Mass of branching Brownian motion in an expanding ball

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Abstract

Consider a d-dimensional branching Brownian motion starting with a single particle at the origin and let n_t be the number of particles at time t whose ancestral lines have remained up to t within a ball of radius $r(t)$ centered at the origin, where $r(t)$ increases sublinearly with t. We obtain a full limit large-deviation result as time tends to infinity on the probability that n_t is aytpically small. A phase transition is identified, at which the nature of the optimal strategy to realize the aforementioned large-deviation event changes, and the Lyapunov exponent giving the decay rate of the associated large-deviation probability is continuous. As a corollary, we also obtain a kind of law of large numbers for n_t under the stronger assumption that $r(t)$ increases subdiffusively with t.

Keywords: Branching Brownian motion, large deviations, killing boundary

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1. Introduction

In this work, we consider a model of branching Brownian motion (BBM) in $\mathbb{R}^d$, where at each time the process consists of the particles of a normal BBM whose ancestral lines at that time are fully inside a ball of radius $r(t)$ centered at the starting position of the BBM, where $r(t)$ increases sublinearly with time. Let n_t be the number of particles of this process at time t. Our main objective is to prove a full limit large-deviation result as time tends to infinity on the probability that n_t is aytpically small, and hence prove [10, Conjecture 2.8]. Inter alia, we identify a phase transition in the aforementioned limit and the optimal strategies to realize the associated large-deviation events.

1.1. Formulation of the problem and background

Let $Z = (Z_t)_{t\geq 0}$ be a strictly dyadic d-dimensional BBM with branching rate $\beta > 0$, where t represents time. The process starts with a single particle, which performs a Brownian motion in $\mathbb{R}^d$ for a random lifetime, at the end of which it dies and simultaneously gives birth to two offspring. Similarly, starting from the position where their parent dies, each offspring repeats the same procedure as their parent independently of others and the parent, and the process evolves through time in this way. All particle lifetimes are exponentially distributed with constant parameter $\beta > 0$. For each $t \geq 0$, Z_t can be viewed as a finite discrete measure on $\mathbb{R}^d$, which is supported at the positions of the particles at time t. We use P_x and E_x, respectively, to denote the law and corresponding expectation of a BBM starting with a single particle at $x \in \mathbb{R}^d$. For simplicity, we set $P = P_0$.

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For any particle u of the BBM, let S_u and σ_u, respectively, denote its fission time and lifetime, and let the function $X_u : [S_u - \sigma_u, S_u) \to \mathbb{R}^d$ give the position of u while alive. For two particles u and v, we write $v < u$ to mean that v is an ancestor of u. Now, extend the definition of X_u as

$$X_u(t) := \begin{cases} X_u(t) & \text{if } S_u - \sigma_u \leq t < S_u, \\ X_v(t) & \text{if } v < u \text{ and } S_v - \sigma_v \leq t < S_v \end{cases}$$

to be able to talk about the path of u for all times $t < S_u$. For $t \geq 0$, let $\mathcal{N}_t$ be the set of particles of BBM alive at time t. For $u \in \mathcal{N}_t$, we will refer to

$$X_u^{(t)} := \{X_u(s) : 0 \leq s \leq t\}$$

as the *ancestral line of u up to t*. Now let $B(x, a)$ denote the open ball of radius $a > 0$ centered at $x \in \mathbb{R}^d$, and let $r : \mathbb{R}_+ \to \mathbb{R}_+$ be a *radius* function which is sublinearly increasing without bound, that is,

$$\lim_{t\to\infty} r(t) = \infty \quad \text{and} \quad r(t) = o(t), \quad t \to \infty. \tag{1}$$

The main quantity of interest in this work is

$$n_t := \# \left\{ u \in \mathcal{N}_t : X_u^{(t)} \subseteq B(0, r(t)) \right\}. \tag{2}$$

The integer-valued process $(n_t)_{t\geq 0}$ was introduced in [10] as the mass of a *BBM with deactivation at a moving boundary* as follows. For a Borel set $A \subseteq \mathbb{R}^d$, denote by ∂A the boundary of A. Consider a family of Borel sets $B = (B_t)_{t\geq 0}$. For each $t \geq 0$, start with Z_t, and delete from it any particle whose ancestral line up to t has exited B_t to obtain $Z_t^{B_t}$, which denotes a BBM with deactivation at ∂B by $Z^B = (Z_t^{B_t})_{t\geq 0}$. The boundary is called *deactivating* in the following sense: once a particle of the BBM hits the boundary of B at that time, it is instantly deactivated and otherwise continues its life normally but is reactivated later if and when its ancestral line becomes fully inside B at that later time. That is, $Z_t^{B_t}$ consists of 'active' particles at time t; these are particles whose ancestral lines have been confined to B_t up to time t but may have left B_s at an earlier time s. Similar to Z_t, $Z_t^{B_t}$ can be viewed as a finite discrete measure on B_t. Observe that the process $Z^B = (Z_t^{B_t})_{t\geq 0}$ is non-Markovian; one can see this by noticing that particles that have disappeared or been deactivated earlier may suddenly reappear or be reactivated at a later time. On the other hand, the process can be recovered from a single BBM as described above.

For $t > 0$, now let $B_t := B(0, r(t))$ with $r(t)$ as in (1) so that $B = (B_t)_{t\geq 0}$ may be viewed as an expanding ball, and p_t be the probability of confinement to B_t of a standard Brownian motion (starting at the origin) over $[0, t]$. That is, if we let $X = (X_t)_{t\geq 0}$ be a standard Brownian motion with corresponding probability $\mathbf{P}_0$, and $\sigma_A = \inf\{s \geq 0 : X_s \notin A\}$ be the first exit time of X out of A for any Borel set $A \subseteq \mathbb{R}^d$, then

$$p_t := \mathbf{P}_0 \left(\sigma_{B_t} \geq t \right).$$

For a Borel set $A \subseteq \mathbb{R}^d$ and $t \geq 0$, denote by $Z_t(A)$ the mass of Z inside A at time t, and use $N_t := Z_t(\mathbb{R}^d)$ to denote the total mass of Z at time t. Similarly, $Z_t^{B_t}(A)$ denotes the mass of $Z_t^{B_t}$ in A at time t. Observe by (2) that $n_t = Z_t^{B_t}(\mathbb{R}^d) = Z_t^{B_t}(B_t)$. The main objective of this work is, for a suitably decreasing function $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ with $\lim_{t\to\infty} \gamma(t) = 0$, to find the asymptotic behavior as $t \to \infty$ of the large-deviation probability

$$P\left(n_t < \gamma_t p_t e^{\beta t} \right), \tag{3}$$

where we have set $\gamma_t = \gamma(t)$. It is easy via a many-to-one argument (see Proposition 1) to show that

$$E[n_t] = p_t e^{\beta t}. \tag{4}$$

Therefore, since $\lim_{t\to\infty} \gamma_t = 0$, for large t one guesses that $\gamma_t p_t e^{\beta t}$ is atypically small for n_t. Theorem 1 verifies that this is indeed so, and at the same time proves Conjecture 2.8 from [10]. We note that only certain bounds for the asymptotics of (3) as $t \to \infty$ were proved in [10] (see Theorem 2.4 therein). Theorem 2 under the stronger assumption $r(t) = o(\sqrt{t})$ further shows that $p_t e^{\beta t}$ is the typical growth of mass in a certain sense. The reason why we call (3) a large-deviation probability is explained in Remark 2.

