

ON GRÜN'S LEMMA FOR PERFECT SKEW BRACES

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ABSTRACT. By previous work of Cedó, Smoktunowicz, and Vendramin, one already knows that the analog of Grün's lemma fails to hold for perfect skew left braces when the socle is used as an analog of the center of a group. In this paper, we use the annihilator instead of the socle. We shall show that the analog of Grün's lemma holds for perfect two-sided skew braces but not in general.

CONTENTS

1. Introduction	1
2. Derived ideal and second annihilator	5
2.1. $\text{Ann}_2(A) * (A * A) = 1$	5
2.2. $[\text{Ann}_2(A), A * A] = 1$	6
2.3. $(A * A) * \text{Ann}_2(A) = 1$	7
3. Constructing counterexamples	9
Acknowledgments	17
References	17

1. INTRODUCTION

A *skew brace* is any set $A = (A, \cdot, \circ)$ equipped with two group operations $\cdot$ and $\circ$ such that left brace relation

$$a \circ (b \cdot c) = (a \circ b) \cdot a^{-1} \cdot (a \circ c)$$

holds for all $a, b, c \in A$, where a^{-1} denotes the inverse of a with respect to $\cdot$. It is easy to see that $(A, \cdot)$ and $(A, \circ)$ must share the same identity element, which we denote by 1. Due to their relations with the set-theoretic solutions to the Yang–Baxter equation, understanding the structure of skew braces is a problem of interest; we refer the reader to [6, 7, 11, 13] for more details.

Given a group $(A, \cdot)$, we can construct a skew brace $(A, \cdot, \circ)$ by defining $\circ$ to be the same operation $\cdot$, or its opposite operation $\cdot^{\text{op}}$, that is $a \cdot^{\text{op}} b = b \cdot a$. The skew braces of the forms $(A, \cdot, \cdot)$ and $(A, \cdot, \cdot^{\text{op}})$ are said to be *trivial* and *almost trivial*, respectively, because they are essentially just groups.

Skew braces may therefore be regarded as an extension of groups. Indeed, there are many similarities between skew braces and groups (see [8, 9, 12, 15] for some examples). The purpose of this paper is to continue research in this direction and explore analogs of Grün’s lemma [5, Satz 4] for skew braces.

A group G is said to be *perfect* if it equals its derived subgroup $[G, G]$. For any group G , let us denote its center by $Z(G)$.

Theorem 1.1 (Grün’s lemma). *For any perfect group G , we have*

$$Z(G/Z(G)) = 1.$$

To consider the analog of Grün’s lemma in the context of skew braces, we need to first define “quotient”, “perfect”, and “center” for skew braces.

In what follows, let $A = (A, \cdot, \circ)$ be a skew brace.

Definition 1.2. A subset I of A is said to be an *ideal* if

- (1) I is a normal subgroup of $(A, \cdot)$;
- (2) I is a normal subgroup of $(A, \circ)$;
- (3) $a \cdot I = a \circ I$ for all $a \in A$.

In this case, we can naturally endow the coset space

$$A/I = \{a \cdot I : a \in A\} = \{a \circ I : a \in A\}$$

with a quotient skew brace structure from that of A .

To measure the difference between the group operations $\cdot$ and $\circ$, define

$$a * b = a^{-1} \cdot (a \circ b) \cdot b^{-1}.$$

For example, we have $a * b = 1$ when A is trivial, and $a * b = a^{-1} \cdot b \cdot a \cdot b^{-1}$ when A is almost trivial. We then see that $*$ may be viewed as an analog of the commutator $[,]$, and the notion of “trivial” for skew braces is a natural analog of “abelian” for groups. For any subsets X, Y of A , we define $X * Y$ to be the subgroup of $(A, \cdot)$ generated by the elements $x * y$ for $x \in X, y \in Y$.

Definition 1.3. The *derived ideal* of A is defined to be the subset $A * A$. It is known (see [3, Proposition 2.1]) that $A * A$ is indeed an ideal of A , in fact the smallest ideal of A for which the quotient skew brace is trivial. We shall say that A is *perfect* if it equals its derived ideal $A * A$.

As for the analog of “center”, there are two natural candidates.

Definition 1.4. The *socle* of A is defined as

$$\text{Soc}(A) = \{a \in A \mid \forall x \in A : a * x = 1\} \cap Z(A, \cdot),$$

which is an ideal of A by [6, Lemma 2.5]. The *annihilator* of A is defined as

$$\begin{aligned} \text{Ann}(A) &= \{a \in A \mid \forall x \in A : a * x = 1\} \cap Z(A, \cdot) \cap Z(A, \circ) \\ &= \{a \in A \mid \forall x \in A : a * x = 1 = x * a\} \cap Z(A, \cdot), \end{aligned}$$

which is easily checked to be an ideal of A .

