FREE NOVIKOV ALGEBRAS AND THE HOPF ALGEBRA OF DECORATED MULTI-INDICES

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Abstract. We propose a combinatorial formula for the coproduct in a Hopf algebra of decorated multi-indices that recently appeared in the literature, which can be briefly described as the graded dual of the enveloping algebra of the free Novikov algebra generated by the set of decorations. Similarly to what happens for the Hopf algebra of rooted forests, the formula can be written in terms of admissible cuts. We also prove a combinatorial formula for the extraction-contraction coproduct for undecorated multi-indices, in terms of a suitable notion of covering subforest.

Keywords: Multi-indices, Novikov algebra, Hopf algebra, symmetry factor, pre-Lie algebra.
MSC classification: 05C05, 16T30, 17A30.

1. Introduction

A Hopf algebra of multi-indices appeared in the recent work of P. Linares, F. Otto and M. Tempelmayr on regularity structures [28], as a new combinatorial tool for handling rough partial differential equations. This new approach was continued in [27, 10, 6], and also in [9], where the role of post-Lie algebras has been highlighted. Multi-indices have also been adapted to the framework of rough paths and rough differential equations in [26, 7]. We will stick to this somewhat simpler framework, which permits us to look at A-decorated multi-indices: contrarily to what happens in general with regularity structures, the set A of decorations does not play an active role in the definition of the coproduct.

Recalling that the free pre-Lie algebra generated by A is the linear span of A-decorated rooted trees [14, 17], the canonical surjective map Φ from the free pre-Lie algebra $\mathrm{PL}(A)$ generated by A onto the free Novikov algebra $\mathrm{N}(A)$ yields, by transposition and multiplicative extension, a canonical embedding $\jmath$ of the multi-index Hopf algebra $\mathcal{H}^A_{\mathrm{LOT}}$ into the Hopf algebra $\mathcal{H}^A_{\mathrm{BCK}}$ of A-decorated rooted forests[1]. This conceptually simple algebraic fact allows a grouping of the terms in the tree expansion [22] of the solution of a rough differential equation driven by a Hölder continuous path $X : [0;T] \to \mathbb{R}^d$ corresponding to any choice of a rough path or branched rough path $\mathbb{X}$ over X [30, 31, 32, 22]. The grouping is performed according to multi-indices, considering for each multi-index M the set of trees t such that $\Phi(t) = M$. Along these lines, a multi-index rough path is a two-parameter family of characters of $\mathcal{H}^A_{\mathrm{LOT}}$ subject to Chen's lemma and suitable estimates [28, 34].

The above description as a graded dual, although perfectly rigorous, is not completely explicit, as it depends on a choice of pairing on the symmetric algebra[2] of $\mathrm{N}(A)$. We propose here a pairing, carefully chosen in order to take symmetry factors into account, which yields an explicit formula for the coproduct. The formula (33) thus obtained, reminiscent to the Connes-Kreimer formula for rooted forests, involves a suitable notion of admissible cut for multi-indices.

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[1]In the undecorated case $A = \{*\}$, the Hopf algebra $\mathcal{H}^A_{\mathrm{LOT}}$ is a quotient of the original Hopf algebra of [28]. A family of generators is discarded due to the fact that we do not deal with regularity structures.

[2]The enveloping algebra is identified with the symmetric algebra by the Guin-Oudom construction [23] associated to the pre-Lie structure.

The paper is organised as follows: after a quick reminder on combinatorial Hopf algebras (Paragraph 2.1), we recall the construction of the rooted forest Hopf algebra in Paragraph 2.2. The multi-index Hopf algebra is introduced in Paragraph 2.3. We introduce the pairing in Paragraph 3.1 and the corresponding Hopf algebra embedding in Section 3. Section 4 introduces graftings and cuts for trees and forests with free edges. The main result of this section is Proposition 12, which carefully counts cuts and graftings with the help of symmetry factors. We prove a recursive formula for the coproduct in Paragraph 5.1, and finally state and prove our main result (Theorem 14 and Equation (33)) in the last Paragraph 5.2. The last section is devoted to a combinatorial formula for the extraction-contraction coproduct (Theorem 15).

Our work is close in spirit to [25], where an algebraic formula for the coproduct for the full multi-index Hopf algebra of [28] is given in terms of a different pairing: see (2.16) and (3.12) therein. Our choice of pairing yields a completely combinatorial interpretation of the coproduct of the multi-index Hopf algebra $\mathcal{H}^A_{\mathrm{LOT}}$. It would be interesting to see whether such a combinatorial interpretation for the regularity structure Hopf algebra of [28] and [25] is available.

Let us finally mention that Y. Bruned and Y. Hou [8] were first in giving explicit formulae for both coproducts. They start from explicit expressions for both corresponding Grossman-Larson products and dualize them. Their approach also covers the regularity structure case. Their choice of pairing uses a different notion of symmetry factor for a multi-index: although both approaches give equivalent results, different coefficients therefore appear in the explicit expressions. Our purely combinatorial interpretation of both coproducts in terms of admissible cuts, and respectively covering subforests, namely Formulae (33) and (37), however relies on our convention for symmetry factors in an essential way.

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2. The Hopf algebras of decorated multi-indices

2.1. **Reminder on combinatorial Hopf algebras.** There seems to be no consensus on what a combinatorial Hopf algebra is, despite the vitality and fruitfulness of this research topic. The following *ad hoc* definition comes from [16]. A combinatorial Hopf algebra is

- A graded connected Hopf algebra on a field $\mathbb{K}$ containing the rationals

$$\mathcal{H} = \bigoplus_{n\geq 0} \mathcal{H}_n, \qquad \mathcal{H}_0 = \mathbb{K}.\mathbf{1},$$

- together with a homogeneous linear basis $\mathcal{B}$ such that the structure constants κ^c_{ab} and κ^{bc}_a of both multiplication and comultiplication are non-negative integers:

$$ab = \sum_{c\in\mathcal{B}} \kappa^c_{ab} c, \qquad \Delta(a) = \sum_{b,c\in\mathcal{B}} \kappa^{bc}_a b\otimes c.$$

Some more requirements can be added, such as

- Moderate growth condition, e.g.

$$\dim \mathcal{H}_n \leq CK^n \text{ for some } C, K > 0.$$

- Nondegeneracy condition:

$$\mathcal{B} \cap \operatorname{Prim}(\mathcal{H}) = \mathcal{B} \cap \mathcal{H}_1, \tag{1}$$

where $\operatorname{Prim}(\mathcal{H})$ is the set of primitive elements. Recall that an element $x \in \mathcal{H}$ is primitive if and only if $\Delta(x) = x \otimes \mathbf{1} + \mathbf{1} \otimes x$.

A morphism between two combinatorial Hopf algebras $(\mathcal{H}, \mathcal{B})$ and $(\mathcal{H}', \mathcal{B}')$ is a graded Hopf algebra morphism $\varphi : \mathcal{H} \to \mathcal{H}'$ such that, for any $b \in \mathcal{B}$, the image $\varphi(\mathcal{B})$ is a finite linear combination of elements of $\mathcal{B}'$ with nonnegative integer coefficients. This organises those combinatorial Hopf algebras into a category.

2.2. **Pre-Lie algebras and the Butcher–Connes–Kreimer Hopf algebra.** A left pre-Lie algebra (here, over a field $\mathbb{K}$ containing the rationals) is a $\mathbb{K}$-vector space P together with a bilinear product $\triangleright$ such that

$$x \triangleright (y \triangleright z) - (x \triangleright y) \triangleright z = y \triangleright (x \triangleright z) - (y \triangleright x) \triangleright z \tag{2}$$

holds for any $x, y, z \in P$. Pre-Lie algebras, which can be traced back to the work of A. Cayley [13], are sometimes called *Vinberg algebras*, as they appeared explicitly for the first time in the work of E. B. Vinberg [37] under the name *left-symmetric algebras* on the classification of homogeneous cones. They appeared independently at the same time in the work of M. Gerstenhaber [20] on Hochschild cohomology and deformations of algebras, under the name "pre-Lie algebras" which is now the standard terminology. The term *chronological algebras* has also been sometimes used, e.g. in [1]. Antisymmetrising the pre-Lie product gives rise to a Lie algebra P, the universal enveloping algebra $\mathcal{U}(P)$ of which is isomorphic (as a Hopf algebra) to the symmetric algebra $S(P)$ endowed with the deshuffle coproduct

$$\Delta_{\sqcup\!\sqcup}(x_1 \cdots x_n) = \sum_{I \sqcup J = \{1,\ldots,n\}} x_I \otimes x_J$$

(with the notation $x_J := x_{j_1} \cdots x_{j_k}$ for $J = \{j_1, \ldots, j_k\}$) and the Grossman-Larson product

$$X \star Y := \sum_{(X)} X_1 (X_2 \triangleright Y),$$

where the Guin-Oudom product $\triangleright$ [23, Proposition 2.7] is the unique extension of the pre-Lie product to $S(P)$ such that

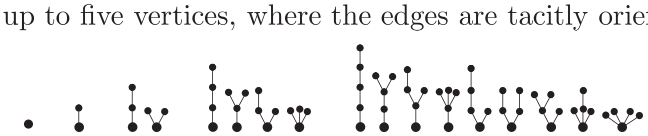
- $1 \triangleright X = X$,
- $X \triangleright YZ = \sum_{(X)} (X_1 \triangleright Y)(X_2 \triangleright Z)$,
- $(xY) \triangleright Z = x \triangleright (Y \triangleright Z) - (x \triangleright Y) \triangleright Z$

for any $X, Y, Z \in S(L)$ and $x \in L$. A key property of the Grossman-Larson product is given by

$$X \triangleright (Y \triangleright Z) = (X \star Y) \triangleright Z$$

for any $X, Y, Z \in S(P)$. For a short survey on pre-Lie algebras, see [11, 33].

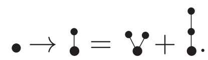
F. Chapoton and M. Livernet provided the explicit description of the free pre-Lie algebra $\operatorname{PL}(A)$ in terms of A-decorated rooted trees endowed with grafting [14]. Recall that a rooted tree is a connected oriented graph with a finite number of vertices, one among them being distinguished as the root, such that any vertex admits exactly one outgoing edge, except the root which has no outgoing edges. Here is the list of rooted trees up to five vertices, where the edges are tacitly oriented from top to bottom:



A rooted forest is a finite collection of rooted trees, possibly with repetitions. For any set A, an A-decorated rooted forest f is a rooted forest together with a map $d : \mathcal{V}(f) \to A$, where $\mathcal{V}(f)$ is the set of vertices of f. The pre-Lie product $s \to t$ of two rooted trees is obtained by grafting the root of s on a vertex of t, summing up over all choices of vertices:

$$s \to t = \sum_{v \in \mathcal{V}(t)} s \to_v t. \tag{3}$$

For example,



The symmetric algebra $S\big(\mathrm{PL}(A)\big)$ is the linear span of A-decorated rooted forests. Denoting also by $\to$ the Guin-Oudom extension of the grafting product, the corresponding Grossman-Larson product [21] is given by the *graft-or-fall* formula

$$F \star G = B_-\big(F \to B_+^c(G)\big).$$

Here $B_+^c(F) := F \to \bullet_c$ for some $c \in A$, and B_- is the root removal, left inverse to B_+^c.