Branching diffusions in frozen or dynamic restricted domains in $\mathbb{R}^d$ have been widely studied over the past decades. Starting with Sevast'yanov [12], most of the models involved absorbing boundaries, where particles were immediately absorbed by the boundary upon hitting it. In [7], Kesten studied a BBM with negative drift in one dimension with absorption at the origin, starting with a particle at position $x > 0$. This model proved to be rich, leading to various fine results in subsequent works. Note that the one-dimensional model of a BBM with drift and a fixed barrier is equivalent to the case of no drift and a linearly moving barrier. More recently in [5], Harris et al. studied a BBM with drift in a fixed-size interval in $\mathbb{R}$, which is in effect a two-sided barriered version of Kesten's model. We emphasize that in all of the aforementioned works, the process studied is Markovian contrary to the non-Markovian nature of the process Z^B introduced here.

1.2. Motivation

The motivation to introduce and study the model of BBM with deactivation at a moving boundary arised from its intimate relation with the problem of BBM among mild Poissonian obstacles. Here, we briefly describe the connection. Let Π be a homogeneous Poisson point process in $\mathbb{R}^d$, $(\Omega, \mathbb{P})$ be the associated probability space, and for $\omega \in \Omega$ define the *trap field* with radius $a > 0$ as the random set

$$K = K(\omega) = \bigcup_{x_i \in \mathrm{supp}(\Pi)} \bar{B}(x_i, a),$$

where $\bar{B}(x, a)$ denotes the closed ball of radius a centered at $x \in \mathbb{R}^d$. The mild obstacle rule for BBM is that when particles are inside K they branch at a lower rate (possibly zero) than when they are outside K, where they branch at the normal rate. Hence, the random trap field serves as a mass-suppressing mechanism and in a typical environment one expects the mass of BBM to grow slower than that of a *free* BBM, that is, a BBM in $\mathbb{R}^d$ without any obstacles.

In [10], a quenched strong law of large numbers for the mass of BBM among mild obstacles was proved (see Theorem 2.1 therein). It was shown that in almost every environment with respect to the Poisson point process, certain trap-free regions (called *clearings*) exist, which may be suitably indexed by time t in regard to the evolution of the BBM. More precisely, it was shown that for all large t, with 'high' probability at least one particle of the BBM is able to hit a spherical clearing of a certain radius $R = R(t)$ soon enough, and the sub-BBM emanating from this particle and confined to this clearing over the remaining time, determines to the leading order the overall growth of mass of the entire BBM among the mild obstacles. Therefore, understanding the growth of particles whose ancestral lines don't escape these clearings is essential for the proofs. We refer the reader to [10, Section 5] for details.

Notation: We use c as a generic positive constant, whose value may change from line to line. If we wish to emphasize the dependence of c on a parameter p, then we write $c(p)$. We denote by $f : A \to B$ a function f from a set A to a set B. For two functions $f, g : \mathbb{R}_+ \to \mathbb{R}_+$, we write

$g(t) = o(f(t))$ if $g(t)/f(t) \to 0$ as $t \to \infty$, and write $g(t) = O(f(t))$ if there exist $M > 0$ and $t_0 > 0$ such that $g(t) \leq Mf(t)$ for all $t \geq t_0$.

We denote by $X = (X_t)_{t \geq 0}$ a generic standard Brownian motion in d-dimensions, and use $\mathbf{P}_x$ and $\mathbf{E}_x$, respectively, as the law of X started at position $x \in \mathbb{R}^d$, and the corresponding expectation. Also, we denote by λ_d the principal Dirichlet eigenvalue of $-\frac{1}{2}\Delta$ on the unit ball in d dimensions.

Outline: The rest of the paper is organized as follows. In Section 2, we present our results. In Section 3, we develop the preparation needed for the proof of Theorem 1, which is our main result. In Section 4 and Section 5, we present, respectively, the proofs of Theorem 1 and Theorem 2.

2. Results

Our main result is on the large-time asymptotic behavior of the probability that the mass of BBM inside a sublinearly expanding ball $B = (B_t)_{t \geq 0}$ with deactivation at the boundary of the ball, is atypically small. For a generic standard Brownian motion $X = (X_t)_{t \geq 0}$ and a Borel set $A \subseteq \mathbb{R}^d$, let $\sigma_A = \inf\{s \geq 0 : X_s \notin A\}$ as before. Denote by $a \wedge b$ the minimum of the numbers a and b.

Theorem 1 (Lower large-deviations for mass of BBM in an expanding ball)**.** *Let $r : \mathbb{R}_+ \to \mathbb{R}_+$ be increasing as in* (1)*. In addition, suppose that $r(t) = o(\sqrt{t})$* (subdiffusive) *or $\sqrt{t} = O(r(t))$* (diffusive *or* superdiffusive)*. Let $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ be defined by $\gamma(t) = e^{-\kappa r(t)}$, where $\kappa > 0$ is a constant. For $t > 0$, set $B_t = B(0, r(t))$, $p_t = \mathbf{P}_0(\sigma_{B_t} \geq t)$, and let n_t be as in* (2)*. Then,*

$$\lim_{t \to \infty} \frac{1}{r(t)} \log P\left(n_t < \gamma_t p_t e^{\beta t}\right) = -(\kappa \wedge \sqrt{2\beta}).$$

Remark 1. *Theorem 1 says that there is a continuous phase transition at a critical value $\kappa = \sqrt{2\beta}$ in the asymptotic behavior of $P\left(n_t < \gamma_t p_t e^{\beta t}\right)$. This is revealed by the rate function given by $I(\kappa) := \kappa \wedge \sqrt{2\beta}$. In terms of the optimal strategies for the BBM to realize the large-deviation event $\{n_t < \gamma_t p_t e^{\beta t}\}$, the phase transition can be explained as follows.*

- *When $\kappa > \sqrt{2\beta}$, the BBM simultaneously suppresses the branching of the initial particle and moves it outside $B(0, r(t))$ over the time interval $[0, r(t)/\sqrt{2\beta}]$. Once the initial particle is moved outside $B(0, r(t))$, the event $\{n_t = 0\}$ is realized and there is no need for further atypical behavior that could incur a probabilistic cost.*

- *When $0 < \kappa \leq \sqrt{2\beta}$, the BBM suppresses the branching completely over the time interval $[0, (\kappa/\beta)r(t)]$, and then behaves 'normally' in the remaining interval $[(\kappa/\beta)r(t), t]$. This means, the parameter κ is low enough so that there is no additional need to move the initial particle outside $B(0, r(t))$ over a time interval of order $r(t)$.*

Remark 2. *We call $P\left(n_t < \gamma_t p_t e^{\beta t}\right)$ with $\gamma_t = e^{-\kappa r(t)}$ a large-deviation probability, because both $P(n_t = 0)$ and $P\left(n_t < \gamma_t p_t e^{\beta t}\right)$ decay as $e^{-cr(t)}$, where the values $c > 0$ may differ, to the leading order for large t. The significance of the choice $\gamma_t = e^{-\kappa r(t)}$ is as follows. It can be shown that if $\gamma_t \to 0$ as $t \to \infty$, then for all large t, $P(n_t < \gamma_t p_t e^{\beta t}) \geq \delta \gamma_t$ for some $\delta > 0$. If γ_t decays slower than $e^{-cr(t)}$ so that $(\log \gamma_t)/r(t) \to 0$ as $t \to \infty$, this would imply $\liminf_{t \to \infty} (r(t))^{-1} \log P\left(n_t < \gamma_t p_t e^{\beta t}\right) = 0$. Therefore, in that case, in view of Theorem 1, the event $\{n_t < \gamma_t p_t e^{\beta t}\}$ would not be a large-deviation event. (See the proof of the lower bound of Theorem 1 in Section 4 for details.)*

