VOLUME GROWTH OF KÄHLER-EINSTEIN METRIC OVER QUASI-PROJECTIVE MANIFOLDS WITH BOUNDARY OF MAXIMAL OR MINIMAL KODAIRA DIMENSION

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ABSTRACT. In this paper, we make some progress about a boundary behavior of the almost-complete Kähler-Einstein metric of negative Ricci curvature on a quasi-projective manifold with semiample log-canonical bundle. First its volume growth near the boundary is investigated in terms of the Kodaira dimension of the boundary, and then we characterize the boundary to be of general type via the volume growth. Moreover the volume growth is determined in the case of a Calabi-Yau boundary. We also affirmatively solve a modified version of the conjecture suggested previously by the author about the residue of the Kähler-Einstein metric if the boundary is a smooth finite quotient of an abelian variety.

1. INTRODUCTION

This paper is a continuation of our work in [27] about a boundary behavior of the almost-complete Kähler-Einstein metric of negative Ricci curvature on a quasi-projective algebraic manifold with semiample log-canonical bundle. Our concrete aim is to find completely a relation between the boundary behavior of the metric and a degeneration of positivity for the log-canonical bundle on the boundary divisor. For such purpose, in [27], we propose a conjecture that the residue of the metric along the divisor coincides with the generalized Kähler-Einstein metric on the boundary in the sense of Song-Tian and H. Tsuji, and actually confirmed the truth when the boundary is of general type. In this paper, we further focus on its volume growth in the case of a Calabi-Yau boundary as well as a boundary of general type. It is already known that the volume of the Kähler-Einstein quasi-projective manifold with a general boundary divisor is explicitly given by the self-intersection number of the log-canonical divisor. So it is quite necessary to investigate the volume form of the Kähler-Einstein metric near the boundary. The author hopes that it might eventually lead to provide useful differential geometric techniques for the theory of quasi-projective algebraic manifolds in order to find out more applications to algebraic geometry such as a logarithmic version of Miyaoka-Yau inequality and a numerical characterization of ball quotients ([28], [42], [45], [3]).

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For an n-dimensional projective algebraic manifold $\overline{X}$ and a smooth prime divisor D of $\overline{X}$, it was shown by R. Kobayashi [28] that if its log-canonical bundle $K_{\overline{X}}+D$ is ample, there exists the unique complete Kähler-Einstein metric ω_X of negative Ricci curvature on the quasi-projective manifold $X = \overline{X} \setminus D$ with Poincaré growth near the boundary D. In his discussion, the volume growth is also determined as follows:

$$(\omega_X)^n = \frac{V}{\|\sigma\|^2(-\log\|\sigma\|^2)^2} = V\, e^{-\log\|\sigma\|^2 - 2\log(-\log\|\sigma\|^2)}$$

where V is a continuous nondegenerate volume form on $\overline{X}$, σ is a canonical section of D and $\|\sigma\|$ means its norm by a smooth Hermitian metric. Note that the induced distance from a fixed point is equivalent to the function $\log(-\log\|\sigma\|^2)$. In [37], G. Schumacher investigated the boundary asymptotics of ω_X itself. It was proved there that after the metric is restricted to a (local) complex hypersurface parallel to D, the limit of the metric equals the Kähler-Einstein metric ω_D of negative Ricci curvature on D as the hypersurface approaches D. Even in a more general situation, we consider the limit and call it the residue of ω_X along D in this paper if the limit exists. We also denote the limit by $\mathrm{Res}_D\,\omega_X$ as in [53]. Hence the above result due to G. Schumacher is formulated as

$$\mathrm{Res}_D\,\omega_X = \omega_D.$$

This is the 0-th order term of the asymptotic behavior, but generalizations to higher order asymptotics were established by D. Wu [48], [49], Rochon-Zhang [36], Jiang-Shi [24]. Moreover interesting asymptotic properties were also studied by H. Auvray [2] for constant scalar curvature or extremal Kähler metrics of Poincaré type.

On the other hand, Tian-Yau [42], H. Tsuji [45], S. Bando [3] and D. Wu [49] generalized the existence and uniqueness result of the almost-complete Kähler-Einstein metric ω_X to the case when the positivity of the log-canonical bundle $K_{\overline{X}}+D$ may be degenerate on the boundary, that is, $K_{\overline{X}}+D$ is nef, big and ample modulo D. In this case, the Kodaira dimension of the canonical bundle K_D of D actually measures how degenerate the positivity is because of the adjunction formula $(K_{\overline{X}}+D)\,|_D = K_D$. We would like to clarify geometric properties of ω_X in terms of the Kodaira dimension such as the completeness, the residue along D and the volume growth near D. However it is not trivial since the metric may not have bounded geometry.

When D is of general type, the author generalized the above volume formula and the residue formula by using the normalized Kähler-Ricci flow in [27]. Precisely speaking, the usual Kähler-Einstein metric ω_D on D never exists in this case, and so ω_D has to be replaced with the singular Kähler-Einstein metric or current of D. In the volume growth formula, we need to allow V to be non-smooth and have zeros in the non-ample locus of K_D. Such a semi-positive top form possibly vanishing somewhere is said to be a pseudo-volume form, and we do not assume any regularities for a pseudo-volume form here. In this paper, we prove conversely that this volume growth formula characterizes the boundary to be of general type as follows.

Theorem 1.1 (Proposition 2.1 in Section 2.1)**.** *Let $\overline{X}$ be a projective manifold of dimension n and D be a smooth prime divisor on $\overline{X}$. Set $X = \overline{X} \setminus D$. Suppose that the log-canonical bundle $K_{\overline{X}} + D$ is semiample and ample modulo D.*

Assume that the Kähler-Einstein metric ω_X has the volume growth near D of the form

$$(\omega_X)^n = \frac{V}{\|\sigma\|^2(-\log\|\sigma\|^2)^2},$$

where V is a bounded pseudo-volume form on $\overline{X}$ whose zero locus is contained in a proper pluripolar subset of D, σ is a canonical section of D and $\|\sigma\|$ means its norm by a smooth Hermitian metric. Then the Kodaira dimension of the canonical bundle of D is equal to $n-1$, that is, D is of general type.

We will also attempt to generalize these formulae due to R. Kobayashi and G. Schumacher to other degenerate cases of positivity on the boundary for the log-canonical bundle. In this paper, we mainly treat the Calabi-Yau boundary D.

The key point to deal with this case is to construct a suitable reference metric for the Kähler-Einstein metric ω_X using a structure of neighborhood of D and the Calabi-Yau metric on D. A similar construction is also studied in [16]. It is shown in Proposition 3.1 that the reference metric has bounded geometry in the sense of Cheng-Yau if and only if D is a finite quotient of an abelian variety or equivalently flat. The notion of bounded geometry in the sense of Cheng-Yau, which is also called quasi-bounded geometry, is related to such a property that curvatures are bounded. Therefore when D is flat, we can apply the technique of analysis for the complex Monge-Ampère equation, which is developed by G. Schumacher [37], and can determine the volume growth of ω_X as

$$(\omega_X)^n = \frac{V}{\|\sigma\|^2(-\log\|\sigma\|^2)^{n+1}},$$

where the notation is the same as in the nondegenerate case. In fact, when $\overline{X}$ be a smooth toroidal compactification of an n-dimensional complex hyperbolic manifold X, the Kähler-Einstein metric induced from the Poincaré-Bergman metric on the complex unit ball satisfies this volume growth ([33]). A higher order asymptotics in a local situation of ours is also established by Fu-Hein-Jiang [18].

Our next result in this paper is that the same volume growth holds for non-flat Calabi-Yau boundaries.

Theorem 1.2 (Theorem 2.3 in Section 2.1)**.** *Let $\overline{X}$ be an n-dimensional projective manifold and D be a smooth prime divisor on $\overline{X}$. Set $X = \overline{X} \setminus D$. Suppose that the log-canonical bundle $K_{\overline{X}} + D$ is semiample and ample modulo D.*

If the Kodaira dimension of D is zero, the Kähler-Einstein metric ω_X has the following volume growth :

$$(\omega_X)^n = \frac{V}{\|\sigma\|^2(-\log\|\sigma\|^2)^{n+1}},$$

where V is a bounded volume form on $\overline{X}$, σ is a canonical section of D and $\|\sigma\|$ means its norm by a smooth Hermitian metric.

Since the reference metric never has bounded geometry in the sense of Cheng-Yau unless D is flat, we consider to deform the Kähler-Einstein metric by the Kähler-Ricci flow or the Monge-Ampère flow according to [27], which has bounded geometry in the sense of Cheng-Yau at any finite time. This deformation corresponds to making $\alpha > 0$ approach 0 for $K_{\overline{X}} + D - \alpha D$. Since for small $\alpha > 0$, $K_{\overline{X}} + D - \alpha D$ is ample on $\overline{X}$, even on the boundary D, the Kähler-Ricci flow at every finite time has the same volume growth as in the nondegenerate case. It is necessary to estimate the solution to the Monge-Ampère flow in a uniform manner with respect to $\alpha > 0$ to establish the above volume growth. Here we focus on connecting carefully the volume growth in the nondegenerate case and the expected volume growth stated in this theorem. In a local situation of ours, several related results are also proved by Datar-Fu-Song [11], but the author has not known yet that our theorem is derived directly from their results.

The author hopes that the Kodaira dimension $\kappa = \kappa(K_D)$ of D influences a power of $-\log\|\sigma\|^2$ in a volume growth of the Kähler-Einstein metric, and some conjecture will be also suggested in this paper.

Another main goal of this paper is to solve affirmatively a certain stronger version of the conjecture on the residue, which is proposed in [27], when the boundary is flat. To be more precise, it is stated as follows:

Theorem 1.3 (Theorem 2.6 in Section 2.2)**.** *Let $\overline{X}$ be a projective manifold of dimension n and D be a smooth prime divisor on $\overline{X}$. Set $X = \overline{X} \setminus D$. Suppose that the log-canonical bundle $K_{\overline{X}} + D$ is semiample and ample modulo D.*

We additionally assume that D is a smooth finite quotinet of an abelian variety. Then the weighted residue $\mathrm{Res}_D \{(-\log\|\sigma\|^2)\omega_X\}$ along D coincides with the Ricci-flat Kähler metric on D which corresponds to the $(n+1)$-times tensor product of the dual of the normal bundle of D. Here σ is a canonical section of D and $\|\sigma\|$ means its norm by a smooth Hermitian metric.

As mentioned above, the additional assumption on D is equivalent to the bounded geometry condition in the sense of Cheng-Yau so that Schumacher's technique is also applicable to get this.

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Throughout this paper, let $\overline{X}$ be an n-dimensional projective manifold and D its smooth prime divisor, and set the quasi-projective manifold $X = \overline{X} \setminus D$. We fix a nonzero holomorphic section σ of the line bundle $\mathcal{O}_{\overline{X}}(D)$ associated with D such that $\mathrm{div}(\sigma) = D$. $K_{\overline{X}}$ and K_D denote the canonical bundles of $\overline{X}$ and D, respectively.

Furthermore in this paper, we always assume the following on positivity of the log-canonical bundle $K_{\overline{X}} + D$:

$$K_{\overline{X}} + D \text{ is semiample, big and ample modulo } D.$$

Here we say that a line bundle L over $\overline{X}$ is ample modulo D if its non-ample locus $\mathbf{B}_+(L)$ is contained in D. Note that the bigness automatically follows from this condition (refer to Tian-Yau [42], S. Bando [3], Di Cerbo-Di Cerbo [13], [14] for further properties of positivity modulo D). According to [42, Lemma 2.7], it follows from the positivity assumption that for an arbitrarily small positive number $\alpha > 0$, $(K_{\overline{X}} + D) - \alpha D = K_{\overline{X}} + (1-\alpha)D$ is ample on $\overline{X}$. We often consider the case when the Kodaira dimension of D vanishes, namely D is so-called Calabi-Yau, and then the dual of the normal bundle N_D is ample. Moreover Kawamata's theorem ([25, Theorem 8.2]) implies that $(K_D)^{\otimes m}$ is a trivial line bundle for some integer $m > 0$.

2. Properties of Kähler-Einstein metric near the boundary

In this section, we summarize several results about a boundary behavior of the almost-complete Kähler-Einstein metric of negative Ricci curvature on X in our specific situation, and some of them might be unknown or not written anywhere. The metric actually exists in more general settings, and see [3], [28], [29], [42], [48], [49] or [51] for further information on existence results.

2.1. Volume growth of Kähler-Einstein metric.

In this subsection, we focus on a volume growth of the almost-complete Kähler-Einstein metric ω_X on X, that is, the growth of the Kähler-Einstein volume form $(\omega_X)^n$ near the boundary D. Especially, our aim is to discover a relation between the growth and the Kodaira dimension $\kappa = \kappa(K_D) \in \{0, 1, \cdots, n-1\}$ of the variety D. Afterward h denotes a smooth Hermitian metric of $\mathcal{O}_{\overline{X}}(D)$ such that $\rho = -\log \|\sigma\|^2 > 0$, where $\|\cdot\|$ stands for the norm with respect to h. We use h hereafter also as a coefficient function with respect to a local trivialization.

