K3 SURFACES AND WEYL GROUP OF TYPE E_6

by

Cédric Bonnafé

Abstract. — Adapting methods of previous papers by A. Sarti and the author, we construct K3 surfaces from invariants of the Weyl group of type E_6. We study in details one of these surfaces, which turns out to have Picard number 20: for this example, we describe an elliptic fibration (and its singular fibers), the Picard lattice and the transcendental lattice.

This paper can be seen as a continuation of previous works by A. Sarti and the author on the construction of *singular* K3 surfaces by using invariants of finite reflection groups [**BoSa2, BoSa3**]. We consider here quotients of complete intersections defined by fundamental invariants of a group of rank 6, namely the Weyl group of type E_6.

Let W be the Weyl group of type E_6 acting on $V = \mathbb{C}^6$ and let W' denote its derived subgroup (which has index 2 in W and is equal to $W \cap \mathbf{SL}_{\mathbb{C}}(V)$). Then the algebra $\mathbb{C}[V]^W$ of polynomial functions on V invariant under the action of W is generated by six homogeneous and algebraically independent polynomials f_2, f_5, f_6, f_8, f_9 and f_{12} (of respective degrees 2, 5, 6, 8, 9 and 12).

We denote by $\mathfrak{X} \subset \mathbb{P}(V)$ the surface defined by $f_2 = f_6 = f_8 = 0$ and, for λ, $\mu \in \mathbb{C}^2$, we denote by $\mathcal{Y}_{\lambda,\mu}$ the surface defined by $f_5 = f_6 + \lambda f_2^3 = f_8 + \mu f_2^4 = 0$. It turns out that $\mathfrak{X}$ is smooth and that $\mathcal{Y}_{\lambda,\mu}$ is smooth for generic values of (λ, μ). Our first main result in this paper is the following (see Theorem 3.3):

Theorem A.

(a) *The minimal resolution of the singular surface $\mathfrak{X}/W'$ is a smooth K3 surface.*

(b) *If λ, μ are such that $\mathcal{Y}_{\lambda,\mu}$ has at most ADE singularities*$^{(*)}$*, then the minimal resolution of $\mathcal{Y}_{\lambda,\mu}/W'$ is a smooth K3 surface.*

In the rest of the paper, we investigate further properties of $\mathfrak{X}/W'$ and of its minimal resolution $\tilde{\mathfrak{X}}$. The second main result of the paper is the following (see Theorem 5.1):

Theorem B.

(a) *The K3 surface $\tilde{\mathfrak{X}}$ admits an elliptic fibration $\tilde{\varphi} : \tilde{\mathfrak{X}} \to \mathbb{P}^1(\mathbb{C})$ whose singular fibers are of type $E_7 + E_6 + A_2 + 2\,A_1$.*

(b) *The Picard lattice of $\tilde{\mathfrak{X}}$ has rank 20 and discriminant $-228 = -2^2 \cdot 3 \cdot 19$.*

(c) *The transcendental lattice of $\tilde{\mathfrak{X}}$ is given by the matrix* $\begin{pmatrix} 2 & 0 \\ 0 & 114 \end{pmatrix}$.

$^{(*)}$We do not know if there are values of (λ, μ) such that $\mathcal{Y}_{\lambda,\mu}$ has more complicated singularities.

In the course of the proof of Theorem B, we obtain a complete description of the Picard lattice: it is generated by 22 smooth rational curves whose intersection graph is given by Theorem 5.1(b).

The paper is organized as follows. In Section 1, we fix general notation and prove some preliminary results about group actions on tangent spaces. In Section 2, we fix the context, recall properties of the Weyl group of type E_6 and recall results from Springer theory [**Spr**]. Section 3 is mainly devoted to the proof of Theorem A. Section 4 gathers many geometric properties of the quotient variety $\mathcal{X}/W'$ (singularities, smooth rational curves, explicit equations in a weighted projective space). Theorem B is proved in Section 5. As a complement to all these data, the last section 6 contains the decomposition of the character of the representation $\mathrm{H}^2(\mathcal{X}, \mathbb{C})$ of W into a sum of irreducible characters (note that $\dim \mathrm{H}^2(\mathcal{X}) = 9502$).

Comments. (1) Many of the results of this paper (mostly the ones from Section 4 to the end) rely on computer calculations done with the software MAGMA [**BCP97**]. To simplify the exposition, we have removed the details and the codes of these computations: they are available on the `arXiv` version of this paper [**Arxiv**], and precise references to this `arXiv` version will be given at each stage such a computational check is needed.

(2) It is fair to say that most of the ideas of this paper come from our previous work with A. Sarti [**BoSa2, BoSa3**]. The main difference is that, here, we start with a complex reflection group of bigger rank (namely, 6 instead of 4) and so we need to consider quotients of complete intersections instead of quotients of hypersurfaces. In some sense, the main purpose of this paper is to show that the methods of [**BoSa2, BoSa3**] can be extended to this case.

(3) It is a rather easy task to determine which complete intersections defined by fundamental invariants of general complex reflection groups might lead, after taking the quotient by the derived subgroup, to a K3 surface. However, describing properties of these surfaces (Picard group, elliptic fibrations, transcendental lattices,...) can become a long and fastidious program. We decided to focus here on the case of the Weyl group of type E_6: it turns out that the choices we made for the degrees of the fundamental invariants defining the complete intersections are the only possible ones if one wants to obtain a K3 surface after taking the quotient by the derived subgroup (see Remark 3.8 for more details).

(4) The K3 surfaces obtained in [**BoSa2**] have big Picard numbers and interesting elliptic fibrations: this allows [**BoSa3**] to give original constructions of some K3 surfaces with Picard number 20 (i.e., the biggest possible). In a forthcoming third paper in this series (with A. Sarti), we will obtain many other K3 surfaces with Picard number 20 by investigating the case of the Weyl group of type F_4.

This is already interesting, but it is quite remarkable that the K3 surface $\tilde{\mathcal{X}}$ studied in this paper has also Picard number 20 and admits an elliptic fibration with unusual singular fibers. We believe that the two-parameters family of K3 surfaces obtained as minimal resolutions of $\mathcal{Y}_{\lambda,\mu}/W'$ might have similar properties: as it is a two-parameters family, we conjecture that a general member of this family will have Picard number 18 and we can hope to find explicit particular members of this family with Picard number 20. This would require much more involved computations.

(5) It is difficult to have a full overview of all the K3 surfaces of Picard number 20 that have been given a concrete description (i.e., not only through their transcendental lattice) but, as far as we know, the K3 surface $\tilde{\mathcal{X}}$ has not been investigated elsewhere, as well as its elliptic fibration and its description as minimal resolution of a quotient by a finite group of a surface of general type.

1. General notation, preliminaries

All vector spaces, all algebras, all algebraic varieties will be defined over $\mathbb{C}$. Algebraic varieties will always be reduced and quasi-projective, but not necessarily irreducible. If $\mathcal{X}$ is an algeraic variety and if $x \in \mathcal{X}$, we denote by $\mathrm{T}_x(\mathcal{X})$ the tangent space of $\mathcal{X}$ at x. If $\mathcal{X}$ is moreover affine, we denote by $\mathbb{C}[\mathcal{X}]$ its algebra of regular functions.

We fix a square root i of -1 in $\mathbb{C}$. If V is a vector space, g is an element of $\mathrm{End}_{\mathbb{C}}(V)$ and $\zeta \in \mathbb{C}$, we denote by $V(g,\zeta)$ the ζ-eigenspace of g. The list of eigenvalues of an element of $\mathrm{End}_{\mathbb{C}}(V)$ will always be given with multiplicities (and will be seen as a multiset: a multiset will be always written with double brackets, e.g. $\{\{a, b, \dots\}\}$). If $d \in \mathbb{N}^*$, we denote by $\boldsymbol{\mu}_d$ the group of d-th roots of unity in $\mathbb{C}^\times$ and we set $\zeta_d = \exp(2i\pi/d)$.

If V is a finite dimensional vector space and if $v \in V \setminus \{0\}$, we denote by $[v]$ its image in the projective space $\mathbb{P}(V)$. If $p \in \mathbb{P}(V)$, we denote by G_p its stabilizer in $G \subset \mathrm{GL}_{\mathbb{C}}(V)$. In other words, $G_{[v]}$ is the set of elements of G admitting v as an eigenvector.

If X is a subset of V and if G is a subgroup of $\mathrm{GL}_{\mathbb{C}}(V)$, we denote by G_X^{set} (resp. G_X^{pt}) the setwise (resp. pointwise) stabilizer of X and we set $G[X] = G_X^{\mathrm{set}}/G_X^{\mathrm{pt}}$ (so that $G[X]$ acts faithfully on the set X and on the vector space generated by X).

If $d_1,\dots, d_n$ are positive integers, we denote by $\mathbb{P}(d_1, \dots, d_n)$ the associated weighted projective space. If $f_1,\dots, f_r \in \mathbb{C}[X_1, \dots, X_n]$ are homogeneous (with X_i endowed with the degree d_i), we denote by $\mathfrak{Z}(f_1, \dots, f_r)$ the (possibly non-reduced) closed subscheme defined by $f_1 = \cdots = f_r = 0$.

The next three lemmas will be used throughout this paper. The first one is trivial, but very useful [**BoSa2**, Lem. 2.2]:

Lemma 1.1. — *Let V be a finite dimensional vector space, let $f \in \mathbb{C}[V]$, let $g \in \mathrm{GL}_{\mathbb{C}}(V)$ and let $v \in V \setminus \{0\}$ be such that:*

(1) *$g(v) = \xi v$, with $\xi \in \mathbb{C}^\times$.*
(2) *f is homogeneous of degree d with $\xi^d \neq 1$.*
(3) *f is g-invariant.*

Then $f(v) = 0$.

The next one is certainly well-known and might have its own interest:

Lemma 1.2. — *Let V be a finite dimensional vector space, let $n = \dim_{\mathbb{C}} V$, let G be a subgroup of $\mathrm{GL}_{\mathbb{C}}(V)$, let $v \in V \setminus \{0\}$ and let $f_1,\dots, f_r \in \mathbb{C}[V]$ be G-invariant and homogeneous of respective degrees $d_1,\dots, d_r$. We assume that:*

(1) *$\mathfrak{Z}(f_1, \dots, f_r)$ is a global complete intersection in $\mathbb{P}(V)$.*
(2) *G stabilizes the line $[v]$ (let $\theta_v : G \to \mathbb{C}^\times$ denote the linear character defined by $g(v) = \theta_v(g)v$ for all $g \in G$).*
(3) *$[v]$ is a smooth point of $\mathfrak{Z}(f_1, \dots, f_r)$.*

Then the r-dimensional semisimple representation $\theta_v^{d_1-1} \oplus \cdots \oplus \theta_v^{d_r-1}$ is isomorphic to a subrepresentation E of the $(n-1)$-dimensional representation $(V/[v])^*$ of G and, as representations of G, we have an isomorphism

$$\mathrm{T}_{[v]}(\mathfrak{Z}(f_1, \dots, f_r)) \simeq E^\perp \otimes \theta_v^{-1},$$

where $E^\perp = \{x \in V/[v] \mid \forall\, \varphi \in E,\ \varphi(x) = 0\}$.