Theorem 2 (Law of large numbers for mass of BBM in an expanding ball)**.** *Let $r:\mathbb{R}_+\to\mathbb{R}_+$ be subdiffusively increasing without bound, that is, $r(t)\to\infty$ as $t\to\infty$ and $r(t)=o(\sqrt{t})$. For $t>0$, set $B_t=B(0,r(t))$, $p_t=\mathbf{P}_0(\sigma_{B_t}\geq t)$, and let n_t be as in (2). Then,*

$$\lim_{t\to\infty}(r(t))^2\left(\frac{\log n_t}{t}-\beta\right)=-\lambda_d \quad \textit{in } P\textit{-probability.}$$

Remark 3. *A subdiffusive expansion means, the ball is expanding slower than the typical rate at which a standard Brownian motion moves away from the origin, and therefore for large t it would be a rare event for the Brownian motion to be confined in B_t. The large-time behavior of p_t when $r=r(t)$ is subdiffusive is given in Proposition C.*

Remark 4. *Theorem 2 is called a law of large numbers, because it says that in a sense the process n_t grows as its expectation as $t\to\infty$. That is, in a loose sense,*

$$\frac{\log n_t}{t}\approx\frac{\log E(n_t)}{t},\quad t\to\infty.$$

Remark 5. *For radius functions of powerlike growth $r(t)=t^\alpha$, $0<\alpha<1/2$, it seems possible with further work to improve the convergence in probability in Theorem 2 to almost sure convergence, and hence obtain a strong law of large numbers for n_t. A careful look at the probability estimates in Section 5 suggests that the decay in t of the probabilities of the unlikely events therein could be sufficient for a Borel-Cantelli argument. It should however be noted that n_t is not almost surely increasing in t; therefore, it is not clear how one would pass from integer (or countable) times to continuous time once the Borel-Cantelli lemma is applied.*

On the other hand, for arbitrarily growing radius functions with $r(t)\to\infty$ as $t\to\infty$ and $r(t)=o(\sqrt{t})$, it is not clear whether the probability estimates in Section 5 are sufficient to pass to a strong law for n_t via a Borel-Cantelli argument.

3. Preparations

In this section, we first list three well-known results, one concerning the distribution of mass in branching systems and the other two on the hitting times of a d-dimensional Brownian motion. Then, we state and prove two propositions which can be obtained from existing results in a somewhat straightforward way. The results of this section will be useful in the proof of the upper bound of Theorem 1.

The following proposition is well-known from the theory of continuous-time branching processes. For a proof, see for example [6, Section 8.11].

Proposition A (Distribution of mass in branching systems)**.** *For a strictly dyadic continuous-time branching process $N=(N_t)_{t\geq 0}$ with constant branching rate $\beta>0$, the probability distribution at time t is given by $P(N_t=k)=e^{-\beta t}(1-e^{-\beta t})^{k-1}$ for $k\geq 1$, from which it follows that*

$$P(N_t>k)=(1-e^{-\beta t})^k \quad \textit{and} \quad E[N_t]=e^{\beta t}.$$

As before, we use $X=(X_t)_{t\geq 0}$ to denote a generic Brownian motion in d dimensions, and use $\mathbf{P}_x$ and $\mathbf{E}_x$, respectively, for the associated probability and expectation for a process that starts at position $x\in\mathbb{R}^d$. Proposition B below is on the large-time asymptotic probability of atypically large (linear) Brownian displacements. For a proof, see for example [8, Lemma 5].

Proposition B (Linear Brownian displacements)**.** *For $k > 0$,*

$$\mathbf{P}_0\left(\sup_{0\le s\le t}|X_s| > kt\right) = \exp\left[-\frac{k^2t}{2}(1+o(1))\right].$$

The following is a standard result on the large-time Brownian confinement in balls, and for instance can be deduced from [2, Proposition 1.6], along with Brownian scaling. Recall that $\sigma_A = \inf\{s \ge 0 : X_s \notin A\}$ denotes the first exit time of X out of A.

Proposition C (Brownian confinement in balls)**.** *For $t > 0$, let $B_t = B(0, r(t))$, where $r : \mathbb{R}_+ \to \mathbb{R}_+$ is such that $r(t) \to \infty$ as $t \to \infty$ and $r(t) = o(\sqrt{t})$. Then, as $t \to \infty$,*

$$\mathbf{P}_0\left(\sigma_{B_t} \ge t\right) = \exp\left[-\frac{\lambda_d t}{r^2(t)}(1+o(1))\right].$$

The following result is an easy consequence of the classical many-to-one lemma for branching processes.

Proposition 1.

$$E[n_t] = p_t e^{\beta t}.$$

Proof. By the many-to-one lemma (using for instance [1, Lemma 25] with the choice $F(X_u(s), s \le t) = \mathbb{1}_{\{X_u(s)\subseteq B(0,r(t))\,\forall\, s\le t\}}$ therein), for any $t \ge 0$,

$$E[n_t] = E\left[\sum_{u\in\mathcal{N}_t}\mathbb{1}_{\{X_u(s)\subseteq B(0,r(t))\,\forall\, s\le t\}}\right] = e^{\beta t}\mathbf{E}_0\left[\mathbb{1}_{\{X_s\subseteq B(0,r(t))\,\forall\, s\le t\}}\right] = p_t e^{\beta t}.$$

□

For $a > 0$, we introduce the notation $\mathbf{P}^a$ to stand for the law of a Brownian motion that starts at a distance a from the origin. Also, for $a > 0$, set $\tau_a = \sigma_{B(0,a)}$ for ease of notation. The following result compares the probabilities of confinement in balls for Brownian motions starting at the center of the ball and at any other point inside the ball.

Proposition 2 (Brownian confinement in balls, comparison)**.** *Let $a, b \in \mathbb{R}$ such that $0 < a < b$. Then, there exists a positive constant $D = D(b/a, d)$ such that for all large t,*

$$D\,\mathbf{P}^a(\tau_b \ge t) \ge \mathbf{P}^0(\tau_b \ge t).$$

Proof. Let $\tau_b^{(\nu)}$ be the first hitting time to b of the Bessel process with index ν. (This process coincides with what is called the Bessel process of dimension $2\nu + 2$ in some standard texts such as [11].) It is well-known that if $2\nu + 2$ is a positive integer, then the Bessel process is identical in law to the radial component of a $(2\nu + 2)$-dimensional Brownian motion. It follows from (2.7) and (2.8) of [4], respectively, that as $t \to \infty$,

$$\mathbf{P}^0(\tau_b \ge t) = \frac{1}{2^{\nu-1}\Gamma(\nu+1)}e^{-\frac{j_{\nu,1}^2 t}{2b^2}}(1+o(1)) \tag{5}$$

and

$$\mathbf{P}^a(\tau_b \ge t) = 2\left(\frac{b}{a}\right)^{\nu}\frac{J_\nu(aj_{\nu,1}/b)}{j_{\nu,1}J_{\nu+1}(j_{\nu,1})}e^{-\frac{j_{\nu,1}^2 t}{2b^2}}(1+o(1)), \tag{6}$$

where $0 < a < b$, J_μ is the Bessel function of the first kind of order μ, $\{j_{\mu,k}\}_{k=1}^{\infty}$ is the increasing sequence of positive zeros of J_μ, and we suppress the dependence of τ on ν (hence on d) in notation. It then follows from (5) and (6) that there exist $t_0 > 0$ and a positive constant $D = D(b/a, d)$ such that for all $t \geq t_0$,

$$D\,\mathbf{P}^a(\tau_b \geq t) \geq \mathbf{P}^0(\tau_b \geq t).$$

□

For $0 < \delta < 1$, $k \geq 0$ (k need not be an integer), and $B_t = B(0, r(t))$, let

$$\widetilde{p}_t = \widetilde{p}_t(k) = \mathbf{P}^{\delta r(t)}\left(\sigma_{B_t} \geq t - kr(t)\right). \tag{7}$$

Observe that by Brownian scaling, we have $\widetilde{p}_t = \mathbf{P}^{\delta}\left(\sigma_{B(0,1)} \geq (t - kr(t))/(r(t))^2\right)$. The following result is on the aytpically small growth of mass of a BBM with deactivation at the boundary of a sublinearly expanding ball, where the BBM is started with a single particle at an interior point of the ball whose distance to the center is on the scale of the radius of the ball. For $a > 0$, denote by P^a the law of a branching Brownian motion that starts with a single particle at distance a from the origin.

