Submitted to
Journal of the Mathematical Society of Japan

The realization spaces of certain conic-line arrangements of degree 7

By Shinzo Bannai, Hiro-o Tokunaga and Emiko Yorisaki

(Received Dec. 14, 2024)
(Revised June 29, 2025)

Abstract. We study the embedded topology of certain conic-line arrangements of degree 7. Two new examples of Zariski pairs are given. Furthermore, we determine the number of connected components of the realization spaces of the conic-line arrangements with prescribed combinatorics. We also calculate the fundamental groups using `SageMath` and the package `Sirocco` in the appendix.

1. Introduction

In this paper, we consider the embedded topology of reducible plane curves with irreducible components of low degree. More precisely we study the embedded topology of certain conic-line arrangements. A collection of a finite number of conics and lines in the complex projective plane $\mathbb{P}^2 := \mathbb{P}^2(\mathbb{C})$ is said to be a conic-line arrangement (a CL arrangement, for short). If it contains no lines, it is said to be a conic arrangement. Let $\mathcal{CL} := \{C_1, \ldots, C_m, L_1, \ldots, L_n\}$ be a CL arrangement of m-conics and n-lines in $\mathbb{P}^2$. By the combinatorics of $\mathcal{CL}$ (see **[5, 6]** for the definition of the combinatorics), we mean that of the reduced curve $B_{\mathcal{CL}} := \sum_{i=1}^m C_i + \sum_{j=1}^n L_j$ and denote it by $\mathrm{Cmb}_{\mathcal{CL}}$. More generally, we denote the combinatorics for a reduced plane curve B by Cmb_B.

Line arrangements have been studied by many mathematicians from various points of view. Some results related to our interests are **[15, 16, 24]**, where the connected components of the moduli spaces of line arrangements are studied. On the other hand, there have not been so many results for CL arrangements before 2000 except some results on conic arrangements by Naruki (**[23]**). Since 2000, they have been studied by various mathematicians. For example, in **[2, 3, 4, 12, 13, 22]**, the fundamental groups of their complements are studied. Also, from the viewpoint of free divisors, we find results such as **[11, 18, 26, 27]**.

In **[34]**, the second author studied the embedded topology of certain CL arrangements of degree 7 and gave examples of Zariski pairs for such arrangements. He also raised the question (see **[34**, Remark 6**]**) whether or not a Zariski triple exists for the CL arrangements considered in **[34]**. This is closely related to determining the number

2020 *Mathematics Subject Classification.* Primary 14H50, 14H10, 14J27; Secondary 14H30.
Key Words and Phrases. conic-line arrangements, realization spaces, Zariski pairs, elliptic surfaces, splitting types.
The first named author is partially supported by JSPS KAKENHI Grant Number JP23K03042. The second author is partially supported by JSPS KAKENHI Grant Numbers JP20K03561, JP24K06673.

of connected components of the *realization space* of plane curves with fixed combinatorial data. Here, the realization space of CL arrangements with given combinatorics $\mathrm{Cmb}_{\mathcal{CL}}$, which we denote by $\mathcal{R}(\mathrm{Cmb}_{\mathcal{CL}})$, means the set of all CL arrangements having the combinatorics $\mathrm{Cmb}_{\mathcal{CL}}$. Since all conics and lines are determined by their equations up to non-zero constants, $\mathcal{R}(\mathrm{Cmb}_{\mathcal{CL}})$ can be regarded as a subset of $\mathbb{P}^{d(d+3)/2}$, where $d = \deg B_{\mathcal{CL}}$. In [**1**], another Zariski pair was given for a CL arrangement of degree 7 and the number of connected components of its realization space was determined. This paper can be considered as a continuation of [**1**] and [**34**]. We generalize the method of studying the realization space used in [**1**] and apply it to a wider range of CL arrangements, which are given by a similar manner to those in [**1, 34**]. Namely we consider CL arrangements of degree 7 obtained by gluing two CL arrangements of degree 3 and 4 in a non-trivial manner, and we completely determine the number of connected components of the realization spaces. This gives a negative answer to the above mentioned question of [**34**, Remark 6]. We also obtain two new Zariski pairs. We now give a precise description of the CL arrangements $\mathcal{CL}_{ij}$ of degree 7 considered in this article. We assume that they have the following combinatorics:

(i) $\mathcal{CL}_{ij} = \mathcal{P}_i \bigsqcup \mathcal{A}_j$ $(i, j = 1, 2)$ where $\mathcal{P}_i$ and $\mathcal{A}_j$ are subarrangements of degree 4 and 3 respectively such that (P1) $\mathcal{P}_1 = \{C, L_1, L_2\}$, $\deg C = 2$, $\deg L_i = 1\,(i = 1, 2)$ with $C \pitchfork (L_1 + L_2)$ and $C \cap L_1 \cap L_2 = \emptyset$, (P2) $\mathcal{P}_2 = \{C_1, C_2\}$, $\deg C_i = 2\,(i = 1, 2)$ with $C_1 \pitchfork C_2$, (A1) $\mathcal{A}_1 = \{M_1, M_2, M_3\}$, non-concurrent three lines, and (A2) $\mathcal{A}_2 = \{D, M\}, \deg D = 2, \deg M = 1$ with $D \pitchfork M$. We call $\mathcal{P}_i$ a plinth for $\mathcal{CL}_{ij}$.

(ii) Let M and D be a line and a conic in $\mathcal{A}_j$, respectively. Then any point in $M \cap B_{\mathcal{P}_i}$ and $D \cap B_{\mathcal{P}_i}$ gives rise to an ordinary triple point or a tacnode of $M + B_{\mathcal{P}_i}$ and $D + B_{\mathcal{P}_i}$, respectively.

(iii) The singularities of $B_{\mathcal{CL}_{ij}}$ are at most nodes, tacnodes or ordinary triple points.

For CL arrangements as above, we have a list as follows: Here Cmb_{ijk} denotes the k-th combinatorics given by the set $\mathcal{CL}_{ij}$. Note that D and M meet at two distinct points, although $D \cap M$ does not appear in some of the real pictures below.

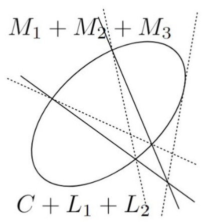

Figure 1. Cmb_{111}

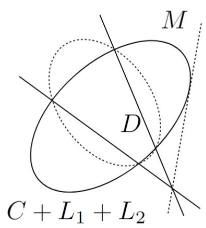

Figure 2. Cmb_{121}

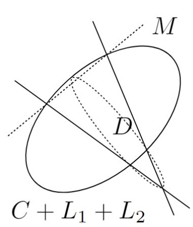

Figure 3. Cmb_{122}

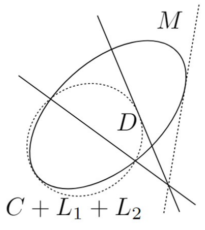

Figure 4. Cmb_{123}

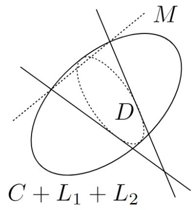

Figure 5. Cmb_{124}

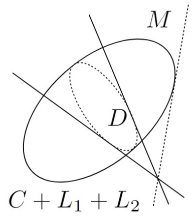

Figure 6. Cmb_{125}

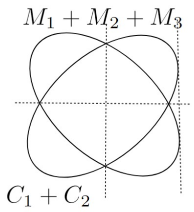

Figure 7. Cmb_{211}

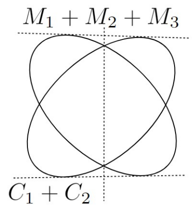

Figure 8. Cmb_{212}

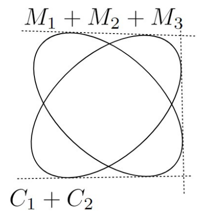

Figure 9. Cmb_{213}

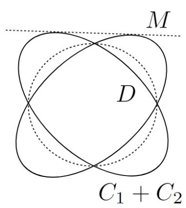

Figure 10. Cmb_{221}

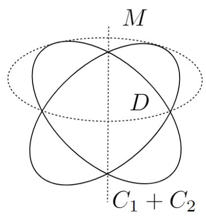

Figure 11. Cmb_{222}

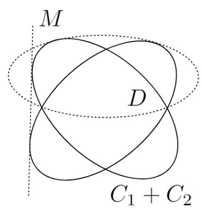

Figure 12. Cmb_{223}

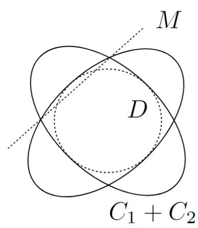

Figure 13. Cmb_{224}

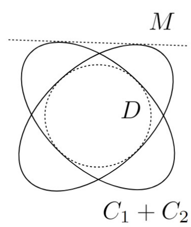

Figure 14. Cmb_{225}

Now our main statement is as follows:

Theorem 1.1. *Let* Cmb_{ijk} *be the combinatorics as in Figures 1-14. Then the following statements hold:*

(i) *The space* $\mathcal{R}(\mathrm{Cmb}_{ijk})$ *is connected for* $ijk = 111$, *121, 122, 125, 211, 213, 221, 222, 225.*

(ii) *Each* $\mathcal{R}(\mathrm{Cmb}_{ijk})$ $ijk = 123$, *124, 212, 223, 224 has exactly two connected components. Moreover, if we choose* $B_1, B_2 \in \mathcal{R}(\mathrm{Cmb}_{ijk})$ *so that* B_1 *and* B_2 *belong to distinct components,* (B_1, B_2) *is a Zariski pair.*

Remark 1.2.

- Fix a conic in $\mathcal{P}_i$. As we explain in Subsection 2.2, some of the above arrangements are canonically constructed from 5 points on the conic by using the theory of rational elliptic surfaces. This fact plays an important role to prove Theorem 1.1.
- Zariski pairs for the combinatorics Cmb_{123} and Cmb_{212} are new, while the one for the case Cmb_{223} was studied in [**1**] where the second statement above for this case, i.e. the fact that there exists a Zariski pair and that the number of connected components of the realization space is exactly two, was proved.
- Zariski pairs for the cases Cmb_{124} and Cmb_{224} were given in [**34**]. Theorem 1.1 shows that the maximum number of possible Zariski n-tuples is $n = 2$ for these cases. Hence, Theorem 1.1 disproves the existence of a Zariski triple for Cmb_{224}, which was expected in [**34**, Remark 6].

Acknowledgments: The authors thank the anonymous referee for his/her comments on the first draft of this paper.

2. Preliminaries

2.1. Some rational elliptic surface

In this article, the theory of elliptic surfaces plays an important role in both our construction of plane curves and our proof of Theorem 1.1. Our main references are [**8, 9, 17, 21, 28, 34**] and we make use of the results given there freely. Here we summarize our convention, notation and terminology. For an elliptic surface $\varphi : S \to C$ over a smooth projective curve C, we always assume the following:

(i) The fibration φ is relatively minimal.

(ii) There exists a section $O : C \to S$. Here we identify O with its image.

(iii) There exists at least one singular fiber.

Let C_o be a smooth conic. Choose five distinct points z_o, p_1, p_2, p_3 and p_4 on C_o. We denote the line passing through p_i and p_j by L_{ij}. Consider a pencil of conics $\{C_\lambda\}_{\lambda \in \Lambda}$ passing through p_1, p_2, p_3 and p_4. There exist three distinct values λ_1, λ_2 and λ_3 in Λ such that each C_{λ_i} $(i = 1, 2, 3)$ becomes two distinct lines. For each case, we denote the intersection point between these two lines by the same label p_0. Note that we are abusing notation and that we are labeling the distinct points $L_{12} \cap L_{34}$, $L_{13} \cap L_{24}$, $L_{14} \cap L_{23}$ with the same p_0. For these values, $C_o + C_{\lambda_i}$ $(i = 1, 2, 3)$ give rise to conic-line arrangements $\mathcal{P}_1$. We may assume that

$$C_{\lambda_1} = L_{12} + L_{34}, \quad C_{\lambda_2} = L_{13} + L_{24}, \quad C_{\lambda_3} = L_{14} + L_{23}.$$

For other values of λ, $C_o + C_\lambda$ gives rise to conic arrangements $\mathcal{P}_2$. Put $\mathcal{Q}_\lambda = C_o + C_\lambda$, $\lambda \in \Lambda$. Likewise we did in our previous articles [**8, 9**], we associate $(\mathcal{Q}_\lambda, z_o)$ with the rational elliptic surface $\varphi_{\mathcal{Q}_\lambda, z_o} : S_{\mathcal{Q}_\lambda, z_o} \to \mathbb{P}^1$, which comes from the double cover $f'_{\mathcal{Q}_\lambda} : S'_{\mathcal{Q}_\lambda} \to \mathbb{P}^2$ branched along $\mathcal{Q}_\lambda$. In the following, we always choose λ and z_o such that

($*$) The tangent line to C_o at z_o meets C_λ at two distinct points.

Also the diagram below is the one introduced in [**8, 9**]

$$\begin{array}{ccccc}
S'_{\mathcal{Q}_\lambda} & \xleftarrow{\ \mu\ } & S_{\mathcal{Q}_\lambda} & \xleftarrow{\ \nu_{z_o}\ } & S_{\mathcal{Q}_\lambda, z_o} \\
{\scriptstyle f'_{\mathcal{Q}_\lambda}} \downarrow & & \downarrow {\scriptstyle f_{\mathcal{Q}_\lambda}} & & \downarrow {\scriptstyle f_{\mathcal{Q}_\lambda, z_o}} \\
\mathbb{P}^2 & \xleftarrow[q]{} & \widehat{\mathbb{P}^2} & \xleftarrow[q_{z_o}]{} & (\widehat{\mathbb{P}^2})_{z_o},
\end{array}$$

where μ is the canonical resolution of singularities, q is a composition of a finite number of blowing-ups so that the branch locus becomes smooth and $f_{\mathcal{Q}_\lambda}$ is the induced double cover. The pencil of lines through z_o gives rise to a pencil Λ_{z_o} of curves of genus 1 on $S_{\mathcal{Q}_\lambda}$. We denote the resolution of indeterminacy of Λ_{z_o} by ν_{z_o} and q_{z_o} is the composition of two blowing-ups induced by ν_{z_o}. We also have an induced double cover $f_{\mathcal{Q}_\lambda, z_o}$: $S_{\mathcal{Q}_\lambda, z_o} \to (\widehat{\mathbb{P}^2})_{z_o}$. The generic fiber $E_{\mathcal{Q}_\lambda, z_o}$ can be considered as an elliptic curve over $\mathbb{C}(\mathbb{P}^1)(\simeq \mathbb{C}(t))$. It is well known that the set $E_{\mathcal{Q}_\lambda, z_o}(\mathbb{C}(t))$ of $\mathbb{C}(t)$ rational points of $E_{\mathcal{Q}_\lambda, z_o}$ can be endowed with a group structure. The induced double cover $f_{\mathcal{Q}_\lambda, z_o}$ coincides with the quotient morphism determined by the involution $[-1]$ on $S_{\mathcal{Q}_\lambda, z_o}$, which is given by the inversion with respect to the group law on $E_{\mathcal{Q}_\lambda, z_o}$. Let $\mathrm{MW}(S_{\mathcal{Q}_\lambda, z_o})$ be the set of sections of $\varphi_{\mathcal{Q}_\lambda, z_o}$. By an integral section, we mean a section s with $s \cdot O = 0$. In [**28**], Shioda defined a $\mathbb{Q}$-valued bilinear form $\langle\ ,\ \rangle$ on $E_{\mathcal{Q}_\lambda, z_o}(\mathbb{C}(t))$ called the height pairing, by which the free part of $E_{\mathcal{Q}_\lambda, z_o}(\mathbb{C}(t))$ becomes a lattice. We make use of this lattice structure in order to find elements in $\mathcal{A}_j$ $(j = 1, 2)$. When we describe $E_{\mathcal{Q}, z_o}(\mathbb{C}(t))$, we take this structure into account. It is known that there is a bijection between $\mathrm{MW}(S_{\mathcal{Q}_\lambda, z_o})$ and $E_{\mathcal{Q}_\lambda, z_o}(\mathbb{C}(t))$. For $s \in \mathrm{MW}(S_{\mathcal{Q}_\lambda, z_o})$, we denote the rational point corresponding to s by P_s and for $P \in E_{\mathcal{Q}_\lambda, z_o}(\mathbb{C}(t))$, we denote the section corresponding to P by s_P. For $P_1, P_2 \in E_{\mathcal{Q}_\lambda, z_o}(\mathbb{C}(t))$, we denote their sum by $P_1 \dot{+} P_2$. This group structure induces a group structure on $\mathrm{MW}(S_{\mathcal{Q}_\lambda, z_o})$ which we also denote by $\dot{+}$. Under the above correspondence, we have $s_{P_1 \dot{+} P_2} = s_{P_1} \dot{+} s_{P_2}$. For an integer $m \in \mathbb{Z}$, we put $[m]P = P \dot{+} \ldots \dot{+} P$ (m terms) for $m > 0$, $[0]P = O$ and $[m]P = [-m]([-1]P)$ for $m < 0$, following the notation in [**30**].