We will consider a Kähler-Einstein metric on X as a main subject in this work. In this paper, we restrict ourselves to the case of negative Ricci curvature and so the word "Kähler-Einstein metric" is always used to mean a Kähler metric ω satisfying the following normalized equation:

$$-\mathrm{Ric}(\omega) = \omega,$$

where $\mathrm{Ric}(\omega) = -\dfrac{1}{2\pi}\sqrt{-1}\partial\overline{\partial}\log \omega^n$. In fact, Tian-Yau [42], H. Tsuji [45], S. Bando [3] and D. Wu [49] established the existence of such a canonical metric in our setting.

Theorem 2.1 (Tian-Yau [42], H. Tsuji [45], S. Bando [3], D. Wu [49])**.** *There exists a unique almost-complete Kähler-Einstein metric ω_X whose cohomology class on $\overline{X}$ is equal to $c_1(K_{\overline{X}} + D)$ as a current. Moreover the volume of the non-compact Riemannian manifold (X, ω_X) is finite and actually equal to the intersection number $\frac{1}{n!}(K_{\overline{X}} + D)^n$.*

The completeness of the resultant metric ω_X has not been known to hold yet, although S.-T. Yau [51] provided a very rough and insufficient argument for the proof. We note that the results of this paper are independent of the completeness.

The proof of the existence of ω_X due to Tian-Yau [42] and S. Bando [45] only yields that the Kähler-Einstein volume form is of the form

$$(\omega_X)^n = \frac{V_0}{\|\sigma\|^2\rho^2}.$$

Here V_0 is a possibly non-smooth pseudo-volume form on $\overline{X}$ which possibly has zeros along D owing to the degeneracy of the positivity of $K_{\overline{X}} + D$ on D. A crucial point is that their arguments are not enough to clarify what shape zeros of V_0 takes. If the shape is understood well, it is possible to calculate also the residue $\mathrm{Res}_D\,\omega_X$ along D, which appeared in [27, Theorem 1.1] as the limit along the directions tangential to D. Indeed, since

$$\begin{aligned}\omega_X = -\mathrm{Ric}\,(\omega_X) &= \frac{1}{2\pi}\sqrt{-1}\partial\overline{\partial}\log\frac{V_0}{\|\sigma\|^2\rho^2}\\ &= \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log\frac{V_0}{h} + 2\left(\frac{\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log h}{\rho} + \frac{\sqrt{-1}}{2\pi}\partial\log\rho\wedge\overline{\partial}\log\rho\right),\end{aligned}$$

we can calculate the residue as

$$\mathrm{Res}_D\,\omega_X = \lim_{\epsilon\to 0}\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log\frac{V_0}{h}\bigg|_{(\sigma=\epsilon)}$$

if the limit exists. Refer to Section 2.2 for its definition and more details about the residue.

First let us present several examples in which a volume growth or a shape of zeros of V is adequately known.

Example 2.1 (R. Kobayashi [28], G. Schumacher [37])**.** When $K_{\overline{X}} + D$ is ample on $\overline{X}$, $K_{\overline{X}} + D$ is nondegenerate and $\kappa(K_D) = n-1$. Then it follows that for an appropriate nowhere vanishing volume form V_0, the Kähler-Einstein volume form is described as

$$(\omega_X)^n = \frac{V_0}{\|\sigma\|^2\rho^2} = \frac{V_0}{\|\sigma\|^2\rho^{n+1-(n-1)}}.$$

As explained in Example 2.6, $\mathrm{Res}_D\,\omega_X = \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log\frac{V_0}{h}\big|_D$ is the smooth Kähler-Einstein metric on D.

Example 2.2 (R. Kobayashi [29])**.** When $\overline{X}$ is 2-dimensional, it holds whether the log-canonical bundle $K_{\overline{X}} + D$ is ample on $\overline{X}$ or D is an elliptic curve. The former case is included in the previous example. In the latter case, i.e., $\kappa(K_D) = 0$, it is proved that $\Omega = \rho V_0$ is bounded and vanishes nowhere and

$$(\omega_X)^2 = \frac{\rho V_0}{\|\sigma\|^2\rho^3} = \frac{\Omega}{\|\sigma\|^2\rho^3} = \frac{\Omega}{\|\sigma\|^2\rho^{2+1-0}}.$$

Hence Ω will appear again in Example 2.7, and it will be stated there that the residue $\mathrm{Res}_D\,\omega_X = \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log\frac{\Omega}{h}\,|_D = 0$ holds.

Example 2.3 (N. Mok [33])**.** Let $\overline{X}$ be a smooth toroidal compactification of an n-dimensional complex hyperbolic manifold with an exceptional divisor D. Then D is a disjoint union of abelian varieties, namely any component of D satisfies $\kappa(K_D) = 0$. Note that this pair $(\overline{X}, D)$ satisfies our positivity condition on $K_{\overline{X}} + D$ (for example see Main Theorem in [33] or [14]).

Since the Kähler-Einstein metric ω_X is a metric induced by the Poincaré-Bergman metric on the complex unit ball, it is possible to calculate explicitly $(\omega_X)^n$ near D. N. Mok carried out it in [33, Section 1] to show that $\Omega = \rho^{n-1}V_0$ is bounded and vanishes nowhere, and that the volume form associated with ω_X can be written as

$$(\omega_X)^n = \frac{\rho^{n-1}V_0}{\|\sigma\|^2\rho^{n+1}} = \frac{\Omega}{\|\sigma\|^2\rho^{n+1}} = \frac{\Omega}{\|\sigma\|^2\rho^{n+1-0}}.$$

Furthermore as mentioned in Example 2.8, $\mathrm{Res}_D\,\omega_X = \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log\frac{\Omega}{h}\Big|_D = 0$ and $\mathrm{Res}_D\,(\rho\,\omega_X) = (n+1)\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log h\Big|_D$.

Example 2.4 ([27])**.** Suppose $\overline{X}$ satisfies that $(K_{\overline{X}} + D)|_D = K_D$ is big, namely $\kappa(K_D) = n-1$. In this case, the author approximated ω_X by a normalized Kähler-Ricci flow to obtain a volume growth. In fact, it is proved in the proof of [27, Theorem 1.1] that V_0 degenerates only on the non-ample locus $\mathbf{B}_+(K_D) \subseteq D$ of K_D and the volume form grows as

$$(\omega_X)^n = \frac{V_0}{\|\sigma\|^2\rho^2} = \frac{V_0}{\|\sigma\|^2\rho^{n+1-(n-1)}}.$$

Moreover it will be explained in Example 2.9 that $\mathrm{Res}_D\,\omega_X = \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log\frac{V_0}{h}\Big|_D$ is equal to the singular Kähler-Einstein metric ω_D on D.

Example 2.5 (Wang [47], Yau-Zhang [53])**.** Let $\overline{X}$ be a smooth toroidal compactification with simple normal crossing boundary divisor D for the Siegel modular variety $X = \mathbb{H}_g/\Gamma$ $(g \geq 2)$. Here $\mathbb{H}_g$ is the Siegel upper half space of degree g and $\Gamma \subset \mathrm{Sp}(g,\mathbb{Q})$ is a neat arithmetic group . The compactification of X depends on a family of fans (or polyhedral decompositions) for the open cones consisting of positive definite symmetric matrices, which is compatible with the induced group action. It is known that this pair $(\overline{X}, D)$ fulfills our positivity condition on $K_{\overline{X}} + D$ ([34, Theorem 3.2]). See [1], [34], [35], [17] for several definitions and fundamental properties concerning the toroidal compactification. Further the Kähler-Einstein metric ω_X on X is a metric induced from the Poincaré-Bergman metric on $\mathbb{H}_g$

$$\begin{aligned}\omega_X &= \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log\frac{1}{(\det\,\mathrm{Im}\,\tau)^{g+1}}\\ &= \frac{g+1}{8\pi}\mathrm{tr}\left(\sqrt{-1}(\mathrm{Im}\,\tau)^{-1}d\tau\wedge(\mathrm{Im}\,\tau)^{-1}d\overline{\tau}\right),\quad \tau = [\tau_{ij}] \in \mathbb{H}_g.\end{aligned}$$

From the invariance by $\mathrm{Sp}(g,\mathbb{R})$, we have

$$(\omega_X)^{\frac{g(g+1)}{2}} = \frac{\left(\frac{g+1}{8\pi}\right)^{\frac{g(g+1)}{2}}}{\frac{g(g+1)}{2}!} \frac{1}{(\det \mathrm{Im}\, \tau)^{g+1}} \bigwedge_{i\le j} \sqrt{-1}d\tau_{ij} \wedge d\overline{\tau}_{ij}$$

and also

$$-\mathrm{Ric}\,(\omega_X) = \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log\,(\omega_X)^{\frac{g(g+1)}{2}} = \omega_X.$$

Notice that the boundary divisor has several components $D_1, D_2, \cdots, D_N$ and they possibly intersect each other differently from our situation. A behavior of ω_X near intersection points of boundary components is studied in [47], [53] from a combinatorial point of view about the family of fans. On the other hand, we consider at present a behavior of the Kähler-Einstein volume form $(\omega_X)^{\frac{g(g+1)}{2}}$ only near $D_i \setminus \bigcup_{j\neq i}(D_i \cap D_j)$ for each component D_i. Let us explain it for readers' convenience, following [47] and [53, Section 3].

Thanks to the resolution of the congruence subgroup problem ([5], [4]), Γ contains the principal congruence subgroup $\Gamma_g(k) = \{\gamma \in \mathrm{Sp}(g,\mathbb{Z})\ ;\ \gamma \equiv I_g \mod k\}$ of some level $k \geq 3$ as a subgroup of Γ with finite index. Hereafter take k to be minimal among numbers satisfying this property. Therefore we have a new quotient $Y = \mathbb{H}_g/\Gamma_g(k)$ and its toroidal compactification $\overline{Y}$ with respect to the same family of fans. There also exists a natural finite holomorphic maps $p : \overline{Y} \to \overline{X}$ which extends the canonical étale covering map $Y = \mathbb{H}_g/\Gamma_g(k) \to X = \mathbb{H}_g/\Gamma$ across the boundaries. On the other hand, we have another compactification X^* (or Y^*) of X (or Y) which is called the Baily-Borel-Satake compactification. The canonical étale covering map $Y = \mathbb{H}_g/\Gamma_g(k) \to X = \mathbb{H}_g/\Gamma$ can be also extended across the boundaries and we denote the map by $p^* : Y^* \to X^*$. Then it is known that we have canonical holomorphic maps $\pi_X : \overline{X} \to X^*$ and $\pi_Y : \overline{Y} \to Y^*$ between these different compactifications. These maps satisfy the commutative diagram :

$$\begin{array}{ccc} \overline{Y} & \xrightarrow{p} & \overline{X} \\ \downarrow \pi_Y & & \pi_X \downarrow \\ Y^* & \xrightarrow{p^*} & X^* \end{array}.$$

After recalling several facts about the boundary divisor in [47] and [53], we will investigate the volume growth of ω_X (and also the residue of ω_X in Example 2.10).

(i) For each $i = 1, 2, \cdots, N$, D_i has a fibration structure over the boundary component $M_i = \pi_X(D_i)$ of X^*. Moreover any singular fiber is contained in $\bigcup_{j\neq i}(D_i \cap D_j) \subset D_i$. Set $D_i^o = D_i \setminus \bigcup_{j\neq i}(D_i \cap D_j)$. We should note that D_i^o and its open neighborhood does not depend on a choice of families of fans. Thus if $(\sigma = o) \subset D_i^o$, the limit $\lim_{\epsilon\to 0} \omega_X|_{(\sigma=\epsilon)}$ can be observed without using families of fans. The same notation is also used and the same properties also hold for $\overline{Y}$.

(ii) For each boundary component E of $\overline{Y}$ with $N = \pi(E)$, a neighborhood of E^o is constructed as follows: assume that $N^o = \pi(E^o)$ is equal to $\gamma \cdot \mathbb{H}_{g-1}/\Gamma_{g-1}(k)$ for some $\gamma \in \mathrm{Sp}(g, \mathbb{Z})$. Here $\mathbb{H}_{g-1}/\Gamma_{g-1}(k)$ is the standard cusp with depth 1, that is, $\mathbb{H}_{g-1}$ is regarded as a boundary component of $\mathbb{H}_g$ consisting of all matrices of type $\begin{bmatrix} z' & 0 \\ 0 & 1 \end{bmatrix}$, $z' \in \mathbb{H}_{g-1}$ under an identification through the Cayley transformation. Therefore since a neighborhood of E^o is induced from that of $\mathbb{H}_{g-1}/\Gamma_{g-1}(k)$ via the action by γ, it suffices to consider only the case $\gamma = I_g$, and especially $N^o = \mathbb{H}_{g-1}/\Gamma_{g-1}(k)$. By the same reason, also for $\overline{X}$, we only treat the boundary component corresponding to the standard cusp of depth 1 and denote it by D in the rest of this example for simplicity unless otherwise stated (D does not mean the whole boundary divisor from now).