The Butcher–Connes–Kreimer Hopf algebra ([15, 18], see also [5]) $\mathcal{H}_{\mathrm{BCK}}^A = \bigoplus_{n \geq 0} \big(\mathcal{H}_{\mathrm{BCK}}^A\big)_n$ is a commutative Hopf algebra of A-decorated rooted forests over $\mathbb{K}$, graded by the number of vertices, obtained as the graded dual of the cocommutative Grossman-Larson Hopf algebra described above. Normalising the dual forest basis by the symmetry factor (see e.g. [4, 35]), namely

$$\langle u, v \rangle := \sigma(u)\delta_u^v \tag{4}$$

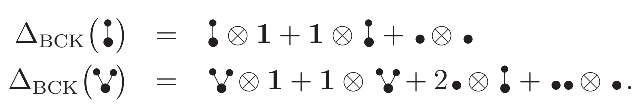
where $\sigma(u)$ is the cardinal of the automorphism group of the forest u, the coproduct on a rooted forest u is described as follows: the set $\mathcal{V}(u)$ of vertices of a forest u is endowed with a partial order defined by $x \leq y$ if and only if there is a path from y to a root passing through x. Any subset W of $\mathcal{V}(u)$ defines a subforest $u_{|_W}$ of u in an obvious manner, i.e. by keeping the edges of u which link two elements of W. The coproduct is then defined by:

$$\Delta_{\mathrm{BCK}}(u) = \sum_{\substack{V \amalg W = \mathcal{V}(u) \\ W < V}} u_{|_V} \otimes u_{|_W}. \tag{5}$$

Here the notation $W < V$ means that $y \not> x$ for any vertex x in V and any vertex y in W. Such a couple $c = (V, W)$ is also called an *admissible cut*, with crown (or pruning) $P^c(u) = u_{|_V}$ and trunk $R^c(u) = u_{|_W}$, so that the coproduct often appears under the form

$$\Delta_{\mathrm{BCK}}(u) = \sum_{c \in \mathrm{Adm}(u)} P^c(u) \otimes R^c(u) \tag{6}$$

in the literature. We have for example:



Recall that the symmetry factor of a rooted forest is the cardinal of its automorphism group. For later use, the symmetry factor of a forest $u = t_1^{\ell_1} \cdots t_n^{\ell_n}$, where $t_1, \ldots, t_n$ are pairwise distinct trees, can be expressed in terms of the symmetry factor of its tree components as

$$\sigma(u) = \ell_1! \cdots \ell_n! \sigma(t_1)^{\ell_1} \cdots \sigma(t_n)^{\ell_n}. \tag{7}$$

Remark 1. The symmetry factor $\sigma(u)$ is obtained by multiplying the *external symmetry factor* $\sigma^{\mathrm{ext}}(u) := \ell_1! \cdots \ell_n!$ by the *internal symmetry factor* $\sigma^{\mathrm{int}}(u) := \sigma(t_1)^{\ell_1} \cdots \sigma(t_n)^{\ell_n}$. The internal symmetry factor is the cardinal of the internal automorphism group, i.e. the normal subgroup of automorphisms which leaves each connected component unchanged. The external symmetry factor is the cardinal of the external automorphism group, which is a subgroup of the permutation group of the connected components.

2.3. **Novikov algebras and the Linares–Otto–Tempelmayr Hopf algebra of multi-indices.** A Novikov algebra [17] is a vector space N over a base field $\mathbb{K}$, together with a bilinear product $\rhd : N \times N \to N$ such that, for any $x, y, z \in N$, the following identities hold:

$$x \rhd (y \rhd z) - (x \rhd y) \rhd z = y \rhd (x \rhd z) - (y \rhd x) \rhd z, \tag{8}$$

$$(x \rhd y) \rhd z = (x \rhd z) \rhd y. \tag{9}$$

The first is left pre-Lie identity, the second is right-NAP identity[3]. Novikov algebras seem to appear for the first time in the article [19] by I. M. Gelfand and I. J. Dorfman in the study of Hamiltonian operators in the formal calculus of variations (see Equation (6.3) therein, where the left pre-Lie identity appears in a disguised form). They have been rediscovered by A. A. Balinskii and S. P. Novikov in [2] (in the right pre-Lie and left NAP form, see Equation (4) therein). The terminology was proposed by J. M. Osborn in [36]. An important example is given by a commutative associative algebra $\mathcal{A}$ endowed with a derivation $D : \mathcal{A} \to \mathcal{A}$. From Leibniz' rule $D(xy) = D(x)y + xD(y)$, it is easily seen that $(\mathcal{A}, \rhd)$ is Novikov with $x \rhd y := xD(y)$.

With our conventions, the free Novikov algebra $\mathrm{N}(A)$ generated by a set A is described as follows [17, Definition 7.7]: let $\overline{\mathrm{N}}(A)$ be the commutative algebra of polynomials with variables x^a_j, $(a, j) \in A \times \{-1, 0, 1, 2, \ldots\}$, let ∂ be the unique derivation of $\overline{\mathrm{N}}(A)$ such that $\partial x^a_j = x^a_{j+1}$. The integer $j \geq -1$ is the *weight* of the variable x^a_j.

A basis of $\overline{\mathrm{N}}(A)$ is given by the monomials

$$\mathbf{x}^{\mathbf{k}} := \prod_{j \geq -1,\, a \in A} (x^a_j)^{k^a_j},$$

where the exponents k^a_j are non-negative integers, equal to zero except a finite number of them. The weight induces a unique $\mathbb{Z}$-grading of the algebra $\overline{\mathrm{N}}(A)$, for which the derivation ∂ is homogeneous of degree one. By Leibniz' rule, the expression of the derivation ∂ is given by

$$\partial \mathbf{x}^{\mathbf{k}} = \sum_{j \geq -1,\, a \in A} k^a_j \mathbf{x}^{\mathbf{k} - \mathbf{e}^a_j + \mathbf{e}^a_{j+1}}, \tag{10}$$

where $\mathbf{e}^a_j$ is the multi-index in which all coordinates are equal to zero except the one in position (j, a) which is equal to one. In other words, $\mathbf{x}^{\mathbf{e}^a_j} = x^a_j$. We have for example in the undecorated case $A = \{*\}$:

$$\partial (x_{-1})^2 = 2x_{-1}x_0, \qquad \partial(x_0 x_1) = (x_1)^2 + x_0 x_2,$$

and in the decorated case

$$\partial\big((x^a_1)^2 x^b_1\big) = 2x^a_1 x^a_2 x^b_1 + (x^a_1)^2 x^b_2.$$

The bilinear product $P \rhd Q := P.\partial Q$ endows $\overline{\mathrm{N}}(A)$ with a Novikov structure, and the free Novikov algebra $\mathrm{N}(A)$ turns out to be the homogeneous component of $\overline{\mathrm{N}}(A)$ of weight -1. This is Theorem 7.8 in [17], recently generalised to multi-Novikov algebras [6, Theorem 3.3]. The embedding of A into $\mathrm{N}(A)$ is given by $a \mapsto x^a_{-1}$.

We shall also use the unique derivation $\overline{\partial} : \overline{N}(A) \to \overline{N}(A)$ defined on the variables by

$$\overline{\partial} x^a_j = x^a_{j-1} \text{ if } j \geq 0, \qquad \overline{\partial} x^a_{-1} = 0. \tag{11}$$

[3] NAP for Non-Associative Permutative [29]. Novikov algebras are right pre-Lie and left NAP in the definition given in [17] (right-symmetric and left-commutative in the terminology employed therein).

Its expression on the basis of monomials is given by

$$\overline{\partial}\mathbf{x}^{\mathbf{k}} = \sum_{j\geq 0,\, a\in A} k_j^a \mathbf{x}^{\mathbf{k}-\mathbf{e}_j^a+\mathbf{e}_{j-1}^a}. \tag{12}$$

We have for example

$$\overline{\partial}(x_2x_3) = (x_2)^2 + x_1x_3, \qquad \overline{\partial}(x_{-1}x_0x_1) = (x_{-1})^2x_1 + x_{-1}(x_0)^2.$$

The canonical surjective map Φ from $\mathrm{PL}(A)$ to $\mathrm{N}(A)$ is the unique pre-Lie morphism extending the embedding of the generators, and can be understood as the *fertility map*: for any A-decorated rooted tree t, we have

$$\Phi(t) = \prod_{v\in\mathcal{V}(t)} x_{f(v)-1}^{d(v)}, \tag{13}$$

where $d(v)\in A$ is the decoration of vertex v, and where $f(v)$ is its fertility, i.e. its number of incoming edges. The map Φ is obviously surjective. From (13) and (3), it is easily seen to be a pre-Lie morphism, namely

$$\Phi(s\to t) = \Phi(s).\partial\Phi(t) \tag{14}$$

for any rooted trees s,t, due to the fact that grafting any tree at vertex v increases its fertility by 1, leaving the fertility of the other vertices unchanged[4] [7, Proposition 2.9].

Remark 2. Recall that $C^\infty(\mathbb{R}^d,\mathbb{R}^d)$, identified with the set of vector fields on $\mathbb{R}^d$, is a pre-Lie algebra on the base field $\mathbb{K}=\mathbb{R}$, with pre-Lie product

$$\left(\sum_{i=1}^d f_i\partial_i\right) \rhd \left(\sum_{j=1}^d g_j\partial_j\right) = \sum_{i=1}^d f_i\left(\sum_{j=1}^d \partial_i g_j\right)\partial_j.$$

When $d=1$, this product specialises to the Novikov product

$$f\rhd g = f.\partial g$$

where ∂g stands for the derivative of g. By freeness universal property, for any family $f:=(f_a)_{a\in A}$ of vector fields on $\mathbb{R}$, there is a unique Novikov algebra morphism $\mathcal{F}_f:\mathrm{N}(A)\to C^\infty(\mathbb{R},\mathbb{R})$ such that $\mathcal{F}_f(\mathbf{x}_{-1}^a)=f_a$. This is the starting point of multi-index B-series introduced in [7]:

$$B_f(\alpha) = \sum_{\mathbf{k},\,\mathrm{wt}(\mathbf{x}^{\mathbf{k}})=-1} \frac{\alpha(\mathbf{x}^{\mathbf{k}})}{\sigma(\mathbf{x}^{\mathbf{k}})}\mathcal{F}_f(\mathbf{x}^{\mathbf{k}}),$$

where α is any linear map from $\mathrm{N}(A)$ into $\mathbb{R}$. Our convention for the symmetry factor $\sigma(\mathbf{x}^{\mathbf{k}})$ is given in Paragraph 3.1 below.