As analogs of Grün’s lemma, it is natural to ask whether

$$\text{Soc}(A/\text{Soc}(A)) = 1 \quad \text{and} \quad \text{Ann}(A/\text{Ann}(A)) = 1$$

hold for all perfect skew braces A . It is already known by [3, Section 3] that there exist perfect skew braces A for which the former equality fails. In this paper, we wish to consider the latter equality instead. Note that

$$\text{Ann}(A/\text{Ann}(A)) = \text{Ann}_2(A)/\text{Ann}(A)$$

for a unique ideal $\text{Ann}_2(A)$ of A by the isomorphism theorems in groups. We shall refer to $\text{Ann}_2(A)$ as the *second annihilator* of A . By Definition 1.4, we are then reduced to investigating whether

$$(1.1) \quad \text{Ann}_2(A) * (A * A) = 1,$$

$$(1.2) \quad [\text{Ann}_2(A), A * A] = 1,$$

$$(1.3) \quad (A * A) * \text{Ann}_2(A) = 1,$$

where $[\cdot, \cdot]$ denotes the commutator in the group $(A, \cdot)$. In the case that A is perfect, we have $A = A * A$, so then (1.1), (1.2), and (1.3) would imply

$$\text{Ann}_2(A) = \text{Ann}(A) \quad \text{that is} \quad \text{Ann}(A/\text{Ann}(A)) = 1.$$

However, as we shall show, while the equalities (1.1) and (1.2) always hold, the equality (1.3) fails in some cases.

Definition 1.5. We shall say that A is *two-sided* if the right brace relation

$$(b \cdot c) \circ a = (b \circ a) \cdot a^{-1} \cdot (c \circ a)$$

also holds for all $a, b, c \in A$.

Our main results are as follows:

Theorem 1.6. *For any skew brace A , we have*

$$\text{Ann}_2(A) * (A * A) = 1 \quad \text{and} \quad [\text{Ann}_2(A), A * A] = 1.$$

Proof. See Propositions 2.2 and 2.3. □

Theorem 1.7. *For any two-sided skew brace A , we have*

$$(A * A) * \text{Ann}_2(A) = 1.$$

Proof. See Corollary 2.6. □

Corollary 1.8. *For any two-sided perfect skew brace A , we have*

$$\text{Ann}(A/\text{Ann}(A)) = 1.$$

Proof. This follows immediately from Theorems 1.6 and 1.7. □

Note that Theorem 1.1 may be recovered from Corollary 1.8 by taking A to be an almost trivial skew brace. Thus, Corollary 1.8 is a genuine generalization of Grün's lemma for skew braces.

In Section 3, we shall describe ways to construct skew braces A for which

$$(A * A) * \text{Ann}_2(A) \neq 1.$$

We shall also see that A may be chosen to be perfect, so as a consequence, we obtain that:

Corollary 1.9. *There exist perfect skew braces A for which*

$$\text{Ann}(A/\text{Ann}(A)) \neq 1.$$

Proof. This follows from Proposition 3.5 and Examples 3.6 and 3.7. □

2. DERIVED IDEAL AND SECOND ANNIHILATOR

In this section, let $A = (A, \cdot, \circ)$ be a skew brace. For each $a \in A$, define

$$\lambda_a : A \longrightarrow A; \quad \lambda_a(x) = a^{-1} \cdot (a \circ x),$$

which is easily checked to be an element of $\text{Aut}(A, \cdot)$. The map

$$\lambda : (A, \circ) \longrightarrow \text{Aut}(A, \cdot); \quad a \mapsto \lambda_a$$

is a morphism of groups (see [6, Proposition 1.9]). For any $a, b \in A$, in terms of this map λ , we have the well-known identities

$$a \circ b = a \cdot \lambda_a(b), \quad a \cdot b = a \circ \lambda_{\bar{a}}(b),$$

$$\bar{a} = \lambda_{\bar{a}}(a^{-1}), \quad a * b = \lambda_a(b) \cdot b^{-1},$$

where $\bar{a}$ denotes the inverse of a with respect to $\circ$. Let us also write

$$[a, b] = a \cdot b \cdot a^{-1} \cdot b^{-1}, \quad [a, b]_{\circ} = a \circ b \circ \bar{a} \circ \bar{b}$$

for the commutators in the groups $(A, \cdot)$ and $(A, \circ)$, respectively. The following identities shall also be useful.

Lemma 2.1. *For any $a, x, y \in A$, we have*

$$(2.1) \quad a * (x \cdot y) = (a * x) \cdot x \cdot (a * y) \cdot x^{-1},$$

$$(2.2) \quad (x \circ y) * a = (x * (y * a)) \cdot (y * a) \cdot (x * a),$$

$$(2.3) \quad \lambda_a(x * y) = (a \circ x \circ \bar{a}) * \lambda_a(y).$$

Proof. Straightforward or see [15, Lemmas 2.1 and 2.3]. □

In the following, we shall investigate the relationship between the derived ideal $A * A$ and the second annihilator $\text{Ann}_2(A)$. In particular, we shall show that (1.1) and (1.2) always hold, and then give a characterization of (1.3).

2.1. $\text{Ann}_2(A) * (A * A) = 1$. For each $a \in A$, consider the map

$$\varphi_a : (A, \cdot) \longrightarrow (A * A, \cdot); \quad \varphi_a(x) = a * x.$$

In the case that $a \in \text{Ann}_2(A)$, we have $\text{Im}(\varphi_a) \subseteq \text{Ann}(A) \subseteq Z(A, \cdot)$, and we deduce from (2.1) that φ_a is a group homomorphism.

Proposition 2.2. *We have $\text{Ann}_2(A) * (A * A) = 1$.*

Proof. It suffices to check that $a * z = 1$ for all $a \in \text{Ann}_2(A)$ and $z \in A * A$. In other words, we want to show that $A * A \subseteq \ker(\varphi_a)$ for each $a \in \text{Ann}_2(A)$. Since φ_a is a homomorphism on $(A, \cdot)$, we only need to show that

$$x * y \in \ker(\varphi_a) \text{ for all } x, y \in A$$

because these elements $x * y$ generate $A * A$ in $(A, \cdot)$.

