as in Fig. 6.

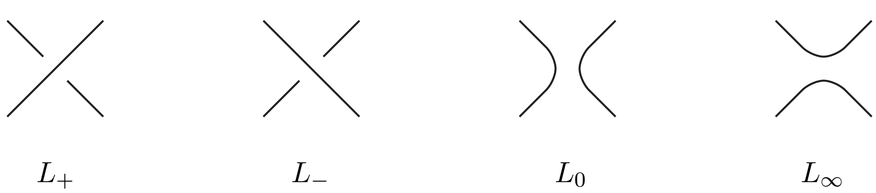


FIGURE 6. Unoriented skein quadruple.

Suppose a link L contains a tangle R. Any link obtained by rotating R about one of the three axes is called a *mutant* of L. We use the following properties of the Q polynomial; see [2, Property 1]:

Proposition 4. (i) *$Q(L_1\#L_2) = Q(L_1)Q(L_2)$, where $L_1\#L_2$ is a connected sum of L_1 and L_2.*
(ii) *$Q(L!) = Q(L)$.*
(iii) *If L_2 is a mutant of L_1, then $Q(L_1) = Q(L_2)$.*

We denote by μ_Q the Q polynomial of the trivial 2-component link, $\mu_Q = 2x^{-1} - 1$.

We denote the Q polynomial of the knot/link $K(k,l,m)$ by $Q(k,l,m)$, $Q(k,l,m) = Q(K(k,l,m))$. Then we obtain the following theorem.

Theorem 5. *For k, l, $m \in \mathbb{Z}$, we have:*

$$Q(k,l,m) = Q(k,0,l+m). \tag{3}$$

Proof. The proof will be divided into four steps:

$$Q(1,-m,m) = Q(1,0,0) \quad (m \in \mathbb{Z}), \tag{4}$$

$$Q(k,-m,m) = Q(k,0,0) \quad (k,m \in \mathbb{Z}), \tag{5}$$

$$Q(k,-m+1,m) = Q(k,0,1) \quad (k,m \in \mathbb{Z}), \tag{6}$$

$$Q(k,-m+n,m) = Q(k,0,n) \quad (k,m,n \in \mathbb{Z}). \tag{7}$$

Proof of Eq. (4). Using Eq. (2) we have:

$$Q(1,-m,m) + Q(-1,-m,m) = x\left(Q(0,-m,m) + Q(\infty,-m,m)\right). \tag{8}$$

By Proposition 2 we have $K(-1,-m,m)! \approx K(1,m,-m) \approx K(1,-m,m)$, and so $Q(-1,-m,m) = Q(1,-m,m)$. Then Eq. (8) is deformed into

$$Q(1,-m,m) = \frac{x}{2}\left(Q(0,-m,m) + Q(\infty,-m,m)\right). \tag{9}$$

By Proposition 3(ii), $K(0,-m,m)$ is isotopic to $J(0)$, which is the square knot, $3_1\#3_1!$, and so

$$Q(0,-m,m) = Q(3_1)^2 = 9 - 12x - 8x^2 + 8x^3 + 4x^4. \tag{10}$$

By Proposition 3(iii), $K(\infty,-m,m)$ is isotopic to either $L_0(0)$ or $L_1(0)$. Since $L_0(0)$ and $L_1(0)$ are mutant and $L_1(0)$ is the pretzel link of type $(3,-3,3,-3)$, $\mathrm{Pz}(3,-3,3,-3)$, and so

$$\begin{aligned} Q(\infty,-m,m) &= Q(\mathrm{Pz}(3,-3,3,-3)) \\ &= 2x^{-1} - 1 + 32x - 80x^2 - 24x^3 + 132x^4 + 20x^5 - 76x^6 - 20x^7 + 12x^8 + 4x^9. \end{aligned} \tag{11}$$

Therefore, from Eq. (9), we obtain

$$\begin{aligned} Q(1,-m,m) &= \frac{x}{2}\big(Q(3_1)^2 + Q(\mathrm{Pz}(3,-3,3,-3))\big) \\ &= 1 + 4x + 10x^2 - 44x^3 - 8x^4 + 68x^5 + 10x^6 - 38x^7 - 10x^8 + 6x^9 + 2x^{10}, \end{aligned}$$

which implies Eq. (4).