As to any pre-Lie algebra, the Guin-Oudom procedure applies to both $\mathrm{N}(A)$ and $\mathrm{PL}(A)$. In particular, the multiplicative extension

$$\Phi: S\big(\mathrm{PL}(A)\big) \longrightarrow S\big(\mathrm{N}(A)\big)$$

of the fertility map to symmetric algebras is a Hopf algebra morphism from $\Big(S\big(\mathrm{PL}(A)\big),\star,\Delta_{\sqcup\!\sqcup},\mathbf{1},\varepsilon\Big)$ onto $\Big(S\big(\mathrm{N}(A)\big),\star,\Delta_{\sqcup\!\sqcup},\mathbf{1},\varepsilon\Big)$, where both Grossman-Larson products are denoted by $\star$, and where $\Delta_{\sqcup\!\sqcup}$ is the usual deshuffle coproduct. The Hopf algebra of multi-indices is defined by

$$\mathcal{H}_{\mathrm{LOT}}^A = \Big(S\big(\mathrm{N}(A)\big),\star,\Delta_{\sqcup\!\sqcup},\mathbf{1},\varepsilon\Big)^\circ,$$

[4]We have adopted the same conventions for the fertility map than [7, Definition 2.12] (called counting map and denoted by Ψ therein). Our conventions for the derivation ∂, denoted by D in [7], are also the same. The fertility map often appears in the recent literature with extra symmetry factors, e.g. [10, Paragraph 2.5], due to a different normalisation of the derivation ∂. See Remark 7 below.

where $(-)^\circ$ stands for the graded dual. A basis of $S\big(\mathrm{N}(A)\big)$ is given by *monomials of monomials* $\mathbb{M} = M_1^{\odot m_1} \odot \cdots \odot M_k^{\odot m_k}$, where the M_j's are distinct monomials in $\mathrm{N}(A)$ (we use the notation $\odot$ for the commutative "external" product in the symmetric algebra, not to be confused with the commutative "internal" product of $\overline{N}(A)$). The explicit formula for the coproduct depends on the choice of a pairing. A proposal will be given in the next section.

3. Pairing and embedding

3.1. **The pairing.** The symmetry factor of any monomial

$$M = \mathbf{x}^{\mathbf{k}} = \prod_{a\in A,\, j\geq -1} (x_j^a)^{k_j^a},$$

where $\mathbf{k}$ stands for the multi-index $(k_j^a)_{a\in A,\, j\geq -1}$, is given by

$$\sigma(M) = \mathbf{k}! = \prod_{a\in A,\, j\geq -1} k_j^a!\,.$$

The degree of the monomial $\mathbf{x}^{\mathbf{k}}$ is

$$|\mathbf{k}| := \sum_{a\in A,\, j\geq -1} k_j^a.$$

Its weight is defined by

$$\mathrm{wt}(\mathbf{x}^{\mathbf{k}}) := \sum_{a\in A,\, j\geq -1} j k_j^a,$$

which we also will denote by $\mathrm{wt}(\mathbf{k})$. Now let us define a symmetric nondegenerate pairing on $S\big(\overline{\mathrm{N}}(A)\big)$ as follows: let $\mathbb{M} = M_1^{\odot \ell_1} \odot \cdots \odot M_n^{\odot \ell_n}$ be a monomial of monomials, and let $\mathbb{M}'$ be another one, and set

$$\langle \mathbb{M},\, \mathbb{M}'\rangle := \sigma(\mathbb{M})\delta_{\mathbb{M}}^{\mathbb{M}'},$$

where the symmetry factor is given by a formula similar to (7):

$$\sigma(\mathbb{M}) := \ell_1! \cdots \ell_n! \sigma(M_1)^{\ell_1} \cdots \sigma(M_n)^{\ell_n}. \tag{15}$$

Under this pairing, the dual of the deshuffle coproduct is the commutative product $\odot$. It remains to compute the dual coproduct of the Grossman-Larson product.

Remark 3. The symmetry factor $\sigma(\mathbb{M})$ is obtained by multiplying the *external symmetry factor* $\sigma^{\mathrm{ext}}(\mathbb{M}) := \ell_1! \cdots \ell_n!$ by the *internal symmetry factor* $\sigma^{\mathrm{int}}(\mathbb{M}) := \sigma(M_1)^{\ell_1} \cdots \sigma(M_n)^{\ell_n}$.

Proposition 4. *Considering the restriction of the above pairing to the polynomial algebra $\overline{\mathrm{N}}(A)$, the derivation $\overline{\partial}$ is the transpose of the derivation ∂.*

Proof. By direct computation:

$$\begin{aligned}
\langle \partial \mathbf{x}^{\mathbf{k}}, \mathbf{x}^{\ell}\rangle &= \sum_{j\geq -1,\, a\in A} k_j^a \langle \mathbf{x}^{\mathbf{k}+\mathbf{e}_{j+1}^a - \mathbf{e}_j^a}, \mathbf{x}^{\ell}\rangle \\
&= \sum_{j\geq -1,\, a\in A} k_j^a\, \ell!\, \mathbb{1}_{(\mathbf{k}+\mathbf{e}_{j+1}^a - \mathbf{e}_j^a = \ell)} \\
&= \sum_{j\geq -1,\, a\in A} (\ell + \mathbf{e}_j^a)!\, \mathbb{1}_{(\mathbf{k}+\mathbf{e}_{j+1}^a = \ell + \mathbf{e}_j^a)}.
\end{aligned}$$

Now

$$\begin{aligned}\langle \mathbf{x}^{\mathbf{k}}, \bar\partial \mathbf{x}^{\ell}\rangle &= \sum_{j\ge 0,\, a\in A} \ell_j^a \langle \mathbf{x}^{\mathbf{k}}, \mathbf{x}^{\ell+\mathbf{e}_{j-1}^a-\mathbf{e}_j^a}\rangle\\ &= \sum_{j\ge 0,\, a\in A} \ell_j^a(\ell+\mathbf{e}_{j-1}^a-\mathbf{e}_j^a)!\, \mathbb{1}_{(\mathbf{k}=\ell+\mathbf{e}_{j-1}^a-\mathbf{e}_j^a)}\\ &= \sum_{j\ge 0,\, a\in A} (\ell+\mathbf{e}_{j-1}^a)!\, \mathbb{1}_{(\mathbf{k}+\mathbf{e}_j^a=\ell+\mathbf{e}_{j-1}^a)},\end{aligned}$$

hence both expressions coincide. □

For any multi-index $\mathbf{k}$, let us denote by $\overleftarrow{\mathbf{k}}$ the corresponding left-shifted multi-index, with coordinates $\overleftarrow{k}_j^a := k_{j+1}^a$. In particular, we have

$$\overleftarrow{\mathbf{e}_j^a} = \mathbf{e}_{j-1}^a.$$

Proposition 5. *For any integer $r\ge 0$ and for any multi-index $\mathbf{k}$ we have*

$$\bar\partial^r \mathbf{x}^{\mathbf{k}} = \sum_{|\ell|=r} C_{\mathbf{k},\ell}\mathbf{x}^{\mathbf{k}-\ell+\overleftarrow{\ell}}, \tag{16}$$

where the coefficients $C_{\mathbf{k},\ell}$ are given by $C_{\mathbf{k},0}=1$ and the recursive formula below:

$$C_{\mathbf{k},\ell} = \sum_{j\ge -1,\, a\in A,\, \ell_j\ge 1} C_{\mathbf{k},\ell-\mathbf{e}_j^a}(k_j^a-\ell_j^a+1+\ell_{j+1}^a). \tag{17}$$

Proof. The definition of $\overline{\partial}$ immediately yields

$$C_{\mathbf{k},\mathbf{e}_j^a} = k_j^a. \tag{18}$$

We therefore can proceed inductively and compute:

$$\begin{aligned}\overline{\partial}^{r+1}\mathbf{x}^{\mathbf{k}} &= \sum_{|\ell'|=r} C_{\mathbf{k},\ell'}\overline{\partial}\mathbf{x}^{\mathbf{k}+\overleftarrow{\ell'}-\ell'}\\ &= \sum_{|\ell'|=r}\sum_{j\ge 0,\, a\in A} C_{\mathbf{k},\ell'}(k_j^a+\ell'^a_{j+1}-\ell'^a_j)\mathbf{x}^{\mathbf{k}+(\overleftarrow{\ell'}+\overleftarrow{\mathbf{e}_j^a})-(\ell'+\mathbf{e}_j^a)}\\ &= \sum_{|\ell|=r+1}\sum_{(\ell',j,a),\, \ell'+\mathbf{e}_j^a=\ell} C_{\mathbf{k},\ell-\mathbf{e}_j^a}(k_j^a+\ell_{j+1}^a-\ell_j^a+1)\mathbf{x}^{\mathbf{k}+\overleftarrow{\ell}-\ell},\end{aligned}$$

which proves the claim. □

Proposition 6. *Let $(M_1,\ldots,M_r)$ be an r-tuple of monomials in $N(A)$, and let $\mathbb{M}=M_1\odot\cdots\odot M_r$ be the monomial of monomials obtained by multiplying the M_j's together. Let F be a rooted decorated forest such that $\Phi(F)=\mathbb{M}$, and let $\mathcal{A}$ be the set of r-tuples of decorated rooted trees given by*

$$\mathcal{A} := \{(t_1,\ldots,t_r),\, t_1\cdots t_r = F \text{ and } \Phi(t_j)=M_j \text{ for any } j=1,\ldots,r\}.$$

Then we have

$$|\mathcal{A}| = \frac{\sigma^{\text{ext}}(\mathbb{M})}{\sigma^{\text{ext}}(F)}.$$

Proof. The external automorphism group of $\mathbb{M}$ acts transitively on $\mathcal{A}$. The stabiliser of the r-tuple $(t_1,\ldots,t_r)$ is the external automorphism group of the forest F. One concludes by the orbit-stabiliser theorem. □

Remark 7. An alternative choice of symmetry factors is frequently used in the literature: in [8] (see also [28, Lemma 6.1] and [7, Section 2]), the authors use (translated in our notations and context)

$$\widetilde{\sigma}(\mathbf{x}^{\mathbf{k}}) = \prod_{a\in A,\, j\geq -1} \big((j+1)!\big)^{k_j^a}.$$