Proposition 3. *Let $0<\delta<1$, $0\leq k_1<k_2$ and $r:\mathbb{R}_+\to\mathbb{R}_+$ be sublinearly increasing as in* (1)*. Let $\gamma_t = e^{-\kappa r(t)}$, where $\kappa > 0$ is a constant, and $\widetilde{p}_t = \widetilde{p}_t(k)$ be as in* (7)*. For $t > 0$, set $B_t = B(0, r(t))$ and let n_t be as in* (2)*. Then, there exists a constant $c = c(\delta, \kappa, \beta) > 0$ such that for all large t,*

$$\sup_{k_1\leq k\leq k_2} P^{\delta r(t)}\left(n_{t-kr(t)} < \gamma_t \widetilde{p}_t e^{\beta(t-kr(t))}\right) \leq e^{-cr(t)}. \tag{8}$$

The following proof follows closely the proof of the upper bound of [10, Theorem 2.4]. The main difference between the current result and the upper bound of [10, Theorem 2.4] is that here, as opposed to starting from the origin, the process starts at an interior point of the ball whose distance to the center is on the scale of the radius of the ball. A minor difference is that here, we investigate the state of the system at times $t - kr(t)$ uniformly over $k \in [k_1, k_2]$, where $0\leq k_1 < k_2$ are fixed. In the following proof, the proof of the upper bound of [10, Theorem 2.4] was suitably modified so as to take into account these two differences. We emphasize that Proposition 3 will be the key ingredient in the proof of the upper bound of Theorem 1 (see Section 4.2).

Proof. Let $k \in [k_1, k_2]$ and set $g_t = 2\gamma_t$. Recall that $N_t = Z_t(\mathbb{R}^d)$, where $Z = (Z_t)_{t\geq 0}$ denotes a BBM. For $t > 0$, start with the estimate

$$P^{\delta r(t)}(\,\cdot\,) \leq P^{\delta r(t)}\left(\,\cdot \;\middle|\; N_{t-kr(t)} > e^{\beta(t-kr(t))}g_t\right) + P^{\delta r(t)}\left(N_{t-kr(t)} \leq e^{\beta(t-kr(t))}g_t\right). \tag{9}$$

Proposition A yields $P(N_{t-kr(t)} \leq n) = 1-(1-e^{-\beta(t-kr(t))})^n \leq ne^{-\beta(t-kr(t))}$ for any $n \geq 1$. Setting $n = \left\lfloor e^{\beta(t-kr(t))}g_t \right\rfloor$, we have for $t > 0$,

$$P\left(N_{t-kr(t)} \leq e^{\beta(t-kr(t))}g_t\right) = P\left(N_{t-kr(t)} \leq \left\lfloor e^{\beta(t-kr(t))}g_t \right\rfloor\right) \leq g_t, \tag{10}$$

which bounds the second term on the right-hand side of (9). Since $k_2 > 0$ is fixed, and $r(t) = o(t)$ by assumption, it is clear that there exists $t_0 > 0$ such that $t - k_2 r(t) > 0$ and $\left\lfloor e^{\beta(t-k_2 r(t))}g_t \right\rfloor \geq 1$ for all $t \geq t_0$. Fix this t_0, and for $t \geq t_0$, define

$$\widetilde{P}_t(\,\cdot\,) = P^{\delta r(t)}\left(\,\cdot \;\middle|\; N_{t-kr(t)} > e^{\beta(t-kr(t))}g_t\right),$$

and let $\widetilde{E}_t$ and $\widetilde{\mathrm{Var}}_t$ denote, respectively, the expectation and variance associated to $\widetilde{P}_t$. Let $\mathcal{N}_t$ denote the set of particles of Z at time t, and for $u \in \mathcal{N}_t$, let $(Y_u(s))_{0\le s\le t}$ denote the ancestral line up to t of particle u. Now, conditional on the event $\{N_{t-kr(t)} > e^{\beta(t-kr(t))}g_t\}$, choose randomly, independent of their genealogy and position, $M_t := \left\lfloor e^{\beta(t-kr(t))}g_t \right\rfloor$ particles out of the particles at time t. Denote the collection of the chosen particles by $\mathcal{M}_t$, and define

$$\hat{n}_t = \sum_{u\in\mathcal{M}_t} \mathbb{1}_{A_u},$$

where $A_u = \{Y_u(s) \in B_t \ \forall\, 0 \le s \le t - kr(t)\}$, and we have suppressed the dependence of A_u on t in notation. Since the collection $\mathcal{M}_t$ is chosen independently of the motion process, the ancestral line of each particle in $\mathcal{M}_t$ is Brownian and the linearity of expectation gives

$$\widetilde{E}_t\left[\hat{n}_t\right] = \widetilde{p}_t \left\lfloor e^{\beta(t-kr(t))}g_t \right\rfloor,$$

where $\widetilde{p}_t$ is as in (7). Now apply Chebyshev's inequality to the random variable $\hat{n}_t$ to obtain, by (9) and (10), for $t \ge t_0$,

$$\begin{aligned} P^{\delta r(t)}\left(n_{t-kr(t)} < \gamma_t \widetilde{p}_t e^{\beta(t-kr(t))}\right) &\le \widetilde{P}_t\left(\hat{n}_t < \gamma_t \widetilde{p}_t e^{\beta(t-kr(t))}\right) + g_t \\ &\le \widetilde{P}_t\left(|\hat{n}_t - \widetilde{E}_t[\hat{n}_t]| > \widetilde{p}_t \left\lfloor e^{\beta(t-kr(t))}g_t \right\rfloor - \gamma_t \widetilde{p}_t e^{\beta(t-kr(t))}\right) + g_t \\ &\le \frac{\widetilde{\mathrm{Var}}_t(\hat{n}_t)}{[(g_t-\gamma_t)\widetilde{p}_t e^{\beta(t-kr(t))} - \widetilde{p}_t]^2} + g_t. \end{aligned} \tag{11}$$

Let $\mathcal{P}$ the probability under which the pair (i,j) is chosen uniformly at random among the $M_t(M_t-1)$ possible pairs in $\mathcal{M}_t$, and let $\mathcal{E}$ be the corresponding expectation. For a generic Brownian motion X, let Var denote its variance and $A := \{X_s \in B_t \ \forall\, 0 \le s \le t\}$. Then,

$$\begin{aligned} \widetilde{\mathrm{Var}}_t\left(\hat{n}_t\right) &= \widetilde{\mathrm{Var}}_t\left(\sum_{u\in\mathcal{M}_t} \mathbb{1}_{A_u}\right) \\ &= M_t \mathrm{Var}\left(\mathbb{1}_A\right) + \sum_{1\le i\ne j\le M_t} \widetilde{\mathrm{Cov}}_t\left(\mathbb{1}_{A_i}, \mathbb{1}_{A_j}\right) \\ &= M_t\left(\widetilde{p}_t - \widetilde{p}_t^2\right) + M_t(M_t-1)\frac{\sum_{1\le i\ne j\le M_t} \widetilde{\mathrm{Cov}}_t\left(\mathbb{1}_{A_i}, \mathbb{1}_{A_j}\right)}{M_t(M_t-1)} \\ &\le g_t e^{\beta(t-kr(t))}\left(\widetilde{p}_t - \widetilde{p}_t^2\right) + g_t^2 e^{2\beta(t-kr(t))}\left[(\mathcal{E}\otimes\widetilde{P}_t)(A_i\cap A_j) - \widetilde{p}_t^2\right], \end{aligned} \tag{12}$$

where $(\mathcal{E}\otimes\widetilde{P}_t)(A_i\cap A_j) = \mathcal{E}[\widetilde{P}_t(A_i\cap A_j)]$ denotes averaging $\widetilde{P}_t(A_i\cap A_j)$ over the $M_t(M_t-1)$ possible pairs in the randomly chosen set $\mathcal{M}_t$. Define

$$\widetilde{p}^{(t)}(x,s,dy) := \mathbf{P}_x(X_s \in dy \mid X_z \in B_t \ \forall\, 0 \le z \le s) \quad \text{and} \quad p^t_{s,x} := \mathbf{P}_x(\sigma_{B_t} \ge s). \tag{13}$$

