Local expansion properties of paracontrolled systems

I. BAILLEUL and N. MOENCH

Abstract. The concept of concrete regularity structure gives the algebraic backbone of the operations involved in the local expansions used in the regularity structure approach to singular stochastic partial differential equations. The spaces and the details of the structures depend on each equation. We introduce here a parameter-dependent universal algebraic regularity structure that can host all the regularity structures used in the study of singular stochastic partial differential equations. This is done by using the correspondence between the notions of model on a regularity structure and the notion of paracontrolled system. We prove that the iterated paraproducts that form the fundamental bricks of paracontrolled systems have some local expansion properties that are governed by this universal structure.

Contents

1 – Introduction

The theory of regularity structures was introduced by M. Hairer [13] as a convenient setting adapted to give sense to, and study, a large class of stochastic partial differential equations that share a common 'singular' feature and involve some a priori ill-defined terms in their formulations, placing them beyond the reach of classical stochastic calculus. Each equation in this class can be formulated as a fixed point problem in a random space of modelled distributions over a deterministic, equation-dependent, regularity structure. A solution to a singular stochastic partial differential equation then comes under the form of a local expansion around each state space point in the setting of regularity structures. One of the remarkable points of this setting is the prominent role played by some algebraic structures. These structures have their origins in two different sides of the story. On the one hand, regularity structures are intimately linked with the choice of representing an unknown function/distribution by its ''jet' in some a priori local expansion system. Some elementary consistency conditions on this representation make appear the Hopf algebra and the comodule over this algebra that define a general (concrete) regularity structure. (See e.g. Section 2.1 of Bailleul & Hoshino's Tourist Guide [4].) On the other hand, the specific needs required to deal with the singular feature of such equations via renormalization comes under the form of another algebraic structure that needs to dovetail nicely with the regularity structure to lead to a clear analysis of a generic singular stochastic partial differential equation.

There is not a unique way of implementing that picture. The initial scope of the theory was mainly about semilinear equations. An iterative fixed point formulation of such equations naturally leads to some regularity structures indexed by some combinatorial trees. The extension by Otto and his co-authors [20, 19, 18] of the theory of regularity structures to the setting of scalar valued quasi-linear equations motivated the introduction of some regularity structures indexed by some multi-indices.

When applied to semi-linear equations, this local expansion device turns out to be greedier than the tree-based expansion device, in the sense that the multi-index based local expansions typically involve less terms than the tree-based expansions. We keep from that picture the fact that for a fixed equation there is not a canonical choice of regularity structure for its study.

Another remarkable feature of the study of singular stochastic partial differential equations is the fact that each equation can be studied/formulated by using an equation-dependent regularity structure, so there is no universal regularity structure that works for all equations at a time. This is in contrast with what happens in the one dimensional case of rough differential equations.

Despite this state of affair, we show in the present work that there is some universal structure behind these different regularity structures. We use for that purpose another set of tools that was developed for the study of singular stochastic partial differential equations: the paracontrolled calculus, introduced first by Gubinelli, Imkeller & Perkowski in [11], and developed in particular by Bailleul & Bernicot [2, 3]. See e.g. [10, 12, 21] and the reference therein for a tiny sample of some important contributions to this setting. A dictionary between the language of regularity structures and the language of paracontrolled calculus was given by Bailleul & Hoshino in their works [5, 6]. Models and modelled distributions are encoded in the notion of paracontrolled system. Such systems involve some non-local operators on functions/distributions. We prove that their pointwise expansions involve some 'universal' algebra that depends only on the number of reference objects in the paracontrolled system and their regularity exponents, not on the reference objects themselves. This is the main result of this work, stated below as Theorem 2.

We now introduce the setting needed to understand this statement. To simplify the exposition, we work in the Euclidean space $\mathbf{R}^{d_0}$ – all that follows has some direct counterpart in a non-isotropic setting. The Besov-Hölder spaces C^{α_1} over $\mathbf{R}^{d_0}$ and their norms $\|\cdot\|_{\alpha_1}$ are defined as usual for any $\alpha_1 \in \mathbf{R}$ from the Littlewood-Paley projectors $\Delta_i : \mathcal{D}'(\mathbf{R}^{d_0}) \to C^\infty(\mathbf{R}^{d_0})$ setting $\|f\|_{\alpha_1} = \sup_{i\geq -1} 2^{i\alpha_1}\|\Delta_i(f)\|_\infty$, we will also write K_i for the kernel associated to the operator Δ_i. Let

$$\Delta_{<j} := \sum_{i\leq j-1} \Delta_i,$$

and define the paraproduct $\mathsf{P}(f,g)$ of any two distributions f,g as

$$\mathsf{P}(f,g) := \sum_{i\geq 1} \Delta_{<i-1}(f)\,\Delta_i(g).$$

For $f\in C^{\alpha_1}$ and $g\in C^{\alpha_2}$ we have the optimal continuity estimates

$$\|\mathsf{P}(f,g)\|_{\alpha_2} \lesssim \|f\|_\infty \|g\|_{\alpha_2} \qquad \text{if } \alpha_1 \geq 0$$

and

$$\|\mathsf{P}(f,g)\|_{\alpha_1+\alpha_2} \lesssim \|f\|_{\alpha_1} \|g\|_{\alpha_2} \qquad \text{if } \alpha_1 < 0.$$

See for instance Section 2.6 in Bahouri, Chemin & Danchin's textbook [1] for a reference.

Our main results involve some iterated paraproduct operators that we introduce in Section 1.1. These operators are non-local and it is non-obvious to give a systematic description of their pointwise expansion properties. We know from Section 2.1 of [4] that such expansions involve a priori some concrete regularity structures. We describe in *§1* of Section 1.2 a particular concrete regularity structure and define in *§2* of that section a pair $(\mathsf{\Pi},\mathsf{g})$ of maps on that regularity structure. Our first main result, Theorem 1, states that $(\mathsf{\Pi},\mathsf{g})$ is a model. Paracontrolled systems are then introduced in Section 1.3, where we state the main result of this work, Theorem 2. Its proof in Section 5 will make it clear that Theorem 2 is a corollary of Theorem 1.

1.1 – Local expansion properties of iterated paraproducts. We define inductively the iterated paraproduct operator by setting $\mathsf{P}(f) = f$, for any distribution $f\in\mathcal{D}'(\mathbf{R}^{d_0})$, and

$$\mathsf{P}(f_1,\dots,f_n) = \mathsf{P}\big(\mathsf{P}(f_1,\dots,f_{n-1}), f_n\big)$$

for any $n\geq 2$ and any distributions $f_1,\dots,f_n$ in $\mathcal{D}'(\mathbf{R}^{d_0})$. If $f_n\in C^{\alpha_n}$, the above continuity estimates on the paraproduct operator imply that the iterated paraproduct $\mathsf{P}(f_1,\dots,f_n)$ is in some C^γ space where $\gamma \leq \alpha_n$. Such distributions may nonetheless have some local descriptions to an accuracy strictly larger than α_n around an arbitrary point. In the case of a paraproduct $\mathsf{P}(f,g)$ where $f \in C^{\alpha_1}, g \in C^{\alpha_1}$ and $0 < \alpha_1 < 1/2$, one has for instance

$$\big|\mathsf{P}(f,g)(y) - \mathsf{P}(f,g)(x) - f(x)\big(g(y) - g(x)\big)\big| \lesssim |y - x|^{2\alpha_1}, \tag{1.1}$$

so one can give in that setting a local description of the behaviour of $\mathsf{P}(f,g)$ around an arbitrary point x up to a precision $|y-x|^{2\alpha_1}$. More generally, for $\alpha_1, \dots, \alpha_n$ in the interval $(0,1)$ and $f_1 \in C^{\alpha_1}, \dots, f_n \in C^{\alpha_n}$, define inductively on n

$$S_{ab}(x,y) = \mathsf{P}(f_a, \dots, f_b)(y) - \mathsf{P}(f_a, \dots, f_b)(x) - \sum_{c=a}^{b-1} \mathsf{P}(f_a, \dots, f_c)\, S_{(c+1)b}(x,y)$$

for any $1 \leq a \leq b \leq n$. M. Hoshino proved in Theorem 3.1 of [15] that if $\alpha_1 + \cdots + \alpha_n < 1$ then $|S_{1n}(x,y)| \lesssim |y-x|^{\alpha_1 + \cdots + \alpha_n}$. This gives the equivalent of the inequality (1.1) for the iterated paraproduct $\mathsf{P}(f_1, \dots, f_n)$ in that case.

One needs an additional ingredient to provide some expansion result at precision larger than 1. For $g \in C^{\alpha_2}$ with $\alpha_2 > 0$ we write

$$R^{\alpha_2}(g)(y,x) := g(y) - \sum_{|k| < \alpha_2} \partial^k g(x)\, \frac{(y-x)^k}{k!}$$

for the Taylor remainder function of g at order α_2. We use here the convention that for $z = (z^1, \dots, z^{d_0}) \in \mathbf{R}^{d_0}$ and $k \in \mathbf{N}^{d_0}$ one sets $z^k = \prod_{1 \leq i \leq d_0} (z^i)^{k_i}$. M. Hoshino extended in [17] the expansion result (1.1) for $\mathsf{P}(f,g)$ to any $f \in C^{\alpha_1}, g \in C^{\alpha_2}$ for $\alpha_1, \alpha_2 > 0$ by proving amongst other things that

$$\left|\mathsf{P}(f,g)(y) - \sum_{|k| < \alpha_1 + \alpha_2} \partial_\star^k \mathsf{P}(f,g)(x)\, \frac{(y-x)^k}{k!} - \sum_{|k| < \alpha_1} \partial^k f(x) R^{\alpha_2}(g)(y,x)\right| \lesssim |y-x|^{\alpha_1 + \alpha_2}, \tag{1.2}$$

where the *generalized derivative*

$$\partial_\star^k \mathsf{P}(f,g) := \partial^k \mathsf{P}(f,g) - \sum_{\substack{k_1 + k_2 = k \\ |k_1| < \alpha_1, |k_2| \geq \alpha_2}} \binom{k}{k_1} (\partial^{k_1} f)(\partial^{k_2} g)$$

is indeed well-defined pointwise. The inequality (1.2) provides a local description of the behaviour of $\mathsf{P}(f,g)$ around an arbitrary point x to a precision $|y-x|^{\alpha_1+\alpha_2}$ when $\alpha_1, \alpha_2 > 0$. Hoshino was able to prove in [17] a local expansion result for $\mathsf{P}(f_1, f_2, f_3)$ when $\alpha_1, \alpha_2, \alpha_3$ are all three positive. Theorem 1 below provides the most general extension of this type of result for some arbitrary iterated paraproducts $\mathsf{P}(f_1, \dots, f_n)$. In the particular case where the $f_k \in C^{\alpha_k}$ with $\alpha_k > 0$ for all $1 \leq k \leq n$, it implies that the function $\mathsf{P}(f_1, \dots, f_n)$ has a local description around an arbitrary point x up to a precision $|y-x|^{\alpha_1 + \cdots + \alpha_n}$. The statement of Theorem 1 does not require that all the α_k be positive and takes a very precise form. Not only does $\mathsf{P}(f_1, \dots, f_n)$ have a local expansion around any point x, but the functions whose values at x give the coefficients of the expansion of $\mathsf{P}(f_1, \dots, f_n)$ also have some local expansion, to a lower precision though. The coefficients that appear in the latter expansion can also be expanded, to an even lower precision, and so on. A reader acquainted with regularity structures will recognize here the verbal description of a modelled distribution over a regularity structure. Theorem 1 states that a certain family of functions and distributions defines a model over a particular regularity structure which we now introduce.

1.2 – Regularity structures associated with iterated paraproducts. The reader will find in Appendix A.1 some basic facts about regularity structures. It suffices to mention here that they involve some pairs of vector spaces (T, T^+) equipped with some algebraic structures

$$\Delta : T \to T \otimes T^+$$

and

$$\Delta^+ : T^+ \to T^+ \otimes T^+.$$

§1. The regularity structure. We need some notations to introduce the structure that is involved in Theorem 1.

We use some gothic letters $\mathfrak{k} = (k_1, \dots, k_c) \in (\mathbf{N}^{d_0})^c$ to denote some tuples of multi-indices $k_i \in \mathbf{N}^{d_0}$ of arbitrary length c. Denote by $|k| = k^1 + \cdots + k^d$ the $\ell^1(\mathbf{N})$-norm of an arbitrary $k = (k^1, \dots, k^{d_0}) \in \mathbf{N}^{d_0}$, and set for $\mathfrak{k} = (k_1, \dots, k_c) \in (\mathbf{N}^{d_0})^c$

$$|\mathfrak{k}| := \big(|k_1|, \cdots, |k_c|\big) \in \mathbf{N}^c$$
$$\|\mathfrak{k}\| := |k_1| + \cdots + |k_c| \in \mathbf{N}.$$

For $k \in \mathbf{N}^{d_0}$ and a non-null integer c we define the set $\mathcal{P}_c(k)$ of partitions of k into c sub-mutli-indices as

$$\mathcal{P}_c(k) := \Big\{(k_1, \dots, k_c) \in (\mathbf{N}^{d_0})^c\,;\; k = k_1 + \cdots + k_c\Big\}.$$

One has

$$\|\mathfrak{k}\| = |k|$$

for any $\mathfrak{k} \in \mathcal{P}_c(k)$ where $k \in \mathbf{N}^{d_0}, c \geq 1$. For some integers $a < b$ we write $[\![a,b]\!]$ for the set of integers in the closed interval $[a,b]$. Let $X = (X^1, \dots, X^{d_0})$ stand for an abstract d_0-dimensional monomial with commutative symbol coordinates. For $p = (p^1, \dots, p^{d_0}) \in \mathbf{N}^{d_0}$ we set

$$X^p := (X^1)^{p^1} \cdots (X^{d_0})^{p^{d_0}}.$$

Denote by $(\varepsilon_1, \dots, \varepsilon_{d_0})$ the canonical basis of $\mathbf{N}^{d_0}$, so $X^{\varepsilon_i} = X^i$.

We fix a tuple of real numbers

$$\alpha = (\alpha_1, \dots, \alpha_n).$$

The following symbols

$$\mathcal{B} := \Big\{[\![a,b]\!]_{\mathfrak{j}}\, X^p\Big\}_{1\leq a<b\leq n,\, \mathfrak{j}\in\mathcal{P}_{b-a}(\ell),\, \ell\in\mathbf{N}^{d_0},\, p\in\mathbf{N}^{d_0}} \cup \big\{X^p\big\}_{p\in\mathbf{N}^{d_0}}$$

form the basis of a vector space denoted by T. Similarly the following symbols

$$\mathcal{B}^+ := \Big\{[\![a,b]\!]^{\mathfrak{k}}_{\mathfrak{j}}\Big\}_{\mathbf{condition}(a,b,\mathfrak{k},\mathfrak{j})} \cup \big\{X^{\varepsilon_i}\big\}_{1\leq i\leq d_0}$$

generate freely an algebra with unit $\mathbf{1}^+$ that we denote by T^+. One says that $(a,b,\mathfrak{k},\mathfrak{j})$ satisfies $\mathbf{condition}(a,b,\mathfrak{k},\mathfrak{j})$ if $1 \leq a < b \leq n, \mathfrak{k} = (k_a, \dots, k_b) \in \mathcal{P}_{b-a+1}(k)$ for some $k \in \mathbf{N}^{d_0}$, and $\mathfrak{j} \in \mathcal{P}_{b-a}(\ell)$ for some $\ell \in \mathbf{N}^{d_0}$, and we have

$$\max(|k|,|\ell|) < \sum_{1\leq j\leq n} |\alpha_j|$$

and

$$|\ell| + \sum_{a\leq j\leq b} \alpha_j > |k|. \tag{1.3}$$

We emphasize that the tuples $\mathfrak{k} = (k_a, \dots, k_b) \in \mathcal{P}_{b-a+1}(k)$ have $b-a+1$ components while the tuples $\mathfrak{j} \in \mathcal{P}_{b-a}(\ell)$ have $b-a$ components. (To have a unified picture in mind one can think of $\mathfrak{j} = (\ell_a, \dots, \ell_{b-1})$ as the tuple $(\ell_a, \dots, \ell_{b-1}, 0)$ with $b-a+1$ components.) The k_i in $\mathfrak{k}$ will represent later some derivatives in some analytic expressions like (1.9) below. The ℓ_j in $\mathfrak{j}$ will represent some polynomial weights in some analytical expressions like (1.4) below. The symbols of $\mathcal{B}$ and $\mathcal{B}^+$ index some analytic quantities that will be described in *§2*. We define an α-dependent grading on T and T^+ by defining the degree of $[\![a,b]\!]_{\mathfrak{j}} X^p \in \mathcal{B}$ as

$$\big|[\![a,b]\!]_{\mathfrak{j}}\, X^p\big|_{\alpha} := \|\mathfrak{j}\| + \sum_{a\leq j\leq b} \alpha_j + |p|,$$

and, requiring that the degree map is multiplicative on T^+, we set $|\varepsilon_i|_{\boldsymbol{\alpha}} = 1$ and define the degree of $[\![a,b]\!]^{\mathfrak{k}}_{\mathfrak{j}} \in \mathcal{B}^+$ as

$$\big|[\![a,b]\!]_{\mathsf{j}}^{\mathfrak{k}}\big|_\alpha := \|\mathsf{j}\| + \sum_{a\le j\le b}\alpha_j - \|\mathfrak{k}\|.$$

We read on the condition (1.3) that the elements of $\mathcal{B}^+$ have a positive degree. We will see in Section 3 that there are some particular splitting maps Δ and Δ^+ that turn the pair

$$\mathscr{T}_\alpha := \big((T,\Delta),(T^+,\Delta^+)\big)$$

into a concrete regularity structure.

§2. A model on the regularity structure. We now define the analytic objects $\mathsf{\Pi}$ and g that we associate to the symbols of the regularity structure. We will see in Theorem 1 below that they define a model $(\mathsf{\Pi},\mathsf{g})$ over a truncated version of $\mathscr{T}_\alpha$ that is parametrized by some distributions $(f_1,\dots,f_n)$, where $f_i\in C^{\alpha_i}$ for all $1\le i\le n$. We make the following assumption on the regularity exponents α_i of the f_i.

Assumption (A) *– One has $\sum_{a\le j\le b}\alpha_j\notin\mathbb{Z}$ for all $1\le a\le b\le n$.*

For $\ell\in\mathbf{N}^{d_0}$ and $i\ge -1$, we define the modified Littlewood-Paley projector Δ_i^ℓ by setting

$$(\Delta_i^\ell f)(x) := f\big((\cdot-x)^\ell K_i(\cdot-x)\big) \tag{1.4}$$

for all $f\in\mathcal{D}'(\mathbf{R}^d)$ and $x\in\mathbf{R}^d$, where $\Delta_i^0=\Delta_i$, so K_i stands above for the smooth kernel of the Littlewood-Paley projector Δ_i. For $j\ge 0$ we define

$$\Delta_{<j}^\ell := \sum_{-1\le j'\le j-1}\Delta_{j'}^\ell$$

and set

$$\mathsf{P}_\ell(f,g) := \sum_{i\ge 1}\big(\Delta_{<i-1}^\ell f\big)\,(\Delta_i g)$$

for any $f,g\in\mathcal{D}'(\mathbf{R}^d)$. For $c\ge 3$, for $\mathsf{j}=(\ell_1,\dots,\ell_{c-1})\in(\mathbf{N}^{d_0})^{c-1}$ and $\mathsf{j}_{\le c-2}=(\ell_1,\dots,\ell_{c-2})\in(\mathbf{N}^{d_0})^{c-2}$ we define recursively

$$\mathsf{P}_{\mathsf{j}}(f_1,\dots,f_c) := \mathsf{P}_{\ell_{c-1}}\big(\mathsf{P}_{\mathsf{j}_{\le c-2}}(f_1,\dots,f_{c-1}),f_c\big).$$

With $\mathsf{j}=(\varepsilon_1,\varepsilon_2,\varepsilon_3)$, we have for instance

$$\mathsf{P}_{\mathsf{j}}(f_1,\dots,f_3,f_4) = \sum_{i_4,i_3,i_2\ge 1}\Delta_{<i_4-1}^{\varepsilon_3}\Big\{\Delta_{<i_3-1}^{\varepsilon_2}\Big(\Delta_{<i_2-1}^{\varepsilon_1}(f_1)\Delta_{i_2}(f_2)\Big)\Delta_{i_3}(f_3)\Big\}\Delta_{i_4}(f_4)$$

For $[\![a,b]\!]_{\mathsf{j}}X^p\in\mathcal{B}$ we define the distribution $\mathsf{\Pi}\big([\![a,b]\!]_{\mathsf{j}}X^p\big)$ by its action on a test function φ

$$\mathsf{\Pi}\big([\![a,b]\!]_{\mathsf{j}}X^p\big)(\varphi) = \mathsf{\Pi}\big([\![a,b]\!]_{\mathsf{j}}\big)(\cdot^p\varphi) \tag{1.5}$$

with $(\cdot^p\varphi)(y)=y^p\varphi(y)$ and

$$\mathsf{\Pi}\big([\![a,b]\!]_{\mathsf{j}}\big) := \mathsf{P}_{\mathsf{j}}\big(f_a,\dots,f_b\big). \tag{1.6}$$

The definition of the character g on T^+ requires a notation. For a tuple $\beta=(\beta_1,\dots,\beta_c)\in\mathbf{R}^c$ of regularity exponents and $\mathsf{j}=(\ell_1,\dots,\ell_{c-1})\in(\mathbf{N}^{d_0})^{c-1}$ we set $\ell_c=0\in\mathbf{N}^{d_0}$ and define the set of j-admissible cuts of β as

$$\mathsf{j}-\mathsf{Cut}(\beta) := \left\{1\le d\le c-1\,;\,\ell_d=0,\ \sum_{1\le e\le d}\big(\beta_e+|\ell_e|\big)>0,\ \sum_{d+1\le e\le c}\big(\beta_e+|\ell_e|\big)<0\right\} \tag{1.7}$$

and for $d\in\mathsf{j}-\mathsf{Cut}(\beta)$ we set

$$r_d = r_d(\beta,\mathsf{j}) := \min\left\{\sum_{1\le e\le d}\big(\beta_e+|\ell_e|\big),\ -\sum_{d+1\le e\le c}\big(\beta_e+|\ell_e|\big)\right\}.$$

Set

$$\beta_{\le e} := (\beta_1,\dots,\beta_e),\qquad \beta_{>e} := (\beta_{e+1},\dots,\beta_c),\qquad \beta_{[\![a,b]\!]} := (\beta_a,\dots,\beta_b)$$

for any $1 \leq e \leq c$ and $e \leq c-1$ and $1 \leq a \leq b \leq c$, respectively. We define recursively

$$
\begin{aligned}
&\widetilde{\mathsf{P}}^{\beta}_{\mathsf{j}}(g_1, \dots, g_c) \\
&\quad := \mathsf{P}_{\mathsf{j}}(g_1, \dots, g_c) \\
&\qquad - \sum_{\substack{d \in \mathsf{j} - \mathsf{Cut}(\boldsymbol{\beta}) \\ |m| < r_d}} \ \sum_{\substack{\mathfrak{m} \in \mathcal{P}_d(m) \\ \mathfrak{m}' \in \mathcal{P}_{c-d}(m)}} \frac{m!}{\mathfrak{m}!\,\mathfrak{m}'!}\, \widetilde{\mathsf{P}}^{\beta_{\leq d} - \mathfrak{m}}_{\mathsf{j}_{\leq d}}\big(\partial^{m_1} g_1, \dots, \partial^{m_d} g_d\big)\, \widetilde{\mathsf{P}}^{\beta_{>d}}_{\mathsf{j}_{>d} + \mathfrak{m}'}\big(g_{d+1}, \dots, g_c\big)
\end{aligned}
\tag{1.8}
$$

where $m \in \mathbf{N}^{d_0}$ and $\mathfrak{m} = (m_1, \dots, m_d) \in \mathcal{P}_d(m)$, and with the convention that $\widetilde{\mathsf{P}}^{\beta_c}_{m}(g_c) = g_c$.

For any $\beta_i \in \mathbf{R}$ we denote by $C^{\beta_i}_{\circ}$ the closure of $C^\infty \cap C^{\beta_i}$ in C^{β_i}, and we assume from now on that $f_i \in C^{\alpha_i}_{\circ}$ for all $1 \leq i \leq n$. In the course of proving Theorem 1 below we will prove that $\widetilde{\mathsf{P}}^{\beta}_{\mathsf{j}}(g_1, \dots, g_c) \in L^\infty$ if $g_i \in C^{\beta_i}_{\circ}$ for all $1 \leq i \leq c$ and $\|\mathsf{j}\| + \sum_{1 \leq i \leq c} \beta_i > 0$. We can then define for $[\![a,b]\!]^{\mathsf{j}}_{\mathfrak{k}} \in \mathcal{B}^+$ with $\mathfrak{k} = (k_a, \dots, k_b)$

$$
\mathsf{g}\big([\![a,b]\!]^{\mathfrak{k}}_{\mathsf{j}}\big) := \widetilde{\mathsf{P}}^{\alpha_{[\![a,b]\!]} - |\mathfrak{k}|}_{\mathsf{j}}\big(\partial^{k_a} f_a, \dots, \partial^{k_b} f_b\big). \tag{1.9}
$$

1 – Theorem. *The pair $(\mathsf{\Pi}, \mathsf{g})$ is a model on the regularity structure $\mathscr{T}_\alpha$. It depends continuously on $(f_1, \dots, f_n) \in \prod_{i=1}^n C^{\alpha_i}_{\circ}$.*

For g to be part of a model, we need to prove that each function $\mathsf{g}\big([\![a,b]\!]^{\mathfrak{k}}_{\mathsf{j}}\big)$ has a local expansion to accuracy $|y-x|^{|[\![a,b]\!]^{\mathfrak{k}}_{\mathsf{j}}|_\alpha}$ around any point x, with the different terms in the expansion indexed by the algebraic structure of the Hopf algebra (T^+, Δ^+). For $\mathsf{\Pi}$ to be part of a model, it also needs to satisfy some local expansion property that involves g as well.

The strategy that we adopt to prove Theorem 1 is first to prove a statement of a similar flavor for some distributions and functions that are built from a simplified version of the iterated paraproducts. The algebra involved in the analysis of these operators is simpler than the algebra associated with the true iterated paraproducts, and their analytical properties are more flexible. At the same time, we will see in Proposition 19 of Section 4.2 that the iterated paraproduct $\mathsf{P}(f_1, \dots, f_n)$ can be written as a sum of simplified iterated paraproducts evaluated on some other functions/distributions built from the f_i. This fact will play a crucial role in transfering the local expansion properties of the simplified iterated paraproducts to the true iterated paraproducts.

1.3 – Local expansion properties of paracontrolled systems. We are interested in iterated paraproducts as they are one of the building blocks of paracontrolled calculus. Paracontrolled systems play within paracontrolled calculus the role that modelled distributions play in the setting of regularity structures.

Assume we are given a finite set of letters $\mathcal{L} = \{l_1, \dots, l_{|\mathcal{L}|}\}$ and a family $[l] \in C^{r_l}$ of distributions on $\mathbf{R}^{d_0}$ indexed by $\mathcal{L}$. We denote by $w_\emptyset$ the empty word and by $w = l_{i_1} \dots l_{i_w}$ a generic word with letters from $\mathcal{L}$. The concatenation of two words w_1 and w_2 is denoted by $w_1 w_2$. If $w = w_1 w_2$ we say that w_1 the a *begining of the word* w. We assume that the letters come with a notion of size $|l_i| \in \mathbf{R}$ and set $|w_\emptyset| = 0$ and

$$
|w| := |l_{i_1}| + \cdots + |l_{i_w}|.
$$

For a positive real number r we denote by $\mathcal{W}_{<r}$ the set of words of size less than r, including the empty word. An r-***paracontrolled system*** is a family $(u_w)_{w \in \mathcal{U}_{<r}}$ of functions/distributions on $\mathbf{R}^{d_0}$ indexed by a subset $\mathcal{U}_{<r}$ of $\mathcal{W}_{<r}$ that contains the empty word $w_\emptyset$ and which has the following properties.

(1) There is a *finite* subset $\mathcal{U}^f_{<r}$ of $\mathcal{U}_{<r}$ made up of words of positive size and such that every word of $\mathcal{U}_{<r}$ is the begining of one of the words of $\mathcal{U}^f_{<r}$. (The exponent f in $\mathcal{U}^f_{<r}$ stands for '*final*'.)

(2) For all $w \in \mathcal{U}_{<r}$ one has

$$
u_w = \sum_{l \in \mathcal{L}} \mathsf{P}\big(u_{wl},\, [l]\big) + u^\sharp_w \tag{1.10}
$$

with $u^\sharp_w \in C^{r-|w|}$.

Condition (1) ensures that the family $(u_w)_{w\in\mathcal{U}_{<r}}$ is finite even if some of the sizes $|l|$ are non-positive. This condition is automatically satisfied if all the $|l|$ are positive. We talk of the $[l]$ as the *reference functions/distributions*.

Here is an example of an $r-$paracontrolled system with two reference functions $[l_1]\in C^{\lfloor l_1\rfloor}, [l_2]\in C^{|l_2|}$ with $|l_1|,|l_2|$ positive and $|l_1|+|l_2|<r$

$$\begin{aligned} u_{w_\emptyset} &= \mathsf{P}(u_{l_1},\lfloor l_1\rfloor)+\mathsf{P}(u_{l_2},\lfloor l_2\rfloor)+u^\sharp_{w_\emptyset} \\ u_{l_1} &= \mathsf{P}(u_{l_1l_1},\lfloor l_1\rfloor)+u^\sharp_{l_1}, \qquad u_2 = \mathsf{P}(u_{l_2l_1},\lfloor l_1\rfloor)+u^\sharp_{l_2} \\ u_{l_1l_1} &= u^\sharp_{l_1l_1}, \qquad\qquad\qquad\quad u_{l_2l_1} = u^\sharp_{l_2l_1}. \end{aligned}$$

One observes that

$$\begin{aligned} u_{w_\emptyset} &= \mathsf{P}\big(u_{l_1l_1},\lfloor l_1\rfloor,\lfloor l_1\rfloor\big)+\mathsf{P}(u^\sharp_{l_1},\lfloor l_1\rfloor)+\mathsf{P}\big(u_{l_2l_1},\lfloor l_1\rfloor,\lfloor l_2\rfloor\big)+\mathsf{P}(u^\sharp_{l_2},\lfloor l_2\rfloor)+\mathsf{P}(u^\sharp_{w_\emptyset}) \\ u_{l_1} &= \mathsf{P}(u^\sharp_{l_1l_1},\lfloor l_1\rfloor)+\mathsf{P}(u^\sharp_{l_1}), \qquad u_{l_2} = \mathsf{P}(u^\sharp_{l_2l_1},\lfloor l_1\rfloor)+\mathsf{P}(u^\sharp_{l_2}). \end{aligned}$$

More generally, for an arbitrary r-paracontrolled system, it follows from (1.10) that each u_w writes as a finite sum of iterated paraproducts of the form $\mathsf{P}\big(u^\sharp_w,[l_{i_1}],\dots,[l_{i_{n_w}}]\big)$, including $u^\sharp_w=\mathsf{P}(u^\sharp_w)$.

Paracontrolled systems were first introduced by Bailleul & Bernicot in [3] in their development of paracontrolled calculus, tailored for its application to some classes of singular stochastic partial differential equations. Under some appropriate conditions, such equations have a unique solution in an equation-dependent space of functions/distributions with a paracontrolled structure (1.10). On can say that paracontrolled calculus replaces the mechanics of local expansions in space that is at the heart of regularity structures by a type of expansion in frequency (Fourier) space.

The notion of paracontrolled system is useful even for the study of regularity structures. Bailleul & Hoshino proved for instance in [5] that, for a model $\mathsf{M}=(\mathsf{\Pi},\mathsf{g})$ on a fixed regularity structure, the distributions/functions $\mathsf{\Pi}(\tau)$ and $\mathsf{g}(\mu)$ can be described by some paracontrolled systems

$$\begin{aligned} \mathsf{\Pi}(\tau) &= \sum_{\sigma<\tau}\mathsf{P}\big(\mathsf{g}(\tau/\sigma),\,[\sigma]^{\mathsf{M}}\big)+[\tau]^{\mathsf{M}} \\ \mathsf{g}(\mu) &= \sum_{1^+<^+\nu<^+\mu}\mathsf{P}\big(\mathsf{g}(\mu/\nu),\,[\nu]^{\mathsf{g}}\big)+[\mu]^{\mathsf{g}} \end{aligned} \tag{1.11}$$

for some reference functions/distributions $[\tau]^{\mathsf{M}}\in C^{|\tau|},[\mu]^{\mathsf{g}}\in C^{|\mu|}$ built from the model M, for some index sets $\sigma<\tau$ and $1^+<^+\nu<^+\mu$ whose precise definition does not matter here – see Section 2.2 of [4] for that point. Furthermore, for any modelled distribution $\boldsymbol{v}=\sum_\tau v_\tau\tau$ of positive regularity r the family $\big(\mathsf{R}^{\mathsf{M}}(\boldsymbol{v}),(v_\tau)_\tau\big)$ is an r-paracontrolled system

$$\begin{aligned} \mathsf{R}^{\mathsf{M}}(\boldsymbol{v}) &= \sum_{|\tau|<r}\mathsf{P}\big(v_\tau,\,[\tau]^{\mathsf{M}}\big)+[\boldsymbol{v}] \\ v_\tau &= \sum_{\tau<\sigma,|\sigma|<r}\mathsf{P}\big(v_\sigma,\,[\sigma/\tau]^{\mathsf{g}}\big)+[v_\tau] \end{aligned} \tag{1.12}$$

with reference functions/distributions the family of brackets $[\tau]^{\mathsf{M}},[\mu]^{\mathsf{g}}$, and for some funtctions $[\boldsymbol{v}]$ and $[v_\tau]$. This is Proposition 12 and Theorem 1 in [5]. Bailleul & Hoshino further proved in Theorem 1 of [6] that a sub-family of these 'brackets' $[\tau]^{\mathsf{M}},[\mu]^{\mathsf{g}}$ parametrizes the set of models over a given regularity structure, providing in particular a linear parametrization of the nonlinear space of models on that regularity structure. These results hold for any reasonable regularity structure. For a particular class of regularity structures $\mathscr{T}$ including the BHZ regularity structures used for the study of subcritical singular stochastic PDEs, they proved that for a given model on $\mathscr{T}$ the set of modelled distributions with regularity r is parametrized by the family of functions/distributions $\big\{[\boldsymbol{v}]\in C^r\big\}\cup\big\{[v_\tau]\in C^{r-|\tau|},\,\tau\text{ in a linear basis of }T\big\}_{|\tau|<r}$ – this is Theorem 5 and Theorem 7 in [6].

In all these results the regularity structure is fixed. In particular, if we are given some placeholders for $[\boldsymbol{v}]$ and the $[v_\tau]$ there is a unique modelled distribution *over the given regularity structure* that has these functions/distributions as its brackets. We have no a priori regularity structure in the more general situation of an arbitrary paracontrolled system. Our second main result means informally that we can lift a paracontrolled system into a modelled distribution on some universal regularity structure and for some system-dependent model. Recall we assume $[l] \in C^{r_l}$.

2 – Theorem. *Pick $r > 0$. Given an r-paracontrolled system $(u_w)_{w\in\mathcal{U}_{<r}}$ as in* (1.10) *there is an explicit regularity structure $\mathscr{T}_{\mathcal{L}}$ that depends only on $|\mathcal{L}|$, r and the regularity exponents r_l, a model M on $\mathscr{T}_{\mathcal{L}}$ and a modelled distribution $\boldsymbol{u}$ of regularity r such that $u_{w_\emptyset} = \mathsf{R}^{\mathsf{M}}(\boldsymbol{u})$.*

We call the regularity structure $\mathscr{T}_{\mathcal{L}}$ universal as it only depends on the numbers $|\mathcal{L}|, r, r_l$ and not on the reference objects $[l]$ themselves.

Bailleul & Hoshino's works [5, 6] established a correspondence between modelled distributions and paracontrolled systems building on (1.11) and (1.12). One associates to an equation a regularity structure $\mathscr{T}$, to a model M on $\mathscr{T}$ the paracontrolled system (1.11) and to a model distribution $\boldsymbol{v}$ defined of M the paracontrolled system (1.12). The inverse map consists in getting back the model M over $\mathscr{T}$ from (1.11) and the modelled distribution $\boldsymbol{v}$ from (1.12). In that particular context, Theorem 2 does something of a different nature. Starting from a regularity structure $\mathscr{T}$, a model M on $\mathscr{T}$, a modelled distribution $\boldsymbol{v}$ and their associated paracontrolled systems (1.11) and (1.12), it introduces

- another regularity structure $\mathscr{T}_{\mathcal{L}}$ that retains little information about the initial regularity structure $\mathscr{T}$,
- a model and a modelled distribution on $\mathscr{T}_{\mathcal{L}}$,

whose associated paracontrolled systems are also given by (1.11) and (1.12). This situation is somewhat reminiscent of the study by Hairer & Kelly [14] of the links between the notions of geometric and branched rough paths.

Note the important fact that Theorem 2 applies to any paracontrolled system. It does not need to come as the system associated with a model on a regularity structure or a modelled distribution.

Organisation of the article. A simplified iterated paraproduct operator $\mathsf{P}_<(f_1,\dots,f_n)$ is introduced in Section 2, and we provide in Section 2.3 its local expansion properties. The latter involve some functions $\partial_\star^k\mathsf{P}(f_1,\dots,f_n)$ that are introduced in Section 2.2. These functions also have some local expansion properties which we investigate in Section 2.4. We leave aside the simplified iterated paraproducts in Section 3 and describe in this section the regularity structure $\mathscr{T}_\alpha$ that is involved in the statement of Theorem 1. This statement is proved in Section 4. We build in Section 4.1 a number of functions/distributions that will be used to represent an iterated paraproduct $\mathsf{P}(f_1,\dots,f_n)$ as a sum of simplified $\mathsf{P}_<$ iterated paraproducts. The representation formula itself is proved in Section 4.2. We prove Theorem 1 in Section 4.3. Section 5 is dedicated to proving Theorem 2. We describe the universal regularity structure involved in this statement in Section 5.1 and prove Theorem 2 in Section 5.2. A number of technical lemmas are deferred to some appendices. The proof of the local expansion property of the functions $\partial_\star^k\mathsf{P}(f_1,\dots,f_n)$ involves in particular some algebraic results that are proved in Appendix A.4. We also defer to the appendices the proof of some algebraic identities that play a crucial role in our proof of Theorem 1. Appendix A.1 gives some background on regularity structures and Appendix A.2 gives some general and particular analysis results.

