QUESTIONS ON THE CHOW RING OF COMPLETE INTERSECTIONS

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ABSTRACT. We state several questions, and prove some partial results, about the Chow ring $A^*(X)$ of complete intersections in projective space. For one thing, we prove that if X is a general Calabi–Yau hypersurface, the intersection product $A^2(X) \cdot A^i(X)$ is one-dimensional, for any $i > 0$. We also show that quintic threefolds have an multiplicative Chow–Künneth (MCK) decomposition. We wonder whether all Calabi–Yau hypersurfaces might have an MCK decomposition, and prove this is the case conditional to a conjecture of Voisin.

1. INTRODUCTION

Given a complex smooth projective variety X, let $A^*(X) = \oplus_i A^i(X)$ denote the Chow ring with $\mathbb{Q}$-coefficients. Even for the simplest varieties, understanding the Chow ring is not so simple. For instance, motivated by the weak Lefschetz theorem in cohomology, Hartshorne has asked the following:

Conjecture 1.1 (Hartshorne 1974 [7]). *Let $X \subset \mathbb{P}^{n+1}(\mathbb{C})$ be a smooth hypersurface, and let $h \in A^1(X)$ denote the hyperplane class. Then*

$$A^i(X) = \mathbb{Q}[h^i] \ \textit{for all}\ i < \frac{n}{2}\ .$$

Apart from some easy results when X has small degree (and so is Fano), Conjecture 1.1 is completely open for $i \geq 2$, and seems highly challenging.

Since a direct attack on Conjecture 1.1 appears hopeless, let us now investigate some consequences of Conjecture 1.1. As is well-known, the image of intersecting with the hyperplane class on a hypersurface X is one-dimensional, i.e. $h \cdot A^i(X) = \mathbb{Q}[h^{i+1}]$ (this is just because $h \cdot A^i(X) = \iota^*\iota_* A^i(X)$ where $\iota\colon X \hookrightarrow \mathbb{P}^{n+1}(\mathbb{C})$ denotes the embedding). This observation means that if one believes in Conjecture 1.1 one must also believe the following:

Conjecture 1.2. *Let $X \subset \mathbb{P}^{n+1}(\mathbb{C})$ be a smooth hypersurface. Then*

$$A^i(X) \cdot A^j(X) = \mathbb{Q}[h^{i+j}] \ \textit{for all}\ i,j > 0 \ \textit{such that}\ (i,j) \neq (\frac{n}{2}, \frac{n}{2})\ .$$

(It seems likely Conjecture 1.2 holds true more generally for complete intersections $X \subset \mathbb{P}^{n+r}(\mathbb{C})$, cf. Remark 2.3 below.)

Restricting attention to hypersurfaces that are Calabi–Yau, there is a remarkable result proven by Voisin:

Key words and phrases. Chow groups, Chow ring, hypersurface in projective space, MCK decomposition.
2020 Mathematics Subject Classification: 14C15, 14C25, 14C30.
This work is supported by ANR grant ANR-20-CE40-0023.

Theorem 1.3 (Voisin [23])**.** *Let $X \subset \mathbb{P}^{n+1}(\mathbb{C})$ be a smooth hypersurface of degree $n+2$. Then*

$$A^i(X) \cdot A^j(X) = \mathbb{Q}[h^n] \ \ \textit{for all } i,j>0 \textit{ such that } i+j=n \ .$$

Combining Theorem 1.3 and Conjecture 1.2, one obtains the following conjecture about the intersection product on hypersurfaces:

Conjecture 1.4. *Let $X \subset \mathbb{P}^{n+1}(\mathbb{C})$ be a smooth hypersurface. Assume that either the dimension n is odd, or the degree of X is $n+2$ (i.e. X is Calabi–Yau). Then*

$$A^i(X) \cdot A^j(X) = \mathbb{Q}[h^{i+j}] \ \ \textit{for all } i,j>0 \ .$$

By looking into Voisin's proof of Theorem 1.3, we come up with some (very partial) confirmation of Conjecture 1.4:

Theorem (=Theorem 2.1)**.** *Let $X \subset \mathbb{P}^{n+1}(\mathbb{C})$ be a general hypersurface of degree $n+2$ (i.e. X is Calabi–Yau). Then*

$$A^2(X) \cdot A^j(X) = \mathbb{Q}[h^{2+j}] \ \ \textit{for all } j>0 \ .$$

This settles Conjecture 1.4 for Calabi–Yau hypersurfaces of dimension ≤ 6. The same result holds for certain Calabi–Yau complete intersections (cf. Theorem 2.2 below). We also prove a result about the ring $B^*(X)$ of cycles modulo algebraic equivalence:

Theorem (=Theorem 2.4)**.** *Let $X \subset \mathbb{P}^{n+1}(\mathbb{C})$ be a general hypersurface of degree $n+2$. Then*

$$B^i(X) \cdot B^{n-i-1}(X) = \mathbb{Q}[h^{n-1}] \ \ \textit{for all } 0<i<n-1 \ .$$

This settles Conjecture 1.4 modulo algebraic equivalence for Calabi–Yau hypersurfaces of dimension ≤ 8.