Put $\tilde{f}_{\mathcal{Q}_\lambda, z_o} := f'_{\mathcal{Q}_\lambda} \circ \mu \circ \nu_{z_o}$ and we denote the plane curve obtained as the image under $\tilde{f}_{\mathcal{Q}_\lambda, z_o}$ of a section $s \in \mathrm{MW}(S_{\mathcal{Q}_\lambda, z_o})$ by $\mathcal{C}_s := \tilde{f}_{\mathcal{Q}_\lambda, z_o}(s) \subset \mathbb{P}^2$.

Here are some properties of $\varphi_{\mathcal{Q}_\lambda, z_o} : S_{\mathcal{Q}_\lambda, z_o} \to \mathbb{P}^1$ (See [**8, 9, 25, 34**]):

<u>The Case $\lambda = \lambda_1, \lambda_2, \lambda_3$</u>

- There exist 6 singular fibers for $\varphi_{\mathcal{Q}_\lambda, z_o}$. All of them are of type I_2. They arise from the tangent line l_{z_o} at z_o and lines connecting z_o and p_i $(0 \le i \le 4)$. We denote them by F_∞ and F_i $(0 \le i \le 4)$, respectively, and their irreducible decomposition by $F_\bullet = \Theta_{\bullet,0} + \Theta_{\bullet,1}$ $\bullet = \infty, 0, 1, \ldots, 4$.
- The group $E_{S_{\mathcal{Q}_\lambda, z_o}}(\mathbb{C}(t))$ is isomorphic to $(A_1^*)^{\oplus 2} \oplus (\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$.
- In order to describe explicit generators of $E_{\mathcal{Q}_\lambda, z_o}(\mathbb{C}(t))$, we consider the case $\lambda = \lambda_1$ where $C_{\lambda_1} = L_{12} + L_{34}$. In this case, C_o and L_{ij} $(1 \le i < j \le 4)$ give rise to elements of $E_{\mathcal{Q}_\lambda, z_o}(\mathbb{C}(t))$ as follows:
  - (i) C_o, L_{12} and L_{34} give rise to 2-torsions, which we denote by P_{C_o}, P_{12} and P_{34}, respectively. Note that $P_{C_o} = [-1]P_{C_o}$, $P_{12} = [-1]P_{12}$ and $P_{34} = [-1]P_{34}$.
  - (ii) For each $(i,j) \in \{(1,3), (1,4), (2,3), (2,4)\}$, L_{ij} gives rise to two points $P_{ij}, [-1]P_{ij} \in E_{\mathcal{Q}_\lambda, z_o}(\mathbb{C}(t))$. We denote them by $[\pm 1]P_{ij}$, for abbreviation.
  - (iii) We may assume that the free part of $E_{\mathcal{Q}_\lambda, z_o}(\mathbb{C}(t))$ is generated by P_{13} and P_{14}, i.e.,

$$(A_1^*)^{\oplus 2} \cong \mathbb{Z}P_{13} \oplus \mathbb{Z}P_{14}$$

    and $P_{23} = P_{14} \dot{+} P_{C_o}, P_{24} = P_{13} \dot{+} P_{C_o}$.
  - (iv) For each $(i,j) \in \{(1,3), (1,4), (2,3), (2,4)\}$, $\mathcal{C}_{[2]P_{ij}} = \mathcal{C}_{[-2]P_{ij}}$ is a conic inscribed in $\mathcal{Q}_{\lambda_1}$ such that $z_o \in \mathcal{Q}_{\lambda_1} \cap \mathcal{C}_{[2]P_{ij}}$.

The Case $\lambda \neq \lambda_1, \lambda_2, \lambda_3$

- There exist 5 reducible singular fibers. All of them are of types either I_2 or III. They arise from the tangent line l_{z_o} at z_o and lines through z_o and p_i $(1 \le i \le 4)$. We denote them by F_∞ and F_i $(1 \le i \le 4)$, respectively, and their irreducible decomposition by $F_\bullet = \Theta_{\bullet,0} + \Theta_{\bullet,1}$ $\bullet = \infty, 1, \ldots, 4$.
- The group $E_{S_{\mathcal{Q}, z_o}}(\mathbb{C}(t))$ is isomorphic to $(A_1^*)^{\oplus 3} \oplus \mathbb{Z}/2\mathbb{Z}$. The unique 2-torsion point arises from C_o, which we denote by P_{C_o}.
- Each L_{ij} gives two elements in $E_{\mathcal{Q}_\lambda, z_o}$ and we denote them by $[\pm 1]P_{ij}$, which satisfy the following properties:
  - (i) Since $\langle P_{1j}, P_{1j} \rangle = 1/2$ $(2 \le j \le 4)$, $\langle P_{1j}, P_{1k} \rangle = 0$ $(2 \le j < k \le 4)$, we may assume

$$(A_1^*)^{\oplus 3} \cong \mathbb{Z}P_{12} \oplus \mathbb{Z}P_{13} \oplus \mathbb{Z}P_{23}$$

    and $P_{ij} \dot{+} P_{C_o} = P_{kl}$, where $\{i,j,k,l\} = \{1,2,3,4\}$.
  - (ii) $\mathcal{C}_{[2]P_{ij}} = \mathcal{C}_{[-2]P_{ij}}$ is a conic inscribed in $\mathcal{Q}_\lambda$ such that $z_o \in \mathcal{Q}_\lambda \cap \mathcal{C}_{[2]P_{ij}}$.

2.2. Construction of lines and conics in $\mathcal{A}_j$ $(j = 1, 2)$ via $S_{\mathcal{Q}_\lambda, z_o}$

Here we explain our method in constructing lines and conics in $\mathcal{A}_j$ $(j = 1, 2)$. This method plays a crucial role to consider a member of $\mathcal{R}(\mathrm{Cmb}_{ijk})$. Choose $P \in E_{\mathcal{Q}_\lambda, z_o}$ and let s_P be the corresponding section. In [**20**], Masuya introduced a *line point* as follows:

DEFINITION 2.1. P is said to be a line-point if $\tilde{f}_{\mathcal{Q}_\lambda, z_o}(s_P)$ is a line. Also a section $s \in \mathrm{MW}(S_{\mathcal{Q}_\lambda, z_o})$ is said to be a line-section if $\tilde{f}_{\mathcal{Q}_\lambda, z_o}(s)$ is a line.

Any line-point is characterized by the following lemma:

LEMMA 2.2 ([**10**, Lemma 9]). *Let $s \in \mathrm{MW}(S_{\mathcal{Q}_\lambda, z_o})$ be an integral section with $s \cdot \Theta_{\infty,1} = 1$. Then $\tilde{f}_{\mathcal{Q}_\lambda, z_o}(s)$ is a line L_s such that*

(i) *the intersection multiplicity at every intersection point between L_s and $\mathcal{Q}_\lambda$ is even,*

(ii) $z_o \notin L_s$.

Conversely, any line satisfying the above two conditions gives rise to two sections $s_{L^\pm}$ such that $s_{L^\pm} \cdot O = 0$ and $s_{L^\pm} \cdot \Theta_{\infty,1} = 1$.

For an integral section s with $s \cdot \Theta_{\infty,0} = 1$, we have the following lemma:

LEMMA 2.3 ([**20**, Lemma 2.12]). *Let $s \in \mathrm{MW}(S_{\mathcal{Q}_\lambda, z_o})$ be an integral section with $s \cdot \Theta_{\infty,0} = 1$. Then $\tilde{f}_{\mathcal{Q}_\lambda, z_o}(s)$ is a smooth conic satisfying either*

(i) *$\tilde{f}_{\mathcal{Q}_\lambda, z_o}(s)$ is the irreducible component of $\mathcal{Q}_\lambda$ through z_o, or*

(ii) *$\tilde{f}_{\mathcal{Q}_\lambda, z_o}(s)$ is tangent to $\mathcal{Q}_\lambda$ at z_o and the intersection multiplicity at every intersection point between $\tilde{f}_{\mathcal{Q}_\lambda, z_o}(s)$ and $\mathcal{Q}_\lambda$ is even.*

Conversely, any conic C that satisfies (i) *gives rise to a 2-torsion section s_C and that satisfies* (ii) *gives rise to two integral sections $s_{C^\pm}$ such that $s_\bullet \cdot \Theta_{\infty,0} = 1$ $(\bullet = C, C^\pm)$.*

REMARK 2.4. Note that although the "converse" part of Lemma 2.3 was not given in [**20**], it follows from our construction of $S_{\mathcal{Q}_\lambda, z_o}$. By Lemmas 2.2 and 2.3, we see that lines and conics in $\mathcal{A}_j$ $(j = 1, 2)$ are canonically obtained from sections described as above and vice versa. We make use of this construction to obtain desired CL arrangements.

3. Approach to construct CL arrangements with prescribed Cmb_{ijk}

In this section, we give rough ideas for the explicit construction of plane curves with prescribed Cmb_{ijk}. We first give a combinatorial classification of lines and a conic in $\mathcal{A}_j$ $(j = 1, 2)$.

<u>The case of $\mathcal{P}_1$:</u> We may assume $L_{12}, L_{34} \in \mathcal{P}_1$ and put $L_1 = L_{12}, L_2 = L_{34}$.

Let M be a line in $\mathcal{A}_j$ $(j = 1, 2)$. By our assumption (ii) for Cmb_{ijk} given in the Introduction, we infer that M is $L_{13}, L_{14}, L_{23}, L_{24}$ or lines L_0, L_0' through p_0 and tangent to C.

Let D be the smooth conic in $\mathcal{A}_2$. Again by our assumption (ii) for Cmb_{ijk}, we infer that D is a conic of one of the following types $D(1, j)$ $(j = 0, 1, 2, 4)$:

(a) $D(1,0)$ passes through p_1, p_2, p_3 and p_4.

(b) $D(1,1)$ passes through p_0, p_i, p_j $(i \in \{1,2\}, j \in \{3,4\})$ and is tangent to C.

(c) $D(1,2)$ passes through p_1 and p_2 (resp. p_3 and p_4) and is tangent to L_{34} (resp. L_{12}) and C.

(d) $D(1,4)$ is a conic inscribed by $B_{\mathcal{P}_1}$.

The case of $\mathcal{P}_2$: Let M be a line in $\mathcal{A}_j$ $(j = 1,2)$. By our assumption (ii) for Cmb_{ijk} given in the Introduction, we infer that M is a bitangent line L_b to $B_{\mathcal{P}_2}$ or L_{ij} $(1 \le i < j \le 4)$. Note that there exist four bitangent lines to $B_{\mathcal{P}_2}$.

Let D be the conic in $\mathcal{A}_2$. By our assumption (ii) for Cmb_{ijk}, we infer that D is a conic of one of the following types $D(2,j)$ $(j = 0,2,4)$:

(a) $D(2,0)$ passes through p_1, p_2, p_3 and p_4.

(b) $D(2,2)$ passes through p_i, p_j and is tangent to both C_1 and C_2.

(c) $D(2,4)$ is tangent to $B_{\mathcal{P}_2}$ at 4 distinct points.

By Remark 2.4, all of the above lines and conics are characterized by rational points of $E_{\mathcal{Q}_\lambda, z_o}$. Let s be a section in $\mathrm{MW}(S_{\mathcal{Q}_\lambda, z_o})$ and let P_s be the corresponding point in $E_{\mathcal{Q}_\lambda, z_o}(\mathbb{C}(t))$. Recall that we describe the group law of $\mathbb{C}(t)$-rational points as $\dot{\pm}$, and define $[2]P_s := P_s \dot{+} P_s$. If we choose sections as in Lemma 2.2 and Lemma 2.3, then we have the table below:

Line or conic	Points in $E_{\mathcal{Q}_\lambda, z_o}(\mathbb{C}(t))$
L_0	$P_{13} \dot{\pm} P_{14} \dot{+} P_{34}$
$D(1,1)$	$P_{13} \dot{\pm} P_{12}$, $P_{13} \dot{\pm} P_{34}$, $P_{14} \dot{\pm} P_{12}$, $P_{14} \dot{\pm} P_{34}$
$D(1,2)$	$P_{13} \dot{\pm} P_{14}$, $P_{13} \dot{\pm} P_{23}$
$D(1,4)$	$[2]P_{13}$, $[2]P_{14}$
L_b	$P_{12} \dot{\pm} P_{13} \dot{\pm} P_{23}$
$D(2,2)$	$P_{12} \dot{\pm} P_{23}$, $P_{12} \dot{\pm} P_{13}$, $P_{13} \dot{\pm} P_{23}$, $P_{12} \dot{\pm} P_{14}$, $P_{13} \dot{\pm} P_{14}$, $P_{12} \dot{\pm} P_{24}$
$D(2,4)$	$[2]P_{12}$, $[2]P_{13}$, $[2]P_{23}$

Table 1. Line points, conic points and their corresponding curves.

Remark 3.1.

- As $\tilde{f}_{\mathcal{Q}_\lambda, z_o}(s_P) = \tilde{f}_{\mathcal{Q}_\lambda, z_o}(s_{[-1]P})$, we only give one of the corresponding two points.
- Since $P_{ij} = P_{kl} \dot{+} P_{C_o}$, relations $P_{12} \dot{\pm} P_{23} = P_{34} \dot{\pm} P_{14}$, $P_{12} \dot{\pm} P_{14} = P_{34} \dot{\pm} P_{23}$, etc hold. This means that there are several rational points that give a conic of type $D(2,2)$.

We next explain our setting about an explicit Weierstrass equation for $E_{\mathcal{Q}_\lambda, z_o}$, which gives an affine equation of $S_{\mathcal{Q}_\lambda, z_o}$. This setting plays an important role to prove Theorem 1.1. Let C_o and z_o, $p_1, \ldots, p_4$ be the smooth conic and distinct 5 points on it as in Subsection 2.1. We take homogeneous coordinates $[T, X, Z]$ of $\mathbb{P}^2$ such that C_o is given by $XZ - T^2 = 0$ and $z_o = [0, 1, 0]$. Note that l_{z_o} is given by $Z = 0$. Let (t, x), $t = T/Z, x = X/Z$ be affine coordinates of $\mathbb{P}^2 \setminus l_{z_o}$. Put $p_i := (t_i, t_i^2)$, $t_i \in \mathbb{C}$ $(i = 1, 2, 3, 4)$. Let $\boldsymbol{t} = (t_1, t_2, t_3, t_4)$ and $\lambda \in \mathbb{C}$. We define $\mathcal{M}$ as follows:

$$\mathcal{M} := \{\boldsymbol{\tau} = (\lambda, \boldsymbol{t}) \in \mathbb{C} \times \mathbb{C}^4 \mid t_i \neq t_j \ (i \neq j)\}.$$

Under these settings, we consider the pencil of conics passing through p_1, p_2, p_3, p_4 and denote it by $\Lambda_{\boldsymbol{t}}$. A general member $C_{\boldsymbol{\tau}}$ of this pencil is given by

$$C_{\boldsymbol{\tau}} : c_{\boldsymbol{\tau}}(t, x) := \lambda(x - t^2) + (x - (t_1 + t_2)t + t_1 t_2)(x - (t_3 + t_4)t + t_3 t_4), \quad (\lambda, \boldsymbol{t}) \in \mathcal{M}.$$

Let $\mathcal{Q}_\tau = C_o + C_{\boldsymbol{\tau}}$. With the above equation, $E_{\mathcal{Q}_{\boldsymbol{\tau}}, z_o}$ is given by the Weierstrass equation:

$$E_{\mathcal{Q}_{\boldsymbol{\tau}}, z_o} : y^2 = f_{\boldsymbol{\tau}}(t, x), \quad f_{\boldsymbol{\tau}}(t, x) = (x - t^2) c_{\boldsymbol{\tau}}(t, x)$$

Note that $\mathcal{Q}_{\boldsymbol{\tau}}$ and explicit generators of $E_{\mathcal{Q}_{\boldsymbol{\tau}}, z_o}(\mathbb{C}(t))$ are determined by C_o and $z_o, p_1, \ldots, p_4$ by Subsection 2.1.

Remark 3.2. Let A_{c_τ} be a symmetric matrix of the quardratic form corresponding to $c_{\boldsymbol{\tau}}(t, x)$. The conic $C_{\boldsymbol{\tau}}$ becomes two lines if and only if $\det A_{c_\tau} = 0$. It means that $C_{\boldsymbol{\tau}}$ consists of two lines precisely when $\lambda = 0, -(t_1 - t_4)(t_2 - t_3), -(t_1 - t_3)(t_2 - t_4)$.

Now let us explain how we construct CL arrangements with Cmb_{ijk}. First we may assume that quartics $B_{\mathcal{P}_1}, B_{\mathcal{P}_2}$ are given by $\mathcal{Q}_{\boldsymbol{\tau}}$ defined by the equation of the form $f_{\boldsymbol{\tau}}(t, x) = 0$, $\boldsymbol{\tau} \in \mathcal{M}$. Also we keep our notation for lines and conics in $\mathcal{A}_j$ $(j = 1, 2)$ as in the beginning of this section.