Considering that a variable x_j^a corresponds to a vertex decorated by a with $j+1$ edges above it, one may consider that it has an intrinsic symmetry factor $(j+1)!$. For a multi-index $\mathbf{k} = (k_j^a)_{a\in A,\, j\geq -1}$, the symmetry factor above can therefore be considered as the internal part of a symmetry factor

$$\widehat{\sigma}(\mathbf{x}^{\mathbf{k}}) := \sigma(\mathbf{x}^{\mathbf{k}})\widetilde{\sigma}(\mathbf{x}^{\mathbf{k}}) = \mathbf{k}! \prod_{a\in A,\, j\geq -1} \big((j+1)!\big)^{k_j^a},$$

our $\mathbf{k}!$ being the external part. Let us recall that our definition of the derivation ∂ on $\overline{\mathrm{N}}(A)$ coincides with the one adopted in [7], but differs from [28, Paragraph 3.2], where the authors choose

$$\partial x_j^a = (j+1)x_{j+1}^a.$$

3.2. **The embedding.** Define the embedding $\jmath : \mathcal{H}^A_{\mathrm{LOT}} \hookrightarrow \mathcal{H}^A_{\mathrm{BCK}}$ by transposing the multiplicatively extended fertility map Φ. This is an injective Hopf algebra morphism. From $\langle \Phi(t), \mathbf{x}^{\mathbf{k}}\rangle = \langle t, \jmath(\mathbf{x}^{\mathbf{k}})\rangle$ for any rooted tree t and any monomial $\mathbf{x}^{\mathbf{k}}$ of weight -1, and considering the definitions of both pairings, we easily get

$$\jmath(\mathbf{x}^{\mathbf{k}}) = \sum_{t,\, \Phi(t)=\mathbf{x}^{\mathbf{k}}} \frac{\sigma(\mathbf{x}^{\mathbf{k}})}{\sigma(t)} t. \tag{19}$$

For example,

$\jmath(x_{-1}x_0) =$ , $\quad \jmath(x_{-1}^2 x_1) =$ , $\quad \jmath(x_{-1}^2 x_0 x_1) = 2$  $+$ , $\quad \jmath(x_{-1}^3 x_0 x_2) =$  $+ 3$ .

Lemma 8.

$$\jmath(\overline{\partial}^r \mathbf{x}^{\mathbf{k}}) = \sum_{|\ell|=r} \sum_{t,\, \Phi(t)=\mathbf{x}^{\mathbf{k}+\overleftarrow{\ell}-\ell}} \frac{D_{\mathbf{k},\ell}}{\sigma(t)} t, \tag{20}$$

with $D_{\mathbf{k},\ell} = C_{\mathbf{k},\ell}(\mathbf{k}+\overleftarrow{\ell}-\ell)!$. The coefficients $D_{\mathbf{k},\ell}$ are given by $D_{\mathbf{k},0} = \mathbf{k}!$ and the recursive formula

$$D_{\mathbf{k},\ell} = \sum_{j\geq -1,\, a\in A, \ell_j^a\geq 1} D_{\mathbf{k},\ell-\mathbf{e}_j^a}(k_{j-1}^a - \ell_{j-1}^a + \ell_j^a). \tag{21}$$

Proof. Equation (20) is a direct consequence of (16) and (19). From (17) and the expression of $D_{\mathbf{k},\ell}$ in terms of $C_{\mathbf{k},\ell}$, we can compute:

$$\begin{aligned} D_{\mathbf{k},\ell} &= \sum_{j\geq -1,\, a\in A,\, \ell_j^a\geq 1} C_{\mathbf{k},\ell-\mathbf{e}_j^a}(k_j^a - \ell_j^a + 1 + \ell_{j+1}^a)(\mathbf{k}+\overleftarrow{\ell}-\ell)! \\ &= \sum_{j\geq -1,\, a\in A,\, \ell_j^a\geq 1} D_{\mathbf{k},\ell-\mathbf{e}_j^a} \frac{(\mathbf{k}+\overleftarrow{\ell}-\ell)!}{(\mathbf{k}+\overleftarrow{\ell}-\mathbf{e}_{j-1}^a-\ell+\mathbf{e}_j^a)!}(k_j^a - \ell_j^a + 1 + \ell_{j+1}^a) \\ &= \sum_{j\geq -1,\, a\in A, \ell_j^a\geq 1} D_{\mathbf{k},\ell-\mathbf{e}_j^a}(k_{j-1}^a - \ell_{j-1}^a + \ell_j^a). \end{aligned}$$

□

3.3. **Cocycles and mock-cocycles.** Recall the cocycle operator $B_+^a : S\big(\mathrm{PL}(A)\big) \to \mathrm{PL}(A)$ for any $a \in A$, which grafts all trees of a given forest on a common root decorated by a. Any A-decorated rooted tree t is uniquely given by $t = B_+^a(F)$, where F is the forest obtained from t by removing the root, and where a is the decoration of the root of t. Its transpose $B_-^a : \mathcal{H}_{\mathrm{BCK}}^A \to \mathcal{H}_{\mathrm{BCK}}^A$ is given by $B_-^a(t) = F$ if $t = B_+^a(F)$, and by $B_-^a(t) = 0$ if the decoration of the root of t is different from a.

Now consider, for any $a \in A$, the operator $L^a : S\big(\mathrm{N}(A)\big) \to \mathrm{N}(A)$ defined by

$$L^a(\mathbf{x}^{\mathbf{k}^1} \odot \cdots \odot \mathbf{x}^{\mathbf{k}^r}) := \mathbf{x}^{\mathbf{k}^1+\cdots+\mathbf{k}^r} x_{r-1}^a.$$

We obviously have

$$\Phi \circ B_+^a = L^a \circ \Phi. \tag{22}$$

Its transpose $\overline{L}^a : \mathcal{H}_{\mathrm{LOT}}^A \to \mathcal{H}_{\mathrm{LOT}}^A$ is given by

$$\overline{L}^a(\mathbf{x}^{\mathbf{k}}) = \sum_{\mathbb{M},\, L^a(\mathbb{M})=\mathbf{x}^{\mathbf{k}}} \frac{\sigma(\mathbf{x}^{\mathbf{k}})}{\sigma(\mathbb{M})} \mathbb{M}.$$

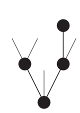
From (22) we immediately get

$$\jmath \circ \overline{L}^a = B_-^a \circ \jmath. \tag{23}$$

4. Free edges, graftings and cuts

4.1. **Rooted trees and forests with free edges.** The free Novikov $N(A)$ is included inside a wider Novikov algebra $\overline{N}(A)$. Similarly, the free pre-Lie algebra $\mathrm{PL}(A)$ is included in a wider pre-Lie algebra $\overline{\mathrm{PL}}(A)$ defined as the linear span of rooted trees with free edges, i.e. edges without upper vertex. It can be seen as the free pre-Lie algebra generated by $A \times \mathbb{N}_0$, where the second component of the decoration of a given vertex indicates the number of free edges attached to it. The idea of considering free edges is borrowed from the Feynman diagrams designed to describe interactions between elementary particles. These graphs contain free edges, also known as external edges, excepted the so-called vacuum graphs describing creation-annihilation of virtual particles. The reader can refer to any textbook in quantum field theory, see also [3].

The linear span of rooted forests with free edges will be denoted by $\overline{\mathcal{H}}_{\mathrm{BCK}}^A$. The weight $\mathrm{wt}(t)$ of t is given by the total number of edges minus the number of vertices. The pairing is naturally extended to $\overline{\mathcal{H}}_{\mathrm{BCK}}^A$, where the symmetry factor of a forest with free edges is understood as the symmetry factor of the corresponding $A \times \mathbb{N}_0$-decorated forest. We obviously have $\mathrm{wt}(t) = -1$ for an ordinary rooted tree without free edges.



A rooted tree of weight 3, with four free edges.

Formula (13) extends the fertility map Φ to a map $\overline{\Phi} : \overline{\mathrm{PL}}(A) \to \overline{N}(A)$ which respects the weight. Similarly, by transposing the map $\overline{\Phi}$, Formula (19) extends the embedding $\jmath$ to a map

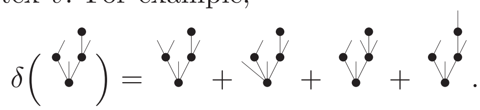
$$\overline{\jmath} : \overline{\mathcal{H}}_{\mathrm{LOT}}^A \longrightarrow \overline{\mathcal{H}}_{\mathrm{BCK}}^A,$$

where $\overline{\mathcal{H}}_{\mathrm{LOT}}^A$ stands for the commutative algebra $S\big(\overline{N}(A)\big)^\circ$. We remark that $\overline{\jmath}$ is not an embedding, as it sends any monomial $\mathbf{x}^{\mathbf{k}}$ of weight ≤ -2 to zero.

Let $\delta : \overline{\mathrm{PL}}(A) \to \overline{\mathrm{PL}}(A)$ be the map given by

$$\delta(t) = \sum_{v \in \mathcal{V}(t)} \delta_v(t),$$

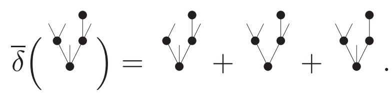
where δ_v adds a free edge to vertex v. For example,



Let us use the symbol $\overline{\delta}$ for the map given by

$$\overline{\delta}(t) = \sum_{v \in \mathcal{V}(t)} \overline{\delta}_v(t),$$

where $\overline{\delta}_v$ removes a free edge at vertex v, and returns zero if v has no free edge. For example,



Proposition 9. *The map $\overline{\delta}$ is the transpose of δ, and we have*

$$\overline{\Phi} \circ \delta = \partial \circ \overline{\Phi}. \tag{24}$$

Proof. Choosing a vertex v (resp. w) in a tree t (resp. u), and considering both sets

$$P := \{v \in \mathcal{V}(t),\, \delta_v(t) = u\}, \qquad Q := \{w \in \mathcal{V}(u),\, \overline{\delta}_w(u) = t\},$$

the orbit-stabiliser theorem states

$$P \simeq \mathrm{Aut}(t)/\,\mathrm{Aut}_v(t), \qquad Q \simeq \mathrm{Aut}(u)/\,\mathrm{Aut}_w(u),$$

where v (resp. w) is any choice of element in P (resp. Q), and where $\mathrm{Aut}_v(t)$ is the stabiliser of v in $\mathrm{Aut}(t)$ (and similarly for Q). If two trees t and u are such that $u = \delta_v(t)$ for some $v \in \mathcal{V}(t)$, then both sets of vertices $\mathcal{V}(t)$ and $\mathcal{V}(u)$ can be naturally identified. In that case, from the obvious isomorphism $\mathrm{Aut}_v(t) \simeq \mathrm{Aut}_v(u)$ we get

$$\sigma(u)|P| = \sigma(t)|Q|.$$

We therefore have

$$\begin{aligned} \langle \delta(t), u \rangle &= |P|\sigma(u) \\ &= |Q|\sigma(t) \\ &= \langle t, \overline{\delta}(u) \rangle, \end{aligned}$$

which proves the first assertion. The second assertion comes from the fact that adding a free edge to a vertex corresponds to shifting the variable associated to it. □

By transposing (24), we immediately get

Corollary 10.

$$\overline{\delta} \circ \overline{\jmath} = \overline{\jmath} \circ \overline{\partial}. \tag{25}$$

Proposition 11. *Let t be a rooted A-decorated tree with r free edges. Let r_v be the number of free edges at a given vertex v of t. The formula for the r-th power of $\overline{\delta}$ is*

$$\overline{\delta}^r(t) = \frac{r!}{\prod_{v \in \mathcal{V}(t)} r_v!} t_0,$$

where t_0 is the tree t with all free edges removed.