Remark 1.3. — Keep the notation of the proposition. Recall that we have a natural identification $\mathrm{T}_{[v]}(\mathbb{P}(V)) \simeq V/[v]$ but that this identification is not G-equivariant. We will recall the construction of this isomorphism in the proof of the proposition, so that we can follow the action of G: as we will see, we have a natural isomorphism of G-representations $\mathrm{T}_{[v]}(\mathbb{P}(V)) \simeq (V/[v]) \otimes \theta_v^{-1}$. Therefore, $\mathrm{T}_{[v]}(\mathcal{X})$ is naturally a subrepresentation of $(V/[v]) \otimes \theta_v^{-1}$, and this is the aim of this proposition to identify this subrepresentation. ■

Proof. — Write $\mathcal{X} = \mathfrak{X}(f_1, \dots, f_r)$ and $p = [v]$. The point p of $\mathbb{P}(V)$ corresponds to the homogeneous ideal $\mathfrak{p}$ of $\mathbb{C}[V]$ generated by $v^\perp \subset V^*$. The local ring $\mathcal{O}$ of $\mathbb{P}(V)$ at p is

$$\mathbb{C}[V]_{(\mathfrak{p})} = \{a/b \mid a, b \in \mathbb{C}[V], \text{ homogeneous and } b \notin \mathfrak{p}\}.$$

We denote by $\mathfrak{m}$ the unique maximal ideal of $\mathcal{O}$. Then

$$\mathfrak{m} = \{a/b \mid a, b \in \mathbb{C}[V], \text{ homogeneous, } a \in \mathfrak{p} \text{ and } b \notin \mathfrak{p}\}.$$

The map

$$\begin{array}{rccc} \mathrm{d}_v : & \mathfrak{m} & \longrightarrow & V^* \\ & f & \longmapsto & \mathrm{d}_v f \end{array}$$

induces an isomorphism

$$\delta : \mathrm{T}_p(\mathbb{P}(V))^* = \mathfrak{m}/\mathfrak{m}^2 \xrightarrow{\sim} v^\perp = (V/[v])^*.$$

(Recall that, if $a \in \mathfrak{p}$ is homogeneous of degree m and $b \notin \mathfrak{p}$ is homogeneous, then $\mathrm{d}_v(a/b)(v) = (\mathrm{d}_v a)(v)/b(v) = 0$ because $(\mathrm{d}_v a)(v) = ma(v) = 0$ by Euler's identity). Note, however, that δ is not G-equivariant for the classical action: indeed, as $g(v) = \theta_v(g)v$ for all $g \in G$, δ induces an isomorphism of G-modules

$$\delta : \mathrm{T}_p(\mathbb{P}(V))^* \xrightarrow{\sim} (V/[v])^* \otimes \theta_v.$$

Through this isomorphism,

$$\mathrm{T}_p(\mathcal{X})^* \simeq \big((V/[v])^*/E\big) \otimes \theta_v, \qquad \text{where} \quad E = \sum_{k=1}^{r} \mathbb{C}\delta(f_k).$$

Now, since $\mathcal{X}$ is smooth at p and $\mathcal{X}$ is a global complete intersection, this means that the elements $\delta(f_1)$,..., $\delta(f_r)$ are linearly independent. To conclude the proof of the proposition, it remains to notice that $g(\delta(f_k)) = \theta_v(g)^{d_k - 1}\delta(f_k)$ for all $g \in G$, which follows immediately from the G-invariance of f_k. □

Lemma 1.2 applied to the case where G is a cyclic group gives the following result:

Corollary 1.4. — *Let V be a finite dimensional vector space, let $n = \dim_{\mathbb{C}} V$, let $g \in \mathrm{GL}_{\mathbb{C}}(V)$, let $v \in V \setminus \{0\}$ and let f_1,..., $f_r \in \mathbb{C}[V]$ be g-invariant and homogeneous of respective degrees d_1,..., d_r. Let $\{\{\xi_1, \dots, \xi_n\}\}$ be the multiset of eigenvalues of g. We assume that:*

(1) *$\mathfrak{X}(f_1, \dots, f_r)$ is a global complete intersection in $\mathbb{P}(V)$.*
(2) *$g(v) = \xi_n v$.*
(3) *$[v]$ is a smooth point of $\mathfrak{X}(f_1, \dots, f_r)$.*

Then $\{\{\xi_n^{-d_1}, \dots, \xi_n^{-d_r}\}\}$ is contained in the multiset $\{\{\xi_n^{-1}\xi_1, \dots, \xi_n^{-1}\xi_{n-1}\}\}$ and the list of eigenvalues of g for its action on the tangent space $\mathrm{T}_{[v]}(\mathfrak{X}(f_1, \dots, f_r))$ is the multiset

$$\{\{\xi_n^{-1}\xi_1, \dots, \xi_n^{-1}\xi_{n-1}\}\} \setminus \{\{\xi_n^{-d_1}, \dots, \xi_n^{-d_r}\}\}.$$

Remark 1.5. — Keep the notation of Lemma 1.2 and assume moreover that G is a closed reductive subgroup of $\mathrm{GL}_{\mathbb{C}}(V)$ (for instance, G might be finite), that $v \in V^G$ (so that $\theta_v = 1$) and that $\mathbb{P}(V^G) \subset \mathfrak{X}(f_1, \dots, f_r)$. We want to show that

(#) *$[v]$ is a singular point of $\mathfrak{X}(f_1, \dots, f_r)$.*

For this, assume that $[v]$ is smooth and let E denote the G-stable subspace of $(V/[v])^*$ of dimension r defined in the proof of Lemma 1.2 and such that the G-module $\mathrm{T}_{[v]}(\mathfrak{X}(f_1, \dots, f_r))$ is identified with $E^{\perp}$. By hypothesis, $V^G/[v] \subset \mathrm{T}_{[v]}(\mathfrak{X}(f_1, \dots, f_r))$ so, since G acts semisimply on V, its orthogonal F in $(V/[v])^*$ satisfies $F^G = 0$. This contradicts the fact that $E \subset F$. This shows (#).

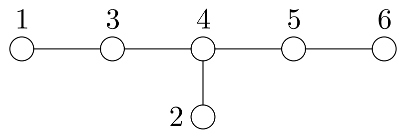
If G is finite, $\dim_{\mathbb{C}} V^G = 1$ (resp. $\dim_{\mathbb{C}} V^G = 2$) and $r = 1$, we retrieve [**BoSa2**, Coro. 2.4] (resp. [**BoSa2**, Coro. 2.9]). ■

2. Set-up

For the classical theory of Coxeter groups, reflection groups, Dynkin diagrams, we mainly refer to Bourbaki's book [**Bou**] or Broué's book [**Bro**]. For Springer theory (and its enhancement by Lehrer-Springer), we refer to [**Spr**], [**LeSp1**], [**LeSp2**] and [**LeMi**].

Let $V = \mathbb{C}^6$ be endowed with its canonical basis $(e_1, e_2, e_3, e_4, e_5, e_6)$. Through this basis, we identify $\mathrm{GL}_{\mathbb{C}}(V)$ and $\mathrm{GL}_6(\mathbb{C})$ and we let it act on V on the left. We denote by $(X_1, X_2, X_3, X_4, X_5, X_6)$ the dual basis of $(e_1, e_2, e_3, e_4, e_5, e_6)$, so that the algebra $\mathbb{C}[V]$ will be identified with the polynomial algebra $\mathbb{C}[X_1, X_2, X_3, X_4, X_5, X_6]$. We endow it with the symmetric bilinear form $\langle , \rangle$ attached to the Dynkin diagram of type E_6 (we follow the strange numbering of nodes given by Bourbaki [**Bou**, Chap. 6, Planche V]):

(E_6)



Recall that this means that

$$\begin{cases} \langle e_k, e_k \rangle = 1 & \text{if } 1 \leqslant k \leqslant 6, \\ \langle e_k, e_l \rangle = -1/2 & \text{if } 1 \leqslant k \neq l \leqslant 6 \text{ and } \{k, l\} \text{ is an edge of the graph } (\mathrm{E}_6), \\ \langle e_k, e_l \rangle = 0 & \text{if } 1 \leqslant k \neq l \leqslant 6 \text{ and } \{k, l\} \text{ is not an edge of the graph } (\mathrm{E}_6). \end{cases}$$

Recall that $\langle , \rangle$ is non-degenerate (in fact, when restricted to $\mathbb{R}^6$, it is positive definite). For $1 \leqslant k \leqslant 6$, we denote by s_k the orthogonal reflection such that $s_k(e_k) = -e_k$. We set

$$W = \langle s_1, s_2, s_3, s_4, s_5, s_6 \rangle.$$

We denote by $\mathrm{O}(V)$ the orthogonal group of V, with respect to the bilinear form $\langle , \rangle$. By construction, W is a sugroup of $\mathrm{O}(V)$ and is called a *Weyl group of type* E_6.

2.A. First properties of W. — The following numerical results concerning W can be found in [**Bou**, Chap. VI, Table V]. The group W is finite and acts irreducibly of V. Moreover,

$$|W| = 51840. \tag{2.1}$$

Also, the center of W is trivial, so W acts faithfully on $\mathbb{P}(V)$. Let $\varepsilon : W \to \boldsymbol{\mu}_2 = \{-1,1\}$, $w \mapsto \det(w)$. Recall that $\operatorname{Ker}\varepsilon$ is the derived (i.e. commutator) subgroup of W, which will be denoted by W'. In particular,

$$|W'| = 25920. \tag{2.2}$$

We denote by $\operatorname{Deg}(W)$ (resp. $\operatorname{Codeg}(W)$) the *degrees* (resp. the *codegrees*) of W, as defined in [**Bro**, Chap. 4]. Moreover,

$$\operatorname{Deg}(W) = \{2,5,6,8,9,12\} \qquad \text{and} \qquad \operatorname{Codeg}(W) = \{0,3,4,6,7,10\}. \tag{2.3}$$

In particular, this means that there exist 6 homogeneous W-invariant polynomials f_2, f_5, f_6, f_8, f_9 and f_{12}, of respective degrees 2, 5, 6, 8, 9 and 12, such that

$$\mathbb{C}[V]^W = \mathbb{C}[f_2, f_5, f_6, f_8, f_9, f_{12}]. \tag{2.4}$$

We set $\mathbf{f} = (f_2, f_5, f_6, f_8, f_9, f_{12})$ and we recall that $\mathbf{f}$ is not uniquely determined. Thanks to (2.4), we get that

$$\begin{array}{rccc} \pi_{\mathbf{f}}: & \mathbb{P}(V) & \longrightarrow & \mathbb{P}(2,5,6,8,9,12) \\ & [v] & \longmapsto & [f_2(v), f_5(v), f_6(v), f_8(v), f_9(v), f_{12}(v)] \end{array}$$

is well-defined and induces an isomorphism of varieties

$$\mathbb{P}(V)/W \xrightarrow{\sim} \mathbb{P}(2,5,6,8,9,12). \tag{2.5}$$

The graded algebra associated with the weighted projective space $\mathbb{P}(2,5,6,8,9,12)$ will be denoted by $\mathbb{C}[Z_2, Z_5, Z_6, Z_8, Z_9, Z_{12}]$, with Z_d being given the degree d (for all $d \in \operatorname{Deg}(W)$).