Notations. *We collect here a number of notations that are used throughout the text.*

- *The letters i, j and a, b, c, d, e will exclusively be used to denote some integers.*
- *The letters k, ℓ, m will denote exclusively some elements of $\mathbf{N}^{d_0}$.*
- *We denote by $\alpha = (\alpha_1, \alpha_2, \dots)$ or $\beta = (\beta_1, \beta_2, \dots)$ some finite tuples of regularity exponents α_i, β_j in $\mathbf{R}$.*
- *For $z = (z^1, \dots, z^{d_0}) \in \mathbf{R}^{d_0}$ and $k \in \mathbf{N}^{d_0}$ we write $z^k = \prod_{1\le i\le d_0}(z^i)^{k_i}$.*
- *For $k = (k^1,\dots,k^{d_0}) \in \mathbf{N}^{d_0}$ we write $k! = \prod_{i=1}^{d_0} k^i!$ and for $m, m'_1, \dots, m'_r$ in $\mathbf{N}^{d_0}$ we set*
$$\binom{m}{(m'_1,\dots,m'_r)} := \frac{m!}{\prod_{1\le i\le r} m'_i!}.$$
- *We write $\lesssim_p$ for an inequality that holds up to a multiplicative positive constant that only depends on some parameter p.*

– *We work here in the Euclidean space $\mathbf{R}^{d_0}$. All that follows has a direct counterpart in an anisotropic version of $\mathbf{R}^{d_0}$. We stick to the Euclidean setting not to distract the reader from the main points of this work.*

2 – Simplified iterated paraproducts $\mathsf{P}_<$ and their local expansion properties

We introduce in this section some simplified iterated paraproducts. It turns out to be convenient to define these operators on a slightly larger class $\mathsf{C}^\alpha_\circ$ of spaces than the usual C^α spaces ($\alpha \in \mathbf{R}$). This extended setting is described in Section 2.1, where the simplified iterated operator $\mathsf{P}_<$ is also introduced. The description of the local expansion properties of the simplified iterated paraproducts involves some generalized derivative operators $\partial^k_\star$ that we introduce in Section 2.2. The main result of this section is Corollary 5, that entails the continuity of the $\partial^k_\star \mathsf{P}_<$ operators on the space $\prod_{j=1}^n \mathsf{C}^{\alpha_j}_\circ$ if $|k| < \sum_{j=1}^n \alpha_j$. Generalizing Hoshino's result (1.2), we state and prove the local expansion property of the $\mathsf{P}_<(\mathsf{f}_1, \dots, \mathsf{f}_n)$ in Section 2.3, for $\mathsf{f}_j \in \mathsf{C}^{\alpha_j}_\circ$ with $\alpha_j > 0$ for all $1 \leq j \leq n$. It takes the form

$$\begin{aligned} \mathsf{P}_<(\mathsf{f}_1, \dots, \mathsf{f}_n)(y) = &\sum_{|k| < \sum_{j=1}^n \alpha_j} \partial^k_\star \mathsf{P}_<\big(\mathsf{f}_1, \dots, \mathsf{f}_n\big)(x) \frac{(y-x)^k}{k!} \\ &+ \sum_{c=1}^{n-1} \sum_{|k| < \sum_{j=1}^c \alpha_j} \partial^k_\star \mathsf{P}_<\big(\mathsf{f}_1, \dots, \mathsf{f}_c\big)(x) \; \frac{(y-x)^k}{k!} \triangle_{yx} \mathsf{P}_<(\mathsf{f}_{c+1}, \dots, \mathsf{f}_n) \\ &+ \big(\triangle_{yx} \mathsf{P}_<\big)(\mathsf{f}_1, \dots, \mathsf{f}_n), \end{aligned}$$

where each term $\triangle_{yx} \mathsf{P}_<(\mathsf{f}_{c+1}, \dots, \mathsf{f}_n)$ is of order $|y-x|^{\sum_{j=c+1}^n \alpha_j}$, for $x, y \in \mathbf{R}^d$ close enough.

2.1 – Simplified iterated paraproducts. We will work through part of this document with the following extension of the classical Besov-Hölder spaces.

Definition – *For $r \in \mathbf{R}$ we define C^r as the vector space of sequences $\mathsf{f} = (f_i)_{i \geq -1}$ of smooth functions to which one can associate a ball $B \subset (\mathbf{R}^{d_0})'$ such that each f_i is spectrally supported in $2^i B$ and*

$$\|\mathsf{f}\|_r := \sup_{i \geq -1} 2^{ir} \, \|f_i\|_{L^\infty} < \infty.$$

This formula defines a norm on C^r. An element of

$$\mathsf{C}^\infty := \bigcap_{r>0} \mathsf{C}^r$$

is said to be smooth, and we set

$$\mathsf{C}^{-\infty} := \bigcup_{r \in \mathbf{R}} \mathsf{C}^r, \qquad \mathsf{C}^{0+} := \bigcup_{r>0} \mathsf{C}^r.$$

We write $\mathsf{C}^r_\circ$ for the closure of C^∞ in C^r.

For $r > 0$ there is a canonical continuous non-injective surjection from C^r onto the classical Besov-Hölder space C^r sending $\mathsf{f} = (f_i)_{i \geq -1}$ to $\sum_{i \geq -1} f_i$. The Littlewood-Paley projectors give a continuous injection ι from C^r into C^r for any $r \in \mathbf{R}$, by setting $\iota(f) := (\Delta_i(f))_{i \geq -1}$. We define for any distribution f on $\mathbf{R}^{d_0}$ and $o \in \mathbf{R}_+$ its Taylor polynomial $T^o_h f$ of order o in the direction $h \in \mathbf{R}^d$ as the distribution

$$(T^o_h f)(\cdot) := \sum_{|k|<o} \frac{h^k}{k!} \, \partial^k f(\cdot).$$

Its associated Taylor remainder $R^o_h f$ is defined from the relation

$$|h|^o (R^o_h f)(\cdot) := f(\cdot + h) - (T^o_h f)(\cdot).$$

The derivation operator ∂^k, the Taylor expansion and remainder maps T^o_h, R^o_h can be applied to any $\mathsf{f} = (f_i)_{i \geq -1} \in \mathsf{C}^r$ by applying the corresponding classical operators to each f_i. These operations behave well in this context. For any $r\in\mathbf{R}$ and $k\in\mathbf{N}^{d_0}$, Bernstein inequalities ensures that the operator ∂^k sends continuously C^r into $\mathsf{C}^{r-|k|}$. We give the proof of the following elementary fact in Appendix A.2.

*3 – **Lemma.*** *For $r\in\mathbf{R}, \mathsf{f}\in\mathsf{C}^r$ with f_i is spectrally supported in 2^iB for all $i\geq -1$, and $o\in\mathbf{R}_+$ we have*

$$\mathsf{f}(\cdot+h)-\sum_{|k|<o}\frac{h^k}{k!}\,(\partial^k\mathsf{f})(\cdot) = |h|^o\,(R_h^o\mathsf{f})(\cdot)$$

with

$$\|R_h^o\mathsf{f}\|_{r-o}\lesssim_B \|\mathsf{f}\|_r\,,$$

uniformly over $|h|\leq 1$.

For $\mathsf{f}_1,\dots,\mathsf{f}_n$ in $\mathsf{C}^{-\infty}$ we define iteratively the simplified iterated paraproducts

$$\mathbf{P}_<(\mathsf{f}_1,\dots,\mathsf{f}_n)=\big(\mathsf{P}_<(\mathsf{f}_1,\dots,\mathsf{f}_n)_i\big)_{i\geq -1}\in\mathsf{C}^{-\infty}$$

as the element of $\mathsf{C}^{-\infty}$ given by $\mathsf{P}_<(\mathsf{f}_1)=\mathsf{f}_1$ and with $\mathsf{f}_n=(f_{ni})_{i\geq-1}$

$$\mathsf{P}_<(\mathsf{f}_1,\dots,\mathsf{f}_n)_i:=\sum_{j<i-1}\mathsf{P}_<(\mathsf{f}_1,\dots,\mathsf{f}_{n-1})_j\,f_{ni}.$$

We write

$$\mathsf{P}_<(\mathsf{f}_1,\dots,\mathsf{f}_n):=\sum_{i\geq-1}\mathsf{P}_<(\mathsf{f}_1,\dots,\mathsf{f}_n)_i$$

for its associated distribution. With $\mathsf{f}_1=(\Delta_i(f_1))_{i\geq-1}$, and similar definitions of $\mathsf{f}_2,\mathsf{f}_3,\mathsf{f}_4$ in terms of some distributions f_2,f_3,f_4, we have for instance

$$\mathsf{P}_<(\mathsf{f}_1,\mathsf{f}_2,\mathsf{f}_3)=\sum_{\substack{i_1<i_2-1\\ i_2<i_3-1}}\Delta_{i_1}(f_1)\Delta_{i_2}(f_2)\Delta_{i_3}(f_3)$$

while

$$\mathsf{P}(f_1,f_2,f_3)=\sum_{\substack{i_1<i_2-1\\ i_2<i_3-1}}\Delta_{<i_3-1}\big(\Delta_{i_1}(f_1)\Delta_{i_2}(f_2)\big)\Delta_{i_3}(f_3),$$

and

$$\mathsf{P}_<\big(\mathsf{f}_1,\mathsf{f}_2,\mathsf{f}_3,\mathsf{f}_4\big)=\sum_{\substack{i_1<i_2-1\\ i_2<i_3-1\\ i_3<i_4-1}}\Delta_{i_1}(f_1)\Delta_{i_2}(f_2)\Delta_{i_3}(f_3)\Delta_{i_4}(f_4)$$

while

$$\mathsf{P}\big(f_1,f_2,f_3,f_4\big)=\sum_{\substack{i_1<i_2-1\\ i_2<i_3-1\\ i_3<i_4-1}}\Delta_{<i_4-1}\Big(\Delta_{<i_3-1}\big(\Delta_{i_1}(f_1)\Delta_{i_2}(f_2)\big)\Delta_{i_3}(f_3)\Big)\Delta_{i_4}(f_4)$$

Recall that ***Assumption (A)*** requires from a tuple $\alpha=(\alpha_1,\dots,\alpha_n)\in\mathbf{R}^n$ that $\sum_{a\leq j\leq b}\alpha_j\notin\mathbb{Z}$ for all $1\leq a\leq b\leq n$.. From now on

*all our tuples $\alpha=(\alpha_1,\dots,\alpha_n),\beta=(\beta_1,\dots,\beta_n)$ in $\mathbf{R}^n$ will satisfy **Assumption (A)**.*

2.2 – Generalized derivative operator $\partial_\star^k$. We define in this section some operators $\partial_\star^k$ that will turn out to be involved in the pointwise expansion of the simplified iterated paraproducts. They are built from some operators $\widetilde{\mathsf{P}}_<^\beta$ that we first introduce in §1. The continuity properties of these operators are stated in §1 in Corollary 5 and proved in §2.

§1 On the operators $\widetilde{\mathsf{P}}^{\beta}_{<}$ – Recall that, for a tuple $\beta=(\beta_1,\dots,\beta_c)\in\mathbf{R}^c$ of regularity exponents and $\mathsf{j}=(\ell_1,\dots,\ell_{c-1})\in(\mathbf{N}^{d_0})^{c-1}$, we set $\ell_c=0\in\mathbf{N}^{d_0}$ and define the set of j-admissible cuts of β as

$$\mathsf{j}-\mathsf{Cut}(\beta):=\Big\{1\leq d\leq c-1\,;\,\ell_d=0,\ \sum_{1\leq e\leq d}\big(\beta_e+|\ell_e|\big)>0,\ \sum_{d+1\leq e\leq c}\big(\beta_e+|\ell_e|\big)<0\Big\}.$$

We define here the set of ***cuts of*** β as

$$\mathsf{Cut}(\beta):=\mathbf{0}-\mathsf{Cut}(\beta)=\Big\{d\in[\![1,n-1]\!],\quad\sum_{j=1}^{d}\beta_j>0\ \text{ and }\ \sum_{j=d+1}^{n}\beta_j<0\Big\}.$$

We also define the following set of ***multi-cuts of*** β

$$\mathsf{MultiCut}(\beta)=\Big\{\boldsymbol{d}=\Big(0=:d_0<d_1<\cdots<d_{n(\boldsymbol{d})}:=n\Big)\,;\,\forall e\in[\![1,n(\boldsymbol{d})-1]\!],\ d_e\in\mathsf{Cut}(\beta)\Big\}.$$

For $\beta\in\mathbf{R}^n$ and $\mathsf{f}_1,\dots,\mathsf{f}_n\in\mathsf{C}^{0+}$ we set

$$\widetilde{\mathsf{P}}^{\beta}_{<}(\mathsf{f}_1,\dots,\mathsf{f}_n):=\sum_{\boldsymbol{d}\in\mathsf{MultiCut}(\beta)}(-1)^{n(\boldsymbol{d})+1}\prod_{e=1}^{n(\boldsymbol{d})}\mathsf{P}_{<}\big(\mathsf{f}_{d_{e-1}+1},\dots,\mathsf{f}_{d_e}\big).\tag{2.1}$$

One has for instance $\widetilde{\mathsf{P}}^{\beta}_{<}(\mathsf{f})=\mathsf{P}_{<}(\mathsf{f})=\sum_{i\geq-1}f_i$ for all $\mathsf{f}=(f_i)_{i\geq-1}\in\mathsf{C}^{-\infty}$, and

$$\widetilde{\mathsf{P}}^{(2,1)}_{<}(\mathsf{f}_1,\mathsf{f}_2)=\mathsf{P}_{<}(\mathsf{f}_1,\mathsf{f}_2),$$
$$\widetilde{\mathsf{P}}^{(1,-2)}_{<}(\mathsf{f}_1,\mathsf{f}_2)=\mathsf{P}_{<}(\mathsf{f}_1,\mathsf{f}_2)-\mathsf{f}_1\mathsf{f}_2,$$
$$\widetilde{\mathsf{P}}^{(1,-2,2,-1)}_{<}(\mathsf{f}_1,\mathsf{f}_2,\mathsf{f}_3,\mathsf{f}_4)=\mathsf{P}_{<}(\mathsf{f}_1,\mathsf{f}_2,\mathsf{f}_3,\mathsf{f}_4)-\mathsf{f}_1\mathsf{P}_{<}(\mathsf{f}_2,\mathsf{f}_3,\mathsf{f}_4)-\mathsf{P}_{<}(\mathsf{f}_1,\mathsf{f}_2,\mathsf{f}_3)\mathsf{f}_4+\mathsf{f}_1\mathsf{P}_{<}(\mathsf{f}_2,\mathsf{f}_3)\mathsf{f}_4,$$
$$\widetilde{\mathsf{P}}^{(1,-1,3/2)}_{<}(\mathsf{f}_1,\mathsf{f}_2,\mathsf{f}_3)=\mathsf{P}_{<}(\mathsf{f}_1,\mathsf{f}_2,\mathsf{f}_3).$$

We also set for $i\geq-1$

$$\widetilde{\mathsf{P}}^{\beta}_{<}(\mathsf{f}_1,\dots,\mathsf{f}_n)\{i\}$$
$$:=\sum_{\boldsymbol{d}\in\mathsf{MultiCut}(\beta)}(-1)^{n(\boldsymbol{d})+1}\Big\{\prod_{c=1}^{n(\boldsymbol{d})-1}\mathsf{P}_{<}\big(\mathsf{f}_{d_{c-1}+1},\dots,\mathsf{f}_{d_c}\big)\Big\}\mathsf{P}_{<}\big(\mathsf{f}_{d_{n(\boldsymbol{d})-1}+1},\dots,\mathsf{f}_n\big)_i.$$

One has the relation

$$\widetilde{\mathsf{P}}^{\beta}_{<}(\mathsf{f}_1,\dots,\mathsf{f}_n)=\sum_{i\geq-1}\widetilde{\mathsf{P}}^{\beta}_{<}(\mathsf{f}_1,\dots,\mathsf{f}_n)\{i\},$$

but beware that $\widetilde{\mathsf{P}}^{\beta}_{<}(\mathsf{f}_1,\dots,\mathsf{f}_n)\{i\}$ does not represent the Littlewood-Paley projection of the distribution $\widetilde{\mathsf{P}}^{\beta}_{<}(\mathsf{f}_1,\dots,\mathsf{f}_n)$ as $\widetilde{\mathsf{P}}^{\beta}_{<}(\mathsf{f}_1,\dots,\mathsf{f}_n)\{i\}$ is not spectrally supported in a ball. We introduce it as it appears naturally in the algebraic manipulations involving the operators $\widetilde{\mathsf{P}}^{\beta}_{<}$.

Equation 2.1 defining the $\widetilde{\mathsf{P}}^{\beta}_{<}$ can be compared to Zimmermann's forest formula. Here, the 'diverging parts' one can extract from a simplified paraproduct $\mathsf{P}_{<}(\mathsf{f}_1,\dots,\mathsf{f}_n)$ are the tails $\mathsf{P}_{<}(\mathsf{f}_j,\dots,\mathsf{f}_n)$ such that $\sum_{j=1}^{n}\beta_j<0$.

We prove below the following statement.

4 – ***Proposition.*** *For any* $\beta\in\mathbf{R}^n$*, set*

$$\mathcal{E}_\beta:=\Big\{c\in[\![1,n]\!]\,;\,\sum_{j=1}^{c}\beta_j>0\ \text{ and }\ \sum_{j=c+1}^{n}\beta_j>0\Big\}$$

and

$$m_0:=\begin{cases}\max\mathcal{E}_\beta, & \text{if } \mathcal{E}_\beta\neq\emptyset,\\ 1, & \text{otherwise.}\end{cases}.$$

One has for every $\big(\mathsf{f}_1,\dots,\mathsf{f}_n\big)\in(\mathsf{C}^{0+})^n$ *the estimate*

$$\big\|\widetilde{\mathsf{P}}^{\beta}_{<}\big(\mathsf{f}_1,\dots,\mathsf{f}_n\big)\{i\}\big\|_{L^\infty}\lesssim 2^{-i\sum_{j=m_0}^{n}\beta_j}\prod_{j=1}^{n}\|\mathsf{f}_j\|_{\beta_j}.$$

5 – *Corollary.* *For any* $\beta\in\mathbf{R}^n$ *such that* $\sum_{j=1}^n\beta_j>0$, *the multilinear map*

$$\big(\mathsf{f}_1,\dots,\mathsf{f}_n\big)\in(\mathsf{C}^\infty)^n\mapsto\widetilde{\mathsf{P}}^{\beta}_{<}\big(\mathsf{f}_1,\dots,\mathsf{f}_n\big)\in L^\infty$$

has a continuous extension as a map from $\prod_{j=1}^n\mathsf{C}_\circ^{\beta_j}$ *into* L^∞.

Proof – From the definition of m_0, the sum $\sum_{j=m_0}^n\beta_j$ is positive, then $\sum_{i\geq-1}2^{-i\sum_{j=m_0}^n\beta_j}<+\infty$, so that the last proposition ensures

$$\big|\widetilde{\mathsf{P}}^{\beta}_{<}\big(\mathsf{f}_1,\dots,\mathsf{f}_n\big)\big|\leq\sum_i\big\|\widetilde{\mathsf{P}}^{\beta}_{<}\big(\mathsf{f}_1,\dots,\mathsf{f}_n\big)\{i\}\big\|_{L^\infty}\lesssim\prod_{j=1}^{n}\|\mathsf{f}_j\|_{\beta_j}.$$

This inequality gives the result. ▷

§2 Proof of Proposition 4 – The following algebraic result will be useful in the proof of Proposition 4. The reader can harmlessly skip its proof on a first reading In the following statement, any constant in the open interval $(1,2)$ could be used in place of the constant $3/2$.

6 – *Lemma.* *Given* $\beta\in\mathbf{R}^n$ *we define*

$$\rho_c:=\left\{\begin{array}{ll}+1 & \text{if } (n-c)\in\mathsf{Cut}(\beta)\\ -1 & \text{otherwise}\end{array}\right.,\qquad \rho:=\prod_{c=1}^{n-1}(-\rho_c).$$

For any $\mathsf{f}_1=(f_{1i})_{i\geq-1},\dots,\mathsf{f}_n=(f_{ni})_{i\geq-1}$ *in* C^∞ *we have*

$$\widetilde{\mathsf{P}}^{\beta}_{<}(\mathsf{f}_1,\dots,\mathsf{f}_n)\{i_1\}=\rho\sum_{\rho_1(i_2-i_1+3/2)>0}\cdots\sum_{\rho_{n-1}(i_n-i_{n-1}+3/2)>0}\prod_{c=1}^{n}f_{ci_{n-c+1}}.$$

Proof – We prove the identity by induction on n. The result holds for $\widetilde{\mathsf{P}}^{\beta}_{<}(\mathsf{f}_1)$. Suppose now that it holds for $(n-1)$ functions and consider first the case that $(n-1)\notin\mathsf{Cut}(\beta)$, so $\rho_1=-1$ and the condition $\rho_1(i_2-i_1+3/2)>0$ reads $i_2<i_1-1$. Then $\widetilde{\mathsf{P}}^{\beta}_{<}\big(\mathsf{f}_1,\dots,\mathsf{f}_n\big)\{i_1\}$ is equal to

$$\begin{aligned}&\sum_{\boldsymbol{d}\in\mathsf{MultiCut}(\beta)}(-1)^{n(\boldsymbol{d})+1}\left\{\prod_{c=1}^{n(\boldsymbol{d})-1}\mathsf{P}_<\big(\mathsf{f}_{d_{c-1}+1},\dots,\mathsf{f}_{d_c}\big)\right\}\mathsf{P}_<\big(\mathsf{f}_{d_{n(\boldsymbol{d})-1}+1},\dots,\mathsf{f}_n\big)_{i_1}\\ &=\sum_{\boldsymbol{d}\in\mathsf{MultiCut}(\beta)}(-1)^{n(\boldsymbol{d})+1}\left\{\prod_{c=1}^{n(\boldsymbol{d})-1}\mathsf{P}_<\big(\mathsf{f}_{d_{c-1}+1},\dots,\mathsf{f}_{d_c}\big)\right\}\mathsf{P}_<\big(\mathsf{f}_{d_{n(\boldsymbol{d})-1}+1},\dots,\mathsf{f}_{n-1}\big)_{<i_1-1}f_{ni_1}\\ &=\sum_{i_2<i_1-1}\widetilde{\mathsf{P}}^{\beta^*}_{<}\big(\mathsf{f}_1,\dots,\mathsf{f}_{n-1}\big)\{i_2\}f_{ni_1}\end{aligned}$$

where

$$\beta^*:=\Big(\beta_1,\,\dots,\,\beta_{n-2},\,\beta_{n-1}+\beta_n\Big).$$

From the induction hypothesis we have

$$\widetilde{\mathsf{P}}^{\beta^*}_{<}\big(\mathsf{f}_1,\dots,\mathsf{f}_{n-1}\big)\{i_2\}=\rho\sum_{\rho_2(i_3-i_2+3/2)>0}\cdots\sum_{\rho_{n-1}(i_n-i_{n-1}+3/2)>0}\prod_{c=1}^{n-1}f_{ci_{n-c+1}},$$

so we can conclude the induction in that case. If now $(n-1)\in\mathsf{Cut}(\beta)$ we have $\rho_1=1$ and the condition $\rho_1(i_2-i_1+3/2)>0$ reads $i_2\geq i_1-1$. We have in that case

$$
\begin{aligned}
&\widetilde{\mathsf{P}}^{\beta}_{<}(\mathsf{f}_1,\dots,\mathsf{f}_n)\{i_1\} \\
&\quad= \sum_{\substack{\boldsymbol{d}\in\mathsf{MultiCut}(\beta)\\(n-1)\in\boldsymbol{d}}} (-1)^{n(\boldsymbol{d})+1}\prod_{c=1}^{n(\boldsymbol{d})-1}\mathsf{P}_{<}\big(\mathsf{f}_{d_{c-1}+1},\dots,\mathsf{f}_{d_c}\big)f_{ni_1} \\
&\qquad+ \sum_{\substack{\boldsymbol{d}\in\mathsf{MultiCut}(\beta)\\(n-1)\notin\boldsymbol{d}}} (-1)^{n(\boldsymbol{d})+1}\prod_{c=1}^{n(\boldsymbol{d})-1}\mathsf{P}_{<}\big(\mathsf{f}_{d_{c-1}+1},\dots,\mathsf{f}_{d_c}\big)\mathsf{P}_{<}\big(\mathsf{f}_{d_{n(\boldsymbol{d})-1}+1},\dots,\mathsf{f}_n\big)_{i_1} \\
&\quad= \sum_{\substack{\boldsymbol{d}\in\mathsf{MultiCut}(\beta)\\(n-1)\in\boldsymbol{d}}} (-1)^{n(\boldsymbol{d})+1}\prod_{c=1}^{n(\boldsymbol{d})-1}\mathsf{P}_{<}\big(\mathsf{f}_{d_{c-1}+1},\dots,\mathsf{f}_{d_c}\big)f_{ni_1} \\
&\qquad+ \sum_{\substack{\boldsymbol{d}\in\mathsf{MultiCut}(\beta)\\(n-1)\notin\boldsymbol{d}}} (-1)^{n(\boldsymbol{d})+1}\prod_{c=1}^{n(\boldsymbol{d})-1}\mathsf{P}_{<}\big(\mathsf{f}_{d_{c-1}+1},\dots,\mathsf{f}_{d_c}\big)\mathsf{P}_{<}\big(\mathsf{f}_{d_{n(\boldsymbol{d})-1}+1},\dots,\mathsf{f}_{n-1}\big)_{<i_1-1}f_{ni_1} \\
&\quad= \sum_{\boldsymbol{d}\in\mathsf{MultiCut}(\beta^*)} (-1)^{n(\boldsymbol{d})+1}\Bigg\{\prod_{c=1}^{n(\boldsymbol{d})-1}\mathsf{P}_{<}\big(\mathsf{f}_{d_{c-1}+1},\dots,\mathsf{f}_{d_c}\big) \\
&\qquad\qquad - \prod_{c=1}^{n(\boldsymbol{d})-2}\mathsf{P}_{<}\big(\mathsf{f}_{d_{c-1}+1},\dots,\mathsf{f}_{d_c}\big)\mathsf{P}_{<}\big(\mathsf{f}_{d_{n(\boldsymbol{d})-2}+1},\dots,\mathsf{f}_{n-1}\big)_{<i_1-1}\Bigg\}f_{ni_1} \\
&\quad= -\sum_{i_2>i_1-2}\widetilde{\mathsf{P}}^{\beta^*}_{<}\big(\mathsf{f}_1,\dots,\mathsf{f}_{n-1}\big)\{i_2\}\,f_{ni_1}.
\end{aligned}
$$

We conclude from the induction hypothesis that

$$
\widetilde{\mathsf{P}}^{\beta}_{<}\big(\mathsf{f}_1,\dots,\mathsf{f}_n\big)\{i_1\} = -\sum_{i_2\geq i_1-1}(-\rho)\sum_{\rho_2(i_3-i_2+3/2)>0}\cdots\sum_{\rho_{n-1}(i_n-i_{n-1}+3/2)>0}\left\{\prod_{c=1}^{n-1}f_{ci_{n-c+1}}\right\}f_{ni_1}
$$

which allows us to close the induction in that case. ▷

Proof of Proposition 4 – For $\mathsf{f}_1,\dots\mathsf{f}_n\in\mathsf{C}^{+\infty}$ we have from Lemma 6 the bound

$$
\big|\widetilde{\mathsf{P}}^{\beta}_{<}\big(\mathsf{f}_1,\dots,\mathsf{f}_n\big)\{i\}\big| \lesssim C_\beta(i)\prod_{j=1}^n\|\mathsf{f}_j\|_{\beta_j},
$$

where

$$
C_\beta(i_1) := \sum_{\rho_1(i_2-i_1+3/2)>0}\cdots\sum_{\rho_{n-1}(i_n-i_{n-1}+3/2)>0}\prod_{c=1}^n 2^{-i_{n+c-1}\beta_c}.
$$

We prove by induction that

$$
C_\beta(i)\lesssim 2^{-i\sum_{j=m_0}^n\beta_j}. \tag{2.2}
$$

– If $\beta_1<0$ we have $\rho_{n-1}=-1$ and

$$
\sum_{i_n;\ \rho_{n-1}(i_n-i_{n-1}+3/2)>0} 2^{-i_n\beta_1}\simeq 2^{-i_{n-1}\beta_1}.
$$

We have in that case

$$
C_{(\beta_1,\dots,\beta_n)}(i)\simeq C_{(\beta_1+\beta_2,\beta_3\dots,\beta_n)}(i).
$$

– If now $\beta_1>0$ and $\sum_{j=2}^n\beta_j<0$, then $\rho_{n-1}=+1$ and we have

$$\sum_{i_n;\ \rho_{n-1}(i_n-i_{n-1}+3/2)>0} 2^{-i_n\beta_1} \simeq 2^{-i_{n-1}\beta_1},$$

so we have again

$$C_{(\beta_1,\dots,\beta_n)}(i) \simeq C_{(\beta_1+\beta_2,\beta_3\dots,\beta_n)}(i).$$

– If finally $\beta_1 > 0$ and $\sum_{j=2}^n \beta_j > 0$, we have this time

$$\sum_{i_n;\ \rho_{n-1}(i_n-i_{n-1}+3/2)>0} 2^{-i_n\beta_1} \simeq 1,$$

so

$$C_{(\beta_1,\dots,\beta_n)}(i) \simeq C_{(\beta_2,\beta_3\dots,\beta_n)}(i).$$

In all the cases the inequality (2.2) follows by induction since $C_{(\beta_1)}(i) = 2^{-i\beta_1}$. ▷

§3 Generalized derivative operators $\partial_\star^k$ – These operators are defined in terms of the operators $\widetilde{\mathsf{P}}_<^\beta$ as follows.

Definition – *Pick some integers* $1 \leq a \leq b \leq n$ *and* $\alpha = (\alpha_a,\dots,\alpha_b) \in \mathbf{R}^{b-a+1}$. *For* $\mathfrak{k} = (k_a,\dots,k_b) \in (\mathbf{N}^{d_0})^{b-a+1}$ *and* $\mathsf{f}_a,\dots,\mathsf{f}_b$ *in* C^∞ *we define*

$$\partial_{\star\alpha}^{\mathfrak{k}} \mathsf{P}_<\big(\mathsf{f}_a,\dots,\mathsf{f}_b\big) := \widetilde{\mathsf{P}}_<^{\alpha_{[a,b]}-|\mathfrak{k}|}\Big(\partial^{k_a}\mathsf{f}_a,\dots,\partial^{k_b}\mathsf{f}_b\Big).$$

and

$$\partial_{\star\alpha}^{k} \mathsf{P}_<\big(\mathsf{f}_a,\dots,\mathsf{f}_b\big) := \sum_{\mathfrak{k}\in\mathcal{P}_{b-a+1}(k)} \binom{k}{\mathfrak{k}}\, \partial_{\star\alpha}^{\mathfrak{k}} \mathsf{P}_<\big(\mathsf{f}_a,\dots,\mathsf{f}_b\big),$$

As a consequence of Corollary 5 the map $\partial_{\star\alpha}^k\mathsf{P}_<$ is continuous from $\prod_{j=a}^b \mathsf{C}_\circ^{\alpha_j}$ into L^∞ if $|k| < \sum_{j=a}^b \alpha_j$. It makes sense in that setting to simply write $\partial_\star^k$ rather than $\partial_{\star\alpha}^k$, as the information on α is already recorded in the domain $\prod_{j=a}^b \mathsf{C}_\circ^{\alpha_j}$ of the extension.

The following lemma gives a recursive definition of the $\widetilde{\mathsf{P}}_<^\beta(\mathsf{f}_1,\dots,\mathsf{f}_n)\{i\}$ and leads in (2.4) below to a similar recursive formula for the operators $\partial_\star\mathsf{P}_<$.

7 – Lemma. *For any* $\beta = (\beta_1,\dots,\beta_n) \in \mathbf{R}^n$ *and any* $\mathsf{f}_1,\dots,\mathsf{f}_n$ *in* C^∞ *we have*

$$\begin{aligned}\widetilde{\mathsf{P}}_<^\beta\big(\mathsf{f}_1,\dots,\mathsf{f}_n\big)\{i\} &= \mathsf{P}_<\big(\mathsf{f}_1,\dots,\mathsf{f}_n\big)_i \\ &\quad - \sum_{d\in\mathsf{Cut}(\beta)} \widetilde{\mathsf{P}}_<^{(\beta_1,\dots,\beta_d)}\big(\mathsf{f}_1,\dots,\mathsf{f}_d\big)\, \widetilde{\mathsf{P}}_<^{(\beta_{d+1},\dots,\beta_n)}\big(\mathsf{f}_{d+1},\dots,\mathsf{f}_n\big)\{i\}.\end{aligned} \tag{2.3}$$

Proof – ***Assumption (A)*** implies in particular that the numbers $\sum_{c=1}^j \beta_c$ are all distinct for different $j \in [\![1,n-1]\!]$. We then have the following partition of $\mathsf{MultiCut}(\beta)$

$$\mathsf{MultiCut}(\beta) = \big\{(0,n)\big\} \sqcup \bigsqcup_{d\in\mathsf{Cut}(\beta)} \mathsf{MultiCut}(\beta)[d],$$

with

$$\mathsf{MultiCut}(\beta)[d] := \left\{\boldsymbol{d} \in \mathsf{MultiCut}(\beta)\,;\, d \in \boldsymbol{d},\quad \sum_{c=1}^d \beta_c = \min_{j\in\mathbf{d}} \sum_{c=1}^j \beta_c\right\}.$$

One can thus write

$$\begin{aligned}\widetilde{\mathsf{P}}_<^\beta(\mathsf{f}_1,\dots\mathsf{f}_n)\{i\} &= \mathsf{P}_<(\mathsf{f}_1,\dots,\mathsf{f}_n)_i \\ &\quad + \sum_{d\in\mathsf{Cut}(\beta)} \sum_{\boldsymbol{d}\in\mathsf{MultiCut}(\beta)[d]} (-1)^{n(\boldsymbol{d})+1} \prod_{c=1}^{n(\boldsymbol{d})-1} \mathsf{P}_<\big(\mathsf{f}_{d_{c-1}+1},\dots,\mathsf{f}_{d_c}\big)\, \mathsf{P}_<\big(\mathsf{f}_{d_{n(\boldsymbol{d})-1}+1},\dots,\mathsf{f}_n\big)\{i\}.\end{aligned}$$

For $d \in \mathsf{Cut}(\beta)$ and $1 < j < d$ we have the equivalence

$$\Big(\exists\, \boldsymbol{d} \in \mathsf{MultiCut}(\beta)[d],\ \ j \in \boldsymbol{d}\Big) \Leftrightarrow \Big(j \in \mathsf{Cut}\big((\beta_1,\dots,\beta_d)\big)\Big).$$

Likewise for $d < j < n$ we have

$$\Big(\exists\, \boldsymbol{d} \in \mathsf{MultiCut}(\beta)[d],\ \ j \in \boldsymbol{d}\Big) \Leftrightarrow \Big(j - d \in \mathsf{Cut}\big((\beta_{d+1}, \dots, \beta_n)\big)\Big).$$

This entails that we have

$$\begin{aligned}&\sum_{\boldsymbol{d}\in \mathsf{MultiCut}(\beta)[d]} (-1)^{n(\boldsymbol{d})+1} \prod_{c=1}^{n(\boldsymbol{d})-1} \mathsf{P}_<\big(\mathsf{f}_{d_{c-1}+1}, \dots, \mathsf{f}_{d_c}\big) \mathsf{P}_<\big(\mathsf{f}_{d_{n(\boldsymbol{d})-1}+1}, \dots, \mathsf{f}_n\big)\{i\} \\ &= -\widetilde{\mathsf{P}}_<^{\beta_{\leq d}}\big(\mathsf{f}_1, \dots, \mathsf{f}_d\big)\, \widetilde{\mathsf{P}}_<^{\beta_{>d}}\big(\mathsf{f}_{d+1}, \dots, \mathsf{f}_n\big)\{i\},\end{aligned}$$

from which the statement of the lemma follows. ▷

Recall that Corollary 5 extends continuously the $\widetilde{\mathsf{P}}_<^{\beta}$ maps to $\prod_{j=1}^n \mathsf{C}_\circ^{\beta_j}$ and justifies that we remove the index α from $\partial_{\star\alpha}^k$ when it is understood that $(\mathsf{f}_1, \dots, \mathsf{f}_n) \in \prod_{j=1}^n \mathsf{C}_\circ^{\alpha_j}$. One can rewrite Lemma 7 in the context of the $\partial_\star$-derivatives. For any multi-indice $k \in \mathbf{N}^{d_0}$ and $(\mathsf{f}_1, \dots, \mathsf{f}_n) \in \prod_{j=1}^n \mathsf{C}_\circ^{\beta_j}$ we have

$$\begin{aligned}&\partial_\star^k \mathsf{P}_<\big(\mathsf{f}_1, \dots, \mathsf{f}_n\big) \\ &\quad= \partial^k \mathsf{P}_<\big(\mathsf{f}_1, \dots, \mathsf{f}_n\big) - \sum_{c=1}^{n-1} \sum_{\substack{|\ell| < \sum_{j=1}^c \alpha_j \\ |k-\ell| > \sum_{j=c+1}^n \alpha_j}} \binom{k}{\ell} \partial_\star^\ell \mathsf{P}_<\big(\mathsf{f}_1, \dots, \mathsf{f}_c\big)\, \partial_\star^{k-\ell} \mathsf{P}_<\big(\mathsf{f}_{c+1}, \dots, \mathsf{f}_n\big).\end{aligned} \tag{2.4}$$

2.3 – Local expansion properties of the $\mathsf{P}_<(\mathsf{f}_1, \dots \mathsf{f}_n)$.

Recall Hoshino's expansion result (1.2) for $\mathsf{P}(f, g)$, for both f and g of positive regularity. We give in Proposition 12 below a similar expansion result for $\mathsf{P}_<(\mathsf{f}_1, \dots, \mathsf{f}_n)$, for any $n \geq 2$ and $\mathsf{f}_j \in \mathsf{C}_\circ^{\alpha_j}$ for all $1 \leq j \leq n$.

Let us make a first naive try at expanding $\mathsf{P}_<(\mathsf{f}_1, \dots \mathsf{f}_n)(\cdot + h)$ as a function of $h \in \mathbf{R}^{d_0}$. For any $o > 0$ we have

$$\begin{aligned}&\mathsf{P}_<\big(\mathsf{f}_1, \dots, \mathsf{f}_n\big)(\cdot + h) = \mathsf{P}_<\big(\mathsf{f}_1(\cdot + h), \dots, \mathsf{f}_n(\cdot + h)\big) \\ &\quad= \mathsf{P}_<\bigg(\sum_{|k_1|<o} \frac{h^{k_1}}{k_1!} \partial^{k_1}\mathsf{f}_1 + |h|^o R_h^o \mathsf{f}_1,\ \mathsf{f}_2(\cdot + h), \dots\bigg) \\ &\quad= \sum_{|k_1|<o} \mathsf{P}_<\bigg(\partial^{k_1}\mathsf{f}_1 \frac{h^{k_1}}{k_1!}, \sum_{|k_2|<o-|k_1|} \frac{h^{k_2}}{k_2!} \partial^{k_2}\mathsf{f}_2 + |h|^{o-|k_1|} R_h^{o-|k_1|}\mathsf{f}_2, \dots\bigg) \\ &\qquad\qquad + \mathsf{P}_<\Big(|h|^o R_h^o \mathsf{f}_1,\ \mathsf{f}_2(\cdot + h), \dots\Big) = (\cdots) \\ &\quad= \sum_{|k|<o} \sum_{\mathfrak{k}\in\mathcal{P}_n(k)} \frac{h^k}{k!} \binom{k}{\mathfrak{k}} \mathsf{P}_<\big(\partial^{k_1}\mathsf{f}_1, \dots, \partial^{k_n}\mathsf{f}_n\big) \\ &\qquad + \sum_{c=1}^n \sum_{\substack{|k|<o \\ \mathfrak{k}\in\mathcal{P}_{c-1}(k)}} \frac{h^k |h|^{o-|k|}}{\mathfrak{k}!} \mathsf{P}_<\Big(\partial^{k_1}\mathsf{f}_1, \dots, \partial^{k_{c-1}}\mathsf{f}_{c-1}, R_h^{o-|k|}\mathsf{f}_c,\ \mathsf{f}_{c+1}(\cdot + h), \dots\Big) \\ &\quad= T_h^o \mathsf{P}_<\big(\mathsf{f}_1, \dots, \mathsf{f}_n\big) \\ &\qquad + \sum_{c=1}^n \sum_{\substack{|k|<o \\ \mathfrak{k}\in\mathcal{P}_{c-1}(k)}} \frac{h^k |h|^{o-|k|}}{\mathfrak{k}!} \mathsf{P}_<\Big(\partial^{k_1}\mathsf{f}_1, \dots, \partial^{k_{c-1}}\mathsf{f}_{c-1}, R_h^{o-|k|}\mathsf{f}_c,\ \mathsf{f}_{c+1}(\cdot + h), \dots\Big).\end{aligned} \tag{2.5}$$

This formula does not give us the kind of expansion we are looking for as the last paraproducts in the right hand side of the equality contain some distributions with negative regularities so these paraproducts have no reason to define some functions. This would be the case if we had instead of some $\mathsf{P}_<$ terms some $\widetilde{\mathsf{P}}_<^{\beta}$ terms, for some appropriate tuples β depending on the arguments. We will get our local expansion for $\mathsf{P}_<(\mathsf{f}_1, \dots \mathsf{f}_n)(\cdot + h)$ by introducing the appropriate terms to force the appearance of these $\widetilde{\mathsf{P}}^{\beta}_{<}$ operators. We proceed gradually and first introduce the quantity that will be the remainder term in this expansion.

