HERZ SPACES AND INTEGRAL OPERATORS ON MORREY SPACES

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Abstract. This paper establishes an approach for extending the mapping properties of integral operators from Lebesgue spaces to Morrey spaces. The main result shows that if the operator is bounded on the Lebesgue space and the kernel of the operator satisfies a condition involving a certain Herz space, then we can extend this operator to be a bounded operator on the corresponding Morrey space. We also obtain a new operator norm estimate of the integral operator on Morrey space in terms of the operator norm on Lebesgue space and the Herz space norm of the kernel of the integral operator. The main result applies to several integral operators including singular integral operators, fractional integral operators and strongly singular integral operators.

1. Introduction

This paper aims to establish a unified approach to the study of some integral operators on Morrey spaces. Our approach applies to singular integral operators, fractional integral operators and strongly singular integral operators.

Morrey spaces were introduced by Morrey in [16]. Since then, the class of Morrey spaces has become one of the most important classes of function spaces in harmonic analysis. The mapping properties of important operators appearing in harmonic analysis, such as singular integral operators and fractional integral operators have been extended to Morrey spaces, see, for example, [17, 20].

While the methods to extend the mapping properties of singular integral operators and fractional integral operators from Lebesgue spaces to Morrey spaces are closely related to each other, some significant modifications are required. In this paper, we obtain the mapping properties of singular integral operators and fractional integral operators by using

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a single approach. Roughly speaking, our main result states that whenever translations and dilations of the kernels of the integral operators belong to a Herz space on $\mathbb{R}^n\backslash B(0,1)$ and their Herz space norms are uniformly bounded, then such integral operators can be extended to a bounded operator on the corresponding Morrey spaces. Furthermore, we obtain some operator norm estimates of the integral operators on Morrey spaces in terms of the operator norms on Lebesgue spaces and the Herz space norms of the kernels of the integral operators. This result is new even for singular integral operators and fractional integral operators on Morrey spaces when compared with the results in [1, 8, 9, 17, 20].

Herz spaces were introduced by Herz in [6]. They are also an important generalization of Lebesgue spaces. For further details on Herz spaces, the reader is referred to [14].

The main result of this paper does not only apply to singular integral operators and fractional integral operators, it also applies to those integral operators with kernels possessing stronger singularity such as strongly singular integral operators. These operators were introduced in [7] and [23] for the study of Fourier multipliers and Fourier series. A more general form of these operators was introduced and studied in [10, 15]. Our method gives the precise definition of strongly singular integral operators on Morrey spaces. It provides a solid and novel foundation for the study of the mapping properties of strongly singular integral operators on Morrey spaces.

This paper is organized as follows. The definitions of Morrey spaces, Herz spaces and the integral operators for our studies are given in Section 2. The main result of this paper, including the definition of the extension of the integral operator to Morrey spaces, the mapping properties of the extended operators on Morrey spaces and the operator norm estimate on Morrey spaces are given in Section 3. As an application of the main result, we also establish the mapping properties of strongly singular integral operators on Morrey spaces in Section 3.

2. Preliminaries and Definitions

For any $x \in \mathbb{R}^n$ and $r > 0$, define $B(x,r) = \{y \in \mathbb{R}^n : |x-y| < r\}$. Write $\mathbb{B} = \{B(x,r) : x \in \mathbb{R}^n, r > 0\}$. For $1 < p < \infty$, L^p denotes the Lebesgue space on $\mathbb{R}^n$ with respect to the Lebesgue measure, the set L^p_c consists of all $f \in L^p$ with compact support.

We now give the definition of Morrey spaces and Herz spaces as follows.

Definition 2.1. Let $1 < p \le q < \infty$. The Morrey space M_p^q consists of all locally integrable functions f satisfying

$$\|f\|_{M_p^q} = \sup_{B\in\mathbb{B}} \frac{1}{|B|^{\frac{1}{p}-\frac{1}{q}}} \|\chi_B f\|_{L^p} < \infty.$$

For the properties of Morrey spaces, especially, the mapping properties of the Hardy-Littlewood maximal function, singular integral operators and fractional integral operators on Morrey spaces, the reader is referred to [1, 17, 20]. For applications of Morrey spaces in partial differential equations, see [3, 4, 5].

Next, we introduce the definition of Herz spaces on $\mathbb{R}^n\backslash B(0,1)$. For any $j \in \mathbb{Z}$, write $\chi_j = \chi_{B(0,2^j)\backslash B(0,2^{j-1})}$.