Proof of Eq. (5). We proceed by induction on k. Equation (5) with $k = 0$ comes from Eq. (10), and Eq. (5) with $k = 1$ is Eq. (4). Suppose that Eq. (5) holds for any integer m and $k = k' - 1$ and k'. Using Eq. (2) we have

$$Q(k'+1,-m,m) + Q(k'-1,-m,m) = x\big(Q(k',-m,m) + Q(\infty,-m,m)\big) \tag{12}$$

for any integer m. In particular, we have:

$$Q(k'+1,0,0) + Q(k'-1,0,0) = x\big(Q(k',0,0) + Q(\infty,0,0)\big). \tag{13}$$

By the inductive assumption $Q(k'-1,-m,m)=Q(k'-1,0,0)$ and $Q(k',-m,m)=Q(k',0,0)$, and by Eq. (11) $Q(\infty,-m,m)=Q(\infty,0,0)$, and so we obtain $Q(k'+1,-m,m)=Q(k'+1,0,0)$. Thus Eq. (5) holds for any integers $k(\geq 0)$ and m. The case $k<0$ is similar.

Proof of Eq. (6). Using Eq. (2) we have

$$\begin{aligned} Q(k,-m+1,m-1)+Q(k,-m+1,m+1)&=x\left(Q(k,-m+1,m)+Q(k,-m+1,\infty)\right),\\ Q(k,-m-1,m+1)+Q(k,-m+1,m+1)&=x\left(Q(k,-m,m+1)+Q(k,\infty,m+1)\right).\end{aligned}$$

Then, since $Q(k,-m+1,m-1)=Q(k,-m-1,m+1)=Q(k,0,0)$ (Eq. (5)) and $Q(k,-m+1,\infty)=Q(k,\infty,m+1)=\mu_Q$ (Proposition 3(i)), we obtain $Q(k,-m+1,m)=Q(k,-m,m+1)$. This holds for any integers k, m, and so we obtain Eq. (6).

Proof of Eq. (7). The cases $n=0$, 1 are Eqs. (5) and (6), respectively. Using Eq. (2) we have

$$\begin{aligned} Q(k,-m+j,m-1)+Q(k,-m+j,m+1)&=x\left(Q(k,-m+j,m)+Q(k,-m+j,\infty)\right),\\ Q(k,-m+j-2,m+1)+Q(k,-m+j,m+1)&=x\left(Q(k,-m+j-1,m+1)+Q(k,\infty,m+1)\right).\end{aligned}$$

From Proposition 3(i) we have $Q(k,-m+j,\infty)=Q(k,\infty,m+1)=\mu_Q$. Thus, by induction on n we obtain Eq. (7). $\square$

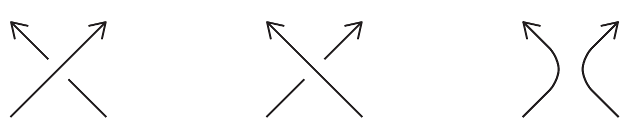
4. HOMFLYPT and Conway polynomials

The *Conway polynomial* $\nabla(L;z)\in\mathbb{Z}[z]$ [3], the *Jones polynomial* $V(L;t)\in\mathbb{Z}[t^{\pm1/2}]$ [7], and the *HOMFLYPT polynomial* $P(L;v,z)\in\mathbb{Z}[v^{\pm1},z^{\pm1}]$ [4, 7, 20] are invariants of the isotopy type of an oriented link L. They are defined by

$$\nabla(U;z)=1,$$
$$\nabla(L_+;z)-\nabla(L_-;z)=z\nabla(L_0;z), V(U;t)=1,$$
$$t^{-1}V(L_+;t)-tV(L_-;t)=\left(t^{1/2}-t^{-1/2}\right)V(L_0;t), \tag{14}$$
$$P(U;v,z)=1,$$
$$v^{-1}P(L_+;v,z)-vP(L_-;v,z)=zP(L_0;v,z), \tag{15}$$

where (L_+,L_-,L_0) is a *skein triple*, an ordered set of three oriented links that are identical except near one point where they are as in Fig. 7. We say that the link L_0 is obtained from L_+ or L_- by *smoothing* the crossing in L_+ or L_-. The Conway polynomial and the Jones polynomial are obtained from the HOMFLYPT polynomial as follows:

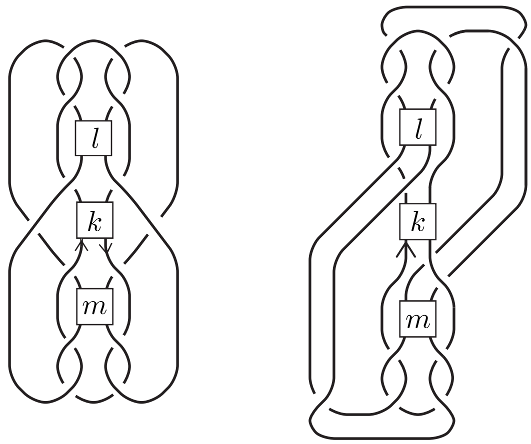
$$\nabla(L;z)=P(L;1,z),$$
$$V(L;t)=P\left(L;t,t^{1/2}-t^{-1/2}\right). \tag{16}$$



L_+ L_- L_0

FIGURE 7. Skein triple.

We calculate the HOMFLYPT polynomial of the oriented knot $\vec{K}(k,l,m)$, k, l, $m\in\mathbb{Z}$ as shown in Fig. 8(a), which we can deform into Figs. 8(b) and 8(c). Let $P(k,l,m)$ be the HOMFLYPT polynomial of $\vec{K}(k,l,m)$.



(a) (b) (c)

FIGURE 8. The oriented knot $\vec{K}(k,l,m)$.

Proposition 6. *For k, l, $m \in \mathbb{Z}$, we have:*

$$P(k,l,m) = \begin{cases} v^{-l-m}\left(P(k,0,0)-1\right)+1 & \textit{if } (l,m) \equiv (0,0) \pmod 2, \\ v^{1-l-m}\left(P(k,0,1)-1\right)+1 & \textit{if } (l,m) \equiv (0,1),(1,0) \pmod 2, \\ v^{-l-m}\left(P(k,-1,1)-1\right)+1 & \textit{if } (l,m) \equiv (1,1), \pmod 2 \end{cases}$$

Proof. The two arcs of the l-tangle in $\vec{K}(k,l,m)$ has anti-parallel orientation. So smoothing a crossing in this tangle, we obtain $K(k,\infty,m)$ with proper orientation, which is a trivial 2-component link U^2 by Proposition 3(i). Thus, we obtain a skein triple:

$$\left(\vec{K}(k,l-2,m), \vec{K}(k,l,m), U^2\right).$$

Similarly, we obtain a skein triple:

$$\left(\vec{K}(k,l,m-2), \vec{K}(k,l,m), U^2\right).$$

Since the HOMFLYPT polynomial of the trivial 2-component link is $(v^{-1}-v)z^{-1}$, using Eq. (15) we have

$$v^{-1}P(k,l-2,m) - vP(k,l,m) = v^{-1}-v,$$
$$v^{-1}P(k,l,m-2) - vP(k,l,m) = v^{-1}-v,$$

which imply

$$P(k,l,m)-1 = v^{-2}\left(P(k,l-2,m)-1\right) = v^{-2}\left(P(k,l,m-2)-1\right).$$

Then, since $K(k,l,m) \approx K(k,m,l)$, we obtain the result. $\square$

The diagram Fig. 1(a) of the knot $K(k,l,m)$, k, l, $m \in \mathbb{Z}$, shows it is a symmetric union of the trefoil; cf. [16, Appendix]. So, we can calculate the Alexander polynomial by applying Lamm's theorem in [15]. Since the Alexander polynomial and the Conway polynomial are essentially the same [19, Theorem 6.2.1], we obtain the following.