Remark 2.6. — The quadratic form $Q : V \to \mathbb{C}$, $v \mapsto \langle v, v\rangle$ is W-invariant. Hence f_2 is a scalar multiple of Q. Since Q is positive definite when restricted to $\mathbb{R}^6$, we have in particular that $f_2(e_1) \neq 0$. ■

Let $\operatorname{Ref}(W)$ be the set of reflections of W and let $\mathscr{A}$ be the hyperplane arrangement of W (i.e. $\mathscr{A} = \{V^s \mid s \in \operatorname{Ref}(W)\}$). Then

$$|\operatorname{Ref}(W)| = |\mathscr{A}| = \sum_{d \in \operatorname{Deg}(W)} (d-1) = 36. \tag{2.7}$$

If $H \in \mathscr{A}$, we denote by α_H an element of V^* such that $H = \operatorname{Ker}\alpha_H$. We set

$$\operatorname{Jac} = \prod_{H \in \mathscr{A}} \alpha_H \in \mathbb{C}[V].$$

Since all the α_H are well-defined up to a non-zero scalar, Jac is also well-defined up to a non-zero scalar. Then

$${}^w\operatorname{Jac} = \varepsilon(w)\operatorname{Jac} \tag{2.8}$$

for all $w \in W$ and

$$\mathbb{C}[V]^{W'} = \mathbb{C}[f_2, f_5, f_6, f_8, f_9, f_{12}, \operatorname{Jac}]. \tag{2.9}$$

Moreover, Jac is homogeneous of degree 36 by (2.7). Also, since $\mathrm{Jac}^2 \in \mathbb{C}[V]^W$ by (2.8), there exists a unique homogeneous polynomial $P_{\mathbf{f}} \in \mathbb{C}[Z_2, Z_5, Z_6, Z_8, Z_9, Z_{12}]$ such that

$$\mathrm{Jac}^2 = P_{\mathbf{f}}(f_2, f_5, f_6, f_8, f_9, f_{12}).$$

Of course, $P_{\mathbf{f}}$ depends heavily on the choice of the family $\mathbf{f}$ and, up to a non-zero scalar, on the choice of the α_H's. It turns out that this relation generates the ideal of relations between the functions f_2, f_5, f_6, f_8, f_9, f_{12} and Jac. In particular, the map

$$\begin{array}{rccc} \pi'_{\mathbf{f}}: & \mathbb{P}(V) & \longrightarrow & \mathbb{P}(2,5,6,8,9,12,36) \\ & [v] & \longmapsto & [f_2(v) : f_5(v) : f_6(v) : f_8(v) : f_9(v) : f_{12}(v) : \mathrm{Jac}(v)] \end{array}$$

is well-defined and induces an isomorphism of varieties

$$\tag{2.10} \begin{array}{rcl} \mathbb{P}(V)/W' & \stackrel{\sim}{\longrightarrow} & \{[z_2 : z_5 : z_6 : z_8 : z_9 : z_{12} : j] \in \mathbb{P}(2,5,6,8,9,12,36) \mid \\ & & \qquad j^2 = P_{\mathbf{f}}(z_2, z_5, z_6, z_8, z_9, z_{12})\}. \end{array}$$

Finally, we denote by $\omega : \mathbb{P}(V)/W' \longrightarrow \mathbb{P}(V)/W$ the natural morphism, which is just the quotient map by $\boldsymbol{\mu}_2 \simeq W/W'$. Through the isomorphisms (2.5) and (2.10), the action of the non-trivial element of $\boldsymbol{\mu}_2$ is given by the involutive automorphism σ given by

$$\sigma[z_2 : z_5 : z_6 : z_8 : z_9 : z_{12} : j] = [z_2 : z_5 : z_6 : z_8 : z_9 : z_{12} : -j]$$

and ω is given by

$$\omega[z_2 : z_5 : z_6 : z_8 : z_9 : z_{12} : j] = [z_2 : z_5 : z_6 : z_8 : z_9 : z_{12}].$$

2.B. Eigenspaces, Springer theory. — As in [**BoSa2**], an important role is played by Springer and Lehrer-Springer theory. We recall briefly the results we will need (this subsection is a simplified version of [**BoSa2**, §3.3], adapted to the particular case of our Weyl group W). All the results stated here can be found in [**Spr**], [**LeSp1**], [**LeSp2**]. Note that some of the proofs have been simplified in [**LeMi**]. Let us fix now a natural number e. We set

$$\Delta(e) = \{d \in \mathrm{Deg}(W) \mid e \text{ divides } d\},$$

$$\Delta^*(e) = \{d^* \in \mathrm{Codeg}(W) \mid e \text{ divides } d^*\},$$

$$\delta(e) = |\Delta(e)| \qquad \text{and} \qquad \delta^*(e) = |\Delta^*(e)|.$$

With this notation, we have

$$\tag{2.11} \delta(e) = \max_{w \in W}\bigl(\dim V(w, \zeta_e)\bigr).$$

In particular, ζ_e is an eigenvalue of some element of W if and only if $\delta(e) \neq 0$ that is, if and only if $e \in \{1,2,3,4,5,6,8,9,12\}$. In this case, we fix an element w_e of W of minimal order such that

$$\dim V(w_e, \zeta_e) = \delta(e).$$

We set for simplification $V(e) = V(w_e, \zeta_e)$ and $W(e) = W[V(e)] = W_{V(e)}^{\mathrm{set}}/W_{V(e)}^{\mathrm{pt}}$.

Remark 2.12. — Note for future reference that w_3 and w_3^{-1} are not conjugate in W': in fact, they are not conjugate in $\mathrm{SO}(V)$ by [**Bon2**, Lemma 1.7]. ■

If $f \in \mathbb{C}[V]$, we denote by $f^{[e]}$ its restriction to $V(e)$. Note that if $d \in \mathrm{Deg}(W)$ is such that $d \notin \Delta(e)$, then $f_d^{[e]} = 0$ by Lemma 1.1.

***Theorem 2.13* (Springer, Lehrer-Springer).** — *Assume that $\delta(e) \neq 0$. Then:*

(a) *If $w \in W$, then there exists $x \in W$ such that $x(V(w, \zeta_e)) \subset V(e)$.*
(b) *$W(e)$ acts (faithfully) on $V(e)$ as a group generated by reflections.*
(c) *The family $(f_d^{[e]})_{d \in \Delta(e)}$ is a family of fundamental invariants of $W(e)$. In particular, the list of degrees of $W(e)$ consists of the degrees of W which are divisible by e.*
(d) *We have*

$$\bigcup_{w \in W} V(w, \zeta_e) = \bigcup_{x \in W} x(V(e)) = \{v \in V \mid \forall\, d \in \mathrm{Deg}(W) \setminus \Delta(e),\ f_d(v) = 0\}.$$

(e) *$\delta^*(e) \geqslant \delta(e)$ with equality if and only if $W_{V(e)}^{\mathrm{pt}} = 1$.*
(f) *If $\delta^*(e) = \delta(e)$, then w_e has order e, $W(e) = W_{V(e)}^{\mathrm{set}} = C_W(w_e)$ and the multiset of eigenvalues (with multiplicity) of w_e is equal to $\{\{\zeta_e^{1-d}\}\}_{d \in \mathrm{Deg}(W)}$. Moreover, if w is such that $\dim V(w, \zeta_e) = \delta(e)$, then w is conjugate to w_e.*

Example 2.14. — Let $e \in \{8, 9, 12\}$. Then $\delta(e) = \delta^*(e) = 1$. So $V(e)$ is a line in V, and can be viewed as an element of $\mathbb{P}(V)$. It follows from the above results that

$$W_{V(e)} = \langle w_e \rangle$$

(see for instance [**BoSa2**, Rem. 3.14] for a proof).

Moreover, by Theorem 2.13(f), we have $\det(w_e) = \zeta_e^{-36}$, so

$$w_9, w_{12} \in W' \qquad \text{and} \qquad w_8 \notin W'.$$

In particular, if we denote by p_e (resp. q_e) the image of $V(e)$ in $\mathbb{P}(V)/W'$ (resp. $\mathbb{P}(V)/W$), then the morphism ω is unramified (resp. ramified) over q_9 and q_{12} (resp. p_8). This implies also that $\sigma(p_9) \neq p_9$ and $\sigma(p_{12}) \neq p_{12}$. ■

Example 2.15. — Note that $\delta^*(5) = 2 > \delta(5) = 1$ while, if $e \in \{1, 2, 3, 4, 6, 8, 9, 12\}$, then $\delta^*(e) = \delta(e)$. If x is an element of order 5, then x admits a primitive fifth root of unity as eigenvalue, so admits ζ_5 as an eigenvalue because x is represented by a matrix with rational coefficients. So we can take $w_5 = x$, and so w_5 has order 5.

The above argument shows that the multiset of eigenvalues of w_5 is $\{\{1, 1, \zeta_5, \zeta_5^2, \zeta_5^3, \zeta_5^4\}\}$. Now, if w is an element of W such that $w(V(5)) = V(5)$, then, since w is defined over $\mathbb{Q}$, it stabilizes the ζ_5^k-eigenspace of w for $k \in \{1, 2, 3, 4\}$. Therefore, it stabilizes the sum E of these 4 eigenspaces. But $E = \mathrm{Ker}(\mathrm{Id}_V + w_5 + w_5^2 + w_5^3 + w_5^4)$ is defined over $\mathbb{Q}$ (and so over $\mathbb{R}$) so its orthogonal $E^\perp$ in V with respect to the bilinear form $\langle , \rangle$ satisfies $E \oplus E^\perp = V$ (because its restriction to $\mathbb{R}^6$ is positive definite). This shows that w and w_5 stabilize also $E^\perp$, which is necessarily equal to $\mathrm{Ker}(w_5 - \mathrm{Id}_V)$. In particular w centralizes w_5. But, by the fourth line of [**Arxiv**, Comput. I.12], we have $C_{W'}(w_5) = \langle w_5 \rangle$. So we have proved that

$$W'_{V(5)} = \langle w_5 \rangle.$$

This fact will be used in the proof of Theorem 3.3. ■

3. K3 surfaces

Let
$$\mathcal{X} = \mathfrak{X}(f_2, f_6, f_8)$$
and, for λ, $\mu \in \mathbb{C}$, set
$$\mathcal{Y}_{\lambda,\mu} = \mathfrak{X}(f_5, f_6 + \lambda f_2^3, f_8 + \mu f_2^4).$$
Note that the invariants f_2 and f_5 are uniquely defined up to a scalar and that, up to a scalar, every fundamental invariant of degree 6 (resp. 8) is of the form $f_6 + \lambda f_2^3$ (resp. $f_8 + \mu f_2^4 + \nu f_6 f_2$) for some $\lambda \in \mathbb{C}$ (resp. μ, $\nu \in \mathbb{C}$). But $\mathfrak{X}(f_2, f_6 + \lambda f_2^3, f_8 + \mu f_2^4 + \nu f_6 f_2) = \mathcal{X}$ and $\mathfrak{X}(f_5, f_6 + \lambda f_2^3, f_8 + \mu f_2^4 + \nu f_6 f_2) = \mathcal{Y}_{\lambda,\mu-\lambda\nu}$. So this shows in particular that $\mathcal{X}$ does not depend on the choices of the fundamental invariants of W. By construction, $\mathcal{X}$ and $\mathcal{Y}_{\lambda,\mu}$ are W-stable.