Definition 2.2. Let $1 \le p, q < \infty$ and $\alpha \in \mathbb{R}$. The Herz space $\mathcal{K}_{p,q}^{\alpha}$ on $\mathbb{R}^n\backslash B(0,1)$ consists of all locally integrable functions f on $\mathbb{R}^n\backslash B(0,1)$ satisfying

$$\|f\|_{\mathcal{K}_{p,q}^{\alpha}} = \left(\sum_{j=1}^{\infty}(2^{j\alpha}\|\chi_j f\|_{L^p})^q\right)^{\frac{1}{q}} < \infty.$$

For the definition and applications of Herz space on $\mathbb{R}^n$, the reader may consult [14]. For some further studies of Herz spaces, such as the applications to partial differential equations, see [11, 13, 18, 19].

We now introduce the integral operators studied in this paper.

Definition 2.3. Let $1 < p_1, p_2 < \infty$. For any bounded linear operator $T : L^{p_1} \to L^{p_2}$, write $T \in \mathcal{L}_{p_1,p_2}$ if there exists a Lebesgue measurable function $K : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ such that for any $f \in L_c^{p_1}$ and $x \notin \operatorname{supp} f$,

$$T(f)(x) = \int_{\mathbb{R}^n} K(x,y)f(y)dy.$$

We say that T is associated with the kernel K.

Whenever T is a Calderón-Zygmund operator in the sense of [2, Definition 1], then for any $p \in (1,\infty)$, $T \in \mathcal{L}_{p,p}$ [2, p.14-18]. Also, for $\alpha \in (0,n)$ and $p_1 \in (1, \frac{n}{\alpha})$, if I_α is the fractional integral operator [12, (3.0.2)], then $I_\alpha \in \mathcal{L}_{p_1,p_2}$ [12, Theorem 3.1.1] where $\frac{1}{p_2} = \frac{1}{p_1} - \frac{\alpha}{n}$.

We now present an estimate for $\int_{|x-y|>r} |K(x,y)f(y)|dy$ where $f \in M_p^q$. We see that the Herz space $\mathcal{K}_{p',1}^{\frac{n}{p}-\frac{n}{q}}$ is involved in this estimate.

As translations and dilations of the kernel are also involved in this estimate, in order to simplify the presentation, we introduce the following notation. Let $r > 0$ and $\theta \in \mathbb{R}$. For any Lebesgue measurable function $K : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, write

$$K_r^{\theta}(x,y) = r^{\theta}K(rx, r(x+y)), \quad x,y \in \mathbb{R}^n.$$

Lemma 2.1. *Let $\theta \in \mathbb{R}$, $1 < p \le q < \infty$, $r > 0$ and $K : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be a Lebesgue measurable function. If for any $x \in \mathbb{R}^n$, $K_r^\theta(x/r, \cdot) \in \mathcal{K}_{p',1}^{\frac{n}{p}-\frac{n}{q}}$, then there is a constant $C > 0$ independent of K such that for any $f \in M_p^q$, $x \in \mathbb{R}^n$ and $r > 0$, we have*

$$\int_{|x-y|>r} |K(x,y)f(y)|dy \le Cr^{-\theta+n-\frac{n}{q}} \|K_r^\theta(x/r,\cdot)\|_{\mathcal{K}_{p',1}^{\frac{n}{p}-\frac{n}{q}}} \|f\|_{M_p^q}.$$

Proof. Let $f \in M_p^q$, $c \in \mathbb{R}^n$ and $r > 0$. For any $j \ge 1$, the Hölder inequality and Definition 2.1 give

$$\begin{aligned}
&\int_{2^{j-1}r\le|y-x|<2^jr} |K(x,y)f(y)|dy \\
&\le \left(\int_{2^{j-1}r\le|y-x|<2^jr} |K(x,y)|^{p'}dy\right)^{\frac{1}{p'}} \left(\int_{2^{j-1}r\le|y-x|<2^jr} |f(y)|^p dy\right)^{\frac{1}{p}} \\
&\le C\left(\int_{2^{j-1}r\le|y-x|<2^jr} |K(x,y)|^{p'}dy\right)^{\frac{1}{p'}} (2^jr)^{\frac{n}{p}-\frac{n}{q}} \|f\|_{M_p^q}
\end{aligned}$$

for some $C > 0$ independent of K, $x \in \mathbb{R}^n$ and $r > 0$.