The quartics having $\mathrm{Cmb}_{B_{\mathcal{P}_1}}$ are given by $\lambda_1 = 0$, $\lambda_2 = -(t_1 - t_4)(t_2 - t_3)$ and $\lambda_3 = -(t_1 - t_3)(t_2 - t_4)$ for a fixed $\boldsymbol{t}$ where we have $\mathcal{P}_1 = \{C_o, L_{12}, L_{34}\}, \{C_o, L_{13}, L_{24}\}$ and $\{C_o, L_{14}, L_{23}\}$, respectively. On the other hand, once we choose one of the three values $\lambda_1, \lambda_2, \lambda_3$ and fix it, by interchanging the coordinates of $\boldsymbol{t}$ continuously, the pair of lines $\{L_{ij}, L_{kl}\}$ $i < j, k < l, \{i, j, k, l\} = \{1, 2, 3, 4\}$ are also continuously interchanged. Hence we may assume that $\mathcal{P}_1$ is given by $\lambda_1 = 0$ and $B_{\mathcal{P}_1} = C_o + L_{12} + L_{34}$. Here are some more remarks:

- $B_{\mathcal{P}_1}$ is determined by a 2-partition of $\{1, 2, 3, 4\}$. Since p_0 is determined by this 2-partition, two tangent lines to C_o that pass through p_0 are also canonically determined by $\boldsymbol{t}$.
- Fix $\boldsymbol{t}$. Then, smooth conics of type $D(1, 0), D(2, 0)$ which pass through p_1, p_2, p_3 and p_4 are given by C_λ for some λ.

In order to describe $\mathcal{R}(\mathrm{Cmb}_{ijk})$, we define four disjoint subsets, $\mathcal{M}_0$ and $\mathcal{M}_i$ $(i = 1, 2, 3)$, of $\mathcal{M}$ as follows:

$$\begin{aligned} \mathcal{M}_0 &:= \{\boldsymbol{\tau} = (\lambda, \boldsymbol{t}) \in \mathcal{M} \mid \lambda \neq \lambda_1, \lambda_2, \lambda_3\}, \\ \mathcal{M}_i &:= \{\boldsymbol{\tau} = (\lambda, \boldsymbol{t}) \in \mathcal{M} \mid \lambda = \lambda_i\} \quad (i = 1, 2, 3). \end{aligned}$$

Note that each $\mathcal{M}_i$ $(i = 0, 1, 2, 3)$ is path connected.

In the following, we explain how we construct conic-line arrangements with Cmb_{ijk}. For $\mathcal{P}_1$, we may assume that $\mathcal{P}_1 = \{C_o, L_{12}, L_{34}\}$ for some fixed $\boldsymbol{\tau} = (0, \boldsymbol{t}) \in \mathcal{M}_1$.

$\underline{\mathrm{Cmb}_{111}}$: In this case, $\mathcal{A}_1$ is one of the following

$$\{L_{13}, L_{24}, L_0\}, \{L_{13}, L_{24}, L_0'\}, \{L_{14}, L_{23}, L_0\}, \{L_{14}, L_{23}, L_0'\}$$

$\underline{\mathrm{Cmb}_{121}}$: Any conic of type $D(1,0)$ is given by $c_{\boldsymbol{\tau}'}(t,x) = 0$ for some $\boldsymbol{\tau}' = (\lambda', \boldsymbol{t}) \in \mathcal{M}_0$, which we denote by D. Hence we may assume $\mathcal{A}_2$ is given by $\{L_0, D\}$ or $\{L_0', D\}$.

$\underline{\mathrm{Cmb}_{122}}$: The conic in $\mathcal{A}_2$ is of type $D(1,1)$. We denote a conic of type $D(1,1)$ passing through p_i, p_j by D_{ij}. Hence $\mathcal{A}_2$ is one of the following:

$$\{L_{13}, D_{24}\}, \{L_{14}, D_{23}\}, \{L_{23}, D_{14}\}, \{L_{24}, D_{13}\}.$$

We may assume that D_{ij} is tangent to C_o at z_o. Since D_{ij} is a conic through p_0, p_i, p_j that is tangent to C_o at z_o, it is uniquely determined. Hence any CL arrangement with Cmb_{122} is determined by p_1, p_2, p_3, p_4 and z_o.

$\underline{\mathrm{Cmb}_{123}}$: The conic in $\mathcal{A}_2$ is of type $D(1,2)$. We denote a conic of type $D(1,2)$ passing through p_i, p_j by D_{ij}. We may assume that D_{ij} is tangent to C_o at z_o. Hence we see that $\mathcal{A}_2$ is one of the following:

$$\{L_0, D_{12}\}, \{L_0', D_{12}\}, \{L_0, D_{34}\}, \{L_0', D_{34}\}.$$

Note that L_0, L_0', and D_{ij} are obtained from sections of $S_{\mathcal{Q}_{\boldsymbol{\tau}}, z_o}$ as in Table 1. Hence, every CL arrangement with Cmb_{123} is determined by p_1, p_2, p_3, p_4 and z_o.

$\underline{\mathrm{Cmb}_{124}}$: The conic in $\mathcal{A}_2$ is of type $D(1,4)$, which we denote by D. We may assume that D is tangent to C_o at z_o. Hence we see that $\mathcal{A}_2$ is one of the following:

$$\{L_{13}, D\}, \{L_{14}, D\}, \{L_{23}, D\}, \{L_{24}, D\}.$$

Note that L_{ij}, D as above are obtained from sections of $S_{\mathcal{Q}_{\boldsymbol{\tau}}, z_o}$ as in Table 1. Hence, every CL arrangement with Cmb_{124} is determined by p_1, p_2, p_3, p_4 and z_o.

$\underline{\mathrm{Cmb}_{125}}$: The conic in $\mathcal{A}_2$ is of type $D(1,4)$, which we denote by D. We may assume that D is tangent to C_o at z_o. Hence we see that $\mathcal{A}_2 = \{L_0, D\}, \{L_0', D\}$. Note that L_0, L_0', D are obtained from sections of $S_{\mathcal{Q}_{\boldsymbol{\tau}}, z_o}$ as in the table in this section. Hence, every CL arrangement with Cmb_{125} is determined by p_1, p_2, p_3, p_4 and z_o.

For $\mathcal{P}_2$, we may assume that $\mathcal{P}_2 = \{C_o, C_{\boldsymbol{\tau}}\}$ for some fixed $\boldsymbol{\tau} \in \mathcal{M}_0$. From Table 1, we see that there exist four bitangent lines for $B_{\mathcal{P}_2}$, which we denote by L_{b_i} $(1 \le i \le 4)$. Table 1 shows that all bitangent lines of $B_{\mathcal{P}_2}$ are determined by p_1, p_2, p_3, p_4 and are canonically constructed if we choose z_o.

$\underline{\mathrm{Cmb}_{211}}$: $\mathcal{A}_1$ consists of three lines as follows: $\{L_{12}, L_{34}, L_b\}$, $\{L_{13}, L_{24}, L_b\}$, $\{L_{14}, L_{23}, L_b\}$, where there are four possibilities for L_b. By Table 1, every bitangent line is given by three line points. Hence, every CL arrangement with Cmb_{211} is determined by $\boldsymbol{\tau}$.

$\underline{\mathrm{Cmb}_{212}}$: $\mathcal{A}_1$ consists of a pair of four bitangent lines and L_{ij}. There are 36 possibilities for such collections. Yet, likewise Cmb_{211}, every CL arrangement with Cmb_{212} is determined by $\boldsymbol{\tau}$.

Cmb$_{213}$: $\mathcal{A}_1$ consists of three of four bitangent lines. Likewise Cmb$_{211}$, every CL arrangement with Cmb$_{213}$ is determined by $\boldsymbol{\tau}$.

Cmb$_{221}$: $\mathcal{A}_2$ consists of a bitangent line and a smooth conic of type $D(2,0)$. Every CL arrangement with Cmb$_{221}$ is determined by $\boldsymbol{\tau} = (\lambda, \boldsymbol{t}), \boldsymbol{\tau}' = (\lambda', \boldsymbol{t})$ having the same $\boldsymbol{t}$ but $\lambda \neq \lambda'$ that give two smooth conics $c_{\boldsymbol{\tau}}(t,x), c_{\boldsymbol{\tau}'}(t,x)$ in the pencil $\Lambda_{\boldsymbol{t}}$.

Cmb$_{222}$: $\mathcal{A}_2$ consists of a line L_{ij} and a conic of type $D(2,2)$. We denote a conic of type $D(2,2)$ passing through p_i, p_j by D_{ij}. Then we infer that $\mathcal{A}_2$ is of the form $\{L_{ij}, D_{kl}\}$, $\{i,j,k,l\} = \{1,2,3,4\}$. We may assume that D_{kl} is tangent to C_o at z_o. By Table 1, every CL arrangement with Cmb$_{222}$ is determined by $\boldsymbol{\tau}$ and z_o with an appropriate choice of P_{ij}'s.

Cmb$_{223}$: $\mathcal{A}_2$ consists of a bitangent line L_b and a conic of type $D(2,2)$. We denote a conic of type $D(2,2)$ passing through p_i, p_j by D_{ij}. Then we infer that $\mathcal{A}_2$ is of the form $\{L_b, D_{ij}\}$. We may assume that D_{ij} is tangent to C_o at z_o. By Table 1, every CL arrangement with Cmb$_{223}$ is determined by $\boldsymbol{\tau}$ and z_o with an appropriate choice of P_{ij}'s.

Cmb$_{224}$: $\mathcal{A}_2$ consists of a line L_{ij} and a conic D of type $D(2,4)$, which we denote by D. We may assume that D is tangent to C_o at z_o. By Table 1, every CL arrangement with Cmb$_{224}$ is determined by $\boldsymbol{\tau}$ and z_o with an appropriate choice of P_{ij}'s.

Cmb$_{225}$: $\mathcal{A}_2$ consists of a bitangent line L_b and a conic of type $D(2,4)$, which we denote by D. We may assume that D is tangent to C_o at z_o. By Table 1, every CL arrangement with Cmb$_{225}$ is determined by $\boldsymbol{\tau}$ and z_o with an appropriate choice of P_{ij}'s.

4. Proof of Theorem 1.1

4.1. Our strategy

Let us explain our strategy to prove Theorem 1.1. Our approach is similar to that we take in [**1**, Section 3]. As we see in Section 3, we first choose homogeneous coordinates $[T, X, Z]$ so that $z_o = [0,1,0]$ and C_o is given by $XZ - T^2 = 0$. Next we choose $\boldsymbol{\tau}$. Then we are able to construct CL arrangements with Cmb$_{ijk}$ in a canonical way via sections of $S_{\mathcal{Q}_\lambda, z_o}$ except those involving conics of type $D(1,0)$ and $D(2,0)$. Our basic idea to prove Theorem 1.1 is to connect two CL arrangements with fixed Cmb$_{ijk}$ by moving $\boldsymbol{\tau}$, which is done in [**1**, Lemma 3.1, Remark 3.2, Corollary 3.3]. Let us explain it more precisely. Let $\mathcal{M}$ and $\mathcal{M}_i$ $(i = 0,1,2,3)$ be as defined in Section 3. We first choose z_o. The deformation of the CL arrangements can be described by moving $\boldsymbol{\tau} = (\lambda, \boldsymbol{t})$ in $\mathcal{M}_i$ which can be described by giving a continuous path in $\mathcal{M}_i$ as in Section 3. We consider $\gamma : [0,1] \to \mathcal{M}_i$, $s \mapsto \gamma(s) = (\lambda(s), (t_1(s), t_2(s), t_3(s), t_4(s)))$ as such a path. Let $\boldsymbol{\tau}_o = (\lambda_o, (a_1, a_2, a_3, a_4)), \boldsymbol{\tau}'_o = (\lambda'_o, (a'_1, a'_2, a'_3, a'_4)) \in \mathcal{M}_i$, $\gamma(0) = \boldsymbol{\tau}_o$ and $\gamma(1) = \boldsymbol{\tau}'_o$. If $t_i(0) = a_i$ and $t_i(1) = a'_j$, we say "a_i goes to a'_j along a path in $\mathcal{M}_i$" and denote it by $a_i \rightsquigarrow a'_j$. Note again that each $\mathcal{M}_i$ $(i = 0,1,2,3)$ is path connected. In our proof of Theorem 1.1, we exploit $\mathcal{M}_1$ to describe connected components of $\mathcal{R}(\mathrm{Cmb}_{ijk})$ for $ijk = 111, 122, 123, 124, 125$, while we exploit $\mathcal{M}_0$ to describe connected components of $\mathcal{R}(\mathrm{Cmb}_{ijk}))$ $ijk = 121, 211, 212, 213, 224$. In this section, we keep the same notation for lines and conics as those given in Section 3. Now we prove Theorem 1.1 based on case-by-case arguments.

4.2. $\mathbf{Cmb_{111}}$

Let C_o be the conic as before and choose $\boldsymbol{\tau} = (0, \boldsymbol{t}) \in \mathcal{M}_1$. We put two elements $B_{\boldsymbol{\tau}}, B'_{\boldsymbol{\tau}} \in \mathcal{R}(\mathrm{Cmb}_{111})$ as follows:

$$\begin{aligned} \mathcal{Q}_{\boldsymbol{\tau}} &= C_o + L_{12,\boldsymbol{\tau}} + L_{34,\boldsymbol{\tau}} \\ B_{\boldsymbol{\tau}} &:= \mathcal{Q}_{\boldsymbol{\tau}} + L_{i_1 j_1,\boldsymbol{\tau}} + L_{i_2 j_2,\boldsymbol{\tau}} + L_{0,\boldsymbol{\tau}}, \\ B'_{\boldsymbol{\tau}} &:= \mathcal{Q}_{\boldsymbol{\tau}} + L_{i_1 j_1,\boldsymbol{\tau}} + L_{i_2 j_2,\boldsymbol{\tau}} + L'_{0,\boldsymbol{\tau}}. \end{aligned}$$

Note that once we choose $L_{i_1 j_1,\boldsymbol{\tau}}$, the second line $L_{i_2 j_2,\boldsymbol{\tau}}$ is automatically determined.

Choose $\boldsymbol{a} = (-2, -1, 1, 2)$, $\boldsymbol{\tau}_o = (0, \boldsymbol{a}) \in \mathcal{M}_1$. In this case, $L_{12,\boldsymbol{\tau}_o} : x + 3t + 2 = 0$, $L_{34,\boldsymbol{\tau}_o} : x - 3t + 2 = 0$, $L_{13,\boldsymbol{\tau}_o} : x + t - 2 = 0$, $L_{24,\boldsymbol{\tau}_o} : x - t - 2 = 0$ and $p_0 = (0, -2)$. As for $L_{0,\boldsymbol{\tau}_o}$ and $L'_{0,\boldsymbol{\tau}_o}$, we have

$$L_{0,\boldsymbol{\tau}_o} : x + 2\sqrt{2}t + 2 = 0, \quad L'_{0,\boldsymbol{\tau}_o} : x - 2\sqrt{2}t + 2 = 0.$$

Define $B_{\boldsymbol{\tau}_o}, B'_{\boldsymbol{\tau}_o} \in \mathrm{Cmb}_{111}$ to be

$$\begin{aligned} B_{\boldsymbol{\tau}_o} &:= \mathcal{Q}_{\boldsymbol{\tau}_o} + L_{13,\boldsymbol{\tau}_o} + L_{24,\boldsymbol{\tau}_o} + L_{0,\boldsymbol{\tau}_o}, \\ B'_{\boldsymbol{\tau}_o} &:= \mathcal{Q}_{\boldsymbol{\tau}_o} + L_{13,\boldsymbol{\tau}_o} + L_{24,\boldsymbol{\tau}_o} + L'_{0,\boldsymbol{\tau}_o}. \end{aligned}$$

Note that $B'_{\boldsymbol{\tau}_o}$ is transformed to $B_{\boldsymbol{\tau}_o}$ by $(t, x) \mapsto (-t, x)$. Now choose $B \in \mathcal{R}(\mathrm{Cmb}_{111})$ arbitrarily. By taking suitable coordinates of $\mathbb{P}^2$ so that the conic in B is given by C_o as before, we may assume that B is realized as $B_{\boldsymbol{\tau}}$ for some $\boldsymbol{\tau} \in \mathcal{M}_1$. Consider a path $\gamma : [0, 1] \to \mathcal{M}_1$ so that (i) $\gamma(0) = \boldsymbol{\tau}, \gamma(1) = \boldsymbol{\tau}_o$, (ii) $B_{\gamma(s)} \in \mathcal{R}(\mathrm{Cmb}_{111})$ for $\forall s \in [0, 1]$ and (iii) $t_{i_1} \rightsquigarrow -2$, $t_{i_2} \rightsquigarrow -1$, $t_{j_1} \rightsquigarrow 1$ and $t_{j_2} \rightsquigarrow 2$. Then $B = B_{\boldsymbol{\tau}}$ is deformed to $B_{\boldsymbol{\tau}_o}$ or $B'_{\boldsymbol{\tau}_o}$. As we remark above, $B_{\boldsymbol{\tau}_o}$ is transformed to $B'_{\boldsymbol{\tau}_o}$. This shows that B is continuously deformed to $B_{\boldsymbol{\tau}_o}$ in $\mathcal{R}(\mathrm{Cmb}_{111})$. Hence $\mathcal{R}(\mathrm{Cmb}_{111})$ is connected.