Let $a \in \text{Ann}_2(A)$ and $x, y \in A$. Note that

$$\varphi_a(x * y) = \varphi_a(\lambda_x(y)) \cdot \varphi_a(y)^{-1} = (a * \lambda_x(y)) \cdot (a * y)^{-1}.$$

Since $[\bar{a}, x]_\circ \in \text{Ann}(A)$, we have $[\bar{a}, x]_\circ * \lambda_x(y) = 1$ and (2.2) yields that

$$a * \lambda_x(y) = (a \circ [\bar{a}, x]_\circ) * \lambda_x(y) = (x \circ a \circ \bar{x}) * \lambda_x(y),$$

which in turn is equal to $\lambda_x(a * y)$ by (2.3). Hence, we obtain

$$\varphi_a(x * y) = \lambda_x(a * y) \cdot (a * y)^{-1} = x * (a * y),$$

which is equal to 1 because $a * y \in \text{Ann}(A)$. This completes the proof. $\square$

2.2. $[\text{Ann}_2(A), A * A] = 1$. For each $a \in A$, consider the map

$$\pi_a : (A, \cdot) \longrightarrow ([A, A], \cdot); \quad \pi_a(x) = [a, x].$$

In the case that $a \in \text{Ann}_2(A)$, we have $\text{Im}(\pi_a) \subseteq \text{Ann}(A) \subseteq Z(A, \cdot)$, and we deduce from the standard identity

$$[a, x \cdot y] = [a, x] \cdot x \cdot [a, y] \cdot x^{-1}$$

(note that (2.1) is an analog of this) that π_a is a group homomorphism.

Proposition 2.3. *We have $[\text{Ann}_2(A), A * A] = 1$.*

Proof. It is enough to show that $[a, z] = 1$ for all $a \in \text{Ann}_2(A)$ and $z \in A * A$. In other words, we want to show that $A * A \subseteq \ker(\pi_a)$ for each $a \in \text{Ann}_2(A)$. Since π_a is a homomorphism on $(A, \cdot)$, we only need to check that

$$x * y \in \ker(\pi_a) \text{ for all } x, y \in A$$

because these elements $x * y$ generate $A * A$ in $(A, \cdot)$.

Let $a \in \text{Ann}_2(A)$ and $x, y \in A$. Note that

$$\pi_a(x * y) = \pi_a(\lambda_x(y)) \cdot \pi_a(y)^{-1} = [a, \lambda_x(y)] \cdot [a, y]^{-1}.$$

Since $x * a \in \text{Ann}(A) \subseteq Z(A, \cdot)$, we have

$$[a, \lambda_x(y)] = [(x * a)a, \lambda_x(y)] = [\lambda_x(a), \lambda_x(y)],$$

which in turn is equal to $\lambda_x([a, y])$ because $\lambda_x \in \text{Aut}(A, \cdot)$. It follows that

$$\pi_a(x * y) = \lambda_x([a, y]) \cdot [a, y]^{-1} = x * [a, y],$$

which is equal to 1 because $[a, y] \in \text{Ann}(A)$. This completes the proof. $\square$

2.3. $(A * A) * \text{Ann}_2(A) = 1$. For each $a \in A$, consider the map

$$\psi_a : (A, \circ) \longrightarrow (A * A, \cdot); \quad \psi_a(x) = x * a.$$

Note that unlike the ϕ_a and π_a considered in the previous subsections, here we use the group operation $\circ$ in domain. In the case that $a \in \text{Ann}_2(A)$, we have $\text{Im}(\psi_a) \subseteq \text{Ann}(A) \subseteq Z(A, \cdot)$, and we see from (2.2) that ψ_a is a group homomorphism since $A * \text{Ann}(A) = 1$.

Now, it is obvious that

$$\begin{aligned} (2.4) \quad (A * A) * \text{Ann}_2(A) = 1 &\iff \forall a \in \text{Ann}_2(A), z \in A * A : z * a = 1 \\ &\iff \forall a \in \text{Ann}_2(A) : A * A \subseteq \ker(\psi_a). \end{aligned}$$

Let us consider when this last inclusion is satisfied.

Proposition 2.4. *For each $a \in \text{Ann}_2(A)$, we have*

$$\psi_a \text{ is a homomorphism on } (A, \cdot) \iff A * A \subseteq \ker(\psi_a).$$

Proof. First, suppose that ψ_a is a homomorphism on $(A, \cdot)$. Then it suffices to check that $x * y \in \ker(\psi_a)$ for all $x, y \in A$ since these elements $x * y$ generate $A * A$ in $(A, \cdot)$. But ψ_a is also a homomorphism on $(A, \circ)$, so clearly

$$\begin{aligned} \psi_a(x * y) &= \psi_a(x^{-1} \cdot (x \circ y) \cdot y^{-1}) \\ &= \psi_a(x)^{-1} \cdot \psi_a(x) \cdot \psi_a(y) \cdot \psi_a(y)^{-1} \\ &= 1. \end{aligned}$$

Conversely, suppose that $A * A \subseteq \ker(\psi_a)$. For any $x, y \in A$, first we write

$$\psi_a(x \cdot y) = \psi_a(x \circ \lambda_{\bar{x}}(y)) = \psi_a(x) \cdot \psi_a(\lambda_{\bar{x}}(y)).$$