We apply the change of variables $w = \frac{x}{r}$ and $z = \frac{y}{r}$ and, then, apply the change of variable $z - w = v$ for the integral for K. We obtain

$$\begin{aligned}
&\int_{2^{j-1}r\le|y-x|<2^jr} |K(x,y)f(y)|dy \\
&\le C\left(\int_{2^{j-1}\le|z-w|<2^j} |K(rw,rz)|^{p'}r^n dz\right)^{\frac{1}{p'}} (2^jr)^{\frac{n}{p}-\frac{n}{q}} \|f\|_{M_p^q} \\
&\le C\left(\int_{2^{j-1}\le|v|<2^j} |K(rw,r(w+v))|^{p'}r^n dv\right)^{\frac{1}{p'}} (2^jr)^{\frac{n}{p}-\frac{n}{q}} \|f\|_{M_p^q} \\
&= C\left(\int_{2^{j-1}\le|v|<2^j} |K_r^\theta(w,v)|^{p'}dv\right)^{\frac{1}{p'}} r^{-\theta+\frac{n}{p'}} (2^jr)^{\frac{n}{p}-\frac{n}{q}} \|f\|_{M_p^q}.
\end{aligned}$$

By taking the summation over j on both sides of the above inequalities, Definition 2.2 yields

$$
\begin{aligned}
&\int_{|x-y|>r} |K(x,y)f(y)|dy \\
&= \sum_{j=1}^{\infty} \int_{2^{j-1}r\le|y-x|<2^j r} |K(x,y)f(y)|dy \\
&\le C\sum_{j=1}^{\infty} \left(\int_{2^{j-1}\le|v|<2^j} |K_r^\theta(w,v)|^{p'}dv\right)^{\frac{1}{p'}} r^{-\theta+\frac{n}{p'}}(2^j r)^{\frac{n}{p}-\frac{n}{q}}\|f\|_{M_p^q} \\
&\le Cr^{-\theta+n-\frac{n}{q}}\left(\sum_{j=1}^{\infty} 2^{j(\frac{n}{p}-\frac{n}{q})}\|\chi_j K_r^\theta(w,\cdot)\|_{L^{p'}}\right)\|f\|_{M_p^q} \\
&= Cr^{-\theta+n-\frac{n}{q}}\|K_r^\theta(x/r,\cdot)\|_{\mathcal{K}_{p',1}^{\frac{n}{p}-\frac{n}{q}}}\|f\|_{M_p^q}
\end{aligned}
$$

for some $C>0$ independent of K, $x\in\mathbb{R}^n$ and $r>0$. $\square$

3. Main result

This section establishes the main result of this paper. We begin with the definition of the integral operators on Morrey spaces.

Definition 3.1. Let $1<p_i<q_i<\infty$, $i=1,2$. Suppose that $T\in\mathcal{L}_{p_1,p_2}$. For any $f\in M_{p_1}^{q_1}$ and $x\in B(z,r)\in\mathbb{B}$, we define

$$
\mathcal{T}(f)(x)=T(\chi_{B(z,2r)}f)(x)+\int_{\mathbb{R}^n\setminus B(z,2r)} K(x,y)f(y)dy. \tag{3.1}
$$

Notice that T is originally defined in the Lebesgue space, and the definition of singular integral operators on Lebesgue spaces is highly delicate, see [21, 22]. Similarly, extending the definition of T to Morrey spaces requires careful justification. In particular, it is necessary to check that the above definition is independent of the ball $B(z,r)$. We use the idea from [8, 9] to verify that the definition of $\mathcal{T}$ is well defined.

The following is the main result of this paper. It consists of two results. We first show that $\mathcal{T}$ is well defined on $M_{p_1}^{q_1}$ and, then we establish the boundedness of $\mathcal{T}$ from $M_{p_1}^{q_1}$ to $M_{p_2}^{q_2}$. We also give an operator norm estimate of the integral operator on Morrey spaces in terms of the operator norm on Lebesgue spaces and the Herz space norm of the kernel of the integral operator.

Theorem 3.1. *Let $1<p_i<q_i<\infty$, $i=1,2$. Suppose that*

$$
\theta=n-\frac{n}{q_1}+\frac{n}{q_2} \tag{3.2}
$$

and $T \in \mathcal{L}_{p_1,p_2}$ is associated with the kernel K. If K satisfies

$$\sup_{x\in\mathbb{R}^n}\sup_{r>0}\|K_r^\theta(x/r,\cdot)\|_{\mathcal{K}^{\frac{n}{p_1}-\frac{n}{q_1}}_{p_1',1}} < \infty, \tag{3.3}$$

then $\mathcal{T}$ is a well defined bounded linear operator from $M^{q_1}_{p_1}$ to $M^{q_2}_{p_2}$ and $\mathcal{T}$ satisfies

$$\|\mathcal{T}\|_{M^{q_1}_{p_1}\to M^{q_2}_{p_2}} \leq \|T\|_{L^{p_1}\to L^{p_2}} + C\sup_{x\in\mathbb{R}^n}\sup_{r>0}\|K_r^\theta(x/r,\cdot)\|_{\mathcal{K}^{\frac{n}{p_1}-\frac{n}{q_1}}_{p_1',1}} \tag{3.4}$$

for some $C>0$ independent of K where $\|\mathcal{T}\|_{M^{q_1}_{p_1}\to M^{q_2}_{p_2}}$ and $\|T\|_{L^{p_1}\to L^{p_2}}$ denote the operator norms of $\mathcal{T} : M^{q_1}_{p_1} \to M^{q_2}_{p_2}$ and $T : L^{p_1} \to L^{p_2}$, respectively.