Remark 3.1. — The choice of the family **f** of fundamental invariants is irrelevant for all the theoretical results stated in this paper. However, for some numerical results (for instance, the equation of $\mathcal{X}/W'$ given by (4.10) or the coordinates of its singular points given by Lemma 4.8), this choice needs to be specified: in this paper, for all computational results, we will choose **f** as in [**Arxiv**, App. I]. ■

Proposition 3.2. — *With the above notation, we have:*

(a) *$\mathcal{X}$ is a smooth irreducible surface, which is a complete intersection in $\mathbb{P}(V)$).*
(b) *If λ, $\mu \in \mathbb{C}$, then $\mathcal{Y}_{\lambda,\mu}$ has pure dimension 2, and so is a complete intersection in $\mathbb{P}(V)$. If it is smooth or has only ADE singularities, then it is irreducible.*
(c) *The set of $(\lambda, \mu) \in \mathbb{C}^2$ such that $\mathcal{Y}_{\lambda,\mu}$ is smooth is a non-empty open subset of $\mathbb{C}^2$.*

Proof. — Through the isomorphism (2.5), we have
$$\mathcal{X}/W \simeq \mathbb{P}(5, 9, 12) \qquad \text{and} \qquad \mathcal{Y}_{\lambda,\mu}/W \simeq \mathbb{P}(2, 9, 12),$$
so $\mathcal{X}/W$ and $\mathcal{Y}_{\lambda,\mu}/W$ are irreducible of dimension 2. Since W is finite, this implies that $\mathcal{X}$ and $\mathcal{Y}_{\lambda,\mu}$ are of pure dimension 2 and so are complete intersections. In particular, they are connected [**Har**, Chap. II, Exer. 8.4(c)].

(a) By [**Arxiv**, Comput. I.1], the open affine chart of $\mathcal{X}$ defined by $x_6 \neq 0$ is smooth. This shows that the singular locus $\mathcal{S}$ of $\mathcal{X}$ is contained in the projective hyperplane $\mathbb{P}(H)$, where $H = \{(x_1, x_2, x_3, x_4, x_5, x_6) \in \mathbb{C}^6 \mid x_6 = 0\}$. Since W acts on $\mathcal{X}$, we have that $\mathcal{S}$ is contained in $\mathbb{P}(\cap_{w\in W} w(H))$. But $\cap_{w\in W} w(H)$ is a W-stable proper subspace of V, so it is equal to $\{0\}$ since W acts irreducibly on V. Hence $\mathcal{S} = \varnothing$. This shows that $\mathcal{X}$ is smooth and, in particular, irreducible (because it is connected).

(b) We already know that $\mathcal{Y}_{\lambda,\mu}$ is connected. If moreover it admits only ADE singularities, then it is necessarily irreducible.

(c) Let U be the set of $(\lambda, \mu) \in \mathbb{C}^2$ such that $\mathcal{Y}_{\lambda,\mu}$ is smooth. It is clear that U is open, so we only need to prove that it is non-empty. With the particular choice of fundamental invariants given in [**Arxiv**, App. I] (see Remark 3.1), we have that the open affine chart of $\mathcal{Y}_{0,0}$ defined by $x_6 \neq 0$ is smooth by [**Arxiv**, Comput. I.1]. The same argument as in (a) allows to conclude that $\mathcal{Y}_{0,0}$ is smooth and irreducible. □

We are now ready to state the first main result of this paper:

Theorem 3.3. — *Let λ, $\mu \in \mathbb{C}$ be such that $\mathcal{Y}_{\lambda,\mu}$ admits only ADE singularities and let $\mathcal{V}$ be one of the varieties $\mathcal{X}$ or $\mathcal{Y}_{\lambda,\mu}$. Then $\mathcal{V}/W'$ is a K3 surface with only ADE singularities and its minimal smooth resolution $\tilde{\mathcal{V}}$ is a smooth K3 surface.*

Proof. — The proof will be given in the next subsections, following the same lines as [**BoSa2**, Theo. 5.4]. More precisely, if $\mathcal{V}$ denotes one of the above varieties $\mathcal{X}$ or $\mathcal{Y}_{\lambda,\mu}$, we will prove in the next subsections the following three facts:

(A) The smooth locus of $\mathcal{V}/W'$ admits a symplectic form (see Lemma 3.6);
(B) The variety $\mathcal{V}/W'$ has only ADE singularities (see Lemma 3.9);
(C) The Euler characteristic of $\mathcal{V}/W'$ is positive (see Lemma 3.12).

Therefore, by (A) and (B), the variety $\mathcal{V}/W'$ is a symplectic singularity and its minimal smooth resolution $\tilde{\mathcal{V}}$ is a crepant resolution, i.e. a symplectic resolution. So $\tilde{\mathcal{V}}$ admits a symplectic form. By the classification of surfaces, this forces $\tilde{\mathcal{V}}$ to be a smooth K3 surface or an abelian variety. But, by (B), the Euler characteristic of $\tilde{\mathcal{V}}$ is greater than or equal to the one of $\mathcal{V}/W'$, so the Euler characteristic of $\tilde{\mathcal{V}}$ is positive by (C). Since an abelian variety has Euler characteristic 0, this shows that $\tilde{\mathcal{V}}$ is a smooth K3 surface. $\square$

So it remains to prove the three facts (A), (B) and (C) used in the above proof.

Notation. For the rest of this section, we fix λ, μ in $\mathbb{C}$ such that $\mathcal{Y}_{\lambda,\mu}$ admits only ADE singularities and we denote by $\mathcal{V}$ a variety which can be $\mathcal{X}$ or $\mathcal{Y}_{\lambda,\mu}$.

Recall that this implies that $\mathcal{V}$ is irreducible and normal and, in particular, that $\mathcal{V}/W'$ is irreducible and normal too.

3.A. Symplectic form. — Through the isomorphism (2.10), we have

$$\mathcal{X}/W' \simeq \{[z_5 : z_9 : z_{12} : j] \in \mathbb{P}(5,9,12,36) \mid j^2 = P_{\mathbf{f}}(0, z_5, 0, 0, z_9, z_{12})$$

and $$\mathcal{Y}_{\lambda,\mu}/W' \simeq \{[z_2 : z_9 : z_{12} : j] \in \mathbb{P}(2,9,12,36) \mid j^2 = P_{\mathbf{f}}(z_2, 0, -\lambda z_2^3, -\mu z_2^4, z_9, z_{12})\}.$$

But note that $\mathbb{P}(5,9,12,36) = \mathbb{P}(5,3,4,12)$, so there exists a unique homogeneous polynomial $Q_{\mathbf{f}} \in \mathbb{C}[Y_5, Y_3, Y_4]$ of degree 24 (where Y_i is endowed with the degree i) such that $P_{\mathbf{f}}(0, z_5, 0, 0, z_9, z_{12}) = Q_{\mathbf{f}}(z_5^3, z_9, z_{12})$. Hence,

$$\mathcal{X}/W' \simeq \{[y_5 : y_3 : y_4 : j] \in \mathbb{P}(5,3,4,12) \mid j^2 = Q_{\mathbf{f}}(y_5, y_3, y_4)\}. \tag{3.4}$$

So the degree of the equation defining $\mathcal{X}/W'$ (namely, 24) is equal to the sum of the weights of the projective space (namely, $5+3+4+12$). By [**BoSa2**, Lem. A.1], this implies that the smooth locus of $\mathcal{X}/W'$ is endowed with a symplectic form.

On the other hand, $\mathbb{P}(2,9,12,36) = \mathbb{P}(1,9,6,18) = \mathbb{P}(1,3,2,6)$ so there exists a unique homogeneous polynomial $Q_{\mathbf{f}}^{\lambda,\mu} \in \mathbb{C}[Y_1, Y_3, Y_2]$ of degree 12 (where Y_i is endowed with the degree i) such that $P_{\mathbf{f}}(z_2, 0, -\lambda z_2^3, -\mu z_2^4, z_9, z_{12}) = Q_{\mathbf{f}}^{\lambda,\mu}(z_2^3, z_9, z_{12})$. Hence,

$$\mathcal{Y}_{\lambda,\mu}/W' \simeq \{[y_1 : y_3 : y_2 : j] \in \mathbb{P}(1,3,2,6) \mid j^2 = Q_{\mathbf{f}}^{\lambda,\mu}(y_1, y_3, y_2)\}. \tag{3.5}$$

So the degree of the equation defining $\mathscr{Y}_{\lambda,\mu}/W'$ (namely, 12) is equal to the sum of the weights of the projective space (namely, $1+3+2+6$). By [**BoSa2**, Lem. 5.4], this implies that the smooth locus of $\mathscr{X}/W'$ is endowed with a symplectic form. Therefore, we have proved the following lemma, which corresponds to the Fact (A) stated in the proof of Theorem 3.3:

Lemma 3.6. — *The smooth locus of $\mathscr{V}/W'$ admits a symplectic form.*

Remark 3.7. — In both cases, the ramification locus of the morphism $\omega : \mathscr{V}/W' \to \mathscr{V}/W$ is given by the equation $j=0$ (in the models given by equations (3.4) or (3.5)). ■

Remark 3.8. — Let $\{d_1,d_2,d_3\}$ be a subset of $\mathrm{Deg}(W)$ of cardinality 3 and consider the complete intersection $\mathscr{D} = \mathscr{Z}(f_{d_1},f_{d_2},f_{d_3})$. Then, through the isomorphism (2.10), one can write $\mathscr{D}/W'$ as a closed subvariety of $\mathbb{P}(d'_1,d'_2,d'_3,36)$ defined by an equation of degree 72, where $\{d'_1,d'_2,d'_3\}$ is the complement of $\{d_1,d_2,d_3\}$ in $\mathrm{Deg}(W)$. After simplifying the weights of $\mathbb{P}(d'_1,d'_2,d'_3,36)$ as above, one finds that the degree of the equation of $\mathscr{D}/W'$ equals the sume of the weights of the weighted projective space if and only if (d_1,d_2,d_3) is equal to $(2,6,8)$, $(5,6,8)$ or $(2,5,8)$. However, in this last case, $\mathscr{D}$ is a union of 80 projective planes (as follows from Theorem 2.13(d) applied to the case where $e=3$) which sometimes intersect along a projective line, so this case is uninteresting for our purpose. So the choices of the degrees of the equations defining $\mathscr{X}$ or $\mathscr{Y}_{\lambda,\mu}$ are the only ones that have a chance to give rise to a K3 surface after taking the quotient by W'. ■

3.B. Singularities. — We aim to prove here the following lemma, which corresponds to the Fact (B) stated in the proof of Theorem 3.3:

Lemma 3.9. — *The surface $\mathscr{V}/W'$ has only ADE singularities.*

Proof. — First, as $\mathscr{V}$ has only ADE singularities and W' has index 2 in W, every point of $\mathscr{V}/W'$ lying above a smooth point of $\mathscr{V}/W$ is smooth or is an ADE singularity by [**BoSa2**, Coro. B.7]. So it remains only to study the points lying above the singular points of $\mathscr{V}/W$.