For $1 \leq a \leq b \leq n-1, k \in \mathbf{N}^{d_0}$ and $\mathfrak{k} = (k_{a+1}, \dots, k_{b-1}) \in \mathcal{P}_{b-a-1}(k)$ set

$$\alpha_a(\mathfrak{k}, o) := \Big(\alpha_{a+1} - |k_{a+1}|, \dots, \alpha_{b-1} - |k_{b-1}|, \, \alpha_b - o + |k|, \, \alpha_{b+1}, \dots, \, \alpha_n\Big).$$

and

$$\begin{aligned}\big(\triangle^{\alpha}_{h,o}\mathsf{P}_{<}\big)(\mathsf{f}_{a+1}, \dots, \mathsf{f}_n) := \sum_{b=a+1}^{n} \sum_{\substack{|k|<o \\ \mathfrak{k}\in\mathcal{P}_{b-a-1}(k)}} \frac{h^k |h|^{o-|k|}}{\mathfrak{k}!} \, \widetilde{\mathsf{P}}^{\alpha_a(\mathfrak{k},o)}_{<} \Big(&\partial^{k_{a+1}}\mathsf{f}_{a+1} \dots, \partial^{k_{b-1}}\mathsf{f}_{b-1}, \\ &R_h^{o-|k|}\mathsf{f}_b, \, \mathsf{f}_{b+1}(\cdot + h), \dots, \mathsf{f}_n(\cdot + h)\Big),\end{aligned} \tag{2.6}$$

and for $i \geq -1$

$$\begin{aligned}\big(\triangle^{\alpha}_{h,o}\mathsf{P}_{<}\big)(\mathsf{f}_{a+1}, \dots, \mathsf{f}_n)\{i\} := \sum_{b=a+1}^{n} \sum_{\substack{|k|<o \\ \mathfrak{k}\in\mathcal{P}_{b-a-1}(k)}} \frac{h^k |h|^{o-|k|}}{\mathfrak{k}!} \, \widetilde{\mathsf{P}}^{\alpha_a(\mathfrak{k},o)}_{<} \Big(&\partial^{k_{a+1}}\mathsf{f}_{a+1} \dots, \partial^{k_{b-1}}\mathsf{f}_{b-1}, \\ &R_h^{o-|k|}\mathsf{f}_b, \, \mathsf{f}_{b+1}(\cdot + h), \dots, \mathsf{f}_n(\cdot + h)\Big)\{i\}.\end{aligned} \tag{2.7}$$

We denote by δ_0 the distance from $\mathbb{Z}$ to the set of all $\sum_{a\leq j\leq b} \alpha_j \notin \mathbb{Z}$ where $1 \leq a \leq b \leq n$; it is positive from ***Assumption (A)***. Proposition 4 and Corollary 5 give us some uniform continuity estimates on $(\triangle^{\alpha}_{h,o}\mathsf{P}_{<})(\mathsf{f}_{a+1}, \dots, \mathsf{f}_n)\{i\}$ and $(\triangle^{\alpha}_{h,o}\mathsf{P}_{<})(\mathsf{f}_{a+1}, \dots, \mathsf{f}_n)$ in the form of the following Lemma.

8 – Lemma. *If $o > \sum_{j=a+1}^{n} \alpha_j - \delta_0$, one has*

$$\Big|\big(\triangle^{\alpha}_{h,o}\mathsf{P}_{<}\big)(\mathsf{f}_{a+1}, \dots, \mathsf{f}_n)\{i\}\Big| \lesssim |h|^o \, 2^{-i(\sum_{j=a+1}^{n}\alpha_j - o)} \prod_{j=a+1}^{n} \|\mathsf{f}_j\|_{\alpha_j}, \tag{2.8}$$

and for $o < \sum_{j=a+1}^{n} \alpha_j$

$$\Big|\big(\triangle^{\alpha}_{h,o}\mathsf{P}_{<}\big)(\mathsf{f}_{a+1}, \dots, \mathsf{f}_n)\Big| \lesssim |h|^o \prod_{j=a+1}^{n} \|\mathsf{f}_j\|_{\alpha_j}. \tag{2.9}$$

Proof – We prove here estimate 2.8, the other one is proven along the same lines. From the assumption on o, for any $k < |o|$ and $\mathfrak{k} \in \mathcal{P}_{b-a-1}(k)$, the sum of the entries of the uplet $\alpha_a(\mathfrak{k}, o)$ is positive, and then Proposition 4 ensures that

$$\begin{aligned}&\Big|\widetilde{\mathsf{P}}^{\alpha_a(\mathfrak{k},o)}_{<}\Big(\partial^{k_{a+1}}\mathsf{f}_{a+1} \dots, \partial^{k_{b-1}}\mathsf{f}_{b-1}, \; R_h^{o-|k|}\mathsf{f}_b, \, \mathsf{f}_{b+1}(\cdot + h), \dots, \mathsf{f}_n(\cdot + h)\Big)\{i\}\Big| \\ &\qquad \lesssim 2^{-i(\sum_{j=a+1}^{n}\alpha_j - o)} \prod_{c=a+1}^{b-1} \big\|\partial^{k_c}\mathsf{f}_c\big\|_{\alpha_c - |k_c|} \, \big\|R_h^{o-|k|}\mathsf{f}_b\big\|_{\alpha_b - |o| + |k|} \prod_{d=b+1}^{n} \|\mathsf{f}_d(\cdot + h)\|_{\alpha_d}.\end{aligned}$$

Using Lemma 3 for estimating $\big\|R_h^{o-|k|}\mathsf{f}_b\big\|_{\alpha_b-|o|+|k|}$ and the Bernstein inequalities for estimating $\big\|\partial^{k_c}\mathsf{f}_c\big\|_{\alpha_c-|k_c|}$, we obtain for this last twisted paraproduct the bound $2^{-i(\sum_{j=a+1}^{n}\alpha_j - o)} \prod_{j=a+1}^{n} \|\mathsf{f}_j\|_{\alpha_j}$. Summing these inequalities over b and $\mathfrak{k}$ gives the estimate 2.8. ▷

9 – Proposition. *Pick $\mathsf{f}_1, \dots, \mathsf{f}_n$ in C^∞. Assume all the α_j are positive and $o > \sum_{j=1}^{n} \alpha_j - \delta_0$. Then we have*

$$\begin{aligned}&\big(\triangle^{\alpha}_{h,o}\mathsf{P}_{<}\big)(\mathsf{f}_1, \dots, \mathsf{f}_n)\{i\} \\ &\qquad = \mathsf{P}_{<}\big(\mathsf{f}_1, \dots, \mathsf{f}_n\big)_i(\cdot + h) - T_h^o \mathsf{P}_{<}\big(\mathsf{f}_1, \dots, \mathsf{f}_n\big)_i \\ &\qquad\quad - \sum_{a=1}^{n} \sum_{|k|<\sum_{i=1}^{a}\alpha_i} \partial^k_{\star\alpha_{\leq a}} \mathsf{P}_{<}(\mathsf{f}_1, \dots, \mathsf{f}_a) \, \frac{h^k}{k!} \, \big(\triangle^{\alpha}_{h,o-|k|}\mathsf{P}_{<}\big)(\mathsf{f}_{a+1}, \dots, \mathsf{f}_n)\{i\}.\end{aligned} \tag{2.10}$$

Proof – We use in the proof the shorthand notation

$$\alpha(\mathfrak{k}) := \alpha_0(\mathfrak{k}, o) = \Big(\alpha_1 - |k_1|, \dots, \alpha_{j-1} - |k_{j-1}|, \alpha_j - o + |k|, \alpha_{j+1}, \dots, \alpha_n\Big).$$

As all the α_j are positve we have $\mathsf{Cut}(\alpha(\mathfrak{k})) \subset [\![1, j-1]\!]$, so (2.3) writes here

$$\begin{aligned}
&\widetilde{\mathsf{P}}_<^{\alpha(\mathfrak{k})}\Big(\partial^{k_1}\mathsf{f}_1, \dots, \partial^{k_{j-1}}\mathsf{f}_{j-1}, R_h^{o-|k|}\mathsf{f}_j, \mathsf{f}_{j+1}(\cdot + h), \dots\Big)\{i\} \\
&\quad = \mathsf{P}_<\Big(\partial^{k_1}\mathsf{f}_1, \dots, \partial^{k_{j-1}}\mathsf{f}_{j-1}, R_h^{o-|k|}\mathsf{f}_j, \mathsf{f}_{j+1}(\cdot + h), \dots\Big)_i \\
&\qquad - \sum_{d \in \mathsf{Cut}(\alpha(\mathfrak{k}))} \widetilde{\mathsf{P}}_<^{\alpha(\mathfrak{k})_{\leq d}}\Big(\partial^{k_1}\mathsf{f}_1, \dots, \partial^{k_d}\mathsf{f}_d\Big) \\
&\qquad\qquad \times \widetilde{\mathsf{P}}_<^{\alpha(\mathfrak{k})_{>d}}\Big(\partial^{k_{d+1}}\mathsf{f}_{d+1}, \dots, \partial^{k_{j-1}}\mathsf{f}_{j-1}, R_h^{o-|k|}\mathsf{f}_j, \mathsf{f}_{j+1}(\cdot + h), \dots\Big)\{i\} \\
&= \mathsf{P}_<\Big(\partial^{k_1}\mathsf{f}_1, \dots, \partial^{k_{j-1}}\mathsf{f}_{j-1}, R_h^{o-|k|}\mathsf{f}_j, \mathsf{f}_{j+1}(\cdot + h), \dots\Big)_i \\
&\quad - \sum_{d \in \mathsf{Cut}(\alpha(\mathfrak{k}))} \partial_\star^{\mathfrak{k}_{\leq d}}\mathsf{P}_<\Big(\mathsf{f}_1, \dots, \mathsf{f}_d\Big) \\
&\qquad\qquad \times \widetilde{\mathsf{P}}_<^{\alpha(\mathfrak{k})_{>d}}\Big(\partial^{k_{d+1}}\mathsf{f}_{d+1}, \dots, \partial^{k_{j-1}}\mathsf{f}_{j-1}, R_h^{o-|k|}\mathsf{f}_j, \mathsf{f}_{j+1}(\cdot + h), \dots\Big)\{i\}.
\end{aligned}$$

Note that as $o > \sum_{j=1}^n \alpha_j - \delta_0$ we have

$$\mathsf{Cut}(\alpha(\mathfrak{k})) = \Big\{d \in [\![1, n]\!]\,;\, \sum_{j=1}^d \alpha(\mathfrak{k})_j > 0\Big\};$$

we will use this fact to invert the sums over m and j below. Summing over j, k and $\mathfrak{k}$ gives

$$\begin{aligned}
&(\triangle_{h,o}^\alpha \mathsf{P}_<)(\mathsf{f}_1, \dots, \mathsf{f}_n)\{i\} - \sum_{j=1}^n \sum_{\substack{|k|<o \\ \mathfrak{k} \in \mathcal{P}_{j-1}(k)}} \frac{h^k |h|^{o-|k|}}{\mathfrak{k}!} \mathsf{P}_<\Big(\partial^{k_1}\mathsf{f}_1, \dots, \partial^{k_{j-1}}\mathsf{f}_{j-1}, R_h^{o-|k|}\mathsf{f}_j, \mathsf{f}_{j+1}(\cdot + h), \dots\Big)_i \\
&= \sum_{j=1}^n \sum_{\substack{|k|<o \\ \mathfrak{k} \in \mathcal{P}_{j-1}(k)}} \frac{h^k |h|^{o-|k|}}{\mathfrak{k}!} \sum_{d \in \mathsf{Cut}(\alpha(\mathfrak{k}))} \partial_\star^{\mathfrak{k}_{\leq d}}\mathsf{P}_<\big(\mathsf{f}_1, \dots, \mathsf{f}_d\big) \\
&\qquad \times \widetilde{\mathsf{P}}_<^{\alpha(\mathfrak{k})_{>d}}\Big(\partial^{k_{d+1}}\mathsf{f}_{d+1}, \dots, \partial^{k_{j-1}}\mathsf{f}_{j-1}, R_h^{o-|k|}\mathsf{f}_j, \mathsf{f}_{j+1}(\cdot + h), \dots\Big)\{i\} \\
&= \sum_{d=1}^n \sum_{\substack{|k|<\sum_{i=1}^d \alpha_i \\ \mathfrak{k} \in \mathcal{P}_{d-1}(k)}} \frac{h^k}{\mathfrak{k}!} \partial_{\star\alpha_{\leq d}}^{\mathfrak{k}}\mathsf{P}_<\big(\mathsf{f}_1, \dots, \mathsf{f}_d\big) \sum_{j=d+1}^n \sum_{\substack{|\ell|<o-|k| \\ \mathfrak{j} \in \mathcal{P}_{j-d-1}(\ell)}} \frac{h^\ell |h|^{o-|k|-|\ell|}}{\mathfrak{j}!} \\
&\qquad \times \widetilde{\mathsf{P}}_<^{\beta_d(\mathfrak{j}, o-|k|)}\Big(\partial^{\ell_1}\mathsf{f}_{d+1}, \dots, \partial^{\ell_{j-d-1}}\mathsf{f}_{j-1}, R_h^{o-|k|-|\ell|}\mathsf{f}_j, \mathsf{f}_{j+1}(\cdot + h), \dots\Big)\{i\}
\end{aligned}$$

$$= \sum_{d=1}^{n} \sum_{|k|<\sum_{i=1}^{d}\alpha_i} \frac{h^k}{k!}\, \partial^k_{\star\alpha_{\le d}} \mathsf{P}_<(\mathsf{f}_1,\dots,\mathsf{f}_d)\, \big(\triangle^{\alpha}_{h,o-|k|}\mathsf{P}_<\big)(\mathsf{f}_{d+1},\dots,\mathsf{f}_n)\{i\}.$$

The identity (2.10) then follows from (2.5). ▷

The terms $\triangle^{\alpha}_{h,o-|k|}\mathsf{P}_<(\mathsf{f}_{d+1},\dots,\mathsf{f}_n)$ for which $o-|k| > \sum_{j=d+1}^{n}\alpha_j$, in (2.10), are still problematic as one cannot use Corollary 5 for them.

10 – Lemma. *Assume all the α_j positive. For $1\le a\le n$ and $\sum_{j=a}^{n}\alpha_j - \delta_0 < o_1 < o_2$, we have*

$$\big(\triangle^{\alpha}_{h,o_2}\mathsf{P}_<\big)\big(\mathsf{f}_a,\dots,\mathsf{f}_n\big)\{i\} - \big(\triangle^{\alpha}_{h,o_1}\mathsf{P}_<\big)\big(\mathsf{f}_a,\dots,\mathsf{f}_n\big)\{i\} = \sum_{o_1<|k|<o_2} \frac{h^k}{k!}\,\partial^k_{\star\alpha_{\ge a}}\mathsf{P}_<\big(\mathsf{f}_a,\dots,\mathsf{f}_n\big)\{i\}.$$

Proof – We prove this identity by induction over $n-a$ with the help of Proposition 9 and the inductive relation (2.4) satisfied by the star derivatives.
The result is true for $a=n$ as in this case the operator $\triangle_{h,o}\mathsf{P}_<$ coincides with the Taylor remainder the operator $|h|^r R^r_h$. To run the induction step we use Proposition 9 to see that $\big(\triangle^{\alpha}_{h,o_2}\mathsf{P}_<\big)\big(\mathsf{f}_a,\dots,\mathsf{f}_n\big)\{i\} - \big(\triangle^{\alpha}_{h,o_1}\mathsf{P}_<\big)\big(\mathsf{f}_a,\dots,\mathsf{f}_n\big)\{i\}$ is equal to

$$\begin{aligned} &= T^{o_2}_h\mathsf{P}_<\big(\mathsf{f}_a,\dots,\mathsf{f}_n\big)_i - T^{o_1}_h\mathsf{P}_<\big(\mathsf{f}_a,\dots,\mathsf{f}_n\big)_i \\ &\quad - \sum_{j=a}^{n-1}\sum_{|p|<\sum_{s=a}^{j}\alpha_s} \partial^p_{\star\alpha_{[\![a,j]\!]}}\mathsf{P}_<(\mathsf{f}_a,\dots,\mathsf{f}_j)\,\frac{h^p}{p!} \\ &\qquad\qquad \times\Big\{\big(\triangle^{\alpha}_{h,o_2-p}\mathsf{P}_<\big)\big(\mathsf{f}_{j+1},\dots,\mathsf{f}_n\big)\{i\} - \big(\triangle^{\alpha}_{h,o_1-p}\mathsf{P}_<\big)\big(\mathsf{f}_{j+1},\dots,\mathsf{f}_n\big)\{i\}\Big\}. \end{aligned}$$

From the induction hypothesis the above quantity is equal to

$$\begin{aligned} &\sum_{o_1<|k|<o_2}\partial^k\mathsf{P}_<(\mathsf{f}_a,\dots,\mathsf{f}_n)_i\,\frac{h^k}{k!} - \sum_{j=a}^{n-1}\sum_{|p|<\sum_{s=a}^{j}\alpha_s}\partial^p_{\star\alpha_{[\![a,j]\!]}}\mathsf{P}_<(\mathsf{f}_a,\dots,\mathsf{f}_j)\,\frac{h^p}{p!} \\ &\qquad\qquad \times\sum_{o_1<|\ell|+|p|<o_2}\partial^{\ell}_{\star\alpha_{[\![j+1,n]\!]}}\mathsf{P}_<\big(\mathsf{f}_{j+1},\dots,\mathsf{f}_n\big)\{i\}\,\frac{h^\ell}{\ell!}. \end{aligned}$$

We conclude using (2.4). ▷

For $0\le c\le n-1$ we let

$$\triangle_{yx}\mathsf{P}_<(\mathsf{f}_{c+1},\dots,\mathsf{f}_n) := \big(\triangle^{\alpha}_{y-x,\sum_{j=c+1}^{n}\alpha_j}\mathsf{P}_<\big)\big(\mathsf{f}_{c+1},\dots,\mathsf{f}_n\big)(x),$$

and for $i\ge -1$

$$\triangle_{yx}\mathsf{P}_<(\mathsf{f}_{c+1},\dots,\mathsf{f}_n)\{i\} := \big(\triangle^{\alpha}_{y-x,\sum_{j=c+1}^{n}\alpha_j}\mathsf{P}_<\big)\big(\mathsf{f}_{c+1},\dots,\mathsf{f}_n\big)\{i\}(x).$$

From Lemma 10 we know that for any $o\in(\sum_{j=c+1}^{n}\alpha_j-\delta_0, \sum_{j=c+1}^{n}\alpha_j+\delta_0)$ one has the equality

$$\triangle_{yx}\mathsf{P}_<(\mathsf{f}_{c+1},\dots,\mathsf{f}_n)\{i\} = \big(\triangle^{\alpha}_{y-x,o}\mathsf{P}_<\big)\big(\mathsf{f}_{c+1},\dots,\mathsf{f}_n\big)\{i\}(x).$$

Then, for any o in a neighborhood of $\sum_{j=c+1}^{n}\alpha_j$, the following estimate

$$\big|\triangle_{yx}\mathsf{P}_<(\mathsf{f}_{c+1},\dots,\mathsf{f}_n)\{i\}\big| \lesssim |y-x|^o \prod_{j=c+1}^{n}\|\mathsf{f}_j\|_{\alpha_j}\; 2^{-i(\sum_{j=c+1}^{n}\alpha_j - o)} \tag{2.11}$$

holds as a consequence of (2.8).

The following elementary fact was already used in Hoshino's work [16] and enables us to get the optimal bound on $|\triangle_{yx}\mathsf{P}_<(\mathsf{f}_{c+1},\dots,\mathsf{f}_n)|$. We reproduce its proof in Appendix A.2.

11* – *Lemma. *Assume we are given an* $(\mathbf{R}^d \times \mathbf{R}^d)$*-indexed family of absolutely convergent series* $\big(X_{yx} = \sum_{i\geq -1} X^i_{yx}\big)_{x,y\in\mathbf{R}^d}$*, such that there exists some positive constants* $C > 0$ *and some exponent* $\gamma > 0$ *such that the uniform bound*

$$|X^i_{yx}| \leq C 2^{-i(\gamma-\theta)}\, |y-x|^\theta$$

holds for any θ *in a neighborhood of* γ*. Then we have*

$$|X_{yx}| \lesssim C|y-x|^\gamma,$$

uniformly over $x, y \in \mathbf{R}^d$ *such that* $|y-x| \leq 1$*.*

From (2.11) and Lemma 11 one has then for $|y-x| \leq 1$

$$\big|\triangle_{yx}\mathsf{P}_<(\mathsf{f}_{c+1},\dots,\mathsf{f}_n)\big| \lesssim \Big\{ \prod_{j=c+1}^n \|\mathsf{f}_j\|_{\alpha_j} \Big\} |y-x|^{\sum_{j=c+1}^n \alpha_j}$$

12* – *Proposition. *Pick* $\alpha = (\alpha_1,\dots,\alpha_n) \in (0,+\infty)^n$*. For all* $\big(\mathsf{f}_j \in \mathsf{C}^{\alpha_j}_\circ\big)_{1\leq j\leq n}$ *we have*

$$\begin{aligned}\mathsf{P}_<(\mathsf{f}_1,\dots,\mathsf{f}_n)(y) = &\sum_{|k|<\sum_{j=1}^n \alpha_j} \partial^k_{\star\alpha}\mathsf{P}_<\big(\mathsf{f}_1,\dots,\mathsf{f}_n\big)(x)\frac{(y-x)^k}{k!} \\ &+ \sum_{c=1}^{n-1}\sum_{|k|<\sum_{j=1}^c \alpha_j} \partial^k_{\star\alpha_{\leq c}}\mathsf{P}_<\big(\mathsf{f}_1,\dots,\mathsf{f}_c\big)(x)\;\frac{(y-x)^k}{k!}\triangle_{yx}\mathsf{P}_<(\mathsf{f}_{c+1},\dots,\mathsf{f}_n) \\ &+ \big(\triangle_{yx}\mathsf{P}_<\big)(\mathsf{f}_1,\dots,\mathsf{f}_n),\end{aligned}$$

where

$$\Big|\big(\triangle_{yx}\mathsf{P}_<\big)(\mathsf{f}_{c+1},\dots,\mathsf{f}_n)\Big| \lesssim \Big\{ \prod_{j=c+1}^n \|\mathsf{f}_j\|_{\alpha_j} \Big\}\, |y-x|^{\sum_{j=c+1}^n \alpha_j} \tag{2.12}$$

for all $x, y \in \mathbf{R}^d$ *with* $|y-x| \leq 1$*.*

Proof – First, for $\mathsf{f}_1,\dots,\mathsf{f}_n$ in C^∞ we have from the propositions 9 and 10

$$\begin{aligned}(\triangle_{yx}\mathsf{P}_<)(\mathsf{f}_1,\cdots,\mathsf{f}_n) = \mathsf{P}_<\big(\mathsf{f}_1,\cdots,\mathsf{f}_n\big)(y) &- \sum_{|k|<\theta}\frac{(y-x)^k}{k!}\partial^k\mathsf{P}_<\big(\mathsf{f}_1,\cdots,\mathsf{f}_n\big)(x) \\ &- \sum_{c=1}^n\sum_{|p|<\sum_{j=1}^c \alpha_i} \partial^p_\star\mathsf{P}_<\big(\mathsf{f}_1,\cdots,\mathsf{f}_c\big)(x)\frac{(y-x)^p}{p!}\big(\triangle_{yx}\mathsf{P}_<\big)(\mathsf{f}_{c+1},\cdots,\mathsf{f}_n) \\ &- \sum_{c=1}^n \sum_{\substack{|p|<\sum_{j=1}^c \alpha_j \\ |\ell|>\sum_{j=c+1}^n \alpha_j}} \frac{(y-x)^p}{p!}\,\frac{(y-x)^\ell}{\ell!} \\ &\qquad\qquad \times \partial^p_\star\mathsf{P}_<\big(\mathsf{f}_1,\cdots,\mathsf{f}_c\big)(x)\partial^\ell_\star\mathsf{P}_<\big(\mathsf{f}_{c+1},\cdots,\mathsf{f}_n\big)(x).\end{aligned}$$

Using the recursive relation 2.4 for the $\partial^k_\star$ operators gives the statement of the proposition. We obtain the fact that one can work with $\mathsf{f}_j \in \mathsf{C}^{\alpha_j}_\circ$ rather than with $\mathsf{f}_j \in \mathsf{C}^\infty$ from the inequality (2.12) by an elementary continuity reasoning. ▷

2.4 – Local expansion properties of the $\partial^k_\star\mathsf{P}_<(f_1,\dots,f_n)$. The quantities $\partial^p_\star\mathsf{P}_<(\mathsf{f}_1,\dots,\mathsf{f}_c)$, with $|p| < \sum_{i=1}^c \alpha_i$, appear in Proposition 12 as some coefficients in the local expansion of the simplified paraproduct $\mathsf{P}_<(\mathsf{f}_1,\dots,\mathsf{f}_n)$. These coefficients also have a local expansion property, described in the proposition below. We state it and defer its proof to Appendix A.3 as it is similar to the proof of Proposition 12.

13* – *Proposition. *Assume $|p| < \sum_{i=1}^{c}\alpha_i$ for some $p\in\mathbf{N}^{d_0}$ and $(\alpha_1,\dots,\alpha_n)\in\mathbf{R}^n$. Take $\mathsf{f}_j\in\mathsf{C}_\circ^{\alpha_j}$ for $1\le j\le n$. For all $0\le c\le n-1$ there are some functions $\triangle_{yx}(\partial_\star^p\mathsf{P}_<)(\mathsf{f}_{c+1},\dots,\mathsf{f}_n)$ such that*

$$\Big|\big(\triangle_{yx}(\partial_\star^p\mathsf{P}_<)(\mathsf{f}_{c+1},\dots,\mathsf{f}_n)\Big| \lesssim \left\{\prod_{j=c+1}^{n}\|\mathsf{f}_j\|_{\alpha_j}\right\}|y-x|^{\sum_{j=c+1}^{n}\alpha_j}$$

for all $x,y\in\mathbf{R}^d$ with $|y-x|\le 1$, and we have

$$\begin{aligned}\partial_\star^p\mathsf{P}_<(\mathsf{f}_1,\dots,\mathsf{f}_n)(y) = &\sum_{|k|<\sum_{j=1}^{n}\alpha_j-|p|}\partial_\star^{k+p}\mathsf{P}_<\big(\mathsf{f}_1,\dots,\mathsf{f}_n\big)(x)\,\frac{(y-x)^k}{k!}\\ &+\sum_{c=1}^{n-1}\sum_{|k|<\sum_{j=1}^{c}\alpha_j-|p|}\partial_\star^{k+p}\mathsf{P}_<\big(\mathsf{f}_1,\dots,\mathsf{f}_c\big)(x)\,\frac{(y-x)^k}{k!}\,\triangle_{yx}(\partial_\star^p\mathsf{P}_<)(\mathsf{f}_{c+1},\dots,\mathsf{f}_n)\\ &+\triangle_{yx}(\partial_\star^p\mathsf{P}_<)(\mathsf{f}_1,\dots,\mathsf{f}_n).\end{aligned}\tag{2.13}$$

3 – The regularity structure of iterated paraproducts

We fix $\alpha\in\mathbf{R}^n$ in this section. We introduced in Section 1.2 the spaces T and T^+ of symbols of the regularity structure that we will associate to some iterated paraproducts. The vector space T is spanned by the basis symbols

$$\mathcal{B} := \Big\{[\![a,b]\!]_{\mathsf{j}}\,X^p\Big\}_{1\le a<b\le n,\,\mathsf{j}\in\mathcal{P}_{b-a}(\ell),\,\ell\in\mathbf{N}^{d_0},\,p\in\mathbf{N}^{d_0}} \cup \big\{X^p\big\}_{p\in\mathbf{N}^{d_0}}$$

and the algebra is generated by the basis symbols

$$\mathcal{B}^+ := \Big\{[\![a,b]\!]_{\mathsf{j}}^{\mathfrak{k}}\Big\}_{\textsf{\textbf{condition}}(a,b,\mathfrak{k},\mathsf{j})} \cup \big\{X^{\varepsilon_i}\big\}_{1\le i\le d},$$

where one says that $(a,b,\mathfrak{k},\mathsf{j})$ satisfies **condition**$(a,b,\mathfrak{k},\mathsf{j})$ if $1\le a<b\le n$, $\mathfrak{k}=(k_a,\dots,k_b)\in\mathcal{P}_{b-a+1}(k)$ for some $k\in\mathbf{N}^{d_0}$, and $\mathsf{j}\in\mathcal{P}_{b-a}(\ell)$ for some $\ell\in\mathbf{N}^{d_0}$, and we have

$$\max(|k|,|\ell|) < \sum_{1\le j\le n}|\alpha_j|$$

and

$$\big|[\![a,b]\!]_{\mathsf{j}}^{\mathfrak{k}}\big|_\alpha > 0.$$

Note that $\mathcal{B}^+$ generates T^+ as an algebra. A linear basis of the vector space T^+ is given by the monomials $X^p(p\in\mathbf{N}^{d_0})$ and the $[\![a,b]\!]_{\mathsf{j}}^{\mathfrak{k}}X^p$ for $[\![a,b]\!]_{\mathsf{j}}^{\mathfrak{k}}\in\mathcal{B}^+$ and $p\in\mathbf{N}^{d_0}$. We call below this basis the *canonical linear basis of* T^+. We use below the notation

$$M\big((\sigma_1\otimes\sigma_2),(X^{m_1}\otimes X^{m_2})\big) := (\sigma_1X^{m_1})\otimes(\sigma_2X^{m_2}).$$

We introduce in this section some splitting maps $\Delta: T\to T\otimes T^+$ and $\Delta^+: T^+\to T\otimes T^+$ and prove in Proposition 14 that $((T,\Delta),(T^+,\Delta^+))$ is indeed a concrete regularity structure. We refer the reader to Appendix A.1 for some basics on the subject.

For $\tau=[\![a,b]\!]_{\mathsf{j}}\in\mathcal{B}$, with $\mathsf{j}=(\ell_a,\dots,\ell_b)$, we define a subset of T^+ setting

$$\oplus(\tau) = \Big\{[\![a,c]\!]_{\mathsf{j}_{<c}}^{\mathfrak{p}}\in\mathcal{B}^+\,;\ a\le c\le b,\ \ell_c=0,\ \mathfrak{p}\in\mathcal{P}_{c-a+1}(p),\ p\in\mathbf{N}^{d_0}\Big\}\cup\{\mathbf{1}^+\},$$

we recall that $\mathsf{j}_{<c}=(\ell_a,\dots,\ell_{c-1})$. For $\tau=[\![a,b]\!]_{\mathsf{j}}\in\mathcal{B}$ and $\sigma=[\![a,c]\!]_{\mathsf{j}_{<c}}^{\mathfrak{p}}\in\oplus(\tau)$ we define $(\tau\backslash\mathbf{1}^+)=\tau$ and if $c\le b-1$ we set

$$(\tau\backslash\sigma) := \sum_{p=p_1+p_2}\ \sum_{\mathfrak{p}_1\in\mathcal{P}_{b-c}(p_1)}\frac{p!}{\mathfrak{p}_1!\,p_2!\,\mathfrak{p}!}\,[\![c+1,b]\!]_{\mathsf{j}_{>c-a+1}+\mathfrak{p}_1}X^{p_2},$$

and for $c=b$ we set $(\tau\backslash\sigma) := \frac{1}{\mathfrak{p}!}\,X^p$. For $p\in\mathbf{N}^{d_0}$ we set

$$\Delta(X^p) = \Delta^+(X^p) := \sum_{p_1+p_2=p} \binom{p}{p_1} X^{p_1} \otimes X^{p_2}.$$

We define the map Δ on T by setting

$$\Delta\big([\![a,b]\!]_{\mathfrak{j}} X^p\big) = M\big(\Delta([\![a,b]\!]_{\mathfrak{j}})\,,\,\Delta(X^p)\big)$$

and for $\tau = [\![a,b]\!]_{\mathfrak{j}} \in \mathcal{B}$

$$\Delta(\tau) = \sum_{\sigma\in\oplus(\tau)} (\tau\backslash\sigma)\otimes\sigma.$$

For $\mu = [\![a,b]\!]_{\mathfrak{j}}^{\mathfrak{k}} \in \mathcal{B}^+$ we set

$$\oplus(\mu) = \Big\{ [\![a,c]\!]_{\mathfrak{j}_{<c}}^{\mathfrak{k}_{\leq c-a+1}+\mathfrak{p}} \in \mathcal{B}^+\,;\, a\leq c\leq b, \ell_c = 0,\, \mathfrak{p}\in\mathcal{P}_{c-a+1}(p),\, p\in \mathbf{N}^{d_0}\Big\} \cup \{\mathbf{1}^+\}.$$

For $\mu = [\![a,b]\!]_{\mathfrak{j}}^{\mathfrak{k}} \in \mathcal{B}^+$ and $\nu = [\![a,c]\!]_{\mathfrak{j}_{<c}}^{\mathfrak{k}_{\leq c-a+1}+\mathfrak{p}} \in \oplus(\mu)$ with $\mathfrak{p}\in\mathcal{P}_{c-a+1}(p)$, we define for $c\leq b-1$

$$(\mu\backslash\nu) := \sum_{p=p_1+p_2} \sum_{\mathfrak{p}_1\in\mathcal{P}_{b-c}(p_1)} \frac{p!}{\mathfrak{p}_1!\,p_2!\,\mathfrak{p}!}\, [\![c+1,b]\!]_{\mathfrak{j}_{>c-a+1}+\mathfrak{p}_1}^{\mathfrak{k}_{>c-a+1}} X^{p_2}, \tag{3.1}$$

and for $c = b$ set $(\tau\backslash\sigma) := \frac{1}{\mathfrak{p}!}X^p$. All the terms in this sum have the same homogeneity $|[\![c+1,b]\!]_{\mathfrak{j}_{>c}}^{\mathfrak{k}_{>c-a+1}}|_\alpha + |p|$. We define the map Δ^+ on T^+ by setting

$$\Delta^+\big([\![a,b]\!]_{\mathfrak{j}}^{\mathfrak{k}} X^p\big) = M\big(\Delta^+([\![a,b]\!]_{\mathfrak{j}}^{\mathfrak{k}})\,,\,\Delta^+(X^p)\big)$$

and for $\mu = [\![a,b]\!]_{\mathfrak{j}}^{\mathfrak{k}} \in \mathcal{B}^+$

$$\Delta^+(\mu) = \sum_{\substack{\nu\in\oplus(\mu)\\ \langle(\mu\backslash\nu)\rangle_\alpha>0}} (\mu\backslash\nu)\otimes\nu.$$

With the notation of (3.1), the condition $\langle(\mu\backslash\nu)\rangle_\alpha > 0$ means that we only consider here those $\nu\in\oplus(\mu)$ such that $|[\![c+1,b]\!]_{\mathfrak{j}_{>c}}^{\mathfrak{k}_{>c-a+1}}|_\alpha > 0$. The regularity structures introduced by Bruned, Hairer & Zambotti in [8] are also built from some deformations of some simple combinatorial structure – the Connes-Kreimer structure on trees therein, the deconcatenation splitting here. This picture of a deformed structure was investigated systematically in Bruned & Manchon's work [9].

***14* – Proposition.** *The space $\big((T,\Delta),(T^+,\Delta^+)\big)$ is a concrete regularity structure.*

In the proof of this proposition we use the following generalisation of the Vandermonde identity, which states that for any integer $i\geq 1$, for any p,q,r in $\mathbf{N}^{d_0}$ such that $p+q=r$, and any $\mathfrak{r}\in\mathcal{P}_i(r)$, one has

$$\sum_{\substack{\mathfrak{p}\in\mathcal{P}_i(p),\mathfrak{q}\in\mathcal{P}_i(q)\\ \mathfrak{p}+\mathfrak{q}=\mathfrak{r}}} \frac{\mathfrak{r}!}{\mathfrak{p}!\,\mathfrak{q}!} = \frac{r!}{p!\,q!}. \tag{3.2}$$

Proof – We prove here that we have the comodule identity

$$(\Delta\otimes\mathrm{Id})\Delta = (\mathrm{Id}\otimes\Delta^+)\Delta.$$

The proof of the coassociativity identity

$$(\Delta^+\otimes\mathrm{Id})\Delta^+ = (\mathrm{Id}\otimes\Delta^+)\Delta^+$$

is almost identical and left to the reader. We also let the reader check the other conditions involved in the definition of a concrete regularity structure spelled out in Definition 23 in Appendix A.1.
It suffices to prove the comodule identity for $\tau = [\![a,b]\!]_{\mathfrak{j}} \in \mathcal{B}$ with $\mathfrak{j} = (\ell_a,\dots,\ell_b)$. To lighten the computations we use the convention $[\![c+1,c]\!] = \mathbf{1}^{(+)}$ for any $a\leq c\leq b$. We have

$$\Delta\big([\![a,b]\!]_{\mathfrak{j}}\big) = \sum_{c,\mathfrak{k}_1,k_2,\mathfrak{k}'} \frac{k!}{\mathfrak{k}'!\,\mathfrak{k}_1!\,k_2!} \Big([\![c+1,b]\!]_{\mathfrak{j}_{>c}+\mathfrak{k}_1} X^{k_2}\Big)\otimes\big([\![a,c]\!]_{\mathfrak{j}_{<c}}^{\mathfrak{k}'}\big)$$

where the sum runs over the $a \le c \le b$ such that $\ell_c = 0$ and the multi-indices $k = k_1 + k_2$ such that $\big|[\![a,c]\!]_{\mathfrak{j}_{<c}}^{\mathfrak{k}}\big|_\alpha > 0$, over $\mathfrak{k}_1 \in \mathcal{P}_{b-c}(k_1)$ and $\mathfrak{k}' \in \mathcal{P}_{c-a+1}(k)$. Then $(\Delta \otimes \mathrm{Id})\Delta(\tau)$ is equal to

$$\sum_{\substack{c,\mathfrak{k}_1,k_2,\mathfrak{k}'\\ d,\mathfrak{p}_1,p_2,\mathfrak{p}'}} \frac{k!\,p!}{\mathfrak{k}'!\,\mathfrak{k}_1!\,k_{21}!\,k_{22}!\,\mathfrak{p}'!\,\mathfrak{p}_1!\,p_2!}\,[\![d+1,b]\!]_{\mathfrak{j}_{>d}+(\mathfrak{k}_1)_{>b-d}+\mathfrak{p}_1} X^{k_{21}+p_2} \otimes [\![c+1,d]\!]^{\mathfrak{p}'}_{\mathfrak{j}_{[\![c+1,d-1]\!]}+(\mathfrak{k}_1)_{\le b-d}} X^{k_{22}} \otimes [\![a,c]\!]^{\mathfrak{k}'}_{\mathfrak{j}_{<c}}.$$

where the sum runs over $1 < c \le d \le b$ such that ℓ_c and $\ell_d + (k_1)_{d-c}$ are null, and the multi-indices $k = k_1 + k_2$, $k_2 = k_{21} + k_{22}$, $p = p_1 + p_2$ such that

$$\big|[\![a,c]\!]^{\mathfrak{k}}_{\mathfrak{j}_{<c}}\big|_\alpha > 0$$

and

$$\big|[\![c+1,d]\!]^{\mathfrak{p}'}_{\mathfrak{j}_{[\![c+1,d-1]\!]}+(\mathfrak{k}_1)_{\le d-c}}\big|_\alpha > 0$$

and $\mathfrak{p}_1 \in \mathcal{P}_{b-d}(p_1)$, $\mathfrak{p}' \in \mathcal{P}_{d-c}(p)$. On the other hand $(\mathrm{Id} \otimes \Delta^+)\Delta(\tau)$ is equal to

$$\sum_{\substack{c,\mathfrak{k}_1,k_2,\mathfrak{k}'\\ d,\mathfrak{p}_1,p_2,\mathfrak{p}'}} \frac{k!\,p!}{\mathfrak{k}'!\,\mathfrak{k}_1!\,k_2!\,\mathfrak{p}'!\,\mathfrak{p}_1!\,p_2!}\,[\![c+1,b]\!]_{\mathfrak{j}_{>c}+\mathfrak{k}_1} X^{k_2} \otimes [\![e+1,c]\!]^{\mathfrak{k}_{>e-a+1}}_{\mathfrak{j}_{[\![e+1,c-1]\!]}+\mathfrak{p}_1} X^{p_2} \otimes [\![a,e]\!]^{\mathfrak{k}_{\le e-a+1}+\mathfrak{p}'}_{\mathfrak{j}_{<e}},$$

where the sum runs over $a < e \le c \le b$ such that $\ell_c = \ell_e = 0$ and multi-indices $p = p_1 + p_2$ such that $\mathfrak{p}' \in \mathcal{P}_{e-a+1}(p)$ and

$$\big|[\![a,e]\!]^{\mathfrak{k}_{\le e-a+1}+\mathfrak{p}'}_{\mathfrak{j}_{<e}}\big|_\alpha > 0$$

and

$$\big|[\![e+1,c]\!]^{\mathfrak{k}_{>t}}_{\mathfrak{j}_{[\![e+1,c-1]\!]}+\mathfrak{p}_1}\big|_\alpha > 0.$$

Both sums take the form

$$\sum_{\substack{c,\mathfrak{k}_1,k_2,\mathfrak{q}\\ d,\mathfrak{p}_1,p_2,\mathfrak{q}'}} C^{\mathfrak{k}_1,k_2,\mathfrak{q}}_{\mathfrak{p}_1,p_2,\mathfrak{q}'} [\![d+1,b]\!]_{\mathfrak{j}_{>d}+\mathfrak{k}_1} X^{k_2} \otimes [\![c+1;d]\!]^{\mathfrak{q}}_{\mathfrak{j}_{[\![c+1,d-1]\!]}+\mathfrak{p}_1} X^{p_2} \otimes [\![a,c]\!]^{\mathfrak{q}'}_{\mathfrak{j}_{<c}},$$

where the sum runs over $a \le c \le d \le b$ such that $\ell_c, \ell_d \neq 0$, over multi-indices and tuples of multi-indices $\mathfrak{k}_1, k_2, \mathfrak{q}, \mathfrak{p}_1, p_2, \mathfrak{q}'$ such that the first two terms in each tensor products are in T^+ and $q + q' = k_1 + k_2 + p_1 + p_2$ and $q \le k_1 + k_2$.
We check that the constants $C^{\mathfrak{k}_1,k_2,q}_{\mathfrak{p}_1,p_2,q'}$ coincide in both expressions using the Vandermonde identity (3.2). Both are equal to

$$C^{\mathfrak{k}_1,k_2,q}_{\mathfrak{p}_1,p_2,q'} = \frac{1}{\mathfrak{k}_1!\,k_2!\,\mathfrak{p}_1!\,p_2!\,\mathfrak{q}!\,\mathfrak{q}'!}\,\frac{k!\,q'!}{q!},$$

which concludes the proof of the statement. ▷

An iterated paraproduct $\mathsf{P}_{\mathfrak{j}}(f_1,\dots,f_n)$ can be represented pictorially by a linear tree where each vertex corresponds to a distribution f_j and the entries of $\mathfrak{j}$ appear as decorations on the edges, idem for generalized corrector $\widetilde{\mathsf{P}}^{\mathfrak{k}}_{\mathfrak{j}}(f_1,\dots,f_n)$ with an additional decoration on the vertices corresponding to the entries of $\mathfrak{k}$. The coproduct defined here bears resemblance with to the one in regularity structures on decorated trees, which is also constructed via admissible cuts. While both frameworks involve extracting branches of positive homogeneity, the role of decorations differs. In regularity structures derivations are attached to edges and polynomials to vertices, in our setting derivations corresponds to the $\mathfrak{k}$ and are attached to vertices, whereas polynomials are encoded in the edge decorations through $\mathfrak{j}$.