4.3. $\mathbf{Cmb_{121}}$

For $\boldsymbol{\tau} = (\lambda, \boldsymbol{t}) \in \mathcal{M}_0$, we define two elements $B_{\boldsymbol{\tau}}, B'_{\boldsymbol{\tau}}$ in $\mathcal{R}(\mathrm{Cmb}_{121})$ by

$$B_{\boldsymbol{\tau}} := C_o + L_{12,\boldsymbol{\tau}} + L_{34,\boldsymbol{\tau}} + D_{\boldsymbol{\tau}} + L_{0,\boldsymbol{\tau}}, \ B'_{\boldsymbol{\tau}} := C_o + L_{12,\boldsymbol{\tau}} + L_{34,\boldsymbol{\tau}} + D_{\boldsymbol{\tau}} + L'_{0,\boldsymbol{\tau}}.$$

Here $D_{\boldsymbol{\tau}}$ is the conic given by $c_{\boldsymbol{\tau}}(t, x) = 0$. Put $\boldsymbol{a} = (-2, -1, 1, 2)$ and choose $\boldsymbol{\tau}_o = (\lambda_o, \boldsymbol{a}) \in \mathcal{M}_0$ so that both of

$$B_{\boldsymbol{\tau}_o} = C_o + L_{12,\boldsymbol{\tau}_o} + L_{34,\boldsymbol{\tau}_o} + D_{\boldsymbol{\tau}_o} + L_{0,\boldsymbol{\tau}_o}, \ B'_{\boldsymbol{\tau}_o} = C_o + L_{12,\boldsymbol{\tau}_o} + L_{34,\boldsymbol{\tau}_o} + D_{\boldsymbol{\tau}_o} + L'_{0,\boldsymbol{\tau}_o}$$

are in $\mathcal{R}(\mathrm{Cmb}_{121})$. Now choose $B \in \mathcal{R}(\mathrm{Cmb}_{121})$ arbitrarily. By taking suitable coordinates of $\mathbb{P}^2$ so that the conic in B is given by C_o as before, we may assume that B is realized as $B_{\boldsymbol{\tau}}$ for some $\boldsymbol{\tau} \in \mathcal{M}_0$. Consider a path $\gamma : [0, 1] \to \mathcal{M}_0$ so that (i) $\gamma(0) = \boldsymbol{\tau}, \gamma(1) = \boldsymbol{\tau}_o$, (ii) $B_{\gamma(s)} \in \mathcal{R}(\mathrm{Cmb}_{121})$ for $\forall s \in [0, 1]$ and (iii) $t_1 \rightsquigarrow -2$, $t_2 \rightsquigarrow -1$, $t_3 \rightsquigarrow 1$ and $t_4 \rightsquigarrow 2$. Then $B = B_{\boldsymbol{\tau}}$ is deformed to $B_{\boldsymbol{\tau}_o}$ or $B'_{\boldsymbol{\tau}_o}$. As $D_{\boldsymbol{\tau}_o}$ is invariant under $(t, x) \mapsto (-t, x)$ for any λ_o, $B_{\boldsymbol{\tau}_o}$ is transformed to $B'_{\boldsymbol{\tau}_o}$. This shows that B is continuously deformed to $B_{\boldsymbol{\tau}_o}$ in $\mathcal{R}(\mathrm{Cmb}_{121})$. Hence $\mathcal{R}(\mathrm{Cmb}_{121})$ is connected.

4.4. Cmb_{122}

Let B be an arbitrary element in $\mathcal{R}(\mathrm{Cmb}_{122})$. By choosing coordinates of $\mathbb{P}^2$ so that the conic in $\mathcal{P}_1$ is given by C_o and D is tangent to C_o at z_o, we may assume that B is deformed to an element in $\mathcal{R}(\mathrm{Cmb}_{122})$ of the form

$$B_{\boldsymbol{\tau}} = C_o + L_{12,\boldsymbol{\tau}} + L_{34,\boldsymbol{\tau}} + L_{i_1j_1,\boldsymbol{\tau}} + D_{i_2j_2,\boldsymbol{\tau}},$$

for some $\boldsymbol{\tau} \in \mathcal{M}_1$, where $\{i_1, i_2\} = \{1, 2\}, \{j_1, j_2\} = \{3, 4\}$. Take $\boldsymbol{a} = (-2, -1, 1, 2)$, $\boldsymbol{\tau}_o = (0, \boldsymbol{a}) \in \mathcal{M}_1$ and consider

$$B_{\boldsymbol{\tau}_o} = C_o + L_{12,\boldsymbol{\tau}_o} + L_{34,\boldsymbol{\tau}_o} + L_{14,\boldsymbol{\tau}_o} + D_{23,\boldsymbol{\tau}_o},$$

where $D_{23,\boldsymbol{\tau}_o}$ is given by $x - 3t^2 + 2 = 0$. Now consider a path $\gamma : [0, 1] \to \mathcal{M}_1$ so that (i) $\gamma(0) = \boldsymbol{\tau}, \gamma(1) = \boldsymbol{\tau}_o$, (ii) $B_{\gamma(s)} \in \mathcal{R}(\mathrm{Cmb}_{122})$ for $\forall s \in [0, 1]$ and (iii) $t_1 \rightsquigarrow -2$, $t_2 \rightsquigarrow -1$, $t_3 \rightsquigarrow 1$ and $t_4 \rightsquigarrow 2$. Then we infer that B is deformed to $B_{\boldsymbol{\tau}_o}$ in $\mathcal{R}(\mathrm{Cmb}_{122})$. Hence $\mathcal{R}(\mathrm{Cmb}_{122})$ is connected.

4.5. Cmb_{123}

As for notation and terminology of this subsection about elliptic surfaces, we use those in Section 2.

We first show that there exists a Zariski pair (B_1, B_2) for the combinatorics Cmb_{123}. Let $\mathcal{CL}_{123} := \mathcal{P}_1 \sqcup \mathcal{A}_2$ ($\mathcal{P}_1 = \{C, L_1, L_2\}, \mathcal{A}_2 = \{D, M\}$) be a CL arrangement with Cmb_{123}. Let $\mathcal{Q} := B_{\mathcal{P}_1}$ and choose the tangent point between C and D as z_o. We assume that D is tangent to L_1 and $L_1 = L_{12}, L_2 = L_{34}$. Let $S_{\mathcal{Q},z_o}$ be the rational elliptic surface as before. Then D and M give rise to a conic point P_D and a line point P_M. By Table 1, we have

$$\begin{aligned} P_D &= [\pm 1](P_{13} \dot{+} P_{14}) \quad \text{or} \quad [\pm 1](P_{13} \dot{-} P_{14}), \\ P_M &= [\pm 1](P_{13} \dot{+} P_{14} \dot{+} P_{34}) \quad \text{or} \quad [\pm 1](P_{13} \dot{-} P_{14} \dot{+} P_{34}). \end{aligned}$$

Our tool to distinguish the embedded topology of CL arrangement with Cmb_{123} is the so called *splitting types* introduced in [**7**] as follows:

DEFINITION 4.1 ([**7**, Definition 2.3]). Let $\phi : X \to \mathbb{P}^2$ be a double cover branched at a plane curve $\mathcal{B}$, and let $D_1, D_2 \subset \mathbb{P}^2$ be two irreducible curves such that $\phi^* D_i$ are reducible and $\phi^* D_i = D_i^+ + D_i^-$. For integers $m_1 \leq m_2$, we say that the triple $(D_1, D_2; \mathcal{B})$ has *splitting type* (m_1, m_2) if for a suitable choice of labels $D_1^+ \cdot D_2^+ = m_1$ and $D_1^+ \cdot D_2^- = m_2$.

The following proposition enables us to distinguish the embedded topology of plane curves by the splitting type.

PROPOSITION 4.2 ([**7**, Proposition 2.5]). *Let $\phi_i : X_i \to \mathbb{P}^2$ ($i = 1, 2$) be two double covers branched along plane curves $\mathcal{B}_i$, respectively. For each $i = 1, 2$, let D_{i1} and D_{i2} be two irreducible plane curves such that $\phi_i^* D_{ij}$ are reducible and $\phi_i^* D_{ij} = D_{ij}^+ + D_{ij}^-$. Suppose that $D_{i1} \cap D_{i2} \cap \mathcal{B}_i = \emptyset$, D_{i1} and D_{i2} intersect transversally, and that $(D_{11}, D_{12}; \mathcal{B}_1)$ and $(D_{21}, D_{22}; \mathcal{B}_2)$ have distinct splitting types. Then there is no* *homeomorphism* $h : \mathbb{P}^2 \to \mathbb{P}^2$ *such that* $h(\mathcal{B}_1) = \mathcal{B}_2$ *and* $\{h(D_{11}), h(D_{12})\} = \{D_{21}, D_{22}\}$.

Under these conditions, we have the following lemma:

LEMMA 4.3. $(D, M; \mathcal{Q}) = (0, 2)$ *if and only if* $P_D \dot{+} P_M \dot{+} P_{34} = O$ *with a suitable choice of* P_D *and* P_M.

PROOF. Let s_D and s_M be the sections corresponding to P_D and P_M, respectively. By [**8**, Lemma 2.3],

$$s_D \cdot s_M = -\langle P_D, P_M \rangle + 1.$$

Hence $(D, M; \mathcal{Q}) = (0, 2)$ if and only if $(\langle P_D, P_M\rangle, \langle P_D, [-1]P_M\rangle) = (1, -1)$ or $(-1, 1)$. Now our statement follows from Table 2. □

P_D	P_M	$\langle P_D, P_M\rangle$	$s_D \cdot s_M$
$P_{13} \dot{\pm} P_{14}$	$P_{13} \dot{\pm} P_{14} \dot{+} P_{34}$	1	0
$P_{13} \dot{\pm} P_{14}$	$[-1](P_{13} \dot{\pm} P_{14} \dot{+} P_{34})$	−1	2
$P_{13} \dot{\pm} P_{14}$	$P_{13} \dot{\mp} P_{14} \dot{+} P_{34}$	0	1
$P_{13} \dot{\pm} P_{14}$	$[-1](P_{13} \dot{\mp} P_{14} \dot{+} P_{34})$	0	1

Table 2. The values of height pairings and intersection numbers.

For Cmb_{123}, we can also take $\{D, M, L_{34}\}$ (resp. $\{C, L_{12}\}$) as $\mathcal{P}_1$ (resp. $\mathcal{A}_2$). Put $\mathcal{Q}' = D + M + L_{34}$. Then we can also consider $(C, L_{12}; \mathcal{Q}')$ and the next lemma holds.

LEMMA 4.4. $(C, L_{12}; \mathcal{Q}') = (0, 2)$ *if and only if* $(D, M : \mathcal{Q}) = (0, 2)$.

PROOF. We choose homogeneous coordinates of $\mathbb{P}^2$ as before. If $(D, M; \mathcal{Q}) = (0, 2)$, then we may assume that $P_D \dot{+} P_M \dot{+} P_{34} = O$. Put $P_D = (x_{P_D}, y_{P_D}), P_M = (x_{P_M}, y_{P_M})$. Since the x-coordinates of P_D and P_M give defining equations of D and M, respectively, we may assume that $x_{P_D}, x_{P_M} \in \mathbb{C}[t]$, $\deg x_{P_D} = 2, \deg x_{P_M} = 1$ and there exist $mx + n \in \mathbb{C}(t)[x]$ such that three points P_D, P_M and P_{34} are on the line $y = mx + n$ in $\mathbb{A}^2_{\mathbb{C}(t)}$. Put

$$f_{\mathcal{Q}', z_o} := (x - x_{P_M})(x - x_{P_D})(x - (t_3 + t_4)t - t_3 t_4).$$

Then we have

$$f_{\boldsymbol{\tau}}(t, x) - (mx + n)^2 = f_{\mathcal{Q}', z_o}, \qquad \boldsymbol{\tau} = (0, \boldsymbol{t}) \in \mathcal{M}_1.$$

Now consider a rational elliptic surface $S_{\mathcal{Q}', z_o}$ whose Weierstrass equation of $E_{\mathcal{Q}', z_o}$ is given by $y^2 = f_{\mathcal{Q}', z_o}$. From the above relation, the three points R_1, R_2 and R_3 given by

$$R_1 := (t^2, \sqrt{-1}(mt^2 + n)),\ R_2 := (x_{P_{13}}, \sqrt{-1}(mx_{P_{13}} + n)),\ R_3 := (x_{P_{34}}, \sqrt{-1}(mx_{P_{34}} + n)),$$

where $x_{P_{ij}} = (t_i + t_j)t - t_i t_j$, are on $y = \sqrt{-1}(mx + n)$. Hence $R_1 \dot{+} R_2 \dot{+} R_3 = O$ (By abuse of notation, we use O for the identity element of $E_{\mathcal{Q}', z_o}$). By Lemma 4.3,

$(C, L_{12}; \mathcal{Q}') = (0, 2)$. The converse statement follows by the same argument. □

Now put

$$B_1 := \mathcal{Q} + D + L_0,\ B_2 := \mathcal{Q} + D + L_0'$$

where $D = \tilde{f}_{\mathcal{Q},z_0}(s_{P_{13}\dot{+}P_{14}})$, $L_0 = \tilde{f}_{\mathcal{Q},z_o}(s_{P_{13}\dot{+}P_{14}\dot{+}P_{34}})$ and $L_0' = \tilde{f}_{\mathcal{Q},z_o}(s_{P_{13}\dot{-}P_{14}\dot{+}P_{34}})$. Then we have

Proposition 4.5. *(B_1, B_2) is a Zariski pair.*

Proof. Suppose that there exists a homeomorphism $h : (\mathbb{P}^2, B_1) \to (\mathbb{P}^2, B_2)$. Then either $h(\mathcal{Q}) = \mathcal{Q}$ or $h(\mathcal{Q}') = \mathcal{Q}$ holds. Since $(D, L_0; \mathcal{Q}) = (C, L_1; \mathcal{Q}') = (0, 2)$, $(D, L_0'; \mathcal{Q}) = (1, 1)$, both cases are impossible by [**7**, Proposition 2.5]. □

Remark 4.6. The $mx + n$ in our proof of Lemma 4.3 is in $\mathbb{C}[t, x]$ and its total degree is 2 as $f_{\boldsymbol{\tau}}(t, x) - (mx + n)^2 = f_{\mathcal{Q}',z_o}$. Since p_3, p_4 and p_0 are on both L_{34} and the conic $\tilde{C}$ given by $mx + n = 0$ in the (t, x)-plane, we see that $\tilde{C}$ contains L_{34}. Hence we infer that the three tangent points between $D + M$ and $C + L_{12}$ are collinear.

Here we give an explicit example of a Zariski pair for Cmb_{123}. We keep the previous notation.

Example 4.7. Let $\mathcal{Q}_{\boldsymbol{\tau}_o}$ be a plane quartic given by $f_{\boldsymbol{\tau}_o} = 0$ as before where $\boldsymbol{a} = (-2, -1, 1, 2)$, $\boldsymbol{\tau}_o = (0, \boldsymbol{a}) \in \mathcal{M}_1$. Let $S_{\mathcal{Q}_{\boldsymbol{\tau}_o},z_o}$ be the rational elliptic surface given by the Weierstrass equation $y^2 = f_{\mathcal{Q}_{\boldsymbol{\tau}_o},z_o}$ and $z_o = [0, 1, 0]$. In this case, we have

$$P_{13} = (-t + 2, 2\sqrt{2}(t - 1)(t + 2)), \quad P_{14} = (4, 3(t - 2)(t + 2))$$

and $P_{D_{\boldsymbol{\tau}_o}} := P_{13}\dot{+}P_{14}$, $P_{D'_{\boldsymbol{\tau}_o}} := P_{13}\dot{-}P_{14}$, $P_{M_{\boldsymbol{\tau}_o}} := P_{D_{\boldsymbol{\tau}_o}}\dot{+}P_{34}$ and $P_{M'_{\boldsymbol{\tau}_o}} := P_{D'_{\boldsymbol{\tau}_o}}\dot{+}P_{34}$ as follows:

$$\begin{aligned}
P_{D_{\boldsymbol{\tau}_o}} &= (x_{D_{\boldsymbol{\tau}_o}}, y_{D_{\boldsymbol{\tau}_o}}) \\
x_{D_{\boldsymbol{\tau}_o}} &= (-12\sqrt{2} + 18)t^2 + (36\sqrt{2} - 51)t + 34 - 24\sqrt{2}, \\
y_{D_{\boldsymbol{\tau}_o}} &= 6(t - 2)(12\sqrt{2}t - 17\sqrt{2} - 17t + 24)(t - 1), \\
P_{D'_{\boldsymbol{\tau}_o}} &= (x_{D'_{\boldsymbol{\tau}_o}}, y_{D'_{\boldsymbol{\tau}_o}}), \\
x_{D'_{\boldsymbol{\tau}_o}} &= (12\sqrt{2} + 18)t^2 + (-36\sqrt{2} - 51)t + 34 + 24\sqrt{2}, \\
y_{D'_{\boldsymbol{\tau}_o}} &= 6(t - 2)(12\sqrt{2}t - 17\sqrt{2} + 17t - 24)(t - 1)
\end{aligned}$$

$$\begin{aligned}
P_{M_{\boldsymbol{\tau}_o}} &= (x_{M_{\boldsymbol{\tau}_o}}, y_{M_{\boldsymbol{\tau}_o}}) = (-2\sqrt{2}t - 2, -t^2 - \sqrt{2}t) \\
P_{M'_{\boldsymbol{\tau}_o}} &= (x_{M'_{\boldsymbol{\tau}_o}}, y_{M'_{\boldsymbol{\tau}_o}}) = (2\sqrt{2}t - 2, t^2 - \sqrt{2}t)
\end{aligned}$$

Note that lines given by $x - x_{M_{\boldsymbol{\tau}_o}}$ and $x - x_{M'_{\boldsymbol{\tau}_o}}$ coincide with $L_{0,\boldsymbol{\tau}_o}$ and $L'_{0,\boldsymbol{\tau}_o}$, respectively. Now put

$$B_{1,\boldsymbol{\tau}_o} := \mathcal{Q}_{\boldsymbol{\tau}_o} + D_{\boldsymbol{\tau}_o} + L_{0,\boldsymbol{\tau}_o}, \quad B_{2,\boldsymbol{\tau}_o} := \mathcal{Q}_{\boldsymbol{\tau}_o} + D_{\boldsymbol{\tau}_o} + L'_{0,\boldsymbol{\tau}_o},$$

where $D_{\boldsymbol{\tau}_o}$ is a conic of type $D(1,2)$ given by $x - x_{D_{\boldsymbol{\tau}_o}} = 0$. We can easily check that $B_{1,\boldsymbol{\tau}_o}, B_{2,\boldsymbol{\tau}_o} \in \mathcal{R}(\mathrm{Cmb}_{123})$ using a computer algebra system. Hence $(B_{1,\boldsymbol{\tau}_o}, B_{2,\boldsymbol{\tau}_o})$ is an explicit example of a Zariski pair given in Proposition 4.5.