(iii) Then we consider a holomorphic map to $\overline{Y}$ defined near $\mathbb{H}_{g-1} \times \mathbb{C}^{g-1} \times \{0\}$ in $\mathbb{H}_{g-1} \times \mathbb{C}^{g-1} \times \mathbb{C}$, which is an extension across $\sigma = 0$ of

$$(\tau', \tau'', \sigma) \mapsto \left[\begin{bmatrix} \tau' & \tau'' \\ {}^t\tau'' & \frac{k}{2\pi\sqrt{-1}} \log \sigma \end{bmatrix}\right] \in Y = \mathbb{H}_g/\Gamma_g(k), \quad \sigma \neq 0.$$

The action on $\mathbb{H}_g$ by $\Gamma_g(k)$ induces the action on the domain $\mathbb{H}_{g-1} \times \mathbb{C}^{g-1} \times \mathbb{C}$ of the map by the group $\Gamma_{g-1}(k) \ltimes (k\mathbb{Z}^{g-1})^2$ in the following way: for any element $\gamma = \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}, \begin{bmatrix} m \\ n \end{bmatrix}\right)$ in $\Gamma_{g-1}(k) \ltimes (k\mathbb{Z}^{g-1})^2$,

$$\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}, \begin{bmatrix} m \\ n \end{bmatrix}\right) \cdot (\tau', \tau'', \sigma) := \left(\frac{a\tau' + b}{c\tau' + d}, \frac{\tau'' + \tau' m + n}{c\tau' + d}, c_\gamma \sigma\right),$$

where $c_\gamma \in \mathbb{C}^\times$ is a constant depending on γ. Hence the above map descends on the quotient space $\mathbb{H}_{g-1} \times \mathbb{C}^{g-1} \times \mathbb{C}/\Gamma_{g-1}(k) \ltimes (k\mathbb{Z}^{g-1})^2$. It is further known that the resultant map is isomorphic from a neighborhood of the abelian fiber bundle $\mathbb{H}_{g-1} \times \mathbb{C}^{g-1} \times \{0\}/\Gamma_{g-1}(k) \ltimes (k\mathbb{Z}^{g-1})^2$ to a neighborhood of E^o. Through this isomorphism, $\mathbb{H}_{g-1} \times \mathbb{C}^{g-1} \times \{0\}/\Gamma_{g-1}(k) \ltimes (k\mathbb{Z}^{g-1})^2 \to N^o = \mathbb{H}_{g-1}/\Gamma_{g-1}(k)$ corresponds to $\pi_Y : E^o \to N^o$ as fiber bundles. Note again that this isomorphism does not rely on a choice of a family of fans although the compactification $\overline{Y}$ of $Y = \mathbb{H}_g/\Gamma_g(k)$ does globally.

(iv) On the other hand, as for $X = \mathbb{H}_g/\Gamma$ and the boundary divisor D corresponding to the standard cusp of depth 1, we need minor changes in the definitions of the isomorphism and the neighborhood of E^o in (iii). We first consider a holomorphic map to $\overline{X}$ defined near $\mathbb{H}_{g-1} \times \mathbb{C}^{g-1} \times \{0\} \subset \mathbb{H}_{g-1} \times \mathbb{C}^{g-1} \times \mathbb{C}$ which is an extension across $\sigma = 0$ of

$$(\tau', \tau'', \sigma) \mapsto \left[\begin{bmatrix} \tau' & \tau'' \\ {}^t\tau'' & \frac{k'}{2\pi\sqrt{-1}} \log \sigma \end{bmatrix}\right] \in X = \mathbb{H}_g/\Gamma, \quad \sigma \neq 0.$$

Here k' is a positive integer by which k is divisible. Secondly, it suffices to replace $\Gamma_{g-1}(k) \ltimes (k\mathbb{Z}^{g-1})^2$ with an arithmetic subgroup Γ' of $\mathrm{Sp}(g-1, \mathbb{Q}) \ltimes (\mathbb{Q}^{g-1})^2$ containing $\Gamma_{g-1}(k) \ltimes (k\mathbb{Z}^{g-1})^2$ as a subgroup with finite index.

Among the constructed neighborhoods of E^o and D^o, the map $p : \overline{Y} \to \overline{X}$ can be described in terms of the constructed isomorphisms as a canonical finite covering map $\mathbb{H}_{g-1} \times \mathbb{C}^{g-1} \times \mathbb{C}/\Gamma_{g-1}(k) \ltimes (k\mathbb{Z}^{g-1})^2 \to \mathbb{H}_{g-1} \times \mathbb{C}^{g-1} \times \mathbb{C}/\Gamma'$ defined by $(\tau', \tau'', \sigma) \mapsto (\tau', \tau'', \sigma^{\frac{k}{k'}})$. Through this description of p over the boundaries E^o and D^o, we have the following commutative diagram:

$$\begin{array}{ccc} E^o = \mathbb{H}_{g-1} \times \mathbb{C}^{g-1}/\Gamma_{g-1}(k) \ltimes (k\mathbb{Z}^{g-1})^2 & \xrightarrow{p} & D^o = \mathbb{H}_{g-1} \times \mathbb{C}^{g-1}/\Gamma' \\ \downarrow \pi_Y & & \pi_X \downarrow \\ N^o = \mathbb{H}_{g-1}/\Gamma_{g-1}(k) & \xrightarrow{p^*} & M^o = \mathbb{H}_{g-1}/\Gamma'_{g-1} \end{array}$$

Here Γ'_{g-1} consists of all parabolic elements of Γ for the standard cusp of depth 1. From this diagram, for any $[\tau'] \in N^o$, $\pi_X^{-1}(p^*([\tau']))$ is a smooth finite quotient of the $(g-1)$-dimensional abelian variety $\pi_Y^{-1}([\tau']) \simeq \mathbb{C}^{g-1}/k\tau'\mathbb{Z}^{g-1} + k\mathbb{Z}^{g-1}$, in particular a $(g-1)$-dimensional abelian variety itself.

Let us start to calculate the volume form $(\omega_X)^{\frac{g(g+1)}{2}}$ near $D_i^o = D_i \setminus \bigcup_{j \neq i}(D_i \cap D_j)$ for every boundary component D_i of $\overline{X}$. The invariance under $\mathrm{Sp}(g, \mathbb{R})$ and the property (ii) lead that it is sufficient to deal only with the boundary component D corresponding to the standard cusp of depth 1. On the neighborhood of D^o defined in (iii) and (iv), we have

$$\begin{aligned} (\omega_X)^{\frac{g(g+1)}{2}} &= \frac{\dfrac{\left(\frac{g+1}{8\pi}\right)^{\frac{g(g+1)}{2}}}{\frac{g(g+1)}{2}!}}{\left(\det \begin{bmatrix} \mathrm{Im}\,\tau' & \mathrm{Im}\,\tau'' \\ \mathrm{Im}\,{}^t\tau'' & -\frac{k'}{2\pi}\log|\sigma| \end{bmatrix}\right)^{g+1}} \bigwedge_{i \le j, (i,j) \neq (g,g)} \sqrt{-1}d\tau_{ij} \wedge d\overline{\tau}_{ij} \\ &\qquad \wedge \sqrt{-1}d\left(\frac{k'}{2\pi\sqrt{-1}}\log\sigma\right) \wedge d\left(-\frac{k'}{2\pi\sqrt{-1}}\overline{\log\sigma}\right) \\ &= \frac{\left(\frac{k'}{2\pi}\right)^2 \dfrac{\left(\frac{g+1}{8\pi}\right)^{\frac{g(g+1)}{2}}}{\frac{g(g+1)}{2}!}}{|\sigma|^2\left(-\frac{k'}{2\pi}(\det \mathrm{Im}\,\tau')\log|\sigma| + H(\tau', \tau'')\right)^{g+1}} \bigwedge_{i \le j, (i,j) \neq (g,g)} \sqrt{-1}d\tau_{ij} \wedge d\overline{\tau}_{ij} \wedge \sqrt{-1}d\sigma \wedge d\overline{\sigma} \\ &= \frac{\left(\frac{2\pi}{k'}\right)^{g-1} \dfrac{\left(\frac{g+1}{8\pi}\right)^{\frac{g(g+1)}{2}}}{\frac{g(g+1)}{2}!} \displaystyle\bigwedge_{i \le j, (i,j) \neq (g,g)} \sqrt{-1}d\tau_{ij} \wedge d\overline{\tau}_{ij} \wedge \sqrt{-1}d\sigma \wedge d\overline{\sigma}}{(\det \mathrm{Im}\,\tau')^{g+1}|\sigma|^2\Big(-\log|\sigma| + \frac{2\pi}{k'}(\det \mathrm{Im}\,\tau')^{-1}H(\tau', \tau'')\Big)^{\frac{g(g+1)}{2}+1-\frac{g(g-1)}{2}}}. \end{aligned}$$

Here $H(\tau', \tau'')$ is a polynomial with variables (τ', τ''). Since D^o has the fibration structure $\pi_X : D^o = \mathbb{H}_{g-1} \times \mathbb{C}^{g-1}/\Gamma' \to M^o = \mathbb{H}_{g-1}/\Gamma'_{g-1}$ with fibers of $(g-1)$-dimensional abelian variety, we might regard the (logarithmic) Kodaira dimension of D^o as $\kappa = \frac{g(g-1)}{2}$.

In a general situation, the degeneration of the positivity for $K_{\overline{X}} + D$ is measured by the Kodaira dimension $\kappa(K_D) \in \{0, 1, \cdots, n-1\}$ since the adjunction formula $(K_{\overline{X}} + D)|_D = K_D$ holds. Therefore the above examples lead us to expect that the Kähler-Einstein volume form satisfies the following growth with $\kappa(K_D)$:

Conjecture 2.1. *For any $\kappa \in \{0, 1, \cdots, n-1\}$, $\kappa(K_D) = \kappa$ if and only if the Kähler-Einstein volume form has an expression of*

$$(\omega_X)^n = \frac{V}{\|\sigma\|^2 \rho^{n+1-\kappa}},$$

where h is a smooth Hermitian metric on $\mathcal{O}_{\overline{X}}(D)$ and V is a bounded pseudo-volume form on $\overline{X}$ whose zero locus is contained in a proper pluripolar subset on D.

Here we consider a pluripolar subset rather than an algebraic subset in anticipation of a generalization to non-projective cases. An algebraic subset might fit our setting in this paper more.

Next we discuss this conjecture for $\kappa = n-1$. Example 2.4 yields one implication of the conjecture that the Kähler-Einstein volume form has the above expression with $\kappa = n-1$ if $\kappa(K_D) = n-1$. The next proposition means that the other implication is also true, and this result is one of main theorems in this paper.

Proposition 2.1 (Theorem 1.1 in Introduction)**.** *If the Kähler-Einstein volume form is of the form*

$$(\omega_X)^n = \frac{V}{\|\sigma\|^2 \rho^2},$$

then $\kappa(K_D) = n-1$, namely D is of general type.

Proof. We have to show the bigness of K_D, which is equivalent to $\kappa(K_D) = n-1$. Thanks to the Kähler-Einstein equation, we have near D

$$\begin{aligned}
\omega_X = -\mathrm{Ric}(\omega_X) &= \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log(\omega_X)^n = \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log\frac{V}{\|\sigma\|^2\rho^2} \\
&= \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log\frac{V}{h} + 2\frac{\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log h}{\rho} + 2\frac{\sqrt{-1}}{2\pi}\frac{\partial\rho\wedge\overline{\partial}\rho}{\rho^2} \\
&= \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log\frac{V}{h} + \frac{\sqrt{-1}\partial\overline{\partial}\log h}{\pi\rho} \\
&\quad + \frac{\sqrt{-1}}{\pi|\sigma|^2\rho^2}(|\sigma|^2\partial\log h\wedge\overline{\partial}\log h + \sigma\partial\log h\wedge d\overline{\sigma} + \overline{\sigma}d\sigma\wedge\overline{\partial}\log h + d\sigma\wedge d\overline{\sigma}).
\end{aligned}$$

We consider local coordinates $(z', \sigma) = (z^1, \cdots z^{n-1}, \sigma)$ defined on the unit polydisc Δ^n near D, where σ is also used as a coefficient function of a section σ. For i, $j = 1, 2, \cdots, n-1$, we write $\omega_{i\bar{j}} = \frac{1}{2\pi}\frac{\partial^2}{\partial z^i\partial\bar{z}^j}\log\frac{V}{h\rho^2}$, $\omega_{i\bar{\sigma}} = \frac{1}{2\pi}\frac{\partial^2}{\partial z^i\partial\bar{\sigma}}\log\frac{V}{h\rho^2}$, $\omega_{\sigma\bar{j}} = \frac{1}{2\pi}\frac{\partial^2}{\partial\sigma\partial\bar{z}^j}\log\frac{V}{h\rho^2}$, $\omega_{\sigma\bar{\sigma}} = \frac{1}{2\pi}\frac{\partial^2}{\partial\sigma\partial\bar{\sigma}}\log\frac{V}{h\rho^2}$.