Proof. Let $f \in M^{q_1}_{p_1}$ and $B(z,R) \in \mathbb{B}$. As $T \in \mathcal{L}_{p_1,p_2}$ and $\chi_{B(z,2R)}f \in L^{p_1}$, $T(\chi_{B(z,2R)}f)$ is well defined. Lemma 2.1 guarantees that

$$\chi_{\mathbb{R}^n\setminus B(z,2R)}|K(x,y)||f(y)|$$

is integrable. Consequently, $\int_{\mathbb{R}^n\setminus B(z,2R)} K(x,y)f(y)dy$ is well defined. We now show that the definition is independent of $B(z,R) \in \mathbb{B}$. That is, for any $x \in B(z,R)\cap B(w,S)$ with $B(z,R), B(w,S) \in \mathbb{B}$ and $B(z,R)\cap B(w,S) \neq \emptyset$, we have

$$\begin{aligned} &T(\chi_{B(z,2R)}f)(x) + \int_{\mathbb{R}^n\setminus B(z,2R)} K(x,y)f(y)dy \\ &\quad= T(\chi_{B(w,2S)}f)(x) + \int_{\mathbb{R}^n\setminus B(w,2S)} K(x,y)f(y)dy. \end{aligned} \tag{3.5}$$

Let $B(u,M) \in \mathbb{B}$ be selected so that $B(z,2R)\cup B(w,2S) \subset B(u,M)$. As $T \in \mathcal{L}_{p_1,p_2}$ and $\chi_{B(u,M)\setminus B(z,2R)}f, \chi_{B(u,M)\setminus B(z,2S)}f \in L^{p_1}_c$, for any $x \in B(z,R)\cap B(w,S)$,

$$T(\chi_{B(u,M)\setminus B(z,2R)}f)(x) = \int_{B(u,M)\setminus B(z,2R)} K(x,y)f(y)dy, \tag{3.6}$$

$$T(\chi_{B(u,M)\setminus B(w,2S)}f)(x) = \int_{B(u,M)\setminus B(w,2S)} K(x,y)f(y)dy. \tag{3.7}$$

Since $\chi_{\mathbb{R}^n\setminus B(z,2R)}(\cdot)f(\cdot)K(x,\cdot)$ is absolutely integrable, (3.6) guarantees that

$$\begin{aligned}(3.8)\qquad & T(\chi_{B(z,2R)}f)(x)+\int_{\mathbb{R}^n\setminus B(z,2R)}K(x,y)f(y)dy\\ &=T(\chi_{B(z,2R)}f)(x)+\int_{B(u,M)\setminus B(z,2R)}K(x,y)f(y)dy\\ &+\int_{\mathbb{R}^n\setminus B(u,M)}K(x,y)f(y)dy\\ &=T(\chi_{B(z,2R)}f)(x)+T(\chi_{B(u,M)\setminus B(z,2R)}f)(x)\\ &+\int_{\mathbb{R}^n\setminus B(u,M)}K(x,y)f(y)dy\\ &=T(\chi_{B(u,M)}f)(x)+\int_{\mathbb{R}^n\setminus B(u,M)}K(x,y)f(y)dy.\end{aligned}$$

Similarly, (3.7) asserts that

$$\begin{aligned}(3.9)\qquad & T(\chi_{B(w,2S)}f)(x)+\int_{\mathbb{R}^n\setminus B(w,2S)}K(x,y)f(y)dy\\ &=T(\chi_{B(u,M)}f)(x)+\int_{\mathbb{R}^n\setminus B(u,M)}K(x,y)f(y)dy.\end{aligned}$$

Therefore, (3.8) and (3.9) yield (3.5). That is, $\mathcal{T}(f)$ is well defined when $f\in M^{q_1}_{p_1}$. According to (3.1), it is easy to see that $\mathcal{T}$ is a linear operator on $M^{q_1}_{p_1}$. Finally, we show that $\mathcal{T}$ is a bounded operator from $M^{q_1}_{p_1}$ to $M^{q_2}_{p_2}$. Since $T:L^{p_1}\to L^{p_2}$ is bounded, we have

$$\begin{aligned}\|\chi_{B(z,R)}\mathcal{T}(f)\|_{L^{p_2}} &\le \|T(\chi_{B(z,2R)}f)\|_{L^{p_2}}+\|\chi_{B(z,R)}F\|_{L^{p_2}}\\ (3.10)\qquad &\le \|T\|_{L^{p_1}\to L^{p_2}}\|\chi_{B(z,2R)}f\|_{L^{p_1}}+\|\chi_{B(z,R)}F\|_{L^{p_2}}\end{aligned}$$

for some $C>0$ where

$$F(x)=\int_{\mathbb{R}^n\setminus B(z,2R)}K(x,y)f(y)dy.$$

As (3.3) gives

$$\sup_{x\in\mathbb{R}^n}\sup_{r>0}\|K^\theta_r(x/r,\cdot)\|_{\mathcal{K}^{\frac{n}{p_1}-\frac{n}{q_1}}_{p_1',1}}<\infty,$$