Proposition 7. *For k, l, $m \in \mathbb{Z}$, we have:*

$$\nabla(k,l,m) = \begin{cases} 1+2z^2+z^4 & \text{if } (k,l,m) \equiv (0,0,0),\ (0,1,1) \pmod 2, \\ 1-2z^2 & \text{if } (k,l,m) \equiv (0,0,1),\ (0,1,0),\ (1,1,1) \pmod 2, \\ 1+2z^2+5z^4+z^6 & \text{if } (k,l,m) \equiv (1,1,0),\ (1,0,1) \pmod 2, \\ 1-2z^2-8z^4-6z^6-z^8 & \text{if } (k,l,m) \equiv (1,0,0) \pmod 2. \end{cases}$$

Proof. If $(k,l,m) \equiv (k',l',m') \pmod 2$, then by Theorem 2.4 in [15], the two knots $K(k,l,m)$ and $K(k',l',m')$ share the same Alexander polynomial, which yields $\nabla(k,l,m) = \nabla(k',l',m')$. From Eq. (1), looking at [17], we obtain the result. □

5. Jones polynomial

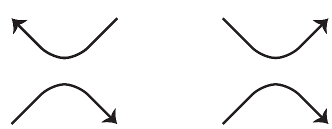
Given a skein triple (L_+, L_-, L_0), we consider another oriented link L_∞ which is one of the diagrams of Fig. 9, the choice being (i) if $c(L_+) < c(L_0)$ and (ii) if $c(L_+) > c(L_0)$, where $c(L)$ is the number of components of a link L. We call the ordered set $(L_+, L_-, L_0, L_\infty)$ an *oriented skein quadruple*, for which we have the following relations of the Jones polynomials [1, Theorem 2]: For the case (i) $c(L_+) < c(L_0)$, we have

$$V(L_+;t) - tV(L_-;t) + t^{3\lambda}(t-1)V(L_\infty;t) = 0, \tag{17}$$

where λ is the linking number of the right-hand component of L_0 with the remainder of L_0, and for the case (ii) $c(L_+) > c(L_0)$, we have

$$V(L_+;t) - tV(L_-;t) + t^{3(\mu-\frac{1}{2})}(t-1)V(L_\infty;t) = 0,$$

where μ is the linking number of the bottom-right and top-left component of L_+ with the remainder of L_+.



(i) (ii)

Figure 9. Oriented link L_∞.

For the case (i), using Eqs. (14) and (17), we have:

$$V(L_+;t) = -t^{1/2}V(L_0;t) - t^{3\lambda+1}V(L_\infty;t), \tag{18}$$

$$V(L_-;t) = -t^{-1/2}V(L_0;t) - t^{3\lambda-1}V(L_\infty;t), \tag{19}$$

which we will use in the proof of the following proposition.

Let $V(k,l,m)$ be the Jones polynomial of $\vec{K}(k,l,m)$.

Proposition 8. *For k, l, $m \in \mathbb{Z}$, we have:*

$$V(k,l,m) = (-t)^{-l-m}\left(V(k,0,0) - 1\right) + 1.$$

Proof. We have an oriented skein quadruple $(\vec{K}(k,0,-1),\vec{K}(k,0,1),\vec{K}(k,0,\infty),\vec{K}(k,0,0))$. Then, since $\vec{K}(k,0,\infty)\approx U^2$ (Proposition 3(i)), by Eq. (19) we have

$$\begin{aligned}V(k,0,1)&=-t^{-1/2}\mu_V-t^{-1}V(k,0,0)\\&=(-t)^{-1}(V(k,0,0)-1)+1,\end{aligned}$$

where $\mu_V=-t^{-1/2}-t^{1/2}$ is the Jones polynomial of the trivial 2-component link. Similarly, from an oriented skein quadruple $(\vec{K}(k,-1,1),\vec{K}(k,1,1),\vec{K}(k,\infty,1),\vec{K}(k,0,1))$, by Eq. (18) we have

$$\begin{aligned}V(k,-1,1)&=-t^{1/2}\mu_V-tV(k,0,1)=(-t)(V(k,0,1)-1)+1\\&=V(k,0,0).\end{aligned}$$