The singular points of $\mathscr{X}/W \simeq \mathbb{P}(5,3,4)$ are $q_5=[1:0:0]$, $q_9=[0:1:0]$ and $q_{12}=[0:0:1]$. The singular points of $\mathscr{Y}_{\lambda,\mu}/W \simeq \mathbb{P}(1,3,2)$ are $q_9=[0:1:0]$ and $q_{12}=[0:0:1]$. Note that the notation q_9 and q_{12} is consistent with Example 2.14, as they correspond to the points q_9 and q_{12} defined in this example through the embeddings $\mathscr{V}/W \hookrightarrow \mathbb{P}(V)/W \simeq \mathbb{P}(2,5,6,8,9,12)$. Still by Example 2.14, the morphism $\mathscr{V}/W' \to \mathscr{V}/W$ is unramified above q_9 and q_{12}. Therefore, the points p_9 and $p'_9=\sigma(p_9)$ (resp. p_{12} and $p'_{12}=\sigma(p_{12})$) of $\mathscr{V}/W'$ are distinct and have the same type of singularities than the point q_9 (resp. q_{12}) of $\mathscr{V}/W$. But q_9 is a singular point of type A_2 of $\mathbb{P}(5,3,4)$ or $\mathbb{P}(1,3,2)$ and q_{12} is a singular point of type A_3 (resp. A_1) of $\mathbb{P}(5,3,4)$ (resp. $\mathbb{P}(1,3,2)$). This shows that the following results holds:

Lemma 3.10. — *We have:*

(a) *The points p_9 and p'_9 are A_2 singularities of $\mathscr{X}/W'$ and the points p_{12} and p'_{12} are A_3 singularities of $\mathscr{X}/W'$.*

(b) *The points p_9 and p'_9 are A_2 singularities of $\mathscr{Y}_{\lambda,\mu}/W'$ and the points p_{12} and p'_{12} are A_1 singularities of $\mathscr{Y}_{\lambda,\mu}/W'$.*

Therefore, it remains to prove that the points of $\mathcal{X}/W'$ lying above q_5 are ADE singularities. For this, note that

$$\pi^{-1}(q_5) = \{x \in \mathbb{P}(V) \mid f_2(x) = f_6(x) = f_8(x) = f_9(x) = f_{12}(x) = 0\}.$$

But, by (2.11) and Theorem 2.13(d) applied to the case where $e = 5$, we get that $V(5)$ is a line in V, so may be viewed as a point of $\mathbb{P}(V)$ and $\pi_{\mathbf{f}}^{-1}(q_5)$ is the W-orbit of $V(5)$. Now, $\delta^*(5) = 2 > \delta(5) = 1$, so it follows from Theorem 2.13(f) that $W_{V(5)}^{\mathrm{pt}} \neq 1$. By Steinberg's Theorem (see for instance [**Bro**, Theo. 4.7]), this shows that $W_{V(5)}^{\mathrm{pt}}$ contains a reflection, and so the stabilizer G of $V(5)$ in W contains a reflection. In particular, G is not contained in W'. This proves that the morphism $\mathcal{X}/W' \to \mathcal{X}/W$ is ramified above q_5: we denote by p_5 the unique point of $\mathcal{X}/W'$ lying above q_5.

Now, Example 2.15 shows that the stabilizer of $V(5)$ in W' is $\langle w_5 \rangle$. So, in order to determine the type of singularity of $\mathcal{X}/W'$ at p_5, we only need to determine the two eigenvalues of w_5 for its action on $\mathrm{T}_{V(5)}(\mathcal{X})$. This is easily done thanks to Corollary 1.4: the two eigenvalues are ζ_5 and ζ_5^{-1}. We have thus proved the following result:

Lemma 3.11. — *The point p_5 is an A_4 singularity of $\mathcal{X}/W'$.*

This completes the proof of Lemma 3.9. □

3.C. Euler characteristic. — Since $\mathcal{V}$ is a complete intersection which is smooth or has only ADE singularities, its cohomology (with coefficients in $\mathbb{C}$) is concentrated in even degree [**Dim**, Theo. 2.1, Lem. 3.2 and Example 3.3]. Now, $\mathrm{H}^k(\mathcal{V}/W', \mathbb{C}) \simeq \mathrm{H}^k(\mathcal{V}, \mathbb{C})^{W'}$, so the cohomology of $\mathcal{V}/W'$ is concentrated in even degree. This implies the next lemma, which corresponds to the Fact (C) stated in the proof of Theorem 3.3, and completes the proof of Theorem 3.3:

Lemma 3.12. — *The Euler characteristic of $\mathcal{V}/W'$ is positive.*

Notation. *The minimal smooth resolution of the surface $\mathcal{X}/W'$ will be denoted by $\rho : \tilde{\mathcal{X}} \to \mathcal{X}/W'$. Theorem 3.3 says that $\tilde{\mathcal{X}}$ is a smooth projective K3 surface.*

4. Some numerical data for the surface $\mathcal{X}/W'$

We complete here the qualitative result given by Theorem 3.3 with some concrete results concerning the surface $\mathcal{X}/W'$ (type of singularities, equation, coordinates of singular points, cohomology,...). These informations will be used in the next section to obtain further properties of the K3 surface $\tilde{\mathcal{X}}$ (Picard lattice, elliptic fibration,...).

4.A. Singularities. — In the course of the proof of Theorem 3.3, we have obtained some quantitative results (see Lemmas 3.10 and 3.11). We complete them by determining all the singularities of $\mathcal{X}/W'$:

Proposition 4.1. — *The surface $\mathcal{X}/W'$ admits $A_4 + 2\,A_3 + 3\,A_2 + 2\,A_1$ singularities.*

Proof. — Note that we have already found $A_4 + 2\,A_3 + 2\,A_2$ singularities in $\mathcal{X}/W'$ (see Lemma 3.10 and 3.11), given by the points p_5, p_9, p'_9, p_{12} and p'_{12} lying respectively above the points $q_5 = [1 : 0 : 0]$, $q_9 = [0 : 1 : 0]$ and $q_{12} = [0 : 0 : 1]$ of $\mathcal{X}/W \simeq \mathbb{P}(5, 3, 4)$.

To determine the other singularities, one must study the fixed points schemes $\mathcal{X}^w$, for $w \in W \setminus \{1\}$ (note that $\mathcal{X}^w$ is in fact smooth because $\mathcal{X}$ is smooth). The details of the arguments and of the computer calculations can be found in [**Arxiv**, Proof of Prop. 4.1 and App. I.B] and are summarized as follows (note that some of the arguments make use of Corollary 1.4):

- There is only one conjugacy class of W' consisting of elements of order 2 whose list of eigenvalues is $\{\{1, 1, 1, 1, -1, -1\}\}$: if v_2 is a representative of this class, then the image of $\mathcal{X}^{v_2}$ in $\mathcal{X}/W'$ consists of two points p_1 and p'_1 which are singular points of type A_1.
- There is one conjugacy class of W' consisting of elements of order 3 whose list of eigenvalues is $\{\{1, 1, 1, 1, \zeta_3, \zeta_3^{-1}\}\}$: if u_3 is a representative of this class, then the image of $\mathcal{X}^{u_3}$ in $\mathcal{X}/W'$ consists of smooth points and one singular point p_2 of type A_2. Moreover, p_2 is different from p_9 and p'_9.
- If $w \neq 1$ does not belong to one of the above two conjugacy classes, then the image of $\mathcal{X}^w$ in $\mathcal{X}/W'$ consists of smooth points or points belonging to $\{p_1, p'_1, p_2, p_5, p_9, p'_9, p_{12}, p'_{12}\}$. They give rise to no new singularity in $\mathcal{X}/W'$.

This is the expected result. □

For the singular points of $\mathcal{X}/W'$, we keep the notation of the proof of Proposition 4.1:

- The two singular points of type A_1 are denoted by p_1 and p'_1. Note that they are both σ-fixed because they lie above smooth points of $\mathcal{X}/W$ (so they must lie on the ramification locus of $\omega : \mathcal{X}/W' \to \mathcal{X}/W$).
- The singular point of type A_2 coming from the fixed point locus $\mathcal{X}^{u_3}$, where u_3 has order 3 and satisfies $\dim V^{u_3} = 4$, is denoted by p_2. Again, $\sigma(p_2) = p_2$.
- The singular points p_5, p_9, p'_9, p_{12} and p'_{12} involved in Lemma 3.9 are of type A_4, A_2, A_2, A_3 and A_3 respectively. We have $\sigma(p_5) = p_5$, $p'_9 = \sigma(p_9)$ and $p'_{12} = \sigma(p_{12})$.

4.B. Two smooth rational curves. — Recall that $V(3)$ is the eigenspace of w_3 for the eigenvalue ζ_3 and denote by $V^-(3)$ the eigenspace of w_3 for the eigenvalue ζ_3^{-1}. By (2.11), they both have dimension 3 and so

$$V = V(3) \oplus V^-(3). \tag{4.2}$$

Let $\mathcal{C}^+ = \mathbb{P}(V(3)) \cap \mathcal{X}$ and $\mathcal{C}^- = \mathbb{P}(V^-(3)) \cap \mathcal{X}$. By Theorem 2.13(f), w_3 and w_3^{-1} are conjugate in W, but it follows from Remark 2.12 that they are not conjugate in W'. Fix $g \in W$ be such that $w_3^{-1} = g w_3 g^{-1}$. Then $g \notin W'$ and $g(V(3)) = V^-(3)$. This shows that

$$\mathcal{C}^- = g(\mathcal{C}^+). \tag{4.3}$$

Lemma 4.4. — *The schemes $\mathcal{C}^+$ and $\mathcal{C}^-$ are smooth irreducible curves of genus* 10.

Proof. — By (4.2), $\mathbb{P}(V)^{w_3} = \mathbb{P}(V(3)) \,\dot\cup\, \mathbb{P}(V^-(3))$, where $\dot\cup$ means a disjoint union. So

$$\mathcal{X}^{w_3} = \mathscr{C}^+ \,\dot\cup\, \mathscr{C}^-. \tag{4.5}$$

Since $\mathcal{X}^{w_3}$ is smooth, this implies that $\mathscr{C}^+$ and $\mathscr{C}^-$ are smooth. Now, by Lemma 1.1, the restriction of f_2, f_5 and f_8 to $V(3)$ are equal to 0. Therefore,

$$\mathscr{C}^+ = \{x \in \mathbb{P}(V(3)) \mid f_6(v) = 0\}.$$

This shows that $\mathscr{C}^+$ is a connected curve of $\mathbb{P}(V(3))$. Since it is smooth, it must be irreducible. Moreover, it is of degree 6, so its genus is equal to 10. □

Let $\mathscr{C}_5^+$ (resp. $\mathscr{C}_5^-$) denote the image of $\mathscr{C}^+$ (resp. $\mathscr{C}^-$) in $\mathcal{X}/W'$ and let $\mathscr{C}_5$ denote the image of $\mathscr{C}^+$ in $\mathcal{X}/W \simeq \mathbb{P}(5,3,4)$. Note that $\mathscr{C}_5$ is also the image of $\mathscr{C}^-$ by (4.3). Moreover,

$$\mathscr{C}_5 = \{[y_5, y_3, y_4] \in \mathbb{P}(5,3,4) \mid y_5 = 0\} = \mathbb{P}(3,4) = \mathbb{P}^1(\mathbb{C}). \tag{4.6}$$

Indeed, $\mathscr{C}_5$ is irreducible, of dimension 1 and contained in $\{[y_5 : y_3 : y_4] \in \mathbb{P}(5,3,4) \mid y_5 = 0\}$ by Lemma 1.1 (see also Theorem 2.13(d)).