For the second term, we compute that

$$\begin{aligned} \psi_a(\lambda_{\bar{x}}(y)) &= \psi_a(y) \cdot \psi_a(y)^{-1} \cdot \psi_a(\lambda_{\bar{x}}(y)) \\ &= \psi_a(y) \cdot \psi_a(\bar{y} \circ \lambda_{\bar{x}}(y)) \\ &= \psi_a(y) \cdot \psi_a(\bar{y} \cdot \lambda_{\bar{y} \circ \bar{x} \circ y}(\bar{y}^{-1})) \\ &= \psi_a(y) \cdot \psi_a(\bar{y} \cdot ((\bar{y} \circ \bar{x} \circ y) * \bar{y}^{-1}) \cdot \bar{y}^{-1}) \\ &= \psi_a(y), \end{aligned}$$

where last equality holds because $A * A$ is normal in $(A, \cdot)$ and so

$$\bar{y} \cdot ((\bar{y} \circ \bar{x} \circ y) * \bar{y}^{-1}) \cdot \bar{y}^{-1} \in A * A \subseteq \ker(\psi_a)$$

(see [15, Corollary 2.2] for example). Thus, we have $\psi_a(x \cdot y) = \psi_a(x) \cdot \psi_a(y)$, whence ψ_a is a homomorphism on $(A, \cdot)$. $\square$

For each $a \in A$, consider the inner automorphism

$$\iota_a : (A, \circ) \longrightarrow (A, \circ); \quad \iota_a(x) = a \circ x \circ \bar{a}.$$

It is basically known in the literature (see [10, Lemma 4.1] or [14, Proposition 2.3] for example) that for any $x, y \in A$, we have

$$\iota_a(x \cdot y) = \iota_a(x) \cdot \iota_a(y) \iff (x \cdot y) \circ \bar{a} = (x \circ \bar{a}) \cdot \bar{a}^{-1} \cdot (y \circ \bar{a}).$$

In other words, we have $\iota_a \in \text{Aut}(A, \cdot)$ if and only if the right brace relation holds when the element on the right of $\circ$ is fixed to be $\bar{a}$. Let us now show that the two equivalent conditions in Proposition 2.4 may be characterized in terms of these inner automorphisms.

Proposition 2.5. *For each $a \in \text{Ann}_2(A)$, we have the relation*

$$\iota_a(x) = a \lambda_a(x) a^{-1} \cdot \psi_{\bar{a}}(x)$$

for all $x \in A$. Moreover, we have the equivalence

$$\psi_{\bar{a}} \text{ is a homomorphism on } (A, \cdot) \iff \iota_a \in \text{Aut}(A, \cdot).$$

Proof. For the equality, we first compute that

$$\begin{aligned}
\iota_a(x) &= a \circ x \circ \bar{a} \\
&= a \cdot \lambda_a(x \cdot (x * \bar{a}) \cdot \bar{a}) \\
&= a \cdot \lambda_a(x) \cdot \lambda_a(x * \bar{a}) \cdot \lambda_a(\bar{a}) \\
&= a \cdot \lambda_a(x) \cdot (a * (x * \bar{a})) \cdot (x * \bar{a}) \cdot a^{-1}.
\end{aligned}$$

Since $x * \bar{a} \in \text{Ann}(A) \subseteq Z(A, \cdot)$, we can move it to the right of a^{-1} , and also the third term vanishes because $A * \text{Ann}(A) = 1$. We thus obtain

$$\iota_a(x) = a\lambda_a(x)a^{-1} \cdot \psi_{\bar{a}}(x),$$

as claimed. For any $x, y \in A$, we then get that

$$\begin{aligned}
\iota_a(x \cdot y) &= a\lambda_a(x \cdot y)a^{-1} \cdot \psi_{\bar{a}}(x \cdot y) \\
&= a\lambda_a(x)\lambda_a(y)a^{-1} \cdot \psi_{\bar{a}}(x \cdot y), \\
\iota_a(x) \cdot \iota_a(y) &= a\lambda_a(x)a^{-1} \cdot \psi_{\bar{a}}(x) \cdot a\lambda_a(y)a^{-1} \cdot \psi_{\bar{a}}(y) \\
&= a\lambda_a(x)\lambda_a(y)a^{-1} \cdot \psi_{\bar{a}}(x) \cdot \psi_{\bar{a}}(y),
\end{aligned}$$

where we have again used the fact that $\psi_{\bar{a}}(x) = x * \bar{a} \in Z(A, \cdot)$. Thus

$$\iota_a(x \cdot y) = \iota_a(x) \cdot \iota_a(y) \iff \psi_{\bar{a}}(x \cdot y) = \psi_{\bar{a}}(x) \cdot \psi_{\bar{a}}(y),$$

and this proves the equivalence. $\square$

Corollary 2.6. *For any skew brace A , we have*

$$(A * A) * \text{Ann}_2(A) = 1 \iff \forall a \in \text{Ann}_2(A) : \iota_a \in \text{Aut}(A, \cdot).$$

*In particular, we have $(A * A) * \text{Ann}_2(A) = 1$ whenever A is two-sided.*

Proof. This follows from (2.4) and Propositions 2.4 and 2.5. $\square$

3. CONSTRUCTING COUNTEREXAMPLES

In this section, we use semidirect products to construct examples of skew braces A for which $(A * A) * \text{Ann}_2(A) \neq 1$. Such skew braces are necessarily non-two-sided by Corollary 2.6. We shall also show that A may be chosen to be perfect, thus yielding counterexamples to the analog of Grün's lemma.