Note that an application of the Markov property of a standard Brownian motion at time s with $0 < s < t$ yields

$$\widetilde{p}_t = p^t_{s,\delta r(t)\mathbf{e}} \int_{B_t} p^t_{t-kr(t)-s,y}\, \widetilde{p}^{(t)}(\delta r(t)\mathbf{e}, s, dy), \tag{14}$$

where $\mathbf{e}$ denotes any unit vector in $\mathbb{R}^d$. Let $Q^{(t)}$ be the distribution of the splitting time of the most recent common ancestor of ith and jth particles under $\mathcal{E}\otimes\widetilde{P}_t$. Then, applying the Markov property at this splitting time, we obtain

$$\left(\mathcal{E}\otimes\widetilde{P}_t\right)(A_i\cap A_j)=\widetilde{p}_t\int_0^{t-kr(t)}\int_{B_t}p^t_{t-kr(t)-s,x}\,\widetilde{p}^{(t)}(\delta r(t)\mathbf{e},s,dx)\,Q^{(t)}(ds). \tag{15}$$

Indeed, conditioning on the event $\{N_{t-kr(t)}>e^{\beta(t-kr(t))}g_t\}$ does not affect the motion of particles; therefore, under the law $\mathcal{E}\otimes\widetilde{P}_t$ the path of each ancestral line is still Brownian. The ancestral line of, say particle i, up to time $t-kr(t)$ contributes a factor of $\widetilde{p}_t$ to the right-hand side of (15), and the ancestral line of particle j starting from the aforementioned splitting time up to time $t-kr(t)$ gives the rest of the right-hand side of (15). Set $p^t_{s,\delta r(t)}=p^t_{s,\delta r(t)\mathbf{e}}$ for simplicity. It then follows from (14) and (15) that

$$\left(\mathcal{E}\otimes\widetilde{P}_t\right)(A_i\cap A_j)=\widetilde{p}_t^2\int_0^{t-kr(t)}\frac{1}{p^t_{s,\delta r(t)}}\,Q^{(t)}(ds). \tag{16}$$

For $t>0$, define

$$J_t(k):=\int_0^{t-kr(t)}\frac{1}{p^t_{s,\delta r(t)}}\,Q^{(t)}(ds).$$

Then, by (12) and (16), we have for all $t\geq t_0$,

$$\widetilde{\mathrm{Var}}_t(\hat{n}_t)\leq g_t\widetilde{p}_te^{\beta(t-kr(t))}+g_t^2\widetilde{p}_t^2e^{2\beta(t-kr(t))}(J_t(k)-1). \tag{17}$$

Observe that $J_t(k)-1\geq 0$. Next, we bound $J_t(k)-1$ from above.

Choose $c>0$, fix it, and for t large enough so that $t-k_2r(t)>cr(t)$, define

$$J_t^{(1)}=\int_0^{cr(t)}\frac{1}{p^t_{s,\delta r(t)}}\,Q^{(t)}(ds),\qquad J_t^{(2)}(k)=\int_{cr(t)}^{t-kr(t)}\frac{1}{p^t_{s,\delta r(t)}}\,Q^{(t)}(ds).$$

Split J_t as $J_t(k)=J_t^{(1)}+J_t^{(2)}(k)$. In what follows, to bound J_t from above for large t, we will use that over the first time interval $[0,cr(t)]$, the integrand $1/p^t_{s,\delta r(t)}$ is small enough; whereas the distribution of $Q^{(t)}$ puts a small enough 'weight' on the second time interval $[cr(t),t-kr(t)]$.

Since $p^t_{s,\delta r(t)}$ is nonincreasing in s, $J_t^{(1)}\leq\left[p^t_{cr(t),\delta r(t)}\right]^{-1}$. Moreover, by Proposition B,

$$1-p^t_{cr(t),\delta r(t)}=\exp\left[-\frac{(1-\delta)^2r(t)}{2c}(1+o(1))\right],$$

from which it follows that

$$J_t^{(1)}-1\leq\exp\left[-\frac{(1-\delta)^2r(t)}{2c}(1+o(1))\right]. \tag{18}$$

Next, we bound $J_t^{(2)}(k)$ from above. It is known from [3, (52)] that $Q^{(t)}$ is absolutely continuous with respect to the Lebesgue measure, which we denote by ds, and its density function, which we denote by $q^{(t)}$, satisfies

$$\exists\,C>0,\ s_0>0\quad\text{such that}\quad\forall\,s\geq s_0,\ q^{(t)}(s)\leq Cse^{-\beta s}.$$

Recall that $j_{\nu,1}$ is the first positive zero of the Bessel function of the first kind of order ν. It follows from [4, (2.8)] and Brownian scaling that there exists a positive constant $D = D(\delta, d)$ such that for all $t > 0$,

$$p^t_{s,\delta r(t)} \geq De^{-\frac{j^2_{\nu,1}}{2(r(t))^2}s}.$$

We may then continue with

$$J^{(2)}_t(k) = \int_{cr(t)}^{t-kr(t)} \frac{1}{p^t_{s,\delta r(t)}} Q^{(t)}(ds) \leq \frac{C}{D} \int_{cr(t)}^{\infty} s \exp\left[-\left(\beta - \frac{j^2_{\nu,1}}{2(r(t))^2}\right) s\right] ds = e^{-\beta c r(t)(1+o(1))}, \tag{19}$$

where the last step follows by an application of integration by parts along with the assumption that $r(t) \to \infty$ as $t \to \infty$. It follows from (18) and (19) that there exists a positive constant $c(\delta, \beta)$ such that for all large t,

$$J_t(k) - 1 = J^{(1)}_t - 1 + J^{(2)}_t(k) \leq e^{-c(\delta,\beta)r(t)}. \tag{20}$$

The bound in (20) has no dependence on k. Moreover, the bounds in (11) and (17) hold for each $k \in [k_1, k_2]$. Then, from (11), (17) and (20), we have

$$P^{\delta r(t)}\left(n_{t-kr(t)} < \gamma_t \widetilde{p}_t e^{\beta(t-kr(t))}\right) \leq \frac{g_t \widetilde{p}_t e^{\beta(t-kr(t))} + g^2_t \widetilde{p}^2_t e^{2\beta(t-kr(t))} e^{-c(\delta,\beta)r(t)}}{[(g_t - \gamma_t)\widetilde{p}_t e^{\beta(t-kr(t))} - \widetilde{p}_t]^2} + g_t.$$

