CLASSIFICATION OF ORIENTABLE TORUS BUNDLES OVER CLOSED ORIENTABLE SURFACES

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Dedicated to Professor Takashi Tsuboi on the occasion of his 70th birthday

Abstract. Let g be a non-negative integer, Σ_g a closed orientable surface of genus g, and $\mathcal{M}_g$ its mapping class group. We classify all the group homomorphisms $\pi_1(\Sigma_g) \to G$ up to the action of $\mathcal{M}_g$ on $\pi_1(\Sigma_g)$ in the following cases; (1) $G = PSL(2;\mathbb{Z})$, (2) $G = SL(2;\mathbb{Z})$. As an application of the case (2), we completely classify orientable T^2-bundles over closed orientable surfaces up to bundle isomorphism. In particular, we show that any orientable T^2-bundle over Σ_g with $g \geq 1$ is isomorphic to the fiber connected sum of g pieces of T^2-bundles over T^2. Moreover, the classification result in the case (1) can be generalized into the case where G is the free product of a finite number of finite cyclic groups. We also apply it to an extension problem of maps from a closed surface to a connected sum of lens spaces.

1. Introduction

Let g be a non-negative integer, Σ_g a closed orientable surface of genus g, and $\mathrm{Diff}_+(\Sigma_g)$ and $\mathcal{M}_g$ denote its orientation preserving diffeomorphism group and mapping class group, respectively. The aim of this paper is to classify orientable T^2-bundles over Σ_g up to bundle isomorphisms.

Orientable T^2-bundles appear in various scenes of geometry as important examples. For example, elliptic bundles over a complex curve like (some kind of) Hopf surfaces and primary Kodaira surfaces have been extensively studied as compact complex surfaces, and they are topologically nothing but orientable T^2-bundles over closed orientable surfaces. In [31], Thurston proved that a primary Kodaira surface admits non-Kähler symplectic structures when regarded as a smooth real 4-dimensional manifold. This example is called a Kodaira–Thurston manifold. In the same paper, he also gave a necessary and sufficient condition for a surface bundle over a surface to admit a compatible symplectic structure. The result was later refined by Geiges [5] and Walczak [35] in the case of T^2-bundles (see also [25]). In the 4-dimensional topology, Seifert fibered 4-manifolds, the analogue of Seifert fibered spaces in dimension 4, have been classified ([36, 32, 33, 14]), where orientable T^2-bundles are treated as the case without multiple fibers. Zieschang's result [36], the starting point of the classification, says that the isomorphism class of an orientable T^2-bundle over Σ_g with $g \geq 2$ is determined only by the fundamental group of the total space (see also [12, 11]). This is an important result, but is not worth being

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called the classification of bundle isomorphism classes. For, it is very difficult to distinguish the isomorphism classes of given two groups in general. Therefore, it is important to approach from the front to the classification of orientable T^2-bundles in terms of their monodromies and Euler classes.

In general, classification of smooth F-bundles up to bundle isomorphisms is one of the most fundamental problems in topology, but at the same time, a very difficult problem. For, the classification problem is reduced to that of homotopy classes of continuous maps from the base space to the classifying space $\mathrm{BDiff}(F)$, whose topology is usually hard to analyze. In our case, however, the situation is not so bad. Indeed, the oriented diffeomorphism group $\mathrm{Diff}_+(T^2)$, which is the structure group of an orientable T^2-bundle, is known to be homotopy equivalent to the affine transformation group $\mathrm{Aff}_+(T^2)$, and in particular, we have

$$\pi_0(\mathrm{Diff}_+(T^2)) \cong SL(2;\mathbb{Z}),\ \pi_1(\mathrm{Diff}_+(T^2)) \cong \mathbb{Z}^2,\ \pi_i(\mathrm{Diff}_+(T^2)) = 0\ (i \geq 2).$$

Then we can show that the isomorphism class of an orientable T^2-bundle over Σ_g is determined only by a group homomorphism $\pi_1(\Sigma_g) \to SL(2;\mathbb{Z})$ called the monodromy and a local coefficient cohomology class in $H^2(\Sigma_g;\{\pi_1(\Sigma_g)\})$ called the Euler class (Proposition 4.6). They are respectively represented by $A_1, B_1, \ldots, A_g, B_g \in SL(2;\mathbb{Z})$ with $[A_1, B_1]\cdots[A_g, B_g] = E_2$ and $(m, n) \in \mathbb{Z}^2$, so we may denote the bundle by $M(A_1, B_1, \ldots, A_g, B_g; m, n)$ (Proposition 4.3). Once the monodromy is fixed, we can easily deal with the Euler class, so what we really have to do is to classify the group homomorphisms $\pi_1(\Sigma_g) \to SL(2;\mathbb{Z})$ representing the monodromies of T^2-bundles up to the action of $\mathcal{M}_g$ on $\pi_1(\Sigma_g)$ and the conjugate action of $SL(2;\mathbb{Z})$ on itself.

When the base is S^2 ($g = 0$), the monodromy is trivial and hence only the Euler class is valid. Consequently, the set of isomorphism classes of T^2-bundles is $\mathbb{Z}_{\geq 0}$ (§ 5.1). When the base is T^2 ($g = 1$), the classification has been settled by Sakamoto–Fukuhara [27] (Theorem 5.1). On the other hand, when $g \geq 2$, the classification of monodromies has not yet been established though Zieschang's result stated above is known.

In this sense, the goal of this paper is to classify all the group homomorphisms

$$\mathrm{Hom}\,(\pi_1(\Sigma_g), SL(2;\mathbb{Z})) = \{\widetilde{\rho}\colon \pi_1(\Sigma_g) \to SL(2;\mathbb{Z}) \mid \widetilde{\rho}\,:\ \text{homomorphism}\}$$

up to the action of $\mathcal{M}_g$ when $g \geq 2$. Instead of trying to classify them directly, our first approach to this problem is to focus on the homomorphism $p \circ \widetilde{\rho}\colon \pi_1(\Sigma_g) \to PSL(2;\mathbb{Z})$ obtained from the monodromy homomorphism $\widetilde{\rho}$ by taking the composition with the quotient map

$$p\colon SL(2;\mathbb{Z}) \to PSL(2;\mathbb{Z}) = SL(2;\mathbb{Z})/\{\pm E_2\}.$$

Namely, our immediate goal is the classification of $\mathrm{Hom}\,(\pi_1(\Sigma_g), PSL(2;\mathbb{Z}))$. Now we regard Σ_g as the boundary of a genus-g handlebody V_g embedded in $\mathbb{R}^3$. This allows us to equip both V_g and its boundary $\Sigma_g = \partial V_g$ with the canonical orientations induced by the standard orientation on $\mathbb{R}^3$. Let $\{\alpha_i, \beta_i\}_{1\leq i\leq g}$ be the canonical generators of $\pi_1(\Sigma_g)$, that is, α_i and β_i are meridian and longitude, respectively, for each i with $1 \leq i \leq g$ (see Figure 1). Then we have

$$\pi_1(\Sigma_g) = \langle \alpha_1, \beta_1, \ldots, \alpha_g, \beta_g \mid [\alpha_1, \beta_1]\cdots[\alpha_g, \beta_g]\rangle.$$

Now we are ready to state our first theorem, which will be a breakthrough to the classification.

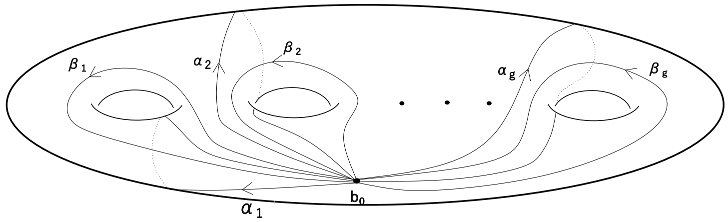

FIGURE 1. Generators of $\pi_1(\Sigma_g)$

Theorem 1.1. *For any homomorphism $\rho\colon \pi_1(\Sigma_g) \to PSL(2;\mathbb{Z})$, there exists a mapping class $f \in \mathcal{M}_g$ such that $(\rho \circ f_*)(\alpha_i) = e$ for each i with $1 \leq i \leq g$, where e denotes the identity element of $PSL(2;\mathbb{Z})$.*

The following is an immediate corollary to Theorem 1.1.

Corollary 1.2. For any homomorphism $\widetilde{\rho}\colon \pi_1(\Sigma_g) \to SL(2;\mathbb{Z})$, there exists a mapping class $f \in \mathcal{M}_g$ such that $(\widetilde{\rho} \circ f_*)(\alpha_i) = \pm E_2$ for each i with $1 \leq i \leq g$.

In the proof of Theorem 1.1, the tool called "charts" plays an essential role. This was originally introduced by Kamada [13] for the study of surface-knots, but is a very useful tool for describing G-representations of surface groups for a given finitely presented group G. It also works very well in our case, since $PSL(2;\mathbb{Z})$ is isomorphic to the free product $\mathbb{Z}_2 * \mathbb{Z}_3$. Then, based on Theorem 1.1 and combining several known results of group theory and mapping class group of handlebody V_g, we can obtain the classification of $\mathrm{Hom}\,(\pi_1(\Sigma_g), PSL(2;\mathbb{Z}))$ as follows. First, any subgroup H of $PSL(2;\mathbb{Z}) \cong \mathbb{Z}_2 * \mathbb{Z}_3$ is isomorphic to the free product of finite number of copies of $\mathbb{Z}$, $\mathbb{Z}_3$ and $\mathbb{Z}_2$ by Kurosh's subgroup theorem (Theorem 2.1). To be more precise, there exist subgroups $H_j \subset H$ $(1 \leq j \leq m)$ such that

$$H_1 \cong \cdots \cong H_k \cong \mathbb{Z}, \quad H_{k+1} \cong \cdots \cong H_l \cong \mathbb{Z}_3, \quad H_{l+1} \cong \cdots \cong H_m \cong \mathbb{Z}_2$$

and

$$H = (H_1 * \cdots * H_k) * (H_{k+1} * \cdots * H_l) * (H_{l+1} * \cdots * H_m),$$

where k, l and m are integers satisfying $0 \leq k \leq l \leq m$. Then our classification theorem is as follows.

Theorem 1.3. *For any homomorphism $\rho\colon \pi_1(\Sigma_g) \to PSL(2;\mathbb{Z}) \cong \mathbb{Z}_2 * \mathbb{Z}_3$, the subgroup $\mathrm{Im}(\rho) \subset \mathbb{Z}_2 * \mathbb{Z}_3$ can be described as*

$$\mathrm{Im}(\rho) = (H_1 * \cdots * H_k) * (H_{k+1} * \cdots * H_l) * (H_{l+1} * \cdots * H_m)$$

by Kurosh's subgroup theorem. Then there exists a mapping class $f \in \mathcal{M}_g$ such that

$$(\rho \circ f_*)(\alpha_i) = e \ \ (1 \leq i \leq g), \quad (\rho \circ f_*)(\beta_i) = \begin{cases} 1 \in H_i \cong \mathbb{Z} & (1 \leq i \leq k), \\ 1 \in H_i \cong \mathbb{Z}_3 & (k+1 \leq i \leq l), \\ 1 \in H_i \cong \mathbb{Z}_2 & (l+1 \leq i \leq m), \\ e & (m+1 \leq i \leq g), \end{cases}$$

where e denotes the identity element of $PSL(2;\mathbb{Z})$. In particular, for any two homomorphisms $\rho_1, \rho_2 \colon \pi_1(\Sigma_g) \to PSL(2;\mathbb{Z})$, the following two conditions are equivalent:

(1) $\mathrm{Im}(\rho_1) = \mathrm{Im}(\rho_2)$.
(2) *There exists a mapping class $f \in \mathcal{M}_g$ such that $\rho_2 = \rho_1 \circ f_*$.*

We call $\rho \circ f_*$ in Theorem 1.3 the normal form of ρ with respect to the action of the mapping class group, or simply, we say that it is of the normal form. Now we define the map

$$p_* \colon \mathrm{Hom}(\pi_1(\Sigma_g), SL(2;\mathbb{Z})) \to \mathrm{Hom}(\pi_1(\Sigma_g), PSL(2;\mathbb{Z}))$$

by $p_*(\widetilde{\rho}) = p \circ \widetilde{\rho}$. Then for each homomorphism $\rho \colon \pi_1(\Sigma_g) \to PSL(2;\mathbb{Z})$, $p_*^{-1}(\rho)$ is the set of all lifts $\widetilde{\rho}$ of ρ with respect to p, and we have $\# p_*^{-1}(\rho) = 2^{2g}$. When ρ is of the normal form, we obtain the following result about the number of orbits of the action of $\mathcal{M}_g$ to all the 2^{2g} lifts of ρ.

Theorem 1.4. *Let $\rho \in \mathrm{Hom}(\pi_1(\Sigma_g), PSL(2;\mathbb{Z}))$ be of the normal form. If $m > l$, then there exist just 2^k $\mathcal{M}_g$-orbits in $p_*^{-1}(\rho)$ corresponding to the choices of the signs of*

$$\widetilde{\rho}(\alpha_1) = \pm E_2, \ \cdots, \ \widetilde{\rho}(\alpha_k) = \pm E_2.$$

On the other hand, if $m = l$, then there exist 2^{k+1} orbits in $p_*^{-1}(\rho)$, whose details are as follows; there are 2^k orbits with $-E_2 \notin \mathrm{Im}(\widetilde{\rho})$ corresponding to the choices of $\widetilde{\rho}(\beta_1), \cdots, \widetilde{\rho}(\beta_k)$, and another 2^k orbits with $-E_2 \in \mathrm{Im}(\widetilde{\rho})$ corresponding to the choices of the signs of

$$\widetilde{\rho}(\alpha_1) = \pm E_2, \ \cdots, \ \widetilde{\rho}(\alpha_k) = \pm E_2.$$

The facts obtained by applying the results so far to the monodromies of orientable T^2-bundles over Σ_g are summarized as follows. First, for any orientable T^2-bundle ξ over Σ_g, the monodromy

$$\widetilde{\rho} \in \mathrm{Hom}(\pi_1(\Sigma_g), SL(2;\mathbb{Z}))$$

is uniquely determined up to the actions of $\mathcal{M}_g$ and $SL(2;\mathbb{Z})$. Applying Corollary 1.2 to this $\widetilde{\rho}$, we can show that ξ is decomposable into the fiber connected sum of g pieces of T^2-bundles over T^2 (Theorem 4.10). According to Theorem 1.3, the decomposition as a fiber connected sum can be taken in the form that respects the free product decomposition of the subgroup $\mathrm{Im}(p \circ \widetilde{\rho}) \leq PSL(2;\mathbb{Z})$. Based on these results, we have Theorem 1.4, which is the classification of $\mathrm{Hom}(\pi_1(\Sigma_g), SL(2;\mathbb{Z}))$ up to the action of $\mathcal{M}_g$. Thus, taking into account of the conjugate action of $SL(2;\mathbb{Z})$, Theorem 1.4 gives the complete classification of monodromies of orientable T^2-bundles. Adding a simple consideration about the Euler classes to it, we obtain the following classification of isomorphism classes of orientable T^2-bundles. This is our main theorem.

Theorem 1.5. *Let ξ_1 and ξ_2 be any two orientable T^2-bundles over Σ_g, and*

$$\widetilde{\rho}_1, \widetilde{\rho}_2 \in \mathrm{Hom}(\pi_1(\Sigma_g), SL(2;\mathbb{Z}))$$

their monodromy representations. Suppose that both $p \circ \widetilde{\rho}_1$ and $p \circ \widetilde{\rho}_2$ are of the normal forms in the sense of Theorem 1.3, and that ξ_1 and ξ_2 are described as

$$M(\varepsilon_1 E_2, B_1, \ldots, \varepsilon_g E_2, B_g; m, n), \quad M(\delta_1 E_2, C_1, \ldots, \delta_g E_2, C_g; k, l),$$

where $k, l, m, n \in \mathbb{Z}$, and for each i, ε_i and δ_i denote either 1 or -1. Then ξ_1 and ξ_2 are isomorphic to each other if and only if there exists $Q \in SL(2;\mathbb{Z})$ satisfying the following conditions:

(1) $\mathrm{Im}(\widetilde{\rho}_1) = Q\mathrm{Im}(\widetilde{\rho}_2)Q^{-1}$.
(2) *The two lifts $\widetilde{\rho}_1$ and $Q\widetilde{\rho}_2 Q^{-1}$ of $p \circ \widetilde{\rho}_1$ are in the same $\mathcal{M}_g$-orbit.*
(3) *When $\varepsilon_1 = \cdots = \varepsilon_g = 1$, there exist $\boldsymbol{x}_1, \dots, \boldsymbol{x}_g \in \mathbb{Z}^2$ such that*

$$\begin{pmatrix} m \\ n \end{pmatrix} - Q \begin{pmatrix} k \\ l \end{pmatrix} = \sum_{i=1}^{g} (B_i - E_2)\boldsymbol{x}_i,$$

and otherwise, there exist $\boldsymbol{x}_0, \boldsymbol{x}_1, \dots, \boldsymbol{x}_g \in \mathbb{Z}^2$ such that

$$\begin{pmatrix} m \\ n \end{pmatrix} - Q \begin{pmatrix} k \\ l \end{pmatrix} = 2\boldsymbol{x}_0 + \sum_{i=1}^{g} (B_i - E_2)\boldsymbol{x}_i.$$

Now let us reconsider about the existence of compatible symplectic structures on a T^2-bundle over a surface under the circumstance that the classification of the isomorphism classes has been done. Then the condition of Geiges and Walczak that we mentioned at the beginning can be briefly summarized as follows.