Proposition 3.1. *Let $B = (B, \cdot, \circ)$ and $C = (C, \cdot, \circ)$ be skew braces. Let*

$$\phi : (C, \circ) \longrightarrow \text{Aut}(B, \cdot) \cap \text{Aut}(B, \circ); \quad c \mapsto \phi_c$$

be any group homomorphism and define

$$\begin{aligned} (b_1, c_1) \cdot (b_2, c_2) &= (b_1 \cdot b_2, c_1 \cdot c_2) \\ (b_1, c_1) \circ (b_2, c_2) &= (b_1 \circ \phi_{c_1}(b_2), c_1 \circ c_2) \end{aligned}$$

on the set $B \times C$. Then $(B \times C, \cdot, \circ)$ is a skew brace. Moreover, we have

$$(3.1) \quad (b_1, c_1) * (b_2, c_2) = (b_1^{-1} \cdot (b_1 \circ \phi_{c_1}(b_2)) \cdot b_2^{-1}, c_1 * c_2)$$

for any $b_1, b_2 \in B$ and $c_1, c_2 \in C$.

The skew brace $(B \times C, \cdot, \circ)$ constructed above is denoted by $B \rtimes_{\phi} C$ and is called the *semidirect product* of B and C with respect to ϕ .

Proof. This construction is well-known (mentioned in [13, Corollary 2.37] for example), and the proof is straightforward. As for (3.1), we compute that

$$\begin{aligned} (b_1, c_1) * (b_2, c_2) &= (b_1, c_1)^{-1} \cdot ((b_1, c_1) \circ (b_2, c_2)) \cdot (b_2, c_2)^{-1} \\ &= (b_1^{-1}, c_1^{-1}) \cdot (b_1 \circ \phi_{c_1}(b_2), c_1 \circ c_2) \cdot (b_2^{-1}, c_2^{-1}) \\ &= (b_1^{-1} \cdot (b_1 \circ \phi_{c_1}(b_2)) \cdot b_2^{-1}, c_1 * c_2), \end{aligned}$$

which is as claimed. □

We now specialize to the case when $B = (B, +, +)$ is a trivial brace, which means that the operation $+$ is taken to be commutative. Here we denote the operation of B by $+$ because we shall take B to be a vector space to produce explicit examples. Note that in this case, the identity (3.1) becomes

$$(3.2) \quad (b_1, c_1) * (b_2, c_2) = ((\phi_{c_1} - \text{id}_B)(b_2), c_1 * c_2)$$

for any $b_1, b_2 \in B$ and $c_1, c_2 \in C$. We shall also take C to be perfect.

Proposition 3.2. *Let $B = (B, +, +)$ be a trivial brace and let $C = (C, \cdot, \circ)$ be a perfect skew brace. Consider the semidirect product $A = B \rtimes_{\phi} C$, where*

$$\phi : (C, \circ) \longrightarrow \text{Aut}(B, +); \quad c \mapsto \phi_c$$

is any group homomorphism. Then we have the equality

$$(3.3) \quad A * A = \left\langle \bigcup_{\xi \in \text{Im}(\phi)} \text{Im}(\xi - \text{id}_B) \right\rangle \times C,$$

and also the inclusions

$$\text{Fix}_\phi(B) \times \{1\} \subseteq \text{Ann}(A) \subseteq \text{Fix}_\phi(B) \times (\ker(\phi) \cap \text{Ann}(C)),$$

where $\text{Fix}_\phi(B)$ is the fixed-point subgroup of $\text{Im}(\phi)$, namely

$$\text{Fix}_\phi(B) = \bigcap_{\xi \in \text{Im}(\phi)} \ker(\xi - \text{id}_B).$$

In particular, we have

$$(3.4) \quad \text{Ann}(A) = \text{Fix}_\phi(B) \times \{1\} \text{ when } \text{Ann}(C) = 1.$$

Proof. From (3.2), it is clear that the left-to-right inclusion of (3.3) holds. By taking $c_2 = 1$ and $b_2 = 0$, respectively, we also see that

$$((\phi_{c_1} - \text{id}_B)(b_2), 1), (0, c_1 * c_2) \in A * A.$$

Since $b_2 \in B$, $c_1, c_2 \in C$ are arbitrary and $C = C * C$, it follows that

$$\left\langle \bigcup_{\xi \in \text{Im}(\phi)} \text{Im}(\xi - \text{id}_B) \right\rangle \times \{1\}, \{0\} \times C \subseteq A * A.$$

This shows that the right-to-left inclusion of (3.3) also holds.