We now go on to study the connected components of $\mathcal{R}(\mathrm{Cmb}_{123})$.

Proposition 4.8. *Any element $B \in \mathcal{R}(\mathrm{Cmb}_{123})$ is deformed to either $B_{1,\boldsymbol{\tau}_o}$ or $B_{2,\boldsymbol{\tau}_o}$ in Example 4.7, i.e., $\mathcal{R}(\mathrm{Cmb}_{123})$ has just two connected components.*

Proof. By Example 4.7, $\mathcal{R}(\mathrm{Cmb}_{123})$ has at least two connected components. Let B be an element in $\mathcal{R}(\mathrm{Cmb}_{123})$. We show that B is continuously deformed to $B_{1,\boldsymbol{\tau}_o}$ or $B_{2,\boldsymbol{\tau}_o}$ in Example 4.7. By taking homogeneous coordinates suitably, we may assume that B is of the form

$$B = B_{\boldsymbol{\tau}} = \mathcal{Q}_{\boldsymbol{\tau}} + D_{\boldsymbol{\tau}} + M_{\boldsymbol{\tau}}, \quad \mathcal{Q}_{\boldsymbol{\tau}} = C_o + L_{12,\boldsymbol{\tau}} + L_{34,\boldsymbol{\tau}}$$

for some $\boldsymbol{\tau} = (0, \boldsymbol{t}) \in \mathcal{M}_1$ and $D_{\boldsymbol{\tau}}$ and $M_{\boldsymbol{\tau}}$ are the conic and line described in Section 3. We may also assume that $D_{\boldsymbol{\tau}}$ passes through p_3 and p_4 and is tangent to $L_{12,\boldsymbol{\tau}}$. Now consider a path $\gamma : [0,1] \to \mathcal{M}_1$ such that (i) $\gamma(0) = \boldsymbol{\tau}$, $\gamma(1) = \boldsymbol{\tau}_o$, (ii) $B_{\gamma(s)} \in \mathcal{R}(\mathrm{Cmb}_{123})$ for $\forall s \in [0,1]$ and (iii) $t_1 \rightsquigarrow -2, t_2 \rightsquigarrow -1, t_3 \rightsquigarrow 1$ and $t_4 \rightsquigarrow 2$. Then $B_{\gamma(0)} = B$ and

$$B_{\gamma(1)} = \mathcal{Q}_{\boldsymbol{\tau}_o} + D_{12,\gamma(1)} + M_{\gamma(1)},$$

where

$$D_{\gamma(1)} = D_{\boldsymbol{\tau}_o} \text{ or } D'_{\boldsymbol{\tau}_o}, \quad M_{\gamma(1)} = L_{0,\boldsymbol{\tau}_o} \text{ or } L'_{0,\boldsymbol{\tau}_o},$$

where $D'_{\boldsymbol{\tau}_o}$ is the conic given by $x - x_{D'} = 0$.

Case (i): $D_{\gamma(1)} = D_{\boldsymbol{\tau}_o}$. In this case, $B_{\gamma(1)}$ is either $B_{1,\boldsymbol{\tau}_o}$ or $B_{2,\boldsymbol{\tau}_o}$.

Case (ii): $D_{\gamma(1)} = D'_{\boldsymbol{\tau}_o}$. In this case, $B_{\gamma(1)}$ is either $\mathcal{Q}_{\boldsymbol{\tau}_o} + D'_{\boldsymbol{\tau}} + L_{0,\boldsymbol{\tau}_o}$ or $\mathcal{Q}_{\boldsymbol{\tau}_o} + D'_{\boldsymbol{\tau}} + L'_{0,\boldsymbol{\tau}_o}$. Consider families of lines and parabolas as follows:

$$L_{u_1u_2}\colon x - (u_1 + u_2)t + u_1u_2 = 0,\ (u_1, u_2) \in \mathbb{C}^2, u_1 \neq u_2,$$
$$D_\mu\colon x - \mu t^2 - (3 - 3\mu)t - 2\mu + 2 = 0,\ \mu \in \mathbb{C}^\times.$$

Namely, $L_{u_1u_2}$ is a line intersecting C_o at (u_1, u_1^2) and (u_2, u_2^2) and D_μ is a parabola passing $(1,1)$ and $(2,4)$. Note that $D_\mu = D_{\boldsymbol{\tau}_o}$ (resp. $D'_{\boldsymbol{\tau}_o}$) when $\mu = 18 - 12\sqrt{2}$ (resp. $\mu = 18 + 12\sqrt{2}$). It can be easily checked that the condition for $L_{u_1u_2}$ and D_μ to be tangent is that (u_1, u_2, μ) satisfies

$$(*) \quad \mu^2 - 4u_1u_2\mu + 6(\mu - 1)(u_1 + u_2) + (u_1 + u_2)^2 - 10\mu + 9 = 0.$$

Note that the surface given by $(*)$ in the (u_1, u_2, μ)-space is irreducible and connected. Now consider a path $\bar{\gamma} : [0,1] \to \mathcal{M}_1 \times \mathbb{C}^\times, \bar{\gamma}(s) = (0, u_1(s), u_2(s), 1, 2, \mu(s))$ such that (i) $(u_1(s), u_2(s), \mu(s))$ satisfies $(*)$ and (ii) $\bar{\gamma}(0) = (0, -2, -1, 1, 2, 18 + 12\sqrt{2})$ and $\bar{\gamma}(1) = (0, -2, -1, 1, 2, 18 - 12\sqrt{2})$. Since (i) $D_{\mu(s)}$ is tangent to $L_{u_1u_2}$ and (ii) the line $M_{\bar{\gamma}(s)}$ is determined by $L_{u_1u_2} \cap L_{34,\boldsymbol{a}}$ and the initial line $M_{\bar{\gamma}(0)}$, we infer that there exists a continuous family $B_{\bar{\gamma}(s)}$ $(0 \le s \le 1)$ in $\mathcal{R}(\mathrm{Cmb}_{123})$ such that $B_{\bar{\gamma}(0)} = B_{\gamma(1)}$ and $B_{\bar{\gamma}(1)} = B_{1,\boldsymbol{\tau}_o}$ or $B_{2,\boldsymbol{\tau}_o}$. Thus our statement follows. □

4.6. $\mathbf{Cmb_{124}}$

In [**34**], we have seen that there exists a Zariski pair for Cmb_{124}. Hence $\mathcal{R}(\mathrm{Cmb}_{124})$ has at least two connected components. In this subsection, we will show that there exist only two components. Let us start with the following example.

Example 4.9. Let $\boldsymbol{\tau}_o = (0, \boldsymbol{a}) \in \mathcal{M}_1$ and $\mathcal{Q}_{\boldsymbol{\tau}_o}$ be as before. We label p_1, p_2, p_3 and p_4 in the same way. Namely, the lines contained in $\mathcal{Q}_{\boldsymbol{\tau}_o}$ are $L_{12,\boldsymbol{\tau}_o}$ and $L_{34,\boldsymbol{\tau}_o}$. In this case, we have

$$[2]P_{13} = \left(\frac{9}{8}t^2, \frac{\sqrt{2}}{32}(-9t^3 + 16t)\right), \ [2]P_{14} = \left(t^2 + \frac{1}{4}, \frac{1}{2}t^2 - \frac{9}{8}\right).$$

Now put

$$D_{\boldsymbol{\tau}_o} : \tilde{f}_{\mathcal{Q}_{\boldsymbol{\tau}_o}, z_o}(s_{[2]P_{13}}) = x - \frac{9}{8}t^2 = 0, L_{13,\boldsymbol{\tau}_o} : \tilde{f}_{\mathcal{Q}_{\boldsymbol{\tau}_o}, z_o}(s_{P_{13}}) = x + t - 2 = 0,$$
$$L_{14,\boldsymbol{\tau}_o} : \tilde{f}_{\mathcal{Q}_{\boldsymbol{\tau}_o}, z_o}(s_{P_{14}}) = x - 4 = 0.$$

Now define B_1 and B_2 to be

$$B_{1,\boldsymbol{\tau}_o} := \mathcal{Q}_{\boldsymbol{\tau}_o} + D_{\boldsymbol{\tau}_o} + L_{13,\boldsymbol{\tau}_o}, \ B_{2,\boldsymbol{\tau}_o} := \mathcal{Q}_{\boldsymbol{\tau}_o} + D_{\boldsymbol{\tau}_o} + L_{14,\boldsymbol{\tau}_o}.$$

Then by [**34**, Theorem 5], $(B_{1,\boldsymbol{\tau}_o}, B_{2,\boldsymbol{\tau}_o})$ is a Zariski pair.

Now we show the following proposition.

Proposition 4.10. *Let B be an arbitrary member in $\mathcal{R}(\mathrm{Cmb}_{124})$. Then B is continuously deformed to either $B_{1,\boldsymbol{\tau}_o}$ or $B_{2,\boldsymbol{\tau}_o}$ in Example 4.9. In particular, $\mathcal{R}(\mathrm{Cmb}_{124})$ has just two connected components.*

Proof. After taking a suitable coordinate change, we may assume that B is of the form

$$B = B_{\boldsymbol{\tau}} = \mathcal{Q}_{\boldsymbol{\tau}} + D_{\boldsymbol{\tau}} + L_{13,\boldsymbol{\tau}},$$

for some $\boldsymbol{\tau} = (0, \boldsymbol{t}) \in \mathcal{M}_1$. Here $D_{\boldsymbol{\tau}}$ is either $\tilde{f}_{\mathcal{Q}_{\boldsymbol{\tau}}, z_o}(s_{[2]P_{13,\boldsymbol{\tau}}})$ or $\tilde{f}_{\mathcal{Q}_{\boldsymbol{\tau}}, z_o}(s_{[2]P_{14,\boldsymbol{\tau}}})$.

<u>Case $D_{\boldsymbol{\tau}} = \tilde{f}_{\mathcal{Q}_{\boldsymbol{\tau}}, z_o}(s_{[2]P_{13,\boldsymbol{\tau}}})$.</u> Consider a path $\gamma : [0,1] \to \mathcal{M}_1$ such that (i) $\gamma(0) = \boldsymbol{\tau}$, $\gamma(1) = \boldsymbol{\tau}_o$, (ii) $B_{\gamma(s)} \in \mathcal{R}(\mathrm{Cmb}_{124})$ for $\forall s \in [0,1]$ and (iii) $t_1 \rightsquigarrow -2$, $t_2 \rightsquigarrow -1$, $t_3 \rightsquigarrow 1$ and $t_4 \rightsquigarrow 2$. This shows that B is continuously deformed to $B_{1,\boldsymbol{\tau}_o}$ while keeping the combinatorics.

<u>Case $D_{\boldsymbol{\tau}} = \tilde{f}_{\mathcal{Q}_{\boldsymbol{\tau}}, z_o}(s_{[2]P_{14,\boldsymbol{\tau}}})$.</u> Consider a path $\gamma : [0,1] \to \mathcal{M}_1$ such that (i) $\gamma(0) = \boldsymbol{\tau}$, $\gamma(1) = \boldsymbol{\tau}_o$, (ii) $B_{\gamma(s)} \in \mathcal{R}(\mathrm{Cmb}_{124})$ for $\forall s \in [0,1]$ and (iii) $t_1 \rightsquigarrow -2$, $t_2 \rightsquigarrow -1$, $t_3 \rightsquigarrow 2$ and $t_4 \rightsquigarrow 1$. Then $L_{13,\boldsymbol{\tau}}$ (resp. $L_{14,\boldsymbol{\tau}}$) is deformed to $L_{14,\boldsymbol{\tau}_o}$ (resp. $L_{13,\boldsymbol{\tau}_o}$) and $D_{\boldsymbol{\tau}}$ is deformed to $\tilde{f}_{\mathcal{Q}_{\boldsymbol{\tau}}, z_o}(s_{[2]P_{13,\boldsymbol{\tau}_o}})$ accordingly. Hence B is continuously deformed to $B_{2,\boldsymbol{\tau}_o}$ while keeping the combinatorics. □

4.7. $\mathbf{Cmb_{125}}$

Choose $B \in \mathcal{R}(\mathrm{Cmb}_{125})$ arbitrarily. By taking appropriate coordinates of $\mathbb{P}^2$, we may assume that $C_1 = C_o$, D is tangent to C_o at $z_o = [0,1,0]$ and there exists $\boldsymbol{\tau} = (0, \boldsymbol{t}) \in \mathcal{M}_1$ such that B is of the form

$$B_{\boldsymbol{\tau}} = C_o + L_{12,\boldsymbol{\tau}} + L_{34,\boldsymbol{\tau}} + D_{\boldsymbol{\tau}} + L_{0,\boldsymbol{\tau}}.$$

By Table 1, $D_{\boldsymbol{\tau}}$ is given by the image of $s_{[2]P_{ij}}$ under $\tilde{f}_{\mathcal{Q}_{\boldsymbol{\tau}},z_o}$. Take $\boldsymbol{a} = (-2,-1,1,2)$, $\boldsymbol{\tau}_o = (0, \boldsymbol{a}) \in \mathcal{M}_1$ and consider an element of $\mathcal{R}(\mathrm{Cmb}_{125})$ given by

$$B_{\boldsymbol{\tau}_o} = C_o + L_{12,\boldsymbol{\tau}_o} + L_{34,\boldsymbol{\tau}_o} + D_{\boldsymbol{\tau}_o} + L_{0,\boldsymbol{\tau}_o},$$

where $D_{\boldsymbol{\tau}_o}$ is given by $[2]P_{23}$. By [**34**, Example 5.2], $D_{\boldsymbol{\tau}_o}$ is given by $x - t^2 - 1/4 = 0$. Now we choose a path $\gamma : [0,1] \to \mathcal{M}_1$ such that (i) $\gamma(0) = \boldsymbol{\tau}, \gamma(1) = \boldsymbol{\tau}_o$, (ii) $B_{\gamma(s)} \in \mathcal{R}(\mathrm{Cmb}_{125})$ for $\forall s \in [0,1]$ and (iii) $t_i \rightsquigarrow -1, t_j \rightsquigarrow 1$. By the deformation along γ, L_{ij} is deformed to L_{23}. Hence $D_{\boldsymbol{\tau}}$ is deformed to $D_{\boldsymbol{\tau}_o}$. If $L_{0,\boldsymbol{\tau}}$ is deformed to $L_{0,\boldsymbol{\tau}_o}$, we see that B is deformed to $B_{\boldsymbol{\tau}_o}$. If $L_{0,\boldsymbol{\tau}}$ is deformed to $L'_{0,\boldsymbol{\tau}_o}$, then we apply the transformation $(t,x) \mapsto (-t,x)$ and see that B is deformed to $B_{\boldsymbol{\tau}_o}$. Thus $\mathcal{R}(\mathrm{Cmb}_{125})$ is connected.

4.8. $\mathbf{Cmb_{211}}$

Let us start with the following remark.

Remark 4.11. Let $B_{\mathcal{P}_2}$ be a quartic given by a conic arrangement $\mathcal{P}_2$. It is known that there exist four bitangent lines for $B_{\mathcal{P}_2}$. When we deform the conics in $\mathcal{P}_2$ continuously, these bitangents are also deformed along with the conics. Note that this observation follows by considering the dual curves of the conics in $\mathcal{P}_2$. We make use of this observation repeatedly in the rest of this article.

Consider two conics C_{o1} and C_{o2} given by

$$C_{o1} : t^2 + x^2 + tx - \frac{27}{4} = 0, \quad C_{o2} : t^2 + x^2 - tx - \frac{27}{4} = 0.$$

We write $C_{o1} \cap C_{o2}$ by $\boldsymbol{p} = \{p_1, p_2, p_3, p_4\}$ whose affine coordinates are given by

$$p_1 = \left(0, \frac{3}{2}\sqrt{3}\right), \quad p_2 = \left(\frac{3}{2}\sqrt{3}, 0\right), \quad p_3 = \left(0, -\frac{3}{2}\sqrt{3}\right), \quad p_4 = \left(-\frac{3}{2}\sqrt{3}, 0\right).$$

The bitangent lines of $C_{o1} + C_{o2}$ are

$$L_{b1,\boldsymbol{p}} : t = 3,\ L_{b2,\boldsymbol{p}} : t = -3,\ L_{b3,\boldsymbol{p}} : x = 3,\ L_{b4,\boldsymbol{p}} : x = -3.$$

Now put

$$B_{oi} := C_{o1} + C_{o2} + L_{13,\boldsymbol{p}} + L_{24,\boldsymbol{p}} + L_{bi,\boldsymbol{p}},\ i = 1,2,3,4.$$

Then $B_{oi} \in \mathcal{R}(\mathrm{Cmb}_{211})$ and all of them are transformed by some projective transformation each other.