Proposition 4.7. — *We have $\mathscr{C}_5^- = \sigma(\mathscr{C}_5^+) \neq \mathscr{C}_5^+$. Moreover, $\mathscr{C}_5^+$ and $\mathscr{C}_5^-$ are both isomorphic to $\mathbb{P}^1(\mathbb{C})$, intersect transversely along only one point and satisfy*

$$\mathscr{C}_5^+ \cup \mathscr{C}_5^- = \{[y_5 : y_3 : y_4 : j] \in \mathcal{X}/W' \mid y_5 = 0\}.$$

Proof. — The fact that $\mathscr{C}_5^- = \sigma(\mathscr{C}_5^+)$ follows from (4.3). Now, the irreducibility of $\mathscr{C}^+$ and $\mathscr{C}^-$ implies the irreducibility of $\mathscr{C}_5^+$ and $\mathscr{C}_5^-$. Also, by (4.6),

$$\mathscr{C}_5^+ \cup \mathscr{C}_5^- \simeq \{[y_3 : y_4 : j] \in \mathbb{P}(3,4,12) \mid j^2 = Q_{\mathbf{f}}(0, y_3, y_4)\}.$$

But $\mathbb{P}(3,4,12) = \mathbb{P}(3,1,3) = \mathbb{P}(1,1,1) = \mathbb{P}^2(\mathbb{C})$, so there exists a polynomial $Q_3 \in \mathbb{C}[T_3, T_4]$, which is homogenous of degree 2 (with T_k of degree 1) and such that $Q_{\mathbf{f}}(0, y_3, y_4) = Q_3(y_3^4, y_4^3)$. Therefore,

$$\mathscr{C}_5^+ \cup \mathscr{C}_5^- \simeq \{[t_3 : t_4 : j] \in \mathbb{P}^2(\mathbb{C}) \mid j^2 = Q_3(t_3, t_4)\}.$$

It just remains to prove that the polynomial Q_3 is the square of a linear form: in other words, we only need to prove that $\mathscr{C}_5^+ \neq \mathscr{C}_5^-$.

Since $\mathscr{C}^+$ is irreducible of degree 6, it cannot be contained in a union of projective lines of $\mathbb{P}(V(3))$. Since moreover $\delta^*(3) = \delta(3) = 3$, Theorem 2.13(f) thus implies that there exists $v \in V(3) \setminus \{0\}$ such that $[v] \in \mathscr{C}^+$ and $W_v = 1$. We now only need to prove that $\pi'_{\mathbf{f}}([v]) \notin \mathscr{C}^-$. So assume that $\pi'_{\mathbf{f}}([v]) \in \mathscr{C}^-$. This would imply that there exists $w \in W'$ such that $w(v) \in V^-(3)$. Consequently,

$$w^{-1} w_3 w w_3(v) = v,$$

and so $w^{-1} w_3 w w_3 = 1$ since $W_v = 1$. Hence, w_3 and w_3^{-1} are conjugate in W', which is impossible by Remark 2.12. □

By exchanging p_9 and p'_9 (resp. p_{12} and p'_{12}) if necessary, we may assume that

$$p_9, p_{12} \in \mathscr{C}_5^+ \qquad \text{and} \qquad p'_9, p'_{12} \in \mathscr{C}_5^-.$$

4.C. Singular points, equation. — Of course, the equation giving $\mathcal{X}/W'$ and the coordinates of the singular points depend on a model for W and on a choice of fundamental invariants. With the choices made in [**Arxiv**, App. I] (see Remark 3.1), we get:

Lemma 4.8. — *The coordinates of the singular points in* $\mathbb{P}(5,3,4,12)$ *are given by*

$$p_5 = [1:0:0:0], \quad p_9, p_9' = [0:1:0:\pm\tfrac{27}{64}\sqrt{-3}], \quad p_{12}, p_{12}' = [0:0:1:\pm\tfrac{16}{243}\sqrt{-3}]$$

$$p_1, p_1' = [\tfrac{1}{9}(\eta 536\sqrt{19}+2336) : \tfrac{1}{9}(-\eta 60\sqrt{19}-260) : \tfrac{1}{3}(\eta 130\sqrt{19}+565) : 0], \textit{ with } \eta = \pm 1,$$

$$p_2 = [4/9 : 10/9 : -5/12 : 0].$$

Proof. — The details of this calculation are given in [**Arxiv**, Proof of Lem. 4.15]. □

The computation of the coordinates of the intersection point p of $\mathscr{C}_5^+$ and $\mathscr{C}_5^-$ is done in [**Arxiv**, §4.C]:

$$p = [0 : 2/9 : 1/4 : 0]. \tag{4.9}$$

As explained in [**Arxiv**, §4.C], knowing that $\mathcal{X}/W'$ contains p and the singular points given in Lemma 4.8 is sufficient for obtaining an explicit equation of $\mathcal{X}/W'$:

$$\mathcal{X}/W' = \{[y_5 : y_3 : y_4 : j] \in \mathbb{P}(5,3,4,12) \mid j^2 = -3(\tfrac{27}{64}\,y_3^4 - \tfrac{16}{243}\,y_4^3)^2 - y_5 R(y_5, y_3, y_4)\} \tag{4.10}$$

with

$$R = y_3 y_4 (\tfrac{207}{32}\,y_3^4 + \tfrac{800}{729}\,y_4^3) + \tfrac{1375}{81}\,y_5 y_3^2 y_4^2 - \tfrac{3125}{864}\,y_5^2 y_3^3 - \tfrac{3125}{108}\,y_5^3 y_4. \tag{4.11}$$

Remark 4.12. — With this equation, it is easy to describe the union $\mathscr{C}_5^+ \cup \mathscr{C}_5^-$, namely

$$\mathscr{C}_5^+ \cup \mathscr{C}_5^- = \{[y_3 : y_4 : j] \in \mathbb{P}(3,4,12) \mid j^2 = -3(\tfrac{27}{64}\,y_3^4 - \tfrac{16}{243}\,y_4^3)^2\}.$$

But $\mathbb{P}(3,4,12) \simeq \mathbb{P}(1,4,4) \simeq \mathbb{P}(1,1,1) = \mathbb{P}^2(\mathbb{C})$ and, through these isomorphisms,

$$\mathscr{C}_5^+ \cup \mathscr{C}_5^- = \{[Y_3 : Y_4 : j] \in \mathbb{P}^2(\mathbb{C}) \mid j^2 = -3(\tfrac{27}{64}\,Y_3 - \tfrac{16}{243}\,Y_4)^2\},$$

which is indeed a union of two smooth rational curves. ■

4.D. Two other smooth rational curves in $\mathcal{X}/W'$. — For $k \in \{3,4\}$, we set

$$\mathscr{C}_k = \{[y_5 : y_3 : y_4 : j] \in \mathcal{X}/W' \mid y_k = 0\}.$$

Proposition 4.13. — *The schemes* $\mathscr{C}_3$ *and* $\mathscr{C}_4$ *are reduced, irreducible and are smooth rational curves. They intersect transversely at* $p_5 = [1:0:0:0]$.

Proof. — From the equation (4.10), we get that

$$\mathscr{C}_3 \simeq \{[y_5 : y_4 : j] \in \mathbb{P}(5,4,12) \mid j^2 = -\tfrac{2^8}{3^9}\, y_4^6 + \tfrac{3125}{108}\, y_5^4 y_4\}.$$

But $\mathbb{P}(5,4,12) = \mathbb{P}(5,1,3)$, so

$$\mathscr{C}_3 \simeq \{[y_5 : y_4 : j] \in \mathbb{P}(5,1,3) \mid j^2 = -\tfrac{2^8}{3^9}\, y_4^6 + \tfrac{3125}{108}\, y_5 y_4\}.$$

The open subset of $\mathscr{C}_3$ defined by $y_4 \neq 0$ is clearly isomorphic to $\mathbb{A}^1(\mathbb{C})$.

Similary, from the equation (4.10), we get that

$$\mathscr{C}_4 \simeq \{[y_5 : y_3 : j] \in \mathbb{P}(5,3,12) \mid j^2 = -\tfrac{3^7}{2^{12}}\, y_3^6 + \tfrac{3125}{864}\, y_5^3 y_3^3\}.$$

But $\mathbb{P}(5,3,12) = \mathbb{P}(5,1,4)$, so

$$\mathscr{C}_4 \simeq \{[y_5 : y_3 : j] \in \mathbb{P}(5,1,4) \mid j^2 = -\tfrac{3^7}{2^{12}}\, y_3^6 + \tfrac{3125}{864}\, y_5 y_3^3\}.$$

The open subset of $\mathscr{C}_4$ defined by $y_3 \neq 0$ is clearly isomorphic to $\mathbb{A}^1(\mathbb{C})$.

So $\mathscr{C}_3$ and $\mathscr{C}_4$ are rational curves, and it remains to prove that $\mathscr{C}_3$ and $\mathscr{C}_4$ are smooth and intersect transversely. Both questions are local around the point $p_5 = [1:0:0:0]$, so we must work in the affine chart of $\mathcal{X}/W'$ defined by $y_5 \neq 0$. The computation is somewhat involved and details are given in [**Arxiv**, Proof of Prop. 4.22]. $\square$

5. The K3 surface $\tilde{\mathfrak{X}}$

Recall that $\rho : \tilde{\mathcal{X}} \longrightarrow \mathcal{X}/W'$ denotes the minimal smooth resolution. We will deduce several properties of $\tilde{\mathcal{X}}$ (Picard lattice, elliptic fibration,...) from the list of properties of $\mathcal{X}/W'$ given in the previous section. Note that since $\tilde{\mathcal{X}}$ is obtained from $\mathcal{X}/W'$ by successively blowing-up the singular locus, the automorphism σ of $\mathcal{X}/W'$ lifts to an automorphism of $\tilde{\mathcal{X}}$ (which will still be denoted by σ).

We denote by Δ_1 and Δ_1' the two smooth rational curves of $\tilde{\mathcal{X}}$ lying above p_1 and p_1' respectively. For $e \in \{2,5,9,12\}$, we denote by $\Delta_e^1, \dots, \Delta_e^{r_e}$ the smooth rational curves of $\tilde{\mathcal{X}}$ lying above p_e (here, r_e is the Milnor number of the singularity p_e[(†)]), and we assume that they are numbered in such a way that $\Delta_e^k \cap \Delta_e^{k+1} \neq \varnothing$. For $e \in \{9,12\}$, the smooth rational curves of $\tilde{\mathcal{X}}$ lying above p_e' are then given by $\sigma(\Delta_e^1), \dots, \sigma(\Delta_e^{r_e})$.