Then using Eq. (16), Proposition 6 yields the result. $\square$

Lemma 9.
(20)

$$V(k,0,0)=\begin{cases}-t^{-3}+t^{-2}-t^{-1}+3-t+t^2-t^3, & \text{if } k=0,\\ t^{-6}-2t^{-5}+3t^{-4}-4t^{-3}+3t^{-2}-2t^{-1}+2+t-2t^2+2t^3-2t^4+t^5, & \text{if } k=1,\\ -t^{-7}+3t^{-6}-5t^{-5}+\cdots+4t^3-3t^4+t^5, & \text{if } k=2,\\ (-t)^{-k-5}+3(-t)^{-k-4}-6(-t)^{-k-3}+\cdots+5t^3-3t^4+t^5, & \text{if } k\geq 3.\end{cases}$$

Proof. First, from Eq. (1) we have $V(0,0,0)=V(3_1)V(3_1!)$ and $V(1,0,0)=V(12n\_605)$, and so we obtain Eq. (20) with $k=0$, 1 from [17]. Suppose $k\geq 1$. Using Eq. (19) at a negative crossing in the k-tangle in the diagram Fig. 1(a), we obtain

$$V(k,0,0)=-t^{-1/2}V(K(\infty,0,0))-t^{-1}V(k-1,0,0). \tag{21}$$

Note that the Jones polynomial $V(K(\infty,0,0))$ does not change depending on the value k. In fact, $K(\infty,0,0)\approx L_0(0)$ (Proposition 3(iii)) and the linking number of $L_0(0)$ is zero. So, we obtain

$$\begin{aligned}-t^{-1/2}V(K(\infty,0,0))&=V(1,0,0)+t^{-1}V(0,0,0)\\&=t^{-6}-2t^{-5}+2t^{-4}-3t^{-3}+2t^{-2}+t^{-1}+1+2t-3t^2+2t^3-2t^4+t^5.\end{aligned}$$

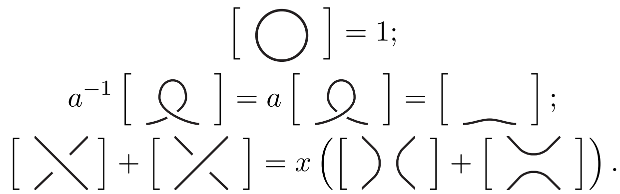
Then by Eq. (21), we obtain Eq. (20) for $k\geq 2$ inductively. $\square$

Proposition 10. *For k, l, m, k', l', $m'\in\mathbb{Z}$, $V(k,l,m)=V(k',l',m')$ if and only if $k=k'$ and $l+m=l'+m'$.*

Proof. The "if" part is trivial. To prove the "only if" part, assume that $V(k,l,m)=V(k',l',m')$. Then by Proposition 8 we have $(-t)^{-l-m}(V(k,0,0)-1)=(-t)^{-l'-m'}(V(k',0,0)-1)$. Note that the Jones polynomial $V(-k,0,0)$ is given by substitution of t^{-1} for t in $V(k,0,0)$. Then considering the difference of the maximal degree and minimal degree of $V(k,0,0)$ in Lemma 9, we have $k=\pm k'$. If $k=k'$, then $l+m=l'+m'$. By Eq. (20) the case $k=-k'$ does not happen. $\square$

6. Kauffman polynomial

Let us review the Kauffman polynomial [11, 12]. Let L be an oriented link and D its diagram. The square bracket polynomial $[D]\in\mathbb{Z}[a^{\pm1},x^{\pm1}]$ defined by the following formulas is a regular isotopy invariant:



(22)