From our assumption about the volume growth, it follows that

$$
\begin{aligned}
\frac{V}{\|\sigma\|^2\rho^2} &= (\omega_X)^n = n(\sqrt{-1}\omega_{i\bar{j}}dz^i \wedge d\bar{z}^j)^{n-1} \wedge \sqrt{-1}\omega_{\sigma\bar{\sigma}}d\sigma \wedge d\bar{\sigma} \\
&+ n(n-1)(\sqrt{-1}\omega_{i\bar{j}}dz^i \wedge d\bar{z}^j)^{n-2} \wedge (\sqrt{-1}\omega_{k\bar{\sigma}}dz^k \wedge d\bar{\sigma}) \wedge (\sqrt{-1}\omega_{\sigma\bar{l}}d\sigma \wedge d\bar{z}^l) \\
&= n(\sqrt{-1}\omega_{i\bar{j}}dz^i \wedge d\bar{z}^j)^{n-1} \wedge \sqrt{-1}\omega_{\sigma\bar{\sigma}}d\sigma \wedge d\bar{\sigma} \\
&- \sqrt{-1}n(n-1)(\sqrt{-1}\omega_{i\bar{j}}dz^i \wedge d\bar{z}^j)^{n-2} \wedge (\omega_{k\bar{\sigma}}dz^k) \wedge (\omega_{\sigma\bar{l}}d\bar{z}^l) \wedge \sqrt{-1}d\sigma \wedge d\bar{\sigma} \\
&\le n(\sqrt{-1}\omega_{i\bar{j}}dz^i \wedge d\bar{z}^j)^{n-1} \wedge \sqrt{-1}\omega_{\sigma\bar{\sigma}}d\sigma \wedge d\bar{\sigma}.
\end{aligned}
$$

The last inequality is related to Hadamard's determinant inequality ([23]). This inequality is further changed into

$$
(1) \qquad \frac{1}{|\sigma|^2\rho^2\omega_{\sigma\bar{\sigma}}}\frac{V}{h}\bigg|_{(\sigma=\epsilon)} \le n\left(\omega_{i\bar{j}}dz^i \wedge d\bar{z}^j\bigg|_{(\sigma=\epsilon)}\right)^{n-1}.
$$

Note that this inequality is regarded as an inequality for volume forms on the level set $(\sigma = \epsilon)$ with $\sigma \neq 0$. For fixed $z' \in \Delta^{n-1}$, $f(z',\sigma) = \log\frac{V}{h\rho^2}$ is subharmonic in $\sigma \in \Delta$. Then by applying the Riesz decomposition theorem in potential theory to the subharmonic function on Δ, we know that

$$
f(z',\sigma) = 2\int_{\tau\in\Delta} G(\tau,\sigma)\frac{\partial^2 f}{\partial\tau\partial\bar{\tau}}(z',\tau)\sqrt{-1}d\tau \wedge d\bar{\tau}
$$

modulo addition of harmonic functions in $\sigma \in \Delta$, where $G(\tau,\sigma)$ is the Green function for the unit disc Δ. Furthermore one can see from a calculation of $\frac{\partial^2 f}{\partial\tau\partial\bar{\tau}}$ that modulo addition of smooth functions,

$$
\begin{aligned}
\log\frac{V}{h} - 2\log\rho &= 2\int_{\tau\in\Delta} G(\tau,\sigma)\left\{\frac{\partial^2\log(V/h)}{\partial\tau\partial\bar{\tau}} + \frac{2}{\rho}\frac{\partial^2\log h}{\partial\tau\partial\bar{\tau}} + \frac{2}{\rho^2}\frac{\partial\log h}{\partial\tau}\frac{\partial\log h}{\partial\bar{\tau}}\right. \\
&\qquad \left. + \frac{2}{\tau\rho^2}\frac{\partial\log h}{\partial\bar{\tau}} + \frac{2}{\bar{\tau}\rho^2}\frac{\partial\log h}{\partial\tau} + \frac{2}{|\tau|^2\rho^2}\right\}\sqrt{-1}d\tau \wedge d\bar{\tau} \\
&= 2\int_{\tau\in\Delta} G(\tau,\sigma)\frac{\partial^2\log(V/h)}{\partial\tau\partial\bar{\tau}}\sqrt{-1}d\tau \wedge d\bar{\tau} \\
&+ 2\int_{\tau\in\Delta} G(\tau,\sigma)\left\{\frac{2}{\rho}\frac{\partial^2\log h}{\partial\tau\partial\bar{\tau}} + \frac{2}{\rho^2}\frac{\partial\log h}{\partial\tau}\frac{\partial\log h}{\partial\bar{\tau}}\right. \\
&\qquad\qquad \left. + \frac{2}{\tau\rho^2}\frac{\partial\log h}{\partial\bar{\tau}} + \frac{2}{\bar{\tau}\rho^2}\frac{\partial\log h}{\partial\tau}\right\}\sqrt{-1}d\tau \wedge d\bar{\tau} \\
&+ 2\int_{\tau\in\Delta} G(\tau,\sigma)\left\{\frac{2}{|\tau|^2\rho^2} - \frac{2}{|\tau|^2(-\log|\tau|^2)^2}\right\}\sqrt{-1}d\tau \wedge d\bar{\tau} \\
&+ 2\int_{\tau\in\Delta} G(\tau,\sigma)\frac{2}{|\tau|^2(-\log|\tau|^2)^2}\sqrt{-1}d\tau \wedge d\bar{\tau}.
\end{aligned}
$$

Since the equality $\frac{\partial^2\{-\log(-\log|\tau|^2)\}}{\partial\tau\partial\overline{\tau}} = \frac{1}{|\tau|^2(-\log|\tau|^2)^2}$ holds, we can apply the formula

$$2\int_{\tau\in\Delta} G(\tau,\sigma)\frac{\partial^2\{-\log(-\log|\tau|^2)\}}{\partial\tau\partial\overline{\tau}}\sqrt{-1}d\tau\wedge d\overline{\tau} = -\log(-\log|\sigma|^2)$$

to the last term. Then since $\log\frac{\log\|\sigma\|^2}{\log|\sigma|^2}$ is also smooth in σ, we get, modulo addition of smooth functions,

$$\begin{aligned}\log\frac{V}{h} &= 2\int_{\tau\in\Delta} G(\tau,\sigma)\frac{\partial^2\log(V/h)}{\partial\tau\partial\overline{\tau}}\sqrt{-1}d\tau\wedge d\overline{\tau}\\ &\quad+2\int_{\tau\in\Delta} G(\tau,\sigma)\left\{\frac{2}{\rho}\frac{\partial^2\log h}{\partial\tau\partial\overline{\tau}}+\frac{2}{\rho^2}\frac{\partial\log h}{\partial\tau}\frac{\partial\log h}{\partial\overline{\tau}}\right.\\ &\qquad\qquad\left.+\frac{2}{\tau\rho^2}\frac{\partial\log h}{\partial\overline{\tau}}+\frac{2}{\overline{\tau}\rho^2}\frac{\partial\log h}{\partial\tau}\right\}\sqrt{-1}d\tau\wedge d\overline{\tau}\\ &\quad+2\int_{\tau\in\Delta} G(\tau,\sigma)\left\{\frac{2}{|\tau|^2\rho^2}-\frac{2}{|\tau|^2(-\log|\tau|^2)^2}\right\}\sqrt{-1}d\tau\wedge d\overline{\tau}.\end{aligned}$$

An important notice is that $G(\tau,\sigma) = -\log|\tau-\sigma|$ holds modulo addition of bounded functions when σ, τ sufficiently close to 0. One can observe that the second term on the right hand side is bounded near the origin $\sigma = 0\in\Delta$. Furthermore, from the calculation

$$\frac{2}{|\tau|^2\rho^2}-\frac{2}{|\tau|^2(-\log|\tau|^2)^2} = \frac{-2\log h(\log h+2\log|\tau|^2)}{|\tau|^2\rho^2(-\log|\tau|^2)^2},$$

the third term on the right hand side is bounded near the origin $\sigma = 0\in\Delta$ because $\int_0^{1/2}\frac{1}{r(\log r)^2}dr < \infty$. Combining them, we have that, modulo addition of bounded functions near the origin,

$$\log\frac{V}{h}(z',\sigma) = 2\int_{\tau\in\Delta} G(\tau,\sigma)\frac{\partial^2\log(V/h)}{\partial\tau\partial\overline{\tau}}(z',\tau)\sqrt{-1}d\tau\wedge d\overline{\tau}.$$

Denote the zero set of V by Z which is contained in some proper pluripolar subset of D. Then for any $z'\notin Z$, $\log\frac{V}{h}(z',\sigma)$ is bounded as $\sigma\to 0$, and so

$$|\tau|^2\rho^2\frac{\partial^2\log(V/h)}{\partial\tau\partial\overline{\tau}}(z',\tau)\to 0$$

as $\tau\to 0$ since $\int_0^{1/2}\frac{1}{r\log r}dr = \infty$. This successively leads to for $z'\notin Z$, a convergence

$$\begin{aligned}(2)\qquad |\sigma|^2\rho^2\omega_{\sigma\overline{\sigma}}(z',\sigma) &= \frac{1}{2\pi}\left(|\sigma|^2\rho^2\frac{\partial^2\log(V/h)}{\partial\sigma\partial\overline{\sigma}}+2|\sigma|^2\rho\frac{\partial^2\log h}{\partial\sigma\partial\overline{\sigma}}\right.\\ &\quad\left.+2|\sigma|^2\frac{\partial\log h}{\partial\sigma}\frac{\partial\log h}{\partial\overline{\sigma}}+2\overline{\sigma}\frac{\partial\log h}{\partial\overline{\sigma}}+2\sigma\frac{\partial\log h}{\partial\sigma}+2\right)\to\frac{1}{\pi}\end{aligned}$$

as $\sigma \to 0$. On the other hand, we have

$$\omega_{i\bar{j}} = \frac{1}{2\pi}\left(\frac{\partial^2 \log(V/h)}{\partial z^i \partial \bar{z}^j} + \frac{2}{\rho}\frac{\partial^2 \log h}{\partial z^i \partial \bar{z}^j} + \frac{2}{\rho^2}\frac{\partial \log h}{\partial z^i}\frac{\partial \log h}{\partial \bar{z}^j}\right).$$

It is immediate from the precompactness of plurisubharmonic functions uniformly bounded above that $\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log(V/h)\big|_D$ makes sense as a closed semipositive $(1,1)$-current on D, and also that passing to a subsequence, we have the weak convergence $\sqrt{-1}\omega_{i\bar{j}}dz^i \wedge d\bar{z}^j\Big|_{(\sigma=\epsilon)} \rightharpoonup \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log(V/h)\big|_D$ on D as $\epsilon \to 0$. Therefore their wedge product behaves from [6, Proposition 2.1] as

$$\limsup_{\epsilon\to 0}\left(\sqrt{-1}\omega_{i\bar{j}}dz^i \wedge d\bar{z}^j\Big|_{(\sigma=\epsilon)}\right)^{n-1} \le \left(\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log\frac{V}{h}\bigg|_D\right)^{n-1}_{\mathrm{ac}} \tag{3}$$

on D, where ac means the absolutely continuous part. If we take a limit in the inequality (1) as $\epsilon \to 0$, then we find from (2) and (3) that

$$\pi\frac{V}{h}\bigg|_D \le n\left(\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log\frac{V}{h}\bigg|_D\right)^{n-1}_{\mathrm{ac}}$$

on $D \setminus Z$. This inequality holds even on the whole boundary D since V vanishes on Z. Here $\frac{V}{h}\big|_D$ can be thought of as a pseudo-volume form, say V_D, on D. Note that V_D has zeros on Z. Then the above inequality means that the semipositive $(1,1)$-current $\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log V_D \in c_1(K_D)$ satisfies

$$\left(\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log V_D\right)^{n-1}_{\mathrm{ac}} \ge \frac{\pi}{n}V_D, \tag{4}$$

and this can conclude the bigness of K_D as desired thanks to the following theorem due to S. Boucksom:

Theorem 2.2 (Theorem 1.2 in [6])**.** *Let L be a pseudoeffective line bundle on a compact Kähler manifold M. Then the volume of L satisfies*

$$\mathrm{vol}(L) = \max_T \int_M (T_{\mathrm{ac}})^n$$

for T ranging among all closed positive $(1,1)$-currents in $c_1(L)$.

Indeed, if this formula is applied with $M = D$, $L = K_D$ and $T = \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log V_D$, then (4) implies

$$\mathrm{vol}(K_D) \ge \int_D \left(\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log V_D\right)^{n-1}_{\mathrm{ac}} \ge \frac{\pi}{n}\int_D V_D > 0.$$

Since the condition $\mathrm{vol}(L) > 0$ is equivalent to the bigness of L, we can finally achieve that K_D is big. □

The inequality (4) can be actually shown to be the equality in [27, Theorem 1.1]. Hence this V_D is the singular Kähler-Einstein volume form, and $\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log V_D$ is the singular Kähler-Einstein metric on D.