Proof. We have

$$\overline{\delta}^r(t) = \sum_{(v_1,\ldots,v_r)\in\mathcal{V}(t)^r} \overline{\delta}_{v_1}\circ\cdots\circ\overline{\delta}_{v_r}(t).$$

A term in the right-hand side is equal to t_0 if and only if each vertex $v\in\mathcal{V}(t)$ appears exactly r_v times in the tuple $(v_1,\ldots,v_r)$, otherwise the term is equal to zero. The number of such r-tuples is equal to the multinomial coefficient above. □

For any admissible cut c of a tree $t\in \mathrm{PL}(A)$, let $\overline{R}_c(t)$ be the associated *full trunk*, which is given by the trunk $R^c(t)$ together with each cut edge replaced by a free edge. The weight of the full trunk is therefore equal to $r-1$, where r is the number of edges belonging to the cut. In view of Proposition 11, we have

$$\overline{\delta}^r\big(\overline{R}^c(t)\big) = \|\overline{t}\| R^c(t), \tag{26}$$

where $\overline{t}$ is a shorthand for $\overline{R}^c(t)$, and

$$\|\overline{t}\| := \frac{r!}{\prod_{v\in\mathcal{V}(\overline{t})} r_v!}. \tag{27}$$

4.2. **Graftings and cuts.**

Definition 1. *Let r be a positive integer, let F be an A-decorated rooted forest with r connected components without free edges, and let $\overline{t}$ be an A-decorated rooted tree with r_v free edges at vertex v, and a total number of r free edges. A grafting of F on $\overline{t}$ consists in grafting a choice of r_v connected components of F on the vertex v for any $v\in\mathcal{V}(\overline{t})$, and by removing the free edges.*

Denoting by $\mathcal{G}(F,\overline{t})$ the set of graftings of F on $\overline{t}$, we obviously have

$$\big|\mathcal{G}(F,\overline{t})\big| = \|\overline{t}\|. \tag{28}$$

For any vertex $x\in\mathcal{V}(\overline{t})$ and for any grafting $b\in\mathcal{G}(F,\overline{t})$, let us denote by $F_b(x)$ the subforest of F attached to x via b.

Proposition 12. *Let r be a positive integer, let F and $\overline{t}$ as in Definition 1, and let t be an A-decorated rooted tree without free edges. Let $\mathcal{G}(t,F,\overline{t})$ be the set of graftings of F on $\overline{t}$ resulting in the tree t. Let $\mathcal{C}(t,F,\overline{t})$ be the set of admissible cuts of t such that $P^c(t)=F$ and $\overline{R}^c(u)=\overline{t}$. Then*

$$|\mathcal{C}(t,F,\overline{t})| = \frac{\sigma(t)}{\sigma(F)\sigma(\overline{t})}|\mathcal{G}(t,F,\overline{t})|.$$

Proof. The group $\mathrm{Aut}\, t$ acts transitively on $\mathcal{C}(t,F,\overline{t})$. The stabiliser of a cut c will be denoted by $\mathrm{Aut}_c\, t$, and its cardinal by $\sigma_c(t)$. On the other hand, the group $\mathrm{Aut}\,\overline{t}\times\mathrm{Aut}\,F$ acts transitively on $\mathcal{G}(t,F,\overline{t})$. To see this, consider two graftings $b,b'\in\mathcal{G}(t,F,\overline{t})$: there exists a permutation α of $\mathcal{V}(\overline{t})$ such that $F_b(x)$ and $F_{b'}\big(\alpha(x)\big)$ are isomorphic, and the permutation α necessarily comes from an automorphism of the tree $\overline{t}$. The stabiliser of b is $\mathrm{Aut}_b\,\overline{t}\times\mathrm{Aut}_b\,F$, where $\mathrm{Aut}_b\,\overline{t}$ is the subgroup of those $\alpha\in\mathrm{Aut}\,\overline{t}$ such that $F_b\big(\alpha(x)\big)$ and $F_b(x)$ are isomorphic for any vertex x of $\overline{t}$, and where $\mathrm{Aut}_b\,F=\prod_{x\in\mathcal{V}(\overline{t})}\mathrm{Aut}\,F_b(x)$ is the subgroup of $\mathrm{Aut}\,F$ which respects the subforests $F_b(x)$. By the orbit-stabiliser theorem, we therefore have

$$\frac{|\mathcal{C}(t,F,\overline{t})|}{|\mathcal{G}(t,F,\overline{t})|} = \frac{\sigma(t)}{|\mathrm{Aut}_c(t)|}\frac{|\mathrm{Aut}_b(\overline{t})|.|\mathrm{Aut}_b(F)|}{\sigma(F)\sigma(\overline{t})}$$

We conclude by noticing the obvious isomorphism

$$\mathrm{Aut}_c(t)\sim\mathrm{Aut}_b(\overline{t})\times\mathrm{Aut}_b(F).$$

□

5. Explicit description of the coproduct

5.1. **A recursive formula for the coproduct.** Recall that the coproduct in $\mathcal{H}^A_{\mathrm{BCK}}$ admits a recursive definition, with respect to the degree, in terms of the cocycle operators B^a_+. This can be rewritten in terms of the operators B^a_- as follows:

$$\Delta_{\mathrm{BCK}}(t) = \sum_{a\in A}(I\otimes B^a_+)\Delta_{\mathrm{BCK}}\big(B^a_-(t)\big) + t\otimes \mathbf{1}. \tag{29}$$

We can now compute, using the Hopf algebra morphism property for $\jmath$:

$$\begin{aligned}
(\jmath\otimes\jmath)\circ\Delta_{\mathrm{LOT}}(\mathbf{x}^{\mathbf{k}}) &= \Delta_{\mathrm{BCK}}\circ\jmath(\mathbf{x}^{\mathbf{k}})\\
&\underset{(29)}{=} \sum_{a\in A}(I\otimes B^a_+)\Delta_{\mathrm{BCK}}\big(B^a_-\circ\jmath(\mathbf{x}^{\mathbf{k}})\big) + \jmath(\mathbf{x}^{\mathbf{k}})\otimes\mathbf{1}\\
&\underset{(23)}{=} \sum_{a\in A}(I\otimes B^a_+)\Delta_{\mathrm{BCK}}\circ\jmath\big(\overline{L}^a(\mathbf{x}^{\mathbf{k}})\big) + \jmath(\mathbf{x}^{\mathbf{k}})\otimes\mathbf{1}\\
&= \sum_{a\in A}(I\otimes B^a_+)\circ(\jmath\otimes\jmath)\circ\Delta_{\mathrm{LOT}}\big(\overline{L}^a(\mathbf{x}^{\mathbf{k}})\big) + \jmath(\mathbf{x}^{\mathbf{k}})\otimes\mathbf{1}. \qquad (30)
\end{aligned}$$

Remark: The map $B^a_+\circ\jmath$ cannot be easily expressed like (23) with $\jmath$ on the left: the above recursive expression therefore cannot be further simplified.

5.2. **An explicit formula for the coproduct.** Let us first define an *admissible cut* of the monomial $\mathbf{x}^{\mathbf{k}}$ as a choice of cutting $\mathbf{k}$ into $r+1$ multi-indices $\mathbf{k}^1,\ldots,\mathbf{k}^r,\overline{\mathbf{k}}$ for some integer $r\ge 0$, with $\mathrm{wt}(\mathbf{k}^j)=-1$ for any $j=1,\ldots,r$ and $\mathrm{wt}\,\overline{\mathbf{k}} = r-1$, so that

$$\mathbf{x}^{\mathbf{k}} = \mathbf{x}^{\mathbf{k}^1}\cdots\mathbf{x}^{\mathbf{k}^r}\mathbf{x}^{\overline{\mathbf{k}}}.$$

In analogy with rooted forests, we set

$$P^{\mathbf{c}}(\mathbf{x}^{\mathbf{k}}) := \mathbf{x}^{\mathbf{k}^1}\odot\cdots\odot\mathbf{x}^{\mathbf{k}^r}, \qquad R^{\mathbf{c}}(\mathbf{x}^{\mathbf{k}}) := \overline{\partial}^r\mathbf{x}^{\overline{\mathbf{k}}}.$$

In view of this, we shall denote by $|\mathbf{c}|$ the set of admissible cuts $\mathbf{c}'$ such that $P^{\mathbf{c}'}(\mathbf{x}^{\mathbf{k}}) = P^{\mathbf{c}}(\mathbf{x}^{\mathbf{k}})$ (and therefore $R^{\mathbf{c}'}(\mathbf{x}^{\mathbf{k}}) = R^{\mathbf{c}}(\mathbf{x}^{\mathbf{k}})$), and by $\|\mathbf{c}\|$ the cardinal of this class. We clearly have

$$\|\mathbf{c}\| = \frac{\mathbf{k}!}{\overline{\mathbf{k}}!\sigma(\mathbf{x}^{\mathbf{k}^1}\odot\cdots\odot\mathbf{x}^{\mathbf{k}^r})} = \frac{\mathbf{k}!}{\mathbf{k}^1!\cdots\mathbf{k}^r!\overline{\mathbf{k}}!\,\sigma^{\mathrm{ext}}(\mathbf{x}^{\mathbf{k}^1}\odot\cdots\odot\mathbf{x}^{\mathbf{k}^r})},$$

whenever $P^{\mathbf{c}}(\mathbf{x}^{\mathbf{k}}) = \mathbf{x}^{\mathbf{k}^1}\odot\cdots\odot\mathbf{x}^{\mathbf{k}^r}$, where σ^{ext} is the external symmetry factor (see Remark 3). In analogy with trees containing free edges, the *full trunk* will be defined by

$$\overline{R}^{\mathbf{c}}(\mathbf{x}^{\mathbf{k}}) := \mathbf{x}^{\overline{\mathbf{k}}}.$$

Definition 2. *Let $\mathbf{c}$ be an admissible cut of the monomial $\mathbf{x}^{\mathbf{k}}$, and let c be an admissible cut of the decorated rooted tree t. We say that c matches $\mathbf{c}$ and write $c\sim\mathbf{c}$ whenever*

- $\mathbf{x}^{\mathbf{k}} = \Phi(t)$,
- $P^{\mathbf{c}}(\mathbf{x}^{\mathbf{k}}) = \Phi\big(P^c(t)\big)$.

An admissible cut c matches $\mathbf{c}$ if and only if it matches any element $\mathbf{c}'\in|\mathbf{c}|$. Note that the second condition implies $\overline{R}^{\mathbf{c}}(\mathbf{x}^{\mathbf{k}}) = \overline{\Phi}\big(\overline{R}^c(t)\big)$.