Looking at Conjecture 1.4, it is natural to wonder whether perhaps Calabi–Yau hypersurfaces might have an MCK decomposition, in the sense of Shen–Vial [18] (roughly speaking, this means that the Chow motive decomposes compatibly with intersection product; cf. Subsection 2.3 below for the precise definition). We show this is the case if one assumes a conjecture made by Voisin (cf. Proposition 2.17 below). We prove an unconditional result in dimension 3:

Theorem (=Theorem 2.18 and Corollary 2.21)**.** *Any smooth quintic threefold $X \subset \mathbb{P}^4(\mathbb{C})$ admits an MCK decomposition.*

In particular, for any $m \in \mathbb{N}$ let

$$R^*(X^m) := \big\langle (p_i)^*(h), (p_{jk})^*(\Delta_X) \big\rangle \ \subset \ A^*(X^m)$$

denote the $\mathbb{Q}$-algebra generated by (pullbacks of) the polarization h and the diagonal Δ_X. Then $R^*(X^m)$ injects into cohomology under the cycle class map for all $m \leq 205$ (and $R^*(X^m)$ injects into cohomology for all m if and only if X is Kimura finite-dimensional, in the sense of [9]*).*

Building on work of Lie Fu [2], we also include a version of Theorem 2.1 that applies to hypersurfaces that are of general type; this is Theorem 3.3 below.

Conventions. *In this paper, the word variety will mean a reduced irreducible scheme of finite type over $\mathbb{C}$. A subvariety will refer to a (possibly reducible) reduced subscheme which is equidimensional.*

All Chow groups will be with rational coefficients: *we denote by $A^i(Y)$ the Chow group of codimension i cycles on Y with $\mathbb{Q}$-coefficients. The notation $A^i_{hom}(Y)$ (resp. $A^i_{AJ}(X)$) will be used to denote the subgroup of homologically trivial (resp. Abel–Jacobi trivial) cycles.*

The contravariant category of Chow motives (i.e., pure motives with respect to rational equivalence and $\mathbb{Q}$-coefficients as described in [15], [11]*) will be denoted $\mathcal{M}_{\rm rat}$.*

2. CALABI–YAU HYPERSURFACES

2.1. Intersecting with codimension 2 cycles.

Theorem 2.1. *Let $X \subset \mathbb{P}^{n+1}(\mathbb{C})$ be a general hypersurface of degree $n+2$. Then*

$$A^2(X) \cdot A^i(X) = \mathbb{Q}[h^{i+2}] \quad \forall i > 0 \ .$$

Proof. In case $n = 4$ this is just Voisin's result (Theorem 1.3). Let us now assume $n \geq 5$.

Let $F := F(X)$ denote the Fano variety of lines in X. By the generality assumption, both X and F are smooth and of the expected dimension, i.e. $\dim F = n - 3$. Let

$$\Delta_X^{sm} := \{(x, x, x) \mid x \in X\} \ \subset \ X \times X \times X$$

denote the small diagonal. Let

$$\Gamma := \bigcup_{t \in F} \mathbb{P}^1_t \times \mathbb{P}^1_t \times \mathbb{P}^1_t \ \subset \ X \times X \times X \ , \tag{1}$$

where $\mathbb{P}^1_t \subset X$ denotes the line corresponding to $t \in F$. Let $o_X := \frac{1}{n+2} h^n \in A^n(X)$ denote the "canonical" zero-cycle of degree 1, where $h \in A^1(X)$ is the hyperplane section class. For any $1 \leq i < j \leq 3$, define

$$\Delta_{ij} := (p_i \times p_j)^*(\Delta_X) \cdot (p_k)^*(o_X) \ \in \ A^{2n}(X \times X \times X) \ ,$$

where $k \in \{1, 2, 3\} \setminus \{i, j\}$, and p_i denotes projection to the ith factor.

In the course of the proof of Theorem 1.3, Voisin has obtained [23, Theorem 3.1] the following equality:

$$\Delta_X^{sm} = \frac{1}{(n+2)!} \Gamma + \Delta_{12} + \Delta_{13} + \Delta_{23} + P(h_1, h_2, h_3) \quad \text{in } A^{2n}(X \times X \times X) \ , \tag{2}$$

where $P(h_1, h_2, h_3)$ is a polynomial in the divisors $h_i := (p_i)^*(h) \in A^1(X \times X \times X)$.

Let $a \in A^2(X)$ and $b \in A^i(X)$ (where $i > 0$) be any cycles. As n is at least 5, we can write

$$a = a_0 + a_2 \ \text{ in } A^2(X) \ ,$$

where $a_0 \in \mathbb{Q}[h^2]$ and $a_2 \in A^2_{hom}(X) = A^2_{AJ}(X)$. Clearly $h \cdot b \in \mathbb{Q}[h^{i+1}]$, and so it will suffice to prove that $a_2 \cdot b$ is in $\mathbb{Q}[h^{i+2}]$. Considering equality (2) as an equality of correspondences from $X \times X$ to X, we find that

$$a_2 \cdot b = (\Delta_X^{sm})_*(a_2 \times b) = \Big(\frac{1}{(n+2)!} \Gamma + \Delta_{12} + \Delta_{13} + \Delta_{23} + P(h_1, h_2, h_3)\Big)_*(a_2 \times b) \ .$$