Next, consider an element (b, c) from the center $Z(A, \cdot) = (B, +) \times Z(C, \cdot)$ of $(A, \cdot)$. By definition, we have $(b, c) \in \text{Ann}(A)$ if and only if

$$(b, c) * (x, y) = ((\phi_c - \text{id}_B)(x), c * y) = (0, 1)$$

$$(x, y) * (b, c) = ((\phi_y - \text{id}_B)(b), y * c) = (0, 1)$$

are satisfied for all $x \in B, y \in C$. They obviously hold when $b \in \text{Fix}_\phi(B)$ and $c = 1$. Conversely, they imply the equalities

$$\forall y \in Y : c * y = y * c = 1, \quad \text{which yields } c \in \text{Ann}(C),$$

$$\forall x \in X : (\phi_c - \text{id}_B)(x) = 0, \quad \text{which yields } c \in \ker(\phi),$$

$$\forall y \in Y : (\phi_y - \text{id}_B)(b) = 0, \quad \text{which yields } b \in \text{Fix}_\phi(B).$$

The inclusions regarding $\text{Ann}(A)$ then follow as well. $\square$

To simplify things further, we shall specialize to the case when

$$B = (\mathbb{F}_p^n, +, +) \quad \text{and} \quad \text{Ann}(C) = 1,$$

where p is a prime and $n \geq 2$ is a positive integer. Below, we shall give three ways to construct the homomorphism ϕ so that

$$(3.5) \quad (A * A) * \text{Ann}_2(A) \neq 1 \text{ for } A = B \rtimes_{\phi} C.$$

In view of (3.3) and (3.4), the only thing that matters is $\text{Im}(\phi)$, and for each $n = 2, 3, 4$, we shall give an example of $\text{Im}(\phi)$ that would realize (3.5).

Proposition 3.3. *Let $C = (C, \cdot, \circ)$ be a perfect skew brace with $\text{Ann}(C) = 1$ and let $A = \mathbb{F}_p^2 \rtimes_{\phi} C$, where p is any prime and $\phi : (C, \circ) \longrightarrow \text{GL}_2(\mathbb{F}_p)$ is a homomorphism with*

$$\text{Im}(\phi) = \langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \rangle \simeq \mathbb{Z}/p\mathbb{Z}.$$

Then the derived ideal and annihilator of A are given by

$$A * A = \langle \begin{bmatrix} 1 \\ 0 \end{bmatrix} \rangle \times C, \quad \text{Ann}(A) = \langle \begin{bmatrix} 1 \\ 0 \end{bmatrix} \rangle \times \{1\},$$

*and we have the non-equality $(A * A) * \text{Ann}_2(A) \neq 1$.*

Proof. The two equalities follow from (3.3), (3.4), and the fact that

$$\text{Im}(Q - I_2) = \ker(Q - I_2) = \langle \begin{bmatrix} 1 \\ 0 \end{bmatrix} \rangle$$

for all $Q \in \text{Im}(\phi)$ with $Q \neq I_2$.

To show the non-equality, consider the element

$$a = \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}, 1 \right) \in Z(A, \cdot).$$

For any $x_1, x_2 \in \mathbb{F}_p$ and $y \in C$, it follows from (3.2) that

$$(3.6) \quad \begin{aligned} \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}, 1 \right) * \left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, y \right) &= ((\phi_1 - I_2) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, 1 * y) = \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, 1 \right) \\ \left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, y \right) * \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}, 1 \right) &= ((\phi_y - I_2) \begin{bmatrix} 0 \\ 1 \end{bmatrix}, y * 1) = \left(\begin{bmatrix} \gamma \\ 0 \end{bmatrix}, 1 \right) \end{aligned}$$

for some $\gamma \in \mathbb{F}_p$. They both lie in $\text{Ann}(A)$ and thus $a \in \text{Ann}_2(A)$. Taking

$$z := \left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, y \right) \text{ with } x_2 = 0 \text{ and } \phi_y = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix},$$

we see that $\gamma = 1$ in (3.6) and so $z * a \neq 1$. Since $a \in \text{Ann}_2(A)$ and $z \in A * A$, this yields $(A * A) * \text{Ann}_2(A) \neq 1$, as desired. $\square$

For $n = 3$, we need to take p to be odd, but the situation is very similar.

Proposition 3.4. *Let $C = (C, \cdot, \circ)$ be a perfect skew brace with $\text{Ann}(C) = 1$ and let $A = \mathbb{F}_p^3 \rtimes_\phi C$, where p is an odd prime and $\phi : (C, \circ) \longrightarrow \text{GL}_3(\mathbb{F}_p)$ is a homomorphism with*

$$\text{Im}(\phi) = \left\langle \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\rangle \times \left\langle \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\rangle \simeq \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}.$$

Then the derived ideal and annihilator of A are given by

$$A * A = \left\langle \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\rangle \times C, \quad \text{Ann}(A) = \left\langle \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\rangle \times \{1\},$$

*and we have the non-equality $(A * A) * \text{Ann}_2(A) \neq 1$.*

Proof. The claim for $\text{Ann}(A)$ holds by (3.4) and the fact that

$$\begin{aligned} \ker \left(\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - I_3 \right) &= \left\langle \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\rangle, \\ \ker \left(\begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - I_3 \right) &= \left\langle \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\rangle. \end{aligned}$$

Here, once we regard elements of $\text{Fix}_\phi(\mathbb{F}_p^3)$ as the fixed points of $\text{Im}(\phi)$, it is clear that it suffices to consider the generators of $\text{Im}(\phi)$. Now, observe that

$$(3.7) \quad \text{Im}(\phi) = \left\{ \begin{bmatrix} 1 & 0 & \gamma \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & \gamma \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} : \gamma \in \mathbb{F}_p \right\}.$$

The claim for $A * A$ then follows from (3.3) and the fact that

$$\begin{aligned} \text{Im} \left(\begin{bmatrix} 1 & 0 & \gamma \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - I_3 \right) &= \left\langle \begin{bmatrix} \gamma \\ 0 \\ 0 \end{bmatrix} \right\rangle, \\ \text{Im} \left(\begin{bmatrix} 1 & 1 & \gamma \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - I_3 \right) &= \left\langle \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} \gamma \\ 0 \\ 0 \end{bmatrix} \right\rangle, \end{aligned}$$

where $2 \neq 0$ because p is assumed to be odd.