Theorem 1.6 (Theorems 6.4 and 6.5)**.** *Let g be a non-negative integer. Then an orientable T^2-bundle*

$$\pi\colon M(A_1, B_1, \dots, A_g, B_g; m, n) \to \Sigma_g$$

admits a compatible symplectic structure if and only if its Euler class is a torsion. This condition is also equivalent to that π is not isomorphic to

$$M(E_2, E_2, \dots, E_2, E_2; m, 0)\ (m \neq 0) \ \text{ nor } \ M(E_2, C^k, \dots, E_2, E_2; m, n)\ (n \neq 0),$$

where $C = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $k \in \mathbb{Z}$.

Finally, we note that Theorem 1.3 can be generalized to the case where the range of homomorphisms is the free product of a finite number of finite cyclic groups $\mathbb{Z}_{k_1} * \cdots * \mathbb{Z}_{k_n}$ (Theorem 3.1). Namely, $\mathrm{Hom}(\pi_1(\Sigma_g), \mathbb{Z}_{k_1} * \cdots * \mathbb{Z}_{k_n})$ can be classified in a similar way as in Theorem 1.3. Since $\mathbb{Z}_{k_1} * \cdots * \mathbb{Z}_{k_n}$ can be seen as the fundamental group of a connected sum of lens spaces, we obtain the following application of it.

Theorem 1.7 (Corollary 3.3)**.** *For any continuous map φ from a closed orientable surface Σ_g to the connected sum of a finite number of lens spaces, there exists a diffeomorphism $f \in \mathrm{Diff}_+(\Sigma_g)$ such that $\varphi \circ f$ can be continuously extended to the handlebody V_g.*

In such a way, our technique is useful also in a situation apart from T^2-bundles, so it is expected that the range of applications expand in the future.

This paper consists of the former part (§2, 3), the latter part (§4, 5) and the applications (§6). Most arguments are the fusion of algebraic viewpoints and topological viewpoints throughout this article, but if we had to say, the former half is the algebraic part and the latter half is the topological one. The specific organization is as follows.

- In §2, as a preliminary to §3, we review some group theory (§2.1), the mapping class group of handlebody V_g (§2.2) and charts for G-monodromies (§2.3). In §3, we prove the classification of $\mathrm{Hom}(\pi_1(\Sigma_g), PSL(2;\mathbb{Z}))$ up to the action of $\mathcal{M}_g$ (Theorem 1.3) using all the tools prepared in §2. We also discuss a generalization of Theorem 1.3 and its application.
- In §4, as a preliminary to §5, we summarize known results about T^2-bundles over surfaces. After recalling about orientable T^2-bundles over S^1 (§4.1), we review the monodromy and the Euler class of orientable T^2-bundles (§4.2), $SL(2;\mathbb{Z})$-bundles (§4.3) and fiber connected sums (§4.4). In §4.5, as an application of Corollary 1.2, we prove that any orientable T^2-bundles over Σ_g is decomposable as the fiber connected sum of g pieces of T^2-bundles over T^2 (Theorem 4.10). Based on these preparations, in §5, we first classify $\mathrm{Hom}(\pi_1(\Sigma_g), SL(2;\mathbb{Z}))$ up to the action of $\mathcal{M}_g$ and use it to prove the classification theorem of isomorphism classes of orientable T^2-bundles over Σ_g (Theorem 1.5).
- In §6, we explain the applications to symplectic geometry (Theorems 6.4, 6.5, 6.6 and 6.7).

2. Preliminaries

2.1. Some group theory. We fix the notations as follows.

- Let G be a group and H its subgroup. In this case, we write $H \leq G$. If H is a normal subgroup of G, then we write $H \trianglelefteq G$.
- We denote by F_n the free group generated by n elements $a_1, \ldots, a_n$, and its automorphism group by $\mathrm{Aut}(F_n)$.
- Let G be a group, and $S = \{s_1, \ldots, s_k\}$ a finite subset of it. In this case, $\langle S \rangle$ denotes the subgroup of G generated by S. If there is no fear of confusion with the presentation of the free group, we also denote it by $\langle s_1, \ldots, s_k \rangle$. On the other hand, we denote the smallest normal subgroup containing S by $N(S)$ or $\langle s_1, \ldots, s_k \rangle^G$.

First we recall about the group structures of $SL(2;\mathbb{Z})$ and $PSL(2;\mathbb{Z})$. As is well-known, the special linear group

$$SL(2;\mathbb{Z}) = \left\{ \begin{pmatrix} a & c \\ b & d \end{pmatrix} \mid a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\}$$

is presented as

$$SL(2;\mathbb{Z}) = \langle s, t \mid s^4, s^2 t^{-3} \rangle,$$

where

$$s = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad t = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}.$$

Thus $SL(2;\mathbb{Z})$ is isomorphic to the free product with amalgamation $\mathbb{Z}_4 *_{\mathbb{Z}_2} \mathbb{Z}_6$. On the other hand, the special projective linear group $PSL(2;\mathbb{Z})$ is the quotient of $SL(2;\mathbb{Z})$ by its center $\{\pm E_2\}$. Let $p\colon SL(2;\mathbb{Z}) \to PSL(2;\mathbb{Z})$ be the quotient map and set $a = p(s)$, $b = p(t)$. Then the presentation of $PSL(2;\mathbb{Z})$ is given by

$$PSL(2;\mathbb{Z}) = \langle a, b \mid a^2, b^3 \rangle.$$

Namely, $PSL(2;\mathbb{Z})$ is isomorphic to the free product $\mathbb{Z}_2 * \mathbb{Z}_3$.

Next we review known facts about free groups and free product of groups.

Theorem 2.1 (Kurosh's subgroup theorem [22]). *Let G_1 and G_2 be groups. Then any subgroup H of the free product $G = G_1 * G_2$ can be described in the form*

$$H = F(X) * (*_{i\in I} g_i A_i g_i^{-1}) * (*_{j\in J} f_j B_j f_j^{-1}),$$

where $F(X)$ is the free group generated by a subset $X \subset G$, and $g_i, f_j \in G$, $A_i \leq G_1$ and $B_j \leq G_2$ for any $i \in I$ and $j \in J$.

Theorem 2.2 (Grushko [8], see also [29]). *For any surjective homomorphism φ from the free group F_n to a free product $H = H_1 * H_2$, there exist subgroups G_1 and G_2 of F_n such that $\varphi(G_1) = H_1$, $\varphi(G_2) = H_2$ and $G_1 * G_2 = F_n$.*

Remark 2.3. By Theorem 2.1, the only decomposition of F_n as a free product of two groups is described as $F_n \cong F_i * F_{n-i}$ for some i with $0 \leq i \leq n$. Thus G_1 and G_2 in Theorem 2.2 are written as $G_1 \cong F_i$ and $G_2 \cong F_{n-i}$. More precisely, there exists an element $\gamma \in \mathrm{Aut}(F_n)$ such that

$$G_1 = \gamma\left(\langle a_1, \ldots, a_i\rangle\right),\ G_2 = \gamma\left(\langle a_{i+1}, \ldots, a_n\rangle\right).$$

Theorem 2.4 (Nielsen [26], see also [1]). *Let i and j be integers with $1 \leq i \leq n$ and $1 \leq j \leq n-1$. We define three automorphisms σ_i, τ_j, η of the free group F_n as follows:*

$$\sigma_i\colon \begin{cases} a_i \mapsto a_i^{-1} \\ a_k \mapsto a_k \quad (k \neq i), \end{cases} \qquad \tau_j\colon \begin{cases} a_j \mapsto a_{j+1} \\ a_{j+1} \mapsto a_j \\ a_k \mapsto a_k \quad (k \neq j, j+1), \end{cases} \qquad \eta\colon \begin{cases} a_1 \mapsto a_2^{-1} a_1 \\ a_2 \mapsto a_2^{-1} \\ a_k \mapsto a_k \quad (k > 2). \end{cases}$$

Then σ_i $(1 \leq i \leq n)$, τ_j $(1 \leq j \leq n-1)$ and η form a generating system of $\mathrm{Aut}(F_n)$.

Remark 2.5. Theorem 2.4 itself can be deduced from a result of Nielsen [26]. On the other hand, Armstrong–Forrest–Vogtmann [1] determined all the relations among σ_i, τ_j and η to provide an explicit presentation of $\mathrm{Aut}(F_n)$.

Since the abelianization of F_n is $\mathbb{Z}^n$, each element γ of $\mathrm{Aut}(F_n)$ canonically induces an element $\bar{\gamma}$ of $GL(n;\mathbb{Z})$. Notice that $\bar{\sigma}_i$, $\bar{\tau}_j$, $\bar{\eta}$ which are induced by the generators σ_i, τ_j, η obtained in the above theorem generate $GL(n;\mathbb{Z})$, since they are nothing but the three types of elementary transformations. Therefore, for any element in $GL(n;\mathbb{Z})$, there exists an element in $\mathrm{Aut}(F_n)$ that induces it.

Proposition 2.6. For any surjective homomorphism $\varphi\colon F_n \to \mathbb{Z}_k$, there exists an element $\gamma \in \mathrm{Aut}(F_n)$ such that $(\varphi \circ \gamma)(a_1) = 1$ and $(\varphi \circ \gamma)(a_i) = 0$ $(2 \leq i \leq n)$.

Proof. We discuss only the case where the range is $\mathbb{Z}$ $(k = 0)$. The $\mathbb{Z}$-linear map $\bar{\varphi}\colon \mathbb{Z}^n \to \mathbb{Z}$ induced by φ is represented by a matrix

$$\begin{pmatrix} p_1 & p_2 & \cdots & p_n \end{pmatrix},$$

where p_i $(1 \leq i \leq n)$ are integers such that their greatest common divisor is 1. It can be transformed to

$$\begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix}$$

by a finite sequence of elementary column operations. By composing those operations, we obtain an invertible matrix $\bar{\gamma} \in GL(n;\mathbb{Z})$. Then an element $\gamma \in \mathrm{Aut}(F_n)$ that induces $\bar{\gamma}$ satisfies desired conditions. The same argument works even when $k \neq 0$. $\square$

2.2. **Mapping class group of handlebody of genus g.**

Let $\mathcal{H}_g$ denote the mapping class group of the handlebody V_g. Since a self-diffeomorphism of V_g induces that of Σ_g when restricted to the boundary $\partial V_g = \Sigma_g$, $\mathcal{H}_g$ is naturally embedded into $\mathcal{M}_g$ because V_g is an irreducible 3-manifold. We denote the subgroup of $\mathcal{M}_g$ obtained as the image by $\mathcal{H}_g^*$.

In this paper, the product of two elements $[f_1], [f_2] \in \mathcal{M}_g$ is defined by

$$[f_1][f_2] = [f_2 \circ f_1].$$

Accordingly, we regard the action of $\mathcal{M}_g$ on Σ_g as a right action. This convention is necessary in order to make each monodromy a homomorphism (see § 4). In line with this, we also define the product on $\mathcal{H}_g$ and $\mathrm{Aut}(F_g)$ in the reverse order of composition, and regard their actions on V_g and F_g as right actions. Furthermore, $\mathcal{M}_g$ and $\mathcal{H}_g$ also act from the right on $\pi_1(\Sigma_g)$ and $\pi_1(V_g)$, respectively. Note that $\pi_1(V_g)$ is isomorphic to the free group F_g, since V_g is homotopy equivalent to the bouquet of g circles.

Let $\iota\colon \Sigma_g \to V_g$ be the inclusion and $\iota_*\colon \pi_1(\Sigma_g) \to \pi_1(V_g) \cong F_g$ the induced homomorphism. Then $\ker(\iota_*)$ is the smallest normal subgroup of $\pi_1(\Sigma_g)$ containing $\{\alpha_1, \dots, \alpha_g\}$, and ι_* maps β_i^{-1} to the element a_i of F_g.

Proposition 2.7 (Griffiths [7]). For an element $f \in \mathcal{M}_g$, the following conditions are equivalent.

(1) f is an element of $\mathcal{H}_g^*$.
(2) f_* preserves the smallest normal subgroup of $\pi_1(\Sigma_g)$ containing $\{\alpha_1, \dots, \alpha_g\}$.

Now let us see several examples.

Example 2.8. The following three examples all belong to $\mathcal{H}_g$.

(1) We take a separating curve γ_i as depicted in Figure 2. Then the half twist k_i about γ_i belongs to $\mathcal{H}_g$, and its action on $\pi_1(V_g) \cong F_g$ coincides with that of σ_i.
(2) Let $\gamma_{j,j+1}$ be a separating curve depicted in Figure 3. Then the half twist d_j about it belongs to $\mathcal{H}_g$, and its action on $\pi_1(V_g) \cong F_g$ coincides with that of $\sigma_j\sigma_{j+1}\tau_j$.
(3) Let δ_1 and δ_2 be the curves depicted in Figure 4. Then we can define the half twist t_1 about them, which belongs to $\mathcal{H}_g$. Its action on $\pi_1(V_g) \cong F_g$ coincides with that of $\tau_1\sigma_2\eta\sigma_2\tau_1$.

The next proposition guarantees that the action of $\mathrm{Aut}(F_g)$ on $\pi_1(V_g) \cong F_g$ is recovered by that of $\mathcal{H}_g$.

Proposition 2.9. For any $\gamma \in \mathrm{Aut}(F_g)$, there exists a mapping class $h \in \mathcal{H}_g$ such that $h_*\colon \pi_1(V_g) \to \pi_1(V_g)$ coincides with $\gamma\colon F_g \to F_g$ under the identification $\pi_1(V_g) \cong F_g$.

Proof. When we regard the generators σ_i, τ_j, η of $\mathrm{Aut}(F_g)$ given in Theorem 2.4 as actions on $\pi_1(V_g)$, the elements $h_{\sigma_i}, h_{\tau_j}, h_\eta \in \mathcal{H}_g$ that realize them are given by

$$h_{\sigma_i} = k_i, \quad h_{\tau_j} = k_j k_{j+1} d_j, \quad h_\eta = k_1 d_1 t_1 d_1 k_1,$$

respectively. □

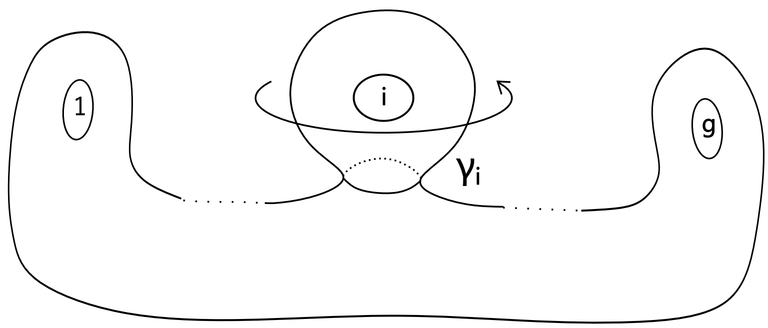

FIGURE 2. γ_i and the half twist k_i

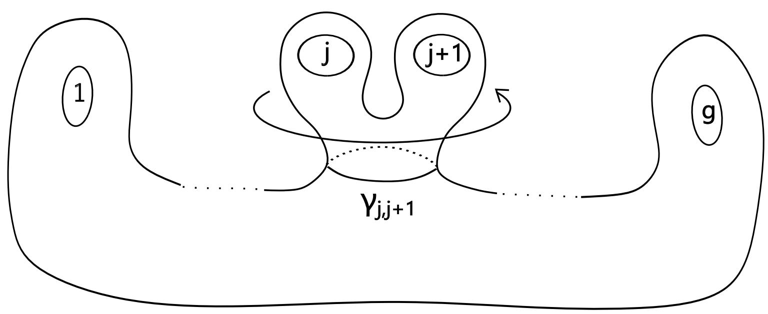

FIGURE 3. $\gamma_{j,j+1}$ and the half twist d_j

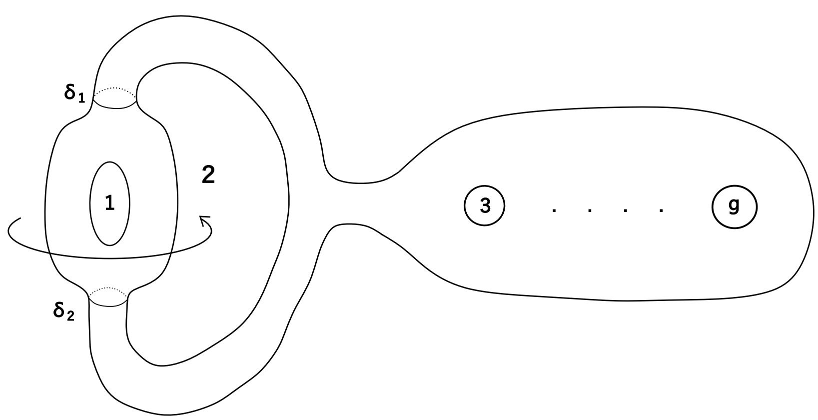

FIGURE 4. δ_1, δ_2 and the half twist t_1

2.3. **G-monodromy and charts.**

Definition 2.10. A group homomorphism from a surface group $\pi_1(\Sigma_g)$ to a group G is called a *G-monodromy representation*.

From now on, let G be a group with a finite presentation $G = \langle \mathcal{X} \mid \mathcal{R} \rangle$. Moreover, let Γ be an oriented graph on Σ_g such that each edge is labelled by an element of $\mathcal{X}$.