To complete the proof, recall the choice $g_t = 2\gamma_t$ and that $\gamma_t = e^{-\kappa r(t)}$ for some $\kappa > 0$. This yields, there exists $t_1 > 0$ such that for all $t \geq t_1$ and $k \in [k_1, k_2]$,

$$P^{\delta r(t)}\left(n_{t-kr(t)} < \gamma_t \widetilde{p}_t e^{\beta(t-kr(t))}\right) \leq \frac{4}{\gamma_t \widetilde{p}_t} e^{-\beta(t-kr(t))} + 8e^{-c(\delta,\beta)r(t)} + g_t \leq 11e^{-(c(\delta,\beta)\wedge\kappa)r(t)}.$$

The last inequality follows, because $r(t) = o(t)$ but $r(t) \to \infty$ as $t \to \infty$, and therefore the decay of $\widetilde{p}_t$ to zero (if at all) is at most subexponential in t. □

4. Proof of Theorem 1

4.1. Proof of the lower bound

The proof of the lower bound of Theorem 1 is based on finding an optimal strategy to realize the event $\{n_t < \gamma_t p_t e^{\beta t}\}$ for each of the low κ regime $0 < \kappa \leq \sqrt{2\beta}$ and the high κ regime $\kappa > \sqrt{2\beta}$, and it was given in [10]. Here, we review the proof for completeness.

Note that $\{n_t = 0\} \subseteq \{n_t < \gamma_t p_t e^{\beta t}\}$, and one way to realize the event $\{n_t = 0\}$ is to completely suppress the branching of the initial particle and move it outside $B_t = B(0, r(t))$ over the time interval $[0, kr(t)]$ for some $k > 0$. By Proposition B, the probability of this joint strategy is

$$\exp\left[-\beta k r(t) - \frac{r(t)}{2k}(1 + o(1))\right]. \tag{21}$$

Optimizing the exponent in (21) over $k > 0$ gives $k = 1/\sqrt{2\beta}$, and with this choice of k, we obtain

$$P(n_t < \gamma_t p_t e^{\beta t}) \geq P(n_t = 0) \geq \exp\left[-\sqrt{2\beta}\, r(t)(1 + o(1))\right]. \tag{22}$$

Now let $f:\mathbb{R}_+\to\mathbb{R}_+$ be such that $f(t)=o(t)$, and denote by τ_1 and $(Y_1(s))_{0\le s\le \tau_1}$, respectively, the lifetime and the path of the initial particle. For $t>0$, define the events

$$A_t=\{N_{f(t)}=1\},\quad E_t=\{n_t<\gamma_t p_t e^{\beta t}\},\quad D_t=\{Y_1(z)\in B_t\ \forall\, 0\le z\le f(t)\}.$$

Estimate

$$P(E_t)\ge P(E_t\cap A_t)=P(E_t\mid A_t)P(A_t). \tag{23}$$

Conditional on A_t, it is clear that $\tau_1\ge f(t)$ and that $n_t=0$ if $Y_1(z)\notin B_t$ for some $z\in[0,f(t)]$. Hence,

$$E\left[n_t\mid A_t\right]=E\left[n_t\mathbb{1}_{D_t}\mid A_t\right]=E\left[n_t\mid A_t,D_t\right]P(D_t\mid A_t). \tag{24}$$

Using the notation from (13) and applying the Markov property of the BBM at time $f(t)$, we have

$$\begin{aligned}E\left[n_t\mid A_t,D_t\right]&=\int_{B_t}E_y\left[n_{t-f(t)}\right]P(Y_1(f(t))\in dy\mid A_t,D_t)\\&=\int_{B_t}E_y\left[n_{t-f(t)}\right]\widetilde{p}^{(t)}(0,f(t),dy)\end{aligned} \tag{25}$$

and

$$P(D_t\mid A_t)=p^t_{f(t),0}. \tag{26}$$

Then, using the many-to-one lemma similarly as in the proof of Proposition 1, we obtain

$$E_y\left[n_{t-f(t)}\right]=p^t_{t-f(t),y}\,e^{\beta(t-f(t))},\quad y\in B_t. \tag{27}$$

It then follows from (24)-(27) that

$$E\left[n_t\mid A_t\right]=e^{\beta(t-f(t))}p^t_{f(t),0}\int_{B_t}p^t_{t-f(t),y}\,\widetilde{p}^{(t)}(0,f(t),dy)=e^{\beta(t-f(t))}p_t,$$

where the last equality follows by applying the Markov property of Brownian motion at time $f(t)$, similar in spirit to (14). Then, by the Markov inequality,

$$P(E_t^c\mid A_t)\le\frac{E\left[n_t|A_t\right]}{\gamma_t p_t e^{\beta t}}=\gamma_t^{-1}e^{-\beta f(t)}. \tag{28}$$

Choose $f(t)=-(1/\beta)\log((1-\delta)\gamma_t)$ with $0<\delta<1$ in (28), which leads to $P(E_t\mid A_t)\ge\delta$. Noting that $P(A_t)=e^{-\beta f(t)}$, the estimate in (23) then yields

$$P(E_t)\ge\delta e^{-\beta f(t)}=\delta(1-\delta)\gamma_t=e^{-\kappa r(t)(1+o(1))}.$$

In view of (22), this completes the proof of the lower bound of Theorem 1.

4.2. Proof of the upper bound

We will follow a discretization method similar to the proof of the upper bound of Theorem 2.1 in [9]. Proposition 2 and Proposition 3 will be the key ingredients in the proof.

Remark 6. *We emphasize that the proof of the upper bound of [10, Theorem 2.4] finds a loose upper bound on the large-deviation probability* $P\left(n_t<\gamma_t p_t e^{\beta t}\right)$ *in the following sense: there exists* *a positive constant c such that*

$$\limsup_{t\to\infty} \frac{1}{r(t)} \log P\left(n_t < \gamma_t p_t e^{\beta t}\right) \le -c. \tag{29}$$

The proof below aims for a sharp upper bound on $P\left(n_t < \gamma_t p_t e^{\beta t}\right)$, and hence finds the largest constant c such that (29) holds. Also, the proof below has a bootstrap nature in that it uses the loose bound in Proposition 3 as a first step to obtain the sharp upper bound on $P\left(n_t < \gamma_t p_t e^{\beta t}\right)$ mentioned above. On the other hand, the proof of the upper bound of [10, Theorem 2.4] is based on a second moment argument for n_t, which is a totally different method than that of the proof below.

Recall that $N_t = Z_t(\mathbb{R}^d)$ is the total mass at time t, and define the random variable

$$\rho_t = \sup\{\rho \in [0,1] : N_{\rho(\kappa/\beta)r(t)} \le \lfloor r(t) \rfloor\}.$$

Observe that for $x \in [0,1]$, we have $\{\rho_t \ge x\} \subseteq \{N_{x(\kappa/\beta)r(t)} \le \lfloor r(t)\rfloor + 1\}$, and that $\rho_t \kappa r(t)/\beta$ is a stopping time. Recall that $\gamma_t = e^{-\kappa r(t)}$ and $p_t = \mathbf{P}_0(\sigma_{B_t} \ge t)$. For $t > 0$, define the event

$$A_t = \{n_t < \gamma_t p_t e^{\beta t}\}.$$

We condition on ρ_t as follows. For every $n = 2, 3, \dots$

$$\begin{aligned} P(A_t) &= \sum_{i=0}^{n-2} P\left(A_t \cap \left\{\frac{i}{n} \le \rho_t < \frac{i+1}{n}\right\}\right) + P\left(A_t \cap \left\{\rho_t \ge 1 - \frac{1}{n}\right\}\right) \\ &\le \sum_{i=0}^{n-2} \exp\left[-\frac{i}{n}\kappa r(t) + o(r(t))\right] P_t^{(i,n)}(A_t) + \exp\left[-\kappa r(t)\left(1 - \frac{1}{n}\right) + o(r(t))\right], \end{aligned} \tag{30}$$

where we have used Proposition A to bound $P(\frac{i}{n} \le \rho_t < \frac{i+1}{n})$ and $P(\rho_t \ge 1 - 1/n)$ from above, and introduced the conditional probabilities

$$P_t^{(i,n)}(\,\cdot\,) = P\left(\cdot \,\middle|\, \frac{i}{n} \le \rho_t < \frac{i+1}{n}\right), \quad i = 0, 1, \dots, n-2.$$