Hence it is enough to show that an arbitrary element $B \in \mathcal{R}(\mathrm{Cmb}_{211})$ can be continuously deformed to $B_{oi} \in \mathcal{R}(\mathrm{Cmb}_{211})$ for some i.

We may assume that B is given in the following form:

$$B_{\boldsymbol{\tau}} = \mathcal{Q}_{\boldsymbol{\tau}} + L_{13,\boldsymbol{\tau}} + L_{24,\boldsymbol{\tau}} + L_{b1,\boldsymbol{\tau}},$$

where $\mathcal{Q}_{\boldsymbol{\tau}} = C_o + C_{\boldsymbol{\tau}}$ for some $\boldsymbol{\tau} = (\lambda, \boldsymbol{t}) \in \mathcal{M}_0$. Let $\phi : \mathbb{P}^2 \to \mathbb{P}^2$ be a projective transformation such that $\phi(C_o) = C_{o1}$. Then there exists $\boldsymbol{\tau_c} = (\lambda_{\boldsymbol{c}}, \boldsymbol{c}) \in \mathcal{M}_0$ such that $\phi(C_{\boldsymbol{\tau_c}}) = C_{o2}$ and points in $C_o \cap C_{\boldsymbol{\tau_c}}$ are labeled so that $L_{ij,\boldsymbol{\tau_c}} = L_{ij,\boldsymbol{p}}$ holds. Now we choose a path $\gamma : [0,1] \to \mathcal{M}_0$ such that (i) $\gamma(0) = \boldsymbol{\tau}, \gamma(1) = \boldsymbol{\tau_c}$, (ii) $B_{\gamma(s)} \in \mathcal{R}(\mathrm{Cmb}_{211})$ for $\forall s \in [0,1]$ and (iii) $t_i \rightsquigarrow c_i$ $(i = 1,2,3,4)$. We see that B can be continuously deformed along γ in $\mathcal{R}(\mathrm{Cmb}_{211})$ to $B_1 := C_o + C_{\boldsymbol{\tau_c}} + L_{13,\boldsymbol{\tau_c}} + L_{24,\boldsymbol{\tau_c}} + L_{b,\boldsymbol{\tau_c}}$. Here $L_{b,\boldsymbol{\tau_c}}$ denotes a bitangent to $C_o + C_{\boldsymbol{\tau_c}}$. As $\phi(B_1) = B_{oi}$ for some i, we infer that B is continuously deformed to B_{oi} and that $\mathcal{R}(\mathrm{Cmb}_{211})$ is connected.

4.9. $\mathbf{Cmb_{212}}$

We first show that there exists a Zariski pair for Cmb_{212}. Let $\mathcal{Q}_{\boldsymbol{\tau}} = C_o + C_{\boldsymbol{\tau}}$ and $B = \mathcal{Q}_{\boldsymbol{\tau}} + L_{ij} + L_{bk} + L_{bl} \in \mathcal{R}(\mathrm{Cmb}_{212})$. Choose $z_o \in C_o$ so that the tangent line at z_o meets $C_{\boldsymbol{\tau}}$ at two distinct points. Let $\varphi_{\mathcal{Q}_{\boldsymbol{\tau}},z_o} : S_{\mathcal{Q}_{\boldsymbol{\tau}},z_o} \to \mathbb{P}^1$ and $\tilde{f}_{\mathcal{Q}_{\boldsymbol{\tau}},z_o} : S_{\mathcal{Q}_{\boldsymbol{\tau}},z_o} \to \mathbb{P}^2$ as in Subsection 2.1. As we have seen in Table 1 or [**8**, Section 3.2], if we put

$$Q_1 := P_{12} \dot{+} P_{13} \dot{+} P_{23}, \quad Q_2 := [-1]P_{12} \dot{+} P_{13} \dot{+} P_{23},$$
$$Q_3 := P_{12} \dot{+} [-1]P_{13} \dot{+} P_{23}, Q_4 := P_{12} \dot{+} P_{13} \dot{+} [-1]P_{23},$$

then we may assume that $L_{bi} := \tilde{f}_{\mathcal{Q}_{\boldsymbol{\tau}},z_o}(s_{Q_i})$ $(i = 1,2,3,4)$ are the four bitangent lines of $\mathcal{Q}_{\boldsymbol{\tau}}$. Then by [**34**, Theorem 3.2 and 3.3] and the argument in p.629-630 in [**34**], we have the following proposition:

Proposition 4.12. *Let p be an odd prime. There exists a D_{2p}-cover of $\mathbb{P}^2$ branched at $2\mathcal{Q}_{\boldsymbol{\tau}} + p(L_{ij} + L_{bk} + L_{bl})$ if and only if the images of P_{ij}, Q_k, Q_l in $E_{\mathcal{Q}_\lambda,z_o}$ are linearly dependent over $\mathbb{Z}/p\mathbb{Z}$.*

By Proposition 4.12, we have:

Corollary 4.13. *Let $B_{kl} := \mathcal{Q}_{\boldsymbol{\tau}} + L_{13} + L_{bk} + L_{bl}$. Then (B_{13}, B_{kl}) (resp.(B_{24}, B_{kl})) is a Zariski pair where $(k,l) \neq (2,4)$ (resp.$(k,l) \neq (1,3)$).*

Proof. If a homeomorphism $h : (\mathbb{P}^2, B_{13}) \to (\mathbb{P}^2, B_{kl})$ exists, it satisfies $h(\mathcal{Q}_{\boldsymbol{\tau}}) = \mathcal{Q}_{\boldsymbol{\tau}}$. Hence our statement follows from Proposition 4.12. □

Remark 4.14. We may use the connected number for $L_{13} + L_{bk} + L_{bl}$ with respect to $f'_{Q_{\boldsymbol{\tau}}}$ to prove our statement. In fact, for example, the connected number is 2 for $(k,l) = (1,3)$, while it is 1 for $(k,l) = (1,2)$. This shows (B_{12}, B_{13}) is a Zariski pair. As for connected numbers, see [**29**] for details.

Let us now consider an explicit example.

Example 4.15. Let $\mathcal{Q}_{\boldsymbol{\tau}_o} = C_o + C_{\boldsymbol{\tau}_o}$ be a plane quartic given by $f_{\mathcal{Q}_{\boldsymbol{\tau}_o}} = 0$ where $\boldsymbol{\tau}_o = (\lambda_o, \boldsymbol{a}) = (-10, -2, -1, 1, 2) \in \mathcal{M}_0$. Let $S_{\mathcal{Q}_{\boldsymbol{\tau}_o}}$ be the rational elliptic surface given by the Weierstrass equation $y^2 = f_{\mathcal{Q}_{\boldsymbol{\tau}_o}}$ and $z_o = [0,1,0]$. In this case, we have

$$\begin{aligned}
P_{12} &= \left(-3t-2, -i\sqrt{10}t^2 - 3i\sqrt{10}t - 2i\sqrt{10}\right),\\
P_{13} &= \left(-t+2, -i\sqrt{2}t^2 - i\sqrt{2}t + 2i\sqrt{2}\right),\\
P_{23} &= \left(1, -i\,t^2 + i\right).
\end{aligned}$$

Under this setting, $P_{L_{b1}} := P_{12}\dot{+}P_{13}\dot{+}P_{23}$, $P_{L_{b2}} := P_{12}\dot{+}P_{13}\dot{-}P_{23}$, $P_{L_{b3}} := P_{12}\dot{-}P_{13}\dot{+}P_{23}$ and $P_{L_{b4}} := [-1]P_{12}\dot{+}P_{13}\dot{+}P_{23}$ are given as follows:

$$\begin{aligned}
P_{L_{b1}} &= \left(\sqrt{2}(\sqrt{5}+3)t - 3\sqrt{5} - 7, (2\sqrt{5}+3)it^2 - \frac{\sqrt{2}}{2}(15\sqrt{5}+29)it + 2(7\sqrt{5}+15)i\right),\\
P_{L_{b2}} &= \left(-\sqrt{2}(\sqrt{5}+3)t - 3\sqrt{5} - 7, (2\sqrt{5}+3)it^2 + \frac{\sqrt{2}}{2}(15\sqrt{5}+29)it + 2(7\sqrt{5}+15)i\right),\\
P_{L_{b3}} &= \left(\sqrt{2}(\sqrt{5}-3)t + 3\sqrt{5} - 7, -(2\sqrt{5}-3)it^2 - \frac{\sqrt{2}}{2}(15\sqrt{5}-29)it - 2(7\sqrt{5}-15)i\right),\\
P_{L_{b4}} &= \left(-\sqrt{2}(\sqrt{5}-3)t + 3\sqrt{5} - 7, -(2\sqrt{5}-3)it^2 + \frac{\sqrt{2}}{2}(15\sqrt{5}-29)it - 2(7\sqrt{5}-15)i\right).
\end{aligned}$$

Now put $L_{bi,\boldsymbol{\tau}_o} : f_{\mathcal{Q}_{\boldsymbol{\tau}_o}, z_o}(s_{P_{bi}}) = 0$. Then we have

$$\begin{aligned}
&L_{b1,\boldsymbol{\tau}_o} : x - \sqrt{2}(\sqrt{5}+3)t + 3\sqrt{5} + 7 = 0, L_{b2,\boldsymbol{\tau}_o} : x + \sqrt{2}(\sqrt{5}+3)t + 3\sqrt{5} + 7 = 0,\\
&L_{b3,\boldsymbol{\tau}_o} : x - \sqrt{2}(\sqrt{5}-3)t - 3\sqrt{5} + 7 = 0, L_{b4,\boldsymbol{\tau}_o} : x + \sqrt{2}(\sqrt{5}-3)t - 3\sqrt{5} + 7 = 0.
\end{aligned}$$

We put

$$B_{ij,\boldsymbol{\tau}_o} := \mathcal{Q}_{\boldsymbol{\tau}_o} + L_{13,\boldsymbol{\tau}_o} + L_{bi,\boldsymbol{\tau}_o} + L_{bj,\boldsymbol{\tau}_o}, \quad i,j = 1,2,3,4, i \neq j.$$

Then $(B_{13,\boldsymbol{\tau}_o}, B_{ij,\boldsymbol{\tau}_o})$ (resp.$(B_{24,\boldsymbol{\tau}_o}, B_{ij,\boldsymbol{\tau}_o})$) are Zariski pairs for $(i,j) \neq (2,4)$ (resp.$(i,j) \neq (1,3)$) by Corollary 4.13.

We give another example of a CL arrangement in Cmb_{212}, which plays an important role to study the connectivity for $\mathcal{R}(\mathrm{Cmb}_{212})$.

Example 4.16. Let C_{o1} and C_{o2} be conics given by

$$C_{o1} : t^2 + x^2 + tx - \frac{27}{4} = 0, \qquad C_{o2} : 676t^2 + 764tx + 676x^2 - 4563 = 0.$$

We write $C_{o1} \cap C_{o2}$ by $\boldsymbol{p} = \{p_1, p_2, p_3, p_4\}$ whose affine coordinates are given by

$$p_1 = \left(\frac{3}{2}\sqrt{3}, 0\right), \quad p_2 = \left(0, -\frac{3}{2}\sqrt{3}\right), \quad p_3 = \left(-\frac{3}{2}\sqrt{3}, 0\right), \quad p_4 = \left(0, \frac{3}{2}\sqrt{3}\right).$$

The bitangent lines of $C_{o1} + C_{o2}$ are

$$L_{b1,\boldsymbol{p}} : 15t + 8x - 39 = 0, \quad L_{b2,\boldsymbol{p}} : 8t + 15x + 39 = 0,$$
$$L_{b3,\boldsymbol{p}} : 15t + 8x + 39 = 0, \quad L_{b4,\boldsymbol{p}} : 8t + 15x - 39 = 0.$$

Now put

$$B_{ij,\boldsymbol{p}} := C_{o1} + C_{o2} + L_{13,\boldsymbol{p}} + L_{bi,\boldsymbol{p}} + L_{bj,\boldsymbol{p}}, \quad i, j = 1, 2, 3, 4.$$

Then $B_{ij,\boldsymbol{p}} \in \mathcal{R}(\mathrm{Cmb}_{212})$.

Now we show the following proposition.

PROPOSITION 4.17. *The curve $\mathcal{B}_{13,\boldsymbol{p}}$ can be continuously deformed to $B_{24,\boldsymbol{p}}$ while preserving the combinatorics* Cmb_{212}.

PROOF. Let β be a parameter and $C_{o2,\beta}$ be a conic defined by

$$\begin{aligned} C_{o2,\beta} : 4\left(\beta^2 + \beta + 1\right)^2 t^2 + 4\left(2\beta^4 + 4\beta^3 - 6\beta^2 - 8\beta - 1\right)tx \\ +4\left(\beta^2 + \beta + 1\right)^2 x^2 - 27\left(\beta^2 + \beta + 1\right)^2 = 0. \end{aligned}$$

The conic $C_{o2,\beta}$ passes through p_1, p_2, p_3, p_4 and furthermore, $C_{o2,\beta} = C_{o2}$ for $\beta = -4, -\frac{5}{7}, -\frac{2}{7}, 3$. Note that $L_{13,\boldsymbol{p}}$ is fixed since p_i does not depend on the parameter β. Also, $C_{o2,\beta} = C_{o1}$ if $(\beta^2 + 4\beta + 1)(\beta^2 - 2\beta - 2) = 0$ and $C_{o2,\beta}$ has singular points if $(\beta^2 + \beta + 1)(2\beta^2 + 2\beta - 1)(2\beta + 1) = 0$.

The three lines $L_{13,\boldsymbol{p}}, L_{bi,\boldsymbol{p},\beta}, L_{bj,\boldsymbol{p},\beta}$ $(i, j \in \{1, 2, 3, 4\})$ intersect at one point if $\beta = -2, -1, 0, 1$. Now we have the following bitangent lines $L_{bi,\boldsymbol{p},\beta}$ of $C_{o1} + C_{o2,\beta}$ when $\beta(\beta - 1)(\beta + 1)(\beta + 2)(\beta^2 + 4\beta + 1)(\beta^2 - 2\beta - 2)(\beta^2 + \beta + 1)(2\beta^2 + 2\beta - 1)(2\beta + 1) \neq 0$:

$$\begin{aligned} L_{b1,\boldsymbol{p},\beta} &: (\beta^2 + 2\beta)t + (\beta^2 - 1)x - (3\beta^2 + 3\beta + 3) = 0 \\ L_{b2,\boldsymbol{p},\beta} &: (\beta^2 - 1)t + (\beta^2 + 2\beta)x + (3\beta^2 + 3\beta + 3) = 0 \\ L_{b3,\boldsymbol{p},\beta} &: (\beta^2 + 2\beta)t + (\beta^2 - 1)x + (3\beta^2 + 3\beta + 3) = 0 \\ L_{b4,\boldsymbol{p},\beta} &: (\beta^2 - 1)t + (\beta^2 + 2\beta)x - (3\beta^2 + 3\beta + 3) = 0. \end{aligned}$$

For $\beta = -4, -\frac{5}{7}, -\frac{2}{7}, 3$, we have the following table:

β	-4	$-\frac{5}{7}$	$-\frac{2}{7}$	3
$L_{b1,\boldsymbol{p},\beta}$	$L_{b4,\boldsymbol{p}}$	$L_{b3,\boldsymbol{p}}$	$L_{b2,\boldsymbol{p}}$	$L_{b1,\boldsymbol{p}}$
$L_{b2,\boldsymbol{p},\beta}$	$L_{b3,\boldsymbol{p}}$	$L_{b4,\boldsymbol{p}}$	$L_{b1,\boldsymbol{p}}$	$L_{b2,\boldsymbol{p}}$
$L_{b3,\boldsymbol{p},\beta}$	$L_{b2,\boldsymbol{p}}$	$L_{b1,\boldsymbol{p}}$	$L_{b4,\boldsymbol{p}}$	$L_{b3,\boldsymbol{p}}$
$L_{b4,\boldsymbol{p},\beta}$	$L_{b1,\boldsymbol{p}}$	$L_{b2,\boldsymbol{p}}$	$L_{b3,\boldsymbol{p}}$	$L_{b4,\boldsymbol{p}}$

By considering $C_{o1} + C_{o2,\beta} + L_{13,\boldsymbol{p}} + L_{b1,\boldsymbol{p},\beta} + L_{b3,\boldsymbol{p},\beta}$ and $C_{o1} + C_{o2,\beta} + L_{13,\boldsymbol{p}} + L_{b2,\boldsymbol{p},\beta} + L_{b4,\boldsymbol{p},\beta}$ for $\beta = -4, -\frac{5}{7}, -\frac{2}{7}, 3$, we see that

$$C_{o1} + C_{o2} + L_{13,\boldsymbol{p}} + L_{b1,\boldsymbol{p}} + L_{b3,\boldsymbol{p}} \rightsquigarrow C_{o1} + C_{o2} + L_{13,\boldsymbol{p}} + L_{b2,\boldsymbol{p}} + L_{b4,\boldsymbol{p}}$$

since we can deform while avoiding the finite number of exceptional values of β where the combinatorics becomes degenerated. Hence our statement follows. $\square$

Remark 4.18. By the proof in the above proposition, we see that there also exists deformations $B_{ij,\boldsymbol{p}} \rightsquigarrow B_{kl,\boldsymbol{p}}$ that preserves the combinatorics for $(i,j,k,l) = (1,2,3,4)$, $(1,4,2,3)$ or $(1,3,2,4)$.