Definition 2.11 (Intersection word)**.** Let $\eta\colon [0,1] \to \Sigma_g$ be a path transverse to the edges of Γ. The path η transversely intersects with Γ at a finite number of points, say $b_1, b_2, \dots, b_n$, in this order when the parameter goes from 0 to 1. Let x_i denote the label of the edge containing b_i. When the edge intersects with η from left to right (resp. right to left), we determine the signature ε_i by $\varepsilon_i = +1$ (resp. $\varepsilon_i = -1$). Then we define the word $w_\Gamma(\eta)$ consisting of letters in $\mathcal{X} \cup \mathcal{X}^{-1}$ by $w_\Gamma(\eta) = x_1^{\varepsilon_1} x_2^{\varepsilon_2} \cdots x_n^{\varepsilon_n}$, which is called the intersection word of η with respect to Γ.

For example, in Figure 5 below, the path l embedded in Σ_g transversely intersects with three edges of Γ labeled as b, b and a, respectively. With respect to the orientation of path l, the first edge intersects with l from left to right, the second one from left to right, and the third one from right to left. Thus the intersection word of l with respect to Γ is given by $w_\Gamma(l) = b^2 a^{-1}$.

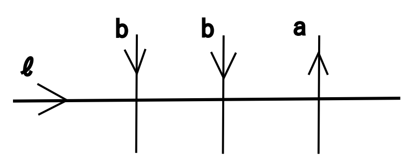

FIGURE 5. Intersection word of l

Definition 2.12 (Charts)**.** A finite oriented graph Γ on Σ_g is called a *chart* with respect to the finitely presented group $G = \langle \mathcal{X} \mid \mathcal{R} \rangle$ if it satisfies the following conditions:

(1) Each edge is labeled by an element of $\mathcal{X}$, i.e., a generator.
(2) For each vertex v, the intersection word of a small simple closed curve m_v rotating around v counterclockwise is a cyclic permutation of an element of $\mathcal{R} \cup \mathcal{R}^{-1}$, i.e., a relator.

However, we allow a loop without vertices (given an orientation and labeled by an element of $\mathcal{X}$) as an edge of Γ. Such an edge is called a *hoop*.

Remark 2.13. An edge of graph is usually defined as a connection between two vertices, so a chart with hoops is not a graph in the normal sense.

The graph depicted in Figure 6 is an example of a chart on the 2-torus T^2 with respect to the group $G = \langle a, b \mid a^2, b^3 \rangle$, where the square shown in the figure is regarded as T^2 by identifying its opposite edges. Taking a vertex v as in the figure,

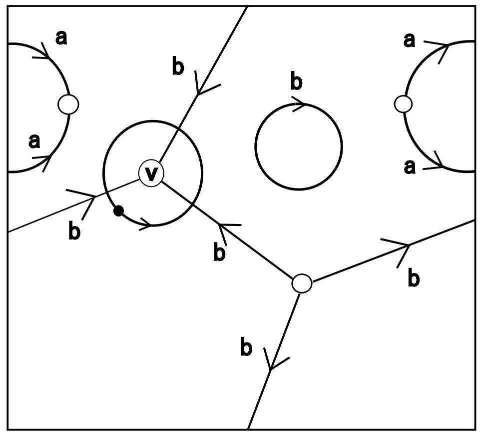

FIGURE 6. Example of Γ and $w_\Gamma(m_v)$

then the intersection word $w_\Gamma(m_v)$ of a small loop m_v around it is b^{-3}, which is indeed (a cyclic permutation of) an element of $\mathcal{R} \cup \mathcal{R}^{-1}$.

For a given chart Γ on Σ_g, we can define a G-monodromy representation

$$\rho_\Gamma \colon \pi_1(\Sigma_g, b_0) \to G$$

by corresponding a loop l with $l(0) = l(1) = b_0$ to its intersection word $w_\Gamma(l)$. Here we need to slightly perturb the loop l so that it becomes transverse to Γ, if necessary. Now we explain why ρ_Γ is well-defined. Let l_0 and l_1 be two loops such that $l_0 \simeq l_1$, and l_t a homotopy connecting them. If l_t is transverse to Γ for each $t \in [0,1]$, then we have $w_\Gamma(l_0) = w_\Gamma(l_1)$, since l_t intersects with Γ in topologically the same way for all $t \in [0,1]$. The way of intersection essentially changes only when the homotopy l_t passes through a vertex v. In this case, the difference between $w_\Gamma(l_0)$ and $w_\Gamma(l_1)$ is just $w_\Gamma(m_v)$, which is a relator of $G = \langle \mathcal{X} \mid \mathcal{R} \rangle$. Therefore, $w_\Gamma(l_0)$ equals to $w_\Gamma(l_1)$ as an element of G, and thus, ρ_Γ is well-defined. (It is clear that ρ_Γ is homogeneous.)

Example 2.14. Let us see some examples of a chart Γ on T^2 and the corresponding homomorphism ρ_Γ.

(1) First we consider the case where $G = F_2 = \langle a, b \rangle$. Since there is no relation between the two generators of the free group F_2, every F_2-chart has no vertex, but only hoops. The chart on T^2 shown in Figure 7 is such an example. The intersection words of the loops α and β depicted in the figure are given by

$$\rho_\Gamma(\alpha) = e, \quad \rho_\Gamma(\beta) = b^{-1}a,$$

which determine the homomorphism $\rho_\Gamma \colon \pi_1(T^2) \to F_2$.

(2) Figure 8 describes the same G-chart as that shown in Figure 6, where $G = \mathbb{Z}_2 * \mathbb{Z}_3 = \langle a, b \mid a^2, b^3 \rangle$. The intersection words of the loops α and β are

$$\rho_\Gamma(\alpha) = b, \quad \rho_\Gamma(\beta) = ba^2 = b,$$

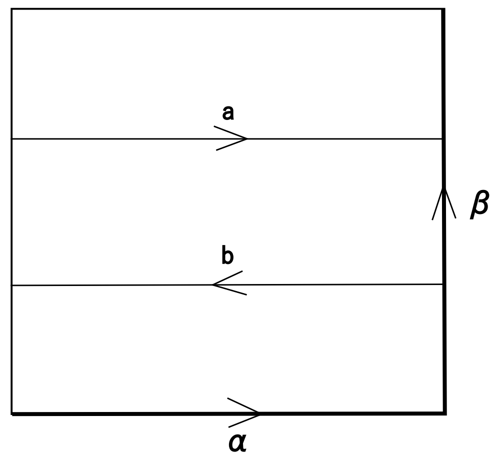

FIGURE 7. Example of a G-chart on T^2 (when $G = F_2 = \langle a, b \rangle$)

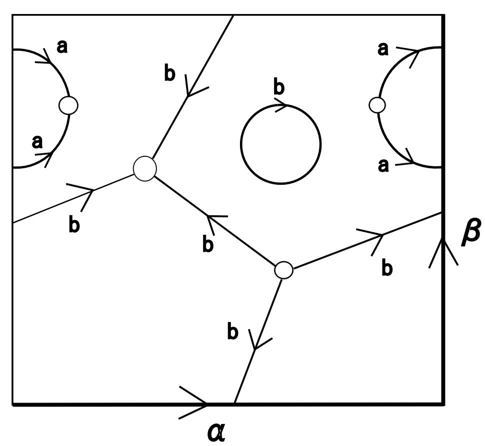

FIGURE 8. Example of a G-chart on T^2 ($G = \langle a, b \mid a^2, b^3 \rangle$)

respectively. From these, the homomorphism $\rho_\Gamma \colon \pi_1(T^2) \to \mathbb{Z}_2 * \mathbb{Z}_3$ is determined.

Thus, for a given chart Γ, the corresponding G-monodromy representation ρ_Γ is determined. In fact, the converse is also true.

Theorem 2.15 (Kamada [13], Hasegawa [10], see also [4])**.** *For any G-monodromy representation ρ, there exists a G-chart Γ such that $\rho_\Gamma = \rho$.*

Outline of the proof. For a given ρ, we construct Γ as follows. Let b_0 be the base point of Σ_g. We take a generating system $\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g$ of the fundamental group $\pi_1(\Sigma_g, b_0)$ as depicted in Figure 1, and the standard cell decomposition of Σ_g corresponding to it. Namely, the cell decomposition has b_0 as a unique 0-cell e^0, 1-cells $e^1_1, e^1_2, \ldots, e^1_{2g-1}, e^1_{2g}$ corresponding to $\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g$, respectively, and a unique 2-cell e^2 attached along the loop $[\alpha_1, \beta_1]\cdots[\alpha_g, \beta_g]$. For each i with $1 \leq i \leq g$, let $w(\alpha_i)$ and $w(\beta_i)$ be the words of letters in $\mathcal{X} \cup \mathcal{X}^{-1}$ corresponding to $\rho(\alpha_i)$ and $\rho(\beta_i)$, respectively. Then we arrange parallel oriented arcs labelled by elements of $\mathcal{X} \cup \mathcal{X}^{-1}$ on a neighborhood of each 1-cell so that they are transverse to the 1-cell and the intersection word of e^1_{2i-1} (resp. e^1_{2i}) coincides with $w(\alpha_i)$ (resp. $w(\beta_i)$). Since ρ is a homomorphism, the intersection word of the loop ∂e^2 with respect to these arcs represents

$$[\rho(\alpha_1), \rho(\beta_1)]\cdots[\rho(\alpha_g), \rho(\beta_g)] = \rho\big([\alpha_1, \beta_1]\cdots[\alpha_g, \beta_g]\big) = \rho(e) = e$$

in G. Hence, this word can be transformed into an empty word by a finite iteration of the following operations;

(1) deletion or insertion of trivial relation $x^{\varepsilon}x^{-\varepsilon}$ ($x \in \mathcal{X}, \varepsilon \in \{\pm 1\}$),
(2) insertion of a relation r^{ε} ($r \in \mathcal{R}, \varepsilon \in \{\pm 1\}$).

The operation (1) corresponds either connecting two adjacent oriented arcs labelled by the same letter x or inserting a new oriented arc labelled by x. The operation (2) corresponds to the insertion of a vertex representing the relation r^{ε}. Therefore, we can extend the arcs arranged on a neighborhood of the 1-skeleton according to the algebraic operations that transform the intersection word of ∂e^2 into an empty word. Then there exists a 2-disk D inside the 2-cell e^2 whose boundary does not intersect with extended arcs. This implies that we have already obtained a graph on $\Sigma_g \setminus D$. Now gluing the 2-disk D without any vertex or edge, then we obtain a chart Γ on Σ_g. It is clear by construction that $\rho_\Gamma = \rho$. □

Remark 2.16. In fact, operations of a chart that do not change its monodromy, which are called chart moves of type W, have been completely classified in [13]. Moreover, it has been proven in the same paper that a chart Γ satisfying $\rho_\Gamma = \rho$ uniquely exists up to those operations. In particular, deletion of a contractible loop from a chart Γ does not change the monodromy. In what follows, we always assume that a chart Γ does not contain contractible loops.

Remark 2.17. Changing the base point b_0 affects the G-monodromy by conjugate action of G. When $G = \mathrm{Diff}(F)$ and $\rho\colon \pi_1(\Sigma_g) \to \mathrm{Diff}(F)$ is the monodromy of an F-bundle, such an ambiguity can be recovered by changing the identification between the fiber over the base point and F. Hence, in the following, we always omit the base point b_0.

When $G = PSL(2;\mathbb{Z}) \cong \mathbb{Z}_2 * \mathbb{Z}_3 = \langle a, b \mid a^2, b^3\rangle$, any vertex of a G-chart Γ is either degree-2 or degree-3 according as it represents a^2 or b^3. Since a degree-2 vertex and a degree-3 vertex are never connected by an edge, the connected component of a degree-2 vertex always forms a simple closed curve on Σ_g. This fact plays an important role in the proof of Theorem 1.1.

3. Classification of $\mathrm{Hom}(\pi_1(\Sigma_g), PSL(2;\mathbb{Z}))$

In this section, we first prove Theorem 1.1 by an ingenious usage of charts. Then we prove Theorem 1.3 and its generalization (Theorem 3.1) by combining several known results in group theory and consideration about the mapping class group $\mathcal{H}_g$ of the handlebody V_g.

Proof of Theorem 1.1. We prove it by induction on g. For any homomorphism

$$\rho\colon \pi_1(\Sigma_g) \to PSL(2;\mathbb{Z}) \cong \mathbb{Z}_2 * \mathbb{Z}_3,$$

we take a chart Γ of ρ with respect to the finite presentation $PSL(2;\mathbb{Z}) = \langle a, b \mid a^2, b^3 \rangle$. Let X be the set of simple closed curves on Σ_g consisiting of the connected components of degree-2 vertices and hoops.

(A) The case $g = 1$.

(1) If $X = \emptyset$, then ρ can be treated as a homomorphism to the abelian group $\mathbb{Z}_3$, and thus, it can be described as

$$\rho(\alpha) = b^i, \;\; \rho(\beta) = b^j \quad (i, j \in \{0, 1, 2\}),$$

where b is the canonical generator of $\mathbb{Z}_3$. Hence, there exist coprime integers p and q such that $\rho(\alpha^p\beta^q) = e$. Then we obtain integers r and s such that $ps - qr = 1$ by the Euclidean algorithm. Now taking the mapping class $f \in \mathcal{M}_1$ represented by $\begin{pmatrix} p & r \\ q & s \end{pmatrix} \in SL(2;\mathbb{Z})$, we have

$$(\rho \circ f_*)(\alpha) = \rho(f_*(\alpha)) = \rho(\alpha^p\beta^q) = e.$$

(2) If $X \neq \emptyset$, we take an element $c \in X$. Then c is a simple closed curve on T^2, which is not contractible by Remark 2.16. Hence, there exist coprime integers p and q such that $c = \alpha^p\beta^q \in \pi_1(T^2)$. Since c is a connected component of a finite graph Γ, there is a simple closed curve c' parallel to c such that $\Gamma \cap c' = \emptyset$. Then the monodromy along c' is trivial, and we have $\rho(\alpha^p\beta^q) = \rho(c') = e$. Now by the same argument as in (1), we obtain $r, s \in \mathbb{Z}$ with $\begin{pmatrix} p & r \\ q & s \end{pmatrix} \in SL(2;\mathbb{Z})$ and the corresponding mapping class $f \in \mathcal{M}_1$ so that we have

$$(\rho \circ f_*)(\alpha) = \rho(f_*(\alpha)) = \rho(\alpha^p\beta^q) = e.$$

(B) The case $g \geq 2$. We are going to prove that the assertion holds for Σ_g under the assumption that it is true for Σ_{g-1}. In order for that, it is enough to show the existence of a separating curve γ with $\rho(\gamma) = e$ that separates Σ_g into the connected sum of Σ_{g-1} and T^2. We will discuss the following three cases:

(1) $X = \emptyset$,
(2) $X \neq \emptyset$ and X contains a non-separating curve,
(3) $X \neq \emptyset$ and all the members of X are separating curves.

(1) In this case, ρ can be treated as a homomorphism into the abelian group $\mathbb{Z}_3$. On the other hand, for any separating curve γ that separates Σ_g into the connected sum of Σ_{g-1} and T^2, there exist $l_1, l_2 \in \pi_1(\Sigma_g)$ such that

$$\gamma = [l_1, l_2] = l_1 l_2 l_1^{-1} l_2^{-1}.$$

Then we have the following by the commutativity of $\mathbb{Z}_3$:

$$\rho(\gamma) = \rho(l_1)\rho(l_2)\rho(l_1)^{-1}\rho(l_2)^{-1} = e.$$

(2) Let $c \in X$ be a non-separating curve. Since c can be mapped by an element of $\mathrm{Diff}_+(\Sigma_g)$ to the location as depicted in Figure 9, we may assume that it is originally in that place. Since c is a connected component of a finite

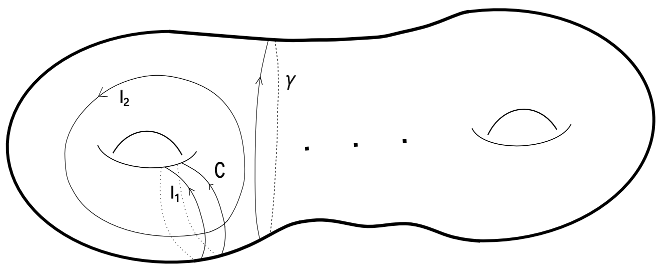

FIGURE 9. How to take l_1, l_2 and γ (when c is non-separating)

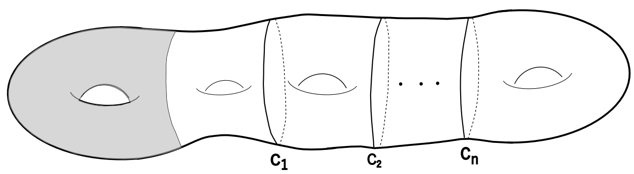

FIGURE 10. When all the elements of X are separating curves

graph Γ, we can take a simple closed curve l_1 parallel to c and satisfying $\Gamma \cap l_1 = \emptyset$. Then it follows that $\rho(l_1) = e$. Moreover, taking l_2 and γ as in Figure 9, we have

$$\rho(\gamma) = \rho([l_2, l_1^{-1}]) = [\rho(l_2), \rho(l_1)^{-1}] = [\rho(l_2), e] = e.$$

(3) When all the members $c_1, \dots, c_n$ of X are separating curves, there are no degree-2 vertices nor hoops in the shaded region in Figure 10. Then taking γ as the boundary curve of that region, we have $\rho(\gamma) = e$ by the same argument as in (1).