The main theorem for volume growth mentioned at Introduction provides an affirmative solution for one implication of Conjecture 2.1 in the case of $\kappa = 0$. We restate it again with our notation.

Theorem 2.3 (Theorem 1.2 in Introduction)**.** *If D is Calabi-Yau, namely the Kodaira dimension $\kappa(K_D) = 0$, then the Kähler-Einstein volume form is described as*

$$(\omega_X)^n = \frac{V}{\|\sigma\|^2\rho^{n+1}},$$

where V is a bounded volume form on $\overline{X}$ which vanishes nowhere.

The proof will be given in Section 4.2. To show the converse statement has not succeeded yet, and we will investigate it in a forthcoming paper.

2.2. **Residue of Kähler-Einstein metric.** In this subsection, we consider the limit of the Kähler-Einstein metric after restriction to a complex hypersurface tangent to the boundary as the hypersurface is made to approach the boundary. This limit is investigated in [37], [53], [27].

Here and hereafter we use σ also as a coefficient function of a section σ with respect to a local trivialization of the line bundle $\mathcal{O}_{\overline{X}}(D)$. We take any holomorphic local coordinates $(z^1, z^2, \cdots, z^{n-1}, z^n)$ of $\overline{X}$ near D with $z^n = \sigma$, and denote a local smooth hypersurface parallel to D by $(\sigma = \epsilon) = \{(z^1, z^2, \cdots, z^{n-1}, \sigma)\,;\, \sigma = \epsilon\}$ for every small complex number ϵ. We would like to determine what is the limit of $\omega_X|_{(\sigma=\epsilon)}$ as $\epsilon \to 0$. The limit is called as follows in this paper:

Definition 2.1. *If the limit of $\omega_X|_{(\sigma=\epsilon)}$ as $\epsilon \to 0$ exists for any holomorphic local coordinates $(z^1, z^2, \cdots, z^{n-1}, z^n)$ of $\overline{X}$ near D with $z^n = \sigma$, the limit form or current on D is called the residue along D and denoted by* $\operatorname{Res}_D \omega_X$.

It is immediate to know that $\operatorname{Res}_D \omega_X$ is a closed semi-positive $(1,1)$-form or $(1,1)$-current on D. The name of the limit due to Yau-Zhang [53] is the restriction to D although their notation $\operatorname{Res}_D \omega_X$ is the same as ours. However it is impossible to restrict ω_X to D in a usual sense since ω_X generally diverges along D. The reasons of our naming are that the restriction map

$$\mathcal{O}_{\overline{X}}(K_{\overline{X}} + D) \to \mathcal{O}_D(K_D)$$

which induces the adjunction formula is called the Poincaré residue, and that the image of ω_X is expected to be the limit $\operatorname{Res}_D \omega_X$.

The cohomology class $c_1(K_D)$ is known to contain the canonical metric, so called the generalized Kähler-Einstein metric in the sense of Song-Tian and H. Tsuji. Let $\kappa = \kappa(K_D) \in \{0, 1, \cdots, n-1\}$ be the Kodaira dimension of K_D, which measures how the positivity of K_D degenerates. For sufficiently large m, the pluri-log-canonical morphism $\Phi_X = \Phi_{|m(K_{\overline{X}}+D)|}$ is isomorphism on X, and the restriction $\Phi_X|_D$ to D

of Φ_X coincides with the pluri-canonical morphism $\Phi_D = \Phi_{|mK_D|} : D \to D_{\mathrm{can}}$ ([27, Lemma 2.1]), where D_{can} is the canonical model of D whose dimension is actually κ. Define a smooth $(1,1)$-form $\theta_0 = \Phi_X^* \chi \in c_1(K_{\overline{X}} + D)$ for some smooth Kähler form χ on $\Phi_X(\overline{X})$, and then θ_0 is semipositive on $\overline{X}$ and positive on X.

Theorem 2.4 (Theorem 3.1 in [40], [46])**.** *There exists a unique closed semipositive* $(1,1)$*-current* $\omega_D \in c_1(K_D)$ *on* D *which is expressed as* $\omega_D = \Phi_D^* \omega_{\mathrm{can}}$ *for some closed semipositive current* $\omega_{\mathrm{can}} = \chi|_{D_{\mathrm{can}}} + \sqrt{-1}\partial\overline{\partial} u_{\mathrm{can}}$ *on* D_{can} *with the following conditions*:

(i) ω_{can} *is a smooth Kähler form on* $D_{\mathrm{can}}^\circ = D_{\mathrm{can}} \setminus S$*, where* S *is an analytic subset of* D_{can} *consisting of singular values of* Φ_D *and the singular locus of* D_{can}*,*

(ii) ω_{can} *satisfies the equation on* D_{can}°:

$$-\mathrm{Ric}(\omega_{\mathrm{can}}) + \omega_{\mathrm{WP}} = \omega_{\mathrm{can}},$$

where ω_{WP} *is the Weil-Petersson metric on* D_{can}° *associated with the Ricci-flat fibration* Φ_D*,*

(iii) u_{can} *is continuous on* D_{can}*.*

In the present paper, this current ω_{can} *and also* $\omega_D = \Phi_D^* \omega_{\mathrm{can}}$ *are both called the generalized Kähler-Einstein metric of* D_{can} *and* D*, respectively.*

They call ω_{can} the canonical metric or the canonical Kähler current in [40] or [46], respectively.

As we said, we expect that the residue $\mathrm{Res}_D\, \omega_X$ is also a representative of $c_1(K_D)$, and it seems likely that $\mathrm{Res}_D\, \omega_X$ should be canonical. Comparing with the generalized Kähler-Einstein metric, we make a conjecture concretely about canonicity of the residue of the Kähler-Einstein metric, which appears also in [27] without use of the residue.

Conjecture 2.2 (Conjecture in [27])**.**

$$\mathrm{Res}_D\, \omega_X = \omega_D.$$

There exist several examples supporting the truth of this conjecture.

Example 2.6 (G. Schumacher [37])**.** G. Schumacher investigated this problem in [37] when $K_{\overline{X}} + D$ is ample on $\overline{X}$, namely its positivity is nondegenerate even on D. He did not use the terminology "residue" there. Note that in this case, there also exists the usual Kähler-Einstein metric ω_D on D thanks to the adjunction formula $(K_{\overline{X}} + D)\,|_D = K_D$ and the resolution of the Calabi conjecture due to T. Aubin and S.-T. Yau. Schumacher's theorem is stated as follows:

Theorem 2.5 (Theorem 1 in [37])**.** *If* $K_{\overline{X}} + D$ *is ample on the whole space* $\overline{X}$*, then we have* $\mathrm{Res}_D\, \omega_X = \omega_D$*.*

His proof is based on analysis of the corresponding complex Monge-Ampère equation for decay estimates of the potential function. Needless to say, Theorem 2.5 assures that the above conjecture for the residue would be true.

Example 2.7 (R. Kobayashi [29]). When $\overline{X}$ is 2-dimensional, it holds whether the log-canonical bundle $K_{\overline{X}} + D$ is ample on $\overline{X}$ or D is an elliptic curve. The former case is included in the previous example. In the latter case, there exists a smooth Hermitian metric h of $\mathcal{O}_{\overline{X}}(D)$ such that $\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log h|_D$ is a flat metric on D which represents $c_1(-D|_D) = c_1(-N_D)$, where N_D means the normal bundle of D in $\overline{X}$. $\|\cdot\|$ stands for the norm with respect to h and set $\rho = -\log\|\sigma\|^2$. Then it is proved that ω_X can be written as

$$\omega_X = \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log\frac{\Omega}{h} + 3\left(\frac{\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log h}{\rho} + \frac{\sqrt{-1}}{2\pi}\partial\log\rho\wedge\overline{\partial}\log\rho\right)$$

for some bounded volume form Ω on $\overline{X}$. This representation leads to the existence of the limit $\mathrm{Res}_D\,\omega_X$, and the equality

$$\mathrm{Res}_D\,\omega_X = \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log\frac{\Omega}{h}\Big|_D = 0$$

holds indeed because of $\mathcal{O}_D(K_D) = \mathcal{O}_D$. The last equation means that Conjecture 2.2 is correct for this example since 0 can be thought of as the generalized Kähler-Einstein metric in this case. Moreover if we apply Theorem 1.3, we find more strongly

$$\mathrm{Res}_D\,(\rho\,\omega_X) = 3\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log h|_D \in c_1(-3N_D).$$

Example 2.8 (N. Mok [33]). Let $\overline{X}$ be a smooth toroidal compactification of an n-dimensional complex hyperbolic manifold with a boundary divisor D.

In this case, as in Example 2.7, there exists a smooth Hermitian metric h of $\mathcal{O}_{\overline{X}}(D)$ such that $\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log h|_D$ is a flat metric on D which represents $c_1(-D|_D) = c_1(-N_D)$, where N_D means the normal bundle of D in $\overline{X}$. $\|\cdot\|$ stands for the norm with respect to h and set $\rho = -\log\|\sigma\|^2$. N. Mok proved in [33, Section 1.2, 1.3] by examining carefully the explicit construction of toroidal compactification that ω_X can be expressed as near the boundary D,

$$\omega_X = (n+1)\left(\frac{\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log h}{\rho} + \frac{\sqrt{-1}}{2\pi}\partial\log\rho\wedge\overline{\partial}\log\rho\right).$$

From this formula, we observe

$$\mathrm{Res}_D\,\omega_X = 0,$$

$$\mathrm{Res}_D\,(\rho\,\omega_X) = (n+1)\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log h|_D \in c_1(-(n+1)N_D).$$

Note that as in Example 2.7, the first equation supports the truth of Conjecture 2.2.

According to Example 2.7 and 2.8, it is plausible to replace Conjecture 2.2 in the case of a Calabi-Yau boundary with the following stronger version:

Conjecture 2.3. *When D is Calabi-Yau, then $\mathrm{Res}_D\,(\rho\,\omega_X)$ coincides with the Ricci-flat Kähler metric in $c_1(-(n+1)N_D)$, where N_D is the normal bundle of D in $\overline{X}$.*

Next we consider the residue in several cases when $K_{\overline{X}} + D$ is not ample on $\overline{X}$ and the Kodaira dimension $\kappa = \kappa(K_D)$ is positive.

Example 2.9 ([27])**.** Suppose $\overline{X}$ satisfies that $(K_{\overline{X}} + D)|_D = K_D$ is big, namely D is of general type, in addition to our positivity assumption. In this case, the generalized Kähler-Einstein metric ω_D is also called the (canonical) Kähler-Einstein current in [44] or the singular Kähler-Einstein metric in [43], [15]. Moreover it is older and has been studied more than the other cases. Let us provide a definition of this object again tailored to the present situation. A closed semipositive $(1,1)$-current $\omega_D \in c_1(K_D)$ on D is said to be the singular Kähler-Einstein metric if $\omega_D = \theta_0|_D + \sqrt{-1}\partial\overline{\partial}u_D$ satisfies the following:

(i) ω_D is a smooth Kähler form outside the non-ample locus $\mathbf{B}_+(K_D)$,

(ii) ω_D satisfies the usual Kähler-Einstein equation $-\mathrm{Ric}(\omega_D) = \omega_D$ outside $\mathbf{B}_+(K_D)$,

(iii) the potential function u_D is continuous on D.

In this case, the author approximated ω_X by a normalized Kähler-Ricci flow to compare the residue of ω_X with that of the Kähler-Ricci flow. Consequently, it is proved in [27, Theorem 1.1, Theorem 2.5] that

$$\mathrm{Res}_D\, \omega_X = \omega_D.$$

This also solves Conjecture 2.2 affirmatively in this case.

Example 2.10 (cf. Wang [47], Yau-Zhang [53])**.** Let $X = \mathbb{H}_g/\Gamma$ $(g \geq 2)$ be a Siegel modular variety given by a neat arithmetic group $\Gamma \subset \mathrm{Sp}(g,\mathbb{Q})$. Take a smooth toroidal compactification $\overline{X}$ with a simple normal crossing boundary divisor with respect to an admissible family of fans. Since D has generally several components and they possibly intersect each other, instead of the residue along the whole boundary, we consider the residue $\mathrm{Res}_{D^o}\, \omega_X$ for each component D. Yau-Zhang [53] and W. Wang [47] investigate the boundary behavior of ω_X and obtain various properties about the Hodge structure, the log-canonical bundle and an explicit description of the volume form near an intersection point of boundary components. However it is not explored at all there what the residue actually is, and so we will calculate the residue here with the same notation as in Example 2.5.