To show the non-equality, consider the element

$$a := \left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, 1 \right) \in Z(A, \cdot).$$

Observe that (3.7) implies

$$\forall y \in C : (\phi_y - I_3) \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \in \left\langle \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\rangle.$$

For any $x_1, x_2, x_3 \in \mathbb{F}_p$ and $y \in C$, we then see from (3.2) that

$$(3.8) \quad \begin{aligned} \left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, 1 \right) * \left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, y \right) &= \left((\phi_1 - I_3) \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, 1 * y \right) = \left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, 1 \right) \\ \left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, y \right) * \left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, 1 \right) &= \left((\phi_y - I_3) \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, y * 1 \right) = \left(\begin{bmatrix} \gamma \\ 0 \\ 0 \end{bmatrix}, 1 \right) \end{aligned}$$

for some $\gamma \in \mathbb{F}_p$. They both lie in $\text{Ann}(A)$ and thus $a \in \text{Ann}_2(A)$. Taking

$$z := \left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, y \right) \text{ with } x_3 = 0 \text{ and } \phi_y = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

we see that $\gamma = 1$ in (3.8) and so $z * a \neq 1$. Since $a \in \text{Ann}_2(A)$ and $z \in A * A$, this yields $(A * A) * \text{Ann}_2(A) \neq 1$, as desired. $\square$

Note that the skew braces constructed in Propositions 3.3 and 3.4 are not perfect. To obtain a perfect skew brace without losing the desired condition (3.5), we shall use $B = (\mathbb{F}_2^4, +, +)$. The suitable subgroup $\text{Im}(\phi)$ of $\text{GL}_4(\mathbb{F}_2)$ that we found is isomorphic to the symmetric group S_4 , and we acknowledge the use of MAGMA [2] in the search of this subgroup.

Proposition 3.5. *Let $C = (C, \cdot, \circ)$ be a perfect skew brace with $\text{Ann}(C) = 1$ and let $A = \mathbb{F}_2^4 \rtimes_\phi C$, where $\phi : (C, \circ) \longrightarrow \text{GL}_4(\mathbb{F}_2)$ is a homomorphism with*

$$\text{Im}(\phi) = \left\langle \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \right\rangle \rtimes \left\langle \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \right\rangle \simeq S_4.$$

Then the derived ideal and annihilator of A are given by

$$A * A = A, \quad \text{Ann}(A) = \left\langle \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\rangle \times \{1\},$$

*and we have the non-equality $A * \text{Ann}_2(A) \neq 1$.*

Proof. The claim for $\text{Ann}(A)$ holds by (3.4) and the fact that

$$\begin{aligned} \ker \left(\begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} - I_4 \right) &= \left\langle \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\rangle, \\ \ker \left(\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} - I_4 \right) &= \left\langle \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\rangle, \\ \ker \left(\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix} - I_4 \right) &= \left\langle \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\rangle, \\ \ker \left(\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} - I_4 \right) &= \left\langle \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\rangle. \end{aligned}$$

Here, again we only need to look at the generators of $\text{Im}(\phi)$ because $\text{Fix}_\phi(\mathbb{F}_2^4)$ is simply the set of fixed points of $\text{Im}(\phi)$. The claim for $A * A = A$ holds by (3.3) because we have

$$\begin{aligned} \text{Im} \left(\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} - I_4 \right) &\ni \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \\ \text{Im} \left(\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix} - I_4 \right) &\ni \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \end{aligned}$$

and the four vectors on the right are linearly independent.

To show that non-equality, consider the element

$$a := \left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, 1 \right) \in Z(A, \cdot).$$

Let U denote the subspace of $\mathbb{F}_2^4$ generated by $(1, 0, 1, 0)$. Observe that

$$\left(\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} - I_4 \right) \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad \text{and} \quad (Q - I_4) \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

for the other three generators Q of $\text{Im}(\phi)$. Since U is fixed by all four of the generators, the above implies that

$$\forall y \in C : \phi_y \left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right) \equiv \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \pmod{U}, \text{ that is } (\phi_y - I_4) \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \in U.$$

For any $x_1, x_2, x_3, x_4 \in \mathbb{F}_2$ and $y \in C$, we then see from (3.2) that

$$\begin{aligned} \left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, 1 \right) * \left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, y \right) &= \left((\phi_1 - I_4) \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, 1 * y \right) = \left(\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, 1 \right) \\ (3.9) \quad \left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, y \right) * \left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, 1 \right) &= \left((\phi_y - I_4) \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, y * 1 \right) = \left(\begin{bmatrix} \gamma \\ 0 \\ \gamma \\ 0 \end{bmatrix}, 1 \right) \end{aligned}$$

for some $\gamma \in \mathbb{F}_2$. They both lie in $\text{Ann}(A)$ and thus $a \in \text{Ann}_2(A)$. Since we can choose $y \in C$ such that $\gamma = 1$ in (3.9), it follows that $z * a \neq 1$ for some $z \in A$. Since $a \in \text{Ann}_2(A)$, we get $A * \text{Ann}_2(A) \neq 1$, as desired. $\square$