Thus we obtain a desired separating curve γ in any case. If we separates Σ_g into T^2 and Σ_{g-1} along this curve, the homomorphism ρ can be decomposed into the two homomorphisms $\rho' \colon \pi_1(T^2) \to PSL(2;\mathbb{Z})$ and $\rho'' \colon \pi_1(\Sigma_{g-1}) \to PSL(2;\mathbb{Z})$. Therefore, the assertion of this theorem holds for Σ_g if it does for Σ_{g-1}. $\square$

Proof of Theorem 1.3. Let $\rho \colon \pi_1(\Sigma_g) \to PSL(2;\mathbb{Z})$ be any homomorphism. By Theorem 1.1, there exists a diffeomorphism $\varphi \in \mathrm{Diff}_+(\Sigma_g)$ such that $(\rho \circ \varphi_*)(\alpha_i) = e$ for each i. Then the homomorphism $\rho' = \rho \circ \varphi_*$ factorizes so that the following diagram commutes;

(3.1)
$$\begin{tikzcd} & \pi_1(V_g) \arrow[dr, "\eta"] & \\ \pi_1(\Sigma_g) \arrow[ur, "\iota_*"] \arrow[rr, "\rho'"'] & & PSL(2;\mathbb{Z}) \end{tikzcd}$$

that is, there exists a homomorphism

$$\eta \colon F_g \cong \pi_1(V_g) \to PSL(2;\mathbb{Z}) \cong \mathbb{Z}_2 * \mathbb{Z}_3$$

such that $\rho' = \eta \circ \iota_*$, where $\iota\colon \Sigma_g \to V_g$ is the inclusion map. Now applying Kurosh's subgroup theorem to the subgroup $\eta(F_g) = \rho'(\pi_1(\Sigma_g)) \subset \mathbb{Z}_2 * \mathbb{Z}_3$, it turns out that $\eta(F_g)$ is isomorphic to the free product of some copies of $\mathbb{Z}_2$, $\mathbb{Z}_3$ and $\mathbb{Z}$. Namely, there exist subgroups $H_j \subset \eta(F_g)$ such that

$$H_1 \cong \cdots \cong H_k \cong \mathbb{Z}, \quad H_{k+1} \cong \cdots \cong H_l \cong \mathbb{Z}_3, \quad H_{l+1} \cong \cdots \cong H_m \cong \mathbb{Z}_2$$

and

$$\eta(F_g) = (H_1 * \cdots * H_k) * (H_{k+1} * \cdots * H_l) * (H_{l+1} * \cdots * H_m).$$

Then we apply Grushko's theorem to the surjective homomorphism $\eta\colon F_g \to \eta(F_g)$ $(m-1)$ times, and obtain subgroups G_j of F_g $(1 \leq j \leq m)$ such that

$$\eta(G_j) = H_j \ \text{ and } \ G_1 * \cdots * G_m = F_g.$$

Since G_j is a finitely generated free group and H_j is isomorphic to either $\mathbb{Z}$, $\mathbb{Z}_3$ or $\mathbb{Z}_2$, we obtain $\gamma \in \mathrm{Aut}(F_g)$ such that

$$(\eta \circ \gamma)(a_i) = \begin{cases} 1 \in H_i \cong \mathbb{Z} \quad (1 \leq i \leq k), \\ 1 \in H_i \cong \mathbb{Z}_3 \ \ (k+1 \leq i \leq l), \\ 1 \in H_i \cong \mathbb{Z}_2 \ \ (l+1 \leq i \leq m), \\ e \quad (m+1 \leq i \leq g) \end{cases}$$

by applying Proposition 2.6 repeatedly. Moreover, there exists $h \in \mathrm{Diff}_+(V_g)$ satisfying $h_* = \gamma$ by Proposition 2.9. Then we have

$$\begin{aligned}(\eta \circ \gamma) \circ \iota_* &= \eta \circ h_* \circ \iota_* = \eta \circ (h \circ \iota)_* = \eta \circ (\iota \circ h|_{\Sigma_g})_* \\ &= \eta \circ \iota_* \circ (h|_{\Sigma_g})_* = \rho' \circ (h|_{\Sigma_g})_* = \rho \circ (\varphi \circ h|_{\Sigma_g})_*.\end{aligned}$$

Since $[h|_{\Sigma_g}] \in \mathcal{H}_g^*$, we have $(h|_{\Sigma_g})_*(\alpha_i) \in \langle \alpha_1, \ldots, \alpha_g \rangle^{\pi_1(\Sigma_g)}$ by Proposition 2.7. Then it follows that

$$\left(\rho \circ (\varphi \circ h|_{\Sigma_g})_*\right)(\alpha_i) \in \Big\langle (\rho \circ \varphi_*)(\alpha_1), \ldots, (\rho \circ \varphi_*)(\alpha_g) \Big\rangle^{PSL(2;\mathbb{Z})} = \{e\},$$

that is, $\left(\rho \circ (\varphi \circ h|_{\Sigma_g})_*\right)(\alpha_i) = e$. Now denote the mapping class $[\varphi \circ h|_{\Sigma_g}] \in \mathcal{M}_g$ by f, then the assertion of the theorem follows. $\square$

Theorem 1.3 can be generalized into the case where the range of homomorphisms is the free product of finite cyclic groups $\mathbb{Z}_{k_1} * \mathbb{Z}_{k_2} * \cdots * \mathbb{Z}_{k_n}$. The following arguments (Theorem 3.1 and Corollary 3.3) are due to Masayuki Asaoka's advice.

By Kurosh's subgroup theorem, any subgroup H of $\mathbb{Z}_{k_1} * \mathbb{Z}_{k_2} * \cdots * \mathbb{Z}_{k_n}$ can be described as

$$H = H_1 * H_2 * \cdots * H_g.$$

Here each H_j is a subgroup of H with $H_i \cong \mathbb{Z}_{\chi(i)}$, where

$$\chi\colon \{1, 2, \ldots, g\} \to \left(\bigcup_{j=1}^{n} \{\text{positive divisors of } k_j\} \right) \cup \{0\}$$

is a map and $\mathbb{Z}_0 = \mathbb{Z}$, $\mathbb{Z}_1 = \{e\}$.

Theorem 3.1. *Let $k_1, k_2, \ldots, k_n$ be integers greater than 1. For any homomorphism*

$$\rho\colon \pi_1(\Sigma_g) \to \mathbb{Z}_{k_1} * \mathbb{Z}_{k_2} * \cdots * \mathbb{Z}_{k_n},$$

the subgroup $\mathrm{Im}(\rho) \subset \mathbb{Z}_{k_1} * \mathbb{Z}_{k_2} * \cdots * \mathbb{Z}_{k_n}$ *can be described as*

$$\mathrm{Im}(\rho) = H_1 * H_2 * \cdots * H_g$$

by the same argument above. Then there exists a mapping class $f \in \mathcal{M}_g$ such that

$$(\rho \circ f_*)(\alpha_i) = e \ \ (1 \leq i \leq g), \ \ (\rho \circ f_*)(\beta_i) = \begin{cases} 1 \in H_i \cong \mathbb{Z}_{\chi(i)} & (\chi(i) \neq 1), \\ e & (\chi(i) = 1). \end{cases}$$

In particular, for any two homomorphisms $\rho_1, \rho_2 \colon \pi_1(\Sigma_g) \to \mathbb{Z}_{k_1} * \mathbb{Z}_{k_2} * \cdots * \mathbb{Z}_{k_n}$, the following two conditions are equivalent.

(1) $\mathrm{Im}(\rho_1) = \mathrm{Im}(\rho_2)$.
(2) *There exists a mapping class $f \in \mathcal{M}_g$ such that $\rho_2 = \rho_1 \circ f_*$.*

Outline of the proof. First we prove the following claim, which is a generalization of Theorem 1.1.

Claim 3.2. For any homomorphism $\rho\colon \pi_1(\Sigma_g) \to \mathbb{Z}_{k_1} * \mathbb{Z}_{k_2} * \cdots * \mathbb{Z}_{k_n}$, there exists a mapping class $f \in \mathcal{M}_g$ such that $(\rho \circ f_*)(\alpha_i) = e$ for each i with $1 \leq i \leq g$.

Since $\mathbb{Z}_{k_1} * \mathbb{Z}_{k_2} * \cdots * \mathbb{Z}_{k_n} = \langle a_1, a_2, \ldots, a_n \mid a_1^{k_1}, a_2^{k_1}, \cdots, a_n^{k_n} \rangle$, a degree-$k_i$ vertex and a degree-k_j vertex of a chart Γ of ρ, corresponding to the relators $a_i^{k_i}$ and $a_j^{k_j}$ respectively, are never connected by an edge if $i \neq j$. Thus, for any i with $1 \leq i \leq g$, each connected component of a degree-k_i vertex is isolated. Now we take the ribbon graph obtained by thickening Γ and denote the set of its boundary components by X. Then a parallel argument to the proof of Theorem 1.1 works. However, we cannot exclude the case where X contains contractible loops, since elements of X are not connected components of Γ this time. Thus we have to discuss the following three cases.

(1) All the elements of X are contractible.
(2) X contains a non-contractible separating curve.
(3) X contains a non-separating curve.

In the case (1), it is easily proven that Γ is connected. Then ρ can be seen as a homomorphism to a single abelian group $\mathbb{Z}_{k_i}$, and thus, the same argument as in the proof of Theorem 1.1 works. In the case (2), we take a non-contractible separating curve $c \in X$. Then it follows that $\Gamma \cap c = \emptyset$, so we separates Σ_g along c. Here we have to notice that Σ_g is not necessarily decomposed into T^2 and Σ_{g-1}, and the assumption of induction should be modified in the form that the assertion holds for all the closed orientable surfaces of genus smaller than g. In the case (3), the same argument as in the proof of Theorem 1.1 (B) (2) works. Thus Claim 3.2 is proven. A parallel argument to the proof of Theorem 1.3 works when we replace $\mathbb{Z}_2 * \mathbb{Z}_3$ by $\mathbb{Z}_{k_1} * \mathbb{Z}_{k_2} * \cdots * \mathbb{Z}_{k_n}$, this completes the proof of Theorem 3.1. $\square$

We obtain the following application as a corollary to this theorem.

Corollary 3.3. For any continuous map φ from a closed orientable surface Σ_g to the connected sum of a finite number of lens spaces, there exists a diffeomorphism $f \in \mathrm{Diff}_+(\Sigma_g)$ such that $\varphi \circ f$ can be continuously extended to the handlebody V_g.

Proof. The assertion follows from Theorem 3.1 (in fact, Claim 3.2 is enough) and the fact that the connected sum of lens spaces $L(p_1, q_1), L(p_2, q_2), \cdots, L(p_n, q_n)$ satisfies

$$\begin{aligned}\pi_1\Big(L(p_1,q_1)\#L(p_2,q_2)\#\cdots\#L(p_n,q_n)\Big) &\cong \mathbb{Z}_{p_1} * \mathbb{Z}_{p_2} * \cdots * \mathbb{Z}_{p_n},\\ \pi_2\Big(L(p_1,q_1)\#L(p_2,q_2)\#\cdots\#L(p_n,q_n)\Big) &= 0.\end{aligned}$$

□

4. Orientable T^2-bundles

In this section, we first review basic facts and known results about orientable T^2-bundles over S^1 in § 4.1, and about orientable T^2-bundles over Σ_g in §4.2, 4.3, and 4.4. Moreover, in § 4.5, we show a new result obtained as an application of Corollary 1.2.

Let B be a connected orientable C^∞-manifold and $\pi\colon E \to B$ an orientable T^2-bundle over B. Then its structure group $\mathrm{Diff}_+(T^2)$ is known to be homotopy equivalent to $\mathrm{Aff}_+(T^2) = T^2 \rtimes SL(2;\mathbb{Z})$ ([9]). In particular, we have

$$\pi_0\left(\mathrm{Diff}_+(T^2)\right) \cong SL(2;\mathbb{Z}),\ \pi_1\left(\mathrm{Diff}_+(T^2)\right) \cong \mathbb{Z}^2,$$

and thus, the mapping class group of T^2 is $\mathcal{M}_1 \cong SL(2;\mathbb{Z})$.

Now we take a base point b_0 on the base space B of the bundle π and fix a diffeomorphism

$$\Psi\colon T^2 \to F_0 := \pi^{-1}(b_0).$$

Let $l\colon [0,1] \to B$ be a loop with $l(0) = l(1) = b_0$. Since the pullback of the bundle $\pi\colon E \to B$ by l is a T^2-bundle over $[0,1]$, which is trivial, there exists a bundle map $\varphi\colon [0,1]\times T^2 \to E$ that covers $l\colon [0,1] \to B$ and satisfies $\varphi_0 = \Psi$, where we set $\varphi(t,p) = \varphi_t(p)$. Then the composition $\Psi^{-1}\circ\varphi_1\colon T^2 \to T^2$ is a diffeomorphism and its isotopy class $[\Psi^{-1}\circ\varphi_1] \in \mathcal{M}_1$ depends only on the isotopy class of Ψ and the homotopy class $[l]$ of l. The mapping class $[\Psi^{-1}\circ\varphi_1]$ is called the monodromy along $[l]$ with respect to Ψ. The map

$$\rho\colon \pi_1(B) \to \mathcal{M}_1 \cong SL(2;\mathbb{Z})$$

defined by sending $[l]$ to $[\Psi^{-1}\circ\varphi_1]$ is called the monodromy of $\pi\colon E \to B$ with respect to Ψ. Since we have chosen to regard the action of $\mathcal{M}_1$ on T^2 as a right action (see § 2.2), ρ becomes a group homomorphism.

Remark 4.1. The mapping class group $\mathcal{M}(B)$ acts on $\pi_1(B)$ by $\alpha\cdot[f] = f_*(\alpha)$, and $SL(2;\mathbb{Z})$ acts on itself by $Q\cdot A = QAQ^{-1}$. Notice that the monodromy representation $\rho\colon \pi_1(B) \to SL(2;\mathbb{Z})$ is uniquely determined by the isomorphism class of π up to these actions of $\mathcal{M}(B)$ and $SL(2;\mathbb{Z})$. Here $[f]\in\mathcal{M}(B)$ and $Q \in SL(2;\mathbb{Z})$ correspond to the base map of a bundle isomorphism and the ambiguity of the isotopy class of an identification $\Psi\colon T^2 \to F_0$, respectively.

Remark 4.2. Since we have decided that the product in $SL(2;\mathbb{Z})$ is defined in the reverse order of composition and the action of $SL(2;\mathbb{Z})$ on T^2 is a right action, one would, strictly speaking, have to rewrite all the notation concerning products of matrices by taking transposes. That is, conventions such as " vectors are regarded as row vectors on which matrices act from the right " and " the product AB of matrices corresponds, under the usual definition, to ${}^tB{}^tA$ " would have to be adopted. However, this would be highly confusing. Moreover, since in the subsequent discussion there will be no point at which the order of multiplication in $SL(2;\mathbb{Z})$ causes any issue, we shall henceforth employ the standard notation for products of matrices.

4.1. **Orientable T^2-bundles over S^1.** Before dealing with T^2-bundles over surfaces, we first consider the case $B = S^1$. In this case, the total space of a T^2-bundle is given by a mapping torus. Here the mapping torus $M(A)$ of $A \in SL(2;\mathbb{Z})$ is defined as follows;

$$M(A) = \mathbb{R} \times T^2 / \big(t, [A\boldsymbol{x}]\big) \sim \big(t+1, [\boldsymbol{x}]\big).$$

The map $\pi\colon M(A) \to S^1 = \mathbb{R}/\mathbb{Z}$ defined by $\pi\big(\big[t, [\boldsymbol{x}]\big]\big) = [t]$ is indeed a T^2-bundle over S^1 with monodromy A. Conversely, any orientable T^2-bundle over S^1 can be described in this way, and its isomorphism class is determined by the conjugacy class of the monodromy $A \in SL(2;\mathbb{Z})$ (see Remark 4.1).

Next we review on bundle automorphisms of $M(A)$. The identity map

$$\mathrm{id}_{M(A)}\colon M(A) \to M(A)$$

is clearly a bundle isomorphism covering id_{S^1}. On the other hand, for any $\boldsymbol{u} \in \mathbb{Z}^2$, the map $\widetilde{f}_{\boldsymbol{u}}\colon M(A) \to M(A)$ defined by

$$\widetilde{f}_{\boldsymbol{u}}\big(\big[t, [\boldsymbol{x}]\big]\big) = \big[t, [\boldsymbol{x} + t\boldsymbol{u}]\big]$$

is also a bundle isomorphism covering id_{S^1}. This is nothing but the bundle isomorphism that sends the trivial section $s_0\colon S^1 \to M(A)$ defined by $s_0([t]) = \big[t, [\mathbf{0}]\big]$ to another one $s_{\boldsymbol{u}}\colon S^1 \to M(A)$ defined by $s_{\boldsymbol{u}}([t]) = \big[t, [t\boldsymbol{u}]\big]$.