The correspondences Δ_{12} and $P(h_1,h_2,h_3)$ being decomposable, they act as zero on homologically trivial cycles. The correspondences Δ_{13} and Δ_{23} act as zero on $A^j(X)\otimes A^i(X)$ for all $j,i>0$. Thus, the above equality boils down to

$$a_2\cdot b=(\Delta_X^{sm})_\ast(a_2\times b)=\frac{1}{(n+2)!}\,\Gamma_\ast(a_2\cdot b)\ .$$

Writing $P\in A^{n-1}(F\times X)$ for the (class of the) universal line, we have equality

$$\begin{aligned}\Gamma&=(P\times P\times P)_\ast(\Delta_F^{sm})\\&=P\circ(\Delta_F^{sm})\circ({}^tP\times{}^tP)\ \text{ in } A^{2n}(X\times X\times X)\ ,\end{aligned}$$

where the second equality is an instance of Lieberman's lemma [11, Lemma 2.1.3]. This means that the action of Γ on $A^2(X)\otimes A^i(X)$ factors as

$$\begin{array}{ccc}A^2(X)\otimes A^i(X) & \xrightarrow{\Gamma_\ast} & A^{i+2}(X)\\ \downarrow{\scriptstyle (P^\ast,P^\ast)} & & \uparrow{\scriptstyle P_\ast}\\ A^1(F)\otimes A^{i-1}(F) & \xrightarrow{(\Delta_F^{sm})_\ast} & A^i(F)\end{array}$$

But since $a_2\in A^2_{AJ}(X)$ we know that $P^\ast(a_2)\in A^1_{AJ}(F)=0$ (here we have used the fact that Abel–Jacobi equivalence is an adequate equivalence relation, hence is compatible with the action of correspondences, cf. [10, p. 134] or [14, Example 1.11]), and so we find

$$\Gamma_\ast(a_2\times b)=0\ .$$

This concludes the proof. ∎

The argument of Theorem 2.1 can be extended to certain Calabi–Yau complete intersections:

Theorem 2.2. *Let $X\subset\mathbb{P}^{n+r}(\mathbb{C})$ be a general complete intersection of dimension $n>r$ that is Calabi–Yau, and assume X is not a complete intersection of quadrics. Then*

$$A^2(X)\cdot A^i(X)=\mathbb{Q}[h^{i+2}]\ \textit{for all}\ \ i>0\ .$$

Proof. The argument is similar to that of Theorem 2.1. In case $n=\dim X=4$ this is a result of Lie Fu [2, Theorem 0.7]. Let us now assume $n\geq 5$.

In proving his result, Fu has established [2, Theorem 1.17] the following equality:

$$\Delta_X^{sm}=\alpha\,\Gamma+(j_{12})_\ast(Z)+(j_{13})_\ast(Z)+(j_{23})_\ast(Z)+P(h_1,h_2,h_3)\ \text{ in } A^{2n}(X\times X\times X)\ , \tag{3}$$

where $\alpha\in\mathbb{Q}^\ast$, where Γ is defined as in (1), where $j_{12}\colon X\times X\to X^3$ is the diagonal embedding $(x,x')\mapsto(x,x,x')$ (and similar definitions for j_{13} and j_{23}), and $Z:=Q(h_1,h_2)$ is a polynomial in the divisors $h_i:=(p_i)^\ast(h)\in A^1(X\times X)$.

Let $a\in A^2(X)$ and $b\in A^i(X)$ (where $i>0$) be any cycles. As n is at least 5, we can write

$$a=a_0+a_2\ \text{ in } A^2(X)\ ,$$

where $a_0\in\mathbb{Q}[h^2]$ and $a_2\in A^2_{hom}(X)=A^2_{AJ}(X)$. A nice recent result of Mboro [12, Theorem 1.2] ensures that $h\cdot b\in\mathbb{Q}[h^{i+1}]$ (this uses the assumptions that $n>r$ and that X is not a complete intersection of quadrics), and so it will suffice to prove that $a_2 \cdot b$ is in $\mathbb{Q}[h^{i+2}]$. Considering equality (3) as an equality of correspondences from $X \times X$ to X, we find that

$$\begin{aligned} a_2 \cdot b &= (\Delta_X^{sm})_*(a_2 \times b) \\ &= \Big(\alpha\,\Gamma + (j_{12})_*(Z) + (j_{13})_*(Z) + (j_{23})_*(Z) + P(h_1,h_2,h_3)\Big)_*(a_2 \times b)\ . \end{aligned}$$

The correspondences $(j_{12})_*(Z)$ and $P(h_1,h_2,h_3)$ being decomposable, they act as zero on homologically trivial cycles. The correspondences $(j_{13})_*(Z)$ and $(j_{23})_*(Z)$ send $A^j(X) \otimes A^i(X)$ to $\mathbb{Q}[h^{i+j}]$, as can be easily seen (cf. [2, Proof of Corollary 1.13]). Thus, the above equality boils down to

$$a_2 \cdot b = (\Delta_X^{sm})_*(a_2 \times b) = \alpha\,\Gamma_*(a_2 \cdot b)\ .$$

Writing $P \in A^{n-1}(F \times X)$ for the (class of the) universal line, as before we have equality

$$\Gamma = P \circ (\Delta_F^{sm}) \circ ({}^tP \times {}^tP) \ \text{ in } A^{2n}(X \times X \times X)\ .$$