Then the Kauffman polynomial of L, $F(L;a,x) \in \mathbb{Z}[a^{\pm 1}, x^{\pm 1}]$, is given by:

$$F(L;a,x) = a^{-w(D)}[D],$$

where $w(D)$ is the writhe of D; the writhe is the algebraic sum of the crossings of D, counting $+1$ for a positive crossing, and -1 for a negative crossing. The Q polynomial is obtained from the Kauffman polynomial:

$$Q(L;x) = F(L;1,x).$$

In what follows, we use the polynomials $\sigma_n \in \mathbb{Z}[x]$ and $\tau_n \in \mathbb{Z}[a^{\pm 1}, x]$, $n \in \mathbb{Z}$, defined by:

$$\begin{gathered} \sigma_{n-1} + \sigma_{n+1} = x\sigma_n, \\ \tau_{n-1} + \tau_{n+1} = x\tau_n + a^{-n}x, \\ \sigma_0 = \tau_0 = \tau_1 = 0, \quad \sigma_1 = 1. \end{gathered} \tag{23}$$

Then we have

$$\sigma_{-n} = -\sigma_n, \tag{24}$$

$$\sigma_{n-1}\sigma_{n+1} = \sigma_n^2 - 1, \tag{25}$$

$$a\tau_{n+1} = \tau_n + x\sigma_n, \tag{26}$$

$$\tau_{1-n}(a,x) = a^{-1}\tau_n(a^{-1},x), \tag{27}$$

where $\tau_{1-n}(a,x) = \tau_{1-n}$ and $\tau_n(a^{-1},x)$ is obtained from τ_n by substituting a into a^{-1}. Moreover, the polynomial σ_n, $n > 0$, has the following forms:

$$\sigma_{2j} = (-1)^{j-1}jx + \alpha_{2j,3}x^3 + \alpha_{2j,5}x^5 + \cdots + \alpha_{2j,2j-5}x^{2j-5} - (2j-2)x^{2j-3} + x^{2j-1}, \tag{28}$$

$$\sigma_{2j+1} = (-1)^j + \alpha_{2j+1,2}x^2 + \alpha_{2j+1,4}x^4 + \cdots + \alpha_{2j+1,2j-4}x^{2j-4} - (2j-1)x^{2j-2} + x^{2j}, \tag{29}$$

where $\alpha_{n,k} \in \mathbb{Z}$.

For $n \in \mathbb{Z} \cup \{\infty\}$, let $[n]$ be the square bracket polynomial of a link diagram containing an n-tangle as shown in Fig. 2. Then we have the following [10, Proposition 2.2]:

$$[n] = \sigma_n[1] - \sigma_{n-1}[0] + \tau_n[\infty]. \tag{30}$$

Since $\sigma_{-1} = -1$, $\sigma_{-2} = -x$, $\tau_{-1} = x$, the case $n = -1$ coincides with Eq. (22). This formula yields the next formula.

Let $D(k,l,m)$ be the diagram Fig. 1(a) and $[k,l,m]$ denote its square bracket polynomial. Let $F(k,l,m)$ be the Kauffman polynomial of $\vec{K}(k,l,m)$.

Lemma 11. *The maximal degree of* $[1,-m,m]$ *in* x *is* 10 *if* $m = 0$, 1, 2, *and* $2m+5$ *if* $m \geq 3$. *Furthermore, comparing the Kauffman polynomials* $F(1,-m,m)$ *for* $m = 0$, 1, 2, *it holds that* $F(1,-m,m) = F(1,-m',m')$ *if and only if* $m = \pm m'$.