By the same reason as in Example 2.5, it suffices to deal with the boundary component D corresponding to the standard cusp with depth 1. For the purpose, we use the coordinates $(\tau', \tau'', \sigma) \in \mathbb{H}_{g-1} \times \mathbb{C}^{g-1} \times \mathbb{C}$ near D^o introduced in (iii) and (iv) to describe $\mathrm{Res}_{D^o}\, \omega_X$. First we find a relation between two residues $\mathrm{Res}_{D^o}\, \omega_X$ and $\mathrm{Res}_{E^o}\, \omega_X$ as

$$(p)^* \mathrm{Res}_{D^o}\, \omega_X = \lim_{\epsilon \to 0} (p)^* \omega_X|_{(\sigma=\epsilon)} = \lim_{\epsilon \to 0} \omega_Y|_{(\sigma=\epsilon)} = \mathrm{Res}_{E^o}\, \omega_Y.$$

So we calculate $\mathrm{Res}_{E^o}\,\omega_Y$ to have

$$\mathrm{Res}_{E^o}\,\omega_Y = \lim_{\epsilon\to 0}\omega_Y|_{(\sigma=\epsilon)} = \lim_{\epsilon\to 0}\frac{g+1}{8\pi}\mathrm{tr}\,\left(\sqrt{-1}(\mathrm{Im}\;\tau)^{-1}d\tau(\mathrm{Im}\;\tau)^{-1}d\overline{\tau}\right)\Big|_{(\sigma=\epsilon)}$$

$$= \lim_{\epsilon\to 0}\frac{(g+1)\sqrt{-1}}{8\pi}\mathrm{tr}\left(\begin{bmatrix}\mathrm{Im}\,\tau' & \mathrm{Im}\,\tau'' \\ \mathrm{Im}\,{}^t\tau'' & -\frac{k}{2\pi}\log|\epsilon|\end{bmatrix}^{-1}\begin{bmatrix}d\tau' & d\tau'' \\ d\,{}^t\tau'' & 0\end{bmatrix}\right.$$

$$\left.\times\begin{bmatrix}\mathrm{Im}\,\tau' & \mathrm{Im}\,\tau'' \\ \mathrm{Im}\,{}^t\tau'' & -\frac{k}{2\pi}\log|\epsilon|\end{bmatrix}^{-1}\begin{bmatrix}d\overline{\tau'} & d\overline{\tau''} \\ d\,\overline{{}^t\tau''} & 0\end{bmatrix}\right).$$

If we use the Landau symbol $O(1)$ as $\epsilon\to 0$ and denote by Δ' the adjugate matrix of $\mathrm{Im}\,\tau'$, then

$$\mathrm{Res}_{E^o}\,\omega_Y = \lim_{\epsilon\to 0}\frac{(g+1)\sqrt{-1}}{8\pi}\frac{1}{(\frac{k}{2\pi}\log|\epsilon|\times\det\mathrm{Im}\,\tau' + O(1))^2}$$

$$\times\,\mathrm{tr}\left(\begin{bmatrix}-\frac{k}{2\pi}\log|\epsilon|\Delta' + O(1) & O(1) \\ O(1) & O(1)\end{bmatrix}\begin{bmatrix}d\tau' & d\tau'' \\ d\,{}^t\tau'' & 0\end{bmatrix}\right.$$

$$\left.\times\begin{bmatrix}-\frac{k}{2\pi}\log|\epsilon|\Delta' + O(1) & O(1) \\ O(1) & O(1)\end{bmatrix}\begin{bmatrix}d\overline{\tau'} & d\overline{\tau''} \\ d\,\overline{{}^t\tau''} & 0\end{bmatrix}\right)$$

$$= \frac{(g+1)\sqrt{-1}}{8\pi}\frac{1}{(\det\mathrm{Im}\,\tau')^2}\mathrm{tr}\left(\begin{bmatrix}\Delta' & 0 \\ 0 & 0\end{bmatrix}\begin{bmatrix}d\tau' & d\tau'' \\ d\,{}^t\tau'' & 0\end{bmatrix}\begin{bmatrix}\Delta' & 0 \\ 0 & 0\end{bmatrix}\begin{bmatrix}d\overline{\tau'} & d\overline{\tau''} \\ d\,\overline{{}^t\tau''} & 0\end{bmatrix}\right)$$

$$= \frac{(g+1)\sqrt{-1}}{8\pi}\mathrm{tr}\left(\begin{bmatrix}(\mathrm{Im}\,\tau')^{-1} & 0 \\ 0 & 0\end{bmatrix}\begin{bmatrix}d\tau' & d\tau'' \\ d\,{}^t\tau'' & 0\end{bmatrix}\begin{bmatrix}(\mathrm{Im}\,\tau')^{-1} & 0 \\ 0 & 0\end{bmatrix}\begin{bmatrix}d\overline{\tau'} & d\overline{\tau''} \\ d\,\overline{{}^t\tau''} & 0\end{bmatrix}\right)$$

$$= \frac{g+1}{8\pi}\mathrm{tr}\left(\sqrt{-1}(\mathrm{Im}\,\tau')^{-1}d\tau'(\mathrm{Im}\,\tau')^{-1}d\overline{\tau'}\right)$$

$$= (\pi_Y)^*\left(\frac{g+1}{g}\omega_{N^o}\right).$$

Here $\omega_{N^o} = \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log\frac{1}{(\det\mathrm{Im}\;\tau')^g}$ is the Poincaré-Bergman metric on the Siegel modular variety $N^o = \mathbb{H}_{g-1}/\Gamma_{g-1}(k)$ of low degree. Notice that $\omega_{N^o} = (p^*)^*\omega_{M^o}$ holds for the Poincaré-Bergman metric ω_{M^o} on $M^o = \mathbb{H}_{g-1}/\Gamma'_{g-1}$. Hence from the commutative diagram of maps and the relation between two residues, we finally conclude that the residue $\mathrm{Res}_{D^o}\,\omega_X$ is written as

$$\mathrm{Res}_{D^o}\,\omega_X = (\pi_X)^*\left(\frac{g+1}{g}\omega_{M^o}\right).$$

On the other hand, it easily follows from (iv) that the Weil-Petersson metric ω_{WP} for $\pi_X : D^o \to M^o$ is given by

$$\omega_{\mathrm{WP}} = -\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log(k^{g-1}\det\mathrm{Im}\,\tau') = \frac{1}{g}\omega_{M^o}.$$

Therefore we obtain

$$-\mathrm{Ric}\left(\frac{g+1}{g}\omega_{M^o}\right)+\omega_{\mathrm{WP}}=\omega_{M^o}+\frac{1}{g}\omega_{M^o}=\frac{g+1}{g}\omega_{M^o}.$$

Namely this means that the residue $\omega_{D^o}=\mathrm{Res}_{D^o}\,\omega_X$ satisfies the generalized Kähler-Einstein equation

$$-\mathrm{Ric}\,(\omega_{D^o})+\omega_{\mathrm{WP}}=\omega_{D^o}$$

if we neglect the pull-back by π_X. We should remark that $\mathrm{Res}_{D^o}\,\omega_X$ is not generally the generalized Kähler-Einstein metric of D since $\mathrm{Res}_{D^o}\,\omega_X\notin \mathrm{c}_1(K_D)$ might happen. The author guesses $\mathrm{Res}_{D^o}\,\omega_X\in \mathrm{c}_1(K_D+\sum_{D_j\neq D}D\cap D_j)$. Note further that there does not always exist the generalized Kähler-Einstein metric on N since E is rational for $k=3$ or $K3$ for $k=4$ in the case that $g=2$ ([39, Example 5.4, 7.7]).

The main theorem for residue stated at Introduction treats the special case of Conjecture 2.2 and 2.3. We restate it again with our notations.

Theorem 2.6 (Theorem 1.3 in Introduction)**.** *If D is a smooth finite quotient of an abelian variety, then the residue* $\mathrm{Res}_D\,\omega_X=0$ *holds, and moreover the weighted residue* $\mathrm{Res}_D\,(\rho\,\omega_X)$ *is the Ricci-flat Kähler metric in* $\mathrm{c}_1(-(n+1)N_D)$ *of D.*

A proof will be given in Section 4.1.

3. Complex Monge-Ampère equation for Calabi-Yau boundary

We assume hereafter that D is Calabi-Yau. In this section, we construct a nice complete Kähler metric which will be regarded as a reference metric for the Kähler-Einstein metric ω_X, and some important property is provided to get suitable estimates of a potential function of ω_X with respect to the reference metric.

3.1. Construction of reference complete Kähler metric.

Take a positive integer m such that $(K_D)^{\otimes m}$ is a trivial line bundle. From our assumption on positivity of $K_{\overline{X}}+D$ and Kodaira's lemma, $(K_{\overline{X}}+D)-\alpha D=K_{\overline{X}}+(1-\alpha)D$ is ample on the whole space $\overline{X}$ for an arbitrarily small positive number $\alpha>0$. Therefore the adjunction formula yields that the normal bundle $N_D=D|_D=-\dfrac{1}{\alpha}(K_{\overline{X}}+(1-\alpha)D)|_D$ is negative. Then from Grauert's theorem ([21]), there is a neighborhood N of D which is holomorphically isomorphic to a neighborhood of the zero section in N_D. Note that at any point on D (which is identified with the zero section), we can take a holomorphic local coordinate system $(z^1,z^2,\cdots,z^{n-1},z^n)$ of N_D near the point such that z^1, z^2, $\cdots$, z^{n-1} are local coordinates of D and z^n is a fiber coordinate of the line bundle N_D. We denote $z'=(z^1,z^2,\cdots,z^{n-1})$ for simplicity. Further z^n becomes a local expression of σ, and so we write $\sigma=z^n$ as before. Therefore via Grauert's isomorphism, this coordinate system $(z^1,z^2,\cdots,z^{n-1},z^n)=(z',\sigma)$ can be considered on $\overline{X}$ near D.

Next we take a special smooth volume form Ω_D on D and a special smooth Hermitian metric h_D on $\mathcal{O}_{\overline{X}}(D)|_D$ using the Calabi-Yau structure of D. Since $(K_D)^{\otimes m}$

is a trivial bundle, there exists a nowhere vanishing m-ple holomorphic volume form $\Omega_D^{(m)}$ on D, and it induces a smooth volume form $\Omega_D = \left(\sqrt{-1}^{(n-1)^2 m}\Omega_D^{(m)} \wedge \overline{\Omega_D^{(m)}}\right)^{1/m}$ on D. This notation means that if $\Omega_D^{(m)} = \omega(dz^1 \wedge dz^2 \wedge \cdots \wedge dz^{n-1})^{\otimes m}$ locally, then

$$\Omega_D = \sqrt{-1}^{(n-1)^2}|\omega|^{2/m} dz^1 \wedge dz^2 \wedge \cdots \wedge dz^{n-1} \wedge d\bar{z}^1 \wedge d\bar{z}^2 \wedge \cdots \wedge d\bar{z}^{n-1}.$$

We may assume that after some constant multiple with Ω_D,

$$n(n+1)^n c_1(-N_D)^{n-1} = 2\pi \int_D \Omega_D.$$

Then thanks to the resolution of the Calabi conjecture due to S.-T. Yau ([50]), there exists a unique Kähler metric $\omega_D \in c_1(-N_D)$ on D such that

$$n(n+1)^n(\omega_D)^{n-1} = 2\pi\Omega_D. \tag{5}$$

Then since $\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log\Omega_D = 0$, one can easily find that ω_D is a Ricci-flat Kähler metric. Moreover we can take a smooth Hermitian metric h_D on $\mathcal{O}_{\overline{X}}(D)|_D = \mathcal{O}_D(N_D)$ such that $\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log h_D = \omega_D$, where h_D is also regarded as a local smooth function $h_D = h_D(z^1, \cdots, z^{n-1})$. We also treat Ω_D similarly.

We consider to extend these objects Ω_D and h_D to the whole $\overline{X}$ appropriately. First remark that h_D can be regarded as a smooth metric of $\mathcal{O}_{\overline{X}}(D)$ on a neighborhood N. It means that $h_D = h_D(z^1, \cdots, z^{n-1})$ is thought of simply as a metric independent of $z^n = \sigma$ in the above special coordinates. We extend further the metric to $\overline{X}$ arbitrarily, and denote by h the resultant metric of $\mathcal{O}_{\overline{X}}(D)$ on $\overline{X}$. Then the curvature form $\Theta = \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log h$ is semipositive on N, and further $\Theta > 0$ in the directions parallel to D over N, i.e., in the subspace spanned by $\frac{\partial}{\partial z^1}, \cdots, \frac{\partial}{\partial z^{n-1}}$. As for Ω_D, $\Omega_D^{(m)}$ can be also considered as a holomorphic section of $m(K_{\overline{X}} + D)$ on N. From our assumption on positivity of $K_{\overline{X}} + D$, there is a smooth $(1,1)$-form $\theta_0 \in c_1(K_{\overline{X}} + D)$ satisfying $\theta_0 > 0$ on X and $\theta_0|_D = 0$. In fact, if we write $\theta_0 = \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log\Omega_X$ for a Hermitian metric Ω_X on $-(K_{\overline{X}} + D)$, then Ω_X is obtained as follows. We can produce holomorphic sections $\{\Omega_i^{(km)}\}_{i=1}^N$ of $km(K_{\overline{X}} + D)$ for sufficiently large k such that they separate points and tangent vectors in X and all the restrictions to D coincide with $(\Omega_D^{(m)})^{\otimes k}$. Let Ω_X be a Bergman kernel composed of them

$$\Omega_X = \frac{1}{N}\sum_{i=1}^N \left(\sqrt{-1}^{n^2 km}\Omega_i^{(km)} \wedge \overline{\Omega_i^{(km)}}\right)^{1/km}.$$

It has an important property that

$$\Omega_X|_D = \frac{1}{N}\sum_{i=1}^N \left(\sqrt{-1}^{(n-1)^2 m}\Omega_D^{(m)} \wedge \overline{\Omega_D^{(m)}}\right)^{1/m} = \frac{1}{N}\sum_{i=1}^N \Omega_D = \Omega_D,$$

which implies $\theta_0|_D = \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log\Omega_X|_D = 0$.