It follows that the action of Γ on $A^2(X) \otimes A^i(X)$ factors as

$$\begin{array}{ccc} A^2(X) \otimes A^i(X) & \xrightarrow{\Gamma_*} & A^{i+2}(X) \\ \downarrow {\scriptstyle (P^*,P^*)} & & \uparrow {\scriptstyle P_*} \\ A^1(F) \otimes A^{i-1}(F) & \xrightarrow{(\Delta_F^{sm})_*} & A^i(F) \end{array}$$

But since $a_2 \in A^2_{AJ}(X)$ we know that $P^*(a_2) \in A^1_{AJ}(F) = 0$, and so we conclude that

$$\Gamma_*(a_2 \times b) = 0\ .$$

This ends the proof. □

Remark 2.3. *One might expect that Conjecture 1.2 holds true for all smooth complete intersections in projective space. Indeed, as is well-known Hartshorne's conjecture (Conjecture 1.1) for complete intersections would follow from the truth of the Bloch–Beilinson conjectures (cf.* [2, Remark 2.14]*). Furthermore, one would expect that all smooth complete intersections $X \subset \mathbb{P}^*$ satisfy $A^i(X) \cdot h = \mathbb{Q}[h^{i+1}]$ for all i (indeed, for many complete intersections this is proven in* [12, Theorem 1.2] *as mentioned above). Combining these two conjectural properties, one obtains Conjecture 1.2 for complete intersections.*

2.2. **A result modulo algebraic equivalence.** In this subsection, we consider the cycle groups $B^*(X) := A^*(X)/A^*_{alg}(X)$ (where $A^i_{alg}(X)$ denotes the subgroup of algebraically trivial cycles).

Theorem 2.4. *Let $X \subset \mathbb{P}^{n+1}(\mathbb{C})$ be a general hypersurface of degree $n+2$. Then*

$$B^i(X) \cdot B^{n-i-1}(X) = \mathbb{Q}[h^{n-1}] \ \text{ for all } 0 < i < n-1\ .$$

Proof. This is similar to the proof of Theorem 2.1. The statement being trivially true for $n = 4$, we may assume $n \geq 5$. Given any $a \in B^i(X)$ and $b \in B^{n-i-1}(X)$, applying once more Voisin's equality (2), their intersection can be expressed as

$$a \cdot b = (\Delta_X^{sm})_*(a \times b) = \Big(\frac{1}{(n+2)!}\,\Gamma + \Delta_{12} + \Delta_{13} + \Delta_{23} + P(h_1,h_2,h_3)\Big)_*(a \times b) \ \text{ in } B^{n-1}(X)\ .$$

The correspondences Δ_{ij} act as zero for dimensional reasons, and it is readily seen that

$$P(h_1,h_2,h_3)_\ast A^\ast(X\times X)=\langle h\rangle\ .$$

It follows that

$$a\cdot b=\frac{1}{(n+2)!}\,\Gamma_\ast(a\times b)\quad\text{in }B^{n-1}(X)\ .$$

Either i or $n-i-1$ is strictly smaller than $n/2$; without loss of generality, let us assume $i<n/2$. This implies that we can write

$$a=a_0+a_1\ \ \text{in }B^i(X)\ ,$$

where $a_0\in\mathbb{Q}[h^i]$ and $a_1\in B^i_{hom}(X)$. Clearly $a_0\cdot b\in\mathbb{Q}[h^{n-1}]$, and so we now restrict attention to the product $a_1\cdot b$. As in the proof of Theorem 2.1 above, the action of Γ factors as

$$\begin{array}{ccc} B^i(X)\otimes B^{n-i-1}(X) & \xrightarrow{\Gamma_\ast} & B^{n-1}(X)\\ \downarrow{\scriptstyle (P^\ast,P^\ast)} & & \uparrow{\scriptstyle P_\ast}\\ B^{i-1}(F)\otimes B^{n-i-2}(F) & \xrightarrow{(\Delta_F^{sm})_\ast} & B^{n-3}(F)\end{array}$$

But since $a_1\in B^i_{hom}(X)$ we know that $P^\ast(a_1)\in B^{i-1}_{hom}(F)$ and so

$$P^\ast(a_1)\cdot P^\ast(b)\in B^{n-3}_{hom}(F)=0$$

(indeed, $\dim F=n-3$ and homological and algebraic equivalence coincide for zero-cycles). We conclude that

$$\Gamma_\ast(a_1\times b)=0\ \ \text{in }B^{n-1}(X)\ ,$$

which ends the proof. □

Remark 2.5. *Theorem 2.4 can be extended to complete intersections as in Theorem 2.2; we leave this as an exercice to the reader.*

2.3. **MCK.**

Definition 2.6 (Murre [13]). *Let X be a smooth projective variety of dimension n. We say that X has a* CK decomposition *if there exists a decomposition of the diagonal*

$$\Delta_X=\pi^0_X+\pi^1_X+\cdots+\pi^{2n}_X\quad\textit{in }A^n(X\times X)\ ,$$

such that the π^i_X are mutually orthogonal idempotents and $(\pi^i_X)_\ast H^\ast(X,\mathbb{Q})=H^i(X,\mathbb{Q})$.

Please note that "CK decomposition" is shorthand for "Chow–Künneth decomposition".

Remark 2.7. *The existence of a CK decomposition for any smooth projective variety is part of Murre's conjectures* [13], [8].