Proof. For k, l, $m \in \mathbb{Z}$ using Eq. (30), we have

$$\begin{aligned}[k,l,m] &= \sigma_l[k,1,m] - \sigma_{l-1}[k,0,m] + \tau_l[k,\infty,m]\\ &= \sigma_l(\sigma_m[k,1,1] - \sigma_{m-1}[k,1,0] + \tau_m[k,1,\infty])\\ &\quad - \sigma_{l-1}(\sigma_m[k,0,1] - \sigma_{m-1}[k,0,0] + \tau_m[k,0,\infty]) + \tau_l[k,\infty,m].\end{aligned}$$

Since $K(k,\infty,m) \approx K(k,l,\infty) \approx U^2$ and the writhes of the diagrams $D(k,\infty,m)$ and $D(k,l,\infty)$ with any orientation are $-k-m$ and $-k-l$, respectively, we have $a^{k+m}[k,\infty,m] = a^{k+l}[k,l,\infty] = \mu_F$, where μ_F is the Kauffman polynomial of the trivial 2-component link, $\mu_F = (a^{-1}-a)x^{-1}-1$. Also, since $K(k,1,0) \approx K(k,0,1)$ and the writhes of $D(k,0,1)$ and $D(k,1,0)$ are $-k-1$, we have $[k,1,0] = [k,0,1]$.

Then we have

$$\begin{aligned}[k,l,m] = \sigma_l\sigma_m[k,1,1] &- (\sigma_l\sigma_{m-1} + \sigma_{l-1}\sigma_m)[k,1,0] + \sigma_{l-1}\sigma_{m-1}[k,0,0]\\ &+a^{-k}(a^{-1}\sigma_l\tau_m - \sigma_{l-1}\tau_m + a^{-m}\tau_l)\mu_F.\end{aligned}$$

If $l=-m$, then using Eqs. (23), (24) and (25), we have

$$\begin{aligned}[k,-m,m] = &-\sigma_m^2[k,1,1] + x\sigma_m^2[k,1,0] - (\sigma_m^2 - 1)[k,0,0]\\ &+ a^{-k}(-a^{-1}\sigma_m\tau_m + \sigma_{m+1}\tau_m + a^{-m}\tau_{-m})\mu_F.\end{aligned}$$

By Eq. (22) we have $[k,1,1] = -[k,1,-1] + x[k,1,0] + xa^{-k-1}\mu_F$, and so

$$\begin{aligned}[k,-m,m] =&\sigma_m^2([k,1,-1] - [k,0,0]) + [k,0,0]\\ &+ a^{-k}(-\sigma_m^2 xa^{-1} - a^{-1}\sigma_m\tau_m + \sigma_{m+1}\tau_m + a^{-m}\tau_{-m})\mu_F.\end{aligned}$$

Let $\max\deg f$ denote the maximal degree of $f \in \mathbb{Z}[a^{\pm1}, x^{\pm1}]$ in x. Then we see for $m \geq 3$ $\max\deg[1,-m,m] = 2m+5$. In fact, we have the following:

$$\max\deg\sigma_m = m-1 \quad (m \geq 1), \tag{31}$$

$$\max\deg([1,1,-1] - [1,0,0]) = 7, \tag{32}$$

$$\max\deg[1,0,0] = 10, \tag{33}$$

$$\max\deg\tau_m = \begin{cases} m-1 & \text{if } m \geq 2,\\ -m & \text{if } m \leq -1.\end{cases} \tag{34}$$

Equation (31) follows from Eqs. (28) and (29). Equation (32) follows from

$$\begin{aligned}a([1,1,-1] - [1,0,0]) &= F(12\text{n\_}268) - F(12\text{n\_}605)\\ &= (a^{-4} + 3a^{-2} + 2 - 2a^2 - 3a^4 - a^6) + \cdots + (-a^{-1} + a^3)x^7;\end{aligned}$$

see [17]. Also for Eq. (33) see [17]. Equation (34) follows from Eqs. (26)–(29).

Now, we see the three Kauffman polynomials $F(1,-m,m)$ ($m = 0$, 1, 2) are mutually distinct. In fact, the coefficient polynomials of x^2 of $F(1,-m,m)$ are mutually distinct as shown in Table 1, where we calculated $F(1,-2,2)$ using the program of Kodama [14].