Lemma 13. *For any monomial $\mathbf{x}^{\mathbf{k}}$ in $\mathrm{N}(A)$ and for any admissible cut $\mathbf{c}$ of $\mathbf{x}^{\mathbf{k}}$, the following holds:*

$$\|\mathbf{c}\|(\jmath\otimes\jmath)\Big(P^{\mathbf{c}}(\mathbf{x}^{\mathbf{k}})\otimes R^{\mathbf{c}}(\mathbf{x}^{\mathbf{k}})\Big) = \sum_{t,\,\Phi(t)=\mathbf{x}^{\mathbf{k}}}\ \sum_{c\in\mathrm{Adm}(t),\,c\sim\mathbf{c}} \frac{\sigma(\mathbf{x}^{\mathbf{k}})}{\sigma(t)}P^c(t)\otimes R^c(t). \tag{31}$$

Proof. Fix an admissible cut $\mathbf{c}$ of the monomial $\mathbf{x}^{\mathbf{k}}$. Denoting the left-hand side and the right-hand side of (31) by $\mathcal{L}$ and $\mathcal{R}$ respectively, we can compute:

$$\begin{aligned}
\mathcal{L} &= \frac{\mathbf{k}!}{\mathbf{k}^1!\cdots\mathbf{k}^r!\overline{\mathbf{k}}!\,\sigma^{\mathrm{ext}}(\mathbf{x}^{\mathbf{k}^1}\odot\cdots\odot\mathbf{x}^{\mathbf{k}^r})}\jmath(\mathbf{x}^{\mathbf{k}^1})\cdots\jmath(\mathbf{x}^{\mathbf{k}^r})\otimes\jmath\circ\overline{\partial}^r(\mathbf{x}^{\overline{\mathbf{k}}})\\
&= (\mathrm{Id}\otimes\overline{\delta}^r)\left(\frac{\mathbf{k}!}{\mathbf{k}^1!\cdots\mathbf{k}^r!\overline{\mathbf{k}}!\,\sigma^{\mathrm{ext}}(\mathbf{x}^{\mathbf{k}^1}\odot\cdots\odot\mathbf{x}^{\mathbf{k}^r})}\jmath(\mathbf{x}^{\mathbf{k}^1})\cdots\jmath(\mathbf{x}^{\mathbf{k}^r})\otimes\jmath(\mathbf{x}^{\overline{\mathbf{k}}})\right)\ \text{(from Corollary 10)}\\
&= \frac{\mathbf{k}!}{\mathbf{k}^1!\cdots\mathbf{k}^r!\overline{\mathbf{k}}!\,\sigma^{\mathrm{ext}}(\mathbf{x}^{\mathbf{k}^1}\odot\cdots\odot\mathbf{x}^{\mathbf{k}^r})}(\mathrm{Id}\otimes\overline{\delta}^r)\left(\sum_{\substack{(t^j)_{1,\ldots,r,}\\ \Phi(t^j)=\mathbf{x}^{\mathbf{k}^j}}}\ \sum_{\overline{t},\,\Phi(\overline{t})=\mathbf{x}^{\overline{\mathbf{k}}}}\frac{\mathbf{k}^1!}{\sigma(t^1)}\cdots\frac{\mathbf{k}^r!}{\sigma(t^r)}\frac{\overline{\mathbf{k}}!}{\sigma(\overline{t})}\,t^1\cdots t^r\otimes\overline{t}\right)\\
&= \frac{\mathbf{k}!}{\sigma^{\mathrm{ext}}(\mathbf{x}^{\mathbf{k}^1}\odot\cdots\odot\mathbf{x}^{\mathbf{k}^r})}(\mathrm{Id}\otimes\overline{\delta}^r)\left(\sum_{F,\,\Phi(F)=\mathbf{x}^{\mathbf{k}^1}\odot\cdots\odot\mathbf{x}^{\mathbf{k}^r}}\ \sum_{\overline{t},\,\Phi(\overline{t})=\mathbf{x}^{\overline{\mathbf{k}}}}\frac{\sigma^{\mathrm{ext}}(\mathbf{x}^{\mathbf{k}^1}\odot\cdots\odot\mathbf{x}^{\mathbf{k}^r})}{\sigma^{\mathrm{ext}}(F)}\frac{1}{\sigma^{\mathrm{int}}(F)\sigma(\overline{t})}\,F\otimes\overline{t}\right)\\
&\qquad \text{(from Proposition 6)}\\
&= \mathbf{k}!(\mathrm{Id}\otimes\overline{\delta}^r)\left(\sum_{F,\,\Phi(F)=\mathbf{x}^{\mathbf{k}^1}\odot\cdots\odot\mathbf{x}^{\mathbf{k}^r}}\ \sum_{\overline{t},\,\Phi(\overline{t})=\mathbf{x}^{\overline{\mathbf{k}}}}\frac{1}{\sigma(F)\sigma(\overline{t})}\,F\otimes\overline{t}\right).
\end{aligned}$$

From (26), (28) and Proposition 12, we therefore get

$$\begin{aligned}
\mathcal{L} &= \mathbf{k}!\sum_{F,\,\Phi(F)=\mathbf{x}^{\mathbf{k}^1}\odot\cdots\odot\mathbf{x}^{\mathbf{k}^r}}\ \sum_{\overline{t},\,\Phi(\overline{t})=\mathbf{x}^{\overline{\mathbf{k}}}}\|\overline{t}\|\frac{1}{\sigma(F)\sigma(\overline{t})}\,F\otimes\overline{t}_0\\
&= \mathbf{k}!\sum_{F,\,\Phi(F)=\mathbf{x}^{\mathbf{k}^1}\odot\cdots\odot\mathbf{x}^{\mathbf{k}^r}}\ \sum_{\overline{t},\,\Phi(\overline{t})=\mathbf{x}^{\overline{\mathbf{k}}}}\frac{|\mathcal{G}(F,\overline{t})|}{\sigma(F)\sigma(\overline{t})}\,F\otimes\overline{t}_0\\
&= \mathbf{k}!\sum_{t,\,\Phi(t)=\mathbf{x}^{\mathbf{k}}}\ \sum_{F,\,\Phi(F)=\mathbf{x}^{\mathbf{k}^1}\odot\cdots\odot\mathbf{x}^{\mathbf{k}^r}}\ \sum_{\overline{t},\,\Phi(\overline{t})=\mathbf{x}^{\overline{\mathbf{k}}}}\frac{|\mathcal{G}(t,F,\overline{t})|}{\sigma(F)\sigma(\overline{t})}\,F\otimes\overline{t}_0\\
&= \mathbf{k}!\sum_{t,\,\Phi(t)=\mathbf{x}^{\mathbf{k}}}\ \sum_{F,\,\Phi(F)=\mathbf{x}^{\mathbf{k}^1}\odot\cdots\odot\mathbf{x}^{\mathbf{k}^r}}\ \sum_{\overline{t},\,\Phi(\overline{t})=\mathbf{x}^{\overline{\mathbf{k}}}}\frac{|\mathcal{C}(t,F,\overline{t})|}{\sigma(t)}\,F\otimes\overline{t}_0\\
&= \sum_{t,\,\Phi(t)=\mathbf{x}^{\mathbf{k}}}\frac{\mathbf{k}!}{\sigma(t)}\sum_{c\in\mathrm{Adm}(t),\,c\sim\mathbf{c}}P^c(t)\otimes R^c(t)\\
&= \mathcal{R}.
\end{aligned}$$

□

Theorem 14. *The coproduct* $\Delta_{\mathrm{LOT}}:\mathcal{H}^A_{\mathrm{LOT}}\to\mathcal{H}^A_{\mathrm{LOT}}\otimes\mathcal{H}^A_{\mathrm{LOT}}$ *is the unique unital algebra morphism defined on the monomials by*

$$\Delta_{\mathrm{LOT}}(\mathbf{x}^{\mathbf{k}})=\sum_{r\geq 0}\ \sum_{\mathbf{k}=\mathbf{k}^1+\cdots+\mathbf{k}^r+\overline{\mathbf{k}},\,\mathrm{wt}(\mathbf{k}^j)=-1}\frac{\mathbf{k}!}{\sigma(\mathbf{x}^{\mathbf{k}^1}\odot\cdots\odot\mathbf{x}^{\mathbf{k}^r})\overline{\mathbf{k}}!}\mathbf{x}^{\mathbf{k}^1}\odot\cdots\odot\mathbf{x}^{\mathbf{k}^r}\otimes\overline{\partial}^r\mathbf{x}^{\overline{\mathbf{k}}},\tag{32}$$

where the inner sum runs over multisets $\{\mathbf{k}^1,\ldots,\mathbf{k}^r\}$ *of multi-indices such that* $\mathrm{wt}(\mathbf{k}^j)=-1$ *for any* $j=1,\ldots,r$ *and such that the remainder* $\overline{\mathbf{k}}$*, of weight* $r-1$*, has only nonnegative components* $\overline{k}^a_j$*. Formula (32) for the coproduct admits the following alternative presentation:*

$$\Delta_{\mathrm{LOT}}(\mathbf{x}^{\mathbf{k}})=\sum_{\mathbf{c}\in\mathrm{Adm}(\mathbf{x}^{\mathbf{k}})}P^{\mathbf{c}}(\mathbf{x}^{\mathbf{k}})\otimes R^{\mathbf{c}}(\mathbf{x}^{\mathbf{k}}),\tag{33}$$

Proof. By applying Lemma 13 and summing over all classes $|\mathbf{c}|$ of admissible cuts $\mathbf{c}$ of the monomial $\mathbf{x}^{\mathbf{k}}$, the coproduct defined by (32), or equivalently by (33), which we denote temporarily by Δ'_{LOT}, verifies

$$(\jmath \otimes \jmath) \circ \Delta'_{\mathrm{LOT}} = \Delta_{\mathrm{BCK}} \circ \jmath.$$

In view of the injectivity of $\jmath$, it therefore coincides with Δ_{LOT}, and the theorem is proven. □

Let us illustrate Theorem 14 on an example: applying (32) to the monomial $x_{-1}^2 x_0 x_1$ yields

$$\begin{aligned}
\Delta_{\mathrm{LOT}}(x_{-1}^2 x_0 x_1) &= x_{-1}^2 x_0 x_1 \otimes \mathbf{1} + \mathbf{1} \otimes x_{-1}^2 x_0 x_1 + 2x_{-1} \otimes \overline{\partial}(x_{-1}x_0x_1) + 2x_{-1}x_0 \otimes \overline{\partial}(x_{-1}x_1) \\
&\quad + x_{-1}^2 x_1 \otimes \overline{\partial} x_0 + x_{-1} \odot x_{-1} \otimes \overline{\partial}^2 (x_0 x_1) + 2x_{-1} \odot x_{-1} x_0 \otimes \overline{\partial}^2 x_1 \\
&= x_{-1}^2 x_0 x_1 \otimes \mathbf{1} + \mathbf{1} \otimes x_{-1}^2 x_0 x_1 + 2x_{-1} \otimes x_{-1}^2 x_1 + 2x_{-1} \otimes x_{-1} x_0^2 + 2x_{-1}x_0 \otimes x_{-1}x_0 \\
&\quad + x_{-1}^2 x_1 \otimes x_{-1} + 3x_{-1} \odot x_{-1} \otimes x_{-1}x_0 + 2x_{-1} \odot x_{-1}x_0 \otimes x_{-1}.
\end{aligned}$$

We leave it to the reader to check that a repeated application of the recursive formula (30), starting from

$$\Delta_{\mathrm{LOT}}(x_{-1}) = x_{-1} \otimes \mathbf{1} + \mathbf{1} \otimes x_{-1}$$

and computing successively the coproducts of $x_{-1}x_0$, $x_{-1}x_0^2$, $x_{-1}^2 x_1$ and $x_{-1}^2 x_0 x_1$, gives the same result. The reader is invited to compare with Example 3.2 in [8]. The differences between their coefficients and ours comes from the different convention for the symmetry factor.