Lemma 2.1 and (3.2) guarantee that

$$\begin{aligned}
&\frac{1}{R^{\frac{n}{p_2}-\frac{n}{q_2}}}\|\chi_{B(z,R)}F\|_{L^{p_2}}\\
&\leq \frac{1}{R^{\frac{n}{p_2}-\frac{n}{q_2}}}\|\chi_{B(z,R)}\|_{L^{p_2}}\|F\|_{L^\infty}\\
&\leq C\frac{1}{R^{\frac{n}{p_2}-\frac{n}{q_2}}}\|\chi_{B(z,R)}\|_{L^{p_2}}R^{-\theta+n-\frac{n}{q_1}}\|K^\theta_{2R}(x/2R,\cdot)\|_{\mathcal{K}^{\frac{n}{p_1}-\frac{n}{q_1}}_{p_1',1}}\|f\|_{M^{q_1}_{p_1}}\\
&\leq C\frac{1}{R^{\frac{n}{p_2}-\frac{n}{q_2}}}R^{\frac{n}{p_2}}R^{-\theta+n-\frac{n}{q_1}}\sup_{x\in\mathbb{R}^n}\sup_{r>0}\|K^\theta_r(x/r,\cdot)\|_{\mathcal{K}^{\frac{n}{p_1}-\frac{n}{q_1}}_{p_1',1}}\|f\|_{M^{q_1}_{p_1}}
\end{aligned}$$

(3.11)

$$\leq C\sup_{x\in\mathbb{R}^n}\sup_{r>0}\|K^\theta_r(x/r,\cdot)\|_{\mathcal{K}^{\frac{n}{p_1}-\frac{n}{q_1}}_{p_1',1}}\|f\|_{M^{q_1}_{p_1}}$$

for some $C>0$ independent of K. Consequently, (3.10) and (3.11) yield

$$\begin{aligned}
&\frac{1}{R^{\frac{n}{p_2}-\frac{n}{q_2}}}\|\chi_{B(z,R)}\mathcal{T}(f)\|_{L^{p_2}}\\
&\leq \frac{1}{R^{\frac{n}{p_2}-\frac{n}{q_2}}}\|T\|_{L^{p_2}\to L^{p_1}}\|\chi_{B(z,2R)}f\|_{L^{p_1}}+\frac{1}{R^{\frac{n}{p_2}-\frac{n}{q_2}}}\|\chi_{B(z,R)}F\|_{L^{p_2}}\\
&\leq \left(\|T\|_{L^{p_1}\to L^{p_2}}+C\sup_{x\in\mathbb{R}^n}\sup_{r>0}\|K^\theta_r(x/r,\cdot)\|_{\mathcal{K}^{\frac{n}{p_1}-\frac{n}{q_1}}_{p_1',1}}\right)\|f\|_{M^{q_1}_{p_1}}.
\end{aligned}$$

Taking the supremum over $B(z,R)\in\mathbb{B}$ on both sides of the above inequality, we find that

$$\begin{aligned}
\|\mathcal{T}(f)\|_{M^{q_2}_{p_2}}&=\sup_{B(z,R)\in\mathbb{B}}\frac{1}{R^{\frac{n}{p_2}-\frac{n}{q_2}}}\|\chi_{B(z,R)}\mathcal{T}(f)\|_{L^{p_2}}\\
&\leq \left(\|T\|_{L^{p_1}\to L^{p_2}}+C\sup_{x\in\mathbb{R}^n}\sup_{r>0}\|K^\theta_r(x/r,\cdot)\|_{\mathcal{K}^{\frac{n}{p_1}-\frac{n}{q_1}}_{p_1',1}}\right)\|f\|_{M^{q_1}_{p_1}}
\end{aligned}$$

which establishes the boundedness of $\mathcal{T}:M^{q_1}_{p_1}\to M^{q_2}_{p_2}$. Furthermore, we also have

$$\|\mathcal{T}\|_{M^{q_1}_{p_1}\to M^{q_2}_{p_2}}\leq\|T\|_{L^{p_1}\to L^{p_2}}+C\sup_{x\in\mathbb{R}^n}\sup_{r>0}\|K^\theta_r(x/r,\cdot)\|_{\mathcal{K}^{\frac{n}{p_1}-\frac{n}{q_1}}_{p_1',1}}$$

for some $C>0$ independent of K.