Table 1. Coefficient polynomials of x^2

$K(1,0,0) = 12\text{n\_}605$	$-5a^{-4} + a^{-2} + 10 + 4a^6$
$K(1,-1,1) = 12\text{n\_}268$	$-5a^{-4} - 8a^{-2} + 1 + 9a^2 + 9a^4 + 4a^6$
$K(1,-2,2)$	$-4a^{-4} + 4a^{-2} + 12 - 2a^2 - 3a^4 + 3a^6$

□

Proof of Theorem 1. The sets of knots $\{K(1,-2n,2n)\,|\,n\in\mathbb{N}\}$ and $\{K(1,1-2n,2n-1)\,|\,n\in\mathbb{N}\}$ satisfy the conditions. In fact, $Q(1,-2n,2n)=Q(1,1-2n,2n-1)=Q(1,0,0)$ by Theorem 5, and $P(1,-2n,2n)=P(1,0,0)$, $P(1,1-2n,2n-1)=P(1,-1,1)$ by Proposition 6. These knots are classified by the Kauffman polynomial by Lemma 11. □

We give some properties of $K(k,l,m)$ using the above results.

Proposition 12. *The knot $K(k,l,m)$, k, l, $m\in\mathbb{Z}$, has the following properties.*
(i) *$K(k,l,m)$ is a ribbon knot.*
(ii) *The determinant of $K(k,l,m)$ is* 9.
(iii)

$$K(k,l,m)\approx\begin{cases}3_1!\#3_1 & \textit{if } k=l+m=0,\\ 6_1 & \textit{if } k=0,\ l+m=-1,\\ 6_1! & \textit{if } k=0,\ l+m=1,\\ (\textit{a hyperbolic } 3\textit{-bridge knot}) & \textit{otherwise}.\end{cases}$$

Proof. (i) The knot $K(k,l,m)$ is presented as a symmetric union of the trefoil knot, and so it is a ribbon knot [15, Theorem 5.1].

(ii) Since $K(k,l,m)$ is a symmetric union of the trefoil knot, applying the formula of Lamm [15, Theorem 2.6], we obtain the result.

(iii) The number of local maxima of the diagrams in Fig. 1 of $K(k,l,m)$ is 3, and so the bridge number of $K(k,l,m)$ is at most 3; cf. [19, Sect. 4.3]. If $K(k,l,m)$ is a 2-bridge knot, then since its determinant is 9, $K(k,l,m)\approx 6_1$, $6_1!$, 9_1, or $9_1!$ by the classification of the 2-bridge knots; cf. [19, Theorem 9.3.3]. Since $\nabla(9_1)=1+10z^2+15z^4+7z^6+z^8$ [17], $K(k,l,m)\not\approx 9_1$ nor $9_1!$ by Proposition 7. Suppose $K(k,l,m)\approx 6_1$ or $6_1!$. From Eq. (1) we have $V(6_1)=V(K(0,-1,0))$ and $V(6_1!)=V(K(0,1,0))$, and so by Proposition 10 if $K(k,l,m)\approx 6_1$, then $k=0$ and $l+m=-1$ and if $K(k,l,m)\approx 6_1!$, then $k=0$ and $l+m=1$.

Next, if $K(k,l,m)$ is a composite knot, then it is a connected sum of two 2-bridge knots. Since the determinant is 9 and it is a ribbon knot, it should be the square knot $3_1!\#3_1$. From Eq. (1) we have $V(3_1!\#3_1)=V(K(0,0,0))$ and so by Proposition 10 $K(k,l,m)\approx 3_1!\#3_1$ if and only if $k=l+m=0$.

Lastly, we show $K(k,l,m)$ is not a torus knot. Since the knot 6_1 is hyperbolic, we consider $K(k,l,m)$ with bridge number 3. If it is a torus knot, then it should be one of type $(3,q)$, $|q|\geq 4$, $T(3,q)$. The determinant of $T(3,q)$ is 1 (if q is odd) or 3 (if q is even), so it is not the case. □

Acknowledgments

The author was partially supported by JSPS KAKENHI, Grant Number JS21K03251 and MEXT Promotion of Distinctive Joint Research Center Program JPMXP0723833165.

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