We note that Hoshino was the first to investigate in [15] the algebraic structure behind the iterated paraproducts, in a restricted setting compared to the present general setting.

For $\tau \in T$ one can re-index the sum defining $\Delta(\tau)$ by its different components τ_1 on the canonical basis $\mathcal{B}$ of the T factor in $T \otimes T^+$ and write

$$\Delta(\tau) =: \sum_{\tau_1 \leq \tau} \tau_1 \otimes (\tau/\tau_1). \tag{3.3}$$

(This identity defines the notations $\tau_1 \leq \tau$ and (τ/τ_1).) Below we write $\tau_1 < \tau$ to mean that τ_1 appears in this decomposition and $\tau_1 \neq \tau$. Similarly we can rewrite

$$\Delta^+(\mu) =: \sum_{\mu_1 \leq^+ \mu} \mu_1 \otimes (\mu/^+\mu_1).$$

(This identity defines the notations $\mu_1 \leq^+ \mu$ and $(\mu/^+\mu_1)$.) Below we write $\mu_1 <^+ \mu$ to mean that μ_1 appears in this decomposition and $\mu_1 \neq \mu$.

4 – Local expansion properties of iterated paraproducts

We prove Theorem 1 in this section. This proof involves the local expansion properties of the operators $\mathsf{P}_<$ and the regularity structure from Proposition 14. The core of the proof relates these two ingredients and rests on a representation

$$\mathsf{P}_{\mathrm{j}}(f_a, \dots, f_b) = \sum_{c \geq 0} \sum_{\tau_1, \dots, \tau_c} \mathsf{P}_<\Big([\tau/\tau_1]^{\mathfrak{f}}, \dots, [\tau_c]^{\mathfrak{f}}\Big)$$

of the P_{j} operators in terms of the simplified iterated paraproduct operators $\mathsf{P}_<$ and some functions $[\sigma]^{\mathfrak{f}}$ that we build from the tuple $\mathfrak{f} = (f_a, \dots, f_b)$. The symbol τ is here equal to $[\![a, b]\!]_{\mathrm{j}} \in \mathcal{B}$ and the notation τ/τ_1 is the notation from (3.3). Once proved such a representation formula, one can infer the local expansion properties of $\mathsf{P}_{\mathrm{j}}(f_a, \dots, f_b)$ from the local expansion properties of the operators $\mathsf{P}_<$ obtained in Section 2 and Section 2.4.

We describe in Section 4.1 the generic construction of some bracket maps $[\sigma]$ from some a priori given pair of maps $(\mathsf{\Pi}, \mathsf{g})$ of a particular type. We construct such a pair of maps in Section 4.2 from a fixed tuple $\mathfrak{f} = (f_1, \dots, f_n)$ of distributions. The actual proof of Theorem 1 occupies all of Section 4.3. The inductive mechanics of this proof is detailed at the begining of this section.

4.1 – Building blocks for a representation of P in terms of $\mathsf{P}_<$.

Recall from Appendix A.1 the basic notions and notations on regularity structures. We work in this section with the concrete regularity structure of Proposition 14. Let $\mathsf{\Pi}$ be a linear map from T into $\mathcal{D}'(\mathbf{R}^{d_0})$ and g be a map from $\mathbf{R}^{d_0}$ into the set of characters on the algebra T^+. For any $x \in \mathbf{R}^{d_0}$ we denote by g_x^{-1} the convolution inverse of the character g_x, uniquely characterized by the property $(\mathsf{g}_x \otimes \mathsf{g}_x^{-1})\Delta^+ = \mathbf{1}'_+\mathbf{1}_+$, where $\mathbf{1}'_+$ stands for the dual vector of the vector $\mathbf{1}_+$ in the canonical linear basis of T^+. We define

$$\mathsf{\Pi}_x := (\mathsf{\Pi} \otimes \mathsf{g}_x^{-1})\Delta$$

for any $x \in \mathbf{R}^{d_0}$. We have

$$\mathsf{\Pi}(\tau) = \sum_{\tau_1 \leq \tau} \mathsf{g}_x(\tau/\tau_1)\, \mathsf{\Pi}_x(\tau_1),$$

that is

$$\mathsf{\Pi}_x(\tau) = \mathsf{\Pi}(\tau) - \sum_{\tau_1 < \tau} \mathsf{g}_x(\tau/\tau_1)\, \mathsf{\Pi}_x(\tau_1).$$

Iterating we obtain the formula

$$\mathsf{\Pi}_x(\tau) = \mathsf{\Pi}(\tau) - \sum_{e \geq 1} (-1)^{e-1} \sum_{\tau_e < \cdots < \tau_1 < \tau} \mathsf{g}_x(\tau/\tau_1) \cdots \mathsf{g}_x(\tau_{q-1}/\tau_e)\, \mathsf{\Pi}(\tau_e), \tag{4.1}$$

where the sum over e is finite as the sets A and A^+ that index the homogeneities of the regularity structure $((T,\Delta),(T^+,\Delta^+))$ are locally finite and bounded from below. Likewise for $\tau/\rho \in T^+$

$$\begin{aligned} \mathsf{g}_{yx}\big(\tau/\rho\big) &= \mathsf{g}_y\big(\tau/\rho\big) - \mathsf{g}_x\big(\tau/\rho\big) \\ &\quad - \sum_{e\geq 1}(-1)^{e-1} \sum_{\rho<\tau_e<\cdots<\tau_1<\tau} \mathsf{g}_x\big(\tau/\tau_1\big)\cdots \mathsf{g}_x\big(\tau_{e-1}/\tau_e\big)\Big(\mathsf{g}_y\big(\tau_e/\tau_1\big) - \mathsf{g}_x\big(\tau_e/\rho\big)\Big). \end{aligned} \tag{4.2}$$

Recall the Δ_i stand for the Littlewood-Paley projectors and $\Delta_{<i-1} = \sum_{-1\leq j<i-1}\Delta_j$. For $\tau\in T$ we define an element $[\tau] = ([\tau]_i)_{i\geq -1}$ of $\mathsf{C}^{-\infty}$ setting

$$[\tau]_i := \Delta_i\big(\mathsf{\Pi}(\tau)\big) - \sum_{e\geq 1}(-1)^{e-1} \sum_{\tau_e<\cdots<\tau_1<\tau} \Delta_{<i-1}\big(\mathsf{g}\big(\tau/\tau_1\big)\big)\cdots\Delta_{<i-1}\big(\mathsf{g}\big(\tau_{q-1}/\tau_e\big)\big)\Delta_i\big(\mathsf{\Pi}\big(\tau_e\big)\big).$$

Likewise, for $\tau/\rho\in T^+$, we define an element $[\tau/\rho] = ([\tau/\rho]_i)_{i\geq -1}$ of $\mathsf{C}^{-\infty}$ setting

$$\begin{aligned} [\tau/\rho]_i &:= \Delta_i\big(\mathsf{g}(\tau/\rho)\big) \\ &\quad - \sum_{e\geq 1}(-1)^{e-1} \sum_{\rho<\tau_e<\cdots<\tau_1<\tau} \Delta_{<i-1}\big(\mathsf{g}\big(\tau/\tau_1\big)\big)\cdots\Delta_{<i-1}\big(\mathsf{g}\big(\tau_{q-1}/\tau_e\big)\big)\Delta_i\big(\mathsf{g}\big(\tau_e/\rho\big)\big). \end{aligned}$$

We now introduce an appropriate notion of size of the pair $(\!|\mathsf{\Pi},\mathsf{g}|\!)$ to quantify the regularity of the $[\tau]$ and $[\tau/\rho]$. For any integer n_0 define $\mathcal{F}_{n_0}$ as the set of C^{n_0} functions φ supported in the unit ball of $\mathbf{R}^{d_0}$ and such that $\|\varphi\|_{C^{n_0}}\leq 1$. For a real-valued function φ on $\mathbf{R}^{d_0}$, $x\in\mathbf{R}^{d_0}$ and $\varepsilon>0$ we define

$$\varphi_x^\varepsilon(y) := \varepsilon^{-d_0}\varphi\big(\varepsilon^{-1}(y-x)\big).$$

To define the size $(\!|\mathsf{\Pi},\mathsf{g}|\!)$ of $(\mathsf{\Pi},\mathsf{g})$ we first for $\tau\in T_{|\tau|}$ and $\nu\in T^+_{|\nu|}$

$$\begin{aligned} \|\tau\|_{(\mathsf{\Pi},\mathsf{g})} &:= \sup_{x\in\mathbf{R}^{d_0}}\ \sup_{\varphi\in\mathcal{F}_{n_0}}\ \sup_{\varepsilon\in(0,1]} \varepsilon^{-|\tau|}\big|\langle\mathsf{\Pi}_x\tau,\varphi_x^\varepsilon\rangle\big|, \\ \|\nu\|_{(\mathsf{\Pi},\mathsf{g})} &:= \sup_{x,y\in\mathbf{R}^{d_0}} \frac{\mathsf{g}_{yx}(\nu)}{|y-x|^{|\nu|}}, \end{aligned}$$

and recursively for $\tau\in\mathcal{B}$ and $\mu\in\mathcal{B}^+$

$$\begin{aligned} \|\tau\|^*_{(\mathsf{\Pi},\mathsf{g})} &:= \max\Big(\|\tau\|_{(\mathsf{\Pi},\mathsf{g})},\ \max_{\sigma<\tau}\|\tau/\sigma\|_{(\mathsf{\Pi},\mathsf{g})}\|\sigma\|^*_{(\mathsf{\Pi},\mathsf{g})}\Big) \\ \|\mu\|^*_{(\mathsf{\Pi},\mathsf{g})} &:= \max\Big(\|\mu\|_{(\mathsf{\Pi},\mathsf{g})},\ \max_{\nu<^+\mu}\|\mu/\nu\|_{(\mathsf{\Pi},\mathsf{g})}\|\nu\|^*_{(\mathsf{\Pi},\mathsf{g})}\Big). \end{aligned}$$

We then set

$$(\!|\mathsf{\Pi},\mathsf{g}|\!) := \max_{\tau\in\mathcal{B},\mu\in\mathcal{B}^+}\Big(\|\tau\|^*_{(\mathsf{\Pi},\mathsf{g})}\,,\ \|\mu\|_{(\mathsf{\Pi},\mathsf{g})}\Big).$$

15 – Proposition. *For any $\tau\in T_{|\tau|}$ and $\tau/\rho\in T^+_{|\tau/\rho|}$ we have*

$$\|[\tau]\|_{|\tau|} + \|[\tau/\rho]\|_{|\tau/\rho|} \lesssim (\!|\mathsf{\Pi},\mathsf{g}|\!),$$

so $[\tau]\in\mathsf{C}^{|\tau|}$ and $[\tau/\rho]\in\mathsf{C}^{|\tau/\rho|}$ if $(\!|\mathsf{\Pi},\mathsf{g}|\!)<\infty$.

The proof of this statement uses the following result stated in Proposition 8 of Bailleul & Hoshino's work [5]. We denote below by $K_j(x-y)$ the translation-invariant kernel of the Littlewood-Paley projector Δ_j and set

$$K_{<i-1} := \sum_{-1\leq j<i-1} K_j.$$

16 – Lemma. *Let $F=(F_x)_{x\in\mathbf{R}^{d_0}}$ be a family of distributions on $\mathbf{R}^{d_0}$ indexed by $\mathbf{R}^{d_0}$. Set*

$$(Q_iF)(z) := \int K_{<i-1}(z-x)F_x\big(K_i(z-\cdot)\big)dx$$

and assume that

$$\|Q_iF\|_\infty \leq C_F 2^{-ir_1}$$

for some positive constant C_F and $r_1 \in \mathbf{R}$. Let G be a function on $(\mathbf{R}^{d_0})^2$ such that we have

$$|F(x,y)| \leq C_G |y-x|^{r_2}$$

for all x, y, for some exponent $r_2 > 0$ and some positive constant C_G. Set

$$\big(Q_i^+ F\big)(z) := \iint K_{<i-1}(z-x) K_{<i-1}(z-y) F(x,y)\, dxdy.$$

Then $QF = (Q_i F)_{i \geq -1} \in \mathsf{C}^{r_1}$ and $Q^+ G = (Q_i^+ G)_{i \geq -1} \in \mathsf{C}^{r_2}$ with

$$\|QF\|_{r_1} + \big\|Q^+ G\big\|_{r_2} \lesssim C_F + C_G.$$

Proof of Proposition 15 – We proceed by induction. For $\tau \in T_{|\tau|}$ and $\tau/\rho \in T^+_{|\tau/\rho|}$ we set $F_{\tau x} = \mathsf{\Pi}_x \tau$ and $G_{\tau/\rho}(x,y) = \mathsf{g}_{yx}(\tau/\rho)$, for all $x, y \in \mathbf{R}^{d_0}$. Writing

$$\mathsf{\Pi}_x \tau = \mathsf{\Pi}_z \tau + \sum_{\sigma < \tau} \mathsf{g}_{zx}(\tau/\sigma)\, \mathsf{\Pi}_z \sigma$$

we see that

$$\big(Q_i F_\tau\big)(z) = \int K_{<i-1}(z-x)(\mathsf{\Pi}_z \tau)\big(K_i(z-\cdot)\big) dx + \sum_{\sigma<\tau} \int K_{<i-1}(z-x)\, \mathsf{g}_{zx}(\tau/\sigma)\, (\mathsf{\Pi}_z \sigma)\big(K_i(z-\cdot)\big) dx$$

with

$$\big|(\mathsf{\Pi}_z \tau)\big(K_i(z-\cdot)\big)\big| \lesssim 2^{-i|\tau|} \|\tau\|_{(\mathsf{\Pi},\mathsf{g})}$$

uniformly in z, with a similar estimate with σ in place of τ, and

$$\int \big|K_{<i-1}(z-x)\, \mathsf{g}_{zx}(\tau/\sigma)\big| dx \lesssim 2^{i|\tau/\sigma|} \|\tau/\sigma\|_{(\mathsf{\Pi},\mathsf{g})}.$$

It follows that

$$\|Q_i F_\tau\|_\infty \lesssim 2^{-i|\tau|} \max\big\{\|\tau/\sigma\|_{(\mathsf{\Pi},\mathsf{g})}\,;\, \sigma \leq \tau\big\},$$

so we get from Lemma 16 that $QF_\tau \in \mathsf{C}^{|\tau|}$ with $\|QF_\tau\|_{|\tau|} \lesssim \max\big\{\|\tau/\sigma\|_{(\mathsf{\Pi},\mathsf{g})}\,;\, \sigma \leq \tau\big\}$. Note that

$$Q_i F_\tau = \Delta_i(\mathsf{\Pi}\tau) - \sum_{e \geq 1} (-1)^{e-1} \sum_{\sigma_e < \cdots < \sigma_1 < \tau} \Delta_{i-1}\Big(\mathsf{g}(\tau/\sigma_1) \cdots \mathsf{g}(\sigma_{e-1}/\sigma_e)\Big)\, \Delta_i(\mathsf{\Pi}\sigma_e).$$

On the other hand one has directly from Lemma 16 that $Q^+ G_{\tau/\rho} \in \mathsf{C}^{|\tau/\rho|}$ with norm bounded above by a constant multiple of $\|\tau/\rho\|_{(\mathsf{\Pi},\mathsf{g})}$. We actually have from (4.2) the following formula for

$$\begin{aligned}
\big(Q^+ G_{\tau/\rho}\big)_i = \Delta_{<i-1}\big(\mathsf{g}(\tau/\sigma)\big) \\
- \sum_{e \geq 1} (-1)^{e-1} \sum_{\sigma < \sigma_e < \cdots < \sigma_1 < \tau} \Big\{ \Delta_{<i-1}\Big(\mathsf{g}(\tau/\sigma_1) \cdots \mathsf{g}(\sigma_{e-1}/\sigma_e)\Big)\, \Delta_{<i-1}\big(\mathsf{g}(\sigma_e/\sigma)\big) \\
- \Delta_{<i-1}\Big(\mathsf{g}(\tau/\sigma_1) \cdots \mathsf{g}(\sigma_{e-1}/\sigma_e)\, \mathsf{g}(\sigma_e/\sigma)\Big) \Big\}
\end{aligned}$$

It follows from induction that $\big((Q^+ G_{\tau/\rho})_i [\rho]_i\big)_{i \geq -1}$ defines an element of $\mathsf{C}^{|\tau|}$ with norm bounded by a constant multiple of $(\!|\mathsf{\Pi}, \mathsf{g}|\!)$. The conclusion of Proposition 15 follows after we check that

$$[\tau]_i = Q_i F_\tau + \sum_{\sigma < \tau} \big(Q_i^+ G_{\tau/\sigma}\big)[\sigma]_i. \tag{4.3}$$

To see that one has this identity we notice that for any $\sigma < \tau$ one has

$$\begin{aligned}
\big(Q_i^+ G_{\tau/\sigma}\big)[\sigma]_i = \sum_{\substack{e_1, e_2 \geq 0 \\ \sigma_{e_2} < \cdots < \sigma < \nu_{e_1} < \cdots < \tau}} (-1)^{e_1+e_2} \Delta_{<i-1}\Big(\mathsf{g}(\tau/\nu_1) \cdots \mathsf{g}(\nu_{e_1-1}/\nu_{e_1})\big) \Delta_{<i-1}\big(\mathsf{g}(\nu_{e_1}/\sigma)\Big) \\
\times \Delta_{<i-1}\big(\mathsf{g}(\sigma/\sigma_1)\big) \cdots \Delta_{<i-1}\big(\mathsf{g}(\sigma_{e_2-1}/\sigma_{e_2}) \Delta_i\big(\mathsf{\Pi}(\sigma_{e_2})\big)
\end{aligned}$$

$$- \sum_{\substack{e_1,e_2\geq 0\\ \sigma_{e_2}<\cdots<\sigma<\nu_{e_1}<\cdots<\tau}} (-1)^{e_1+e_2}\Delta_{<i-1}\big(\mathsf{g}(\tau/\nu_1)\cdots\mathsf{g}(\nu_{e_1-1}/\nu_{e_1})\mathsf{g}(\nu_{e_1}/\sigma)\big)$$
$$\times\, \Delta_{<i-1}\big(\mathsf{g}(\sigma/\sigma_1)\big)\cdots\Delta_{<i-1}\big(\mathsf{g}(\sigma_{e_2-1}/\sigma_{e_2})\big)\Delta_i\big(\Pi(\sigma_{e_2})\big),$$

so summing over $\sigma<\tau$ one recognizes a telescopic sum which simplifies indeed to (4.3). ▷

4.2 – A representation formula. We fix here a tuple $\mathfrak{f}=(f_1,\dots,f_n)$ of smooth functions and $\alpha=(\alpha_1,\dots,\alpha_n)\in\mathbf{R}^n$. In §1 we associate to $\mathfrak{f}$ and α a pair $(\Pi^{\mathfrak{f}},\mathsf{g}^{\mathfrak{f}})$ of maps as in Section 4.1, with associated bracket functions $[\cdot]^{\mathfrak{f}}$. Proposition 19 in §2 is the main result of this section. In its simplest form, with $\tau=[\![a,b]\!]_{\mathrm{j}}$, it tells us that

$$\mathsf{P}_\ell(f_a,\dots,f_b)=\sum_{e\geq 0}\sum_{\sigma_e\prec\cdots\prec\sigma_1\prec\tau}\mathsf{P}_<\big([\tau/\sigma_1]^{\mathfrak{f}},\dots,[\sigma_e]^{\mathfrak{f}}\big)$$

can be represented as a sum of simplified iterated paraproducts. It also provides a similar representation formula for $\widetilde{\mathsf{P}}_{\mathrm{j}}^{[\![a,b]\!]-|\mathfrak{k}|}(\partial^{k_a}f_a,\dots,\partial^{k_b}f_b)$.

To make everything plain recall from Section 2.1 the following notational point. For $\mathsf{h}_1,\dots,\mathsf{h}_e$ in the sequence space $\mathsf{C}^{-\infty}$ we first define $\mathsf{P}_<(h_1,\dots,h_e)_i$ by induction, for all $i\geq -1$, and then set $\mathsf{P}_<(h_1,\dots,\mathsf{h}_e)=\sum_{i\geq -1}\mathsf{P}_<(h_1,\dots,h_e)_i$. The term $\mathsf{P}_<\big([\tau/\sigma_1]^{\mathfrak{f}},\dots,[\sigma_e]^{\mathfrak{f}}\big)$ above has that meaning.

§1. A pair of maps $(\Pi^{\mathfrak{f}},\mathsf{g}^{\mathfrak{f}})$ *associated to* $\mathfrak{f}$. We associate to $\mathfrak{f}$ the pair of maps

$$\mathsf{M}^{\mathfrak{f},\alpha}=\mathsf{M}^{\mathfrak{f}}=(\Pi^{\mathfrak{f}},\mathsf{g}^{\mathfrak{f}})$$

on T and T^+, respectively, where

$$\Pi^{\mathfrak{f}}\big([\![a,b]\!]_{\mathrm{j}}X^p\big)(y):=y^p\,\mathsf{P}_{\mathrm{j}}\big(f_a,\dots,f_b\big)(y)$$

and

$$\mathsf{g}^{\mathfrak{f}}\big([\![a,b]\!]_{\mathrm{j}}^{\mathfrak{k}}X^q\big)(y):=y^q\,\widetilde{\mathsf{P}}_{\mathrm{j}}^{\alpha_{[\![a,b]\!]}-|\mathfrak{k}|}\big(\partial^{k_a}f_a,\dots,\partial^{k_b}f_b\big)(y).$$

(We do not record the dependence of these quantities on α in the notation.) We denote by $[\cdot]^{\mathfrak{f}}$ the bracket maps associated to the pair of maps $(\Pi^{\mathfrak{f}},\mathsf{g}^{\mathfrak{f}})$ as in Section 4.1. Recall from (1.4) the definition of the operators Δ_i^p, for $p\in\mathbf{N}^{d_0}$. For each $[\![a,b]\!]_{\mathrm{j}}X^p\in\mathcal{B}$ we define an element

$$\pi^{\mathfrak{f}}([\![a,b]\!]_{\mathrm{j}}X^p)=\big(\pi^{\mathfrak{f}}([\![a,b]\!]_{\mathrm{j}}X^p)_i\big)_{i\geq -1}$$

of C^{0+} by setting

$$\pi^{\mathfrak{f}}([\![a,b]\!]_{\mathrm{j}}X^p)_i:=\Delta_i^p\big(\mathsf{P}_{\mathrm{j}}(f_a,\cdots,f_b)\big).$$

Likewise, for $[\![a,b]\!]_{\mathrm{j}}^{\mathfrak{k}}X^q\in\mathcal{B}^+$ with $\mathfrak{k}=(k_a,\dots,k_b)\in(\mathbf{N}^{d_0})^{b-a+1}$, we define an element

$$g^{\mathfrak{f}}([\![a,b]\!]_{\mathrm{j}}^{\mathfrak{k}}X^q)=\big(g^{\mathfrak{f}}([\![a,b]\!]_{\mathrm{j}}^{\mathfrak{k}}X^q)_i\big)_{i\geq -1}$$

of C^{0+} by setting

$$g^{\mathfrak{f}}([\![a,b]\!]_{\mathrm{j}}^{\mathfrak{k}}X^q)_i:=\Delta_i^q\Big(\widetilde{\mathsf{P}}_{\mathrm{j}}^{\alpha_{[\![a,b]\!]}-|\mathfrak{k}|}\big(\partial^{k_a}f_a,\cdots,\partial^{k_b}f_b\big)\Big).$$

For $j\geq 0$ we set

$$g^{\mathfrak{f}}([\![a,b]\!]_{\mathrm{j}}^{\mathfrak{k}}X^q)_{<j}:=\sum_{i=-1}^{j-1}g^{\mathfrak{f}}([\![a,b]\!]_{\mathrm{j}}^{\mathfrak{k}}X^q)_i.$$

The statement of Proposition 19 below, and the next two preparatory results, require a notation that we now introduce. For $\tau=[\![a,b]\!]_{\mathrm{j}}X^p\in\mathcal{B}$ we write

$$\sigma\prec\tau\text{ if }\sigma<\tau\text{ and }\sigma=[\![c,b]\!]_{\mathrm{j}'}X^{p'}\text{ with }c>a.$$

We also write

$$\sigma\leqq\tau\text{ if }\sigma<\tau\text{ but not }\sigma\prec\tau.$$

For a descending sequence $\sigma_e \leqq \cdots \leqq \sigma_1 \leqq \tau$ we have $\sigma_j = [\![a,b]\!]_{\mathrm{j}} X^{p_j}$ with $0 \leq p_e < \cdots < p_1 < p$, and $\sigma_j/\sigma_{j+1} = \binom{p_j}{p_{j+1}} X^{p_j - p_{j+1}}$. For $\mu = [\![a,b]\!]_{\mathrm{j}}^{\mathfrak{k}} X^q \in \mathcal{B}^+$ we write

$$\nu \prec \mu \text{ if } \nu < \mu \text{ and } \nu = [\![c,b]\!]_{\mathrm{j}'}^{\mathfrak{k}'} X^{q'} \text{ with } c > a.$$

The next statement relates $\pi^{\mathfrak{f}} \in \mathsf{C}^{0+}$ to the maps $\Pi^{\mathfrak{f}}$ and $\mathsf{g}^{\mathfrak{f}}$ on the one hand, and $g^{\mathfrak{f}}$ to $\mathsf{g}^{\mathfrak{f}}$ on the other hand.

17* – *Lemma. *We have for $\tau \in T$ and $i \geq 1$*

$$\pi^{\mathfrak{f}}(\tau)_i = \Delta_i(\Pi^{\mathfrak{f}}(\tau)) - \sum_{e\geq 1}(-1)^e \sum_{\sigma_e \leqq \cdots \leqq \sigma_1 \leqq \tau} \Delta_{<i-1}\big(\mathsf{g}^{\mathfrak{f}}(\tau/\sigma_1)\big) \cdots \Delta_{<i-1}\big(\mathsf{g}^{\mathfrak{f}}(\sigma_{e-1}/\sigma_e)\big)\, \Delta_i\big(\Pi^{\mathfrak{f}}(\sigma_e)\big) \tag{4.4}$$

and for $\tau/\sigma \in T^+$ with $\sigma \prec \tau$

$$g^{\mathfrak{f}}(\tau/\sigma)_{<i-1} =$$

$$\Delta_{<i-1}\big(\mathsf{g}^{\mathfrak{f}}(\tau/\sigma)\big) - \sum_{e\geq 1}(-1)^e \sum_{\substack{\sigma_e \leqq \cdots \leqq \sigma_1 \leqq \tau \\ \sigma \prec \sigma_e}} \Delta_{<i-1}\big(\mathsf{g}^{\mathfrak{f}}(\tau/\sigma_1)\big) \cdots \Delta_{<i-1}\big(\mathsf{g}^{\mathfrak{f}}(\sigma_{e-1}/\sigma_e)\big)\, \Delta_{<i-1}\big(\mathsf{g}^{\mathfrak{f}}(\sigma_e/\sigma)\big). \tag{4.5}$$

Proof – *1)* We consider first the identity (4.4). Denote by $(\star)_i(\cdot)$ the function on $\mathbf{R}^d$ defined by the right hand side of (4.4). It suffices to treat the case of $\tau = [\![1,n]\!]_{\mathrm{j}} X^p$. One has $\tau/\sigma_1 = \binom{p}{p_1} X^{p-p_1}$ and $\sigma_j/\sigma_{j+1} = \binom{p_j}{p_{j+1}} X^{p_j - p_{j+1}}$ for $1 \leq j \leq e-1$; moreover for $k \in \mathbf{N}^{d_0}$ we have $\Delta_{<i-1}\big(\mathsf{g}^{\mathfrak{f}}(X^k)\big)(x) = x^k$ for all $i \geq 1$. It follows that $(\star)_i(x)$ is equal to

$$\begin{aligned}
&\Delta_i(\Pi^{\mathfrak{f}}([\![1,n]\!]_{\mathrm{j}} X^p))(x) - \sum_{e\geq 1}(-1)^e \sum_{0\leq p_e < \cdots < p_1 < p} \prod_{j=1}^{e-1} \binom{p_j}{p_{j+1}} x^{p_j - p_{j+1}} \Delta_i\Big(\Pi^{\mathfrak{f}}\big([\![1,n]\!]_{\mathrm{j}} X^{p_q}\big)\Big)(x) \\
&= \Delta_i\Big(\Pi^{\mathfrak{f}}\big([\![1,n]\!]_{\mathrm{j}} X^p\big)\Big)(x) - \sum_{r<p} C_{pr}\, x^{p-r} \Delta_i\Big(\Pi^{\mathfrak{f}}\big([\![1,n]\!]_{\mathrm{j}} X^r\big)\Big)(x)
\end{aligned}$$

where

$$C_{pr} := \sum_{e\geq 1}(-1)^e \sum_{r<p_{e-1}<\cdots<p_1<p} \prod_{j=0}^{e-1} \binom{p_j}{p_{j+1}}.$$

We note that the constants C_{pr} satisfy the inductive relation

$$C_{pr} = -\binom{p}{r} + \sum_{r<s<p} \binom{p}{s} C_{sr},$$

so

$$C_{pr} = (-1)^{p-r+1}\binom{p}{r}.$$

One then has

$$\begin{aligned}
(\star)_i(x) &= \Delta_i\Big(\Pi^{\mathfrak{f}}\big([\![1,n]\!]_{\mathrm{j}} X^p\big)\Big)(x) + \sum_{r<p}(-1)^{p-r}\binom{p}{r} x^{p-r} \Delta_i\Big(\Pi^{\mathfrak{f}}\big([\![1,n]\!]_{\mathrm{j}} X^r\big)\Big)(x) \\
&= \int_{\mathbf{R}^{d_0}} K_i(y-x) \sum_{r=0}^{p} (-1)^{p-r}\binom{p}{r} x^{p-r} y^r\, \mathsf{P}_{\mathrm{j}}(f_1,\cdots,f_n)(y)\, dy = \pi^{\mathfrak{f}}(\tau)_i(x).
\end{aligned}$$

2) One uses a similar reasoning to prove identity (4.5). It suffices to treat the case $\tau = [\![1,a]\!]_{\mathrm{j}} X^p$ and $\sigma = [\![n+1,a]\!]_{\mathrm{j}_{>n}+\boldsymbol{s}} X^q$. One has in that case

$$\tau/\sigma = \sum_{s_1,r} \frac{1}{\boldsymbol{s}!(s_1 - |\boldsymbol{s}|)!} \binom{p}{r} [\![1;n]\!]_{\mathrm{j}_{<n}}^{s_1} X^r$$

where the sum runs over the multi-indices s_1, r such that $p = q + \|\boldsymbol{s}\| + r - s_1$ and such that $[\![1;n]\!]_{\mathrm{j}_{<n}}^{s_1} \in T^+$ and $r \geq 0$. We write $D_{p,q}$ for the set of such s_1, r. Writing $(\star\star)_i(\cdot)$ for the right hand side of (4.5),

we have this time

$$
\begin{aligned}
(\star\star)_i(x) = &\sum_{(s_1,r)\in D_{p,q}} \frac{1}{\boldsymbol{s}!(s_1-|\boldsymbol{s}|)!}\binom{p}{r}\Delta_{<i-1}\Big(\mathsf{g}^{\mathfrak{f}}\big([\![1,n]\!]_{\mathfrak{j}}^{s_1}X^r\big)\Big)(x) \\
&- \sum_{p'<p}\sum_{(s_1,r')\in D_{p',q}} C_{pp'}x^{p-p'}\binom{p'}{r'}\frac{1}{\boldsymbol{s}!(s_1-|\boldsymbol{s}|)!}\Delta_{<i-1}\Big(\mathsf{g}^{\mathfrak{f}}\big([\![1,n]\!]_{\mathfrak{j}}^{s_1}X^{r'}\big)\Big)(x) \\
= &\sum_{(s_1,r)\in D_{p,q}} \frac{1}{\boldsymbol{s}!(s_1-|\boldsymbol{s}|)!}\binom{p}{r}\Delta_{<i-1}\Big(\mathsf{g}^{\mathfrak{f}}\big([\![1,n]\!]_{\mathfrak{j}}^{s_1}X^r\big)\Big)(x) \\
&- \sum_{(s_1,r)\in D_{p,q}}\sum_{r'<r}\frac{1}{\boldsymbol{s}!(s_1-|\boldsymbol{s}|)!}\binom{p}{r}(-1)^{r-r'}x^{r-r'}\binom{r}{r'}\Delta_{<i-1}\Big(\mathsf{g}^{\mathfrak{f}}\big([\![1,n]\!]_{\mathfrak{j}}^{s_1}X^{r'}\big)\Big)(x).
\end{aligned}
$$

where we used that, for any fixed s_1, if $(s_1,r')\in D_{p',q}$ and $(s_1,r)\in D_{p',q}$ then $p-p'=r-r'$. This gives indeed equal to $g^{\mathfrak{f}}(\tau/\sigma)_{<i-1}(x)$. ▷

18 – Corollary. *For any $\tau\in\mathcal{B}$ and $i\geq 1$ we have the relation*

$$
[\tau]_i^{\mathfrak{f}} = \pi^{\mathfrak{f}}(\tau)_i - \sum_{e\geq 1}(-1)^{e-1}\sum_{\sigma_e\prec\cdots\prec\sigma_1\prec\tau}\mathsf{g}^{\mathfrak{f}}\big(\tau/\sigma_1\big)_{<i-1}\cdots\mathsf{g}^{\mathfrak{f}}\big(\sigma_e/\sigma_{e-1}\big)_{<i-1}\pi^{\mathfrak{f}}\big(\sigma_e\big)_i.
$$

Likewise for $\tau/\sigma\in\mathcal{B}^+$ we have

$$
[\tau/\sigma]_i^{\mathfrak{f}} = g^{\mathfrak{f}}(\tau/\sigma)_i - \sum_{e\geq 1}(-1)^{e-1}\sum_{\sigma\prec\sigma_e\prec\cdots\prec\sigma_1\prec\tau}\mathsf{g}^{\mathfrak{f}}\big(\tau/\sigma_1\big)_{<i-1}\cdots\mathsf{g}^{\mathfrak{f}}\big(\sigma_e/\sigma_{e-1}\big)_{<i-1}g^{\mathfrak{f}}\big(\sigma_e/\sigma\big)_i.
$$

Proof – Plugging the identities of Lemma 17 giving $\Pi^{\boldsymbol{f}}$ and $\mathsf{g}^{\boldsymbol{f}}$ into the right hand of the identity to prove, developing the products, one recovers the definition of $[\tau]_i^{\boldsymbol{f}}$ and $[\tau/\sigma]_i^{\boldsymbol{f}}$ by noting that any descending sequence $\tau_e<\cdots<\tau_1<\tau$ takes the form

$$
\cdots\leqq\tau_{3,0}\prec\tau_{2,e_2}\leqq\cdots\leqq\tau_{2,0}\prec\tau_{1,e_1}\leqq\cdots\leqq\tau_{1,1}\leqq\tau.
$$

The conclusion follows. ▷

§2. A representation formula of P_ℓ in terms of the $\mathsf{P}_<$ operators. We are now ready to state and prove the main result of this section. Each bracket $[\cdot]^{\mathfrak{f}}$ that appears below is an element of the extended function space C^{0+} so the quantities $\mathsf{P}_<(\dots)_i$ in (4.6) and (4.7) is not the i-th term in a Lilttewood-Paley decomposition but rather the i-th element that defines the corresponding sequence $\mathsf{P}_<(\dots)$.