Definition 2.8 (Shen–Vial [18]). *Let X be a smooth projective variety of dimension n. Let $\Delta^{sm}_X\in A^{2n}(X\times X\times X)$ be the class of the small diagonal*

$$\Delta^{sm}_X:=\big\{(x,x,x)\mid x\in X\big\}\ \subset\ X\times X\times X\ .$$

An MCK decomposition *is a CK decomposition* $\{\pi^i_X\}$ *of* X *that is* multiplicative, *i.e. it satisfies*

$$\pi^k_X \circ \Delta^{sm}_X \circ (\pi^i_X \times \pi^j_X) = 0 \quad \text{in } A^{2n}(X \times X \times X) \quad \text{for all } i + j \neq k \ .$$

Please note that "MCK decomposition" is shorthand for "multiplicative Chow–Künneth decomposition".

Remark 2.9. *Only certain special varieties have an MCK decomposition. For instance, hyperelliptic curves have an MCK, while the general curve of genus* ≥ 3 *does* not *have an MCK. For more on MCK decompositions, cf.* [18], [5] *and the references given there.*

2.4. **Franchetta property.**

Definition 2.10. *Let* $\mathcal{Y} \to B$ *be a smooth projective morphism, where* $\mathcal{Y}, B$ *are smooth quasi-projective varieties. We say that* $\mathcal{Y} \to B$ *has the* Franchetta property in codimension j *if the following holds: for every* $\Gamma \in A^j(\mathcal{Y})$ *such that the restriction* $\Gamma|_{Y_b}$ *is homologically trivial for all* $b \in B$*, the restriction* $\Gamma|_{Y_b}$ *is zero in* $A^j(Y_b)$ *for all* $b \in B$*.*

We say that $\mathcal{Y} \to B$ *has the* Franchetta property *if* $\mathcal{Y} \to B$ *has the Franchetta property in codimension* j *for all* j*.*

This property is studied in [3], [4].

Definition 2.11. *Given a family* $\mathcal{Y} \to B$ *as above, with* $Y := Y_b$ *a fiber, we write*

$$GDA^j_B(Y) := \mathrm{Im}\Big(A^j(\mathcal{Y}) \to A^j(Y)\Big)$$

for the subgroup of generically defined cycles. *In a context where it is clear which family is being referred to, the index* B *will sometimes be suppressed from the notation.*

With this notation, the Franchetta property amounts to saying that $GDA^*_B(Y)$ injects into cohomology, under the cycle class map.

Proposition 2.12. *Let* $B \subset \mathbb{P}H^0(\mathbb{P}^{n+1}, \mathcal{O}_{\mathbb{P}^{n+1}}(d))$ *be the open subset parameterizing smooth hypersurfaces of degree* $d \geq 3$*, and let* $\mathcal{X} \to B$ *denote the universal family. The families* $\mathcal{X} \to B$ *and* $\mathcal{X} \times_B \mathcal{X} \to B$ *have the Franchetta property.*

Proof. This is the same argument as [5, Proposition 5.6], where this is proven for cubic hypersurfaces. ☐

2.5. **Franchetta and MCK.** As is well-known, complete intersections have a "standard" CK decomposition:

Definition 2.13. *Let* $X \subset \mathbb{P}^{n+r}(\mathbb{C})$ *be a smooth complete intersection of dimension* n *and degree* $d := \prod_{i=1}^r d_i$*. Then*

$$\pi^i_X := \begin{cases} \frac{1}{d}\, h^{n-i/2} \times h^{i/2} & \text{if } i \neq n \text{ and } i \text{ is even} \ , \\ 0 \quad \text{if } i \neq n \text{ and } i \text{ is odd} \ , \\ \Delta_X - \sum_{i \neq n} \pi^i_X \quad \text{if } i = n \end{cases}$$

defines a CK decomposition.

Remark 2.14. *In the set-up of Definition 2.13, in case n is even one can further decompose*

$$\pi_X^n = \pi_X^{n,alg} + \pi_X^{n,prim}\ ,$$

where $\pi_X^{n,alg} := \frac{1}{d}\, h^{n/2} \times h^{n/2}$.

Corollary 2.15. *Let $X \subset \mathbb{P}^{n+1}(\mathbb{C})$ be a smooth hypersurface of degree at least 3, and $\{\pi_X^i\}$ as in Definition 2.13. Assume that*

$$\pi_X^{n,prim} \circ \Delta_X^{sm} \circ (\pi_X^{n,prim} \times \pi_X^{n,prim}) = 0 \ \text{ in } A^{2n}(X \times X \times X)\ . \tag{4}$$

Then $\{\pi_X^i\}$ is an MCK decomposition

Proof. The point is that (by the very construction of the projectors π_X^i) one has isomorphisms of Chow motives

$$(X, \pi_X^{2i}, 0) = \mathbb{1}(-i)\ \ \forall\, i \neq n\ ,\ \ (X, \pi_X^{n,alg}, 0) = \mathbb{1}(-\frac{n}{2}) \ \ \text{in } \mathcal{M}_{\rm rat}\ . \tag{5}$$

Let $\pi_X^i, \pi_X^j, \pi_X^k$ be three projectors such that $i + j \neq k$, so that

$$\pi_X^k \circ \Delta_X^{sm} \circ (\pi_X^i \times \pi_X^j) \ \in\ A^{2n}_{hom}(X \times X \times X)\ .$$