Next, we bound $P_t^{(i,n)}(A_t)$ from above. To that end, let $0 < \delta < 1$ and F_t be the event that there is at least one particle outside the ball $B(0, \delta r(t))$ at some instant s with $0 \le s \le \rho_t(\kappa/\beta)r(t)$, and continue with the estimate

$$P_t^{(i,n)}(A_t) \le P_t^{(i,n)}(F_t) + P_t^{(i,n)}(A_t \cap F_t^c). \tag{31}$$

Under the law $P_t^{(i,n)}$, we have $\rho_t < (i+1)/n$, and by definition of ρ_t there are exactly $\lfloor r(t) \rfloor + 1$ particles present at time $\rho_t(\kappa/\beta)r(t)$. Therefore, the first term on the right-hand side of (31) can be estimated via Proposition B and the union bound as

$$\begin{aligned} P_t^{(i,n)}(F_t) &\le (\lfloor r(t)\rfloor + 1)\mathbf{P}_0\left(\sup_{0 \le s \le \frac{i+1}{n}\frac{\kappa}{\beta}r(t)} |X_s| > \delta r(t)\right) \\ &= \exp\left[-\frac{1}{2}\left(\frac{\delta\beta n}{(i+1)\kappa}\right)^2 \frac{i+1}{n}\frac{\kappa}{\beta} r(t)(1+o(1))\right] = \exp\left[-\frac{\delta^2 n\beta r(t)}{2(i+1)\kappa}(1+o(1))\right]. \end{aligned} \tag{32}$$

We now bound the term $P_t^{(i,n)}(A_t \cap F_t^c)$ in (31) from above. Observe that under the law $P_t^{(i,n)}$ on the event F_t^c, there exists an instant, namely $\rho_t(\kappa/\beta)r(t)$, inside the time interval $[i/n(\kappa/\beta)r(t),(i+1)/n(\kappa/\beta)r(t)]$, at which there are exactly $\lfloor r(t)\rfloor+1$ particles, all of which are inside $B(0,\delta r(t))$ and have ancestral lines confined to $B(0,\delta r(t))$ over $[0,\rho_t(\kappa/\beta)r(t)]$. For an upper bound on $P_t^{(i,n)}(A_t\cap F_t^c)$, we suppose the 'worst case', that is, suppose that each of these particles is on the boundary of $B(0,\delta r(t))$ at time $\rho_t(\kappa/\beta)r(t)$. Then, an application of the strong Markov property of the BBM at time $\rho_t(\kappa/\beta)r(t)$ along with the independence of the sub-BBMs initiated at that time yields

$$P_t^{(i,n)}(A_t\cap F_t^c)\le\left(\sup_{\frac{i}{n}\le x\le\frac{i+1}{n}}\left[P^{\delta r(t)}\left(n_{t-x(\kappa/\beta)r(t)}<\gamma_t p_t e^{\beta t}\right)\right]\right)^{\lfloor r(t)\rfloor}. \tag{33}$$

Next, we seek to find a suitable upper bound for $P^{\delta r(t)}\left(n_{t-x(\kappa/\beta)r(t)}<\gamma_t p_t e^{\beta t}\right)$ uniformly over $x\in[i/n,(i+1)/n]$. If $r(t)=o(\sqrt{t})$, then there exists $D_1=D_1(\delta,d)>0$ such that for all large t,

$$\mathbf{P}_0\left(\sigma_{B_t}\ge t\right)=\mathbf{P}_0\left(\sigma_{B(0,1)}\ge t/(r(t))^2\right)\le D_1\mathbf{P}^{\delta}\left(\sigma_{B(0,1)}\ge t/(r(t))^2\right)=D_1\mathbf{P}^{\delta r(t)}\left(\sigma_{B_t}\ge t\right),$$

where we have used Proposition 2 in the inequality since $t/(r(t))^2\to\infty$ as $t\to\infty$ by assumption. On the other hand, if $\sqrt{t}=O(r(t))$, then there exist $M>0$ and $D_2=D_2(\delta,d,M)>0$ such that for all large t,

$$D_2\mathbf{P}^{\delta r(t)}\left(\sigma_{B_t}\ge t\right)=D_2\mathbf{P}^{\delta}\left(\sigma_{B(0,1)}\ge t/(r(t))^2\right)\ge D_2\mathbf{P}^{\delta}\left(\sigma_{B(0,1)}\ge M^2\right)\ge 1.$$

Hence either when $r(t)=o(\sqrt{t})$ or $\sqrt{t}=O(r(t))$, there exists a positive constant $D=D(\delta,d)$ such that $\mathbf{P}_0\left(\sigma_{B_t}\ge t\right)\le D\mathbf{P}^{\delta r(t)}\left(\sigma_{B_t}\ge t\right)$ for all large t. Now let $0\le k_1<k_2$. Provided $\kappa-k_2\beta>0$, there exists $t_0>0$ such that for all $t\ge t_0$ and $k\in[k_1,k_2]$, the following bound on $\gamma_t p_t e^{\beta t}$ holds:

$$\begin{aligned}\gamma_t p_t e^{\beta t}&=e^{-\kappa r(t)}\mathbf{P}_0\left(\sigma_{B_t}\ge t\right)e^{\beta t}\\&=e^{-(\kappa-k\beta)r(t)}\mathbf{P}_0\left(\sigma_{B_t}\ge t\right)e^{\beta(t-kr(t))}\\&\le\bar{\gamma}_t\,D\,\mathbf{P}^{\delta r(t)}\left(\sigma_{B_t}\ge t\right)e^{\beta(t-kr(t))}\\&\le\bar{\gamma}_t\,D\,\mathbf{P}^{\delta r(t)}\left(\sigma_{B_t}\ge t-kr(t)\right)e^{\beta(t-kr(t))}\\&\le e^{-(\kappa-k_2\beta)r(t)/2}\widetilde{p}_t e^{\beta(t-kr(t))},\end{aligned}\tag{34}$$

where $D=D(\delta,d)>0$, we have set $\bar{\gamma}_t:=e^{-(\kappa-k\beta)r(t)}$, used the monotonicity in s of $\mathbf{P}^{\cdot}\left(\sigma_{B_t}\ge s\right)$ in the second inequality, and used that $D\bar{\gamma}_t\le De^{-(\kappa-k_2\beta)r(t)}\le e^{-(\kappa-k_2\beta)r(t)/2}$ for all large t since D does not depend on t. Now set $k_1=(i/n)(\kappa/\beta)$ and $k_2=((i+1)/n)(\kappa/\beta)$. Observe that $\kappa-k_2\beta=\kappa(1-(i+1)/n)>0$ since $i\le n-2$. With these choices of k_1 and k_2, it then follows from (34) and Proposition 3 upon replacing γ_t by $\widetilde{\gamma}_t:=e^{-(\kappa-k_2\beta)r(t)/2}$ in (8) and setting $x=k\beta/\kappa$ that there exist $c=c(\delta,\frac{\kappa-k_2\beta}{2},\beta)>0$ and $t_1>0$ such that for all $x\in[i/n,(i+1)/n]$ and $t\ge t_1$,

$$P^{\delta r(t)}\left(n_{t-x(\kappa/\beta)r(t)}<\gamma_t p_t e^{\beta t}\right)\le P^{\delta r(t)}\left(n_{t-x(\kappa/\beta)r(t)}<\widetilde{\gamma}_t\widetilde{p}_t e^{\beta(t-x(\kappa/\beta)r(t))}\right)\le e^{-cr(t)}. \tag{35}$$