19 – Proposition. *For any $\tau=[\![a,b]\!]_{\mathfrak{j}}X^p\in T$ we have*

$$
\Delta_i^p\big(\mathsf{P}_{\mathfrak{j}}(f_a,\cdots,f_b)\big) = \sum_{e\geq 0}\sum_{\sigma_e\prec\cdots\prec\sigma_1\prec\tau}\mathsf{P}_<\big([\tau/\sigma_1]^{\mathfrak{f}},\dots,[\sigma_e]^{\mathfrak{f}}\big)_i \tag{4.6}
$$

and for $\sigma\leq\tau$ with $\tau/\sigma=[\![c,d]\!]_{\mathfrak{j}'}^{\mathfrak{k}}X^q\in T^+$ we have

$$
\Delta_i^q\Big(\widetilde{\mathsf{P}}_{\mathfrak{j}'}^{\alpha_{[\![c,d]\!]}-|\mathfrak{k}|}\big(\partial^{k_c}f_c,\cdots,\partial^{k_d}f_d\big)\Big) = \sum_{e\geq 0}\sum_{\sigma\prec\sigma_e\prec\cdots\prec\sigma_1\prec\tau}\mathsf{P}_<\big([\tau/\sigma_1]^{\mathfrak{f}},\dots,[\sigma_e/\sigma]^{\mathfrak{f}}\big)_i \tag{4.7}
$$

Proof – We prove (4.6) and let the reader prove (4.7) as its proof is almost identical. We proceed by developing the sum

$$
\sum_{e\geq 0}\sum_{\sigma_e\prec\cdots\prec\sigma_1\prec\tau}\mathsf{P}_<\big([\tau/\sigma_1]^{\mathfrak{f}},\cdots,[\sigma_e]^{\mathfrak{f}}\big)_i
$$

and use the identities of Corollary 18 to see that a number of cancellations give in the end $\pi^{\mathfrak{f}}(\tau)_i$.

A non-increasing map $\mathsf{a}:[\![0,e]\!]\to\mathbf{N}$ is said to be *admissible* if it is such that $\mathsf{a}(e)=0,\mathsf{a}(e-1)=1$ and $\mathsf{a}(j)-\mathsf{a}(j+1)\in\{0;1\}$ for every $0\leq j\leq e-1$. For any such a and any integer $0\leq m\leq\mathsf{a}(0)$ we define $j_{\mathsf{a}}(m)$ as the smallest integer j such that $\mathsf{a}(j)=m$.

We associate to any $i \geq -1$, to any descending chain $\boldsymbol{\nu} : \nu_e \prec \cdots \prec \nu_0 = \tau$, and to any admissible a the element of $\mathsf{C}^{-\infty}$

$$\mathsf{Q}_\pi\big(\tau/\nu_1, \ldots, \nu_{e-1}/\nu_e, \nu_e\big)_{i_e} := \sum_{(i_j)_{0\leq j\leq e-1} \in D_{\mathsf{a},i_e}} \prod_{j=0}^{e-1} \mathsf{g}^{\mathfrak{f}}\big(\nu_j/\nu_{j+1}\big)_{i_j}\, \pi^{\mathfrak{f}}(\nu_e)_{i_e},$$

where

$$D_{\mathsf{a},i_e} := \Big\{(i_j)_{0\leq j\leq e-1} \in [\![-1,+\infty[\![^e, \quad \forall j \in [\![0, e-1]\!],\ i_j < i_{j_{\mathsf{a}}(\mathsf{a}(j)-1)} - 1\Big\}.$$

For every descending chain $\boldsymbol{\sigma} : \sigma_{e'} \prec \cdots \prec \tau$, from the identity of Lemma 17 giving $[\sigma_j/\sigma_{j+1}]$ and $[\sigma_{e'}]$ in terms of $\mathsf{g}^{\mathfrak{f}}$ and $\Pi^{\mathfrak{f}}$, developing the products gives the identity

$$\mathsf{P}_<\big([\tau/\sigma_1], \ldots, [\sigma_{e'}]\big)_i = \sum_{\boldsymbol{\nu},\mathsf{a}} \lambda_{\boldsymbol{\sigma}}^{\boldsymbol{\nu},\mathsf{a}} \mathsf{Q}_{\mathsf{a}}\big(\tau/\nu_1, \ldots, \nu_e\big)_i$$

where the sum runs over the set of descending sequences $\nu_e \prec \cdots \prec \nu_0 = \tau$ and the set of admissible maps a, and where $\lambda_{\boldsymbol{\sigma}}^{\boldsymbol{\nu},\mathsf{a}} = 0$ except if $\boldsymbol{\sigma}$ is a subsequence of $\boldsymbol{\nu}$ of size e' such that $\mathsf{a}(0) - e' \in \{0,1\}$ and $\sigma_{e'-m} = \nu_{j_{\mathsf{a}}(m)}$ for every $0 \leq m \leq e'$, in which case we have $\lambda_{\boldsymbol{\sigma}}^{\boldsymbol{\nu},\mathsf{a}} = (-1)^{e-e'}$. Then

$$\sum_{e'\geq 0} \sum_{\sigma_{q'} \prec\cdots\prec\sigma_1\prec\tau} \mathsf{P}_<\big([\tau/\sigma_1], \cdots, [\sigma_{e'}]\big)_i = \sum_{e\geq 0} \sum_{\nu_e\prec\cdots\prec\nu_1\prec\tau} \sum_{\mathsf{a}} \lambda^{\boldsymbol{\nu},\mathsf{a}} \mathsf{Q}_{\mathsf{a}}\big(\tau/\nu_1, \cdots, \nu_e\big)$$

where $\lambda^{\boldsymbol{\nu},\mathsf{a}} = \sum_{\boldsymbol{\sigma}} \lambda_{\boldsymbol{\sigma}}^{\boldsymbol{\nu},\mathsf{a}}$, for a sum over the set of finite descending sequences $\boldsymbol{\sigma} : \sigma_{e'} \prec \cdots \prec \tau$. We actually have $\lambda^{\boldsymbol{\nu},\mathsf{a}} = 0$ for every non-empty sequence $\boldsymbol{\nu}$. Indeed for any given $\boldsymbol{\nu} \neq \emptyset$ of size e and any admissible a there are only two descending sequences such that $\lambda_{\boldsymbol{\sigma}}^{\boldsymbol{\nu},\mathsf{a}} \neq 0$. These sequences $\boldsymbol{\sigma}^1$ and $\boldsymbol{\sigma}^2$ are of size $\mathsf{a}(0)$ and $\mathsf{a}(0) - 1$, respectively, and

$$\sigma_m^1 = \nu_{j_{\mathsf{a}}(\mathsf{a}(0)-m)}$$

and

$$\sigma_m^2 = \nu_{j_{\mathsf{a}}(\mathsf{a}(0)-1-m)}.$$

The two coefficient $\lambda_\sigma^{\boldsymbol{\nu},\mathsf{a}}$ for these two $\boldsymbol{\sigma}$ are of opposite sign, which implies indeed that $\lambda^{\boldsymbol{\nu},\mathsf{a}} = 0$. ▷

4.3 – Proof of Theorem 1. Theorem 1 states that $(\Pi^{\mathfrak{f}}, \mathsf{g}^{\mathfrak{f}})$ is a model on the regularity structure from Section 3. To put our proof strategy in context, we recall a variation on Lemma 6.6 of Gubinelli, Imkeller & Perkowski's work [11].

Lemma – *Let Π be a linear map from T to $\mathcal{D}'(\mathbf{R}^{d_0})$ and g be a map from $\mathbf{R}^{d_0}$ into the set of characters of the algebra T^+. The pair (Π, g) is a model if and only if one has both*

$$\big|\langle \Pi_x \tau, K_{<i,x}\rangle\big| \lesssim 2^{-i|\tau|} \qquad (\forall\, \tau \in \mathcal{B}) \tag{4.8}$$

uniformly over $i \geq -1$ and $x \in \mathbf{R}^{d_0}$, and

$$|\mathsf{g}_{yx}(\mu)| \lesssim |y-x|^{|\mu|} \qquad (\forall\, \mu \in \mathcal{B}^+) \tag{4.9}$$

uniformly on (x,y) in any compact subset of $\mathbf{R}^{d_0}$.

For $\tau, \sigma \in T$ and a descending sequence $\boldsymbol{\sigma}(e) = (\sigma_e \prec \cdots \prec \sigma_1)$ we write

$$\sigma \prec \boldsymbol{\sigma}(e) \prec \tau \text{ if } \big(\sigma \prec \sigma_e \text{ and } \sigma_1 \prec \tau\big)$$

and set

$$|\tau/\boldsymbol{\sigma}(e)|_\alpha := \big(|\tau/\sigma_1|_\alpha, \cdots, |\sigma_e/\sigma|_\alpha\big) \in \mathbf{R}^{e+1}.$$

Strategy. We prove by *induction* on n the following three facts at a time.

(a)$_n$ For any tuple $\alpha = (\alpha_j)_{1\leq j\leq n} \in \mathbf{R}^n$ such that $\sum_{j=1}^n \alpha_j > 0$ the map

$$\big(g_1, \ldots, g_n\big) \mapsto \widetilde{\mathsf{P}}_{\mathsf{j}}^{\alpha}(g_1, \ldots, g_n)$$

has a continuous extension from $\prod_{j=1}^n C_\circ^{\alpha_j}$ into L^∞.

For any $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbf{R}^n$ and any tuple $\mathfrak{f} = (f_1, \ldots, f_n)$ of smooth functions one has *(b)$_n$* *for any homogeneous* $\tau = [\![a,b]\!]_{\mathsf{j}} X^p \in T$ *we have*

$$\big|\langle \Pi^{\mathfrak{f}}_x \tau, K_{<i,x}\rangle\big| \lesssim \|f_a\|_{\alpha_a}\cdots\|f_b\|_{\alpha_b}\, 2^{-i|\tau|_\alpha},$$

uniformly over $x \in \mathbf{R}^{d_0}$ *and* $i \geq 0$*;*

(c)$_n$ *for any homogeneous* $\tau = [\![a,b]\!]^{\mathfrak{k}}_{\mathsf{j}} X^p \in T^+$ *we have*

$$\big|\mathsf{g}^{\mathfrak{f}}_{y,x}(\tau)\big| \lesssim \|f_a\|_{\alpha_a}\cdots\|f_b\|_{\alpha_b}\, |y-x|^{|\tau|_\alpha},$$

uniformly over $x, y \in \mathbf{R}^{d_0}$.

Theorem 1 follows as a consequence.

- The result holds true for $n = 1$.
- We will use in the induction the following two algebraic identities proved in Appendix A.4.2.

20 – Lemma. *We fix a tuple* $\mathfrak{f} = (f_1, \dots, f_n)$ *of smooth functions.*

(i) *Pick* $k \in \mathbf{N}^{d_0}$ *with* $\mathfrak{k} \in \mathcal{P}_n(k)$*. Set* $\partial^k \mathfrak{f} := (\partial^{k_1} f_1, \dots, \partial^{k_n} f_n)$*. We work in this item in the regularity structure* $\mathscr{T}_{\alpha - |\mathfrak{k}|}$ *with the pair of maps* $(\Pi^{\partial^k \mathfrak{f}}, \mathsf{g}^{\partial^k \mathfrak{f}})$ *and its associated bracket maps* $[\cdot]^{\partial^k \mathfrak{f}}$*. For* $\tau = [\![1,n]\!]_{\mathsf{j}} X^m \in T$ *with* $|k| < |[\![1,n]\!]_{\mathsf{j}}|_\alpha$ *we have*

$$\sum_{e\geq 0}\ \sum_{\sigma_e \prec \cdots \prec \sigma_1 \prec \tau} \widetilde{\mathsf{P}}^{|\tau/\boldsymbol{\sigma}(e)|_{\alpha-|\mathfrak{k}|}}_{<}\Big([\tau/\sigma_1]^{\partial^k \mathfrak{f}}, \dots, [\sigma_e]^{\partial^k \mathfrak{f}}\Big) = \mathbf{1}_{m=0}\, \widetilde{\mathsf{P}}^{\alpha - |\mathfrak{k}|}_{\mathsf{j}}\big(\partial^{k_1} f_1, \cdots, \partial^{k_n} f_n\big).$$

(ii) *We work in this item in the regularity structure* $\mathscr{T}_\alpha$ *with the pair of maps* $(\Pi^{\mathfrak{f}}, \mathsf{g}^{\mathfrak{f}})$ *and its associated bracket maps* $[\cdot]^{\mathfrak{f}}$*. For* $\tau/\sigma = [\![1,n]\!]^{\mathfrak{k}}_{\mathsf{j}} X^m \in T^+$ *and* $|p| < |\tau/\sigma|_\alpha$ *we have the identity*

$$\sum_{e\geq 0}\ \sum_{\sigma \prec \boldsymbol{\sigma}(e) \prec \tau}\ \sum_{\mathfrak{p} \in \mathcal{P}_{e+1}(p)} \binom{p}{\mathfrak{p}} \widetilde{\mathsf{P}}^{|\tau/\boldsymbol{\sigma}(e)|_\alpha - |\mathfrak{p}|}_{<}\Big(\partial^{p_1}[\tau/\sigma_1]^{\mathfrak{f}}, \cdots, \partial^{p_{q+1}}[\sigma_e/\sigma]^{\mathfrak{f}}\Big)$$
$$= \mathbf{1}_{m=0} \left\{ \sum_{\mathfrak{k}\in\mathcal{P}_n(k)}\ \sum_{\mathfrak{p}\in\mathcal{P}_n(p)} \binom{k}{\mathfrak{k}}\binom{p}{\mathfrak{p}} \widetilde{\mathsf{P}}^{\alpha - |\mathfrak{k}+\mathfrak{p}|}_{\mathsf{j}}\Big(\partial^{k_1+p_1} f_1, \cdots, \partial^{k_n+p_n} f_n\Big)\right\}.$$

We proceed with the induction step

$$\big((a)_{n-1}, (b)_{n-1}, (c)_{n-1}\big) \Longrightarrow \big((a)_n, (b)_n, (c)_n\big).$$

– ***We begin by proving (a)$_n$.*** Pick $\beta = (\beta_1, \dots, \beta_n) \in \mathbf{R}^n$ with $\sum_{i=1}^n \beta_i > 0$. We work with the regularity structure $\mathscr{T}_\beta$. For $\mathsf{j} = (\ell_1, \dots, \ell_{n-1}) \in (\mathbf{R}^{d_0})^{n-1}$ set $\mathsf{j}^- := (\ell_1, \dots, \ell_{n-2})$ and

$$\tau_n(\mathsf{j}) := [\![1, n-1]\!]_{\mathsf{j}^-} X^{\ell_{n-1}}.$$

Write $[\![n]\!]$ for $[\![n,n]\!] \in T$ and $[n]^{\boldsymbol{g}}$ for its associated bracket map. From the continuity result of Proposition 5 for the $\widetilde{\mathsf{P}}^\gamma_<$ it suffices to prove that

$$\widetilde{\mathsf{P}}^\beta_{\mathsf{j}}(g_1, \dots, g_n) = \sum_{e\geq 0}\ \sum_{\sigma_e \prec \cdots \prec \tau_n(\mathsf{j})} \widetilde{\mathsf{P}}^{|\tau_n(\mathsf{j})/\boldsymbol{\sigma}^+|_\beta}_{<}\big([\tau_n(\mathsf{j})/\sigma_1]^{\boldsymbol{g}}, \dots, [\sigma_e]^{\boldsymbol{g}}, [n]^{\boldsymbol{g}}\big) \tag{4.10}$$

where

$$\big|\tau_n(\mathsf{j})/\boldsymbol{\sigma}^+\big|_\beta := \Big(|\tau_n(\mathsf{j})/\sigma_1|_\beta,\, |\sigma_1/\sigma_2|_\beta, \dots, |\sigma_e|_\beta,\, \beta_n\Big).$$

The symbol + meaning that we added β_n at the end of the uplet $|\tau_n(\mathsf{j})/\boldsymbol{\sigma}|_\beta$. Indeed, if one has (4.10), the induction hypothesis and Proposition 15 ensure that any term $[\nu]^{\boldsymbol{g}}$ appearing in the right hand side of (4.10) is an element of $\mathsf{C}^{|\nu|}$ that depends continuously on $\boldsymbol{g} \in \prod_{j=1}^n C^{\beta_j}_\circ$. Since

$$|\tau_n(\mathsf{j})/\sigma_1|_\beta + |\sigma_1/\sigma_2|_\beta + \cdots + |\sigma_e|_\beta + \beta_n = \sum_{j=1}^n \beta_j > 0$$

we can use Proposition 5 to conclude that $(a)_n$ holds true.

The remaining of this paragraph is dedicated to proving (4.10) by induction. The recursive definition (2.3) of the $\widetilde{\mathsf{P}}_<^\alpha$ given in Lemma 7 writes here

$$\begin{aligned}\sum_{e\geq 0}\sum_{\sigma_e\prec\cdots\prec\tau_n(\mathrm{j})}&\widetilde{\mathsf{P}}_<^{|\tau_n(\mathrm{j})/\boldsymbol{\sigma}^+|_\beta}\big([\tau_n(\mathrm{j})/\sigma_1]^{\boldsymbol{g}},\dots,[\sigma_e]^{\boldsymbol{g}},[n]^{\boldsymbol{g}}\big)\\ &=\sum_{e\geq 0}\sum_{\sigma_e\prec\cdots\prec\tau_n(\mathrm{j})}\mathsf{P}_<\big([\tau_n(\mathrm{j})/\sigma_1]^{\boldsymbol{g}},\dots,[\sigma_e]^{\boldsymbol{g}},[n]^{\boldsymbol{g}}\big)\\ &\quad-\sum_{\substack{\sigma\prec\tau_n(\mathrm{j})\\ |\sigma|_\beta+\beta_n<0}}\sum_{\sigma\prec\sigma_{e_1}\prec\cdots\prec\tau_n(\mathrm{j})}\widetilde{\mathsf{P}}_<^{|\tau_n(\mathrm{j})/\boldsymbol{\sigma}|_\beta}\big([\tau_n(\mathrm{j})/\sigma_1]^{\boldsymbol{g}},\dots,[\sigma_{e_1}/\sigma]^{\boldsymbol{g}}\big)\\ &\qquad\times\sum_{\nu_{e_2}\prec\cdots\prec\sigma}\widetilde{\mathsf{P}}_<^{|\sigma/\boldsymbol{\nu}^+|_\beta}\big([\sigma/\nu_1]^{\boldsymbol{g}},\dots,[\nu_{e_2}]^{\boldsymbol{g}},[n]^{\boldsymbol{g}}\big).\end{aligned}\tag{4.11}$$

From Proposition 19 one has

$$\sum_{e\geq 0}\sum_{\sigma_e\prec\cdots\prec\tau_n(\mathrm{j})}\mathsf{P}_<\big([\tau_n(\mathrm{j})/\sigma_1]^{\boldsymbol{g}},\dots,[\sigma_e]^{\boldsymbol{g}},[n]^{\boldsymbol{g}}\big)=\mathsf{P}_{\mathrm{j}}\big(g_1,\dots,g_n\big).$$

Since any $\sigma\prec\tau_n(\mathrm{j})$ has the form $\sigma=[\![m+1,n-1]\!]_{\mathrm{j}+p_1}X^{p_2+s_2}$, one has $\tau_n(\mathrm{j})/\sigma=[\![1,m]\!]_{\mathrm{j}}^pX^{s_1}$, with $\ell_{n-1}=s_1+s_2$ and $p=p_1+p_2$. If $s_1=0$, item *(ii)* of Lemma 20 ensures that the sum over the descending sequences $\sigma\prec\sigma_{e_1}\prec\cdots\prec\tau_n(\mathrm{j})$ is null. The terms $\sigma\prec\tau_n(\mathrm{j})$ that may give some non-trivial contributions to the sum (4.11) are thus of the form $\sigma=[\![m+1,n-1]\!]_{\mathrm{j}+p_1}X^{p_2+l_{n-1}}$, for which $\tau_n(\mathrm{j})/\sigma=[\![1,m]\!]_{\mathrm{j}}^p$ with $p_1+p_2=p$. For such σ, item *(ii)* of Lemma 20 gives

$$\sum_{\sigma\prec\sigma_{e_1}\prec\cdots\prec\tau_n(\mathrm{j})}\widetilde{\mathsf{P}}_<^{|\tau_n(\mathrm{j})/\boldsymbol{\sigma}|_\beta}\big([\tau_n(\mathrm{j})/\sigma_1]^{\boldsymbol{g}},\dots,[\sigma_{e_1}/\sigma]^{\boldsymbol{g}}\big)=\sum_{\mathfrak{p}\in\mathcal{P}_m(p)}\binom{p}{\mathfrak{p}}\widetilde{\mathsf{P}}_{\mathrm{j}<m}^{\beta_{\leq m}-|\mathfrak{k}|}\big(\partial^{p_1}g_1,\dots,\partial^{p_m}g_m\big)$$

and we have from the induction hypothesis

$$\sum_{\nu_{e_2}\prec\cdots\prec\sigma}\widetilde{\mathsf{P}}_<^{|\sigma/\boldsymbol{\nu}^+|_{\beta>m}}\big([\sigma/\nu_1]^{\boldsymbol{g}},\dots,[\nu_{e_2}]^{\boldsymbol{g}},[n]^{\boldsymbol{g}}\big)=\widetilde{\mathsf{P}}^\beta_{J_{p_1,p_2}(\mathrm{j}_{>m})}\big(g_{m+1},\dots,g_n\big)$$

where

$$J_{p_1,p_2}(\mathrm{j}_{>m})=\sum_{\mathfrak{a}\in\mathcal{P}_{n-m-2}(p_1)}\binom{p_1}{\mathfrak{a}}\binom{k}{p_1}\big(\ell_{m+1}+a_1,\dots,\ell_{n-2}+a_{n-m-2},\ell_{n-1}+p_2\big).$$

We recognize then in (4.11) the recursive relation satisfied by the $\widetilde{\mathsf{P}}_{\mathrm{j}}^\beta$, which proves (4.10).

– ***We now turn to (b)$_n$.*** We would like to implement the same strategy as in point $(a)_n$: Write an iterated paraproduct as a sum of simplified iterated paraproducts and use their local expansion properties. The problem with this strategy is that Proposition 5 requires some positivity assumption on some regularity exponents to hold – which does not necessarily hold true here. To circumvent this issue, for any $r\geq -1$, we look at the expansion properties of the iterated paraproduct $\mathsf{P}_{\mathrm{j}}\big(f_1,\dots,f_{n-1},\Delta_r(f_n)\big)$ and treat $\Delta_r(f_n)$ as a function of high enough regularity in the estimates. We verify a posteriori that the remainders are summable over $r\geq -1$ and provide the right expression.

We use the same notations as in the proof of point $(a)_n$. Pick $\alpha_n^+>\alpha_n$ big enough such that $\sum_{s=j}^{n-1}\alpha_s+\alpha_n^+>0$ for all $1\leq j<n$. Set

$$\boldsymbol{\alpha}^+:=(\alpha_1,\dots,\alpha_{n-1},\alpha_n^+)$$

and, for any $r\geq -1$, let

$$\mathfrak{f}^r:=\big(f_1,\dots,f_{n-1},\Delta_r(f_n)\big)$$

and

$$\mathsf{M}^{r+}=\big(\Pi^{r+},\mathsf{g}^{r+}\big):=\mathsf{M}^{\mathfrak{f}^r,\boldsymbol{\alpha}^+}$$

(The last notation was introduced at the beginning of Section 4.2.) One has

$$
\begin{aligned}
\mathsf{P}_{\mathfrak{j}}\big(f_1,\dots,\Delta_r f_n\big) &= \sum_{i\geq -1} \Delta^{\ell_{n-1}}_{<i-1}\big(\mathsf{P}_{\mathfrak{j}^-}\big(f_1,\dots,f_{n-1}\big)\big)\Delta_i\big(\Delta_r(f_n)\big) \\
&= \sum_{i\geq -1}\sum_{e\geq 0}\sum_{\sigma_e\prec\cdots\prec\tau_n(\mathfrak{j})} \mathsf{P}_<\Big([\tau_n(\mathfrak{j})/\sigma_1]^{\mathsf{M}},\dots,[\sigma_e]^{\mathsf{M}}\Big)_{<i-1}\Delta_i\big(\Delta_r(f_n)\big) \\
&= \sum_{e\geq 0}\sum_{\sigma_e\prec\cdots\prec\tau_n(\mathfrak{j})} \mathsf{P}_<\Big([\tau_n(\mathfrak{j})/\sigma_1]^{\mathsf{M}^{r+}},\dots,[\sigma_e]^{\mathsf{M}^{r+}},[n]^{\mathsf{M}^{r+}}\Big).
\end{aligned}
\tag{4.12}
$$

From Proposition 15 and the induction hypothesis, every term $[\sigma]$ appearing in the paraproduct $\mathsf{P}_<$ is an element of $\mathsf{C}^{|\sigma|}$ that depends continuously on $\mathfrak{f}\in\big\{\prod_{j=1}^{n-1} C_\circ^{\alpha_j}\big\}\times C_\circ^{\alpha_n^+}$. The assumption on α_n^+ ensures that the homogeneities of the element in the iterated paraproducts

$$
\mathsf{P}_<\big([\tau_n(\mathfrak{j})/\sigma_1]^{\mathsf{M}^{r+}},\dots,[\sigma_e]^{\mathsf{M}^{r+}},[n]^{\mathsf{M}^{r+}}\big)
$$

add up to a positive quantity, however $|\sigma_e|_{\boldsymbol\alpha}$ may be non-positive. This is cured by noticing that the assumption on α_n^+ ensures that

$$
\mathsf{P}_<\Big([\tau_n(\mathfrak{j})/\sigma_1]^{\mathsf{M}^{r+}},\dots,[\sigma_e]^{\mathsf{M}^{r+}},[n]^{\mathsf{M}^{r+}}\Big) = \widetilde{\mathsf{P}}_<^{|\tau_n(\mathfrak{j})/\boldsymbol\sigma^+|_{\boldsymbol\alpha^+}}\Big([\tau_n(\mathfrak{j})/\sigma_1]^{\mathsf{M}^{r+}},\dots,[\sigma_e]^{\mathsf{M}^{r+}},[n]^{\mathsf{M}^{r+}}\Big), \tag{4.13}
$$

where

$$
|\tau_n(\mathfrak{j})/\boldsymbol\sigma^+|_{\boldsymbol\alpha^+} := \big(\,|\tau_n(\mathfrak{j})/\sigma_1|_{\boldsymbol\alpha^+},\,|\sigma_1/\sigma_2|_{\boldsymbol\alpha^+},\dots,|\sigma_e|_{\boldsymbol\alpha^+},\,\alpha_n^+\,\big),
$$

so one can use Proposition 29 on the local expansion of terms of the type $\widetilde{\mathsf{P}}_<^{\gamma}$. The remainder term in (A.1) is $(\triangle_{y-x,\theta}\widetilde{\mathsf{P}}_<^{\gamma})(\dots)(x)$ with $\theta=\sum_{j=1}^n\gamma_j$. We infer from this generic expansion property, (4.12) and (4.13), that $\mathsf{P}_{\mathfrak{j}}(f_1,\dots,\Delta_r(f_n))$ has a corresponding expansion with remainder

$$
\sum_{e\geq 0}\sum_{\sigma_e\prec\cdots\prec\tau_n(\mathfrak{j})}\big(\triangle_{y-x,\theta}\widetilde{\mathsf{P}}_<^{|\tau_n(\mathfrak{j})/\boldsymbol\sigma^+|_{\boldsymbol\alpha^+}}\big)\Big([\tau_n(\mathfrak{j})/\sigma_1]^{\mathsf{M}^{r+}},\dots,[\sigma_e]^{\mathsf{M}^{r+}},[n]^{\mathsf{M}^{r+}}\Big)(x),
$$

with $\theta=\sum_{j=1}^n\alpha_j'-\delta$, here. The following result is proved in Appendix A.2 by induction on n.

21 – Lemma. *For every point $x\in\mathbf{R}^{d_0}$ and $i\geq -1$ one has the identity*

$$
\begin{aligned}
\int_{\mathbf{R}^{d_0}} K_{<i}(h)\sum_{e\geq 0}\sum_{\sigma_e\prec\cdots\prec\tau_n(\mathfrak{j})}\big(\triangle_{h,\theta}\widetilde{\mathsf{P}}_<^{|\tau_n(\mathfrak{j})/\boldsymbol\sigma^+|_{\boldsymbol\alpha^+}}\big)\Big([\tau_n(\mathfrak{j})/\sigma_1]^{\mathsf{M}^{r+}},\dots,[\sigma_e]^{\mathsf{M}^{r+}},[n]^{\mathsf{M}^{r+}}\Big)(x)\,dh \\
= \int_{\mathbf{R}^{d_0}} K_{<i}(h)\big(\mathsf{\Pi}_x^{r\prime}[\![1,n]\!]_{\mathfrak{j}}\big)(x+h)\,dh.
\end{aligned}
\tag{4.14}
$$

We then have from (4.14) and Lemma 6.3 in [11]

$$
\begin{aligned}
\left|\int_{\mathbf{R}^{d_0}} K_{<i}(x-y)\big(\mathsf{\Pi}_x^{\prime(r)}[\![1,n]\!]_{\mathfrak{j}}\big)(y)\,dy\right| &\lesssim \left\{\prod_{j=1}^{n-1}\|f_j\|_{\alpha_j}\right\}\|\Delta_r f_n\|_{\alpha_n^+}\,2^{-i\theta} \\
&\lesssim 2^{-i\theta}\,2^{r(\alpha_n^+-\alpha_n)}\left\{\prod_{j=1}^{n}\|f_j\|_{\alpha_j}\right\}.
\end{aligned}
\tag{4.15}
$$

There is an integer $i(n)$ depending only on n such that we have for $j\leq n$ and $i\geq -1$

$$
\Delta_i\big(\mathsf{P}_{\mathfrak{j}}(f_1,\dots,f_j)\big) = \sum_{r\leq i+i(n)}\Delta_i\big(\mathsf{P}_{\mathfrak{j}}\big(f_1,\dots,f_{j-1},\Delta_r(f_j)\big)\big).
$$

Using the identity (4.1) on $\mathsf{\Pi}_x(\tau)$, we see that we have

$$
\Big\langle \mathsf{\Pi}_x([\![1,n]\!]_{\mathfrak{j}}),K_{<i,x}\Big\rangle = \sum_{r\leq i+i(n)}\Big\langle \mathsf{\Pi}_x^{r\prime}([\![1,n]\!]_{\mathfrak{j}}),K_{<i,x}\Big\rangle,
$$

so the expected bound

$$\Big|\Big\langle \Pi_x(\llbracket 1,n\rrbracket_{\mathfrak{j}}), K_{<i,x}\Big\rangle\Big| \lesssim \Big\{ \prod_{j=1}^n \|f_j\|_{\alpha_j} \Big\} \sum_{r=-1}^{i+i(n)} 2^{r(\alpha_n^+-\alpha_n)}\, 2^{-i\theta} \lesssim \Big\{ \prod_{j=1}^n \|f_j\|_{\alpha_j} \Big\} 2^{-i\sum_{j=1}^n \alpha_j}$$

follows from (4.15).

– ***We finally prove (c)$_n$.*** Pick $\alpha = (\alpha_1,\dots,\alpha_n) \in \mathbf{R}^n$, a multi-indice $k \in \mathbf{R}^{d_0}$ such that $|k| < \sum_{j=1}^n \alpha_j$ and $\mathfrak{k} \in \mathcal{P}_n(k)$. We work in the regularity structure $\mathscr{T}_{\alpha-|\mathfrak{k}|}$. From item *(ii)* of Lemma 20, we have for any smooth functions $f_1,\dots f_n$ the equality

$$\widetilde{\mathsf{P}}_{\mathfrak{j}}^{\alpha-|\mathfrak{k}|}\big(\partial^{k_1}f_1,\dots,\partial^{k_n}f_n\big) = \sum_{e\geq 0}\ \sum_{\sigma_e\prec\cdots\prec\sigma_1\prec\llbracket 1,n\rrbracket_{\mathfrak{j}}} \widetilde{\mathsf{P}}_<^{|\llbracket 1,n\rrbracket_{\mathfrak{j}}/\boldsymbol{\sigma}|_{\alpha-|\mathfrak{k}|}}\Big([\llbracket 1,n\rrbracket_{\mathfrak{j}}/\sigma_1]^{\mathsf{M}_\mathfrak{k}},\dots,[\sigma_e]^{\mathsf{M}_\mathfrak{k}}\Big),$$

where $\mathsf{M}_\mathfrak{k} = \mathsf{M}_{\partial^\mathfrak{k}\mathfrak{f},\alpha-|\mathfrak{k}|}$. Proposition 15 and point $(b)_n$ ensure by induction that all the terms $[\nu]^{\mathsf{M}_\mathfrak{k}}$ are some elements of $\mathsf{C}^{|\nu|_{\alpha-\mathfrak{k}}}$ that depend continuously on all the $f_j \in C_\circ^{\alpha_j}$. As above it follows from Proposition 29 that $\widetilde{\mathsf{P}}_{\mathfrak{j}}^{\alpha-|\mathfrak{k}|}\big(\partial^{k_1}f_1,\dots,\partial^{k_n}f_n\big)$ has a local expansion with remainder

$$R_{\mathfrak{f},\alpha}\big(\llbracket 1,n\rrbracket_{\mathfrak{j}}^{\mathfrak{k}}\big)(x,h) := \sum_{e\geq 0}\ \sum_{\sigma_e\prec\cdots\prec\llbracket 1,n\rrbracket_{\mathfrak{j}}} \big(\triangle_{h,\theta}\widetilde{\mathsf{P}}_<^{|\llbracket 1,n\rrbracket_{\mathfrak{j}}/\boldsymbol{\sigma}|_{\alpha-|\mathfrak{k}|}}\big)\Big(\llbracket 1,n\rrbracket_{\mathfrak{j}}/\sigma_1]^{\mathsf{M}_\mathfrak{k}},\dots,[\sigma_e]^{\mathsf{M}_\mathfrak{k}}\Big)(x)$$

where $\theta = |\llbracket 1,n\rrbracket_{\mathfrak{j}}|_{\alpha-|\mathfrak{k}|}$. From Proposition 29 this remainder has $|h|^\theta \prod_{j=1}^n \|f_j\|_{\alpha_j}$ as an x-uniform upper bound. Point $(c)_n$ will thus be proved after we show that for any $\tau = \llbracket 1,n\rrbracket_{\mathfrak{j}} X^s$ one has

$$\sum_{e\geq 0}\ \sum_{\sigma_e\prec\cdots\prec\tau} \big(\triangle_{h,\theta}\widetilde{\mathsf{P}}_<^{|\tau/\boldsymbol{\sigma}|_{\alpha-|\mathfrak{k}|}}\big)\Big([\tau/\sigma_1]^{\mathsf{M}_\mathfrak{k}},\dots,[\sigma_e]^{\mathsf{M}_\mathfrak{k}}\Big)(x) = \mathbf{1}_{s=0}\ \mathsf{g}_{x+h,x}^{\mathfrak{f}}\big(\llbracket 1,n\rrbracket_{\mathfrak{j}}^{\mathfrak{k}}\big). \tag{4.16}$$

The remainder of this paragraph is dedicated to proving this identity by induction on n. Recall that we write

$$\begin{aligned}\partial_\star^p\mathsf{P}_<\big([\tau/\sigma_1]^{\mathsf{M}_\mathfrak{k}},\dots,&[\sigma_{m-1}/\sigma_m]^{\mathsf{M}_\mathfrak{k}}\big)\\ &= \sum_{\mathfrak{p}\in\mathcal{P}_m(p)} \binom{p}{\mathfrak{p}} \widetilde{\mathsf{P}}_<^{|\tau/\boldsymbol{\sigma}_{\leq m}|_{\alpha-|\mathfrak{k}|}-|\mathfrak{p}|}\big([\tau/\sigma_1]^{\mathsf{M}_\mathfrak{k}},\dots,[\sigma_{m-1}/\sigma_m]^{\mathsf{M}_\mathfrak{k}}\big).\end{aligned}$$

From the definition of $\triangle_{h,\theta}\widetilde{\mathsf{P}}_<$ the left hand side of (4.16) is equal to

$$\begin{aligned}&\sum_{e\geq 0}\ \sum_{\sigma_e\prec\cdots\prec\tau} \widetilde{\mathsf{P}}_<^{|\tau/\boldsymbol{\sigma}|_{\alpha-|\mathfrak{k}|}}\big([\tau/\sigma_1]^{\mathsf{M}_\mathfrak{k}},\dots,[\sigma_e]^{\mathsf{M}_\mathfrak{k}}\big)(x+h)\\
&-\sum_{e\geq 0}\ \sum_{\sigma_e\prec\cdots\prec\tau}\ \sum_{|p|<|\tau|_{\alpha-|\mathfrak{k}|}} \partial_\star^p\widetilde{\mathsf{P}}_<\big(\,[\tau/\sigma_1]^{\mathsf{M}_\mathfrak{k}},\dots,[\sigma_e]^{\mathsf{M}_\mathfrak{k}}\big)(x)\,h^p\\
&-\sum_{\substack{e\geq 0\\ \sigma_e\prec\cdots\prec\tau}}\ \sum_{m=1}^{e}\ \sum_{|p|<|\tau/\sigma_m|_{\alpha-|\mathfrak{k}|}} \partial_\star^p\mathsf{P}_<\Big([\tau/\sigma_1]^{\mathsf{M}_\mathfrak{k}},\dots,[\sigma_{m-1}/\sigma_m]^{\mathsf{M}_\mathfrak{k}}\Big)(x)\\
&\qquad\times\frac{h^p}{p!}\big(\triangle_{h,|\sigma_m|_{\alpha-|\mathfrak{k}|}}\widetilde{\mathsf{P}}_<^{|\sigma_m/\boldsymbol{\sigma}_{>m}|_{\alpha-|\mathfrak{k}|}}\big)\Big([\sigma_m/\sigma_{m-1}]^{\mathsf{M}_\mathfrak{k}},\dots,[\sigma_e]^{\mathsf{M}_\mathfrak{k}}\big)(x),\end{aligned} \tag{4.17}$$

From item *(i)* of Lemma 20, the first double sum in (4.17) is equal to

$$\mathbf{1}_{s=0}\,\widetilde{\mathsf{P}}_{\mathfrak{j}}^{\alpha-|\mathfrak{k}|}\big(\partial^{k_1}f_1,\dots,\partial^{k_n}f_n\big)(x+h) = \mathbf{1}_{s=0}\ \mathsf{g}_{x+h}^{\mathfrak{f}}\big(\llbracket 1,n\rrbracket_{\mathfrak{j}}^{\mathfrak{k}}\big).$$

Lemma 20 also gives that the second line of (4.17) is equal to

$$\mathbf{1}_{s=0} \sum_{|p|<|\tau|_{\alpha-|\mathfrak{k}|}} \mathsf{g}_x^{\mathfrak{f}}\big(\llbracket 1,n\rrbracket_{\mathfrak{j}}^{\mathfrak{k}+p}\big)\,h^p.$$

The $\sigma \in T$ such that $\sigma \prec \tau = [\![1,n]\!]_{\mathfrak{j}} X^s$ have a form $\sigma = [\![m+1,n]\!]_{\mathfrak{j}+v_1} X^{v_2+s_2}$, in which case $\tau/\sigma = [\![1,m]\!]_{\mathfrak{j}}^{v} X^{s_1}$ with $s = s_1 + s_2$ and $v = v_1 + v_2$. For such $\sigma \in T$ Lemma 20 gives

$$\sum_{e_1 \geq 0} \sum_{\sigma_{e_1} \prec \cdots \prec \tau} \partial_\star^p \mathsf{P}_< \Big([\tau/\sigma_1]^{\mathsf{M}_\mathfrak{k}}, \dots, [\sigma_{e_1}/\sigma]^{\mathsf{M}_\mathfrak{k}} \Big)(x) = \mathbf{1}_{s_1=0}\, \mathsf{g}_x^{\mathfrak{f}} \big([\![1,m]\!]_{\mathfrak{j}}^{\mathfrak{k}+v+p} \big).$$

Also, we have by induction that

$$\sum_{e_2 \geq 0} \sum_{\nu_{e_2} \prec \cdots \prec \sigma} \triangle_{h, |\sigma|_{\alpha - |\mathfrak{k}|}} \mathsf{P}_< \Big([\sigma/\nu_1]^{\mathsf{M}_\mathfrak{k}}, \dots, [\nu_{e_2}]^{\mathsf{M}_\mathfrak{k}} \Big) = \mathbf{1}_{v_2+s_2=0}\, \mathsf{g}_{x,x+h}^{\mathfrak{f}} \big([\![m+1,n]\!]_{\mathfrak{j}+v_1}^{\mathfrak{k}} \big).$$

If $s \neq 0$, then either $s_1 \neq 0$ or $s_2 + p_2 \neq 0$, and all the terms of (4.17) add up to 0. Suppose now that $s = 0$. The $\sigma \in T$ we have to consider are of the form $\sigma = [\![m+1,n]\!]_{\mathfrak{j}+v}$, for which $\tau/\sigma = [\![1,m]\!]_{\mathfrak{j}}^{v}$ and the right hand side of (4.17) writes as

$$\begin{aligned} R_{\mathfrak{f},\alpha}\big([\![1,n]\!]_{\mathfrak{j}}^{\mathfrak{k}}\big) = \mathsf{g}_{x+h}^{\mathfrak{f}}\big([\![1,n]\!]_{\mathfrak{j}}^{\mathfrak{k}}\big) &- \sum_{|p| < |[\![1,n]\!]_{\mathfrak{j}}^{\mathfrak{k}}|_\alpha} \mathsf{g}_x^{\mathfrak{f}}\big([\![1,n]\!]_{\mathfrak{j}}^{\mathfrak{k}+p}\big)\, h^p \\ &- \sum_{m,p,v} \mathsf{g}_x^{\mathfrak{f}}\big([\![1,m]\!]_{\mathfrak{j}}^{\mathfrak{k}+p+v}\big)\, \mathsf{g}_{x+h,x}^{\mathfrak{f}}\big([\![m+1,n]\!]_{\mathfrak{j}+v}^{\mathfrak{k}} X^p\big), \end{aligned}$$

where the sum over m, p, v runs over $1 \leq m < n$ and multi-indices p, v such that $\big|[\![1,m]\!]_{\mathfrak{j}}^{v+p}\big|_{\alpha - |\mathfrak{k}|} > 0$ and $\big|[\![m+1,n]\!]_{\mathfrak{j}+v}\big|_{\alpha-|\mathfrak{k}|} > 0$. This sum corresponds to a sum over $\sigma \in T^+$ such that $\sigma \prec [\![1,n]\!]_{\mathfrak{j}}^{\mathfrak{k}}$ in the regularity structure $\mathscr{T}_\alpha$. It follows that we finally have

$$R_{\mathfrak{f},\alpha}\big([\![1,n]\!]_{\mathfrak{j}}^{\mathfrak{k}}\big) = \mathsf{g}_{x+h}^{\mathfrak{f}}\big([\![1,n]\!]_{\mathfrak{j}}^{\mathfrak{k}}\big) - \sum_{\sigma < [\![1,n]\!]_{\mathfrak{j}}^{\mathfrak{k}}} \mathsf{g}_{x+h,x}^{\mathfrak{f}}(\sigma) \mathsf{g}_x^{\mathfrak{f}}(\tau/\sigma) = \mathsf{g}_{x+h,x}^{\mathfrak{f}}\big([\![1,n]\!]_{\mathfrak{j}}^{\mathfrak{k}}\big),$$

which concludes the proof of (4.16), and closes the induction step in the proof of point *(c)*$_n$.