4.2. **Monodromy and Euler class (the case $B = \Sigma_g$).** In the following, we deal with the case where $B = \Sigma_g$. The meaning of the monodromy of an orientable T^2-bundle $\pi\colon M^4 \to \Sigma_g$ is interpreted as follows. First we take the standard cellular decomposition

$$\Sigma_g = \big(e^0 \cup (e^1_1 \cup e^1_2 \cup \cdots \cup e^1_{2g-1} \cup e^1_{2g})\big) \cup e^2$$

used in the proof of Theorem 2.15. Then let

$$\{U^0, U^1_1, U^1_2, \cdots, U^1_{2g-1}, U^1_{2g}, U^2\}$$

be its associating open covering of Σ_g that satisfies the following conditions:

(1) U^0 is an open neighborhood of b_0, and U^1_k an open neighborhood of $e^1_k \setminus U^0$ for each k with $1 \le k \le 2g$.
(2) U^0, U^2 and each U^1_k are all diffeomorphic to an open 2-disk.
(3) Each $U^0 \cap U^1_k$ is the disjoint union of two open 2-disks, and $U^1_i \cap U^1_j = \emptyset$ $(i \ne j)$.
(4) $N := U^0 \cup (U^1_1 \cup \cdots \cup U^1_{2g})$ is a regular neighborhood of the 1-skeleton.
(5) $N \cap U^2$ is a collar neighborhood of ∂N.

By the condition (2), the T^2-bundle π is trivial over U^0, U^2 and each U^1_k. We take a local trivialization $\varphi_0\colon \pi^{-1}(U^0) \cong U^0 \times T^2$ so that $\varphi_0|_{F_0} = \Psi^{-1}$. Then the transition function g^k_{01} over $U^0 \cap U^1_k$ is defined for each k. By the condition (3), we have open sets V_k and W_k diffeomorphic to an open 2-disk such that

$$U^0 \cap U^1_k = V_k \sqcup W_k,$$

If we take the transition function $g^k_{01}\colon V_k \sqcup W_k \to \mathrm{Diff}_+(F)$ that coincides with the constant map to id_{T^2} over V_k, then $g^k_{01}|_{W_k}$ corresponds to the monodromy along e^1_k.

Since a homotopy of transition functions does not change the bundle isomorphism class, we may retake $g_{01}^k|_{W_k}$ as a constant map valued in $\pi_0(\mathrm{Diff}_+(T^2)) = SL(2;\mathbb{Z})$. Thus we obtain an element in $SL(2;\mathbb{Z})$ for each e_k^1 with $1 \leq k \leq 2g$. Denote it by A_i if $k = 2i-1$, and by B_i if $k = 2i$. Then we have

$$\rho(\alpha_1) = A_1,\ \rho(\beta_1) = B_1,\ \cdots,\ \rho(\alpha_g) = A_g,\ \rho(\beta_g) = B_g$$

and $[A_1, B_1] \cdots [A_g, B_g] = E_2$. In other words, the restriction of π over e_k^1 is isomorphic to $M(A_i)$ if $k = 2i-1$ and to $M(B_i)$ if $k = 2i$. Since these data determine the isomorphism class of the T^2-bundle $\pi|_{\pi^{-1}(N)}$ over N, the monodromy ρ can be considered as the description of nontriviality of the bundle over the 1-skeleton.

What is left for us is to describe how to glue the trivial bundle $U^2 \times T^2 \to U^2$ and $\pi|_{\pi^{-1}(N)} \colon \pi^{-1}(N) \to N$. To do so, we can restore the isomorphism class of the T^2-bundle π. Since this information of the gluing is described by the transition function $N \cap U^2 \to \mathrm{Diff}_+(T^2)$, we obtain the corresponding element in $\pi_1\left(\mathrm{Diff}_+(T^2)\right) \cong \mathbb{Z}^2$, say (m, n). Thus the isomorphism class of an orientable T^2-bundle $\pi \colon M^4 \to \Sigma_g$ is represented by $A_1, B_1, \ldots, A_g, B_g \in SL(2;\mathbb{Z})$ with $[A_1, B_1] \cdots [A_g, B_g] = E_2$, and $(m, n) \in \mathbb{Z}^2$. Hence, we may denote this T^2-bundle by $M(A_1, B_1, \ldots, A_g, B_g; m, n)$. Now the following is obvious.

Proposition 4.3. For any orientable T^2-bundle ξ over Σ_g, there exist A_1, B_1, $\ldots$, A_g, $B_g \in SL(2;\mathbb{Z})$ with $[A_1, B_1] \cdots [A_g, B_g] = E_2$ and $(m, n) \in \mathbb{Z}^2$ such that ξ is isomorphic to the bundle

$$\pi \colon M(A_1, B_1, \ldots, A_g, B_g; m, n) \to \Sigma_g.$$

Next, we explain the definition of the Euler class of an orientable T^2-bundle

$$\xi = \left(M(A_1, B_1, \ldots, A_g, B_g; m, n), \pi, \Sigma_g, T^2\right).$$

In order for that, we consider whether the bundle

$$\pi \colon M(A_1, B_1, \ldots, A_g, B_g; m, n) \to \Sigma_g$$

admits a cross section. There exists a trivial cross section over the 1-skeleton corresponding to s_0. It extends over the 2-cell e^2 if and only if $(m, n) = (0, 0)$, since it is necessary and sufficient for a continuous map $\partial D^2 \to T^2$ to be extendable over D^2 that it is trivial as an element of $\pi_1(T^2) \cong \mathbb{Z}^2$. In this sense, it seems possible to consider $(m, n) \in \mathbb{Z}^2$ as the obstruction to the existence of a cross section of π, but this is inaccurate. For, even when $(m, n) \neq (0, 0)$, if we retake another cross section over the 1-skeleton, then it might be extendable over e^2. Such an inconvenience is resolved by the local system $\{\pi_1(T^2)\}$, that is, the locally constant sheaf associated with the monodromy ρ whose stalk over each point $b \in \Sigma_g$ is isomorphic to $\pi_1(F_b) \cong \mathbb{Z}^2$. By obstruction theory, the obstruction to the existence of a cross section of ξ is determined as a second cohomology class with coefficients in $\{\pi_1(T^2)\}$ (see [30, 23] for details). Denote it by $e(\xi)$. Then it lies in

$$H^2\left(\Sigma_g; \{\pi_1(T^2)\}\right) \cong \mathbb{Z}^2/\sim,$$

where the equivalence relation $\sim$ is defined by

$$(k_1, k_2) \sim (l_1, l_2) \iff \text{there exists } \gamma \in \pi_1(\Sigma_g) \text{ such that } \begin{pmatrix} l_1 \\ l_2 \end{pmatrix} = \rho(\gamma) \begin{pmatrix} k_1 \\ k_2 \end{pmatrix}.$$

Thus the obstruction class $e(\xi)$ is not an element of $\mathbb{Z}^2$, but that of the module $\mathbb{Z}^2/\sim$ represented by $(m,n) \in \mathbb{Z}^2$. This is the Euler class of an orientable T^2-bundle ξ.

Definition 4.4 (the Euler class)**.** Let ξ be an orientable T^2-bundle over Σ_g. Then the local coefficient cohomology class

$$e(\xi) \in H^2\left(\Sigma_g; \{\pi_1(T^2)\}\right)$$

defined as the obstruction class to the existence of a cross section of ξ is called the Euler class of ξ.

Remark 4.5. As is implied by the notation, the Euler class $e(\xi)$ is uniquely determined by the isomorphism class of ξ. Notice that the Euler classes of two orientable T^2-bundles ξ and ξ' are comparable as cohomology classes only when its monodromies ρ and ρ' coincide. Here we mean by the coincidence of ρ and ρ' that there exist $f \in \mathrm{Diff}_+(\Sigma_g)$ and $Q \in SL(2;\mathbb{Z})$ such that $\rho \circ f_* = Q\rho' Q^{-1}$ (see Remark 4.1).

Now we are ready to give a necessary and sufficient condition for two orientable T^2-bundles over Σ_g to be isomorphic.

Theorem 4.6. *Let $\xi = (E, \pi, \Sigma_g, T^2)$ and $\xi' = (E', \pi', \Sigma_g, T^2)$ be orientable T^2-bundles over Σ_g, whose monodromies and Euler classes we denote by $\rho, e(\xi)$ and $\rho', e(\xi')$, respectively. Then, the bundles ξ and ξ' are isomorphic if and only if there exist $f \in \mathrm{Diff}_+(\Sigma_g)$ and $Q \in SL(2;\mathbb{Z})$ such that $\rho \circ f_* = Q\rho' Q^{-1}$ and $e(\xi) = f^* e(\xi')$.*

Proof. The only if part is obvious from Remarks 4.1 and 4.5, so we will prove the if part. Since f and Q correspond to the base map of a bundle isomorphism and the ambiguity of the choice of the mapping class $[\Psi] \in \mathcal{M}_1$ determined by an identification diffeomorphism $\Psi\colon T^2 \to F_0$, respectively, we only have to deal with the case where $\rho = \rho'$ and $e(\xi) = e(\xi')$. We construct a bundle isomorphism between ξ and ξ' under these assumptions. Since ρ and ρ' coincide, ξ and ξ' can be described as

$$\pi\colon M(A_1, B_1, \ldots, A_g, B_g; m, n) \to \Sigma_g, \quad \pi'\colon M(A_1, B_1, \ldots, A_g, B_g; k, l) \to \Sigma_g.$$

By the condition $e(\xi) = e(\xi')$, there exists an element $\gamma \in \pi_1(\Sigma_g)$ such that

$$\begin{pmatrix} k \\ l \end{pmatrix} = \rho(\gamma) \begin{pmatrix} m \\ n \end{pmatrix}.$$

Since $\rho(\gamma)$ can be described as $\rho(\gamma) = C_1 \cdots C_r$, where r is a positive integer and $C_i \in \{E_2, A_1, \ldots, A_g, B_1, \ldots, B_g\}$ for each $1 \le i \le r$, we have

$$\rho(\gamma) = E_2 + \sum_{i=1}^{r} (C_i - E_2) C_{i+1} \cdots C_r.$$

Putting $\boldsymbol{w}_i = C_{i+1} \cdots C_r \begin{pmatrix} m \\ n \end{pmatrix}$ for each $1 \le i \le r$, then we obtain

$$\begin{pmatrix} k \\ l \end{pmatrix} = \begin{pmatrix} m \\ n \end{pmatrix} + \sum_{i=1}^{r} (C_i - E_2) \boldsymbol{w}_i.$$

Hence, there exist $\boldsymbol{u}_i, \boldsymbol{v}_i \in \mathbb{Z}^2$ $(1 \leq i \leq g)$ such that

$$\begin{pmatrix} k \\ l \end{pmatrix} = \begin{pmatrix} m \\ n \end{pmatrix} + \sum_{i=1}^{g} \left((A_i - E_2)\boldsymbol{u}_i + (B_i - E_2)\boldsymbol{v}_i \right).$$

Then we can construct a bundle isomorphism

$$\widetilde{f}\colon M(A_1, B_1, \ldots, A_g, B_g; m, n) \to M(A_1, B_1, \ldots, A_g, B_g; k, l)$$

covering the identity map id_{Σ_g} by Proposition 4.8 proved later. □

Putting $g = 1$ in Theorem 4.6, we obtain the following corollary.

Corollary 4.7. Let

$$\xi = \left(M(A, B; m, n), \pi, T^2, T^2\right), \quad \xi' = \left(M(A', B'; m', n'), \pi', T^2, T^2\right)$$

be two orientable T^2-bundles over T^2. Then they are bundle isomorphic if and only if there exist $P = \begin{pmatrix} p & r \\ q & s \end{pmatrix}, Q \in SL(2; \mathbb{Z})$ and $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{Z}^2$ such that $QA'Q^{-1} = A^p B^q$, $QB'Q^{-1} = A^r B^s$ and

$$\begin{pmatrix} m \\ n \end{pmatrix} - Q \begin{pmatrix} m' \\ n' \end{pmatrix} = (A - E_2)\boldsymbol{x} + (B - E_2)\boldsymbol{y}.$$

4.3. $SL(2; \mathbb{Z})$**-bundles.** In this subsection, we introduce a nice model for the T^2-bundle $M(A_1, B_1, \ldots, A_g, B_g; m, n)$ following the manner of Sakamoto–Fukuhara [27], and complete the proof of Theorem 4.6. First we consider the case $(m, n) = (0, 0)$. In this case, we can reduce the structure group to $SL(2; \mathbb{Z})$, since the transition function $N \cap U^2 \to \mathrm{Diff}_+(T^2)$ can be taken as the constant map to id_{T^2}. Such a T^2-bundle is called an $SL(2; \mathbb{Z})$-bundle (in particular, it is a flat T^2-bundle). An $SL(2; \mathbb{Z})$-bundle $M(A_1, B_1, \ldots, A_g, B_g; 0, 0)$ can be described as follows. Let $\begin{pmatrix} x \\ y \end{pmatrix}$ be the standard coordinates on $\mathbb{R}^2$, and $\begin{bmatrix} x \\ y \end{bmatrix}$ denote the corresponding point on $T^2 = \mathbb{R}^2/\mathbb{Z}^2$.

(1) The case $g = 0$. Let $F = T^2$ and $M(0, 0) = \mathbb{C}P^1 \times F$. Then the projection $\pi\colon M(0, 0) \to \mathbb{C}P^1$ to the first factor is the trivial T^2-bundle over S^2. In particular, it is an $SL(2; \mathbb{Z})$-bundle.

(2) The case $g = 1$. Let $F = T^2$ and $A, B \in SL(2; \mathbb{Z})$ satisfy $AB = BA$. We define

$$M(A, B; 0, 0) = \mathbb{R}^2 \times F/\sim,$$

where

$$\left(\begin{pmatrix} x+1 \\ y \end{pmatrix}, \begin{bmatrix} s \\ t \end{bmatrix} \right) \sim \left(\begin{pmatrix} x \\ y \end{pmatrix}, \left[A \begin{pmatrix} s \\ t \end{pmatrix} \right] \right),$$

$$\left(\begin{pmatrix} x \\ y+1 \end{pmatrix}, \begin{bmatrix} s \\ t \end{bmatrix} \right) \sim \left(\begin{pmatrix} x \\ y \end{pmatrix}, \left[B \begin{pmatrix} s \\ t \end{pmatrix} \right]. \right)$$

Then the map $\pi\colon M(A, B; 0, 0) \to \mathbb{R}^2/\mathbb{Z}^2$ defined by

$$\pi\left(\left[\begin{pmatrix} x \\ y \end{pmatrix}, \begin{bmatrix} s \\ t \end{bmatrix} \right] \right) = \begin{bmatrix} x \\ y \end{bmatrix}$$

is an $SL(2; \mathbb{Z})$-bundle over T^2.

(3) The case $g \geq 2$. Let $\mathbb{D}$ be the Poincaré disk and z the complex coordinate on it. Since $\mathbb{D}$ is the universal cover of Σ_g, $\pi_1(\Sigma_g)$ acts on it from the right by the deck transformation. Let $F = T^2$ and $A_1, B_1, \ldots, A_g, B_g \in SL(2;\mathbb{Z})$ satisfy $[A_1, B_1] \cdots [A_g, B_g] = E_2$. We define

$$M(A_1, B_1, \ldots, A_g, B_g; 0, 0) = \mathbb{D} \times F / \sim,$$

where

$$\left(z, \begin{bmatrix} s \\ t \end{bmatrix}\right) \sim \left(z \cdot \gamma^{-1}, \left[\rho(\gamma) \begin{pmatrix} s \\ t \end{pmatrix}\right]\right) \quad (\gamma \in \pi_1(\Sigma_g)).$$

Then the map $\pi \colon M(A_1, B_1, \ldots, A_g, B_g; 0, 0) \to \mathbb{D}/\pi_1(\Sigma_g)$ defined by

$$\pi\left(\left[z, \begin{bmatrix} s \\ t \end{bmatrix}\right]\right) = [z]$$

is an $SL(2;\mathbb{Z})$-bundle over Σ_g.

Next, we prepare a model for the case where $(m, n) \neq (0, 0)$. In the following, we denote $M(A_1, B_1, \ldots A_g, B_g; 0, 0)$ by M_0. Moreover, D denotes a 2-disk of a sufficiently small radius ε that is centered at the origin $0 = [0 : 1]$ of $\mathbb{C} \subset \mathbb{C}P^1$, at $\begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$ on the torus $T^2 = \mathbb{R}^2/\mathbb{Z}^2$, or at $[0]$ on $\Sigma_g = \mathbb{D}/\pi_1(\Sigma_g)$ according as $g = 0, 1$, or $g \geq 2$. Then we define

$$M(A_1, B_1, \ldots, A_g, B_g; m, n) = \left(M_0 - \pi^{-1}(\mathrm{Int} D)\right) \cup_h (D \times F).$$

Here $h \colon \pi^{-1}(\partial D) \to \partial D \times F$ is the diffeomorphism given by

$$h\left(\left[r(\theta), \begin{bmatrix} s \\ t \end{bmatrix}\right]\right) = \left(r(\theta), \left[\begin{pmatrix} s \\ t \end{pmatrix} + (\theta/2\pi)\begin{pmatrix} m \\ n \end{pmatrix}\right]\right),$$

where $r \colon S^1 \to \mathbb{D}$ is the map defined by

$$r(\theta) = \begin{cases} [\varepsilon e^{i\theta} : 1] \ \ (g = 0), \\ \left(1/2 + \varepsilon\cos\theta,\ 1/2 + \varepsilon\sin\theta\right) \ \ (g = 1), \\ \varepsilon e^{i\theta} \ \ (g \geq 2). \end{cases}$$

Now we redefine the map $\pi \colon M(A_1, B_1, \ldots, A_g, B_g; m, n) \to \Sigma_g$ by

$$\pi\left([z_1 : z_2], \begin{bmatrix} s \\ t \end{bmatrix}\right) = [z_1 : z_2] \ \ (g = 0),$$
$$\pi\left(\left[\begin{pmatrix} x \\ y \end{pmatrix}, \begin{bmatrix} s \\ t \end{bmatrix}\right]\right) = \begin{bmatrix} x \\ y \end{bmatrix} \quad (g = 1),$$
$$\pi\left(\left[z, \begin{bmatrix} s \\ t \end{bmatrix}\right]\right) = [z] \quad (g \geq 2).$$

Then it is an orientable T^2-bundle over Σ_g. Thus we obtain the following conclusion: Any orientable T^2-bundle can be obtained from an $SL(2;\mathbb{Z})$-bundle by some T^2-surgery. In the subsequent sections, we always denote an $SL(2;\mathbb{Z})$-bundle by $M(A_1, B_1, \ldots, A_g, B_g)$ omitting $(m, n) = (0, 0)$.