□

Next, we show that $\mathcal{T}$ is an extension of T.

Theorem 3.2. *Let $1<p_i<q_i<\infty$, $i=1,2$. If $T\in\mathcal{L}_{p_1,p_2}$ is associated with the kernel K and K satisfies (3.3), then for any $f\in L^{p_1}\cap M^{q_1}_{p_1}$, we have $\mathcal{T}(f)=T(f)$.*

Proof. Let $f \in L^{p_1} \cap M^{q_1}_{p_1}$. For any fixed $B(z,r) \in \mathbb{B}$, write $f_0 = \chi_{B(z,2r)} f$ and $f_i = \chi_{B(z,2^{i+1}r)\setminus B(z,2^i r)} f$, $i \in \mathbb{N}\setminus\{0\}$.

Since $|f - \sum_{i=0}^N f_i| \le |f|$, $N \in \mathbb{N}$ and $\lim_{N\to\infty} |f - \sum_{i=0}^N f_i| = 0$, the dominated convergence theorem guarantees that $\lim_{N\to\infty} \|f - \sum_{i=0}^N f_i\|_{L^{p_1}} = 0$. As $T \in \mathcal{L}_{p_1,p_2}$, we have

$$\lim_{N\to\infty} \left\| T(f - \sum_{i=0}^N f_i) \right\|_{L^{p_2}} \le C \lim_{N\to\infty} \left\| f - \sum_{i=0}^N f_i \right\|_{L^{p_1}} = 0.$$

Thus, we have

$$T(f) = T(f_0) + \sum_{i=1}^\infty T(f_i)$$

on L^{p_2} and hence, almost everywhere. As $f_i \in L^{p_1}_c$ and $\mathrm{supp} f_i \cap B(z,r) = \emptyset$, for any $x \in B(z,r)$, we have

$$T(f)(x) = T(f_0)(x) + \sum_{i=1}^N \int_{2^i r \le |x-y| < 2^{i+1} r} K(x,y) f(y) dy.$$

Since Lemma 2.1 guarantees that $\chi_{\mathbb{R}^n\setminus B(z,2R)} |K(x,y)||f(y)|$ is integrable, we find that for any $x \in B(z,r)$

$$T(f)(x) = T(f_0)(x) + \int_{2r \le |x-y|} K(x,y) f(y) dy = \mathcal{T}(f)(x).$$

As $B(z,r)$ is arbitrary, we have $T(f)(x) = \mathcal{T}(f)(x)$ for almost all x. $\square$

We now apply the main results to some concrete operators, especially to those operators associated with kernel possessing singularity on $\{(x,x) : x \in \mathbb{R}^n\}$.

Let $\mathbb{S}^{n-1}$ denote the unit sphere on $\mathbb{R}^n$. We say that a Lebesgue measurable function Ω is a homogeneous function if $\Omega(\lambda x) = \Omega(x)$ for all $\lambda > 0$ and $x \in \mathbb{R}^n$.

Theorem 3.3. *Let $\gamma \in \mathbb{R}$, $1 < p_i < q_i < \infty$, $i = 1,2$. Suppose that*

$$\gamma = n - \frac{n}{q_1} + \frac{n}{q_2} \tag{3.12}$$

and Ω is a homogeneous function satisfying

$$\|\Omega\|_{L^{p_1'}(\mathbb{S}^{n-1})} = \left(\int_{\mathbb{S}^{n-1}} |\Omega(\theta)|^{p_1'} d\theta \right)^{\frac{1}{p_1'}} < \infty \tag{3.13}$$

where $d\theta$ is the induced Lebesgue measure on $\mathbb{S}^{n-1}$. If $T \in \mathcal{L}_{p_1,p_2}$ is associated with kernel K and K satisfies

$$|K(x,y)| \le |x-y|^{-\gamma} |\Omega(x-y)|, \quad x \ne y,$$

then $\mathcal{T}$ is a bounded linear operator from $M^{q_1}_{p_1}$ to $M^{q_2}_{p_2}$ and

$$\|\mathcal{T}\|_{M^{q_1}_{p_1}\to M^{q_2}_{p_2}} \leq \|T\|_{L^{p_1}\to L^{p_2}} + C\|\Omega\|_{L^{p_1'}(\mathbb{S}^{n-1})} \tag{3.14}$$

for some $C>0$ independent of Ω.