5 – Back to paracontrolled systems

This section is dedicated to proving Theorem 2. We set ourselves in the setting of Section 1.3, with its finite alphabet $\mathcal{L} = (l_1, \dots, l_{|\mathcal{L}|})$ and its associated set $\mathcal{W}$ of finite words $w = l_{i_1} \dots l_{i_j}$. An a priori notion of size $|\cdot|_\mathcal{L}$ is given on $\mathcal{L}$ and extended to $\mathcal{W}$ setting

$$|l_{i_1} \dots l_{i_j}|_\mathcal{L} := |l_{i_1}|_\mathcal{L} + \cdots + |l_{i_j}|_\mathcal{L}.$$

5.1 – The regularity structure $\mathscr{T}_\mathcal{L}$.

The following construction is identical to the construction of Section 3. We define a set of symbols

$$\mathcal{B} := \Big\{ [\![w]\!]_{\mathfrak{j}} X^p\,;\, w = l_{i_1} \dots l_{i_j} \in \mathcal{W},\, p, \ell \in \mathbf{N}^{d_0},\, \mathfrak{j} \in \mathcal{P}_{j-1}(\ell) \Big\} \cup \{X^k\}_{k \in \mathbf{N}^{d_0}},$$

and

$$\mathcal{B}^+ := \Big\{ [\![w]\!]_{\mathfrak{j}}^{\mathfrak{k}}\,;\, w = l_{i_1} \dots l_{i_j} \in \mathcal{W},\, k, \ell \in \mathbf{N}^{d_0},\, \mathfrak{j} \in \mathcal{P}_{j-1}(\ell),\, \mathfrak{k} \in \mathcal{P}_j(k),\ |w|_\mathcal{L} - |k| + |\ell| > 0 \Big\} \cup \big\{X^{e_i}\big\}_{1 \leq i \leq d_0}.$$

We let T be the vector space freely generated by $\mathcal{B}$, and T_+ be the algebra freely generated by $\mathcal{B}^+$, with unit $\mathbf{1}^+$. We also set

$$\big|[\![w]\!]_{\mathfrak{j}} X^p\big|_\mathcal{L} := |w|_\mathcal{L} + |\ell| + |p|$$

and define $|\cdot|_\mathcal{L}$ on T^+ as a multiplicative function such that $|X^{e_i}|_\mathcal{L} = 1$ and

$$\big|[\![w]\!]_{\mathfrak{j}}^{\mathfrak{k}}\big|_\mathcal{L} := |w|_\mathcal{L} + |\ell| - |k|.$$

Proceding as in Section 3, for $\tau = [\![l_{i_1} \dots l_{i_n}]\!]_{\mathfrak{j}} \in T$ we set

$$\oplus(\tau) := \Big\{ [\![l_{i_1} \dots l_{i_j}]\!]_{\mathfrak{j}_{<j}}^{\mathfrak{p}} \in \mathcal{B}^+\,;\, 1 \leq j \leq n,\ \ell_j = 0, \mathfrak{p} \in \mathcal{P}_j(p), p \in \mathbf{N}^{d_0} \Big\} \cup \{\mathbf{1}^+\},$$

and for $\mu = [\![l_{i_1}\dots l_{i_n}]\!]_{\mathrm{j}}^{\mathfrak{k}} \in T^+$ we set

$$\oplus(\mu) := \Big\{ [\![l_{i_1}\dots l_{i_j}]\!]_{\mathrm{j}_{<j}}^{\mathfrak{k}+\mathfrak{p}} \in \mathcal{B}^+ \,;\, 1\leq j\leq n,\ \ell_j = 0, \mathfrak{p}\in\mathcal{P}_j(p), p\in \mathbf{N}^{d_0} \Big\} \cup \{\mathbf{1}^+\}.$$

Set for $\tau = [\![l_{i_1}\dots l_{i_n}]\!]_{\mathrm{j}} \in T$ and $\sigma = [\![l_{i_1}\dots l_{i_j}]\!]_{\mathrm{j}}^{\mathfrak{p}} \in \oplus(\tau)$ with $j<n$

$$(\tau\backslash\sigma) := \sum_{p=p_1+p_2} \binom{p}{p_1} [\![l_{i_{j+1}}\dots l_{i_n}]\!]_{\mathrm{j}+p_1} X^{p_2},$$

and for $j=n$ set $(\tau\backslash\sigma) := \frac{1}{\mathfrak{p}!}X^p$. For $\mu = [\![l_{i_1}\dots l_{i_n}]\!]_{\mathrm{j}}^{\mathfrak{k}}$ and $\nu = [\![l_{i_1}\dots l_{i_j}]\!]_{\mathrm{j}}^{\mathfrak{k}+\mathfrak{p}} \in \oplus(\mu)$ with $j<n$

$$(\mu\backslash\nu) := \sum_{p=p_1+p_2} \binom{p}{p_1} [\![l_{i_{j+1}}\dots l_{i_n}]\!]_{\mathrm{j}+p_1}^{\mathfrak{k}} X^{p_2},$$

and for $j=n$ set $(\mu\backslash\nu) := \frac{1}{\mathfrak{p}!}X^p$. Finally set

$$\Delta(\tau) := \sum_{\sigma\in\oplus(\tau)} (\tau\backslash\sigma)\otimes\sigma,$$

and

$$\Delta^+(\mu) := \sum_{\substack{\nu\in\oplus(\mu)\\ |(\mu\backslash\nu)|_{\mathcal{L}}>0}} (\mu\backslash\nu)\otimes\nu.$$

Proceding as in Section 3 shows that

$$\mathscr{T}_{\mathcal{L}} = \big((T,\Delta),(T^+,\Delta^+)\big)$$

is a concrete regularity structure. Given $\alpha = (\alpha_1,\dots,\alpha_{|\mathcal{L}|})$ with $\sum_{j=1}^{|\mathcal{L}|}\alpha_j > 0$, and some functions $([l]\in C_\circ^{\alpha_l})_{l\in\mathcal{L}}$ given a priori, we define from Theorem 1 a model on $\mathscr{T}_{\mathcal{L}}$ setting

$$\begin{aligned}
\mathsf{\Pi}([\![l_{i_1}\dots l_{i_n}]\!]_{\mathrm{j}}) &:= \mathsf{P}_{\mathrm{j}}\big([l_{i_1}],\dots,[l_{i_n}]\big),\\
\mathsf{g}\big([\![l_{i_1}\dots l_{i_n}]\!]_{\mathrm{j}}^{\mathfrak{k}}\big) &:= \widetilde{\mathsf{P}}_{\mathrm{j}}^{(|l_{i_1}|_{\mathcal{L}},\dots,|l_{i_n}|_{\mathcal{L}})-|\mathfrak{k}|}\big(\partial^{k_1}[l_{i_1}],\dots,\partial^{k_n}[l_{i_n}]\big),
\end{aligned}$$

and $\mathsf{\Pi}([\![l_{i_1}\dots l_{i_n}]\!]_{\mathrm{j}}X^p)(\cdot) = \cdot^p\, \mathsf{\Pi}([\![l_{i_1}\dots l_{i_n}]\!]_{\mathrm{j}})(\cdot)$, with the notation (1.5).

5.2 – Paracontrolled systems and modelled distributions. We prove Theorem 2 in the refined form of Theorem 22 below. Recall from the introductory Section 1.3 the definition of an r-paracontrolled system. (Paracontrolled systems in the generality of Section 1.3 were first introduced in Bailleul & Mouzard's work [7].)

Let $r>0$ and $(u_w)_{w\in\mathcal{U}_{<r}}$ be a system r-paracontrolled by the $([l])_{l\in\mathcal{L}}$, as in (1.10). For each

$$\tau = [\![w]\!]_{\mathrm{j}}X^p \in \mathcal{B}$$

with $w = l_{j_1}\dots l_{j_m}$ and $\mathrm{j}\in\mathcal{P}_{m-1}(\ell)$ with $l,p\in\mathbf{N}^{d_0}$ such that $|\tau|_{\mathcal{L}} < r$, set

$$u_\tau := \sum_{\substack{w'=l_{i_2}\dots l_{i_n}\in\mathcal{W}\\ ww'\in\mathcal{W}_{<r}}} \ \sum_{\mathfrak{k}\in\mathcal{P}_n(\ell+p)} \binom{k}{\mathfrak{k}} \widetilde{\mathsf{P}}^{(\gamma-|ww'|,|l_{i_2}|,\dots,|l_{i_n}|)-|\mathfrak{k}|}\Big(\partial^{k_1}u^\sharp_{ww'},\, \partial^{k_2}[l_{i_2}],\dots,\partial^{k_n}[l_{i_n}]\Big),$$

From Theorem 1, each u_τ defines a bounded function as $r-|\tau|_{\mathcal{L}}>0$. We define the T-valued function

$$\boldsymbol{u}(x) := \sum_{\tau\in\mathcal{B}} u_\tau(x)\,\tau.$$

Theorem 2 is a direct consequence of the following result.

*22 – **Theorem.** One has $\boldsymbol{u}\in\mathcal{D}^r(T,\mathsf{g})$, and its reconstruction $\mathsf{R}^{\mathsf{M}}(\boldsymbol{u})$ is equal to $u_{w_\emptyset}$*

Proof – We use Theorem 1 to prove that statement, but in a regularity structure that takes into account the $u_w^\sharp$ on the same footing as the $[l]$. We introduce for that purpose a new alphabet

$$\mathcal{A} := \mathcal{L}\sqcup\mathcal{W}$$

and set $|\lambda|_{\mathcal{A}} := |\lambda|_{\mathcal{L}}$ for $\lambda \in \mathcal{L}$ and $|\lambda|_{\mathcal{A}} := r - |\lambda|_{\mathcal{L}}$ for $\lambda \in \mathcal{W}$. We write $\mathcal{W}_{\mathcal{A}}$ for the set of words written with the alphabet $\mathcal{A}$. To avoid any confusion when writing words with the alphabet $\mathcal{A}$ we will write (w), with the parentheses, the letter of $\mathcal{A}$ associated with $w \in \mathcal{W}$. We extend our collection $([l])_{l\in\mathcal{L}}$ into $([\lambda])_{\lambda\in\mathcal{A}}$ setting

$$[w] := u_w^\sharp \in C^{r-|w|}$$

for $\lambda = w \in \mathcal{W}$. As above, Theorem 1 provides a regularity structure associated with $\mathcal{A}$ and a model $\overline{\mathsf{M}} = (\overline{\Pi}, \overline{\mathsf{g}})$ on it associated with $([\lambda])_{\lambda\in\mathcal{A}}$. There is a canonical injection $\iota : \mathscr{T}_{\mathcal{L}} \hookrightarrow \mathscr{T}_{\mathcal{A}}$ that commutes with the coproducts, and $\overline{\mathsf{M}}$ is an extension M. We can thus freely pass from $\overline{\mathsf{g}}$ to g in some computations below.

Within $\mathscr{T}_{\mathcal{A}}$, working with $\overline{\mathsf{M}}$, for $\tau = [\![w]\!]_{\mathrm{j}} X^p$ one can rewrite

$$\overline{u}_\tau(x) = \sum_{w'\in\mathcal{W}} \overline{\mathsf{g}}_x\big([(w'w)w']^{l+p}\big).$$

For $w \in \mathcal{W}$ we let

$$\rho(w) := [(w)w]$$

where $(w)w \in \mathcal{A}$ is the word beginning with the letter $(w) \in \mathcal{A}$ followed by $w \in \mathcal{A}$ – for $w = l_{i_1} \cdots l_{i_n}$ it represents the function $\mathsf{P}\big(u_w^\sharp, [l_{i_1}], \dots, [l_{i_n}]\big)$. Then the function $\boldsymbol{u}$ can be re-written in $\mathscr{T}_{\mathcal{A}}$ under the form

$$\boldsymbol{u}(x) = \sum_{w\in\mathcal{U}_{<r}} \sum_{\sigma<\rho(w)} \overline{\mathsf{g}}_x(\rho(w)/\sigma)\,\sigma,$$

Note that any $\sigma < \rho(w)$ has form $[w']_{\mathrm{j}} X^p$ where w' is a subword of w. We now prove that $\boldsymbol{u} \in \mathcal{D}^r(T, \mathsf{g})$ by proving that for any $w \in \mathcal{U}_{<r}$ the map $\boldsymbol{h}_w(x) = \sum_{\sigma<\rho(w)} \overline{\mathsf{g}}_x(\rho(w)/\sigma)\,\sigma$ is an element of $\mathcal{D}^{|\rho(w)|}(T_{\mathcal{A}}, \overline{g})$. For any $x, y \in \mathbf{R}^{d_0}$, and for $\tau = [\![w]\!]_{\mathrm{j}} X^p$, one has

$$\begin{aligned}\widehat{\overline{\mathsf{g}}_{yx}}(\boldsymbol{h}_w(x)) &= \sum_{\nu\le\sigma<\rho(w)} \overline{\mathsf{g}}_x\big(\rho(w)/\sigma\big)\,\mathsf{g}_{yx}(\sigma/\nu)\,\nu = \sum_{\nu<\tau}\Big(\overline{\mathsf{g}}_y(\rho(w)/\nu) - \overline{\mathsf{g}}_{yx}(\rho(w)/\nu)\Big)\,\nu \\ &= \boldsymbol{h}_w(y) - \sum_{\nu<\tau} \overline{\mathsf{g}}_{yx}\big(\rho(w)/\nu\big)\,\nu.\end{aligned}$$

Theorem 1 ensures that $\big|\overline{\mathsf{g}}_{yx}(\rho(w)/\nu)\big| \lesssim |y-x|^{|\rho(w)/\nu|}$, with $|\rho(w)/\nu| = r - |\nu|_{\mathcal{A}}$, hence $\boldsymbol{h}_w \in \mathcal{D}^r(T, \mathsf{g})$. As

$$\overline{\Pi}_x(\boldsymbol{h}_w(x)) = \sum_{\sigma<\rho(w)} \overline{\mathsf{g}}_x(\rho(w)/\sigma)\,\overline{\Pi}_x\sigma = \overline{\Pi}(\rho(w)) - \overline{\Pi}_x(\rho(w)),$$

the reconstruction of $\boldsymbol{h}_w$ is

$$\overline{\Pi}(\rho(w)) = \mathsf{P}\big(u_w^\#,\, [l_{j_1}], \dots, [l_{j_m}]\big),$$

where $w = l_{j_1} \dots l_{j_m}$. And finally $\mathsf{R}^{\mathsf{M}}(\boldsymbol{u}) = \sum_{w\in\mathcal{U}_{<r}} \mathsf{R}^{\mathsf{M}}(\boldsymbol{h}_w) = u_{w_\emptyset}$.

▷

A – Appendix

A.1 – Basics in regularity structures. We recall here some basic facts about regularity structure. We refer the reader to [4] for a thorough introduction to the subject, and to [13] for the original work of M. Hairer on the subject.

23 – Definition. *A **concrete regularity structure** is a pair $\mathscr{T} = (T, T^+)$ of graded vector spaces*

$$T = \bigoplus_{r\in A} T_r, \qquad T^+ = \bigoplus_{s\in A^+} T_s^+,$$

such that the following holds.

- *The spaces T_r and T_s^+ are finite dimensional for any $r \in A$ and $s \in A^+$. One has $A^+ \subset [0, +\infty)$ and both A and A^+ are bounded from below and have no accumulation points.*
- *The vector space T^+ is an Hopf algebra with coproduct Δ^+ and grading $A^+ \subset [0, +\infty[$.*

– *The vector space is endowed with a linear splitting map* $\Delta : T \to T \otimes T^+$ *such that*
$$(\Delta \otimes \mathrm{Id})\Delta = (\mathrm{Id} \otimes \Delta^+)\Delta.$$
– *We have*
$$\Delta T_{r_1} \subset \bigoplus_{r_2 \in A} T_{r_2} \otimes T^+_{r_1 - r_2}, \qquad \Delta^+ T^+_{s_1} \subset \bigoplus_{s_2 \in A^+} T^+_{s_2} \otimes T^+_{s_1 - s_2}.$$

We suppose here that the vector spaces T and T^+ come with some bases $\mathcal{B}$ and $\mathcal{B}^+$. Then for any $\tau \in T$ we have a decomposition
$$\Delta\tau = \sum_{\sigma \in \mathcal{B}} (\tau/\sigma) \otimes \sigma$$
for some elements $\tau/\sigma \in T$. Likewise we define $\tau/\sigma \in T^+$ for $\tau \in T^+$ and $\sigma \in \mathcal{B}^+$ from the identity
$$\Delta^+\tau = \sum_{\sigma \in \mathcal{B}^+} (\tau/\sigma) \otimes \sigma.$$
For $\sigma, \tau \in \mathcal{B}$ we write $\sigma \leq \tau$ if $\tau/\sigma \neq 0$ and $\sigma < \tau$ if σ and τ are distinct and $\sigma \leq \tau$. For $\tau, \sigma, \nu \in \mathcal{B}$ we have
$$\Delta^+(\tau/\sigma) = \sum_{\sigma \leq \nu \leq \tau} \tau/\nu \otimes \nu/\sigma.$$
We denote by G^+ the set of real-valued characters of the algebra T^+. We endow G^+ with a group structure by defining the convolution product of g_1 and g_2 as
$$(\mathsf{g}_1 * \mathsf{g}_2)(\tau) = (\mathsf{g}_1 \otimes \mathsf{g}_2)\Delta^+\tau,$$
for all $\tau \in T$. We write g^{-1} for the inverse of a character $\mathsf{g} \in G^+$ in this group structure. For any map $x \in \mathbf{R}^{d_0} \mapsto \mathsf{g}_x \in G^+$ we define for any $x, y \in \mathbf{R}^{d_0}$ the character
$$\mathsf{g}_{yx} := \mathsf{g}_y * \mathsf{g}_x^{-1}.$$
Similarly we define for any map $\Pi : T \to \mathcal{D}'(\mathbf{R}^{d_0})$, any point $x \in \mathbf{R}^{d_0}$, a new map $\Pi_x : T \to \mathcal{D}'(\mathbf{R}^{d_0})$ by setting
$$\Pi_x = \big(\Pi \otimes \mathsf{g}_x^{-1}\big)\Delta.$$
For any function φ, point $x \in \mathbf{R}^{d_0}$ and $\varepsilon > 0$ we set
$$\varphi_x^\lambda(\cdot) := \varepsilon^{-d}\,\varphi\Big(\frac{\cdot - x}{\varepsilon}\Big).$$
Finally for any integer n_0 also define $\mathcal{F}_{n_0}$ as the set of C^{n_0} functions φ supported in the unit ball of $\mathbf{R}^{d_0}$ and such that $\|\varphi\|_{C^{n_0}} \leq 1$.

24* – Definition.** *Pick* $n \geq |\beta_0|$. *A* ***model $\mathsf{M} = (\Pi, \mathsf{g})$ *over a regularity structure* $\mathcal{T}$ *is a pair of maps*
$$\Pi : T \to C^{\beta_0}(\mathbf{R}^{d_0}), \qquad \mathsf{g} : \mathbf{R}^{d_0} \to G^+$$
with the following properties.

– *For any* $x \in \mathbf{R}^{d_0}$ *and* $\tau \in T_{|\tau|}$ *we have*
$$\big|\Pi_x(\tau)(\varphi_x^\varepsilon)\big| \lesssim \varepsilon^{|\tau|}$$
uniformly in x *in compact subsets of* $\mathbf{R}^{d_0}$, *in* $\varepsilon \in (0,1)$ *and in* $\varphi \in \mathcal{F}_{n_0}$.

– *For any* $x, y \in \mathbf{R}^{d_0}$ *and* $\mu \in T^+_{|\mu|}$ *we have*
$$|\mathsf{g}_{yx}(\mu)| \lesssim |y - x|^{|\mu|}$$
uniformly for x, y *in compact subsets of* $\mathbf{R}^{d_0}$.

Definition – *Let* $\mathscr{T}$ *be a regularity structure and* $\mathsf{M} = (\Pi, \mathsf{g})$ *be a model on it. For any* $r \in \mathbf{R}$, *a modelled distributions* $\mathfrak{f} \in \mathcal{D}^r(T, \mathsf{g})$ *is a map* $\mathfrak{f} : \mathbf{R}^{d_0} \to \bigoplus_{r' < r} T_{r'}$ *such that*

$$\max_{r'<r}\sup_{x\in\mathbf{R}^{d_0}}\|\mathfrak{f}(x)\|_{r'}<+\infty,$$
$$\max_{r'<r}\sup_{x,y\in\mathbf{R}^{d_0}}\frac{\big\|\mathfrak{f}(y)-\widehat{\mathfrak{g}}_{yx}(\mathfrak{f}(x))\big\|_{r'}}{|y-x|^{r-r'}}<+\infty.$$

A.2 – Basics on analysis and proofs of three lemmas. For any function f and any multi-index $\ell\in\mathbf{N}^{d_0}$ we define the modified Littlewood-Paley projector

$$(\Delta_i^\ell f)(x):=\int_{\mathbf{R}^{d_0}}K_i(x-y)(y-x)^\ell f(y)\,dy.$$

25 – Lemma. *For $f\in C^r$ with $r>0$ one has $\big(\Delta_i^\ell f\big)_{i\geq -1}\in\mathsf{C}^{r+|\ell|}$ and*

$$\big\|(\Delta_i^\ell f)_{i\geq-1}\big\|_{\mathsf{C}^{r+|\ell|}}\lesssim\|f\|_r\,.$$

Proof – If $|i-j|\geq 2$ we have $\Delta_i^\ell(\Delta_j f)=\Delta_j(\Delta_i^\ell f)=0$, so $\Delta_i^k f$ is spectrally supported in a ball 2^iB and

$$(\Delta_i^\ell f)(x)=\sum_{|j-i|\leq1}\Delta_i^\ell(\Delta_j f)(x)=\sum_{|j-i|\leq1}\int K_i(x-y)(x-y)^\ell(\Delta_j f)(y)\,dy$$

Then we get

$$\big|(\Delta_i^\ell f)(x)\big|\leq\Big|\int K_i(z)z^\ell\mathrm{d}z\Big|\sum_{|j-i|\leq1}\|\Delta_j f\|_{L^\infty}\leq 2^{-ir}\Big|\int K_i(z)z^\ell\mathrm{d}z\Big|\,\|f\|_r\leq 2^{-i(r+|\ell|)}\,\|f\|_r\,,$$

using the scaling property of the kernel K_i for the last inequality. ▷

Note that the sequence $(\Delta_i^\ell f)_{i\geq-1}$ does not represent the Littlewood-Paley blocks of any distribution as $\sum_i\Delta_i^\ell f=0$ for any $\ell\neq0$.

Proof of Lemma 3. Pick $\mathsf{f}=(f_i)_{i\geq-1}\in\mathsf{C}^r$ and $o>0$ with integer part $\lfloor o\rfloor$. If f_i is spectrally supported in a ball 2^iB, then $f_i(\cdot+h)-\sum_{|k|<o}\partial^k f_i\frac{h^k}{k!}$ is spectrally supported in the same ball 2^iB. From Taylor Young inequality applied to f_i at order $\lfloor o\rfloor+1$ and Bernstein inequality we have for any $x\in\mathbf{R}^{d_0}$

$$\begin{aligned}\Big|f_i(x+h)-\sum_{|k|<o}\partial^k f_i(x)\frac{h^k}{k!}\Big|&\lesssim|h|^{\lfloor o\rfloor+1}\Big\|D^{\lfloor o\rfloor+1}f_i\Big\|_{L^\infty}\\&\lesssim|h|^{\lfloor o\rfloor+1}2^{i(\lfloor o\rfloor+1)}\,\|f_i\|_{L^\infty}\,.\end{aligned}$$

Similarly Taylor-Young inequality at order $\lfloor o\rfloor$ gives

$$\Big|f_i(x+h)-\sum_{|k|<\lfloor o\rfloor}\partial^k f_i(x)\frac{h^k}{k!}\Big|\lesssim|h|^{\lfloor o\rfloor}2^{i\lfloor o\rfloor}\,\|f_i\|_{L^\infty}\,,$$

from which we see that

$$\begin{aligned}\Big|f_i(x+h)-\sum_{|k|<o}\partial^k f_i(x)\frac{h^k}{k!}\Big|&\leq\Big|f_i(x+h)-\sum_{|k|<\lfloor o\rfloor}\partial^k f_i(x)\frac{h^k}{k!}\Big|+\Big|\sum_{|k|=\lfloor o\rfloor}\partial^k f_i(x)\frac{h^k}{k!}\Big|\\&\lesssim|h|^{\lfloor o\rfloor}2^{i\lfloor o\rfloor}\,\|f_i\|_{L^\infty}+|h|^{\lfloor o\rfloor}\Big\|D^{\lfloor o\rfloor}f_i\Big\|_{L^\infty}\lesssim|h|^{\lfloor o\rfloor}2^{i\lfloor o\rfloor}\,\|f_i\|_{L^\infty}\,.\end{aligned}$$

We conclude by interpolation that we have

$$\Big|f_i(x+h)-\sum_{|k|<o}\partial^k f_i(x)\frac{h^k}{k!}\Big|\lesssim|h|^o2^{io}\,\|f_i\|_{L^\infty}\lesssim|h|^o\,2^{-i(r-o)}\,\|\mathsf{f}\|_r\,.$$

Proof of Lemma 11. Let $\delta > 0$ such that the estimate holds for $\theta \in [\gamma-\delta, \gamma+\delta]$. For $x, y \in \mathbf{R}^d$ with $|y-x| \le 1$, one has for any integer N

$$\sum_{i\le N} X^i_{yx} \le C \sum_{i\le N} |y-x|^{\gamma+\eta} 2^{i\eta} \lesssim C 2^{N\eta} |y-x|^{\gamma+\eta},$$
$$\sum_{i> N} X^i_{yx} \le C \sum_{i> N} |y-x|^{\gamma-\eta} 2^{-i\eta} \lesssim C 2^{-N\eta} |y-x|^{\gamma-\eta},$$

Choosing N such that $|y-x| \simeq 2^{-N}$ gives the required bound.

Proof of Lemma 21. Using the definition of $\triangle_{h,r'}\mathsf{P}_<$ we have

$$\sum_{\sigma_e\prec\cdots\prec\tau_n(\mathfrak{j})} \big(\triangle_{h,r'}\widetilde{\mathsf{P}}^{|\tau_n(\mathfrak{j})/\boldsymbol{\sigma}^+|_{\boldsymbol{\alpha}}+}_<\big)\Big([\tau_n(\mathfrak{j})/\sigma_1]^{\mathsf{M}^{r+}},\dots,[\sigma_e]^{\mathsf{M}^{r+}},[n]^{\mathsf{M}^{r+}}\Big)(x)$$
$$= \sum_{e\ge0}\sum_{\sigma_e\prec\cdots\prec\tau_n(\mathfrak{j})} \widetilde{\mathsf{P}}^{|\tau_n(\mathfrak{j})/\boldsymbol{\sigma}^+|_{\boldsymbol{\alpha}}+}_<\Big([\tau_n(\mathfrak{j})/\sigma_1]^{\mathsf{M}^{r+}},\dots,[\sigma_e]^{\mathsf{M}^{r+}},[n]^{\mathsf{M}^{r+}}\Big)(x+h)$$
$$-\sum_{\substack{e\ge0\\ \sigma_e\prec\cdots\prec\tau_n(\mathfrak{j})}}\sum_{|k|<r'} \partial^k_\star\mathsf{P}_<\Big([\tau_n(\mathfrak{j})/\sigma_1]^{\mathsf{M}^{r+}},\dots,[\sigma_e]^{\mathsf{M}^{r+}},[n]^{\mathsf{M}^{r+}}\Big)(x)\,h^p$$
$$-\sum_{\substack{e\ge0\\ \sigma_e\prec\cdots\prec\tau_n(\mathfrak{j})}}\sum_{\substack{1\le m\le e\\ |k|<|\tau_n(\mathfrak{j})/\sigma_m|}} \partial^k_\star\mathsf{P}_<\Big([\tau_n(\mathfrak{j})/\sigma_1]^{\mathsf{M}^{r+}},\dots,[\sigma_{m-1}/\sigma_m]^{\mathsf{M}^{r+}}\Big)$$
$$\times\frac{h^k}{k!}\,\big(\triangle_{h,|\sigma_m|_\alpha+\alpha_n^+}\mathsf{P}_<\big)\Big([\sigma_m/\sigma_{m-1}]^{\mathsf{M}^{r+}},\dots,[n]^{\mathsf{M}^{r+}}\Big),$$

where we use the shorthand

$$\partial^k_\star\mathsf{P}_<\Big([\tau_n(\mathfrak{j})/\sigma_1]^{\mathsf{M}^{r+}},\dots,[\sigma_{m-1}/\sigma_m]^{\mathsf{M}^{r+}}\Big)$$
$$=\sum_{\mathfrak{k}\in\mathcal{P}_m(k)}\binom{k}{\mathfrak{k}}\widetilde{\mathsf{P}}^{D^{\mathfrak{k}}|\tau_n(\mathfrak{j})/\boldsymbol{\sigma}_{\le m}|_\alpha}_<\Big(\partial^{k_1}[\tau_n(\mathfrak{j})/\sigma_1]^{\mathsf{M}^{r+}},\dots,\partial^{k_m}[\sigma_{m-1}/\sigma_m]^{\mathsf{M}^{r+}}\Big).$$

The first line of the right hand side gives $\mathsf{P}_{\mathfrak{j}}\big(f_1,\dots,\Delta_r f_n\big)(x+h)$. As

$$\int_{\mathbf{R}^{d_0}} K_{<i}(h)h^p\,dh = 0$$

for $p\neq0$, the second line of the right hand side gives a zero contribution when integrated against $K_{<i}$ except for $p=0$, in which case it gives $\mathsf{P}_{\mathfrak{j}}\big(f_1,\dots,\Delta_r f_n\big)(x)$. Then

$$\int_{\mathbf{R}^{d_0}}\sum_{\sigma_e\prec\cdots\prec\tau_n(\mathfrak{j})} \big(\triangle_{h,r'}\widetilde{\mathsf{P}}^{|\tau_n(\mathfrak{j})/\boldsymbol{\sigma}^+|_{\boldsymbol{\alpha}}+}_<\big)\Big([\tau_n(\mathfrak{j})/\sigma_1]^{\mathsf{M}^{r+}},\dots,[\sigma_e]^{\mathsf{M}^{r+}},[n]^{\mathsf{M}^{r+}}\Big)(x)\,dh$$
$$=\int_{\mathbf{R}^{d_0}} K_{<i}(h)\,\mathsf{P}_{\mathfrak{j}}\big(f_1,\dots,\Delta_r f_n\big)(x+h)\,dh - \mathsf{P}_{\mathfrak{j}}\big(f_1,\dots,\Delta_r f_n\big)(x)$$
$$-\sum_{\sigma\prec\tau_n(\mathfrak{j})}\sum_{|k|<|\tau_n(l)/\sigma|}\sum_{\substack{e_1\ge0\\ \sigma\prec\sigma_{e_1}\prec\cdots\prec\tau_n(\mathfrak{j})}} \partial^k_\star\mathsf{P}_<\Big([\tau_n(l)/\sigma_1]^{\mathsf{M}^{r+}},\dots,[\sigma]^{\mathsf{M}^{r+}}\Big)(x)$$
$$\times\int_{\mathbf{R}^{d_0}} K_{<i}(h)\,\frac{h^k}{k!}\sum_{e_2\ge0}\sum_{\nu_{e_2}\prec\cdots\prec\sigma}\triangle_{h,|\sigma|+\alpha_n^+}\mathsf{P}_<\Big([\sigma/\nu_1]^{\mathsf{M}^{r+}},\dots,[n]^{\mathsf{M}^{r+}}\Big)\,dh.$$

The $\sigma\in\mathcal{B}$ such that $\sigma\prec\tau_n(\mathfrak{j})$ have form $\sigma=[\![m+1,n-1]\!]_{\mathfrak{j}+p_1}X^{p_2+s_2}$ and $\tau/\sigma=[\![1,m]\!]^p_{\mathfrak{j}}X^{s_1}$ where $p=s_1+p_2$ and $l_{n-1}=s_1+s_2$. For such σ, using Lemma 20 the sum

$$\sum_{\sigma\prec\cdots\prec\tau_n(\mathfrak{j})}\partial^k_\star\mathsf{P}_<\Big([\tau_n(\mathfrak{j})/\sigma_1]^{\mathsf{M}^{r+}},\dots,[\sigma_{m-1}/\sigma]^{\mathsf{M}^{r+}}\Big)(x)$$

is 0 if $s_1 \neq 0$, otherwise ($s_1 = 0$) this sum is equal to

$$\sum_{\mathfrak{p}\in\mathcal{P}_m(p)} \binom{p}{\mathfrak{p}} \widetilde{\mathsf{P}}^{D^{\mathfrak{k}+\mathfrak{p}}\alpha}_{\mathsf{j}_{\leq m}}\big(\partial^{k_1+p_1} f_1,\dots,\partial^{k_m+p_m} f_m\big)(x) = \mathsf{g}^{\mathfrak{f}}\big([\![1,m]\!]^{\mathfrak{k}+p}_{\mathsf{j}_{\leq m}}\big).$$

On the other hand for $\sigma = [\![m+1,n]\!]_{\mathsf{j}+p_1} X^{p_2+l_{n-1}}$, we have from the induction hypothesis

$$\int_{\mathbf{R}^{d_0}} K_{<i}(h)\,\frac{h^k}{k!} \sum_{e_2\geq 0}\sum_{\nu_{e_2}\prec\cdots\prec\sigma} \big(\triangle_{h,|\sigma|+\alpha_n^+}\mathsf{P}_<\big)\Big([\sigma/\nu_1]^{\mathsf{M}^{r+}},\dots,[n]^{\mathsf{M}^{r+}}\Big)\,dh$$

$$= \int_{\mathbf{R}^{d_0}} K_{<i}(h)\,\Pi^{r'}_x\big([\![m+1,n]\!]_{\mathsf{j}+p}X^k\big)(x+h).$$

One finally gets

$$\int_{\mathbf{R}^{d_0}} \sum_{\sigma_e\prec\cdots\prec\tau_n(\mathsf{j})} \Big(\triangle_{h,r'}\widetilde{\mathsf{P}}^{|\tau_n(\mathsf{j})/\boldsymbol{\sigma}^+|_{\boldsymbol{\alpha}}+}_<\Big)\Big([\tau_n(\mathsf{j})/\sigma_1]^{\mathsf{M}^{r+}},\dots,[\sigma_e]^{\mathsf{M}^{r+}},[n]^{\mathsf{M}^{r+}}\Big)(x)\,dh$$

$$= \int_{\mathbf{R}^{d_0}} K_{<i}(h)\Big\{\mathsf{P}_{\mathsf{j}}\big(f_1,\dots,\Delta_r f_n\big)(x+h) - \mathsf{P}_{\mathsf{j}}\big(f_1,\dots,\Delta_r f_n\big)(x)$$
$$- \sum_{m,k,p} \mathsf{g}^{r'}_x\big([\![1,m]\!]^{k+p}_{\mathsf{j}}\big)\Pi^{r'}_x\big(X^k[\![m+1,n]\!]_{\mathsf{j}+p}\big)(x+h)\,dh\Big\}$$

$$= \int_{\mathbf{R}^{d_0}} K_{<i}(h)\Big\{\Pi^{r'}\big([\![1,n]\!]_{\mathsf{j}}\big)(x+h) - \sum_{\sigma<[\![1,n]\!]_{\mathsf{j}}} \mathsf{g}^{r'}_x\big([\![1,n]\!]_{\mathsf{j}}/\sigma\big)\Pi^{r'}_x(\sigma)(x+h)\Big\}\,dh$$

$$= \int_{\mathbf{R}^{d_0}} K_{<i}(h)\,\Pi^{r'}_x\big([\![1,n]\!]_{\mathsf{j}}\big)(x+h)\,dh,$$

so we have indeed (4.14).

A.3 – Proofs for Section 2.4 Proposition 13 gives the expansion property of the functions $\partial^p_\star\mathsf{P}_<$. The proof of this proposition requires that we introduce some operators.

§1. The operators $\widetilde{\mathsf{P}}^{\beta^1,\beta^2}_<$ – It will be convenient to introduce as an intermediate tool some operators $\widetilde{\mathsf{P}}^{\beta^1,\beta^2}_<$ indexed by two tuples of integers. This operator will be useful to obtain the local expansion of the $\widetilde{\mathsf{P}}^{\beta^1}_<$, the uplet β^2 will play a different role, similar to the one of the β when we used the $\widetilde{\mathsf{P}}^{\beta}$ operator to obtain the expansion of the simplified paraproduct in Section 2.3

Their definition requires the following notation.