In view of (4), we may assume that at least one of $\pi_X^i, \pi_X^j, \pi_X^k$ is different from $\pi_X^{n,prim}$. Then we have that

$$\begin{aligned} \pi_X^k \circ \Delta_X^{sm} \circ (\pi_X^i \times \pi_X^j) &= ({}^t\pi_X^i \times {}^t\pi_X^j \times \pi_X^k)_* \Delta_X^{sm} \\ &= (\pi_X^{2n-i} \times \pi_X^{2n-j} \times \pi_X^k)_* \Delta_X^{sm} \\ &\hookrightarrow \bigoplus A^*(X \times X)\ . \end{aligned}$$

Here the first equality is an application of Lieberman's lemma [11, Lemma 2.1.3], the second equality is by self-duality of the $\{\pi_X^*\}$, and the inclusion follows from property (5). The resulting cycle in $\bigoplus A^*(X \times X)$ is generically defined (since the π_X^* and Δ_X^{sm} are generically defined) and homologically trivial (since $i + j \neq k$). By the Franchetta property for $X \times X$ (Proposition 2.12), the resulting cycle in $\bigoplus A^*(X \times X)$ is rationally trivial, and so

$$\pi_X^k \circ \Delta_X^{sm} \circ (\pi_X^i \times \pi_X^j) = 0 \ \ \text{in } A^{2n}(X \times X \times X)\ ,$$

as desired. This proves the corollary. ☐

2.6. **A conditional result.** In her paper proving Theorem 1.3, Voisin has stated the following conjecture:

Conjecture 2.16 (Voisin [23])**.** *Let $X \subset \mathbb{P}^{n+1}(\mathbb{C})$ be a general hypersurface of degree $n+2$, and let $\Gamma \in A^{2n}(X \times X \times X)$ be the cycle defined in* (1). *Then*

$$\Gamma \in \mathrm{Im}\big(A^*(\mathbb{P}^{n+1} \times \mathbb{P}^{n+1} \times \mathbb{P}^{n+1}) \to A^*(X \times X \times X)\big)\ .$$

(This is [23, Conjecture 3.5].)

Voisin's conjecture has strong consequences:

Proposition 2.17. *Assume Conjecture 2.16. Let $X \subset \mathbb{P}^{n+1}(\mathbb{C})$ be a smooth hypersurface of degree $n+2$.*
(i) Conjecture 1.4 is true for X, i.e.
$$A^i(X)\cdot A^j(X)=\mathbb{Q}[h^{i+j}]\ \textit{for all}\ \ i,j>0\ .$$
(ii) X has an MCK decomposition.

Proof. Conjecture 2.16 together with the equality (2) imply that for a general hypersurface X there is equality

$$\Delta_X^{sm}=\Delta_{12}+\Delta_{13}+\Delta_{23}+P(h_1,h_2,h_3)\ \ \text{in } A^{2n}(X\times X\times X)\ , \tag{6}$$

where $P(h_1,h_2,h_3)$ is a new polynomial in the divisor classes h_i. In view of the spread lemma [24, Lemma 3.2], equality (6) then holds for *all* smooth Calabi–Yau hypersurfaces X. This immediately implies (i) (cf. the proof of Theorem 2.1 above).

To prove (ii), in view of Corollary 2.15, we just need to check the vanishing

$$\pi_X^{n,prim}\circ\Delta_X^{sm}\circ(\pi_X^{n,prim}\times\pi_X^{n,prim})=0\ . \tag{7}$$

In view of equality (6), we can write

$$\pi_X^{n,prim}\circ\Delta_X^{sm}\circ(\pi_X^{n,prim}\times\pi_X^{n,prim})=\pi_X^{n,prim}\circ\Big(\sum\Delta_{ij}+P(h_1,h_2,h_3)\Big)\circ(\pi_X^{n,prim}\times\pi_X^{n,prim})\ .$$

As for the Δ_{ij}, it follows from their construction that there is equality

$$\Delta_{12}=\pi_X^{2n}\circ\Delta_{12}\ ,\ \ \Delta_{13}=\Delta_{13}\circ(\Delta_X\times\pi_X^0)\ ,\ \ \Delta_{23}=\Delta_{23}\circ(\pi_X^0\times\Delta_X)\ .$$

The projectors π_X^j being orthogonal, we thus find that

$$\pi_X^{n,prim}\circ(\sum\Delta_{ij})\circ(\pi_X^{n,prim}\times\pi_X^{n,prim})=0\ .$$

Likewise, any monomial in the h_i satisfies

$$(h_1^ih_2^jh_3^{2n-i-j})=\pi_X^{i+j-n,alg}\circ(h_1^ih_2^jh_3^{2n-i-j})\circ(\pi_X^{2n-i,alg}\times\pi_X^{2n-j,alg})$$

(NB: the superscript “alg” is only operative for π_X^n, n even, i.e. we use the convention $\pi_X^{i,alg}=\pi_X^i$ for $i\neq n$). Again by orthogonality of the projectors, we thus find that

$$\pi_X^{n,prim}\circ\big(P(h_1,h_2,h_3)\big)\circ(\pi_X^{n,prim}\times\pi_X^{n,prim})=0\ .$$

This proves the required vanishing (7), and ends the proof of the proposition. □

2.7. **Quintic threefolds.** In dimension 3, we can prove an unconditional result:

Theorem 2.18. *Any smooth quintic threefold $X\subset\mathbb{P}^4(\mathbb{C})$ admits an MCK decomposition.*

Proof. The point is that for the general quintic threefold, the Fano variety of lines $F=F(X)$ is zero-dimensional (more precisely, F consists of 2875 reduced points [16], [17], [1, Corollary 6.35]).