Proof. Write

$$H(x,y) = |x-y|^{-\gamma}|\Omega(x-y)|\chi_{\mathbb{R}^n\times\mathbb{R}^n\setminus\{(z,z):z\in\mathbb{R}^n\}}(x,y), \quad x,y\in\mathbb{R}^n.$$

For any $r>0$, $x\in\mathbb{R}^n$ and $y\in\mathbb{R}^n\setminus\{0\}$, we have

$$\begin{aligned} H^\gamma_r(x/r,y) &= r^\gamma H(x,x+ry) = r^\gamma|ry|^{-\gamma}|\Omega(ry)| \\ &= |y|^{-\gamma}|\Omega(y)| \end{aligned}$$

because Ω is a homogeneous function. Moreover, (3.13) assures that there is a constant $C>0$ such that for any $j\in\mathbb{N}$

$$\begin{aligned} \|\chi_j H^\gamma_r(x/r,\cdot)\|_{L^{p_1'}} &= \left(\int_{2^{j-1}\leq|y|<2^j}|\Omega(y)|^{p_1'}|y|^{-\gamma p_1'}dy\right)^{\frac{1}{p_1'}} \\ &= C\left(\int_{2^{j-1}}^{2^j}\left(\int_{\mathbb{S}^{n-1}}|\Omega(t\theta)|^{p_1'}d\theta\right)t^{-\gamma p_1'}t^{n-1}dt\right)^{\frac{1}{p_1'}} \\ &= C\left(\int_{2^{j-1}}^{2^j}\left(\int_{\mathbb{S}^{n-1}}|\Omega(\theta)|^{p_1'}d\theta\right)t^{-\gamma p_1'}t^{n-1}dt\right)^{\frac{1}{p_1'}} \\ &\leq C2^{-j(\gamma-\frac{n}{p_1'})}\|\Omega\|_{L^{p_1'}(\mathbb{S}^{n-1})} \end{aligned}$$

for some $C>0$ independent of Ω. Consequently, (3.12) and the above inequalities yield some $C,C_0>0$ such that for any $x\in\mathbb{R}^n$ and $r>0$

$$\begin{aligned} \|H^\gamma_r(x/r,\cdot)\|_{\mathcal{K}^{\frac{n}{p_1}-\frac{n}{q_1}}_{p_1',1}} &= \sum_{j=1}^\infty 2^{j(\frac{n}{p_1}-\frac{n}{q_1})}\|\chi_j H^\gamma_r(x/r,\cdot)\|_{L^{p_1'}} \\ &\leq C\|\Omega\|_{L^{p_1'}(\mathbb{S}^{n-1})}\sum_{j=1}^\infty 2^{-j(-\frac{n}{p_1}+\frac{n}{q_1}+\gamma-\frac{n}{p_1'})} \\ &= C\|\Omega\|_{L^{p_1'}(\mathbb{S}^{n-1})}\sum_{j=1}^\infty 2^{-j(\gamma-n+\frac{n}{q_1})} \\ &= C\|\Omega\|_{L^{p_1'}(\mathbb{S}^{n-1})}\sum_{j=1}^\infty 2^{-j\frac{n}{q_2}} \leq C_0\|\Omega\|_{L^{p_1'}(\mathbb{S}^{n-1})}. \end{aligned}$$

Thus, by taking the supremum over $x \in \mathbb{R}^n$ and $r > 0$ on both sides of the above inequalities, we have

$$\sup_{x\in\mathbb{R}^n}\sup_{r>0}\|H_r^\gamma(x/r,\cdot)\|_{\mathcal{K}^{\frac{n}{p_1}-\frac{n}{q_1}}_{p_1',1}} \leq C_0\|\Omega\|_{L^{p_1'}(\mathbb{S}^{n-1})}. \tag{3.15}$$

Consequently, H satisfies (3.3) and, hence, K also satisfies (3.3). Theorem 3.1 and (3.15) yield the boundedness of $\mathcal{T}$ from $M^{q_1}_{p_1}$ to $M^{q_2}_{p_2}$ and (3.14). □

When $\gamma = n$, the above result gives the boundedness of rough singular integral operators on Morrey spaces When $\gamma = n - \alpha$ with $0 < \alpha < n$, as Theorems 3.1 and Theorem 3.3 apply to the case when $p_1 \neq p_2$, they yield the mapping properties of rough fractional integral operators on Morrey spaces. In particular, it gives the boundedness of singular integral operators and fractional integral operators on Morrey spaces. Notice that the boundedness of singular integral operators and fractional integral operators on Morrey spaces are well known results, see [8, 9, 17, 20]. Theorems 3.1 and 3.3 give a unified approach to study singular integral operators and fractional integral operators on Morrey spaces.

Moreover, inequality (3.14) gives operator norm estimates of singular integral operators, rough singular integral operators, fractional integral operators and rough fractional integral operators on Morrey spaces which are new compared with the results in [8, 9, 17, 20].

When $\gamma = n + \alpha$ with $\alpha > 0$, the above theorem yields the following result for the integral operators associated with kernel K satisfying

$$|K(x,y)| \leq |x-y|^{-n-\alpha}|\Omega(x-y)|, \quad x \neq y \tag{3.16}$$

for some homogeneous function Ω.