Definition – *For β^1,β^2 in $\mathbf{R}^n$ such that $\beta^1_i \geq \beta^2_i$ for all $1\leq i\leq n$ we set*

$$\mathsf{MultiCut}\big(\beta^1,\beta^2\big) := \Big\{\boldsymbol{d} = \big(0=d_0<d_1<\cdots<d_{n(\boldsymbol{d})}=n\big)\,;\,\forall e\in[\![1,n(\boldsymbol{d})-1]\!],\ d_e\in\mathsf{Cut}(\beta^1)\cup\mathsf{Cut}(\beta^2)\Big\}.$$

For $(\mathsf{h}_i)_{1\leq i\leq n}\subset \mathsf{C}^{0+}$ we set

$$\widetilde{\mathsf{P}}^{\beta^1,\beta^2}_<\big(\mathsf{h}_1,\dots,\mathsf{h}_n\big) := \sum_{\boldsymbol{d}\in\mathsf{MultiCut}(\beta^1,\beta^2)} (-1)^{n(\boldsymbol{d})+1}\prod_{e=1}^{n(\boldsymbol{d})} \mathsf{P}_<\big(\mathsf{h}_{d_{e-1}+1},\dots,\mathsf{h}_{d_e}\big).$$

We will use in the end the operators $\widetilde{\mathsf{P}}^{\beta^1,\beta^2}_<$ in some situations where $\sum_{i=1}^n\beta^2_i>0$. In that case we have $\mathsf{Cut}(\beta^1)\subset\mathsf{Cut}(\beta^2)$, so $\widetilde{\mathsf{P}}^{\beta^1,\beta^2}_<(\mathsf{h}_1,\dots,\mathsf{h}_n) = \widetilde{\mathsf{P}}^{\beta^2}_<(\mathsf{h}_1,\dots,\mathsf{h}_n)$, and we can use the continuity property of Proposition 5. The general $\widetilde{\mathsf{P}}^{\beta^1,\beta^2}_<$ operators will be useful in the algebraic steps below.

For any $k\in\mathbf{N}^{d_0}$ and $\mathfrak{k}=(k_a,\dots,k_b)\in\mathcal{P}_{b-a+1}(k)$ we also set

$$\partial^{\mathfrak{k}}_\star\widetilde{\mathsf{P}}^{\beta_{[\![a,b]\!]}}_<(\mathsf{f}_a,\dots,\mathsf{f}_b) := \widetilde{\mathsf{P}}^{\beta_{[\![a,b]\!]},\beta_{[\![a,b]\!]}-|\mathfrak{k}|}_<\big(\partial^{k_a}\mathsf{f}_a,\dots,\partial^{k_b}\mathsf{f}_b\big)$$

and

$$\partial_\star^k \widetilde{\mathsf{P}}_<^{\beta_{[\![a,b]\!]}}(\mathsf{f}_a,\dots,\mathsf{f}_b) := \sum_{\mathfrak{k}\in\mathcal{P}_{b-a+1}(k)} \binom{k}{\mathfrak{k}} \partial_\star^{\mathfrak{k}} \widetilde{\mathsf{P}}_<^{\beta_{[\![a,b]\!]}}(\mathsf{f}_a,\dots,\mathsf{f}_b).$$

The following statement is proved in Appendix A.4.1.

26 – Lemma. *Pick β^1,β^2 in $\mathbf{R}^d$ satisfying **Assumption (A)**. If $\beta^2_i \leq \beta^1_i$ for all $1\leq i\leq n$ then for any $\mathsf{h}_1,\dots,\mathsf{h}_n$ in C^∞ we have*

$$\widetilde{\mathsf{P}}_<^{\beta^1,\beta^2}\big(\mathsf{h}_1,\dots\mathsf{h}_n\big) = \widetilde{\mathsf{P}}_<^{\beta^1}\big(\mathsf{h}_1,\dots,\mathsf{h}_n\big) - \sum_{d\in\mathsf{Cut}(\beta^2)\setminus\mathsf{Cut}(\beta^1)} \widetilde{\mathsf{P}}_<^{\beta^1_{\leq m},\beta^2_{\leq m}}\big(\mathsf{h}_1,\dots,\mathsf{h}_m\big)\, \widetilde{\mathsf{P}}_<^{\beta^1_{>m},\beta^2_{>m}}\big(\mathsf{h}_{m+1},\dots,\mathsf{h}_n\big).$$

§2 Local expansion properties of the $\widetilde{\mathsf{P}}_<^\beta\big(\mathsf{f}_1,\dots,\mathsf{f}_n\big)$ – Pick $\beta\in\mathbf{R}^n$ such that $\sum_{i=1}^n \beta_i > 0$. Proceding as in (2.5) we see that

$$\begin{aligned}\widetilde{\mathsf{P}}_<^\beta\big(\mathsf{f}_1,\dots,\mathsf{f}_n\big)(\cdot+h) &= \widetilde{\mathsf{P}}_<^\beta\big(\mathsf{f}_1(\cdot+h),\dots,\mathsf{f}_n(\cdot+h)\big)\\ &= T_h^o \widetilde{\mathsf{P}}_<^\beta\big(\mathsf{f}_1,\dots,\mathsf{f}_n\big)\\ &\quad + \sum_{m=1}^n \sum_{\substack{|k|<o\\ \mathfrak{k}\in\mathcal{P}_{m-1}(k)}} \frac{h^k|h|^{o-|k|}}{\mathfrak{k}!} \widetilde{\mathsf{P}}_<^\beta\Big(\partial^{k_1}\mathsf{f}_1,\dots,\partial^{k_{m-1}}\mathsf{f}_{m-1}, R_h^{o-|k|}\mathsf{f}_m, \mathsf{f}_{m+1}(\cdot+h),\dots\Big)\end{aligned}$$

With the same motivations as in Section 2.3 we set here

$$\beta_a(\mathfrak{k},o) := \Big(\beta_{a+1}-|k_{a+1}|,\dots,\beta_{b-1}-|k_{b-1}|,\ \beta_b - o + |k|,\ \beta_{b+1},\dots,\beta_n\Big).$$

and

$$\begin{aligned}\big(\triangle_{h,o}\widetilde{\mathsf{P}}_<^\beta\big)(\mathsf{f}_{a+1},\dots,\mathsf{f}_n) := \sum_{b=a+1}^n \sum_{\substack{|k|<o\\ \mathfrak{k}\in\mathcal{P}_{b-a-1}(k)}} \frac{h^k|h|^{o-|k|}}{\mathfrak{k}!}\, \widetilde{\mathsf{P}}_<^{\beta>a,\beta_a(\mathfrak{k},r)}\Big(&\partial^{k_{a+1}}\mathsf{f}_{a+1}\dots,\partial^{k_{b-1}}\mathsf{f}_{b-1},\\ &R_h^{o-|k|}\mathsf{f}_b,\ \mathsf{f}_{b+1}(\cdot+h),\dots,\mathsf{f}_n(\cdot+h)\Big),\end{aligned}$$

and for $i\geq -1$

$$\begin{aligned}\big(\triangle_{h,o}\widetilde{\mathsf{P}}_<^\beta\big)(\mathsf{f}_{a+1},\dots,\mathsf{f}_n)\{i\} := \sum_{b=a+1}^n \sum_{\substack{|k|<o\\ \mathfrak{k}\in\mathcal{P}_{b-a-1}(k)}} \frac{h^k|h|^{o-|k|}}{\mathfrak{k}!}\, \widetilde{\mathsf{P}}_<^{\beta>a,\beta_a(\mathfrak{k},r)}\Big(&\partial^{k_{a+1}}\mathsf{f}_{a+1}\dots,\partial^{k_{b-1}}\mathsf{f}_{b-1},\\ &R_h^{o-|k|}\mathsf{f}_b,\ \mathsf{f}_{b+1}(\cdot+h),\dots,\mathsf{f}_n(\cdot+h)\Big)\{i\}.\end{aligned}$$

We define

$$\mathsf{I}(\beta) := \Big\{c\in[\![1,n-1]\!]\,;\ \sum_{j=1}^c \beta_j > 0 \ \ \textit{and}\ \ \sum_{j=c+1}^n \beta_j > 0\Big\}.$$

27 – Proposition. *For $o > \sum_{j=1}^n \beta_j - \delta_0$, we have*

$$\begin{aligned}\big(\triangle_{h,r}^\beta\widetilde{\mathsf{P}}_<^\beta\big)(\mathsf{f}_1,\dots,\mathsf{f}_n)\{i\} &= \widetilde{\mathsf{P}}_<^\beta\big(\mathsf{f}_1,\dots,\mathsf{f}_n\big)\{i\}(\cdot+h) - T_h^o\widetilde{\mathsf{P}}_<^\beta\big(\mathsf{f}_1,\dots,\mathsf{f}_n\big)\{i\}\\ &\quad - \sum_{c\in\mathsf{I}(\beta)}\ \sum_{|k|<\sum_{j=1}^c\beta_j} \partial_{\star\beta_{\leq c}}^k \mathsf{P}_<\big(\mathsf{f}_1,\dots,\mathsf{f}_c\big)\,\frac{h^k}{k!}\,\big(\triangle_{h,o-|k|}^\beta\widetilde{\mathsf{P}}_<^{\beta>c}\big)\big(\mathsf{f}_{c+1},\dots,\mathsf{f}_n\big)\{i\},\end{aligned}$$

Proof – Recall that for $k\in\mathbf{N}^{d_0}$ and $\mathfrak{k}\in\mathcal{P}_{j-1}(k)$

$$\beta(\mathfrak{k},o) = \Big(\beta_1-|k_1|,\dots,\beta_{j-1}-|k_{j-1}|,\beta_j-o+|k|,\beta_{j+1},\dots,\beta_n\Big).$$

We are going to apply Lemma 26 with the tuples α and $\beta(\mathfrak{k},o)$, which verify indeed $\beta(\mathfrak{k},o)_a \le \alpha_a$ for any $1 \le a \le n$. Moreover for $a > j$ we have $\beta(\mathfrak{k},o)_a = \alpha_a$, and as consequence $\mathsf{Cut}(\beta(\mathfrak{k},o))\backslash\mathsf{Cut}(\alpha) \subset [\![1, j-1]\!]$. Then Lemma 26 gives

$$\begin{aligned}
&\widetilde{\mathsf{P}}^{\beta(\mathfrak{k},o)}_{<}\Big(\partial^{k_1}\mathsf{f}_1,\dots,\partial^{k_{j-1}}\mathsf{f}_{j-1},\, R_h^{o-|k|}\mathsf{f}_j,\, \mathsf{f}_{j+1}(\cdot+h),\dots\Big)\{i\}\\
&= \widetilde{\mathsf{P}}^{\beta}_{<}\Big(\partial^{k_1}\mathsf{f}_1,\dots,\partial^{k_{j-1}}\mathsf{f}_{j-1}, R_h^{o-|k|}\mathsf{f}_j,\, \mathsf{f}_{j+1}(\cdot+h),\dots\Big)\{i\}\\
&\quad - \sum_{c\in\mathsf{Cut}(\beta(\mathfrak{k},o))\backslash\mathsf{Cut}(\alpha)} \widetilde{\mathsf{P}}^{\alpha_{\le c},\beta(\mathfrak{k},o)_{\le c}}_{<}\Big(\partial^{k_1}\mathsf{f}_1,\dots,\partial^{k_c}\mathsf{f}_c\Big)\\
&\qquad\qquad \times \widetilde{\mathsf{P}}^{\beta_{>c},\beta(\mathfrak{k},o)_{>c}}_{<}\Big(\partial^{k_{c+1}}\mathsf{f}_{c+1},\dots,\partial^{k_{j-1}}\mathsf{f}_{j-1}, R_h^{o-|k|}\mathsf{f}_j,\, \mathsf{f}_{j+1}(\cdot+h),\dots\Big)\{i\}\\
&= \widetilde{\mathsf{P}}^{\beta}_{<}\Big(\partial^{k_1}\mathsf{f}_1,\dots,\partial^{k_{j-1}}\mathsf{f}_{j-1},\;\; R_h^{o-|k|}\mathsf{f}_j,\, \mathsf{f}_{j+1}(\cdot+h),\dots\Big)\{i\}\\
&\quad - \sum_{c\in\mathsf{Cut}(\beta(\mathfrak{k},o))\backslash\mathsf{Cut}(\beta)} \partial_\star^{\mathfrak{k}_{\le c}}\mathsf{P}_{<}\Big(\mathsf{f}_1,\dots,\mathsf{f}_c\Big)\\
&\qquad\qquad \times \widetilde{\mathsf{P}}^{\alpha_{>c},\beta(\mathfrak{k},o)_{>c}}_{<}\Big(\partial^{k_{c+1}}\mathsf{f}_{c+1},\dots,\partial^{k_{j-1}}\mathsf{f}_{j-1}, R_h^{o-|k|}\mathsf{f}_j,\, \mathsf{f}_{j+1}(\cdot+h),\dots\Big)\{i\},
\end{aligned}$$

where we used that

$$\widetilde{\mathsf{P}}^{\beta(\mathfrak{k},o)_{\le c},\beta_{\le c}}_{<}\big(\partial^{k_1}\mathsf{f}_1,\dots,\partial^{k_c}\mathsf{f}_c\big) = \widetilde{\mathsf{P}}^{\beta(\mathfrak{k},o)_{\le c}}_{<}\big(\partial^{k_1}\mathsf{f}_1,\dots,\partial^{k_c}\mathsf{f}_c\big)$$

since $\sum_{i=1}^c \beta(\mathfrak{k},o)_i > 0$ for $c \in \mathsf{Cut}(\beta(\mathfrak{k},o))$.
We now sum over j, k and $\mathfrak{k}$ and invert the sums over c and j. In order to implement this sum inversion we use the inclusion $\mathsf{Cut}\big(\beta(\mathfrak{k},o)\big)\backslash\mathsf{Cut}\big(\beta\big) \subset \mathsf{I}(\beta)$. This gives

$$\begin{aligned}
&\big(\triangle_{h,o}\widetilde{\mathsf{P}}^{\beta}_{<}\big)\big(\mathsf{f}_1,\dots,\mathsf{f}_n\big)\{i\} - \sum_{j=1}^{n}\sum_{\substack{|k|<o\\ \mathfrak{k}\in\mathcal{P}_{j-1}(k)}} \frac{h^k|h|^{o-|k|}}{\mathfrak{k}!}\,\widetilde{\mathsf{P}}^{\beta}_{<}\Big(\partial^{k_1}\mathsf{f}_1,\dots,\partial^{k_{j-1}}\mathsf{f}_{j-1},\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad R_h^{o-|k|}\mathsf{f}_j,\, \mathsf{f}_{j+1}(\cdot+h),\,\dots\Big)\{i\}\\
&\quad= \sum_{j=1}^{n}\sum_{\substack{|k|<o\\ \mathfrak{k}\in\mathcal{P}_{j-1}(k)}} \frac{h^k|h|^{o-|k|}}{\mathfrak{k}!}\sum_{c\in\mathsf{Cut}(\beta(\mathfrak{k},o))\backslash\mathsf{Cut}(\beta)} \partial_\star^{\mathfrak{k}}\mathsf{P}_{<}\big(\mathsf{f}_1,\dots,\mathsf{f}_c\big)\\
&\qquad\quad \times \widetilde{\mathsf{P}}^{\beta_{>c},\beta(\mathfrak{k})_{>c}}_{<}\Big(\partial^{k_{c+1}}\mathsf{f}_{c+1},\dots,\partial^{k_{j-1}}\mathsf{f}_{j-1}, R_h^{o-|k|}\mathsf{f}_j,\, \mathsf{f}_{j+1}(\cdot+h),\,\dots\Big)\{i\}\\
&\quad= \sum_{c\in\mathsf{I}(\beta)}\sum_{\substack{|k|<\sum_{i=1}^c\beta_i\\ \mathfrak{k}\in\mathcal{P}_{c-1}(k)}} \frac{h^k}{\mathfrak{k}!}\,\partial_\star^{\mathfrak{k}}\mathsf{P}_{<}\big(\mathsf{f}_1,\dots,\mathsf{f}_c\big)\\
&\qquad \times \sum_{j=c+1}^{n}\sum_{\substack{|p|<o\\ \mathfrak{p}\in\mathcal{P}_{j-c-1}(p)}} \frac{h^p|h|^{o-|k|-|p|}}{\mathfrak{p}!}\,\widetilde{\mathsf{P}}^{\beta_{>c},\beta(\mathfrak{k},o)_{>c}}_{<}\Big(\partial^{p_1}\mathsf{f}_{c+1},\dots,\partial^{p_{j-c-1}}\mathsf{f}_{j-1},\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad R_h^{o-|p|}\mathsf{f}_j,\, \mathsf{f}_{j+1}(\cdot+h),\,\dots\Big)\{i\}\\
&\quad= \sum_{c\in\mathsf{I}(\beta)}\sum_{|k|<\sum_{i=1}^c\beta_i} \frac{h^k}{k!}\,\partial_\star^{k}\widetilde{\mathsf{P}}^{\beta_{\le c}}_{<}(\mathsf{f}_1,\dots,\mathsf{f}_c)\,\big(\triangle_{h,o-|k|}\widetilde{\mathsf{P}}^{\beta_{>c}}_{<}\big)(\mathsf{f}_{c+1},\dots,\mathsf{f}_n)\{i\}.
\end{aligned}$$

The result follows from this identity. $\triangleright$

***28* – Lemma.** *For* $0 \leq c \leq n-1$, *for* $o_2 > o_1 > \sum_{j=c+1}^n \alpha_j - \delta_0$, *we have*

$$\big(\triangle_{h,o_2} \widetilde{\mathsf{P}}_<^{\beta_{>c}}\big)(\mathsf{f}_{c+1}, \dots, \mathsf{f}_n)\{i\} - \big(\triangle_{h,o_1} \widetilde{\mathsf{P}}_<^{\beta_{>c}}\big)(\mathsf{f}_{c+1}, \dots, \mathsf{f}_n)\{i\} = \sum_{o_1 < |k| < o_2} \frac{h^k}{k!}\, \partial_\star^k \widetilde{\mathsf{P}}_<^{\beta_{>c}}(\mathsf{f}_{c+1}, \dots, \mathsf{f}_n)\{i\}.$$

Proof – The proof follows the same induction as for Lemma 10. The result is true for $c = n-1$ as $\triangle_{h,o} \widetilde{\mathsf{P}}_<^\beta$ coincides still with the Taylor remainder $|h|^o R_h^o$. Suppose it to be true for $(n-c-1)$ functions. Proposition 27 gives then

$$\begin{aligned}
&\big(\triangle_{h,o_2} \widetilde{\mathsf{P}}_<^{\beta_{>c}}\big)\big(\mathsf{f}_{c+1}, \dots, \mathsf{f}_n\big)\{i\} - \big(\triangle_{h,o_1} \widetilde{\mathsf{P}}_<^{\beta_{>c}}\big)\big(\mathsf{f}_{c+1}, \dots, \mathsf{f}_n\big)\{i\} \\
&= T_h^{o_2} \widetilde{\mathsf{P}}_<^{\beta_{>c}}\big(\mathsf{f}_{c+1}, \dots, \mathsf{f}_n\big)_i - T_h^{o_1} \widetilde{\mathsf{P}}_<^{\beta_{>c}}\big(\mathsf{f}_{c+1}, \dots, \mathsf{f}_n\big)_i \\
&\quad - \sum_{j \in \mathsf{I}(\beta_{>c})} \sum_{|p| < \sum_{a=c+1}^j \beta_a} \partial_\star^p \mathsf{P}_<(\mathsf{f}_{c+1}, \dots, \mathsf{f}_j)\, \frac{h^p}{p!} \\
&\qquad \times \Big\{ \big(\triangle_{h,o_2 - |p|} \widetilde{\mathsf{P}}_<^{\beta_{>j}}\big)\big(\mathsf{f}_{j+1}, \dots, \mathsf{f}_n\big)\{i\} - \big(\triangle_{h,o_1 - |p|} \widetilde{\mathsf{P}}_<^{\beta_{>j}}\big)\big(\mathsf{f}_{j+1}, \dots, \mathsf{f}_n\big)\{i\} \Big\}.
\end{aligned}$$

From the induction hypothesis this quantity is equal to

$$\begin{aligned}
&\sum_{o_1 < |k| < o_2} \frac{h^k}{k!}\, \partial^k \widetilde{\mathsf{P}}_<^{\beta_{>c}}(\mathsf{f}_{c+1}, \dots, \mathsf{f}_n)\{i\} - \sum_{j \in \mathsf{I}(\beta_{>c})} \sum_{|p| < \sum_{a=c+1}^j \beta_a} \frac{h^p}{p!}\, \partial_\star^p \mathsf{P}_<(\mathsf{f}_{c+1}, \dots, \mathsf{f}_j) \\
&\qquad \times \sum_{o_1 < |\ell| + |p| < o_2} \frac{h^\ell}{\ell!}\, \partial_\star^\ell \widetilde{\mathsf{P}}_<^{\beta_{>c}}(\mathsf{f}_{j+1}, \dots, \mathsf{f}_n)\{i\} \\
&= \sum_{o_1 < |k| < o_2} \frac{h^k}{k!} \sum_{\mathfrak{k} \in \mathcal{P}_{n-c}(k)} \binom{k}{\mathfrak{k}} \Lambda_{\mathfrak{k},i},
\end{aligned}$$

where

$$\begin{aligned}
\Lambda_{\mathfrak{k},i} &:= \widetilde{\mathsf{P}}_<^{\beta_{>c}}\big(\partial^{k_1} \mathsf{f}_{c+1}, \dots, \partial^{k_{n-c}} \mathsf{f}_n\big)\{i\} - \\
&\sum_{j \in \mathsf{Cut}(\beta_{>c} - |\mathfrak{k}|) \setminus \mathsf{Cut}(\beta_{>c})} \widetilde{\mathsf{P}}_<^{\beta_{[\![c+1,j]\!]} - |\mathfrak{k}|}\big(\partial^{k_1} \mathsf{f}_{c+1}, \dots, \partial^{k_{n-j+1}} \mathsf{f}_n\big)\, \widetilde{\mathsf{P}}_<^{\beta_{>j}, \beta_{>j} - |\mathfrak{k}|}\big(\partial^{k_{n-j}} \mathsf{f}_{j+1}, \dots, \partial^{k_{n-c}} \mathsf{f}_n\big)\{i\}.
\end{aligned}$$

Lemma 26 gives

$$\Lambda_{\mathfrak{k},i} = \widetilde{\mathsf{P}}_<^{\beta_{>c}, \beta_{>c} - |\mathfrak{k}|}\big(\partial^{k_1} \mathsf{f}_{c+1}, \dots, \partial^{k_{n-c}} \mathsf{f}_n\big)\{i\},$$

and the result follows. ▷

For $0 \leq a \leq n-1$ we define

$$\big(\triangle_{yx} \widetilde{\mathsf{P}}_<^\beta\big)(\mathsf{f}_{a+1}, \dots, \mathsf{f}_n) := \big(\triangle_{y-x, \sum_{j=a+1}^n \beta_j} \widetilde{\mathsf{P}}_<^\beta\big)(\mathsf{f}_{a+1}, \dots, \mathsf{f}_n)(x);$$

From the same arguments of Section 2, for o in a neighborhood of $\sum_{j=1}^n \beta_j$ one has the estimate

$$\big|\big(\triangle_{yx} \widetilde{\mathsf{P}}_<^\beta\big)(\mathsf{f}_{a+1}, \dots, \mathsf{f}_n)\{i\}\big| \lesssim \prod_{j=1}^n \|\mathsf{f}_j\|_{\beta_j}\, |y-x|^o 2^{-i(\sum_{j=1}^n \beta_j - o)}$$

where

$$\big(\triangle_{yx} \widetilde{\mathsf{P}}_<^\beta\big)(\mathsf{f}_{a+1}, \dots, \mathsf{f}_n)\{i\} := \big(\triangle_{y-x, \sum_{j=a+1}^n \beta_j} \widetilde{\mathsf{P}}_<^\beta\big)(\mathsf{f}_{a+1}, \dots, \mathsf{f}_n)\{i\}(x).$$

Then Lemma 11 gives the estimate

$$\big|\big(\triangle_{yx} \widetilde{\mathsf{P}}_<^\beta\big)(\mathsf{f}_{a+1}, \dots, \mathsf{f}_n)\big| \lesssim \Big\{ \prod_{j=a+1}^n \|\mathsf{f}_j\|_{\beta_j} \Big\}\, |y-x|^{\sum_{j=a+1}^n \beta_j}.$$

***29* – Proposition.** *Pick* $\beta = (\beta_1, \dots, \beta_n) \in \mathbf{R}^n$ *with* $\sum_{j=1}^n \beta_j > 0$ *and* $\mathsf{f}_j \in \mathsf{C}_\circ^{\beta_j}$ *for* $1 \leq j \leq n$. *Then we have the local expansion*

$$
\begin{aligned}
\widetilde{\mathsf{P}}^{\beta}_{<}(\mathsf{f}_1,\dots,\mathsf{f}_n)(y) = & \sum_{|k|<\sum_{j=1}^n \beta_j} \partial^k_\star \widetilde{\mathsf{P}}^{\beta}_{<}\big(\mathsf{f}_1,\dots,\mathsf{f}_n\big)(x)\,\frac{(y-x)^k}{k!} \\
& + \sum_{c=1}^{n-1} \sum_{|k|<\sum_{j=1}^c \beta_j} \partial^k_\star \widetilde{\mathsf{P}}^{\beta}_{<}\big(\mathsf{f}_1,\dots,\mathsf{f}_c\big)(x)\,\frac{(y-x)^k}{k!}\,\triangle_{yx}\widetilde{\mathsf{P}}^{\beta}_{<}(\mathsf{f}_{c+1},\dots,\mathsf{f}_n) \\
& + \triangle_{yx}\widetilde{\mathsf{P}}^{\beta}_{<}(\mathsf{f}_1,\dots,\mathsf{f}_n).
\end{aligned}
\tag{A.1}
$$

where

$$
\Big|\big(\triangle_{yx}\widetilde{\mathsf{P}}^{\beta}_{<}\big)(\mathsf{f}_{c+1},\dots,\mathsf{f}_n)\Big| \lesssim \Bigg\{\prod_{j=c+1}^n \|\mathsf{f}_j\|_{\beta_j}\Bigg\}\,|y-x|^{\sum_{j=c+1}^n \beta_j}.
$$

Proof – We proceed as in the proof of Proposition 12. Proposition 27 and Lemma 28 give

$$
\begin{aligned}
(\triangle_{yx}\widetilde{\mathsf{P}}^{\beta}_{<})(\mathsf{f}_1,\dots,\mathsf{f}_n) = \widetilde{\mathsf{P}}^{\beta}_{<}\big(\mathsf{f}_1,\dots,\mathsf{f}_n\big)(\cdot+h) & - \sum_{|k|<o} \frac{h^k}{k!}\,\partial^k \widetilde{\mathsf{P}}^{\beta}_{<}\big(\mathsf{f}_1,\dots,\mathsf{f}_n\big) \\
& - \sum_{c\in\mathsf{I}(\beta)} \sum_{|p|<\sum_{j=1}^c \beta_i} \partial^p_\star \widetilde{\mathsf{P}}^{\beta}_{<}(\mathsf{f}_1,\dots,\mathsf{f}_c)\,\frac{h^p}{p!}\,\big(\triangle_{yx}\widetilde{\mathsf{P}}^{\beta}_{<}\big)(\mathsf{f}_{c+1},\dots,\mathsf{f}_n) \\
& - \sum_{c\in\mathsf{I}(\beta)} \sum_{\substack{|p|<\sum_{j=1}^c \beta_j \\ |\ell|>\sum_{j=c+1}^n \beta_j}} \frac{h^p}{p!}\,\frac{h^\ell}{\ell!}\,\partial^p_\star \widetilde{\mathsf{P}}^{\beta}_{<}(\mathsf{f}_1,\dots,\mathsf{f}_c)\,\partial^\ell_\star \widetilde{\mathsf{P}}^{\beta}_{<}(\mathsf{f}_{c+1},\dots,\mathsf{f}_n).
\end{aligned}
$$

From Lemma 26, we have, for any $k\in\mathbf{N}^{d_0}$ such that $|k|<\sum_{j=1}^n \beta_j$, that $\partial^k_\star \mathsf{P}_<\big(\mathsf{f}_1,\dots,\mathsf{f}_n\big)$ is equal to

$$
\partial^k \widetilde{\mathsf{P}}^{\beta}_{<}\big(\mathsf{f}_1,\dots,\mathsf{f}_n\big) - \sum_{c\in\mathsf{I}(\beta)} \sum_{\substack{|p|<\sum_{j=1}^c \beta_j \\ |k-p|>\sum_{j=c+1}^n \beta_j}} \binom{k}{p}\,\partial^p_\star \widetilde{\mathsf{P}}^{\beta}_{<}(\mathsf{f}_1,\dots,\mathsf{f}_c)\,\partial^{k-p}_\star \widetilde{\mathsf{P}}^{\beta}_{<}(\mathsf{f}_{c+1},\dots,\mathsf{f}_n).
$$

This identity concludes the proof of this proposition. ▷

A.4 – Proof of some algebraic lemmas. We prove in this section a number of algebraic results that were used in the main body of the text. We start Section A.4.1 by proving the inductive relation (Lemma 26) on the $\widetilde{\mathsf{P}}^{\beta^1,\beta^2}_{<}$ that we used above. The operators $\widetilde{\mathsf{P}}^{\beta^1,\beta^2}_{<}$ have an analogue $\widetilde{\mathsf{P}}^{\beta^1,\beta^2}_{\mathsf{j}}$ defined from the (true) iterated paraproduct operator. The remainder of Section A.4.1 is dedicated to proving Lemma 31, which is the analogue of Lemma 26 for the operators $\widetilde{\mathsf{P}}^{\beta^1,\beta^2}_{\mathsf{j}}$. Lemma 31 plays a crucial role in our proof of Lemma 20. The later is the main ingredient of our proof of Theorem 1. The proof of Lemma 20 occupies all of Section A.4.2.

A.4.1 – Algebraic properties of the $\widetilde{\mathsf{P}}^{\beta^1,\beta^2}_{<}$. We start with the

Proof of Lemma 26. The proof is very similar to the proof of Lemma 7. From ***Assumption (A)*** we have the following partition of $\mathsf{MultiCut}(\beta^1,\beta^2)$

$$
\mathsf{MultiCut}\big(\beta^1,\beta^2\big) = \mathsf{MultiCut}\big(\beta^1\big) \sqcup \bigsqcup_{m\in\mathsf{Cut}(\beta^2)\backslash\mathsf{Cut}(\beta^1)} \mathsf{MultiCut}(\beta^2)\big[\beta^1,m\big],
$$

where

$$
\mathsf{MultiCut}(\beta^2)\big[\beta^1,m\big] := \Big\{\mathbf{i}\in\mathsf{MultiCut}\big(\beta^1,\beta^2\big)\,;\, m\in\mathbf{i},\quad \sum_{s=1}^m \beta^2_s = \min_{j\in\mathbf{i},\ j\notin\mathsf{Cut}(\beta^1)} \sum_{s=1}^j \beta^2_s\Big\}.
$$

We thus have

$$\widetilde{\mathsf{P}}_<^{\beta^1,\beta^2}(\mathsf{h}_1,\dots \mathsf{h}_n) = \mathsf{P}_<^{\beta^1}\big(\mathsf{h}_1,\dots,\mathsf{h}_n\big) + \sum_{m\in\mathsf{Cut}(\beta^2)\backslash\mathsf{Cut}(\beta^1)}\ \sum_{\mathbf{i}\in\mathsf{MultiCut}(\beta^2)[\beta^1,m]} (-1)^{n(\boldsymbol{d})+1}\prod_{k=1}^{n(\boldsymbol{d})}\mathsf{P}_<\big(\mathsf{h}_{i_{k-1}+1},\dots,\mathsf{h}_{i_k}\big).$$

so it suffices to show that for any $m\in\mathsf{Cut}(\beta^2)\backslash\mathsf{Cut}(\beta^1)$ we have

$$\sum_{\mathbf{i}\in\mathsf{MultiCut}(\beta^2)[\beta^1,m]} (-1)^{n(\boldsymbol{d})+1}\prod_{k=1}^{n(\boldsymbol{d})}\mathsf{P}_<\big(\mathsf{h}_{i_{k-1}+1},\dots,\mathsf{h}_{i_k}\big) = -\widetilde{\mathsf{P}}_<^{\beta^1_{\le m},\beta^2_{\le m}}\big(\mathsf{h}_1,\dots,\mathsf{h}_m\big)\widetilde{\mathsf{P}}_<^{\beta^1_{>m},\beta^2_{>m}}\big(\mathsf{h}_{m+1},\dots,\mathsf{h}_n\big).$$

Pick $m\in\mathsf{Cut}(\beta^2)\backslash\mathsf{Cut}(\beta^1)$. We prove that: For $1<j<m$ we have

$$\Big\{\exists\,\mathbf{i}\in\mathsf{MultiCut}(\beta^2)[\beta^1,m]\,;\,j\in\mathbf{i}\Big\}\Leftrightarrow\Big\{j\in\mathsf{Cut}\big(\beta^1_{\le m}\big)\cup\mathsf{Cut}\big(\beta^2_{\le m}\big)\Big\},$$

and for $m<j<n$ we have

$$\Big\{\exists\,\mathbf{i}\in\mathsf{MultiCut}(\beta^2)[\beta^1,m]\,;\,j\in\mathbf{i}\Big\}\Leftrightarrow\Big\{j-m\in\mathsf{Cut}\big(\beta^1_{>m}\big)\cup\mathsf{Cut}\big(\beta^2_{>m}\Big\}.$$

The proof of Lemma 26 follows from these equivalences as in the proof of Lemma 7.

As a preliminary remark we note that for $m\in\mathsf{Cut}(\beta^2)\backslash\mathsf{Cut}(\beta^1)$ we have $\sum_{s=1}^m\beta^1_s>0$ and $\sum_{s=m+1}^n\beta^1_s>0$. We prove now the first equivalence relation. Suppose $\boldsymbol{i}\in\mathsf{MultiCut}(\beta^2)[\beta^1,m]$ and $1<j<m$ is such that $j\in\boldsymbol{i}$. If $j\in\mathsf{Cut}(\beta^1)$ then $j\in\mathsf{Cut}(\beta^1_{\le m})$. Otherwise $j\in\mathsf{Cut}(\beta^2)\backslash\mathsf{Cut}(\beta^1)$ and $j\in\mathsf{Cut}(\beta^2_{\le m})$. Reciprocally if $j\in\mathsf{Cut}(\beta^1_{\le m})\cup\mathsf{Cut}(\beta^2_{\le m})$ then necessarily $\sum_{s=j+1}^m\beta^2_s<0$ and $j\in\mathsf{Cut}(\beta^2)$ and $\sum_{s=j+1}^n\beta^2_s<\sum_{s=m+1}^n\beta^2_s$.

The second equivalence relation is proved in the same way. ▷

The remainder of this section is dedicated to stating and proving an analogue of Lemma 26 for some operator $\widetilde{\mathsf{P}}_\ell^{\beta^1,\beta_2}$ that we can associate to the iterated paraproduct operators $\widetilde{\mathsf{P}}_\ell$. We first need an ad hoc setting to introduce these operators. It is very close to the setting of Section 3.

Fix $n\ge 1$. Define the set of symbols

$$\widehat{\mathcal{B}} := \Big\{[\![a,b]\!]^{\mathfrak{k}}_{\mathfrak{j}}X^m\,;\,1\le a\le b\le n,\ \ell,k\in\mathbf{N}^{d_0},\mathfrak{j}\in\mathcal{P}_{b-a}(\ell),\mathfrak{k}\in\mathcal{P}_{b-a+1}(k),m\in\mathbf{N}^{d_0}\Big\}\cup\{X^m\}_{m\in\mathbf{N}^{d_0}}.$$

Given $\beta\in\mathbf{R}^n$ and $\tau=[\![a,b]\!]^{\mathfrak{k}}_{\mathfrak{j}}\in\widehat{\mathcal{B}}$ we set

$$|\tau|_\beta=\sum_{j=a}^b\beta_j-|k|+|\ell|.$$

We denote by $\widehat{T}$ the vector space freely spanned by the elements of $\widehat{\mathcal{B}}$, and for $\tau=[\![a,b]\!]^{\mathfrak{k}}_{\mathfrak{j}}$ we set

$$\oplus(\tau):=\Big\{[\![a,c]\!]^{\mathfrak{k}+\mathfrak{p}}_{\mathfrak{j}_{<c-a}}\,;\,a<c<b,\,\ell_{c-a}=0\Big\},$$

for $\sigma=[\![a,c]\!]^{\mathfrak{k}+\mathfrak{p}}_{\mathfrak{j}_{<c-a}}\in\oplus(\tau)$ define an element of $\widehat{T}$ setting

$$(\tau\backslash\sigma):=\sum_{k=p_1+p_2}\binom{k}{p_1}[\![c+1,b]\!]_{\mathfrak{j}_{>j_1-j}+p_1}X^{p_2}$$

Finally we define a coproduct $\widehat{\Delta}:\widehat{T}\to\widehat{T}\otimes\widehat{T}$ setting

$$\widehat{\Delta}(\tau)=\sum_{\sigma\in\oplus(\tau)}(\tau\backslash\sigma)\otimes\sigma.$$

Proceeding as in the proof of Proposition 14 one can see that $\widehat{\Delta}$ is co-associative. We note in particular that all the elements of T in the sum defining $(\tau\backslash\sigma)$ have the same homogeneity. Re-indexing the sum defining $\widehat{\Delta}(\tau)$ we can write

$$\widehat{\Delta}(\tau) =: \sum_{\nu \widehat{<} \tau} \nu \otimes \tau/\nu,$$

with ν running in the basis $\mathcal{B}$ of T and τ/ν defined by this identity. The element τ/ν of $\widehat{T}$ is a sum of terms with the same $|\cdot|_\beta$-homogeneity, so we can abuse notations and write $|\tau/\nu|_\beta$.