Combining (31)-(33) and (35), for all large t,

$$P_t^{(i,n)}(A_t)\le\exp\left[-\frac{\delta^2 n\beta r(t)}{2(i+1)\kappa}(1+o(1))\right]+\left(e^{-cr(t)}\right)^{\lfloor r(t)\rfloor}. \tag{36}$$

Observe that the first term on the right-hand side of (36) is the dominating term for large t. (The positive constant $c = c(\delta, \frac{\kappa - k_2\beta}{2}, \beta)$ in the second term depends on $(\kappa - k_2\beta)/2$, which could be as small as $\kappa/(2n)$. However, this dependence of c on n will not matter at the end since we will first let $t \to \infty$ below.) Substituting (36) into (30), and optimizing over $i \in \{0, 1, \ldots, n-2\}$ yields

$$\limsup_{t\to\infty} \frac{1}{r(t)} \log P(A_t) \leq -\beta \left[\min_{i \in \{0,1,\ldots,n-2\}} \left\{ \frac{i}{n}\frac{\kappa}{\beta} + \frac{\delta^2}{2\kappa(i+1)/n} \right\} \wedge \frac{\kappa}{\beta}\left(1 - \frac{1}{n}\right) \right]. \tag{37}$$

First, let $\delta \to 1$ on the right-hand side of (37); then set $\rho = i/n$ to obtain

$$\limsup_{t\to\infty} \frac{1}{r(t)} \log P(A_t) \leq -\beta \left[\min_{\rho \in \{0,1/n,\ldots,(n-2)/n\}} \left\{ \frac{\rho\kappa}{\beta} + \frac{1}{2\kappa\rho + 2\kappa/n} \right\} \wedge \frac{\kappa}{\beta}\left(1 - \frac{1}{n}\right) \right].$$

Now let $n \to \infty$ and use the continuity of the functional form from which the minimum is taken to obtain

$$\limsup_{t\to\infty} \frac{1}{r(t)} \log P(A_t) \leq -\beta \left[\inf_{\rho \in (0,1]} \left\{ \frac{\rho\kappa}{\beta} + \frac{1}{2\kappa\rho} \right\} \wedge \frac{\kappa}{\beta} \right]. \tag{38}$$

For $\rho \in (0, 1]$, define the function f by

$$f(\rho) = \frac{\rho\kappa}{\beta} + \frac{1}{2\kappa\rho}.$$

One can check that when $\kappa > \sqrt{2\beta}$, f is minimized at $\bar{\rho} = \sqrt{\beta/(2\kappa^2)}$ over $(0, 1]$ where $0 < \bar{\rho} < 1/2$, and the minimum value of f is $\sqrt{2/\beta}$. On the other hand, when $\kappa \leq \sqrt{2\beta}$, the minimum value of f over $(0, \infty)$ is $\sqrt{2/\beta}$, which means, due to $\kappa/\beta \leq \sqrt{2/\beta}$, the second term under the minimum in (38) is the output of this minimum. Collecting all this regarding f, and using (38), we arrive at

$$\limsup_{t\to\infty} \frac{1}{r(t)} \log P(A_t) \leq -(\kappa \wedge \sqrt{2\beta}).$$

This completes the proof of the upper bound of Theorem 1.

5. Proof of Theorem 2

Recall that the statement of Theorem 2 includes the stronger assumption that $r(t) = o(\sqrt{t})$.

5.1. Proof of the upper bound

We use the Markov inequality together with the known formula for $E(n_t)$. Recall from (4) that $E(n_t) = p_t e^{\beta t}$, where p_t stands for the probability of confinement of a Brownian motion to $B(0, r(t))$ over $[0, t]$. It follows from Proposition C that $p_t = \mathbf{P}_0(\sigma_{B_t} \geq t) = \exp\left[-\lambda_d t(1 + o(1))/(r(t))^2\right]$. Then, by the Markov inequality, for any $\varepsilon > 0$,

$$P\left(n_t > \exp\left[\beta t - \frac{(\lambda_d - \varepsilon)t}{(r(t))^2}\right]\right) \leq \frac{E(n_t)}{\exp\left[\beta t - \frac{(\lambda_d-\varepsilon)t}{(r(t))^2}\right]} = \frac{\exp\left[-\frac{\lambda_d t}{(r(t))^2}(1 + o(1)) + \beta t\right]}{\exp\left[\beta t - \frac{(\lambda_d-\varepsilon)t}{(r(t))^2}\right]}$$

$$= \exp\left[-\frac{t}{(r(t))^2}(\varepsilon + o(1))\right] \to 0, \quad t \to \infty.$$

This proves the upper bound of Theorem 2.

5.2. Proof of the lower bound

To prove the lower bound of Theorem 2, we will show that for any $\varepsilon > 0$, as $t \to \infty$, $P\left(n_t < \exp\left[\beta t - (\lambda_d + \varepsilon)t/(r(t))^2\right]\right) \to 0$. We consider two cases: $r(t) \lesssim t^{1/3}$ and $r(t) \gtrsim t^{1/3}$, where we use the notation $f(t) \gtrsim g(t)$ to mean there exists a positive constant $c > 0$ such that $f(t)/g(t) \geq c$ for all $t \geq t_0$ for some $t_0 > 0$.

<u>Case 1</u>: Suppose that $\lim_{t\to\infty} r(t) = \infty$ and $r(t) \lesssim t^{1/3}$. Then, $e^{-\varepsilon t/(r(t))^2} \leq e^{-\kappa r(t)}$ for some $\kappa = \kappa(\varepsilon) > 0$ for all large t, and it follows that for all large t,

$$P\left(n_t < \exp\left[\beta t - \frac{(\lambda_d + \varepsilon)t}{(r(t))^2}\right]\right) \leq P\left(n_t < \exp\left[\beta t - \frac{\lambda_d t}{(r(t))^2} - \kappa r(t)\right]\right) \to 0, \quad t \to \infty,$$

where we have used Theorem 1 and that $p_t = \exp\left[-\frac{\lambda_d t}{(r(t))^2}(1 + o(1))\right]$.

<u>Case 2</u>: Suppose that $r(t) = o(\sqrt{t})$ and $r(t) \gtrsim t^{1/3}$. To handle this case, we revisit the proof of the upper bound of Theorem 2 in [10]. Although the aforementioned result uses that $\gamma_t = e^{-\kappa r(t)}$, its proof up to (6.23) therein does not use the specific form of γ_t at all, but only that $\gamma_t < g_t$ for all $t > 0$, where g_t is an arbitrary function such that $g_t \to 0$. Thus, we can apply [10, (6.23)] with for instance $g_t = 2\gamma_t$ and $\gamma_t = e^{-\frac{\varepsilon t}{2(r(t))^2}}$. Then, for all large t,

$$\begin{aligned} P\left(n_t < \exp\left[\beta t - \frac{(\lambda_d + \varepsilon)t}{(r(t))^2}\right]\right) &\leq P\left(n_t < \gamma_t p_t e^{\beta t}\right) \\ &\leq \frac{4}{\gamma_t p_t} e^{-\beta t} + 8e^{-\sqrt{\beta/2}\, r(t)} + g_t \leq 3g_t. \end{aligned}$$

Since $g_t \to 0$ as $t \to \infty$, this completes the proof of the lower bound of Theorem 2.

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