Corollary 3.4. *Let $\alpha \in (0,\infty)$, $1 < p_i < q_i < \infty$, $i = 1, 2$. Suppose that*

$$\alpha = \frac{n}{q_2} - \frac{n}{q_1} \tag{3.17}$$

and Ω is a homogeneous function satisfying (3.13). If $T \in \mathcal{L}_{p_1,p_2}$ is associated with kernel K and K satisfies (3.16), then $\mathcal{T}$ is a bounded linear operator from $M^{q_1}_{p_1}$ to $M^{q_2}_{p_2}$.

We present some examples of integral operators with kernels satisfying (3.16) from [10, Section 3.1.3] and [15]. Those operators are named as strongly singular integral operators.

Let $\beta \in (0,\infty)$, $\alpha \in [0, \frac{n\beta}{2})$ and $a, \phi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be smooth functions satisfying

$$|\partial_x^\mu \partial_y^\nu a(x,y)| \leq C|x-y|^{-n-\alpha-|\mu|-|\nu|} \tag{3.18}$$

$$|\partial_x^\mu \partial_y^\nu \phi(x,y)| \leq C|x-y|^{-\beta-|\mu|-|\nu|} \tag{3.19}$$

$$C|x-y|^{-\beta-1} \leq |\nabla_x \phi(x,y)| \tag{3.20}$$

$$C|x-y|^{-\beta-1} \leq |\nabla_y \phi(x,y)| \tag{3.21}$$

$$C \leq \left| \det \left(\frac{\partial^2 \phi_\lambda(x,y)}{\partial x_i \partial y_j} \right) \right| \tag{3.22}$$

uniformly in λ where $\phi_\lambda(x,y) = \lambda^\beta \phi(\lambda x, \lambda y)$, $\lambda > 0$, for some $C > 0$.

For any a, ϕ satisfying the above conditions, the operator $T_{a,\phi}$ is defined as the integral operator associated with the kernel

$$K_{a,\phi}(x,y) = e^{i\phi(x,y)} a(x,y).$$

Some examples of operators $T_{a,\phi}$ are pseudodifferential operators with symbols in the Hörmander class and the convolution operators defined by the kernel

$$K_{a,\phi}(x,y) = e^{i|x-y|^\beta} |x-y|^{-n-\alpha} \chi(|x-y|)$$

where χ is a smooth function with compact support in a neighborhood of the origin. These convolution operators are used in [7, 23] for the studies of Fourier multipliers and Fourier series.

We have the following result from [10, Theorem C].

Theorem 3.5. *Let $\beta \in (0,\infty)$, $\alpha \in [0, \frac{n\beta}{2})$ and $a, \phi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ satisfy (3.18)-(3.22). If $p \in (1,\infty)$ satisfies*

$$\left| \frac{1}{p} - \frac{1}{2} \right| \leq \frac{1}{2} - \frac{\alpha}{n\beta}, \tag{3.23}$$

then $T_{a,\phi}$ is bounded on L^p.

For any a, ϕ satisfying (3.18)-(3.22), let $\mathcal{T}_{a,\phi}$ be the extension of $T_{a,\phi}$ given by Definition 3.1. According to Definition 3.1 and Theorem 3.1, the strongly singular integral operator $\mathcal{T}_{a,\phi}$ is well defined on Morrey spaces. Theorem 3.5 and Corollary 3.4 yield the boundedness of $\mathcal{T}_{a,\phi}$ on Morrey spaces.

Corollary 3.6. *Let $\beta \in (0,\infty)$, $\alpha \in (0, \frac{n\beta}{2})$ and $a, \phi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ satisfy (3.18)-(3.22). If $q_1 \in (p,\infty)$, $q_2 \in (p, \frac{n}{\alpha})$ satisfy (3.17) and p satisfies (3.23), then $\mathcal{T}_{a,\phi} : M_p^{q_1} \to M_p^{q_2}$ is bounded.*

We remark that the set of exponents satisfying the conditions in Corollary 3.6 is non-empty as can be seen for example by choosing $\beta = 1$, $\alpha = \frac{n}{4}$, $p = 3$, $q_1 = 20$ and $q_2 = \frac{10}{3}$. Obviously, $\alpha = \frac{n}{4} < \frac{n}{2}$. As $\frac{1}{4} < \frac{1}{3} < \frac{3}{4}$, the condition (3.23) is satisfied. We see that $q_1 = 20 > 3 = p$, $q_2 = \frac{10}{3} > 3 = p$ and $q_2 = \frac{10}{3} < 4 = \frac{n}{\alpha}$. Moreover,

$$\frac{n}{q_2} - \frac{n}{q_1} = \frac{n}{4} = \alpha.$$

Thus, (3.17) is fulfilled.

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