6. Extraction-contraction

After recalling from [12] the extraction-contraction coproduct of (undecorated) rooted forests, we define in this section an extraction-contraction coproduct for multi-indices by Formula (37) below, and we prove (Theorem 15) that the injection $\jmath$ previously defined is a coalgebra morphism for both extraction-contraction coproducts. As a consequence $\jmath$ is a morphism of comodule-bialgebras, i.e. respects both extraction-contraction coproducts in addition to both Hopf algebra structures.

6.1. **Reminder on extraction-contraction of rooted forests.** Recall from [12] that a second coproduct Γ on the algebra of (non-decorated) rooted forests makes it a commutative bialgebra, on which the Hopf algebra $\mathcal{H}_{\mathrm{BCK}}^{\{*\}}$ is a comodule-bialgebra. In particular, both coproducts are linked by a cointeraction diagram, dual to left distributivity. It is given by

$$\Gamma(t) = \sum_{s \subseteq t} s \otimes t/s, \tag{34}$$

where the sum runs over the *covering subforests* of the forest t. A covering subforest is a partition of the set of vertices of t into connected blocks, i.e. blocks in which any vertex can be reached from another by following edges of t. The notation t/s stands for the corresponding contracted forest, obtained from s by shrinking each block to a single vertex. A decorated version of this picture is available provided a commutative semigroup structure on the set of decorations is given, in order to decide how to decorate the vertices of the contracted forest.

6.2. **Free edges and extraction-contraction for multi-indices.** Formula (34) can be precised as follows: the extraction of a covering subforest $s \subseteq t$ with $r(s)+1$ connected components gives rise to a forest $F(s)$ with $r(s)$ free edges, which are naturally in bijection with the edges of the contracted forest t/s. Note that two different covering subforests can give rise to the same forest after extraction. We denote by $F_0(s)$ the same forest without its free edges, given by

$$F_0(s) = \frac{1}{\|F(s)\|} \overline{\delta}^{r(s)} F(s),$$

where the coefficient $\|F(s)\|$ is given by the recipe (27). Formula (34) therefore takes the following form:

$$\Gamma(t)=\sum_{s\subseteq t}F_0(s)\otimes t/s. \tag{35}$$

Such a coproduct Γ does exist on the multi-index Hopf algebra [26]: we give in this section an explicit combinatorial formula for it, similar to (34) the same way (33) is similar to the Connes-Kreimer formula (6). We stick to the nondecorated setting $A=\{*\}$ for simplicity, but a decorated version of this picture is also available when A is endowed with a commutative semigroup structure. Details are left to the reader.

A *covering subforest* of a monomial $\mathbf{x}^{\mathbf{k}}=\prod_{j\geq -1}x_j^{k_j}$ (of weight -1) is a partition π of the corresponding multiset of variables into several multisets of total weight ≥ -1, each of them giving rise to a monomial $\mathbf{x}^{\beta^j}$ for $j=1,\ldots,r(\pi)+1$. Here $r(\pi)+1$ is the number of multisets involved. We write $\pi\subseteq\mathbf{x}^{\mathbf{k}}$ for π being a covering subforest of $\mathbf{x}^{\mathbf{k}}$. The corresponding monomial of monomials can be written as

$$\mathbb{M}(\pi)=\mathbf{x}^{\beta^1}\odot\cdots\odot\mathbf{x}^{\beta^{r(\pi)+1}},$$

with $\beta^1+\cdots+\beta^{r(\pi)+1}=\mathbf{k}$. Its *reduced form* is defined by

$$\mathbb{M}_0(\pi):=\frac{1}{\|\pi\|}\overline{\partial}^{r(\pi)}\mathbb{M}(\pi),$$

with

$$\|\pi\|=\|\mathbb{M}(\pi)\|:=\frac{r(\pi)!}{\prod_{j=1}^{r(\pi)+1}(\mathrm{wt}\,\beta^j+1)!}. \tag{36}$$

Our educated guess for the extraction-contraction coproduct on monomials is, on the model of (35):

$$\Gamma(\mathbf{x}^{\mathbf{k}})=\sum_{\pi\subseteq\mathbf{x}^{\mathbf{k}}}\mathbb{M}_0(\pi)\otimes\mathbf{x}^{\mathbf{k}}/\pi, \tag{37}$$

where the contracted monomial, of weight -1, is given by

$$\mathbf{x}^{\mathbf{k}}/\pi=\mathbf{x}^{\mathbf{k}}/\mathbb{M}(\pi):=\prod_{j=1}^{r(\pi)+1}x_{\mathrm{wt}\,\beta^j}.$$

Theorem 15. *The injection $\jmath$ is a coalgebra morphism with respect to both extraction-contraction coproducts* (34) *and* (37)*.*

The proof of Theorem 15 is postponed to Paragraph 6.4 below.

Corollary 16. *The Hopf algebra $(\mathcal{H}^{\{*\}}_{\mathrm{LOT}},.,\Delta)$ is a comodule-bialgebra over the bialgebra $(\mathcal{H}^{\{*\}}_{\mathrm{LOT}},.,\Gamma)$.*

Proof. From Theorem 15 and from the fact that $\jmath$ is an algebra morphism, the comodule-bialgebra structure of $(\mathcal{H}^{\{*\}}_{\mathrm{BCK}},.,\Delta)$ over the bialgebra $(\mathcal{H}^{\{*\}}_{\mathrm{BCK}},.,\Gamma)$ restricts to a comodule-bialgebra structure on the image of $\jmath$ (endowed with the restriction of the Connes-Kreimer admissible cut coproduct Δ) over itself (endowed with the restriction of the extraction-contraction coproduct Γ). $\square$

6.3. Insertion and extraction of forests revisited.

Definition 3. *Let F be a rooted forest with $r(F)$ free edges and $r(F)+1$ connected components, and let $\overline{t}$ be a rooted tree with $r(F)+1$ vertices and $r(F)$ edges (hence without free edges). An insertion of F inside $\overline{t}$ is a class of bijections*

$$\tau:\{\textit{edges of }\overline{t}\}\longrightarrow\{\textit{free edges of }F\}$$

that induces a bijection $\widetilde{\tau}$ from $\mathcal{V}(\overline{t})$ onto the set of connected components of F, in the sense that the restriction of τ to $E_v(\overline{t})$ (the latter denoting the set of edges of $\overline{t}$ with bottom vertex v) is the set of free *edges of some connected component of F. Two such bijections τ and τ' are in the same class if and only if $\widetilde{\tau}=\widetilde{\tau'}$ and $\tau'=\varepsilon\circ\tau$, where ε is a permutation of the free edges of F that does not change their vertex.*

Proposition 17. *Let F be a rooted forest with $r(F)$ free edges and $r(F)+1$ connected components, let $\overline{t}$ be a rooted tree with $r(F)+1$ vertices and $r(F)$ edges, and let $\mathcal{I}(F,\overline{t})$ be the set of insertions of F inside $\overline{t}$. Let us denote by e_v (resp. r_w) the number of edges of $\overline{t}$ with bottom vertex $v\in\mathcal{V}(\overline{t})$ (resp. the number of free edges of F attached to vertex $w\in\mathcal{V}(F)$). The following formula holds:*

$$|\mathcal{I}(F,\overline{t})|=\frac{\prod_{v\in\mathcal{V}(\overline{t})}e_v!}{\prod_{w\in\mathcal{V}(F)}r_w!}\sigma\big(\Phi(\overline{t})\big).$$

Proof. The product in the numerator is the total number of bijections associated to a particular bijection from $\mathcal{V}(\overline{t})$ onto the set of connected components of F. The denominator is the cardinal of each equivalence class, and the factor $\sigma\big(\Phi(\overline{t})\big)$ is the number of permutations of $\mathcal{V}(\overline{t})$ leaving invariant the fertility of each vertex. Proposition 17 follows. □

Proposition 18. *Let F be a rooted forest with $r(F)$ free edges and $r(F)+1$ connected components, let $\overline{t}$ be a rooted tree with $r(F)+1$ vertices and $r(F)$ edges, and let t be a rooted tree without free edges. Let $\mathcal{I}(t,F,\overline{t})$ be the set of insertions of F inside $\overline{t}$ such that the resulting tree is isomorphic to t, and let $\mathcal{E}(t,F,\overline{t})$ be the set of extractions of F from t with contraction $\overline{t}$, i.e. covering subforests $s\subseteq t$ such that $F(s)\sim F$ and $t/s\sim\overline{t}$. Then we have*

$$\frac{\mathcal{E}(t,F,\overline{t})}{\mathcal{I}(t,F,\overline{t})}=\frac{\sigma(t)}{\sigma(F)\sigma(\overline{t})}.$$

Proof. The group $\operatorname{Aut}t$ acts transitively on $\mathcal{E}(t,F,\overline{t})$. On the other hand, the group $\operatorname{Aut}F\times\operatorname{Aut}\overline{t}$ acts transitively on $\mathcal{I}(t,F,\overline{t})$. Any insertion ι obviously gives rise to a covering subforest $s=s(\iota)$ of t. The stabilizer of $s\in\mathcal{E}(t,F,\overline{t})$ is isomorphic to $\operatorname{Int}F(s)\times\operatorname{Aut}_s\overline{t}$, where $\operatorname{Int}F$ is the group of automorphisms of $F(s)$ leaving each connected component fixed, and where $\operatorname{Aut}_s\overline{t}$ is the group of automorphisms τ of $\overline{t}=t/s$ such that, for any $v\in\mathcal{V}(\overline{t})$, both connected components of s corresponding to v and $\tau(v)$ are isomorphic. This is precisely the stabilizer of ι. By the orbit-stabilizer theorem we therefore get

$$\begin{aligned}\frac{\mathcal{E}(t,F,\overline{t})}{\mathcal{I}(t,F,\overline{t})}&=\frac{\sigma(t)/|\operatorname{Stab}s|}{\sigma(F)\sigma(\overline{t})/|\operatorname{Stab}\iota|}\\&=\frac{\sigma(t)}{\sigma(F)\sigma(\overline{t})}.\end{aligned}$$