We set a smooth volume form $\Omega = \Omega_X h$ on $\overline{X}$ which means the product of their coefficients in a local expression. We define the following reference Kähler form on X with h and Ω (after a suitable small constant multiple of σ if necessary):

$$
\begin{aligned}
\Omega_0 &= \frac{\Omega}{\|\sigma\|^2(-\log\|\sigma\|^2)^{n+1}} = \frac{\Omega}{\|\sigma\|^2\rho^{n+1}}, \\
\omega_0 &= \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log\Omega_0 = \theta_0 + (n+1)\frac{\Theta}{\rho} + (n+1)\frac{\sqrt{-1}}{2\pi}\partial\log\rho\wedge\overline{\partial}\log\rho
\end{aligned}
\tag{6}
$$

where $\rho = -\log\|\sigma\|^2 > 0$ and $\|\cdot\|$ stands for the norm with respect to h. This ω_0 is a complete Kähler metric on X. The reason is that first ω_0 is positive away from D because θ_0 is strictly positive there. ω_0 also keeps positive even on $N \setminus D$ because θ_0 is positive there, the semipositive form $\frac{\Theta}{\rho}$ is strictly positive in the parallel directions to D and $\sqrt{-1}\partial\log\rho\wedge\overline{\partial}\log\rho$ is strictly positive in the normal directions to D. Furthermore this argument enables us to get the completeness since ω_0 dominates $\sqrt{-1}\partial\log\rho\wedge\overline{\partial}\log\rho$ near D. A similar construction was introduced first by Carlson-Griffiths [8, Proposition 2.1], H. Tsuji [45, Lemma 3.1]. In a local situation of ours, an almost same reference Kähler metric appears in [16], [18], [11].

We explore the curvature of this metric ω_0. The next property is also mentioned in [16] and [11] without any rigorous argument, but we provide a detailed proof here for the convenience of the readers.

Proposition 3.1. *The reference metric ω_0 has bounded geometry in the sense of Cheng-Yau* ([42, Definition 1.3], [10, Definition 1.1])*, which is sometimes called quasi-bounded geometry, if and only if D is a smooth finite quotient of an abelian variety.*

Proof. First we suppose that D is a smooth finite quotient of an abelian variety. By taking the pull-back by a finite covering map, we may assume that D is an abelian variety. Then it is easy to check in the same manner with [29, Lemma 4], [16, Section 3] that a neighborhood of D is isomorphic to a neighborhood of a boundary component in a toroidal compactification of a complex hyperbolic manifold. Furthermore as in Example 2.8, the metric h of $\mathcal{O}_{\overline{X}}(D)$ can be written as $h = e^{a|z'|^2}$ for some constant $a > 0$ where $z' = (z^1, z^2, \cdots, z^{n-1})$ is the standard coordinate system of the universal cover of D, namely $\mathbb{C}^{n-1}$. Especially, $\Theta = a\sum_{i=1}^{n-1}\sqrt{-1}dz^i\wedge d\overline{z}^i$ and this is the Euclidean metric multiplied by a. Since the last two terms $(n+1)\frac{\Theta}{\rho} + (n+1)\frac{\sqrt{-1}}{2\pi}\partial\log\rho\wedge\overline{\partial}\log\rho$ in ω_0 is just a metric induced by the Poincaré-Bergman metric and the first term θ_0 in ω_0 does not destroy the boundedness of the curvature, it follows that ω_0 has bounded geometry in the sense of Cheng-Yau. In fact, the quasi-coordinate system near D is given by holomorphic maps $\phi_\eta : W \to \overline{X}$ for $0 < \eta < 1$,

$$
(z^1, z^2, \cdots, z^{n-1}, \sigma) = \left(\sqrt{\frac{1+\eta}{1-\eta}}w^1, \sqrt{\frac{1+\eta}{1-\eta}}w^2, \cdots, \sqrt{\frac{1+\eta}{1-\eta}}w^{n-1}, e^{b\sqrt{-1}\frac{1+\eta}{1-\eta}\tau}\right). \tag{7}
$$

Here $W = \{(w', \tau) = (w^1, \cdots, w^{n-1}, \tau) \in \mathbb{C}^n \mid |\tau - \sqrt{-1}|^2 + 4|w'|^2 < R^2|\tau + \sqrt{-1}|^2\}$ for some fixed $R \in (0,1)$ and b is some real constant. Then we can calculate

$$\phi_\eta^* \rho = \frac{1+\eta}{1-\eta}(2b \operatorname{Im} \tau - a|w'|^2), \tag{8}$$

$$\begin{aligned}
\phi_\eta^* \omega_0 = {} & \phi_\eta^* \theta_0 + \frac{a(n+1)}{2b \operatorname{Im} \tau - a|w'|^2} \sum_{i=1}^{n-1} \sqrt{-1} dw^i \wedge d\overline{w}^i \\
& + \frac{n+1}{(2b \operatorname{Im} \tau - a|w'|^2)^2} \frac{\sqrt{-1}}{2\pi} \left(\frac{b}{\sqrt{-1}} d\tau - a \sum_{i=1}^{n-1} \overline{w}^i dw^i \right) \wedge \left(-\frac{b}{\sqrt{-1}} d\overline{\tau} - a \sum_{i=1}^{n-1} w d\overline{w}^i \right).
\end{aligned}$$

Thanks to $\theta_0|_D = 0$, all derivatives of coefficients of $\phi_\eta^* \omega_0$ in (w', τ) are bounded on W.

Next we show the converse implication. At any point $(o, \sigma) \in N$, we directly calculate the curvature tensor $R_{i\bar{i}j\bar{j}}(o,\sigma)$ of ω_0 for $i,\ j \in \{1, 2, \cdots, n-1\}$. Here $z' = (z^1, z^2, \cdots, z^{n-1})$ is taken to be a normal holomorphic coordinate system of D at o for the Calabi-Yau metric ω_D, and $R^D_{i\bar{i}j\bar{j}}(o)$ denotes the curvature tensor of ω_D. We express $\omega_0 = \sqrt{-1}\hat{g}_{k\bar{l}} dz^k \wedge d\overline{z}^l$ and its inverse by $\hat{g}^{\bar{k}l}$, and use σ also as an index regarding the coordinate σ, for instance $\hat{g}^{\bar{k}\sigma}$. If we use the Landau symbol $O\left(\frac{1}{\rho^k}\right)$, $O\left(\frac{1}{|\sigma|\rho^k}\right)$, etc ... as $\sigma \to 0$, then

$$\begin{aligned}
R_{i\bar{i}j\bar{j}}(o,\sigma) = & - \frac{\partial^2 \hat{g}_{i\bar{j}}}{\partial z^j \partial \overline{z}^i}(o,\sigma) + \sum_{k,l=1}^{n-1} \hat{g}^{\bar{l}k}(o,\sigma) \frac{\partial \hat{g}_{j\bar{i}}}{\partial z^k}(o,\sigma) \frac{\partial \hat{g}_{i\bar{j}}}{\partial \overline{z}^l}(o,\sigma) \\
& + \left(\sum_{l=1}^{n-1} \hat{g}^{\bar{l}\sigma}(o,\sigma) \frac{\partial \hat{g}_{j\bar{i}}}{\partial \sigma}(o,\sigma) \frac{\partial \hat{g}_{i\bar{j}}}{\partial \overline{z}^l}(o,\sigma) + \sum_{k=1}^{n-1} \hat{g}^{\overline{\sigma}k}(o,\sigma) \frac{\partial \hat{g}_{j\bar{i}}}{\partial z^k}(o,\sigma) \frac{\partial \hat{g}_{i\bar{j}}}{\partial \overline{\sigma}}(o,\sigma) \right) \\
& + \hat{g}^{\overline{\sigma}\sigma}(o,\sigma) \frac{\partial \hat{g}_{j\bar{i}}}{\partial \sigma}(o,\sigma) \frac{\partial \hat{g}_{i\bar{j}}}{\partial \overline{\sigma}}(o,\sigma) \\
= & O\left(\frac{1}{\rho^2}\right) + \frac{n+1}{\rho} R^D_{i\bar{i}j\bar{j}}(o) + \sum_{k,l=1}^{n-1} \left(\frac{\rho}{n+1} \delta_{k\bar{l}} + O(1) \right) O\left(\frac{1}{\rho^2}\right) O\left(\frac{1}{\rho^2}\right) \\
& + O(|\sigma|\rho^2) O\left(\frac{1}{\rho^2}\right) O\left(\frac{1}{\rho^2|\sigma|}\right) + \frac{|\sigma|^2\rho^2}{n+1} \left(1 + O\left(\frac{1}{\rho}\right)\right) O\left(\frac{1}{\rho^2|\sigma|}\right) O\left(\frac{1}{\rho^2|\sigma|}\right) \\
= & \frac{n+1}{\rho} R^D_{i\bar{i}j\bar{j}}(o) + O\left(\frac{1}{\rho^2}\right).
\end{aligned}$$

Since $\hat{g}_{i\bar{i}}(o,\sigma) = (n+1)\rho^{-1} + O(\rho^{-2})$, we find that the bisectional curvature of $\frac{\partial}{\partial z^i}$ and $\frac{\partial}{\partial z^j}$ at (o,σ) becomes

$$\frac{R_{i\bar{i}j\bar{j}}(o,\sigma)}{\hat{g}_{i\bar{i}}(o,\sigma)\hat{g}_{j\bar{j}}(o,\sigma)} = \frac{R^D_{i\bar{i}j\bar{j}}(o)}{n+1}\rho + O(1).$$

If D is not a smooth finite quotient of an abelian variety, then $R^D_{i\bar{i}j\bar{j}}(o) \neq 0$ for some point $o \in D$. Thus we conclude from this formula that this bisectional curvature of ω_0 at (o, σ) diverges at a rate equal to ρ as $\sigma \to 0$. □

3.2. **Complex Monge-Ampère equation with a reference metric ω_0.** The Kähler-Einstein metric ω_X on X satisfies the equation $-\mathrm{Ric}(\omega_X) = \omega_X$. In our setting, this equation can be reduced to the complex Monge-Ampère equation

$$(\omega_X)^n = \left(\omega_0 + \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}u_0\right)^n = e^{u_0}\,\Omega_0 \tag{9}$$

for a potential u_0 of ω_X with $\omega_X = \omega_0 + \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}u_0$. Tian-Yau [42] and S. Bando [3] indeed obtained the existence of u_0 applying a certain modified continuity method with 2 parameters. D. Wu [49] developed the ϵ-approximation method due to S.-T. Yau to show the existence of u_0. Notice that the power of ρ in Ω_0 is actually 2 in their papers and their potential is $u_0 - (n-1)\log\rho$. In order to investigate the residue and the volume growth, we need to get suitable estimates near the boundary D for u_0. The reason is that the right hand side of (9) expresses the growth and we have from (6)

$$\begin{aligned} \mathrm{Res}_D\,\omega_X &= \lim_{\epsilon\to 0}\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}u_0|_{(\sigma=\epsilon)}, \\ \mathrm{Res}_D(\rho\,\omega_X) &= (n+1)\omega_D + \lim_{\epsilon\to 0}\rho\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}u_0|_{(\sigma=\epsilon)}. \end{aligned} \tag{10}$$

For such estimates, the following property is important.