As one can see from the proof of Proposition 3.5, the main idea is to take

$$\text{Im}(\phi) = \langle M_1, \dots, M_d \rangle,$$

where we choose the matrices $M_1, \dots, M_d \in \text{GL}_n(\mathbb{F}_p)$ to be such that

(1) the columns of $M_1 - I_n, \dots, M_d - I_n$ generate $\mathbb{F}_p^n$;

(2) there exists $\vec{v} \in \mathbb{F}_p^n$ lying outside of

$$U := \bigcap_{i=1}^d \ker(M_i - I_n)$$

but gets mapped into U under every $M_1 - I_n, \dots, M_d - I_n$.

These conditions, respectively, ensure that for the skew brace $A = \mathbb{F}_p^n \rtimes_{\phi} C$ (with C perfect and $\text{Ann}(C) = 1$) constructed, we have:

- (1) A is perfect (recall (3.3));
- (2) $(\vec{v}, 1) \in \text{Ann}_2(A) \setminus \text{Ann}(A)$ (recall (3.4) and (3.2)).

We can produce more candidates for $\text{Im}(\phi)$ other than the one in Proposition 3.5. However, there might not exist a perfect skew brace C with $\text{Ann}(C) = 1$ such that $(C, \circ)$ has the candidates as quotients. We have taken $\text{Im}(\phi) \simeq S_4$ because thanks to the `YangBaxter` package in `GAP` [4], we have the following examples. Here `SmallSkewbrace(n,k)` denotes the skew brace whose ID is (n, k) in the database of `GAP`, and similarly for `SmallGroup(n,k)`.

Example 3.6. According to `GAP`, the skew brace

$$C = (C, \cdot, \circ) = \text{SmallSkewbrace}(24, 853)$$

is perfect with $(C, \cdot) \simeq \mathbb{F}_3 \times \mathbb{F}_2^3$ and $(C, \circ) \simeq S_4$. We have $\text{Ann}(C) = 1$ since $(C, \circ)$ has trivial center. This skew brace C is among the simple braces that were constructed in [1, Sections 6 and 7]. In our special case, one finds that the lambda map of C is given by

$$\lambda_{(v, x_1, x_2, x_3)} = \left((-1)^{x_3 - x_1 x_2}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}^v \begin{bmatrix} M(x_1, x_2, x_3) & 0 \\ \epsilon(v, x_1, x_2, x_3) & 1 \end{bmatrix} \right),$$

where we define

$$M(x_1, x_2, x_3) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^{x_3 - x_1 x_2}$$

$$\epsilon(v, x_1, x_2, x_3) = [x_1 \ x_2] \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}^v \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^{1 + x_3 - x_1 x_2},$$

for all $v \in \mathbb{F}_3$ and $x_1, x_2, x_3 \in \mathbb{F}_2$. In the notation of [1, Theorem 6.3], we are taking $Q(x_1, x_2) = x_1x_2$, $\gamma = -1$, $C = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$, $F = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, and $z = \begin{bmatrix} 0 & 1 \end{bmatrix}$. The function $q(x_1, x_2, x_3) = x_3 - Q(x_1, x_2)$ is determined by Q , while the matrix B represents the bilinear form $(\vec{x}, \vec{y}) \mapsto Q(\vec{x} + \vec{y}) - Q(\vec{x}) - Q(\vec{y})$ associated to Q , which is given by $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ in this case. Finally, we note that the $c(\vec{x})$ there denotes the linear transformation on $\mathbb{F}_2^2$ induced by C .

Example 3.7. According to **GAP**, the skew brace

$$C = (C, \cdot, \circ) = \text{SmallSkewbrace}(72, 1483)$$

is perfect with $(C, \cdot) \simeq C_9 \times \mathbb{F}_2^3$ and $(C, \circ) \simeq \text{SmallGroup}(72, 15)$, which is the unique non-split extension of C_3 and S_4 . We have $\text{Ann}(C) = 1$ since $(C, \circ)$ has trivial center. This skew brace C is a socle extension of that in Example 3.6, in the sense that $C/\text{Soc}(C)$ is isomorphic to **SmallSkewbrace**(24, 853).

Therefore, for the skew braces $C = (C, \cdot, \circ)$ in Examples 3.6 and 3.7, there exists a homomorphism $\phi : (C, \circ) \longrightarrow \text{GL}_4(\mathbb{F}_2)$ such that $\text{Im}(\phi) \simeq S_4$ is given as in Proposition 3.5. The skew brace $A = \mathbb{F}_2^4 \rtimes_{\phi} C$ is then perfect but

$$\text{Ann}(A/\text{Ann}(A)) \neq 1.$$

This gives counterexamples to the analog of Grün's lemma. We remark that the A here is in fact a brace because $(C, \cdot)$ is abelian for both examples.

Remark 3.8. In the setting of Proposition 3.2, similar to (3.4) we also have

$$\text{Ann}(A) = \text{Fix}_{\phi}(B) \times \{1\} \text{ when } \ker(\phi) = 1.$$

One can try to produce examples with $(A * A) * \text{Ann}_2(A) \neq 1$ by taking ϕ to be injective rather than requiring $\text{Ann}(C) = 1$.

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