Finally, we denote by $\tilde{\mathscr{C}}_5^{\pm}$ the strict transform of $\mathscr{C}_5^{\pm}$ in $\tilde{\mathcal{X}}$. Of course, $\tilde{\mathscr{C}}_5^{-} = \sigma(\tilde{\mathscr{C}}_5^{+})$. As $\tilde{\mathcal{X}}$ is obtained from $\mathcal{X}/W'$ by successive blow-ups of points, $\tilde{\mathscr{C}}_5^{+}$ and $\tilde{\mathscr{C}}_5^{-}$ are smooth rational curves. Also, we denote by $\tilde{\mathscr{C}}_3$ and $\tilde{\mathscr{C}}_4$ the strict transforms of $\mathscr{C}_3$ and $\mathscr{C}_4$: for the same reason, they are also smooth rational curves.

One of the aims of this section is to determine the intersection graph of the 22 smooth rational curves $\tilde{\mathscr{C}}_3$, $\tilde{\mathscr{C}}_4$, $\tilde{\mathscr{C}}_5^{\pm}$, Δ_1, Δ_1', $(\Delta_e^k)_{e \in \{2,5,9,12\}, 1 \leqslant k \leqslant r_e}$ and $({}^{\sigma}\Delta_e^k)_{e \in \{9,12\}, 1 \leqslant k \leqslant r_e}$. For this, we will use the construction of an elliptic fibration on $\tilde{\mathcal{X}}$.

Recall that, for a K3 surface, an *elliptic fibration* is just a morphism to $\mathbb{P}^1(\mathbb{C})$ such that at least one fiber is a smooth elliptic curve. Since $\mathcal{X}/W'$ has $A_4 + 2\,A_3 + 3\,A_2 + 2\,A_1$ singularities, its Picard number is greater than or equal to $1 + (4 + 2\cdot 3 + 3 \cdot 2 + 1) = 19$ (in fact, we will see later that it has Picard number 20). Therefore, it admits an elliptic fibration (because every

[(†)]For the definition of the Milnor number of an isolated hypersurface singularity, see [**Mil**, §7]: recall that the Milnor number of a singularity of type A_k, D_k or E_k is equal to k.

K3 surface with Picard number $\geqslant 5$ admits an elliptic fibration [**Huy**, Chap. 11, Prop. 1.3(ii)]). Another aim of this section is to contruct at least one such fibration. For this, let

$$\begin{array}{rccc} \varphi: & (\mathcal{X}/W')\setminus\{p_5\} & \longrightarrow & \mathbb{P}^1(\mathbb{C}) \\ & [y_5 : y_3 : y_4 : j] & \longmapsto & [y_3^4 : y_4^3]. \end{array}$$

Then φ is a well-defined morphism of varieties, so the map $\varphi\circ\rho : \tilde{\mathfrak{X}}\setminus\rho^{-1}(p_5) \longrightarrow \mathbb{P}^1(\mathbb{C})$ is also a well-defined morphism of varieties. We are now ready to prove the second main result of our paper:

Theorem 5.1. — *With the above notation, we have:*

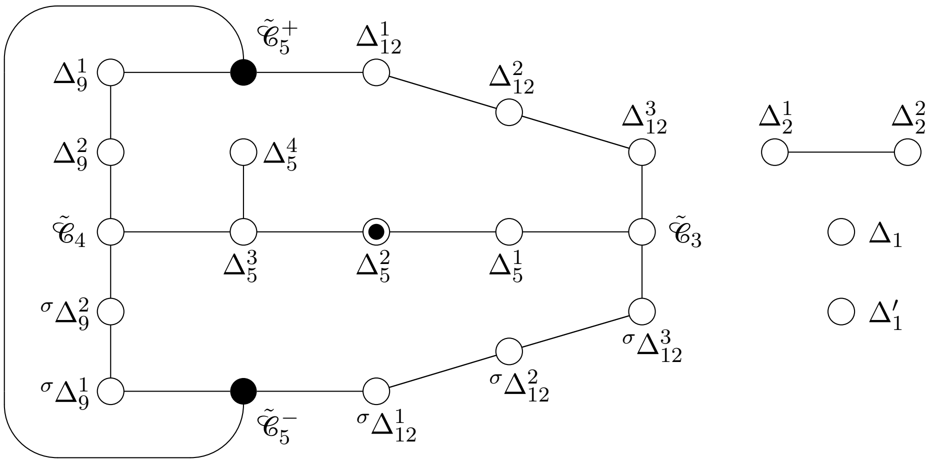
(a) *The morphism $\varphi\circ\rho : \tilde{\mathfrak{X}}\setminus\rho^{-1}(p_5) \longrightarrow \mathbb{P}^1(\mathbb{C})$ extends to a unique morphism*

$$\tilde{\varphi} : \tilde{\mathfrak{X}} \to \mathbb{P}^1(\mathbb{C}),$$

which is an elliptic fibration whose singular fibers are of type $E_7 + E_6 + A_2 + 2\,A_1$.

(b) *There exists a way of numbering the smooth rational curves lying above singular points of $\mathcal{X}/W'$ such that the intersection graph of the 22 smooth rational curves $\tilde{\mathscr{C}}_3$, $\tilde{\mathscr{C}}_4$, $\tilde{\mathscr{C}}_5^{\pm}$, Δ_1, Δ_1', $(\Delta_e^k)_{e\in\{2,5,9,12\},1\,\leqslant\,k\,\leqslant\,r_e}$ and $({}^\sigma\Delta_e^k)_{e\in\{9,12\},1\,\leqslant\,k\,\leqslant\,r_e}$ is given by*[(‡)]

(★)



In this graph:

(b1) *The union of the singular fibers of $\tilde{\varphi}$ of type E_7 and E_6 is given by the white disks in the big connected subgraph of (★).*

(b2) *The singular fibers of $\tilde{\varphi}$ of type A_1 are $\tilde{\mathscr{C}}_1 \cup \Delta_1$ and $\tilde{\mathscr{C}}_1' \cup \Delta_1'$ for some smooth rational curves $\tilde{\mathscr{C}}_1$ and $\tilde{\mathscr{C}}_1'$.*

(b3) *The singular fiber of $\tilde{\varphi}$ of type A_2 is $\tilde{\mathscr{C}}_2 \cup \Delta_2^1 \cup \Delta_2^2$ for some smooth rational curve $\tilde{\mathscr{C}}_2$.*

(b4) *The curves marked with full black disks in (★) are sections of $\tilde{\varphi}$.*

(b5) *The curve Δ_5^2 is a double section of $\tilde{\varphi}$.*

(c) *The* 22 *smooth rational curves in this intersection graph generate the Picard lattice* $\mathrm{Pic}(\tilde{\mathfrak{X}})$. *More precisely,* $\mathrm{Pic}(\tilde{\mathfrak{X}})$ *is generated by the list obtained from these* 22 *smooth rational curves by removing Δ_5^2 and Δ_5^4. Its discriminant is $-228 = -2^2\cdot 3\cdot 19$.*

(d) *The Mordell-Weil group of $\tilde{\varphi}$ is isomorphic to $\mathbb{Z}^2$.*

(e) *The transcendental lattice is given by the matrix* $\begin{pmatrix} 2 & 0 \\ 0 & 114 \end{pmatrix}$.

[(‡)] This means that two smooth rational curves C and C' in this list intersect if and only if they are joined by an edge in the above graph, and that, if so, then $C\cdot C' = 1$.

Remark 5.2. — There are several possible types of singular fibers of type A_1 (resp. A_2) in elliptic fibrations. In the above Theorem 5.1, singular fibers of type A_1 (resp. A_2) are of type I_2 (resp. I_3) in Kodaira's classification. ■

Proof. — The details of the (very computational) proof of the statements (a) and (b) is given in [**Arxiv**, App. III].

For (c), let M denote the incidence matrix of the 18 smooth rational curves belonging to the big connected subgraph of (★). Then M has rank 16 and the greatest common divisor of the diagonal minors of M is equal to 19 by [**Arxiv**, Comput. IV.1]. Moreover, the diagonal minor corresponding to the curves $\tilde{\mathscr{C}}_3$, $\tilde{\mathscr{C}}_4$, $\tilde{\mathscr{C}}_5^{\pm}$, Δ_5^1, Δ_5^3, $(\Delta_e^k)_{e\in\{9,12\},1\leqslant k\leqslant r_e}$ and $({}^\sigma\Delta_e^k)_{e\in\{9,12\},1\leqslant k\leqslant r_e}$ is equal to -19 (see [**Arxiv**, Comput. IV.1]). So, if we denote by Λ the lattice generated by these 16 curves together with Δ_1, Δ_1', Δ_2^1 and Δ_2^2, then Λ has rank 20 and discriminant $-228=-2^2\cdot 3\cdot 19$. This shows that $\mathrm{Pic}(\tilde{\mathfrak{X}})$ has rank $\geqslant 20$, and so has rank 20 as a K3 surface has always Picard number $\leqslant 20$. If we denote by n the index of Λ in $\mathrm{Pic}(\tilde{\mathfrak{X}})$, then n^2 divides 228, which shows that $n\in\{1,2\}$.

But if we denote by $\mathrm{T}(\tilde{\mathfrak{X}})$ the transcendental lattice of $\tilde{\mathfrak{X}}$, then $\mathrm{T}(\tilde{\mathfrak{X}})$ has rank $22-20=2$, is even and definite positive, with discriminant $228/n^2$. Hence it can be represented by a matrix of the form

$$\begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix}$$

with a, $c>0$ and $4ac-b^2=228/n^2$. But $4ac-b^2\equiv 0$ or $3 \mod 4$, so $4ac-b^2\neq 57$. This shows that $n=1$ and that $\mathrm{Pic}(\tilde{\mathfrak{X}})$ is generated by $\tilde{\mathscr{C}}_3$, $\tilde{\mathscr{C}}_4$, $\tilde{\mathscr{C}}_5^{\pm}$, Δ_5^1, Δ_5^3, $(\Delta_e^k)_{e\in\{9,12\},1\leqslant k\leqslant r_e}$ and $({}^\sigma\Delta_e^k)_{e\in\{9,12\},1\leqslant k\leqslant r_e}$, as expected. This concludes the proof of (c).

(d) Since we have determined the Picard lattice of $\tilde{\mathfrak{X}}$ in (c), the structure of the Mordell-Weil group follows (note that it has no torsion, as expected by [**Shi**, Table 1, entry 2420]).