Corollary 4.19. *The curve $B_{12,\boldsymbol{p}}$ can be continuously deformed to $B_{14,\boldsymbol{p}}$ while preserving the combinatorics* Cmb_{212}.

Proof. We use the same example in Proposition 4.17. We put

$$B_{12,\boldsymbol{p},\beta} := C_{o1} + C_{o2,\beta} + L_{13,\boldsymbol{p}} + L_{b1,\boldsymbol{p},\beta} + L_{b2,\boldsymbol{p},\beta},$$
$$B_{14,\boldsymbol{p},\beta} := C_{o1} + C_{o2,\beta} + L_{13,\boldsymbol{p}} + L_{b1,\boldsymbol{p},\beta} + L_{b4,\boldsymbol{p},\beta}.$$

By letting $\beta' = 0$, we see that $C'_{o2} := C_{o2,\beta'}$ is given by

$$C'_{o2} : t^2 + x^2 - tx - \frac{27}{4} = 0.$$

and the bitangent lines of $C_{o1} + C'_{o2}$ are

$$L_{b1,\boldsymbol{p},\beta'} : x - 3 = 0, \quad L_{b2,\boldsymbol{p},\beta'} : t + 3 = 0, \quad L_{b3,\boldsymbol{p},\beta'} : x + 3 = 0, \quad L_{b4,\boldsymbol{p},\beta'} : t - 3 = 0.$$

Then $B_{12,\boldsymbol{p},\beta'}, B_{14,\boldsymbol{p},\beta'} \in \mathcal{R}(\mathrm{Cmb}_{212})$ are transformed to each other by $[T,X,Z] \mapsto [-T,X,Z]$. Hence $B_{12,\boldsymbol{p}}$ can be deformed to $B_{14,\boldsymbol{p}}$, and our assertion follows. $\square$

We are now in position to prove the following proposition:

Proposition 4.20. *Any element $B \in \mathcal{R}(\mathrm{Cmb}_{212})$ is deformed to either $B_{12,\boldsymbol{p}}$ or $B_{13,\boldsymbol{p}}$ in Example 4.16, i.e., $\mathcal{R}(\mathrm{Cmb}_{212})$ has exactly two connected components.*

Proof. Our proof consists of two steps:

(I) Any element $B \in \mathcal{R}(\mathrm{Cmb}_{212})$ is deformed to $B_{ij,\boldsymbol{\tau}_o}$ $(i,j \in \{1,2,3,4\}, i \neq j)$ in Example 4.15.

(II) $B_{ij,\boldsymbol{\tau}_o}$ is deformed to either $B_{12,\boldsymbol{p}}$ or $B_{13,\boldsymbol{p}}$ in Example 4.16.

Since $B_{12,\boldsymbol{p}}$ and $B_{13,\boldsymbol{p}}$ belong to distinct connected components of $\mathcal{R}(\mathrm{Cmb}_{212})$, Steps (I) and (II) imply Proposition 4.20.

Step (I): After taking a suitable coordinate change and labeling the intersection points $C_1 \cap C_2$, we may assume that B is given as follows:

There exists $\boldsymbol{\tau} \in \mathcal{M}_0$ such that

$$B = B_{\boldsymbol{\tau}} = \mathcal{Q}_{\boldsymbol{\tau}} + L_{i_1 i_2,\boldsymbol{\tau}} + L_{bj_1,\boldsymbol{\tau}} + L_{bj_2,\boldsymbol{\tau}}, \quad i_1, i_2, j_1, j_2 \in \{1,2,3,4\},\ i_1 \neq i_2, j_1 \neq j_2$$

where $L_{bj_1,\boldsymbol{\tau}}$ and $L_{bj_2,\boldsymbol{\tau}}$ are given by $\tilde{f}_{\mathcal{Q}_{\boldsymbol{\tau}},z_o}(Q_{j_1})$ and $\tilde{f}_{\mathcal{Q}_{\boldsymbol{\tau}},z_o}(Q_{j_2})$, respectively.

Now consider a path $\gamma : [0,1] \to \mathcal{M}_0$ such that (i) $\gamma(0) = \boldsymbol{\tau}$, $\gamma(1) = \boldsymbol{\tau_o}$ (ii) $B_{\gamma(s)} \in \mathcal{R}(\mathrm{Cmb}_{212})$ for $\forall s \in [0,1]$, and (iii) $t_1 \rightsquigarrow -2$, $t_2 \rightsquigarrow -1$, $t_3 \rightsquigarrow 1$, $t_4 \rightsquigarrow 2$. Then B is deformed to $B_{ij,\boldsymbol{\tau}_o}$. Hence we have the assertion in Step (I).

Step (II): Let $B_{ij,\boldsymbol{\tau}_o}$ be the CL-arrangement as in Example 4.15. By Corollary 4.13, $\mathcal{R}(\mathrm{Cmb}_{212})$ has at least two connected components. Here we show that any $B_{ij,\boldsymbol{\tau}_o}$ which has 6 possibilities can be continuously deformed to either $B_{12,\boldsymbol{p}}$ or $B_{13,\boldsymbol{p}}$.

Let $\phi : \mathbb{P}^2 \to \mathbb{P}^2$ be a projective transformation such that $\phi(C_o) = C_{o1}$. We choose $\boldsymbol{c} = (c_1, c_2, c_3, c_4)$ and $\boldsymbol{\tau_c} = (\lambda_{\boldsymbol{c}}, \boldsymbol{c}) \in \mathcal{M}_0$ such that $\phi(\mathcal{Q}_{\boldsymbol{\tau_c}}) = C_{o1} + C_{o2}$. Now we choose a path γ in $\mathcal{M}_0$ as in Step (I) such that $\gamma(0) = \boldsymbol{\tau_c}$ and $\gamma(1) = \boldsymbol{\tau}_o$. Then we infer that $B_{ij,\boldsymbol{\tau}_o}$ is continuously deformed to $B_{i_1j_1,\boldsymbol{\tau_c}}$ in $\mathcal{R}(\mathrm{Cmb}_{212})$. Since $\phi(B_{i_1j_1,\boldsymbol{\tau_c}}) = B_{i_2j_2,\boldsymbol{p}}$ for some i_2, j_2, we see that $B_{ij,\boldsymbol{\tau}_o}$ is continuously deformed to $B_{i_2j_2,\boldsymbol{p}}$. Now by Proposition 4.17 and Corollary 4.19, $B_{ij,\boldsymbol{p}}$ is deformed to either $B_{12,\boldsymbol{p}}$ or $B_{13,\boldsymbol{p}}$ and we have the assertion in Step (II). □

4.10. Cmb$_{213}$

We keep our notation in Cmb$_{211}$. Let B be an arbitrary element in $\mathcal{R}(\mathrm{Cmb}_{213})$ and we may assume that B is given in the form

$$B = B_{\boldsymbol{\tau}} = \mathcal{Q}_{\boldsymbol{\tau}} + L_{b_1,\boldsymbol{\tau}} + L_{b_2,\boldsymbol{\tau}} + L_{b_3,\boldsymbol{\tau}}$$

for some $\boldsymbol{\tau} = (\lambda, \boldsymbol{t}) \in \mathcal{M}_0$. In other words, B is determined by the residual bitangent $L_{b_4,\boldsymbol{\tau}}$. Hence we infer that it is enough to show that $\mathcal{Q}_{\boldsymbol{\tau}} + L_{b_4,\boldsymbol{\tau}}$ can be continuously deformed to $C_{o1} + C_{o2} + L_{bi,\boldsymbol{p}}$ with keeping the combinatorics. This is done in the same way as in Cmb$_{211}$. Hence $\mathcal{R}(\mathrm{Cmb}_{213})$ is connected.

4.11. Cmb$_{221}$

Let $B = C_1 + C_2 + D + M \in \mathcal{R}(\mathrm{Cmb}_{221})$. As we have seen in Subsection 4.10, $C_1 + C_2 + M$ can be continuously deformed to $C_{o1} + C_{o2} + L_{bi,\boldsymbol{p}}$ while keeping the combinatorics. Since D is a member of the pencil generated by C_1 and C_2, such a conic is deformed simultaneously with keeping Cmb$_{221}$. Hence we infer that B is continuously deformed to $C_{o1} + C_{o2} + C' + L_{bi,\boldsymbol{p}}$, where C' is a member of the pencil generated by C_{o1} and C_{o2}. Hence $\mathcal{R}(\mathrm{Cmb}_{221})$ is connected.

4.12. Cmb$_{222}$

For Cmb$_{222}$, any element $B = C_1 + C_2 + D + M \in \mathcal{R}(\mathrm{Cmb}_{222})$ is determined by $C_1 + C_2 + D$. As we have seen in [**1**, Lemma 3.1], $\mathcal{R}(\mathrm{Cmb}_{C_1+C_2+D})$ is connected and so is $\mathcal{R}(\mathrm{Cmb}_{222})$.

4.13. Cmb$_{223}$

This case was discussed in [**1**] and $\mathcal{R}(\mathrm{Cmb}_{223})$ has exactly two connected components.

4.14. Cmb$_{224}$

Choose $C_1 + C_2 + D + M \in \mathrm{Cmb}_{224}$. Let us start with the following lemma.

LEMMA 4.21. *Let $C_1 + C_2 + D$ be a conic arrangement as above. Then $\mathcal{R}(\mathrm{Cmb}_{C_1+C_2+D})$ is connected.*

PROOF. We label the four tangent points between $(C_1 + C_2) \cap D$ by $C_1 \cap D = \{q_1, q_3\}$ and $C_2 \cap D = \{q_2, q_4\}$. Let $L_{q_1q_3}$ (resp. $L_{q_2q_4}$) be a line connecting q_1 and q_3

(resp. q_2 and q_4). Then C_1 (resp. C_2) is a member of the pencil generated by D and $2L_{q_1q_2}$ (resp. D and $2L_{q_2q_4}$).

Now consider a projective transformation $\phi : \mathbb{P}^2 \to \mathbb{P}^2$ such that $\phi(D) = D_o$ where D_o is a conic given by $T^2 + X^2 = Z^2$. Put $q_{oi} = \phi(q_i)$ $(i = 1, 2, 3, 4)$. Then $\phi(L_{q_1q_3}) = L_{q_{o1}q_{o3}}$ and $\phi(L_{q_2q_4}) = L_{q_{o2}q_{o4}}$. Now we move q_{oi} $(i = 1, 2, 3, 4)$ continuously so that

$$q_{o1} \rightsquigarrow (1, 0),\ q_{o2} \rightsquigarrow (0, 1),\ q_{o3} \rightsquigarrow (-1, 0),\ q_{o4} \rightsquigarrow (0, -1).$$

Since the two pencils of conics are also continuously deformed along with q_{oi} $(i = 1, 2, 3, 4)$, we infer that $C_1 + C_2 + D$ is continuously deformed to $C_{1a} + C_{2a} + D_o$ while keeping the combinatorics $\mathrm{Cmb}_{C_1+C_2+D}$, where

$$C_{1a} : \left(\frac{t}{a}\right)^2 + x^2 = 1, \quad C_{2a} : t^2 + \left(\frac{x}{a}\right)^2 = 1, \quad (a \in \mathbb{R}_{>1}).$$

□

In [**34**], we have seen there exists a Zariski pair for Cmb_{224}. Hence $\mathcal{R}(\mathrm{Cmb}_{224})$ has at least two connected components. In this subsection, we will show that $\mathcal{R}(\mathrm{Cmb}_{224})$ has just two connected components. We denote a member of $\mathcal{R}(\mathrm{Cmb}_{224})$ by $B = B_{\mathcal{P}_2}+D+M$, where D is a conic of type $D(2, 4)$. Next, we consider an explicit example, which gives 'base points' in $\mathcal{R}(\mathrm{Cmb}_{224})$.

Example 4.22. Let $\mathcal{Q}_{\boldsymbol{\tau}_o}$ and $S_{\mathcal{Q}_{\boldsymbol{\tau}_o},z_o}$ ($\boldsymbol{\tau}_o = (\lambda_o, \boldsymbol{a})$) be the quartic and the rational elliptic surface considered in Example 4.15. In this case, we have

$$\begin{aligned} P_{12} &= \left(-3t - 2, -i\sqrt{10}t^2 - 3i\sqrt{10}t - 2i\sqrt{10}\right), \\ P_{13} &= \left(-t + 2, -i\sqrt{2}t^2 - i\sqrt{2}t + 2i\sqrt{2}\right), \\ P_{14} &= \left(4, -i\,t^2 + 4i\right). \end{aligned}$$

and by using the duplication formula of the group law on $E_{\mathcal{Q}_{\boldsymbol{\tau}_o},z_o}$

$$\begin{aligned} [2]P_{12} &= \left(\frac{1}{10}t^2, -\frac{3}{100}i\sqrt{10}(t^2 + 20)t\right), \\ [2]P_{13} &= \left(\frac{1}{2}t^2, -\frac{1}{4}i\sqrt{2}(t + 2)(t - 2)t\right), \\ [2]P_{14} &= \left(t^2 - \frac{9}{4}, -\frac{3}{2}i(t^2 + \frac{19}{4})\right). \end{aligned}$$

Now put

$$\begin{aligned} D_{24,\boldsymbol{\tau}_o} &: \bar{f}_{\mathcal{Q}_{\boldsymbol{\tau}_o},z_o}(s_{[2]P_{12}}) = x - \frac{1}{10}t^2 = 0, \\ L_{12,\boldsymbol{\tau}_o} &: \bar{f}_{\mathcal{Q}_{\boldsymbol{\tau}_o},z_o}(s_{P_{12}}) = x + 3t + 2 = 0, \\ L_{13,\boldsymbol{\tau}_o} &: \bar{f}_{\mathcal{Q}_{\boldsymbol{\tau}_o},z_o}(s_{P_{13}}) = x + t - 2 = 0, \\ L_{14,\boldsymbol{\tau}_o} &: \bar{f}_{\mathcal{Q}_{\boldsymbol{\tau}_o},z_o}(s_{P_{14}}) = x - 4 = 0. \end{aligned}$$

We define $B_{1,\boldsymbol{\tau}_o}, B_{2,\boldsymbol{\tau}_o}$ and $B_{3,\boldsymbol{\tau}_o}$ to be

$$B_{1,\boldsymbol{\tau}_o} := \mathcal{Q}_{\boldsymbol{\tau}_o}+D_{24,\boldsymbol{\tau}_o}+L_{12,\boldsymbol{\tau}_o},\ B_{2,\boldsymbol{\tau}_o} := \mathcal{Q}_{\boldsymbol{\tau}_o}+D_{24,\boldsymbol{\tau}_o}+L_{13,\boldsymbol{\tau}_o},\ B_{3,\boldsymbol{\tau}_o} := \mathcal{Q}_{\boldsymbol{\tau}_o}+D_{24,\boldsymbol{\tau}_o}+L_{14,\boldsymbol{\tau}_o}.$$

Then by [**33**], $(B_{1,\boldsymbol{\tau}_o}, B_{2,\boldsymbol{\tau}_o})$ and $(B_{1,\boldsymbol{\tau}_o}, B_{3,\boldsymbol{\tau}_o})$ are Zariski pairs.

PROPOSITION 4.23. *The curve $B_{2,\boldsymbol{\tau}_o}$ can be deformed to $B_{3,\boldsymbol{\tau}_o}$ while preserving the combinatorics* Cmb_{224}.

PROOF. Let $C_{1a} + C_{2a} + D_o$ be the one as in the proof of Lemma 4.21. By Lemma 4.21, $\mathcal{Q}_{\boldsymbol{\tau}_o} + D_{24,\boldsymbol{\tau}_o}$ is continuously deformed to $C_{1a} + C_{2a} + D_o$ while keeping $\mathrm{Cmb}_{\mathcal{Q}_{\boldsymbol{\tau}_o}+D_{24,\boldsymbol{\tau}_o}}$ such that the points $p_i \in C_1 \cap C_2$ go to $p_{j,a} \in C_{1a} \cap C_{2a}$. Here we label $p_{j,a}$'s counterclockwisely so that p_1 goes to $p_{1,a}$. Let $L_{j,a}$ be lines passing though $p_{1,a}$ and another point $p_{j,a}$ in $C_{1a} \cap C_{2a}$. Since there is a projective transformation ϕ' such that $\phi'(L_{2,a}) = L_{4,a}$ and $\phi'(C_{1a}+C_{2a}+D_o) = C_{1a}+C_{2a}+D_o$, the curve $C_{1a}+C_{2a}+D_o$ can be deformed to $C_{1a} + C_{2a} + D_o$.