Now we are ready to complete the proof of Theorem 4.6.

Proposition 4.8. The two orientable T^2-bundles

$$\pi\colon M(A_1, B_1, \ldots, A_g, B_g; m, n) \to \Sigma_g, \quad \pi'\colon M(A_1, B_1, \ldots, A_g, B_g; k, l) \to \Sigma_g$$

are isomorphic if there exist $\boldsymbol{u}_i, \boldsymbol{v}_i \in \mathbb{Z}^2$ such that

$$\begin{pmatrix} k \\ l \end{pmatrix} = \begin{pmatrix} m \\ n \end{pmatrix} + \sum_{i=1}^{g} \left((A_i - E_2)\boldsymbol{u}_i + (B_i - E_2)\boldsymbol{v}_i \right).$$

Proof. We explicitly construct a bundle isomorphism

$$\widetilde{f}\colon M(A_1, B_1, \ldots, A_g, B_g; k, l) \to M(A_1, B_1, \ldots, A_g, B_g; m, n).$$

In the following, we assume $g \geq 2$, since the claim is trivial when $g = 0$ and the construction has been done in Proposition 2 (3) of [27] when $g = 1$. The idea of our construction is as follows. First we construct a bundle isomorphism over the 1-skeleton of the standard cellular decomposition of Σ_g so that it coincides with $\widetilde{f}_{B_i \boldsymbol{v}_i}\colon M(A_i) \to M(A_i)$ over e^1_{2i-1}, and with $\widetilde{f}_{-\boldsymbol{u}_i}\colon M(B_i) \to M(B_i)$ over e^1_{2i}. Then it extends over the 2-cell e^2 to become a bundle isomorphism $\widetilde{f}$ over Σ_g.

Let us write down $\widetilde{f}$ explicitly by using the above model of orientable T^2-bundles. First we take a fundamental domain P of the action of $\pi_1(\Sigma_g)$ which is a hyperbolic regular $4g$-gon centered at $0 \in \mathbb{D}$. Notice that each vertex of P corresponds to the base point b_0 of Σ_g and the boundary ∂P is written

$$\partial P = [\alpha_1, \beta_1][\alpha_2, \beta_2] \cdots [\alpha_g, \beta_g] = \alpha_1 \beta_1 \alpha_1^{-1} \beta_1^{-1} \alpha_2 \beta_2 \alpha_2^{-1} \beta_2^{-1} \cdots \alpha_g \beta_g \alpha_g^{-1} \beta_g^{-1}$$

as an element of $\pi_1(\Sigma_g, b_0)$. We take a parametrization $w\colon [0, 4g] \to \partial P \cong S^1$ so that for each integer $4i + j$ $(0 \leq i \leq g,\ j = 0, 1, 2, 3)$, the interval $[4i + j, 4i + j + 1]$ is mapped to α_i, β_i, α_i^{-1}, β_i^{-1} according as $j = 0, 1, 2, 3$, and that

$$\begin{aligned} w(u) \cdot \beta_i &= w(8i + 3 - u) \quad (4i \leq u \leq 4i + 1), \\ w(u) \cdot \alpha_i^{-1} &= w(8i + 5 - u) \quad (4i + 1 \leq u \leq 4i + 2). \end{aligned}$$

Now we define

$$\widetilde{f}\colon \pi^{-1}(\partial P) \to \pi'^{-1}(\partial P)$$

by

$$\begin{aligned} \widetilde{f}\left(\left[w(u), \begin{bmatrix} s \\ t \end{bmatrix} \right] \right) &= \left[w(u), \left[\begin{pmatrix} s \\ t \end{pmatrix} + (u - 4i) B_i \boldsymbol{v}_i \right] \right] && (4i \leq u \leq 4i + 1), \\ & \left[w(u), \left[\begin{pmatrix} s \\ t \end{pmatrix} + (4i + 2 - u) \boldsymbol{u}_i \right] \right] && (4i + 1 \leq u \leq 4i + 2), \\ & \left[w(u), \left[\begin{pmatrix} s \\ t \end{pmatrix} + (4i + 3 - u) \boldsymbol{v}_i \right] \right] && (4i + 2 \leq u \leq 4i + 3), \\ & \left[w(u), \left[\begin{pmatrix} s \\ t \end{pmatrix} + (u - 4i - 3) A_i \boldsymbol{u}_i \right] \right] && (4i + 3 \leq u \leq 4i + 4). \end{aligned}$$

This induces a bundle isomorphism over the 1-skeleton and, moreover, specifies a bundle isomorphism over ∂P, which corresponds to an element of $\pi_1(\mathrm{Diff}_+(T^2)) = \mathbb{Z}^2$. In fact, this element is exactly $(k - m, l - n)$, since

$$\sum_{i=1}^{g} (B_i \boldsymbol{v}_i - \boldsymbol{u}_i - \boldsymbol{v}_i + A_i \boldsymbol{u}_i) = \sum_{i=1}^{g} \left((A_i - E_2)\boldsymbol{u}_i + (B_i - E_2)\boldsymbol{v}_i \right) = \begin{bmatrix} k - m \\ l - n \end{bmatrix}$$

by assumption. This resolves the difference in the coefficients of the T^2-surgeries corresponding to the two bundles π and π'. Consequently, $\widetilde{f}$ extends in a natural way over the 2-cell P, giving rise to a bundle isomorphism that covers the identity map of Σ_g. □

4.4. **Fiber connected sum.** In this section, we explain the fiber connected sum, a method of constructing a new T^2-bundle from two T^2-bundles. Let $\xi = (M, \pi, \Sigma_g, T^2)$ and $\xi' = (M', \pi', \Sigma_{g'}, T^2)$ be orientable T^2-bundles. Let b_0 (resp. b_0') be a base point on the surface Σ_g (resp. $\Sigma_{g'}$). We take local trivializations of ξ and ξ' around each base point, and denote them by

$$\varphi_U \colon \pi^{-1}(U) \to U \times T^2, \ \ {\varphi'}_{U'} \colon \pi^{-1}(U') \to U' \times T^2,$$

respectively. Now we take 2-disks D and D' so that $b_0 \in D \subset U$ and $b_0' \in D' \subset U'$. Then ∂D and $\partial D'$ carry natural orientations derived from those of Σ_g and $\Sigma_{g'}$, respectively. We take an orientation reversing diffeomorphism $f \colon \partial D \to \partial D'$ and an orientation preserving diffeomorphism $h \colon T^2 \to T^2$.

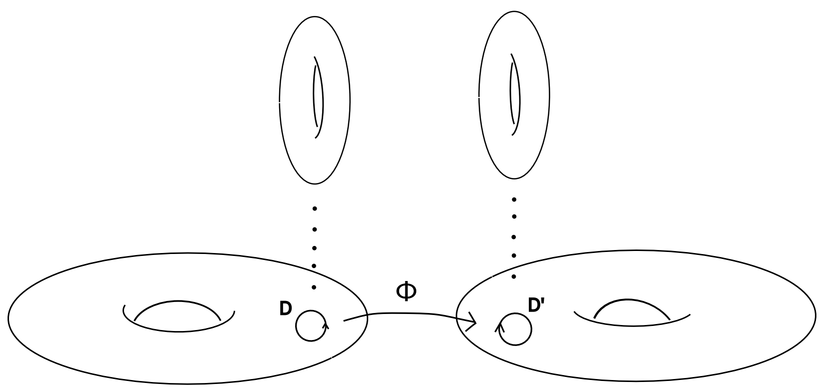

FIGURE 11. Fiber connected sum

Then we obtain an orientable T^2-bundle over $\Sigma_{g+g'}$ from two bundles $\pi| \colon \pi^{-1}(\Sigma_g \setminus D) \to \Sigma_g \setminus D$ and $\pi'| \colon {\pi'}^{-1}(\Sigma_{g'} \setminus D') \to \Sigma_{g'} \setminus D'$ by the fiberwise gluing along their boundaries given by

$$\Phi \colon \pi^{-1}(\partial D) \xrightarrow{\varphi_U} \partial D \times T^2 \xrightarrow{f \times h} \partial D' \times T^2 \xrightarrow{({\varphi'}_{U'})^{-1}} {\pi'}^{-1}(\partial D').$$

This is called the fiber connected sum of ξ and ξ', and denoted by $\xi \# \xi'$ or

$$\pi \# \pi' \colon M \#_f M' \to \Sigma_{g+g'}.$$

By this definition, however, the diffeomorphism type of $M \#_f M'$ depends on the choices of the local trivializations φ_U, ${\varphi'}_{U'}$ and the diffeomorphism h. In order to avoid such an ambiguity, we fix φ_U, ${\varphi'}_{U'}$ and h as follows.

First we describe the two T^2-bundles ξ and ξ' by

$$\pi \colon M(A_1, B_1, \ldots, A_g, B_g; m, n) \to \Sigma_g, \ \ \pi' \colon M(C_1, D_1, \ldots, C_{g'}, D_{g'}; k, l) \to \Sigma_{g'},$$

and take the local trivializations φ_U and $\varphi'_{U'}$ so that $\varphi_U|_{F_0} = \Psi^{-1}$ and $\varphi'_{U'}|_{F_0'} = {\Psi'}^{-1}$, where $\Psi \colon T^2 \to F_0 = \pi^{-1}(b_0)$ and $\Psi' \colon T^2 \to F_0' = {\pi'}^{-1}(b_0')$ are the diffeomorphisms fixed in the definition of the monodromies of π and π'. Now we fix

$h = \mathrm{id}_{T^2}$, then the fiber connected sum of ξ and ξ' can be written as

$$\pi\#\pi' \colon M(A_1, B_1, \dots, A_g, B_g, C_1, D_1, \dots, C_{g'}, D_{g'}; m+k, n+l) \to \Sigma_{g+g'}.$$

In particular,

$$M(A, B; m, n)\#_f M(C, D; k, l) \cong M(A, B, C, D; m+k, n+l)$$

holds when $g = g' = 1$. Thus the monodromy and the Euler class of the T^2-bundle $\xi\#\xi'$ is described by those of ξ and ξ'.

4.5. **Decomposition theorem.** Recall that any orientable T^2-bundle over Σ_g can be described as

$$\pi \colon M(A_1, B_1, \dots, A_g, B_g; m, n) \to \Sigma_g$$

by $(m, n) \in \mathbb{Z}^2$ and $A_1, B_1, \dots, A_g, B_g \in SL(2;\mathbb{Z})$ with $[A_1, B_1] \cdots [A_g, B_g] = E_2$. Then, applying Corollary 1.2 and Proposition 4.6 to the monodromy $\widetilde{\rho}$ of π, we obtain the following.

Theorem 4.9. *Let ξ be an orientable T^2-bundle over Σ_g. Then there exist $B_1, \dots, B_g \in SL(2;\mathbb{Z})$ and $(m, n) \in \mathbb{Z}^2$ such that ξ is isomorphic to either of the 2^g T^2-bundles $M(\pm E_2, B_1, \dots, \pm E_2, B_g; m, n)$.*

Moreover, the following is obtained as a corollary.

Theorem 4.10. *Any orientable T^2-bundle over Σ_g with $g \geq 1$ is isomorphic to the fiber connected sum of g pieces of T^2-bundles over T^2.*

Proof. Since $\pm E_2$ commute with any element of $SL(2;\mathbb{Z})$,

$$M(\pm E_2, B_1; m, n),\ M(\pm E_2, B_2; 0, 0),\ \cdots,\ M(\pm E_2, B_g; 0, 0)$$

are all orientable T^2-bundles over T^2. By the consideration in §4.4, it follows that

$$\begin{aligned} M(\pm E_2, B_1; m, n)\#_f M(\pm E_2, B_2; 0, 0)\#_f \cdots \#_f M(\pm E_2, B_g; 0, 0) \\ \cong M(\pm E_2, B_1, \pm E_2, B_2, \dots, \pm E_2, B_g; m, n). \end{aligned}$$

Therefore, by Theorem 4.9, any orientable T^2-bundle over Σ_g is isomorphic to the fiber connected sum of g pieces of T^2-bundles over T^2. □

5. Classification of orientable T^2-bundles over Σ_g

In this section, we prove the classification theorem of orientable T^2-bundles over Σ_g (Theorem 1.5), which is our main theorem. Since the classification is already known for the case $g = 0, 1$ (§5.1, 5.2), the main part is the case $g \geq 2$ (§5.3). There, we first classify the $\mathcal{M}_g$-orbits of $\mathrm{Hom}(\pi_1(\Sigma_g), SL(2;\mathbb{Z}))$ (Theorem 1.4). This is equivalent, up to the conjugate action of $SL(2;\mathbb{Z})$, to the classification of isomorphism classes of $SL(2;\mathbb{Z})$-bundles over Σ_g. Hence, we can complete the proof of the main theorem by adding a simple argument about Euler classes to it.

5.1. **The case** $g = 0$**.** Since S^2 is simply-connected, any T^2-bundle over S^2 has trivial monodromy, and hence, admits a principal T^2-bundle structure. The isomorphism classes of principal T^2-bundles are classified by their Euler classes

$$(m, n) \in \pi_2(BT^2) = \pi_2(BS^1 \times BS^1) = \pi_2(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) = \mathbb{Z}^2,$$

but as orientable T^2-bundles, $M(m, n)$ and $M(d, 0)$ are isomorphic by Remark 4.5, where d is the greatest common divisor of m and n. Thus the isomorphism classes of orientable T^2-bundles are classified by $d \in \mathbb{Z}_{\geq 0}$.

5.2. **Theorem of Sakamoto and Fukuhara (the case $g = 1$).** The isomorphism classes of orientable T^2-bundles over T^2 have been completely classified as follows.

Theorem 5.1 (Sakamoto-Fukuhara [27]). *Let $\xi = (M^4, \pi, T^2, T^2)$ be an orientable T^2-bundle over T^2. Then there exist $B \in SL(2;\mathbb{Z})$ and $(m, n) \in \mathbb{Z}^2$ such that ξ is isomorphic to either $M(E_2, B; m, n)$ or $M(-E_2, B; m, n)$. Moreover, two orientable T^2-bundles $M(\varepsilon E_2, B; m, n)$ and $M(\delta E_2, C; k, l)$ are isomorphic if and only if one of the following conditions is satisfied, where $\varepsilon, \delta = \pm 1$.*

(1) *$\varepsilon = \delta = 1$ and there exist $Q \in SL(2;\mathbb{Z})$ and $\boldsymbol{x} \in \mathbb{Z}^2$ such that $QCQ^{-1} = B^{\pm 1}$ and*
$$\begin{pmatrix} m \\ n \end{pmatrix} - Q \begin{pmatrix} k \\ l \end{pmatrix} = (B - E_2)\boldsymbol{x}.$$
(2) *$\varepsilon = 1$, $\delta = -1$, $\mathrm{ord}(B)$ is either 2, 4 or 6 and there exist $Q \in SL(2;\mathbb{Z})$ and $\boldsymbol{x} \in \mathbb{Z}^2$ such that $QCQ^{-1} = \pm B^{\pm 1}$ and*
$$\begin{pmatrix} m \\ n \end{pmatrix} - Q \begin{pmatrix} k \\ l \end{pmatrix} = (B - E_2)\boldsymbol{x}.$$
(3) *$\varepsilon = -1$, $\delta = 1$, $\mathrm{ord}(C)$ is either 2, 4 or 6 and there exist $Q \in SL(2;\mathbb{Z})$ and $\boldsymbol{x} \in \mathbb{Z}^2$ such that $QBQ^{-1} = \pm C^{\pm 1}$ and*
$$\begin{pmatrix} k \\ l \end{pmatrix} - Q \begin{pmatrix} m \\ n \end{pmatrix} = (C - E_2)\boldsymbol{x}.$$
(4) *$\varepsilon = \delta = -1$ and there exist $Q \in SL(2;\mathbb{Z})$ and $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{Z}^2$ such that $QCQ^{-1} = \pm B^{\pm 1}$ and*
$$\begin{pmatrix} m \\ n \end{pmatrix} - Q \begin{pmatrix} k \\ l \end{pmatrix} = (B - E_2)\boldsymbol{x} + 2\boldsymbol{y}.$$

The former claim is nothing but a special case of Theorem 4.9. On the other hand, we need another lemma for the proof of the latter claim. As is well known, the order of B in $SL(2;\mathbb{Z})$ is either 1, 2, 3, 4, 6 or ∞. Hence, the cyclic subgroup $\langle B \rangle$ of $SL(2;\mathbb{Z})$ generated by B contains $-E_2$ if and only if $\mathrm{ord}(B)$ is either 2, 4 or 6. In these cases, the following holds.

Proposition 5.2. If the order of $B \in SL(2;\mathbb{Z})$ is equal to 4, all the four T^2-bundles $M(\pm E_2, \pm B; m, n)$ are isomorphic to each other. If it is equal to 2 or 6, the following holds;
$$M(E_2, B; m, n) \cong M(-E_2, B; m, n) \cong M(-E_2, -B; m, n) \not\cong M(E_2, -B; m, n),$$
where $\cong$ and $\not\cong$ mean "isomorphic" and "non-isomorphic" as T^2-bundles, respectively.