□

6.4. **Proof of Theorem 15.** We have

$$\begin{aligned}\mathcal{R}:=\Gamma\big(\jmath(\mathbf{x}^{\mathbf{k}})\big)&=\sum_{t,\,\Phi(t)=\mathbf{x}^{\mathbf{k}}}\frac{k!}{\sigma(t)}\Gamma(t)\\&=\sum_{t,\,\Phi(t)=\mathbf{x}^{\mathbf{k}}}\frac{k!}{\sigma(t)}\sum_{s\subseteq t}\frac{1}{\|F(s)\|}\overline{\delta}^{r(s)}F(s)\otimes t/F(s).\end{aligned}$$

On the other hand,

$$\begin{aligned}\mathcal{L}:=(\jmath\otimes\jmath)\big(\Gamma(\mathbf{x}^{\mathbf{k}})\big)&=\sum_{\pi\subseteq\mathbf{x}^{\mathbf{k}}}\frac{1}{\|\pi\|}\jmath\big(\overline{\partial}^{r(\pi)}\mathbb{M}(\pi)\big)\otimes\jmath(\mathbf{x}^{\mathbf{k}}/\pi)\\&=\sum_{\mathbb{M}}\sum_{\pi\subseteq\mathbf{x}^{\mathbf{k}},\,\mathbb{M}(\pi)=\mathbb{M}}\frac{1}{\|\pi\|}\jmath\big(\overline{\partial}^{r(\pi)}\mathbb{M}\big)\otimes\jmath(\mathbf{x}^{\mathbf{k}}/\mathbb{M}).\end{aligned}$$

Here, the external sum runs over monomials of monomials $\mathbb{M} = \mathbf{x}^{\beta^1} \odot \cdots \odot \mathbf{x}^{\beta^{r(\mathbb{M}(\pi))+1}}$, with $\mathrm{wt}\,\beta^j \geq -1$ and $\beta^1 + \cdots + \beta^{r(\mathbb{M})+1} = \mathbf{k}$. Using $r(\mathbb{M}) = r(\pi)$ whenever $\mathbb{M} = \mathbb{M}(\pi)$ and following the same lines than in the proof of Lemma 13, we compute:

$$\begin{aligned}
\mathcal{L} &= \sum_{\mathbb{M}} \frac{\mathbf{k}!}{\sigma(\mathbb{M})} \frac{1}{\|\mathbb{M}\|} \jmath\big(\overline{\partial}^{r(\mathbb{M})}\mathbb{M}\big) \otimes \jmath(\mathbf{x}^{\mathbf{k}}/\mathbb{M}) \\
&= \sum_{\mathbb{M}} \frac{\mathbf{k}!}{\sigma(\mathbb{M})} \frac{1}{\|\mathbb{M}\|} \overline{\delta}^{r(\mathbb{M})} \jmath(\mathbb{M}) \otimes \jmath(\mathbf{x}^{\mathbf{k}}/\mathbb{M}) \\
&= \sum_{\mathbb{M}} \frac{\mathbf{k}!}{\sigma(\mathbb{M})} \frac{1}{\|\mathbb{M}\|} \overline{\delta}^{r(\mathbb{M})} \Big(\jmath(\mathbf{x}^{\beta^1}) \cdots \jmath(\mathbf{x}^{\beta^{r(\mathbb{M})+1)}})\Big) \otimes \jmath(\mathbf{x}^{\mathbf{k}}/\mathbb{M}) \\
&= \sum_{\mathbb{M}} \frac{\mathbf{k}!}{\sigma(\mathbb{M})\|\mathbb{M}\|} \overline{\delta}^{r(\mathbb{M})} \sum_{t^1,\ldots,t^{r(\mathbb{M})+1},\overline{t},\, \Phi(t^j)=\mathbf{x}^{\beta^j},\, \Phi(\overline{t})=\mathbf{x}^{\mathbf{k}}/\mathbb{M}} \frac{\beta^1! \cdots \beta^{r(\mathbb{M})+1}! \sigma(\mathbf{x}^{\mathbf{k}}/\mathbb{M})}{\sigma(t^1)\cdots\sigma(t^{r(\mathbb{M})+1})\sigma(\overline{t})} t^1 \cdots t^{r(\mathbb{M})+1} \otimes \overline{t} \\
&= \sum_{\mathbb{M}} \frac{\mathbf{k}!}{\sigma(\mathbb{M})\|\mathbb{M}\|} \overline{\delta}^{r(\mathbb{M})} \sum_{F,\overline{t},\, \Phi(F)=\mathbb{M},\, \Phi(\overline{t})=\mathbf{x}^{\mathbf{k}}/\mathbb{M}} \frac{\beta^1! \cdots \beta^{r(\mathbb{M})+1}! \sigma(\mathbf{x}^{\mathbf{k}}/\mathbb{M})}{\sigma(t^1)\cdots\sigma(t^{r(\mathbb{M})+1})\sigma(\overline{t})} \frac{\sigma^{\mathrm{ext}(\mathbb{M})}}{\sigma^{\mathrm{ext}(F)}} F \otimes \overline{t} \\
&= \sum_{\mathbb{M}} \frac{\mathbf{k}!}{\|\mathbb{M}\|} \overline{\delta}^{r(\mathbb{M})} \sum_{F,\overline{t},\, \Phi(F)=\mathbb{M},\, \Phi(\overline{t})=\mathbf{x}^{\mathbf{k}}/\mathbb{M}} \frac{\sigma(\mathbf{x}^{\mathbf{k}}/\mathbb{M})}{\sigma(F)\sigma(\overline{t})} F \otimes \overline{t}.
\end{aligned}$$

From Proposition 17 and Proposition 18 we therefore get

$$\begin{aligned}
\mathcal{L} &= \sum_{\mathbb{M}} \frac{\mathbf{k}!}{\|\mathbb{M}\|} \overline{\delta}^{r(\mathbb{M})} \sum_{F,\overline{t},\, \Phi(F)=\mathbb{M},\, \Phi(\overline{t})=\mathbf{x}^{\mathbf{k}}/\mathbb{M}} \mathcal{I}(F,\overline{t}) \frac{\prod_{w\in\mathcal{V}(F)} r_w!}{\Big(\prod_{v\in\mathcal{V}(\overline{t})} e_v!\Big)\sigma\big(\Phi(\overline{t})\big)} \frac{\sigma(\mathbf{x}^{\mathbf{k}}/\mathbb{M})}{\sigma(F)\sigma(\overline{t})} F \otimes \overline{t} \\
&= \sum_{t,\, \Phi(t)=\mathbf{x}^{\mathbf{k}}} \sum_{\mathbb{M}} \frac{\mathbf{k}!}{\|\mathbb{M}\|} \overline{\delta}^{r(\mathbb{M})} \sum_{F,\overline{t},\, \Phi(F)=\mathbb{M},\, \Phi(\overline{t})=\mathbf{x}^{\mathbf{k}}/\mathbb{M}} \mathcal{I}(t,F,\overline{t}) \frac{\prod_{w\in\mathcal{V}(F)} r_w!}{\Big(\prod_{v\in\mathcal{V}(\overline{t})} e_v!\Big)\sigma\big(\Phi(\overline{t})\big)} \frac{\sigma(\mathbf{x}^{\mathbf{k}}/\mathbb{M})}{\sigma(F)\sigma(\overline{t})} F \otimes \overline{t} \\
&= \sum_{t,\, \Phi(t)=\mathbf{x}^{\mathbf{k}}} \sum_{\mathbb{M}} \frac{\mathbf{k}!}{\|\mathbb{M}\|} \overline{\delta}^{r(\mathbb{M})} \sum_{F,\overline{t},\, \Phi(F)=\mathbb{M},\, \Phi(\overline{t})=\mathbf{x}^{\mathbf{k}}/\mathbb{M}} \mathcal{E}(t,F,\overline{t}) \frac{\prod_{w\in\mathcal{V}(F)} r_w!}{\prod_{v\in\mathcal{V}(\overline{t})} e_v!} \frac{1}{\sigma(F)\sigma(\overline{t})} \frac{\sigma(F)\sigma(\overline{t})}{\sigma(t)} F \otimes \overline{t} \\
&= \sum_{t,\, \Phi(t)=\mathbf{x}^{\mathbf{k}}} \frac{\mathbf{k}!}{\sigma(t)} \sum_{\mathbb{M}} \frac{1}{\|\mathbb{M}\|} \overline{\delta}^{r(\mathbb{M})} \sum_{F,\overline{t},\, \Phi(F)=\mathbb{M},\, \Phi(\overline{t})=\mathbf{x}^{\mathbf{k}}/\mathbb{M}} \mathcal{E}(t,F,\overline{t}) \frac{\prod_{w\in\mathcal{V}(F)} r_w!}{\prod_{v\in\mathcal{V}(\overline{t})} e_v!} F \otimes \overline{t}.
\end{aligned}$$

From (27) and (36), we finally get

$$\begin{aligned}
\mathcal{L} &= \sum_{t,\, \Phi(t)=\mathbf{x}^{\mathbf{k}}} \frac{\mathbf{k}!}{\sigma(t)} \sum_{\mathbb{M}} \frac{\prod_{j=1}^{r(\mathbb{M})+1} (\mathrm{wt}\,\beta^j + 1)!}{r(\mathbb{M})!} \overline{\delta}^{r(\mathbb{M})} \sum_{F,\overline{t},\, \Phi(F)=\mathbb{M},\, \Phi(\overline{t})=\mathbf{x}^{\mathbf{k}}/\mathbb{M}} \frac{r(F)!}{\prod_{w\in\mathcal{V}(F)} r_w!} \frac{1}{\|F\|} \mathcal{E}(t,F,\overline{t}) \frac{\prod_{w\in\mathcal{V}(F)} r_w!}{\prod_{v\in\mathcal{V}(\overline{t})} e_v!} F \otimes \overline{t} \\
&= \sum_{t,\, \Phi(t)=\mathbf{x}^{\mathbf{k}}} \frac{\mathbf{k}!}{\sigma(t)} \sum_{\mathbb{M}} \overline{\delta}^{r(\mathbb{M})} \sum_{F,\overline{t},\, \Phi(F)=\mathbb{M},\, \Phi(\overline{t})=\mathbf{x}^{\mathbf{k}}/\mathbb{M}} \frac{1}{\|F\|} \mathcal{E}(t,F,\overline{t})\, F \otimes \overline{t} \\
&= \mathcal{R},
\end{aligned}$$

which proves Theorem 15.

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