To construct an MCK decomposition for the general quintic threefold, in view of Corollary 2.15, we just need to check the vanishing

$$\pi_X^3\circ\Delta_X^{sm}\circ(\pi_X^3\times\pi_X^3)=0\ \text{ in } A^6(X\times X\times X)\ . \tag{8}$$

In view of equality (2), we can write

$$\pi^3_X \circ \Delta^{sm}_X \circ (\pi^3_X \times \pi^3_X) = \pi^3_X \circ \big(\frac{1}{5!}\,\Gamma + \sum \Delta_{ij} + P(h_1,h_2,h_3)\big) \circ (\pi^3_X \times \pi^3_X)\ .$$

The summands involving Δ_{ij} and $P(h_1,h_2,h_3)$ vanish for general reasons (cf. the proof of Proposition 2.17(ii) above), and so it only remains to analyze the summand involving Γ. As before, we can write

$$\Gamma = P \circ (\Delta^{sm}_F) \circ ({}^tP \times {}^tP) \ \text{ in } A^6(X\times X\times X)\ ,$$

where $P\in A^2(F\times X)$ is the (class of the) universal line. In particular, we have

$$\pi^3_X \circ \Gamma \circ (\pi^3_X \times \pi^3_X) = \pi^3_X \circ P \circ (\Delta^{sm}_F) \circ ({}^tP \times {}^tP) \circ (\pi^3_X \times \pi^3_X) \ \text{ in } A^6(X\times X\times X)\ .$$

But now, Lemma 2.19 below (combined with the orthogonality $\pi^3_X\circ\pi^4_X=0$) implies the vanishing

$$\pi^3_X \circ \Gamma \circ (\pi^3_X \times \pi^3_X) = 0 \ \text{ in } A^6(X\times X\times X)\ .$$

This shows (8) and gives an MCK decomposition for the general quintic threefold.

To extend to *all* smooth quintic threefolds, one observes that all terms in (8) are generically defined, and so the spread lemma [24, Lemma 3.2] implies the vanishing (8) for all smooth quintic threefolds.

Lemma 2.19. *Let X be a general quintic threefold, and $P\in A^2(F\times X)$ the universal line. Then*

$$P = \pi^4_X \circ P \quad \text{in } A^2(F\times X)\ .$$

To prove the lemma, we observe that the equality is true modulo homological equivalence (indeed, the correspondence P sends $H^\ast(F,\mathbb{Q})=H^0(F,\mathbb{Q})$ to $H^4(X,\mathbb{Q})$). All terms being generically defined (and remembering that $A^2(F\times X)=A^2(X)^{\oplus 2875}$), the lemma then follows from the Franchetta property for X (Proposition 2.12). ☐

Remark 2.20. *Unfortunately, the argument proving Theorem 2.18 breaks down for Calabi–Yau hypersurfaces of dimension greater than 3. The reason is that in Lemma 2.19, the argument hinges on the Franchetta property for $F\times X$. This is easy when the dimension of F is zero, but becomes problematic as soon as the dimension of F is greater than 0.*

Corollary 2.21. *Let X be a smooth quintic threefold. For any $m\in\mathbb{N}$ let*

$$R^\ast(X^m) := \big\langle (p_i)^\ast(h), (p_{jk})^\ast(\Delta_X)\big\rangle \ \subset\ A^\ast(X^m)$$

denote the $\mathbb{Q}$-algebra generated by (pullbacks of) the polarization h and the diagonal Δ_X. Then $R^\ast(X^m)$ injects into cohomology under the cycle class map for all $m\le 205$. Moreover, $R^\ast(X^m)$ injects into cohomology for all m if and only if X is Kimura finite-dimensional, in the sense of [9].

Proof. This is an application of [4, Proposition 2.11], using the fact that X has an MCK decomposition and that $\dim H^3(X,\mathbb{Q})=204$. ☐

Remark 2.22. *Conjecturally, all smooth projective varieties are Kimura finite-dimensional* [9]. *An example which is known to be Kimura finite-dimensional is the Fermat quintic threefold*

$$x_0^5 + \cdots + x_4^5 = 0 \ .$$

(This can be proven using Shioda's inductive structure of Fermat hypersurfaces, cf. [19].*)*

Remark 2.23. *As explained in* [4, Section 2.3], *the statement of Corollary 2.21 is inspired by results on the so-called "tautological ring" of hyperelliptic curves* [20], [21] *and of K3 surfaces* [22], [25].