Proof. In each case, we will construct a bundle isomorphism using Corollary 4.7. Since $(m, n) = (m', n')$, we may assume that $Q = E_2$ and $\boldsymbol{x} = \boldsymbol{y} = \boldsymbol{0}$. Thus, what we only have to do is to find out an appropriate $P \in SL(2;\mathbb{Z})$ for each case.

(1) The case where $\mathrm{ord}(B) = 4$. Since we have $B^2 = -E_2$, an isomorphism
$$M(E_2, B; m, n) \cong M(-E_2, B; m, n)$$
is obtained by setting $P = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$. Similarly, setting $P = \begin{pmatrix} -1 & -1 \\ 4 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}$, we obtain isomorphisms
$$M(E_2, B; m, n) \cong M(E_2, -B; m, n), \quad M(E_2, B; m, n) \cong M(-E_2, -B; m, n),$$

respectively.

(2) The case where $\mathrm{ord}(B) = 6$. Since we have $B^3 = -E_2$, we obtain isomorphisms
$$M(E_2, B; m, n) \cong M(-E_2, B; m, n), \quad M(E_2, B; m, n) \cong M(-E_2, -B; m, n)$$
by setting $P = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 3 & 4 \end{pmatrix}$, respectively. On the other hand, since $-E_2 \notin \langle -B \rangle$ and $-E_2 \in \langle B \rangle$, we have
$$\langle E_2, -B \rangle = \langle -B \rangle \neq \langle B \rangle = \langle E_2, B \rangle \,.$$
Then it follows that
$$M(E_2, B; m, n) \ncong M(E_2, -B; m, n),$$
since their monodromy groups are different.

(3) The case where $\mathrm{ord}(B) = 2$, that is, $B = -E_2$. In this case, we obtain isomorphisms
$$M(E_2, -E_2; m, n) \cong M(-E_2, -E_2; m, n), \; M(E_2, -E_2; m, n) \cong M(-E_2, E_2; m, n)$$
by setting $P = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, respectively. On the other hand, the two bundles $M(E_2, -E_2; m, n)$ and $M(E_2, E_2; m, n)$ are not isomorphic to each other, since they have different monodromy groups.

□

By using this proposition, we prove the theorem of Sakamoto and Fukuhara.

Proof of Theorem 5.1. The former claim of Theorem 5.1 follows by putting $g = 1$ in Theorem 4.9.

We prove the latter claim by using Corollary 4.7. First we deal with the cases (2) and (3), but it suffices to argue about (2) since (3) can be replaced by (2) by exchanging the roles of B and C. The images of the monodromies of $M(E_2, B; m, n)$ and $M(-E_2, C; k, l)$ are $\langle B \rangle$ and $-E_2 \in \langle -E_2, C \rangle$, respectively. Hence, it is necessary for the two bundles to be isomorphic that these subgroups of $SL(2; \mathbb{Z})$ coincide up to conjugation. In particular, $\langle B \rangle$ must contain $-E_2$. Therefore, the order of B should be 2, 4 or 6. Moreover, there exists $Q \in SL(2; \mathbb{Z})$ such that $QCQ^{-1} = \pm B^{\pm 1}$, since C must coincide with a power of B up to conjugation. Applying Proposition 5.2, we can find an appropriate $P \in SL(2; \mathbb{Z})$ for each case so that the monodromies of the two bundles coincide. Finally, we consider about the Euler class. Since the Euler class of $M(E_2, B; m, n)$ belongs to the $\mathbb{Z}$-module
$$H^2\left(\Sigma_g; \{\pi_1(T^2)\}\right) \cong \mathbb{Z}^2 / \left\langle (B - E_2)\boldsymbol{e}_1, (B - E_2)\boldsymbol{e}_2 \right\rangle,$$
the condition that the Euler classes of the two bundles coincide can be written as
$$\begin{pmatrix} m \\ n \end{pmatrix} - Q \begin{pmatrix} k \\ l \end{pmatrix} \in \langle (B - E_2)\boldsymbol{e}_1, (B - E_2)\boldsymbol{e}_2 \rangle = \{(B - E_2)\boldsymbol{x} \mid \boldsymbol{x} \in \mathbb{Z}^2\}.$$
This completes the proof of the cases (2) and (3).

What is left to us is to deal with the cases (1) and (4), but they are rather simple. If the monodromies of the two bundles coincide up to conjugation, then there exists $Q \in SL(2; \mathbb{Z})$ satisfying $QCQ^{-1} = B^{\pm 1}$ in the case of (1), and $QCQ^{-1} = \pm B^{\pm 1}$ in the case of (4). Conversely, in all these cases, we can easily find $P \in SL(2; \mathbb{Z})$ that

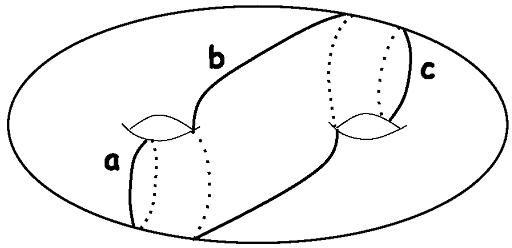

FIGURE 12. Simple closed curves a, b and c

makes the monodromies of the two bundles coincide. Finally, the Euler class of the T^2-bundle belongs to the $\mathbb{Z}$-module

$$H^2\left(\Sigma_g; \{\pi_1(T^2)\}\right) \cong \mathbb{Z}^2 / \langle (B - E_2)\boldsymbol{e}_1, (B - E_2)\boldsymbol{e}_2 \rangle$$

in the case of (1), and to

$$H^2\left(\Sigma_g; \{\pi_1(T^2)\}\right) \cong \mathbb{Z}^2 / \langle 2\boldsymbol{e}_1, 2\boldsymbol{e}_2, (B - E_2)\boldsymbol{e}_1, (B - E_2)\boldsymbol{e}_2 \rangle$$

in the case of (4). Then we can write down the condition that the Euler classes of the two bundles coincide as that on integers k, l, m, n. Thus we have proven the latter claim. □

5.3. **Main Theorem (the case $g \geq 2$).** When $g \geq 2$, the classification of monodromies becomes more complicated since the fundamental group $\pi_1(\Sigma_g)$ is noncommutative. Based on Theorem 1.3, however, $\mathcal{M}_g$-orbits of $\mathrm{Hom}(\pi_1(\Sigma_g), SL(2;\mathbb{Z}))$ can be classified (Theorem 1.4), and then, we obtain the main theorem (Theorem 1.5). Before going into the detail of the arguments, we first prepare some propositions needed later.

Proposition 5.3. For any $B, C \in SL(2;\mathbb{Z})$, the following isomorphism holds;

$$M(-E_2, B, E_2, C) \cong M(-E_2, B, E_2, -C).$$

Proof. Let $f\colon \Sigma_2 \to \Sigma_2$ be the product of the right-handed Dehn twists about a and c and the left-handed Dehn twist about b, where a, b and c are the simple closed curves depicted in Figure 12. The homomorphism $f_*\colon \pi_1(\Sigma_2) \to \pi_1(\Sigma_2)$ satisfies

$$f_*(\alpha_1) = \alpha_1,\ f_*(\beta_1) = \alpha_1^{-1}\alpha_2^{-1}\beta_1\alpha_1,\ f_*(\alpha_2) = \alpha_2,\ f_*(\beta_2) = \alpha_2^{-1}\alpha_1^{-1}\beta_2\alpha_2.$$

Hence we have

$$(\tilde{\rho} \circ f_*)(\alpha_1) = -E_2,\ (\tilde{\rho} \circ f_*)(\beta_1) = B,\ (\tilde{\rho} \circ f_*)(\alpha_2) = E_2,\ (\tilde{\rho} \circ f_*)(\beta_2) = -C.$$

Therefore, pulling back the T^2-bundle $\pi\colon M(-E_2, B, E_2, C) \to \Sigma_2$ by f, we obtain the T^2-bundle $f^*\pi\colon M(-E_2, B, E_2, -C) \to \Sigma_2$. Since f is a self-diffeomorphism of Σ_2, we have an isomorphism $f^*\pi \cong \pi$. □

For any homomorphism $\rho\colon \pi_1(\Sigma_g) \to PSL(2;\mathbb{Z})$, there are 2^{2g} choices of taking a lift $\widetilde{\rho}\colon \pi_1(\Sigma_g) \to SL(2;\mathbb{Z})$ of ρ with respect to the projection $p\colon SL(2;\mathbb{Z}) \to PSL(2;\mathbb{Z})$. We want to determine which two lifts are transformed to each other by the action of the mapping class group $\mathcal{M}_g$. From now on, we suppose that ρ is of the normal form in the sense of Theorem 1.3. Put $\widetilde{\rho}(\alpha_i) = A_i$ and $\widetilde{\rho}(\beta_i) = B_i$.

Then we have $A_i = E_2$ or $A_i = -E_2$ for each i with $1 \leq i \leq g$, so this can be described as $A_i = \varepsilon_i E_2$, where ε_i is equal to either $+1$ or -1 for each i.

Proposition 5.4. The monodromy group of $M(\varepsilon_1 E_2, B_1, \ldots, \varepsilon_g E_2, B_g)$ does not contain $-E_2$ if and only if $\varepsilon_i = 1$ and the order of B_i is neither 2, 4 nor 6 for each i with $1 \leq i \leq g$.

Proof. The only if part is obvious. On the other hand, the if part follows from the condition that ρ is of the normal form. The proof is as follows. Since $\mathrm{ord}(B_i) \neq 2$, we have $B_i = E_2$ for each i with $m+1 \leq i \leq g$. Now suppose that the monodromy group $\mathrm{Im}(\widetilde{\rho})$ contains $-E_2$. Then we can write $-E_2 = B_{\sigma_1}^{n_1} B_{\sigma_2}^{n_2} \cdots B_{\sigma_r}^{n_r}$, where r is a positive integer and $\sigma_i \in \{1, 2, \ldots, m\}$ for each i. Then we have

$$p(B_{\sigma_1})^{n_1} p(B_{\sigma_2})^{n_2} \cdots p(B_{\sigma_r})^{n_r} = e \in \mathrm{Im}(\rho) \subset PSL(2; \mathbb{Z}).$$

Since each $p(B_i)$ $(1 \leq i \leq m)$ is a generator of the factor H_i of the free product decomposition of $\mathrm{Im}(\rho)$, it follows that $n_1 = n_2 = \cdots = n_r = 0$. Thus we have $E_2 = -E_2$, which is a contradiction. Therefore, $\mathrm{Im}(\widetilde{\rho})$ does not contain $-E_2$. $\square$

We note that the monodromy group $\mathrm{Im}(\widetilde{\rho})$ contains an element of order 4 if and only if $l < m$, since the condition that $B \in SL(2; \mathbb{Z})$ is of order 4 is equivalent to the one that $p(B) \in PSL(2; \mathbb{Z})$ is of order 2. Moreover, if $l < m$, then we have $\mathrm{ord}(B_i) = 4$ for any i with $l+1 \leq i \leq m$. Then, applying Proposition 5.2 and using the isomorphisms

$$M(E_2, B_i) \cong M(-E_2, B_i) \cong M(E_2, -B_i) \cong M(-E_2, -B_i),$$

we can freely switch ε_i and the sign of B_i without changing the isomorphism class of the T^2-bundle $M(\varepsilon_1 E_2, B_1, \ldots, \varepsilon_g E_2, B_g)$. Similarly, we can exclude the elements of order 2 and 6 from $B_1, \ldots, B_g$ preserving the isomorphism class of the T^2-bundle. For, if B_i is of order 2 or 6, then B_i can be replaced by $-B_i$ (of order 1 or 3) by using the bundle isomorphisms

$$M(E_2, B_i) \cong M(-E_2, -B_i), \ M(-E_2, B_i) \cong M(-E_2, -B_i).$$

Therefore, we don't have to deal with all the 2^{2g} possibilities of lifts of ρ, but only those satisfying the following three conditions;

(a) if $k+1 \leq i \leq l$, then $\mathrm{ord}(B_i) = 3$,
(b) if $l+1 \leq i \leq m$, then $\varepsilon_i = -1$, and
(c) if $m+1 \leq i \leq g$, then $B_i = E_2$.

In what follows, we assume that the monodromy $\widetilde{\rho}$ of $M(\varepsilon_1 E_2, B_1, \ldots, \varepsilon_g E_2, B_g)$ satisfies these conditions.

Proposition 5.5. The following assertions hold.

(1) When the monodromy group of $M(\varepsilon_1 E_2, B_1, \ldots, \varepsilon_g E_2, B_g)$ contains $-E_2$, all the 2^g T^2-bundles $M(\varepsilon_1 E_2, \pm B_1, \ldots, \varepsilon_g E_2, \pm B_g)$ are isomorphic to each other.
(2) When the monodromy group of $M(E_2, B_1, \ldots, E_2, B_g)$ does not contain $-E_2$, the 2^k T^2-bundles $M(E_2, \pm B_1, \ldots, E_2, \pm B_k, E_2, B_{k+1}, \ldots, E_2, B_g)$ are pairwise non-isomorphic.

Proof. (1) By assumption, there exists i with $1 \leq i \leq g$ such that $\varepsilon_i = -1$. For, if $\varepsilon_i = 1$ for all i, then by Proposition 5.4 and conditions (a) and (c), there must be B_i of order 4, and hence, we have $\varepsilon_i = -1$ by the condition (b), which contradicts to the assumption that $\varepsilon_i = 1$ for all i. Hence, we can take i $(1 \leq i \leq g)$ with $\varepsilon_i = -1$. Then, it follows from the isomorphisms $M(-E_2, B_i) \cong M(-E_2, -B_i)$ and

$$M(-E_2, B_i)\#_f M(E_2, B_j) \cong M(-E_2, B_i)\#_f M(E_2, -B_j)$$

(see Propositions 5.2 and 5.3) that replacing B_j by $-B_j$ $(1 \leq j \leq g)$ does not change the isomorphism class of T^2-bundle.

(2) Let us denote the monodromy group $\widetilde{\rho}(\pi_1(\Sigma_g)) \subset SL(2;\mathbb{Z})$ by H. Since H does not contain $-E_2$, it is impossible for H to contain both $\pm B_i$ for any i with $1 \leq i \leq k$. Hence, by replacing B_i by $-B_i$ or vice versa, we can switch whether $B_i \in H$ or $-B_i \in H$ preserving the condition that H does not contain $-E_2$. Therefore, the 2^k T^2-bundles

$$M(E_2, \pm B_1, \ldots, E_2, \pm B_k, E_2, B_{k+1}, \ldots, E_2, B_g)$$

are indeed pairwise non-isomorphic, since they have different monodromy groups.

□

Proposition 5.6. The two T^2-bundles

$$M(\varepsilon_1 E_2, B_1, \ldots, \varepsilon_k E_2, B_k, \varepsilon_{k+1} E_2, B_{k+1}, \ldots, \varepsilon_g E_2, B_g)$$
$$M(\delta_1 E_2, B_1, \ldots, \delta_k E_2, B_k, \delta_{k+1} E_2, B_{k+1}, \ldots, \delta_g E_2, B_g)$$

are not isomorphic to each other if $(\varepsilon_1, \ldots, \varepsilon_k) \neq (\delta_1, \ldots, \delta_k)$.