(e) The transcendental lattice of $\tilde{\mathfrak{X}}$ is given by a matrix of the form $\begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix}$ whose underlying quadratic form is definite positive and has discriminant 228 by (c). The classification of even integral binary quadratic forms [**Bue**, Theo. 2.3], shows that there are only four such matrices, up to equivalence, namely:

$$M_1=\begin{pmatrix} 2 & 0 \\ 0 & 114 \end{pmatrix},\quad M_2=\begin{pmatrix} 6 & 0 \\ 0 & 38 \end{pmatrix},\quad M_3=\begin{pmatrix} 4 & 2 \\ 2 & 58 \end{pmatrix}\quad\text{and}\quad M_4=\begin{pmatrix} 12 & 6 \\ 6 & 22 \end{pmatrix}.$$

Let $P=\mathrm{Pic}(\tilde{\mathfrak{X}})$ and $T=\mathrm{T}(\tilde{\mathfrak{X}})$. Let

$$P^\perp=\{v\in\mathbb{Q}\otimes_{\mathbb{Z}}P \mid \forall\, v'\in P,\ \langle v,v'\rangle\in\mathbb{Z}\}$$

and let us define $T^\perp$ similarly. Then the quadratic forms on $\mathbb{Q}\otimes_{\mathbb{Z}}P$ and $\mathbb{Q}\otimes_{\mathbb{Z}}T$ induce well-defined maps

$$q_P: P^\perp/P\longrightarrow \mathbb{Q}/2\mathbb{Z}\qquad\text{and}\qquad q_T: T^\perp/T\longrightarrow\mathbb{Q}/2\mathbb{Z}.$$

Since $\mathrm{H}^2(\tilde{\mathfrak{X}},\mathbb{Z})$ is unimodular of signature $(3,19)$, it turns out that there is an isomorphism $\iota: T^\perp/T\xrightarrow{\sim}P^\perp/P$ such that $q_T=-q_P\circ\iota$ (see [**Nik**, Prop. 1.6.1]). In particular, the set of values of q_T and $-q_P$ coincide: the subset $-q_P(P^\perp/P)$ of $\mathbb{Q}/2\mathbb{Z}$ can easily be computed thanks to (b) and (c) using MAGMA, and we only need to compare the corresponding sets for the four rank 2 lattices determined by M_1, M_2, M_3 and M_4. This comparison gives the result (see [**Arxiv**, §IV.B] for detailed computations). □

Remark 5.3. — Theorem 5.1(e) shows that the K3 surface $\tilde{\mathfrak{X}}$ is not a Kummer surface [**Huy**, Chap. 14, Cor. 3.20]. It also shows that $\tilde{\mathfrak{X}}$ is not isomorphic to any of the singular K3 surfaces constructed by Barth-Sarti in [**BaSa**] or to any of the singular K3 surfaces constructed by Brandhorst-Hashimoto in [**BrHa**] (see also [**BoSa1**] for a description of some of these). ■

6. Complement: action of W on the cohomology of $\mathfrak{X}$

The group W acts on $\mathfrak{X}$ so it acts on the cohomology groups $\mathrm{H}^k(\mathfrak{X}, \mathbb{C})$. Since $\mathfrak{X}$ is a complete intersection in $\mathbb{P}^5(\mathbb{C})$, with defining equations of degree 2, 6, and 8, we have:

$$\dim_{\mathbb{C}} \mathrm{H}^k(\mathfrak{X}, \mathbb{C}) = \begin{cases} 1 & \text{if } k \in \{0,4\}, \\ 9502 & \text{if } k = 2, \\ 0 & \text{otherwise.} \end{cases} \tag{6.1}$$

The action of W on $\mathrm{H}^0(\mathfrak{X}, \mathbb{C})$ and $\mathrm{H}^4(\mathfrak{X}, \mathbb{C})$ is trivial. The aim of this subsection is to determine the character of the representation of W afforded by $\mathrm{H}^2(\mathfrak{X}, \mathbb{C})$.

For this, we first need to parametrize the irreducible characters of W. If $\chi \in \mathrm{Irr}(W)$, we denote by b_χ the minimal number k such that χ occurs in the character of the symmetric power $\mathrm{Sym}^k(V)$ of the natural representation V of W. For instance, if we denote by $\mathbf{1}_W$ the trivial character of W and by χ_V the character afforded by the natural representation V, then

$$b_{\mathbf{1}_W} = 0, \qquad b_{\chi_V} = 1 \qquad \text{and} \qquad b_\varepsilon = |\mathcal{A}| = 36 \tag{6.2}$$

(recall that ε denotes the restriction of the determinant to W). Indeed, the first two equalities are immediate from the definition and the last one follows from [**Bou**, Chap. V, §5, Prop. 5] and (2.7). Recall from Molien's formula that the number b_χ an be computed as follows: let t be an indeterminate and let

$$F_\chi(t) = \frac{\prod_{d \in \mathrm{Deg}(W)}(1-t^d)}{|W|} \sum_{w \in W} \frac{\chi(w^{-1})}{\det(1-tw)} \in \mathbb{C}(t).$$

It is a classical fact [**Bro**, §4.5.2] that $F_\chi(t) \in \mathbb{N}[t]$, that $F_\chi(1) = \chi(1)$ and

$$b_\chi = \mathrm{val}\, F_\chi(t). \tag{6.3}$$

The polynomial $F_\chi(t)$ is called the *fake degree* of χ.

A particular feature of the Weyl group of type E_6 is that the map

$$\begin{array}{rccc} \mathrm{DB}: & \mathrm{Irr}(W) & \longrightarrow & \mathbb{N} \times \mathbb{N} \\ & \chi & \longmapsto & (\chi(1), b_\chi) \end{array} \tag{6.4}$$

is injective (see for instance [**Arxiv**, Comput. II.2]). We denote by $\mathcal{DB}(W)$ the image of DB. If $(d, b) \in \mathcal{DB}(W)$, let $\phi_{d,b}$ denote its inverse image in $\mathrm{Irr}(W)$. Note that $\phi_{d,b}$ is the character afforded by an irreducible representation of dimension d. For instance, by (6.2), we get

$$\phi_{1,0} = \mathbf{1}_W, \qquad \phi_{6,1} = \chi_V \qquad \text{and} \qquad \phi_{1,36} = \varepsilon. \tag{6.5}$$

By [**Arxiv**, Comput. II.2], we have that $|\mathrm{Irr}(W)| = 25$ and that

$$\begin{aligned} \mathcal{DB}(W) = \{&(1,0); (1,36); (6,1); (6,25); (10,9); (15,17); (15,4); (15,16); (15,5); \\ &(20,20); (20,10); (20,2); (24,6); (24,12); (30,3); (30,15); (60,11); \\ &(60,8); (60,5); (64,13); (64,4); (80,7); (81,6); (81,10); (90,8)\}. \end{aligned} \tag{6.6}$$

For $i \geqslant 0$, let $\chi_{\mathfrak{X}}^{(i)}$ denote the character afforded by the W-module $\mathrm{H}^i(\mathfrak{X}, \mathbb{C})$. We set

$$\chi_{\mathfrak{X}} = \sum_{i \geqslant 0} (-1)^i \chi_{\mathfrak{X}}^{(i)}.$$

By (6.1), we have

$$\chi_{\mathfrak{X}} = \chi_{\mathfrak{X}}^{(0)} + \chi_{\mathfrak{X}}^{(2)} + \chi_{\mathfrak{X}}^{(4)} = 2 \cdot \mathbf{1}_W + \chi_{\mathfrak{X}}^{(2)}. \tag{6.7}$$

The character $\chi_{\mathfrak{X}}^{(2)}$ is given by the following formula:

$$\begin{aligned} \chi_{\mathfrak{X}}^{(2)} \;=\; & \mathbf{1}_W + 3\,\varepsilon + 8\,\phi_{6,25} + 2\,\phi_{10,9} + 7\,\phi_{15,17} + \phi_{15,4} + 9\,\phi_{15,16} + \phi_{15,5} \\ & +14\,\phi_{20,20} + 4\,\phi_{20,10} + 2\,\phi_{24,6} + 8\,\phi_{24,12} + 14\,\phi_{30,15} + 18\,\phi_{60,11} + 12\,\phi_{60,8} \\ & +4\,\phi_{60,5} + 26\,\phi_{64,13} + 2\,\phi_{64,4} + 12\,\phi_{80,7} + 7\,\phi_{81,6} + 21\,\phi_{81,10} + 12\,\phi_{90,8}. \end{aligned} \tag{6.8}$$

Proof. — Since $\mathfrak{X}$ is smooth and W is finite, it follows from Lefschetz fixed point formula that $\chi_{\mathfrak{X}}(w)$ is equal to the Euler characteristic of the fixed point subvariety $\mathfrak{X}^w$. If $\dim(\mathfrak{X}^w) \geqslant 1$, then w is conjugate to 1, s_1 or w_3 (see [**Arxiv**, Comput. II.3]). But:

- $\chi_{\mathfrak{X}}(1) = 9504$ by (6.1).
- Note that $\mathbb{P}(V)^{s_1} = [e_1] \cup \mathbb{P}(V^{s_1})$. Since $[e_1] \notin \mathfrak{X}$ by Remark 2.6, we have that $\mathfrak{X}^{s_1} = \mathfrak{X} \cap \mathbb{P}(V^{s_1})$. So $\mathfrak{X}^{s_1}$ is a smooth complete intersection in $\mathbb{P}(V^{s_1}) \simeq \mathbb{P}^4(\mathbb{C})$ defined by equations of degree 2, 6 and 8 (the restrictions of f_2, f_6 and f_8 to V^{s_1}), so it has Euler characteristic $-2 \cdot 6 \cdot 8 \cdot (2 + 6 + 8 - 4 - 1) = -1056$. Hence, $\chi_{\mathfrak{X}}(s_1) = -1056$.
- By Lemma 4.4 (and its proof), $\chi_{\mathfrak{X}}(w_3) = -36$.

If $\dim(\mathfrak{X}^w) \leqslant 0$, then $\chi_{\mathfrak{X}}(w)$ is just the cardinality of $\mathfrak{X}^w$ (which might be equal to 0). These last values of $\chi_{\mathfrak{X}}$ as well as the decomposition of $\chi_{\mathfrak{X}}$ as a sum of irreducible characters are computed in [**Arxiv**, Comput. II.4]. The result then follows from (6.7). □

To be fair, knowing the exact character is not that interesting, but at least we will use it for making a sanity check for Proposition 4.1. Indeed, $\mathrm{H}^k(\mathfrak{X}, \mathbb{C})^{W'}$ is the direct sum of $\mathrm{H}^k(\mathfrak{X}, \mathbb{C})^W$ and the ε-isotypic component of $\mathrm{H}^k(\mathfrak{X}, \mathbb{C})$. Then (6.8) and (6.1) show that

$$\sum_{k \in \mathbb{Z}} (-1)^k \dim_{\mathbb{C}} \mathrm{H}^k(\mathfrak{X}, \mathbb{C})^{W'} = 6.$$

In other words, the Euler characteristic of $\mathfrak{X}/W'$ is equal to 6. But the fiber of the map $\tilde{\mathfrak{X}} \to \mathfrak{X}/W'$ above an A_k singularity is the union of k smooth rational curves in A_k-configuration, and this union has Euler characteristic $k + 1$. So the Euler characteristic of $\tilde{\mathfrak{X}}$ is the Euler characteristic of $\mathfrak{X}/W'$ plus the sum of all the Milnor numbers of singularities of $\mathfrak{X}/W'$. So, by Proposition 4.1, the Euler characteristic of $\tilde{\mathfrak{X}}$ is

$$6 + 2 \cdot 1 + 3 \cdot 2 + 2 \cdot 3 + 4 = 24,$$

as expected for a K3 surface.

Remark 6.9. — Since $\mathfrak{X}$ is a smooth complete intersection, its Hodge numbers can be computed from the degrees of the equations and we get that

$$h^{2,0}(\mathfrak{X}) = h^{0,2}(\mathfrak{X}) = 1591 \qquad \text{and} \qquad h^{1,1}(\mathfrak{X}) = 6320.$$

However, we do not know how to compute the character of the representations $\mathrm{H}^{p,q}(\mathfrak{X}, \mathbb{C})$. ■

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Cédric Bonnafé, IMAG, Université de Montpellier, CNRS, Montpellier, France
E-mail : `cedric.bonnafe@umontpellier.fr` • *Url :* `http://imag.umontpellier.fr/~bonnafe/`