Now we show that $L_{12,\boldsymbol{\tau}_o} \rightsquigarrow L_{3,a}$. In fact, suppose that $L_{12,\boldsymbol{\tau}_o} \rightsquigarrow L_{2,a}$. As either $L_{13,\boldsymbol{\tau}_o} \rightsquigarrow L_{4,a}$ or $L_{14,\boldsymbol{\tau}_o} \rightsquigarrow L_{4,a}$, this means that there exists a homeomorphism from $(\mathbb{P}^2, B_{1,\boldsymbol{\tau}_o})$ to $(\mathbb{P}^2, B_{2,\boldsymbol{\tau}_o})$ or $(\mathbb{P}^2, B_{3,\boldsymbol{\tau}_o})$, but this is impossible. By a similar argument, $L_{12,\boldsymbol{\tau}_o} \rightsquigarrow L_{4,a}$ is also impossible. Hence $L_{12,\boldsymbol{\tau}_o} \rightsquigarrow L_{3,a}$. Thus $\{L_{13,\boldsymbol{\tau}_o}, L_{14,\boldsymbol{\tau}_o}\} \rightsquigarrow \{L_{2,a}, L_{4,a}\}$. Therefore our statement follows. □

PROPOSITION 4.24. *Let B be an arbitrary member in $\mathcal{R}(\mathrm{Cmb}_{224})$. Then B is continuously deformed to either $B_{1,\boldsymbol{\tau}_o}$ or $B_{2,\boldsymbol{\tau}_o}$ in Example 4.22. In particular, $\mathcal{R}(\mathrm{Cmb}_{224})$ has exactly two connected components.*

PROOF. After taking a suitable coordinate change, we may assume that B is given as follows:

There exists $\boldsymbol{\tau} \in \mathcal{M}_0$ such that

$$B = B_{\boldsymbol{\tau}} = \mathcal{Q}_{\boldsymbol{\tau}} + D_{\boldsymbol{\tau}} + L_{12,\boldsymbol{\tau}}$$

where $D_{\boldsymbol{\tau}}$ is either $\tilde{f}_{\mathcal{Q}_{\boldsymbol{\tau}},z_o}(s_{[2]P_{12,\boldsymbol{\tau}}})$, $\tilde{f}_{\mathcal{Q}_{\boldsymbol{\tau}},z_o}(s_{[2]P_{13,\boldsymbol{\tau}}})$ or $\tilde{f}_{\mathcal{Q}_{\boldsymbol{\tau}},z_o}(s_{[2]P_{23,\boldsymbol{\tau}}})$.

Case $D_{\boldsymbol{\tau}}=\tilde{f}_{\mathcal{Q}_{\boldsymbol{\tau}},z_o}(s_{[2]P_{12,\boldsymbol{\tau}}})$. Consider a path $\gamma : [0,1] \to \mathcal{M}_0$ such that (i) $\gamma(0) = \boldsymbol{\tau}$, $\gamma(1) = \boldsymbol{\tau}_o$, (ii) $B_{\gamma(s)} \in \mathcal{R}(\mathrm{Cmb}_{224})$ for $\forall s \in [0,1]$, and (iii) $t_1 \rightsquigarrow -2$, $t_2 \rightsquigarrow -1$, $t_3 \rightsquigarrow 1$, $t_4 \rightsquigarrow 2$. Then shows that B is continuously deformed to $B_{1,\boldsymbol{\tau}_o}$ while keeping the combinatorics.

Case $D_{\boldsymbol{\tau}}=\tilde{f}_{\mathcal{Q}_{\boldsymbol{\tau}},z_o}(s_{[2]P_{13,\boldsymbol{\tau}}})$. Consider a path $\gamma : [0,1] \to \mathcal{M}_0$ such that (i) $\gamma(0) = \boldsymbol{\tau}$, $\gamma(1) = \boldsymbol{\tau}_o$, (ii) $B_{\gamma(s)} \in \mathcal{R}(\mathrm{Cmb}_{224})$ for $\forall s \in [0,1]$, and (iii) $t_1 \rightsquigarrow -2$, $t_2 \rightsquigarrow 1$, $t_3 \rightsquigarrow -1$, $t_4 \rightsquigarrow 2$. Note that such that p_2 and p_3 are interchanged under this operation. Then $L_{12,\boldsymbol{\tau}}$ (resp.$D_{\boldsymbol{\tau}}$) is deformed to $L_{13,\boldsymbol{\tau}_o}$ (resp.$D_{\boldsymbol{\tau}_o}$). Hence B is continuously deformed to $B_{2,\boldsymbol{\tau}_o}$ while keeping the combinatorics.

Case $D_{\boldsymbol{\tau}}=\tilde{f}_{\mathcal{Q}_{\boldsymbol{\tau}},z_o}(s_{[2]P_{23,\boldsymbol{\tau}}})$. Consider a path $\gamma : [0,1] \to \mathcal{M}_0$ such that (i) $\gamma(0) = \boldsymbol{\tau}$, $\gamma(1) = \boldsymbol{\tau}_o$ (ii) $B_{\gamma(s)} \in \mathcal{R}(\mathrm{Cmb}_{224})$ for $\forall s \in [0,1]$,and (iii) $t_1 \rightsquigarrow -2$, $t_2 \rightsquigarrow 2$, $t_3 \rightsquigarrow 1$, $t_4 \rightsquigarrow -1$. Note that such that p_2 and p_4 are interchanged under this operation. Then $L_{12,\boldsymbol{\tau}}$ is deformed to $L_{14,\boldsymbol{\tau}_o}$. Since $[2]P_{34,\boldsymbol{\tau}_o} = [2]P_{12,\boldsymbol{\tau}_o}$, $D_{\boldsymbol{\tau}}$ is deformed to $\tilde{f}_{\mathcal{Q}_{\boldsymbol{\tau}_o},z_o}(s_{[2]P_{34,\boldsymbol{\tau}_o}}) =$

$\tilde{f}_{\mathcal{Q}_{\boldsymbol{\tau}_o},z_o}(s_{[2]P_{12,\boldsymbol{\tau}_o}}) = D_{\boldsymbol{\tau}_o}$. Hence B is continuously deformed to $B_{3,\boldsymbol{\tau}_o}$, while preserving the combinatorics.

By Proposition 4.23, $B_{2,\boldsymbol{\tau}_o}$ can be deformed to $B_{3,\boldsymbol{\tau}_o}$ while keeping the combinatorics. Hence our statement follows. □

4.15. $\mathbf{Cmb_{225}}$

Take $B = C_1 + C_2 + D + M \in \mathcal{R}(\mathrm{Cmb}_{225})$ arbitrarily. By Remark 4.11 and Lemma 4.21, $C_1 + C_2 + D + M$ is continuously deformed to $C_{1a} + C_{2a} + D_o + M_o$ keeping with Cmb_{225} where C_{1a},C_{2a},D_o are as in the proof of Lemma 4.21 and M_o is one of $x = t \pm \sqrt{a^2+1}, x = -t \pm \sqrt{a^2+1}$. Since $C_{1a} + C_{2a} + D_o +$ (a bitangent) is transformed to each other by some projective transformation, $\mathcal{R}(\mathrm{Cmb}_{225})$ is connected.

A. A remark on the fundamental groups

In this section, we study the fundamental groups of the arrangements in the Zariski pairs given in Theorem 1.1. We calculate a presentation of the fundamental group for each case using `SageMath 10.4` [**31**] and the package `Sirocco` [**19**]. Then we calculate the number of epimorphisms from the fundamental groups to S_3, the symmetric group of degree 3, using `GAP` [**14**]. The existence of such epimorphisms implies that the group is non-abelian, and the difference in the number of epimorphisms allows us to distinguish non-isomorphic fundamental groups. We use the following commands:

- `ProjectivePlaneCurveArrangements()`

  This command constructs projective plane curve arrangements as a `SageMath` object.

- `fundamental_group()`

  This command computes the fundamental group of the projective plane curve arrangement in terms of generators and relations. The package `Sirocco` must be enabled.

- `meridian()`

  This command returns the information of the meridians of the irreducible components of the arrangement in terms of the generators of the fundamental groups. The package `Sirocco` must be enabled.

- `GQuotients()`

  This is a `GAP` command that computes epimorphisms from a group to a given finite group. The output is given in terms of the images of the generators.

and the results are summarized in the following table:

Combinatorics	Arrangement	abelian/non-abelian	Num. of epi. to S_3
Cmb_{123}	$B_{1,\boldsymbol{\tau}_o}$	non-abelian	5
	$B_{2,\boldsymbol{\tau}_o}$	non-abelian	3
Cmb_{124}	$B_{1,\boldsymbol{\tau}_o}$	non-abelian	7
	$B_{2,\boldsymbol{\tau}_o}$	non-abelian	6
Cmb_{212}	$B_{13,\boldsymbol{p}}$	non-abelian	7
	$B_{12,\boldsymbol{p}}$	non-abelian	6
Cmb_{223}	B_1, B_2	free abelian of rank 3	0
Cmb_{224}	$B_{1,\boldsymbol{\tau}_o}$	non-abelian	7
	$B_{2,\boldsymbol{\tau}_o}$	non-abelian	6

REMARK A.1.

(i) The fundamental groups for Cmb_{124} and Cmb_{224} were computed in [**3**]. Also the fundamental groups for Cmb_{223} were calculated in [**1**].

(ii) For each epimorphism to S_3, the orders of the images of the meridians of the irreducible components can be read off from the output of `GQuotients()`. We can construct S_3-covers of $\mathbb{P}^2$ with the corresponding branch data using the methods in [**32, 33**] which support the correctness of the above calculations.

References

[1] Amram, M., Bannai, S., Shirane, T., Sinichkin, U., Tokunaga, H.o.: The realization space of a certain conic line arrangement of degree 7 and a π_1-equivalent Zariski pair (2023). ArXiv:2307.01736 To appear in Israel J. of Math.

[2] Amram, M., Garber, D., Teicher, M.: Fundamental groups of tangent conic-line arrangements with singularities up to order 6. Math. Z. **256**(4), 837–870 (2007). DOI 10.1007/s00209-007-0109-4

[3] Amram, M., Shwartz, R., Sinichkin, U., Tan, S.L., Tokunaga, H.o.: Zariski pairs of conic-line arrangements of degree 7 and 8 via fundamental groups (2023). ArXiv:2106.0350

[4] Amram, M., Teicher, M., Uludag, A.M.: Fundamental groups of some quadric-line arrangements. Topology Appl. **130**(2), 159–173 (2003). DOI 10.1016/S0166-8641(02)00218-3

[5] Artal Bartolo, E., Bannai, S., Shirane, T., Tokunaga, H.: Torsion divisors of plane curves and Zariski pairs. St. Petersburg Math. J. **34**(5), 721–736 (2023). DOI 10.1090/spmj/1776. Translated from Algebra i Analiz **34** (2022), no. 5

[6] Artal Bartolo, E., Cogolludo, J.I., Tokunaga, H.o.: A survey on Zariski pairs. In: Algebraic geometry in East Asia—Hanoi 2005, *Adv. Stud. Pure Math.*, vol. 50, pp. 1–100. Math. Soc. Japan, Tokyo (2008). DOI 10.2969/aspm/05010001

[7] Bannai, S.: A note on splitting curves of plane quartics and multi-sections of rational elliptic surfaces. Topology Appl. **202**, 428–439 (2016). DOI 10.1016/j.topol.2016.02.005

[8] Bannai, S., Kawana, N., Masuya, R., Tokunaga, H.: Trisections on certain rational elliptic surfaces and families of Zariski pairs degenerating to the same conic-line arrangement. Geom. Dedicata **216**(1), Paper No. 8, 23 (2022). DOI 10.1007/s10711-021-00672-5

[9] Bannai, S., Tokunaga, H.o.: Geometry of bisections of elliptic surfaces and Zariski N-plets for conic arrangements. Geom. Dedicata **178**, 219–237 (2015). DOI 10.1007/s10711-015-0054-z

[10] Bannai, S., Tokunaga, H.o.: Geometry of bisections of elliptic surfaces and Zariski N-plets II. Topology Appl. **231**, 10–25 (2017). DOI 10.1016/j.topol.2017.09.003

[11] Dimca, A., Pokora, P.: On conic-line arrangements with nodes, tacnodes, and ordinary triple points. J. Algebraic Combin. **56**(2), 403–424 (2022). DOI 10.1007/s10801-022-01116-3

[12] Friedman, M., Garber, D.: On the structure of fundamental groups of conic-line arrangements having a cycle in their graph. Topology Appl. **177**, 34–58 (2014). DOI 10.1016/j.topol.2014.05.013

[13] Friedman, M., Garber, D.: On the structure of conjugation-free fundamental groups of conic-line arrangements. J. Homotopy Relat. Struct. **10**(4), 685–734 (2015). DOI 10.1007/s40062-014-0081-8

[14] The GAP Group: GAP – Groups, Algorithms, and Programming, Version 4.13.1 (2024). URL `https://www.gap-system.org`

[15] Guerville-Ballé, B.: On the nonconnectedness of moduli spaces of arrangements: The splitting polygon structure. Kyoto Journal of Mathematics **65**(1), 153 – 170 (2025). DOI 10.1215/21562261-2024-0011

[16] Guerville-Ballé, B.: On the nonconnectedness of moduli spaces of arrangements, ii: construction of nonarithmetic pairs (2024). URL `https://arxiv.org/abs/2409.18022`

[17] Kodaira, K.: On compact analytic surfaces. II, III. Ann. of Math. (2) 77 (1963), 563–626; ibid. **78**, 1–40 (1963). DOI 10.2307/1970500

[18] Macinic, A.: On the deletion and addtion of a conic to a free curve (2024). ArXiv:2408.11714

[19] Marco-Buzunariz, M.Á., Rodríguez, M.: Sirocco: A library for certified polynomial root continuation. In: G.M. Greuel, T. Koch, P. Paule, A. Sommese (eds.) Mathematical Software – ICMS 2016, pp. 191–197. Springer International Publishing, Cham (2016). URL `https://doi.org/10.1007/978-3-319-42432-3_24`

[20] Masuya, R.: Geometry of weak-bitangent lines to quartic curves and sections on certain rational elliptic surfaces. Hiroshima Math. J. **54**(1), 1–31 (2024). DOI 10.32917/h2021060

[21] Miranda, R.: The basic theory of elliptic surfaces. Dottorato di Ricerca in Matematica. [Doctorate in Mathematical Research]. ETS Editrice, Pisa (1989)

[22] Namba, M., Tsuchihashi, H.: On the fundamental groups of Galois covering spaces of the projective plane. Geom. Dedicata **105**, 85–105 (2004). DOI 10.1023/B:GEOM.0000024693.88450.d9

[23] Naruki, I.: Some invariants for conics and their applications. Publ. Res. Inst. Math. Sci. **19**(3), 1139–1151 (1983). DOI 10.2977/prims/1195182023

[24] Nazir, S., Yoshinaga, M.: On the connectivity of the realization spaces of line arrangements. Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **11**(4), 921–937 (2012)

[25] Oguiso, K., Shioda, T.: The Mordell-Weil lattice of a rational elliptic surface. Comment. Math. Univ. St. Paul. **40**(1), 83–99 (1991)

[26] Pokora, P., Szemberg, T.: Conic-line arrangements in the complex projective plane. Discrete Comput. Geom. **69**(4), 1121–1138 (2023). DOI 10.1007/s00454-022-00397-6

[27] Schenck, H., Tohăneanu, c.S.O.: Freeness of conic-line arrangements in $\mathbb{P}^2$. Comment. Math. Helv. **84**(2), 235–258 (2009). DOI 10.4171/CMH/161

[28] Shioda, T.: On the Mordell-Weil lattices. Comment. Math. Univ. St. Paul. **39**(2), 211–240 (1990)

[29] Shirane, T.: Connected numbers and the embedded topology of plane curves. Canad. Math. Bull. **61**(3), 650–658 (2018). DOI 10.4153/CMB-2017-066-5

[30] Silverman, J.H.: The arithmetic of elliptic curves, *Graduate Texts in Mathematics*, vol. 106, second edn. Springer, Dordrecht (2009). DOI 10.1007/978-0-387-09494-6

[31] The Sage Developers: SageMath, the Sage Mathematics Software System (Version 10.4) (2024). `https://www.sagemath.org`

[32] Tokunaga, H.o.: On dihedral Galois coverings. Canadian Journal of Mathematics **46**(6), 1299–1317 (1994). DOI 10.4153/CJM-1994-074-4

[33] Tokunaga, H.o.: Geometry of irreducible plane quartics and their quadratic residue conics. J. Singul. **2**, 170–190 (2010). DOI 10.5427/jsing.2010.2k

[34] Tokunaga, H.o.: Sections of elliptic surfaces and Zariski pairs for conic-line arrangements via dihedral covers. J. Math. Soc. Japan **66**(2), 613–640 (2014). DOI 10.2969/jmsj/06620613

Shinzo Bannai
Department of Applied Mathematics,
Faculty of Science,
Okayama University of Science,
1-1 Ridai-cho, Kita-ku, Okayama 700-0005 JAPAN
E-mail: bannai@ous.ac.jp

Hiro-o Tokunaga
Department of Mathematical Sciences,
Graduate School of Science,
Tokyo Metropolitan University,
1-1 Minami-Ohsawa, Hachiohji 192-0397 JAPAN
E-mail: tokunaga@tmu.ac.jp

Emiko Yorisaki
Department of Mathematical Sciences,
Graduate School of Science,
Tokyo Metropolitan University,
1-1 Minami-Ohsawa, Hachiohji 192-0397 JAPAN
E-mail: sagawa-emiko@ed.tmu.ac.jp