Proof. We will prove the assertion by reduction to the absurd, that is, we suppose $(\varepsilon_1, \ldots, \varepsilon_k) \neq (\delta_1, \ldots, \delta_k)$ and the two bundles are isomorphic, and then lead a contradiction. Without loss of generality we may assume that $\varepsilon_1 = 1$ and $\delta_1 = -1$, so it is enough to consider the case where there exists a bundle isomorphism

$$\tilde{\varphi}\colon M(E_2, B_1, \varepsilon_2 E_2, B_2, \ldots, \varepsilon_g E_2, B_g) \to M(-E_2, B_1, \delta_2 E_2, B_2, \ldots, \delta_g E_2, B_g)$$

covering the base diffeomorphism $\varphi\colon \Sigma_g \to \Sigma_g$. Denote the monodromy representation of $M(-E_2, B_1, \delta_2 E_2, B_2, \ldots, \delta_g E_2, B_g)$ by ρ. Then we have

$$(\rho \circ \varphi_*)(\alpha_1) = E_2, \ \ (\rho \circ \varphi_*)(\alpha_i) = \varepsilon_i E_2 \ \ (2 \leq i \leq g), \ \ (\rho \circ \varphi_*)(\beta_j) = B_j,$$

where α_j, β_j $(1 \leq j \leq g)$ are the canonical generators of $\pi_1(\Sigma_g)$. Then, for each i $(2 \leq i \leq g)$, the loops $\varphi(\alpha_i)$, $\varphi(\beta_i)$ are contained in the smallest normal subgroup N generated by $\alpha_1, \alpha_2, \ldots, \alpha_g, \beta_2, \ldots, \beta_g$, because $\rho(\varphi(\alpha_i)) = \varepsilon_i E_2$, $\rho(\varphi(\beta_i)) = B_i$ are contained in $\rho(N) = \langle -E_2, B_2, \ldots, B_g \rangle^{\mathrm{Im}(\rho)}$ and the homomorphism

$$\bar{\rho}\colon \pi_1(\Sigma_g)/N \cong \langle \beta_1 \rangle \cong \mathbb{Z} \to \rho\left(\pi_1(\Sigma_g)\right)/\rho(N) \cong \langle B_1 \rangle \cong \mathbb{Z}$$

induced by $\rho\colon \pi_1(\Sigma_g) \to SL(2;\mathbb{Z})$ is an isomorphism. Here we note that B_1 does not belong to $\rho(N)$, since $p \circ \rho$ is of normal form and $p(B_1)$ is a generator of one factor of the free product decomposition of $\mathrm{Im}(p \circ \rho)$, where $p\colon SL(2;\mathbb{Z}) \to PSL(2;\mathbb{Z})$ is the canonical projection. Since the loops $\varphi(\alpha_i)$ and $\varphi(\beta_i)$ $(2 \leq i \leq g)$ are elements of N, their intersection numbers with α_1 are all 0. Hence, the loop $\varphi^{-1}(\alpha_1)$ has intersection number 0 with any of α_i and β_i $(2 \leq i \leq g)$. Recalling that the first homology group is the abelianization of the fundamental group, $\varphi^{-1}(\alpha_1)$ is homologous to an integral linear combination of α_1 and β_1. Since $p \circ \rho \circ \varphi_* = p \circ \rho$

is of the normal form and $(\rho\circ\varphi_*)(\varphi^{-1}(\alpha_1)) = \rho(\alpha_1) = -E_2$, the coefficient of β_1 must be 0. Thus we have

$$\varphi^{-1}(\alpha_1) = \alpha_1^n s,$$

where n is an integer and s is an element of the commutator subgroup $[\pi_1(\Sigma_g), \pi_1(\Sigma_g)]$. Then we obtain that

$$(\rho\circ\varphi_*)(s) = (\rho\circ\varphi_*)(\alpha_1^{-n}\varphi^{-1}(\alpha_1)) = (\rho\circ\varphi_*)(\alpha_1)^{-n}\rho(\alpha_1) = -E_2.$$

This implies that $-E_2$ belongs to the commutator subgroup $[SL(2;\mathbb{Z}), SL(2;\mathbb{Z})]$, which is a contradiction. Thus we have obtained that

$$M(E_2, B_1, \varepsilon_2 E_2, B_2, \ldots, \varepsilon_g E_2, B_g) \not\cong M(-E_2, B_1, \delta_2 E_2, B_2, \ldots, \delta_g E_2, B_g).$$

This completes the proof. □

Based on these propositions, we can prove Theorem 1.4 as follows.

Proof of Theorem 1.4. (1) The case where $m > l$. For any lift $\widetilde{\rho}$ of ρ, the monodoromy group $\mathrm{Im}(\widetilde{\rho})$ contains $-E_2$, since each B_i $(l+1 \le i \le m)$ is of order 4. In this case, replacing B_i by $-B_i$ does not change the isomorphism class of the T^2-bundle by Propositions 5.5 (1). Moreover, by applying Proposition 5.2, we can change the signs ε_i $(k+1 \le i \le l,\ m+1 \le i \le g)$ preserving the isomorphism class of the T^2-bundle (recall that ε_i is fixed to be -1 when $l+1 \le i \le m$). Therefore, by Proposition 5.6, there are just 2^k isomorphism classes of T^2-bundles $M(\varepsilon_1 E_2, B_1, \ldots, \varepsilon_g E_2, B_g)$ corresponding to the choices of ε_i $(1 \le i \le k)$.

(2) The case where $m = l$. In this case, the order of B_i is not 4, so each B_i with $k+1 \le i \le g$ is uniquely determined by the conditions (a) and (c). Moreover, if $-E_2 \in \mathrm{Im}(\rho)$, then there exists an integer i with $1 \le i \le g$ and $\varepsilon_i = -1$. Now we argue the following three cases.

(i) If there exists an integer i with $1 \le i \le k$ and $\varepsilon_i = -1$, then the isomorphism class depends only on the choices of ε_i $(1 \le i \le k)$ by the same argument as in (1). By Proposition 5.6, there are just $2^k - 1$ isomorphism classes corresponding to such choices.

(ii) If $\varepsilon_1 = \cdots = \varepsilon_k = 1$ and there exists an integer i with $k+1 \le i \le g$ and $\varepsilon_i = -1$, then such T^2-bundles are all isomorphic to each other by the same argument as in (1). Hence, the isomorphism class is unique in this case.

(iii) If $-E_2 \notin \mathrm{Im}(\rho)$, then we have $\varepsilon_1 = \cdots = \varepsilon_g = 1$. In this case, there is no ambiguity other than the choices of the signs of B_i $(1 \le i \le k)$. By Proposition 5.5 (2), the 2^k choices yield different isomorphism classes. Therefore, there are 2^k isomorphism classes.

Thus there are just 2^{k+1} isomorphism classes of T^2-bundles.

□

Proof of Theorem 1.5. The conditions (1) and (2) in Theorem 1.5 are equivalent to that there exist $f \in \mathcal{M}_g$ and $Q \in SL(2;\mathbb{Z})$ such that

$$Q\widetilde{\rho}_2 Q^{-1} = \widetilde{\rho}_1 \circ f_*.$$

Moreover, the Euler class $e(\xi_1)$ of $M(\varepsilon_1 E_2, B_1, \ldots, \varepsilon_g E_2, B_g; m, n)$ lies in

$$\mathbb{Z}^2 / \langle (B_1 - E_2)\mathbf{e}_1, (B_1 - E_2)\mathbf{e}_2, \cdots, (B_g - E_2)\mathbf{e}_1, (B_g - E_2)\mathbf{e}_2 \rangle$$

if $\varepsilon_1 = \cdots = \varepsilon_g = 1$, and in

$$\mathbb{Z}^2 / \langle 2\mathbf{e}_1, 2\mathbf{e}_2, (B_1 - E_2)\mathbf{e}_1, (B_1 - E_2)\mathbf{e}_2, \cdots, (B_g - E_2)\mathbf{e}_1, (B_g - E_2)\mathbf{e}_2 \rangle$$

otherwise. Hence the condition (3) in Theorem 1.5 is equivalent to the one that $e(\xi_1) = f^* e(\xi_2)$ for the above $f \in \mathcal{M}_g$ and $Q \in SL(2;\mathbb{Z})$. Hence, by Proposition 4.6, the conditions (1), (2), (3) in Theorem 1.5 give a necessary and sufficient condition for ξ_1 and ξ_2 to be isomorphic. □

We obtain the following as a corollary to Theorem 1.5.

Corollary 5.7. Let ξ_1 and ξ_2 be orientable T^2-bundles whose monodromy groups coincide with $SL(2;\mathbb{Z})$. Then ξ_1 and ξ_2 are bundle isomorphic.

Proof. We set $B_1 = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$, $B_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Then the $SL(2;\mathbb{Z})$-bundle

$$M(E_2, B_1, E_2, B_2, E_2, E_2, \ldots, E_2, E_2)$$

satisfies the condition, so it is enough to argue the case where ξ_1 is isomorphic to this bundle. Let $\widetilde{\rho}_1$ and $\widetilde{\rho}_2$ denote the monodromies of ξ_1 and ξ_2, respectively. We assume that $p \circ \widetilde{\rho}_2$ is of the normal form in the sense of Theorem 1.3 (notice that $p \circ \widetilde{\rho}_1$ is automatically so). Then the condition (1) is fulfilled by the assumption $\mathrm{Im}(\widetilde{\rho}_1) = \mathrm{Im}(\widetilde{\rho}_2) = SL(2;\mathbb{Z})$. Since $\mathrm{Im}(p \circ \widetilde{\rho}_1) = \mathrm{Im}(p \circ \widetilde{\rho}_2)$, we may assume that $p \circ \widetilde{\rho}_1 = p \circ \widetilde{\rho}_2$ holds by Theorem 1.3. Putting $\rho = p \circ \widetilde{\rho}_1 = p \circ \widetilde{\rho}_2$, then $\widetilde{\rho}_1$ and $\widetilde{\rho}_2$ are both lifts of ρ with respect to p. Since $\mathrm{Im}(\rho) = PSL(2;\mathbb{Z}) \cong \mathbb{Z}_2 * \mathbb{Z}_3$, the integers k, l, m in Theorem 1.3 are determined as $k = 0$, $l = 1$, $m = 2$. Therefore, the $\mathcal{M}_g$-orbit of lifts of ρ is unique by Theorem 1.4, and in particular, the condition (2) is fulfilled. Then $SL(2;\mathbb{Z})$-bundles corresponding to lifts of ρ are all isomorphic to each other. In particular, ξ_2 is isomorphic to $M(E_2, B_1, E_2, B_2, E_2, E_2, \ldots, E_2, E_2; m, n)$ for some $(m,n) \in \mathbb{Z}^2$. Finally, we can check the condition (3) as follows. Since the four column vectors of the two matrices

$$B_1 - E_2 = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}, \; B_2 - E_2 = \begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix}$$

generates $\mathbb{Z}^2$, there are indeed $\boldsymbol{x}_1$ and $\boldsymbol{x}_2 \in \mathbb{Z}^2$ such that

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} - \begin{pmatrix} m \\ n \end{pmatrix} = (B_1 - E_2)\boldsymbol{x}_1 + (B_2 - E_2)\boldsymbol{x}_2,$$

which means that the condition (3) is fulfilled. Therefore, by Theorem 1.5, the T^2-bundles ξ_1 and ξ_2 are isomorphic. □

6. T^2-bundles with compatible symplectic structures

Let $\pi\colon M^4 \to \Sigma_g$ be an orientable Σ_h-bundle over a closed orientable surface Σ_g. A symplectic structure ω on M^4 is said to be compatible with π if its restriction to each fiber $\pi^{-1}(b)$ ($b \in \Sigma_g$) is also a symplectic form. Whether a given surface bundle over a surface admits a compatible symplectic structure is determined by the following result.

Theorem 6.1 (Thurston [31])**.** *A Σ_h-bundle $\pi\colon M^4 \to \Sigma_g$ admits a compatible symplectic structure if and only if the homology class represented by a fiber Σ_h is nonzero in $H_2(M^4;\mathbb{R})$.*

When $h \neq 1$, a compatible symlectic struture always exists since the above condition is automatically fulfilled. When $h = 1$, namely, in the case of T^2-bundles, the situation is a little more complicated. First we consider when $g = 0$. In this case, no nontrivial T^2-bundle admits a compatible symplectic structure nor even a symplectic form on the total space. For, the second Betti number of the total space of a T^2-bundle over S^2 is 0 unless its Euler class is $(0, 0)$. When $g = 1$, Geiges has given the following answer based on Sakamoto-Fukuhara's classification (Theorem 5.1).

Theorem 6.2 (Geiges [5]). *An orientable T^2-bundle $\pi\colon M^4 \to T^2$ admits a compatible symplectic structure if and only if it is not isomorphic to*

$$M(E_2, E_2; m, 0)\ (m \neq 0)\ \text{ nor }\ M(E_2, C^k; m, n)\ (n \neq 0),$$

where $C = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $k \in \mathbb{Z}$. On the other hand, every orientable T^2-bundle over T^2 admits a symplectic structure on the total space.

The case where $g \geq 2$ has been settled by Walczak as follows.

Theorem 6.3 (Walczak [35]). *An orientable T^2-bundle*

$$\pi\colon M(A_1, B_1, \dots, A_g, B_g; m, n) \to \Sigma_g$$

with $g \geq 2$ admits a compatible symplectic structure if and only if its total space M^4 admits a symplectic structure. Moreover, such T^2-bundles are classified by their monodromies and Euler classes as follows.

(1) *$A_i = B_i = E_2$ for all i $(1 \leq i \leq g)$ and $(m, n) = (0, 0)$.*
(2) *The monodromy is nontrivial and there exists a nonzero vector $\boldsymbol{x} = (x_1, x_2) \in \mathbb{Z}^2$ such that $A_i\boldsymbol{x} = \boldsymbol{x}$, $B_i\boldsymbol{x} = \boldsymbol{x}$ for all i $(1 \leq i \leq g)$ and $nx_1 - mx_2 = 0$.*
(3) *There is not a nonzero vector $\boldsymbol{x} \in \mathbb{Z}^2$ such that $A_i\boldsymbol{x} = \boldsymbol{x}$ and $B_i\boldsymbol{x} = \boldsymbol{x}$ for all i.*

Thus the problem of determining which surface bundle over a surface admits a symplectic structure has already been solved. However, the statements of Theorems 6.2 and 6.3 on the existence of compatible symplectic structures can be briefly summarized as follows. This was pointed out by Yoshihiko Mitsumatsu.

Theorem 6.4. *Let g be a non-negative integer. Then an orientable T^2-bundle*

$$\pi\colon M(A_1, B_1, \dots, A_g, B_g; m, n) \to \Sigma_g$$

admits a compatible symplectic structure if and only if its Euler class is a torsion.

Proof. We put $M_0 = M(A_1, B_1, \dots, A_g, B_g)$ and $C = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Let X be a $2 \times 4g$ matrix consisting of $4g$ column vectors of the following $2g$ matrices;

$$A_1 - E_2,\ B_1 - E_2,\ \dots,\ A_g - E_2,\ B_g - E_2.$$

Then the first Betti number of M_0 can be described as $b_1(M_0) = 2g + 2 - \mathrm{rank}(X)$. Hence, the three cases $b_1(M_0) = 2g + 2,\ 2g + 1,\ 2g$ can be interpreted to the following three conditions, respectively.

(1) $A_1 = B_1 = \cdots = A_g = B_g = E_2$.
(2) there exist $Q \in SL(2; \mathbb{Z})$ and $k_i, l_i \in \mathbb{Z}$ such that $A_i = QC^{k_i}Q^{-1}$, $B_i = QC^{l_i}Q^{-1}$.

(3) otherwise.

Then the necessary and sufficient condition that Geiges and Walczak gave can be rephrased as follows, where $\boldsymbol{q}_1$ denotes the first column vector of Q.

(1) and $(m, n) = (0, 0)$.
(2) and $\begin{pmatrix} m \\ n \end{pmatrix} \in \langle \boldsymbol{q}_1 \rangle$.
(3) and (m, n) is arbitrary.

This condition is equivalent to the one that the Euler class $[(m, n)] \in \mathbb{Z}^2 / \operatorname{Im}(X)$ is a torsion element. Indeed, if (1), $\operatorname{Im}(X) = \{\mathbf{0}\}$, if (2),

$$\operatorname{Im}(X) = \langle k_1 \boldsymbol{q}_1, l_1 \boldsymbol{q}_1, \ldots, k_g \boldsymbol{q}_1, l_g \boldsymbol{q}_1 \rangle = \langle d \boldsymbol{q}_1 \rangle,$$

where d is the greatest common divisor of $k_1, l_1, \ldots, k_g, l_g$, and if (3), $\mathbb{Z}^2 / \operatorname{Im}(X)$ is a finite group. This completes the proof of the theorem. □

From the viewpoint of our main theorem, Theorem 6.4 can be interpreted as follows, which seems a natural generalization of Geiges' condition (the former part of Theorem 6.2).

Theorem 6.5. *Let g be a non-negative integer. An orientable T^2-bundle $\pi\colon M^4 \to \Sigma_g$ admits a compatible symplectic structure if and only if it is not isomorphic to*

$$M(E_2, E_2, \ldots, E_2, E_2; m, 0)\ (m \neq 0) \ \text{ nor } \ M(E_2, C^k, \ldots, E_2, E_2; m, n)\ (n \neq 0),$$

where $C = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $k \in \mathbb{Z}$.

Combining it with the former part of Theorem 6.2 and the latter part of Theorem 6.3, we also obtain the following.

Theorem 6.6. *Let $\pi\colon M^4 \to \Sigma_g$ and C be as in Theorem 6.5. Then M^4 admits a symplectic structure if and only if π is not isomorphic to either of the following.*

(1) $M(E_2, E_2, \ldots, E_2, E_2; m, 0)$ $(g \neq 1,\ m \neq 0)$.
(2) $M(E_2, C^k, \ldots, E_2, E_2; m, n)$ $(g \geq 2,\ k \in \mathbb{Z},\ n \neq 0)$.

On the other hand, Ue ([32, 33, 34]) clarified which orientable T^2-bundle admits a complex structure on its total space. Comparing Theorem 6.6 with his result, we obtain the following theorem.

Theorem 6.7. *Let $\pi\colon M^4 \to \Sigma_g$ and C be as in Theorem 6.5. Then M^4 admits a non-Kähler symplectic structure if π is not isomorphic to either of the following;*

(1) *trivial bundle,*
(2) *hyperelliptic bundles* (*see* [32], *List* I. 1 − 3(a)),
(3) $M(E_2, E_2, \ldots, E_2, E_2; m, 0)$ $(g \neq 1,\ m \neq 0)$,
(4) $M(E_2, C^k, \ldots, E_2, E_2; m, n)$ $(g \geq 2,\ k \in \mathbb{Z},\ n \neq 0)$,
(5) $M(E_2, B, \ldots, E_2, E_2; m, n)$ $(g \geq 2,\ \operatorname{ord}(B) = 2, 3, 4, 6)$.

By this theorem, we can reconfirm the fact that there exist infinitely many non-Kähler closed symplectic 4-manifolds with even first Betti number, which is known as a corollary to Gompf's result [6]. This is in a good contrast with the fact that a compact complex surface admits a Käher metric if and only if its first Betti number is even ([15, 16, 17, 18, 19, 20, 21, 24, 28]).

Remark 6.8. A non-compact complex surface does not necessarily admit a Käher metric even if its first Betti number is even. Indeed, there exist uncountably many non-Kähler complex structures on $\mathbb{R}^4$ ([2]). Moreover, any orientable open 4-manifold admits both Kähler structures and non-Kähler complex structures ([3]).

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