For $\tau \in \widehat{\mathcal{B}}$ we define the set of cuts

$$\widehat{\mathsf{Cut}}(\tau,\beta) := \Big\{\sigma \widehat{<} \tau\,;\, |\sigma|_\beta < 0 \ \text{ and } \ |\tau/\sigma|_\beta > 0\Big\},$$

and the set of multicuts

$$\widehat{\mathsf{MultiCut}}(\tau,\beta) := \Big\{\boldsymbol{\sigma} = (\sigma_1,\cdots,\sigma_{e(\boldsymbol{\sigma})}) \in \widehat{\mathsf{Cut}}(\tau)^{e(\boldsymbol{\sigma})}\,;\, e(\boldsymbol{\sigma}) \geq 1,\ \ \sigma_{e(\boldsymbol{\sigma})} \widehat{<} \cdots \widehat{<} \sigma_1 \widehat{<} \tau\Big\}.$$

For a fixed tuple $\boldsymbol{g} = (g_1,\ldots,g_n)$ of distributions, for $\sigma = [\![a,b]\!]_{\mathrm{j}}^{\mathfrak{k}} \in \widehat{\mathcal{B}}$ we set

$$\Upsilon_{\boldsymbol{g}}(\sigma) := \mathsf{P}_{\mathrm{j}}\big(\partial^{k_a} g_a,\ldots,\partial^{k_b} g_b\big).$$

We note that for any $\mathfrak{p} \in (\mathbf{N}^{d_0})^n$, setting $\partial^{\mathfrak{p}}\boldsymbol{g} = (\partial^{p_1} g_1,\ldots,\partial^{p_n} g_n)$, one has

$$\Upsilon_{\partial^{\mathfrak{p}}\mathbf{g}}\big([\![a,b]\!]_{\mathrm{j}}^{\mathfrak{k}}\big) = \Upsilon_{\boldsymbol{g}}\big([\![a,b]\!]_{\mathrm{j}}^{\mathfrak{k}+\mathfrak{p}}\big). \tag{A.2}$$

30 – Lemma. *For any $\ell \in \mathbf{N}^{d_0}$ and $\mathrm{j} \in \mathcal{P}_{n-1}(\ell)$, letting $\tau = [\![1,n]\!]_{\mathrm{j}}^{0}$, we have*

$$\widetilde{\mathsf{P}}_{\mathrm{j}}^{\beta}\big(g_1,\cdots,g_n\big) = \mathsf{P}_{\mathrm{j}}\big(g_1,\cdots,g_n\big) - \sum_{\boldsymbol{\sigma}\in\widehat{\mathsf{MultiCut}}(\tau,\beta)} (-1)^{e(\boldsymbol{\sigma})+1}\,\Upsilon_{\boldsymbol{g}}(\tau/\sigma_1)\Upsilon_{\boldsymbol{g}}(\sigma_1/\sigma_2)\cdots\Upsilon_{\boldsymbol{g}}(\sigma_{e(\boldsymbol{\sigma})}). \tag{A.3}$$

Proof – We prove (A.3) by induction on n. The result is true for $n=1$. We prove that the right hand side of (A.3) satisfies the same recursive relation as $\widetilde{\mathsf{P}}_{\mathrm{j}}^{\beta}\big(g_1,\ldots,g_n\big)$. The proof is analogous to the proof of Lemma 7.
From ***Assumption (A)*** we have a partition

$$\widehat{\mathsf{MultiCut}}(\tau,\beta) = \bigsqcup_{\sigma\widehat{<}\tau} \widehat{\mathsf{MultiCut}}(\tau,\beta)[\nu],$$

where

$$\widehat{\mathsf{MultiCut}}(\tau,\beta)[\nu] := \Big\{\boldsymbol{\sigma} = (\sigma_1,\cdots,\sigma_{e(\boldsymbol{\sigma})}) \in \widehat{\mathsf{MultiCut}}(\tau,\beta)\,;\, \nu \in \boldsymbol{\sigma},\ |\tau/\nu| = \min_{1\leq j\leq e(\boldsymbol{\sigma})} |\tau/\sigma_j|\Big\}.$$

For any $\nu \in \widehat{\mathsf{Cut}}(\tau,\beta)$ and $\mu \widehat{<} \nu$ we have the equivalence

$$\Big\{\exists\,\boldsymbol{\sigma} \in \widehat{\mathsf{MultiCut}}(\tau,\beta)[\nu]\,;\, \mu \in \boldsymbol{\sigma}\Big\} \Leftrightarrow \Big\{\mu \in \widehat{\mathsf{Cut}}(\nu,\beta)\Big\}.$$

Likewise, for $\nu \widehat{<} \mu \widehat{<} \tau$ we have

$$\Big\{\exists\,\boldsymbol{\sigma} \in \widehat{\mathsf{MultiCut}}(\tau,\beta)[\nu]\,;\, \mu \in \boldsymbol{\sigma}\Big\} \Leftrightarrow \Big\{\mu/\nu \in \widehat{\mathsf{Cut}}(\tau/\nu,\beta)\Big\}.$$

Define

$$\overline{\Upsilon}_{\boldsymbol{g}}(\tau,\beta) := \sum_{\boldsymbol{\sigma}\in\widehat{\mathsf{MultiCut}}(\tau,\beta)} (-1)^{e(\boldsymbol{\sigma})+1}\Upsilon_{\boldsymbol{g}}(\tau/\sigma_1)\Upsilon_{\boldsymbol{g}}(\sigma_1/\sigma_2)\cdots\Upsilon_{\boldsymbol{g}}(\sigma_{e(\boldsymbol{\sigma})}).$$

Using the two equivalence relations above, the same computation as in the proof of Lemma 7 gives that

$$\overline{\Upsilon}_{\boldsymbol{g}}(\tau,\beta) = -\sum_{\sigma\in\widehat{\mathsf{Cut}}(\tau,\beta)} \big(\Upsilon_{\boldsymbol{g}}(\sigma) - \overline{\Upsilon}_{\boldsymbol{g}}(\sigma,\beta)\big)\Big(\Upsilon_{\boldsymbol{g}}(\tau/\sigma) - \overline{\Upsilon}_{\boldsymbol{g}}(\tau/\sigma,\beta)\Big).$$

From the induction hypothesis, for $\sigma = [\![c+1,n]\!]_{\mathrm{j}+p} \in \widehat{\mathsf{Cut}}(\tau,\beta)$ we have

$$\Upsilon_{\boldsymbol{g}}(\sigma) - \overline{\Upsilon}_{\boldsymbol{g}}(\sigma,\beta) = \widetilde{\mathsf{P}}_{\mathrm{j}+p}^{\beta_{>m}}(g_{m+1},\ldots,g_n).$$

Likewise, for $\tau/\sigma = [\![1,c]\!]_{\mathsf{j}}^{p}$, using (A.2) we have

$$\begin{aligned}\Upsilon_{\boldsymbol g}(\tau/\sigma) - \overline{\Upsilon}_{\boldsymbol g}(\tau/\sigma,\beta) &= \sum_{\mathfrak p\in\mathcal P_m(p)} \binom{p}{\mathfrak p}\Big\{\Upsilon_{\boldsymbol g}\big([\![1,c]\!]_{\mathsf j}^{\mathfrak p}\big) - \overline{\Upsilon}_{\boldsymbol g}\Big([\![1,c]\!]_{\mathsf j}^{\mathfrak p},\beta\Big)\Big\}\\ &= \sum_{\mathfrak p\in\mathcal P_m(p)} \binom{p}{\mathfrak p}\Big\{\Upsilon_{\partial^{\mathfrak p}\boldsymbol g}\big([\![1,c]\!]_{\mathsf j}^{0}\big) - \overline{\Upsilon}_{\partial^{\mathfrak p}\boldsymbol g}\Big([\![1,c]\!]_{\mathsf j}^{0},\beta-|\mathfrak p|\Big)\Big\}\\ &= \sum_{\mathfrak p\in\mathcal P_m(p)} \binom{p}{\mathfrak p}\,\widetilde{\mathsf P}_{\mathsf j}^{\beta_{\leq c}-|\mathfrak p|}\big(\partial^{p_1}g_1,\dots,\partial^{p_m}g_m\big).\end{aligned}$$

This closes the induction step. ▷

Define

$$\begin{aligned}&\widehat{\mathsf{MultiCut}}(\tau,\beta^1,\beta^2)\\ &:= \Big\{\boldsymbol\sigma = (\sigma_1,\cdots,\sigma_{e(\boldsymbol\sigma)}) \in \big(\widehat{\mathsf{Cut}}(\tau,\beta^1)\cup\widehat{\mathsf{Cut}}(\tau,\beta^2)\big)^{e(\boldsymbol\sigma)}\,;\ e(\boldsymbol\sigma)\geq 1,\ \ \sigma_{e(\boldsymbol\sigma)}\widehat{<}\cdots\widehat{<}\sigma_1\widehat{<}\tau\Big\},\end{aligned}$$

and set

$$\widetilde{\mathsf P}_{\mathsf j}^{\beta^1,\beta^2}(g_1,\dots,g_n) := \mathsf P_{\mathsf j}(g_1,\dots,g_n) \ - \sum_{\boldsymbol\sigma\in\mathsf{MultiCut}(\tau,\beta^1,\beta^2)} (-1)^{e(\boldsymbol\sigma)+1}\Upsilon_{\boldsymbol g}(\tau/\sigma_1)\,\Upsilon_{\boldsymbol g}(\sigma_1/\sigma_2)\cdots\Upsilon_{\boldsymbol g}(\sigma_{e(\boldsymbol\sigma)}).$$

31 – Lemma. *Suppose β^1,β^2 are two tuples of real numbers such that $\beta^1_s\geq\beta^2_s$ for every $1\leq s\leq n$. Then we have*

$$\widetilde{\mathsf P}_{\mathsf j}^{\beta^1,\beta^2}(g_1,\dots,g_n) = \widetilde{\mathsf P}_{\mathsf j}^{\beta^1}(g_1,\dots,g_n) - \sum \widetilde{\mathsf P}_{\mathsf j}^{\beta^1-|\mathfrak k|,\beta^2-|\mathfrak k|}\big(\partial^{k_1}g_1,\dots,\partial^{k_c}g_c\big)\,\widetilde{\mathsf P}_{\mathsf j+k}^{\beta^1,\beta^2}(g_{c+1},\dots,g_n),$$

for a sum over $[\![c+1,n]\!]_{\mathsf j}^{\mathfrak k}\in\widehat{\mathsf{Cut}}(\tau,\beta^2)\backslash\widehat{\mathsf{Cut}}(\tau,\beta^1)$.

Proof – The proof is the same as the proof of Lemma 26. Using ***Assumption (A)*** we can partition of $\widehat{\mathsf{MultiCut}}(\tau,\beta^1,\beta^2)$ as

$$\widehat{\mathsf{MultiCut}}\big(\tau,\beta^1,\beta^2\big) = \widehat{\mathsf{MultiCut}}(\tau,\beta^1)\sqcup \bigsqcup_{\nu\in\widehat{\mathsf{Cut}}(\tau,\beta^2)\backslash\widehat{\mathsf{Cut}}(\tau,\beta^1)} \widehat{\mathsf{MultiCut}}(\tau,\beta^2)\big[\beta^1,\nu\big],$$

where

$$\widehat{\mathsf{MultiCut}}(\tau,\beta^2)\big[\beta^1,\nu\big] := \Big\{\boldsymbol\sigma\in\widehat{\mathsf{MultiCut}}\big(\tau,\beta^1,\beta^2\big)\,;\ \nu\in\boldsymbol\sigma,\ |\tau/\nu|_{\beta_2} = \min_{\sigma\in\boldsymbol\sigma,\,\sigma\notin\widehat{\mathsf{Cut}}(\tau,\beta^1)} |\tau/\sigma|_{\beta_2}\Big\}.$$

Then we have

$$\begin{aligned}\widetilde{\mathsf P}_{\mathsf j}^{\beta^1,\beta^2}(g_1,\cdots,g_n) =&\widetilde{\mathsf P}_{\mathsf j}^{\beta^1}(g_1,\cdots,g_n)\\ &+\sum_{\substack{\nu\in\widehat{\mathsf{Cut}}(\tau,\beta^2)\backslash\widehat{\mathsf{Cut}}(\tau,\beta^1)\\ \boldsymbol\sigma\in\widehat{\mathsf{MultiCut}}(\tau,\beta^2)[\beta^1,\nu]}} (-1)^{e(\boldsymbol\sigma)+1}\,\Upsilon_{\boldsymbol g}(\tau/\sigma_1)\Upsilon_{\boldsymbol g}(\sigma_1/\sigma_2)\cdots\Upsilon_{\boldsymbol g}(\sigma_{e(\boldsymbol\sigma)}).\end{aligned}$$

It suffices then to show for any $\nu = [\![1,c]\!]_{\mathsf j}^{\mathfrak k}\in\widehat{\mathsf{Cut}}(\tau,\beta^2)\backslash\widehat{\mathsf{Cut}}(\tau,\beta^1)$ we have

$$\begin{aligned}&\sum_{\boldsymbol\sigma\in\widehat{\mathsf{MultiCut}}(\tau,\beta^2)[\beta^1,\nu]} (-1)^{e(\boldsymbol\sigma)+1}\Upsilon_{\boldsymbol g}(\tau/\sigma_1)\cdots\Upsilon_{\boldsymbol g}(\sigma_{e(\boldsymbol\sigma)})\\ &\qquad = -\widetilde{\mathsf P}_{\mathsf j_{\leq m}}^{D^{\mathfrak k}\beta^1_{\leq m},D^{\mathfrak k}\beta^2_{\leq m}}\big(\partial^{k_1}f_1,\cdots,\partial^{k_m}f_m\big)\widetilde{\mathsf P}_{\mathsf j+k}^{\beta^1_{>m},\beta^2_{>m}}\big(f_{m+1},\cdots,f_n\big).\end{aligned}$$

For such a ν, we show below that for $\mu\widehat{<}\nu$ we have

$$\Big\{\exists\,\boldsymbol\sigma\in\widehat{\mathsf{MultiCut}}(\tau,\beta^2)[\beta^1,\nu],\ \ \mu\in\boldsymbol\sigma\Big\}\Leftrightarrow\Big\{\mu\in\widehat{\mathsf{Cut}}(\nu,\beta^1)\cup\widehat{\mathsf{Cut}}(\nu,\beta^2)\Big\},\tag{A.4}$$

and that for $\nu\widehat{<}\mu\widehat{<}\tau$ we have

$$\Big\{\exists\,\boldsymbol\sigma\in\widehat{\mathsf{MultiCut}}(\tau,\beta^2)[\beta^1,\nu],\ \ \mu\in\boldsymbol\sigma\Big\}\Leftrightarrow\Big\{\tau/\mu\in\widehat{\mathsf{Cut}}(\tau/\nu,\beta^1)\cup\widehat{\mathsf{Cut}}(\tau/\nu,\beta^2)\Big\}.\tag{A.5}$$

We can then conclude the proof of our lemma in the same way as in the proof of Lemma 7.
A basic observation we will use is that for $\nu = [\![1,c]\!]_{\mathsf{j}}^{\mathfrak{k}} \in \widehat{\mathsf{Cut}}(\tau,\beta^2)\backslash\widehat{\mathsf{Cut}}(\tau,\beta^1)$ we necessarily have $|\nu|_{\beta^1} > 0$ and $|\tau/\nu|_{\beta^1} > 0$. We now prove (A.4). Suppose $\boldsymbol{\sigma} \in \widehat{\mathsf{MultiCut}}(\tau,\beta^2)[\beta^1,\nu]$ and $\mu \widehat{<} \nu$ such that $\mu \in \boldsymbol{\sigma}$. If $\mu \in \widehat{\mathsf{Cut}}(\tau,\beta^1)$, then $\mu \in \widehat{\mathsf{Cut}}(\nu,\beta^1)$ and otherwise $\mu \in \widehat{\mathsf{Cut}}(\tau,\beta^2)\backslash\widehat{\mathsf{Cut}}(\tau,\beta^1)$, then $\mu \in \widehat{\mathsf{Cut}}(\mu,\beta^2)$. Reciprocally if $\mu \in \widehat{\mathsf{Cut}}(\nu,\beta^1,\beta^2)$, then necessarily $|\nu/\mu|_{\beta^2} < 0$ and $\mu \in \widehat{\mathsf{Cut}}(\tau,\beta^2)$ and $|\tau/\mu|_{\beta^2} < |\tau/\nu|_{\beta^2}$. We proceed similarly to prove the equivalence (A.5). ▷

A.4.2 – Proof of Lemma 20. ***We first prove point (i)*** by induction. From the definition of the operator $\widetilde{\mathsf{P}}_<$ we have

$$\begin{aligned}\widetilde{\mathsf{P}}_<^{|\tau/\boldsymbol{\sigma}|_{\alpha-|\mathfrak{k}|}}\Big([\tau/\sigma_1]^{\partial^{\mathfrak{k}}\mathfrak{f}},\dots,[\sigma_e]^{\partial^{\mathfrak{k}}\mathfrak{f}}\Big) &= \mathsf{P}_<\Big([\tau/\sigma_1]^{\partial^{\mathfrak{k}}\mathfrak{f}},\dots,[\sigma_e]^{\partial^{\mathfrak{k}}\mathfrak{f}}\Big)\\ &\quad - \sum_c \widetilde{\mathsf{P}}_<^{|\tau/\boldsymbol{\sigma}_{\leq c}|_{\alpha-|\mathfrak{k}|}}\Big([\tau/\sigma_1]^{\partial^{\mathfrak{k}}\mathfrak{f}},\cdots,[\sigma_{c-1}/\sigma_c]^{\partial^{\mathfrak{k}}\mathfrak{f}}\Big)\\ &\qquad \times \widetilde{\mathsf{P}}_<^{|\sigma_c/\boldsymbol{\sigma}_{>c}|_{\alpha-|\mathfrak{k}|}}\Big([\sigma_c/\sigma_{c+1}]^{\partial^{\mathfrak{k}}\mathfrak{f}},\cdots,[\sigma_e]^{\partial^{\mathfrak{k}}\mathfrak{f}}\Big),\end{aligned}$$

with a sum over the set of integers $c \in [\![1,e]\!]$ such that $|\tau/\sigma_c|_{\alpha-|\mathfrak{k}|} > 0$ and $|\sigma_c|_{\alpha-|\mathfrak{k}|} < 0$. Summing over the set of descending sequences $\sigma_e \prec \cdots \prec \sigma_1 \prec \tau = [\![1,n]\!]_{\mathsf{j}} X^m$, we obtain that

$$\sum_{e\geq 0}\ \sum_{\sigma_e\prec\cdots\prec\sigma_1\prec\tau} \widetilde{\mathsf{P}}_<^{|\tau/\boldsymbol{\sigma}|_{\alpha-|\mathfrak{k}|}}\Big([\tau/\sigma_1]^{\partial^{\mathfrak{k}}\mathfrak{f}},\dots,[\sigma_e]^{\partial^{\mathfrak{k}}\mathfrak{f}}\Big)$$

is equal to

$$\begin{aligned}&\sum_{e\geq 0}\ \sum_{\sigma_e\prec\cdots\prec\sigma_1\prec\tau} \mathsf{P}_<\Big([\tau/\sigma_1]^{\partial^{\mathfrak{k}}\mathfrak{f}},\cdots,[\sigma_e]^{\partial^{\mathfrak{k}}\mathfrak{f}}\Big) -\\ &\sum_{\substack{\sigma\prec\tau\\ \sigma\in\widehat{\mathsf{Cut}}(\tau,\alpha-|\mathfrak{k}|)}}\ \sum_{\substack{e_1\geq 0\\ \sigma\prec\sigma_{e_1}\prec\cdots\prec\tau}} \widetilde{\mathsf{P}}_<^{|\tau/\boldsymbol{\sigma}|_{\alpha-|\mathfrak{k}|}}\Big([\tau/\sigma_1]^{\partial^{\mathfrak{k}}\mathfrak{f}},\dots,[\sigma_{e_1}/\sigma]^{\partial^{\mathfrak{k}}\mathfrak{f}}\Big)\\ &\qquad \times \sum_{e_2\geq 0}\ \sum_{\nu_{e_2}\prec\cdots\prec\sigma} \widetilde{\mathsf{P}}_<^{|\sigma/\boldsymbol{\nu}|_{\alpha-|\mathfrak{k}|}}\Big([\sigma/\nu_1]^{\partial^{\mathfrak{k}}\mathfrak{f}},\dots,[\nu_{e_2}]^{\partial^{\mathfrak{k}}\mathfrak{f}}\Big)\end{aligned} \tag{A.6}$$

From Lemma 19 the first sum in (A.6) is equal to $\mathsf{P}_{\mathsf{j}}\big(\partial^{k_1}f_1,\cdots,\partial^{k_n}f_n\big)$ if $m=0$ and 0 otherwise.

For the second double sum in the right hand side of (A.6), note first that all the homogeneities in the tuple $|\tau/\boldsymbol{\sigma}|_{\alpha-|\mathfrak{k}|}$ are positive. It follows that we have

$$\widetilde{\mathsf{P}}_<^{|\tau/\boldsymbol{\sigma}|_{\alpha-|\mathfrak{k}|}}\Big([\tau/\sigma_1]^{\partial^{\mathfrak{k}}\mathfrak{f}},\dots,[\sigma_{c-1}/\sigma]^{\partial^{\mathfrak{k}}\mathfrak{f}}\Big) = \mathsf{P}_<\Big([\tau/\sigma_1]^{\partial^{\mathfrak{k}}\mathfrak{f}},\dots,[\sigma_{c-1}/\sigma]^{\partial^{\mathfrak{k}}\mathfrak{f}}\Big).$$

Now, the elements $\sigma \prec \tau$ have the form $\sigma = [\![c+1,n]\!]_{\mathsf{j}_{>c}+p_1} X^{p_2+m_1}$, and $\tau/\sigma = [\![1,c]\!]^{p}_{\mathsf{j}_{<c}} X^{m_2}$ with $m = m_1+m_2$ and $p = p_1+p_2$, so it follows from Lemma 19 that

$$\begin{aligned}&\sum_{e_1\geq 0}\ \sum_{\sigma\prec\sigma_{e_1}\prec\cdots\prec\tau} \mathsf{P}_<\Big([\tau/\sigma_1]^{\partial^{\mathfrak{k}}\mathfrak{f}},\dots,[\sigma_{m-1}/\sigma]^{\partial^{\mathfrak{k}}\mathfrak{f}}\Big)\\ &\qquad = \mathbf{1}_{m_2=0}\sum_{\mathfrak{p}\in\mathcal{P}_c(p)} \binom{p}{\mathfrak{p}} \widetilde{\mathsf{P}}_{\mathsf{j}_{<c}}^{\alpha_{\leq c}-|\mathfrak{k}+\mathfrak{p}|}\big(\partial^{k_1+p_1}f_1,\dots,\partial^{k_c+p_c}f_c\big)\end{aligned}$$

Also, the induction hypothesis gives

$$\sum_{e_2\geq 0}\ \sum_{\nu_{e_2}\prec\cdots\prec\sigma} \widetilde{\mathsf{P}}_<^{|\sigma/\boldsymbol{\nu}|_{\alpha-|\mathfrak{k}|}}\Big([\sigma/\nu_1]^{\partial^{\mathfrak{k}}\mathfrak{f}},\dots,[\nu_{e_2}]^{\partial^{\mathfrak{k}}\mathfrak{f}}\Big) = \mathbf{1}_{p_2+m_1=0}\, \widetilde{\mathsf{P}}_{\mathsf{j}_{>c}+p_1}^{\alpha-|\mathfrak{k}|_{>c}}\big(\partial^{k_{c+1}}f_{c+1},\dots,\partial^{k_n}f_n\big).$$

If $m \neq 0$ then $m_1 \neq 0$ or $m_2+p_1 \neq 0$, and then all the terms in the right hand side of A.6 add up to 0; this closes the induction in that case. If now $m=0$, the non-zero terms in the sum over $\sigma \prec \tau$ are the terms with $\sigma = [\![c+1,n]\!]_{\mathrm{j}+p}$ and $\tau/\sigma = [\![1,c]\!]_{\mathrm{j}}^{p}$, and

$$\sum_{e\geq 0}\sum_{\sigma_e\prec\cdots\prec\sigma_1\prec\tau}\widetilde{\mathsf{P}}_{<}^{|\tau/\boldsymbol{\sigma}|_{\alpha-|\mathfrak{k}|}}\big([\tau/\sigma_1]^{\partial^{\mathfrak{k}}\mathfrak{f}},\dots,[\sigma_e]^{\partial^{\mathfrak{k}}\mathfrak{f}}\big) = \mathsf{P}_{\mathrm{j}}\big(\partial^{k_1}f_1,\dots,\partial^{k_n}f_n\big)$$
$$-\sum_{c,p}\sum_{\mathfrak{p}\in\mathcal{P}_c(p)}\widetilde{\mathsf{P}}_{\mathrm{j}_{<c}}^{\alpha_{\leq c}-|\mathfrak{k}+\mathfrak{p}|}\big(\partial^{k_1+p_1}f_1,\dots,\partial^{k_c+p_c}f_c\big)\,\widetilde{\mathsf{P}}_{\mathrm{j}_{>c}+p}^{\alpha_{>c}-|\mathfrak{k}|}\big(\partial^{k_{c+1}}f_{c+1},\dots,\partial^{k_n}f_n\big),$$

where the sum in the right hand side runs over the paris (c,p) such that $\ell_c = 0$, $|[\![1,c]\!]_{\mathrm{j}_{\leq c}}^{p}|_{\alpha-|\mathfrak{k}|} > 0$ and $|[\![c+1,n]\!]_{\mathrm{j}_{>c}+p}|_{\alpha-|\mathfrak{k}|} < 0$. It follows then from recursive definition of the correctors $\widetilde{P}_{\mathrm{j}}^{\beta}$ that the above quantity is indeed equal to $\widetilde{\mathsf{P}}_{\mathrm{j}}^{\alpha-|\mathfrak{k}|}\big(\partial^{k_1}f_1,\dots,\partial^{k_n}f_n\big)$.

– ***We now prove point*** (**ii**) by proving the following stronger statement: For $\tau = [\![1,n]\!]_{\mathrm{j}}X^m \in T$, and for *any* $p\in\mathbf{N}^{d_0}$, one has

$$\sum_{e\geq 0}\sum_{\sigma\prec\sigma_e\prec\cdots\prec\tau}\partial_\star^p\mathsf{P}_<\Big([\tau/\sigma_1]^{\mathfrak{f}},\dots,[\sigma_e/\sigma]^{\mathfrak{f}}\Big)$$
$$= \mathbf{1}_{m=0}\sum_{\mathfrak{p}\in\mathcal{P}_n(p)}\sum_{\mathfrak{k}\in\mathcal{P}_n(k)}\binom{p}{\mathfrak{p}}\binom{k}{\mathfrak{k}}\widetilde{\mathsf{P}}_{\mathrm{j}}^{\alpha-|\mathfrak{k}|,\alpha-|\mathfrak{k}+\mathfrak{p}|}\Big(\partial^{k_1+p_1}f_1,\dots,\partial^{k_n+p_n}f_n\Big),$$

where

$$\partial_\star^p\mathsf{P}_<\Big([\tau/\sigma_1]^{\mathfrak{f}},\dots,[\sigma_e/\sigma]^{\mathfrak{f}}\Big) = \sum_{\mathfrak{p}\in\mathcal{P}_{e+1}(p)}\binom{p}{\mathfrak{p}}\widetilde{\mathsf{P}}_<^{|\tau/\boldsymbol{\sigma}|_{\alpha}-|\mathfrak{p}|}\Big(\partial^{p_1}[\tau/\sigma_1]^{\mathfrak{f}},\dots,\partial^{p_{e+1}}[\sigma_e/\sigma]^{\mathfrak{f}}\Big).$$

The proof of this fact relies on Lemma 31 and is an induction over n. The result is true for $n=1$; suppose it holds true for $(n-1)$. From the definition of $\partial_\star^p\mathsf{P}_<$ and the recursive relation of Lemma 7, for any descending sequence $\sigma\prec\sigma_e\prec\cdots\prec\tau$, the distribution $\partial_\star^p\mathsf{P}_<\big([\tau/\sigma_1]^{\mathfrak{f}},\cdots,[\sigma_e/\sigma]^{\mathfrak{f}}\big)$ is equal to

$$\sum_{\mathfrak{p}\in\mathcal{P}_e(p)}\binom{p}{\mathfrak{p}}\widetilde{\mathsf{P}}_<^{|\tau/\boldsymbol{\sigma}|-|\mathfrak{p}|}\Big(\partial^{p_1}[\tau/\sigma_1]^{\mathfrak{f}},\dots,\partial^{p_q}[\sigma_e/\sigma]^{\mathfrak{f}}\Big)$$
$$= \sum_{\mathfrak{p}\in\mathcal{P}_e(p)}\binom{p}{\mathfrak{p}}\Big\{\mathsf{P}_<\Big(\partial^{p_1}[\tau/\sigma_1]^{\mathfrak{f}},\dots,\partial^{p_e}[\sigma_e/\sigma]^{\mathfrak{f}}\Big)\quad(\cdots)$$
$$-\sum_{c\in\mathsf{Cut}(|\tau/\boldsymbol{\sigma}|-|\mathfrak{p}|)}\widetilde{\mathsf{P}}_<^{|\tau/\boldsymbol{\sigma}_{\leq c}|-|\mathfrak{p}_{\leq c}|}\Big(\partial^{p_1}[\tau/\sigma_1]^{\mathfrak{f}},\dots,\partial^{p_c}[\sigma_{c-1}/\sigma_c]^{\mathfrak{f}}\Big)$$
$$\times\,\widetilde{\mathsf{P}}_<^{|\sigma_c/\boldsymbol{\sigma}_{>c}|-|\mathfrak{p}_{>c}|}\Big(\partial^{p_{c+1}}[\sigma_c/\sigma_{c+1}]^{\mathfrak{f}},\dots,\partial^{p_e}[\sigma_e/\sigma]^{\mathfrak{f}}\Big)\Big\}.$$

Then, summing over descending sequences $\sigma\prec\sigma_e\prec\cdots\prec\tau$ and inverting the sums over $\mathfrak{p}$ and c, we obtain

$$\sum_{e\geq 0}\sum_{\sigma\prec\sigma_e\prec\cdots\prec\tau}\partial_\star^p\mathsf{P}_<\Big([\tau/\sigma_1]^{\mathfrak{f}},\dots,[\sigma_e/\sigma]^{\mathfrak{f}}\Big) = \partial^p\Big\{\sum_{e\geq 0}\sum_{\sigma\prec\sigma_e\prec\cdots\prec\tau}\mathsf{P}_<\Big([\tau/\sigma_1]^{\mathfrak{f}},\dots,[\sigma_e/\sigma]^{\mathfrak{f}}\Big)\Big\}$$
$$-\sum_{\nu\prec\tau}\sum_{\substack{p=a+b\\(a,b)\in\mathsf{C}(\tau,\nu,\sigma)}}\binom{p}{a}\Big\{\sum_{\substack{e_1\geq 0\\\nu\prec\nu_{e_1}\prec\cdots\prec\tau}}\sum_{\mathfrak{a}\in\mathcal{P}_{e_1}(a)}\binom{a}{\mathfrak{a}}\widetilde{\mathsf{P}}_<^{|\tau/\boldsymbol{\nu}|-|\mathfrak{a}|}\Big(\partial^{a_1}[\tau/\nu_1]^{\mathfrak{f}},\dots,\partial^{a_{e_1}}[\nu_{e_1}/\nu]^{\mathfrak{f}}\Big)$$
$$\times\sum_{\substack{e_2\geq 0\\\sigma\prec\sigma_{e_2}\prec\cdots\prec\nu}}\sum_{\mathfrak{b}\in\mathcal{P}_{e_2}(b)}\binom{b}{\mathfrak{b}}\widetilde{\mathsf{P}}_<^{|\nu/\boldsymbol{\sigma}|-|\mathfrak{b}|}\Big(\partial^{b_1}[\nu/\sigma_1]^{\mathfrak{f}},\dots,\partial^{b_{e_2}}[\sigma_e/\sigma]^{\mathfrak{f}}\Big)\Big\},$$

where

$$\mathsf{C}(\tau,\sigma,\nu) := \Big\{(a,b)\in(\mathbf{N}^{d_0})^2,\quad |\tau/\nu| > |a| \text{ and } |\nu/\sigma| < |b|\Big\}.$$

From Lemma 19 the first line of the right hand side of the last equality is equal 0 if $m\neq 0$, as $\sum_{i\geq -1}\Delta_i^m(g) = 0$ for any function g; it is equal to $\partial^p\mathsf{g}^{\mathfrak{f}}(\tau)$ if $m=0$.

We are able to use induction hypothesis for the remaining terms. Suppose first that $m \neq 0$. For any $\nu \prec \tau$ the elements $\tau/\nu, \nu/\sigma \in T^+$ have the form $\tau/\nu = [\![1,c]\!]_{\mathsf{j}}^{k'_1} X^{m_1}$ and $\nu/\sigma = [\![c+1,n]\!]_{\mathsf{j}+v}^{k'_2} X^{m_2}$ where m_1, m_2 cannot be both equal to 0. The induction assumption ensures in that case that the second term on the right hand side is 0, which closes the induction step.

Suppose now that $m = 0$. In this case for $\nu \prec \tau$, the elements $\tau/\nu, \nu/\sigma \in T^+$ have form $\tau/\nu = [\![1,c]\!]_{\mathsf{j}}^{k_a+v}$ and $\nu/\sigma = [\![c+1,n]\!]_{\mathsf{j}+v_1}^{k_b} X^{v_2}$ with $k = k_a + k_b$ and $v = v_1 + v_2$. For $v_2 \neq 0$ the induction assumption ensures that the sum over e_2 is null. We are thus left with the ν for which $v_2 = 0$. This leads to the equality

$$\begin{aligned}
&\sum_{e\geq 0} \sum_{\sigma\prec\sigma_e\prec\cdots\prec\tau} \partial_\star^p \mathsf{P}_<\Big([\tau/\sigma_1]^{\mathfrak{f}}, \dots, [\sigma_e/\sigma]^{\mathfrak{f}}\Big) \\
&= \sum_{\mathfrak{k}\in\mathcal{P}_n(k)} \binom{k}{\mathfrak{k}} \bigg\{ \partial^p \widetilde{\mathsf{P}}_{\mathsf{j}}^{\alpha-|\mathfrak{k}|}\Big(\partial^{k_1} f_1, \dots, \partial^{k_n} f_n\Big) \\
&\quad - \sum_{\substack{p=q+q' \\ c,v}} \binom{p}{q} \sum_{\mathfrak{q}\in\mathcal{P}_c(q)} \sum_{\mathfrak{v}\in\mathcal{P}_c(v)} \binom{q}{\mathfrak{q}}\binom{v}{\mathfrak{v}} \mathsf{P}_{\mathsf{j}_{\leq m}}^{\alpha-|\mathfrak{k}|_{\leq c}, \alpha_{\leq m}-|\mathfrak{k}+\mathfrak{q}+\mathfrak{v}|}\Big(\partial^{k_1+q_1+v_1} f_1, \dots, \partial^{k_c+q_c+v_c} f_c\Big) \\
&\qquad \times \sum_{\mathfrak{q}'\in\mathcal{P}_{n-c}(q')} \binom{q}{\mathfrak{q}'} \widetilde{\mathsf{P}}_{\mathsf{j}_{>c}+v}^{\alpha-|\mathfrak{k}|_{>c}, \alpha_{>c}-|\mathfrak{k}+\mathfrak{q}|}\Big(\partial^{k_{c+1}+q_{c+1}} f_{c+1}, \dots, \partial^{k_n+q_n} f_n\Big)\bigg\},
\end{aligned}$$

where the sum over q, q in $\mathbf{N}^{d_0}$ subject to $q + q' = p$, and c, v runs over the indices such that $[\![1,c]\!]_{\mathsf{j}}^{\mathfrak{k}_{\leq c}+v}, [\![c+1,n]\!]_{\mathsf{j}+v}^{\mathfrak{k}_{>c}} \in T^+$ and $\ell_c = 0$ and

$$|q| < \Big|[\![1,c]\!]_{\mathsf{j}}^{\mathfrak{k}_{\leq c}+v}\Big|_{\alpha-\mathfrak{q}}, \quad \text{and} \quad |q'| > \Big|[\![c+1,n]\!]_{\mathsf{j}+v}^{\mathfrak{k}_{>c}}\Big|_{\alpha-\mathfrak{q}}.$$

This gives then

$$\begin{aligned}
&\sum_{\mathfrak{p}\in\mathcal{P}_n(p)} \sum_{\mathfrak{k}\in\mathcal{P}_n(k)} \binom{p}{\mathfrak{p}}\binom{k}{\mathfrak{k}} \bigg\{ \widetilde{\mathsf{P}}_{\mathsf{j}}^{\alpha-|\mathfrak{k}|}\Big(\partial^{k_1+p_1} f_1, \dots, \partial^{k_n+p_n} f_n\Big) \\
&\qquad - \sum_{\substack{c,v \\ \mathfrak{v}\in\mathcal{P}_c(v)}} \widetilde{\mathsf{P}}_{\mathsf{j}_{\leq c}}^{\alpha-|\mathfrak{k}|_{\leq c}, \alpha_{\leq c}-|\mathfrak{k}+\mathfrak{p}|}\Big(\partial^{k_1+p_1+v_1} f_1, \dots, \partial^{k_c+p_c+v_c} f_c\Big) \\
&\qquad\qquad \times \widetilde{\mathsf{P}}_{\mathsf{j}_{>c}+v}^{\alpha-|\mathfrak{k}|_{>c}, \alpha_{>c}-|\mathfrak{k}+\mathfrak{b}|}\Big(\partial^{k_{c+1}+p_{c+1}} f_{c+1}, \dots, \partial^{k_n+p_n} f_n\Big)\bigg\},
\end{aligned}$$

where the sum sum over c, v runs over indexes such that $[\![c+1,n]\!]_{\mathsf{j}+v}^{\mathfrak{k}_{>c}} \in \widehat{\mathsf{Cut}}\big(\tau, \alpha-|\mathfrak{k}+\mathfrak{p}|\big) \setminus \widehat{\mathsf{Cut}}\big(\tau, \alpha-|\mathfrak{k}|\big)$. The result follows in that case from Lemma 31.

References

[1] Bahouri H., Chemin J.Y. and Danchin R., *Fourier Analysis and Nonlinear Partial Differential Equations.* Grundlehren der mathematischen Wissenschaften, **343**, Springer (2011).

[2] Bailleul I. and Bernicot F., *Spacetime paraproducts for paracontrolled calculus, 3d-PAM and multiplicative Burgers equations.* Ann. Sci. Éc. Norm. Sup., **51**(6):1399–1457, (2018).

[3] Bailleul I. and Bernicot F., *High order paracontrolled calculus.* Forum Math. Sigma, **7**(e44):1–94, (2019).

[4] Bailleul I. and Hoshino M., *A tourist guide to regularity structures and singular stochastic PDEs.* arXiv:2006.03525, to appear in EMS Surveys Math. Sci. (2025).

[5] Bailleul I. and Hoshino M., *Paracontrolled calculus and regularity structures I.* J. Math. Soc. Japan **73** (2021), no. 2, 553–595.

[6] Bailleul I. and Hoshino M., *Paracontrolled calculus and regularity structures II.* J. Éc. polytech. Math. **8** (2021), 1275–1328.

[7] Bailleul I. and Mouzard A., *Paracontrolled calculus for quasilinear singular PDEs*. Stoch. PDE: Anal. Comp., **11**:599–650 (2023).

[8] Bruned Y., Hairer M. and Zambotti L., *Algebraic renormalisation of regularity structures*. Invent. Math., **215**:1039–1156, (2019).
[9] Bruned Y. and Manchon D., *Algebraic deformation for (S)PDEs*. J. Math. Soc. Japan, **75**(2):485–526, (2023).
[10] Ghosal, P. and Yi, J., *Fractal geometry of the PAM in 2D and 3D with white noise potential*. arXiv:2303.16063, (2023).
[11] Gubinelli M., Imkeller P. and Perkowski N., *Paracontrolled distributions and singular PDEs*. PDEs. Forum Math. Pi, **3**(e6):1–75, (2015).
[12] Gubinelli M. and Perkowski N., *KPZ reloaded*, Comm. Math. Phys., **349**:165–269, (2017).
[13] Hairer M., *A theory of regularity structures*. Invent. Math., **198**:269–504, (2014).
[14] Hairer M. and Kelly D., *Geometric versus non-geometric rough paths*. Ann. Inst. H. Poincaré Probab. Statist., **51**(1):207–251, (2015).
[15] Hoshino M., *Iterated paraproducts and iterated commutator estimates in Besov spaces*. Stochastic analysis, random fields and integrable probability—Fukuoka 2019, 239–259, Adv. Stud. Pure Math., **87**, Math. Soc. Japan, Tokyo.
[16] Hoshino M., *Commutator estimates from a viewpoints of regularity structures* . RIMS Kôkyûroku Bessatsu, **B79**:179–197, (2020).
[17] Hoshino M., *A note on the Taylor estimates of iterated paraproducts*. RIMS Kôkyûroku Bessatsu, **B95**:37–51, (2024).
[18] Linares, P., Otto, F. and Tempelmayr, M., *The structure group for quasi-linear equations via universal enveloping algebras*. Comm. Am. Math. Soc., **3**:1–64, (2023).
[19] Otto, F., Sauer, J., Smith, S.A. and Weber, H., *A priori bounds for quasi-linear SPDEs in the full subcritical regime*. J. Europ. Math. Soc., **27**(1):71–118, (2024).
[20] Otto, F. and Weber, H., *Quasilinear SPDEs via rough paths*. Arch. Ration. Mech. Anal., **232**(2):873–950, (2019).
[21] Shen, H., Zhu, R. and Zhu, X. *Global well-posedness for 2D generalized Parabolic Anderson Model via paracontrolled calculus*. arXiv:2402.19137, (2024).

• **I. Bailleul** – Univ Brest, CNRS UMR 6205, Laboratoire de Mathematiques de Bretagne Atlantique, F- 29200 Brest, France. *E-mail*: ismael.bailleul@univ-brest.fr. Partial support from the ANR-22-CE40-0017 grant.

• **N. Moench** – Univ. Rennes, CNRS, IRMAR - UMR 6625, F-35000 Rennes, France. *E-mail: nicolas.moench@ens-rennes.fr*