3. General type hypersurfaces

In this section, we consider hypersurfaces of degree d strictly larger than $n+2$; these hypersurfaces are of general type. Hartshorne's conjecture (Conjecture 1.1) has the following consequence:

Conjecture 3.1. *Let $X \subset \mathbb{P}^{n+1}(\mathbb{C})$ be a smooth hypersurface. Then*

$$A^{i_1}(X) \cdot A^{i_2}(X) \cdot A^{i_3}(X) = \mathbb{Q}[h^{i_1+i_2+i_3}] \ \textit{for all} \ \ i_1, i_2, i_3 > 0 \ .$$

3.1. **Lie Fu's theorem.** Inspired by Voisin's result for Calabi–Yau hypersurfaces (Theorem 1.3), Lie Fu has proven a nice result about zero-cycles that are intersections on general type hypersurfaces, providing partial confirmation to Conjecture 3.1:

Theorem 3.2 (Fu [2])**.** *Let $X \subset \mathbb{P}^{n+1}(\mathbb{C})$ be a general hypersurface of degree $d \geq n+2$. Assume $i_1, \ldots, i_{d-n}$ are strictly positive integers such that $\sum_j i_j = n$. Then*

$$A^{i_1}(X) \cdot A^{i_2}(X) \ldots \cdot A^{i_{d-n}}(X) = \mathbb{Q}[h^n] \ .$$

3.2. **Main result.** The main result in this section is that in the setting of Theorem 3.2, one can obtain a stronger result if one of the codimensions i_j is equal to 2:

Theorem 3.3. *Let $X \subset \mathbb{P}^{n+1}(\mathbb{C})$ be a general hypersurface of degree $d \geq n+2$. Then*

$$A^2(X) \cdot A^{i_1}(X) \cdot A^{i_2}(X) \cdot \ldots \cdot A^{i_{d-n-1}}(X) = \mathbb{Q}[h^{2+\sum_j i_j}] \ \ \textit{for all} \ \ i_1, \ldots, i_{d-n-1} > 0 \ .$$

Proof. Let $k := d+1-n$ (and so $k \geq 3$), and let

$$\delta_X := \{(x, x, \ldots, x) | x \in X\} \ \ \subset \ X^k$$

denote the smallest diagonal. By the generality assumption, the Fano variety $F := F(X)$ of lines in X is smooth of dimension $n-k$. In the course of proving Theorem 3.2, Fu has obtained [2, Theorem 2.12] the following dichotomy for δ_X: either

$$\delta_X = \frac{(-1)^{k-1}}{d!} \Gamma + \sum_{i=1}^{k} D_i + \sum_j \lambda_j \sum_{|I|=j} D_I + P(h_1, \ldots, h_k) \ \text{ in } A^*(X^k) \ , \tag{9}$$

or there exists $\ell < k$ such that

$$\delta_X = \sum_{i=1}^{k} D_i + \sum_j \lambda_j \sum_{|I|=j} D_I + P(h_1, \ldots, h_k) \ \text{ in } A^*(X^\ell) \ . \tag{10}$$

Here, Γ is defined as

$$\Gamma := \bigcup_{t\in F(X)} \mathbb{P}^1_t \times \cdots \times \mathbb{P}^1_t \ \subset\ X^k\ ,$$

the cycle D_i is defined as $(p_i)^*(o_X)\cdot \Delta_{i^c}$, where Δ_{i^c} is the diagonal of the complementary set $\{1,\ldots,k\}\setminus i$, and similarly, for any subset $I\subset\{1,\ldots,k\}$, the cycle D_I is defined as $\prod_{i\in I}(p_i)^*(o_X)\cdot\Delta_{I^c}$.

Once again, we consider the equality (9) (resp. (10)) as an equality of correspondences from X^{k-1} (resp. $X^{\ell-1}$) to X. Given $d-n$ cycles of positive codimension $a_0,\ldots,a_{d-n-1}\in A^*(X)$, it is readily seen that

$$(D_i)_*(a_0\times\cdots\times a_{d-n-1}) = (D_I)_*(a_0\times\cdots\times a_{d-n-1}) = 0\ \text{ in } A^*(X)\ ,$$

while

$$\big(P(h_1,\ldots,h_k)\big)_*(a_0\times\cdots\times a_{d-n-1})\ \in\ \mathbb{Q}[h^*]\ .$$

Hence, to prove Theorem 3.3 one only needs to worry about the action of Γ. Since (by Lieberman's lemma) Γ can be written as

$$\Gamma = P\circ(\delta_F)\circ({}^tP\times\cdots\times{}^tP)\ \text{ in } A^*(X^k)\ ,$$

the action of Γ factors as

$$\begin{array}{ccc}
A^2(X)\otimes A^{i_1}(X)\otimes\cdots\otimes A^{i_{k-2}}(X) & \xrightarrow{\Gamma_*} & A^{2+\sum_j i_j}(X)\\
\downarrow {\scriptstyle (P^*,\ldots,P^*)} & & \uparrow {\scriptstyle P_*}\\
A^1(F)\otimes A^{i_1-1}(F)\otimes\cdots\otimes A^{i_{k-2}-1}(F) & \xrightarrow{(\delta_F)_*} & A^{\sum_j i_j -k+3}(F)
\end{array}$$

The result now follows from the fact that $A^1_{AJ}(F)=0$ (cf. the proof of Theorem 2.1). $\square$

Remark 3.4. *Similarly to Theorem 2.4, one can prove a result for one-cycles modulo algebraic equivalence for general type hypersurfaces; we leave this as an exercice for the diligent reader.*

Acknowledgements. *Thanks to the highly efficient coffee machine at the Schiltigheim Research Center in Advanced Mathematics.*

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