Proposition 3.2. *The reference metric ω_0 is close to the Kähler-Einstein metric near D. Namely if we set $F = \log\dfrac{(\omega_0)^n}{\Omega_0}$, then $F = O(\|\sigma\|\rho)$ holds near D. Especially, the following convergence holds* :

$$\frac{(\omega_0)^n}{\Omega_0}(z) \to 1 \quad (z \to D).$$

Proof. By a direct calculation, we have near D

$$\begin{aligned} (\omega_0)^n &= \left(\theta_0 + (n+1)\frac{\Theta}{\rho}\right)^n + n(n+1)\left(\theta_0 + (n+1)\frac{\Theta}{\rho}\right)^{n-1} \wedge \frac{\sqrt{-1}}{2\pi}\partial\log\rho \wedge \overline{\partial}\log\rho \\ &= \left(\theta_0 + (n+1)\frac{\Theta}{\rho}\right)^n + n(n+1)\left(\rho\theta_0 + (n+1)\Theta\right)^{n-1} \wedge \\ &\left[\frac{\sqrt{-1}\partial\log h \wedge \overline{\partial}\log h}{2\pi\rho^{n+1}} + \left\{\frac{\sqrt{-1}d\sigma \wedge \overline{\partial}\log h}{2\pi\sigma\rho^{n+1}} + \frac{\sqrt{-1}\partial\log h \wedge d\overline{\sigma}}{2\pi\overline{\sigma}\rho^{n+1}}\right\}\right. \\ &\qquad\qquad \left. + \frac{\sqrt{-1}d\sigma \wedge d\overline{\sigma}}{2\pi|\sigma|^2\rho^{n+1}}\right]. \end{aligned}$$

We should remark that all the terms other than the last one become $O(\|\sigma\|\rho)$ after multiplied by $\|\sigma\|^2\rho^{n+1}$ because $\theta_0|_D = 0$. The last term is expanded as

$$\begin{aligned} &n(n+1)\left(\rho\theta_0 + (n+1)\Theta\right)^{n-1} \wedge \frac{\sqrt{-1}d\sigma \wedge d\overline{\sigma}}{2\pi|\sigma|^2\rho^{n+1}} \\ &= n(n+1)\sum_{k=1}^{n-1}\binom{n-1}{k}(\rho\theta_0)^k \wedge \{(n+1)\Theta\}^{n-1-k} \wedge \frac{\sqrt{-1}d\sigma \wedge d\overline{\sigma}}{2\pi|\sigma|^2\rho^{n+1}} \\ &\qquad + \frac{n(n+1)^n\Theta^{n-1}}{2\pi} \wedge \frac{\sqrt{-1}d\sigma \wedge d\overline{\sigma}}{|\sigma|^2\rho^{n+1}}, \end{aligned}$$

and also note that the first term becomes $O(\|\sigma\|\rho)$ after multiplied by $\|\sigma\|^2\rho^{n+1}$ because $\theta_0|_D = 0$ and any components of θ_0 containing $d\sigma$ or $d\overline{\sigma}$ never appear there. The Calabi-Yau equation (5) is applied for the last term to get

$$\frac{(\omega_0)^n}{\Omega_0} = O(\|\sigma\|\rho) + \frac{n(n+1)^n\Theta^{n-1}}{2\pi\Omega_X} \wedge \sqrt{-1}d\sigma \wedge d\overline{\sigma} = O(\|\sigma\|\rho) + 1.$$

Therefore we get the desired equality. □

When D is a smooth finite quotient of an abelian variety, it is possible from Proposition 3.1 to analyze (9) in the same way with [37, Theorem 1]. Otherwise, there are no general methods to estimate u_0 directly. According to [42, Theorem 5.1], [3, Theorem 2.1], we consider to replace ω_0 with a Kähler metric ω_α of bounded geometry in the sense of Cheng-Yau for sufficiently small $\alpha > 0$. Set actually

$$\theta_\alpha = \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log\frac{\Omega}{h^{1-\alpha}} = \theta_0 + \alpha\Theta \in c_1(K_{\overline{X}} + (1-\alpha)D),$$

whose crucial property is to be positive in the parallel directions to D. A singular volume form Ω_α on $\overline{X}$ and a complete Kähler metric ω_α on X is defined as

$$\begin{aligned} \Omega_\alpha &= \frac{\Omega}{\|\sigma\|^{2-2\alpha}\rho^{n+1}}, \\ \omega_\alpha &= \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log\Omega_\alpha = \theta_\alpha + (n+1)\frac{\Theta}{\rho} + (n+1)\frac{\sqrt{-1}}{2\pi}\partial\log\rho \wedge \overline{\partial}\log\rho. \end{aligned}$$

Then it is well-known that (X, ω_α) has bounded geometry in the sense of Cheng-Yau (see, for example, [28, Lemma 2] or [3, Lemma 2.1]). It is thus useful to replace ω_0 with ω_α in (9) to have

$$(\omega_\alpha + \sqrt{-1}\partial\overline{\partial}u_\alpha)^n = e^{u_\alpha}\,\Omega_\alpha \tag{11}$$

for a potential u_α of ω_X with $\omega_X = \omega_\alpha + \sqrt{-1}\partial\overline{\partial}u_\alpha$. Namely $u_0 = u_\alpha + \alpha\log\|\sigma\|^2$ holds. In fact, the equation (11) for $\alpha > 0$ is investigated in [42], [3] by the standard analysis of bounded geometry in the sense of Cheng-Yau to show that u_α and automatically u_0 exist.

In order to get an estimate of u_0 via that of u_α as $\alpha \to 0$, we introduce the Kähler-Ricci flow which corresponds to taking $\alpha \to 0$ in (11) according to [32], [27]. In these papers, they always discuss metrics and volume forms in which the power of ρ is 2 in a denominator of a singular volume form. So we denote

$$\hat{\Omega}_\alpha = \Omega_\alpha \rho^{n-1},$$
$$\hat{\omega}_\alpha = \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log\hat{\Omega}_\alpha = \omega_\alpha + (n-1)\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log\rho.$$

Hence $\hat{}$ means that the power of ρ is 2 in the denominator of the volume form. The (normalized) Kähler-Ricci flow $\omega_X(t)$ is a smooth 1-parameter family of complete Kähler metrics on X satisfying the following partial differential equation:

$$\begin{cases} \dfrac{\partial\omega_X(t)}{\partial t} = -\mathrm{Ric}(\omega_X(t)) - \omega_X(t), \quad t > 0, \\ \omega_X(0) = \hat{\omega}_\alpha. \end{cases} \tag{12}$$

General results are obtained by Lott-Zhang and the author as follows. One might be able to obtain a similar consequence for an initial condition of $\omega_X(0) = \omega_\alpha$ and it might fit for our situation, but we refer to their result as it is.

Theorem 3.1 (Theorem 4.1 of [32], Theorem 3.3 in [27])**.** *The solution $\omega_X(t)$ exists forever and $(X, \omega_X(t))$ has bounded geometry in the sense of Cheng-Yau for any $t \geq 0$. Moreover it satisfies the following* :

- $\omega_X(t) \in c_1(K_{\overline{X}} + (1-\alpha e^{-t})D)$,
- *$\omega_X(t)$ has uniformly bounded k-th covariant derivatives with respect to $\hat{\omega}_\alpha$ for all k on any finite time interval,*
- *$\omega_X(t)$ is uniformly quasi-isometric to $\hat{\omega}_\alpha$ on any finite time interval,*
- *The limit $\omega_X(\infty) = \lim_{t\to\infty}\omega_X(t)$ exists in the C^∞_{loc}-topology and actually coincides with the Kähler-Einstein metric ω_X.*

Indeed, the equation (12) is reduced to the flow for a potential function $u_\alpha(t)$ if we write $\omega_X(t) = \hat{\omega}_{\alpha e^{-t}} + \sqrt{-1}\partial\overline{\partial}u_\alpha(t)$:

$$\begin{cases} \dot{u}_\alpha(t) = \log\dfrac{\left(\hat{\omega}_{\alpha e^{-t}} + \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}u_\alpha(t)\right)^n}{\hat{\Omega}_0} - u_\alpha(t), \quad t > 0, \\ u_\alpha(0) = 0. \end{cases} \tag{13}$$

Here $\dot{u}_\alpha(t) = \dfrac{\partial u_\alpha(t)}{\partial t}$. Theorem 3.1 is actually proved by estimating $u_\alpha(t)$ for properties of $u_\alpha(t)$ corresponding to the above properties of $\omega_X(t)$. Note that the equality $u_\alpha(\infty) + (n-1)\log\rho = u_0$ holds.

4. Proofs of Theorem 1.2 and 1.3

In this section, we will prove Theorem 1.3 and 1.2 in Introduction. Note that a positive constant, say C or C_m, might be different from place to place in what follows.

4.1. **Residue of Kähler-Einstein metric for flat boundary.** In this subsection, we will show Theorem 1.3 or Theorem 2.6 along the line of the proof of Theorem 2.5.

In the present case of a flat boundary, ω_0 has bounded geometry in the sense of Cheng-Yau. So the standard continuity method leads to the existence of the solution u_0 for (9) with bounded covariant derivatives at any order with respect to the reference metric ω_0. It also follows that $\omega_X = \omega_0 + \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}u_0$ is equivalent to ω_0. In addition, we have the volume growth as

$$(\omega_X)^n = e^{u_0}\Omega_0 = \frac{V}{\|\sigma\|^2\rho^{n+1}},$$

where $V = e^{u_0}\Omega$ is a bounded volume form on $\overline{X}$. This means that Conjecture 2.1 is true for boundaries of smooth finite quotients of abelian varieties. For our purpose on residue, it is enough from (10) to estimate uniformly the weighted potential $w := \frac{1}{2\pi}\rho^\lambda\, u_0$ with some appropriate exponent $0 < \lambda \leq 1$.

Proposition 4.1. *If we choose λ to be sufficiently small, then for any integer $m \geq 0$, there exists a positive constant C_m such that*

$$|\nabla^m w| \leq C_m$$

holds on X. Here ∇ denotes the covariant derivative with respect to ω_0

In fact, if this is true, then using the quasi-coordinate map ϕ_η in (7), we have that for any integer $m \geq 0$, there exists a positive constant C_m such that

$$\left|D_{w',\overline{w}'}\{(\rho^\lambda u_0)\circ\phi_\eta\}\right| \leqq C_m$$

holds for any derivative $D_{w',\overline{w}'}$ of order m with respect to w' and $\overline{w}'$. This can be translated into an expression with the original coordinates as

$$\left|\left(\frac{1+\eta}{1-\eta}\right)^{\frac{m}{2}} D_{z',\overline{z}'}(\rho^\lambda u_0)\circ\phi_\eta\right| \leqq C_m,$$

where $D_{z',\overline{z}'}$ means such a derivative that w' is replaced with z' in $D_{w',\overline{w}'}$. Moreover taking (8) into this inequality, we have

$$|D_{z',\overline{z}'}\, u_0| \leqq \frac{C_m}{\rho^{\frac{m}{2}+\lambda}}.$$

Applying the last inequalities for $m \geq 2$, we can consequently show that in (10), the limits $\lim_{\epsilon\to 0}$ become 0 in the C^∞-topology, and we get the desired residue formulas stated in Theorem 1.3 and 2.6.

It is possible to prove Proposition 4.1 in the same manner as a proof of Theorem 2.5, [26, Theorem 1.3] or [27, Theorem 1.1], and so we give only a sketch of proof.

Proof. From (9) and the inequality in [28, page 408], we have

$$\square_{\omega_X}\frac{u_0}{2\pi} \leq u_0 - F = \log\frac{\left(\omega_0 + \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}u_0\right)^n}{(\omega_0)^n} \leq \square_{\omega_0}\frac{u_0}{2\pi},$$

where $\Box_{\omega_0}$ or $\Box_{\omega_X}$ is the Laplacian of ω_0 or ω_X, respectively. This inequality can be made translate into the following two inequalities for the weighted function $w = \frac{\rho^\lambda}{2\pi} u_0$:

$$-\Box_{\omega_X} w + \lambda n \frac{\sqrt{-1}(\partial\rho \wedge \overline{\partial} w + \partial w \wedge \overline{\partial}\rho) \wedge (\omega_X)^{n-1}}{\rho\,(\omega_X)^n} + \left(2\pi - \rho^\lambda \Box_{\omega_X} \rho^{-\lambda}\right) w \geq \rho^\lambda F,$$

$$-\Box_{\omega_0} w + \lambda n \frac{\sqrt{-1}(\partial\rho \wedge \overline{\partial} w + \partial w \wedge \overline{\partial}\rho) \wedge (\omega_0)^{n-1}}{\rho\,(\omega_0)^n} + \left(2\pi - \rho^\lambda \Box_{\omega_0} \rho^{-\lambda}\right) w \leq \rho^\lambda F.$$

Proposition 3.2 implies that $\rho^\lambda F$ and also other coefficients in these inequalities have bounded covariant derivatives at any order. Further similarly to [37, Lemma 5], it is possible to take λ so small that $\left|\rho^\lambda \Box_{\omega_0} \rho^{-\lambda}\right| < \pi$ and $\left|\rho^\lambda \Box_{\omega_X} \rho^{-\lambda}\right| < \pi$ hold on X. This fact enables us to achieve uniform C^0-estimates for w by using Yau's generalized maximum principle.

From similar calculations as in [37, Section 6], or [26, Section 4.4] for an orbifold version, or [27, Section 5.2] for a Kähler-Ricci flow version, it follows that (9) becomes

$$-\frac{1}{G}\Box_{\omega_0}\frac{u_0}{2\pi} - \frac{n e^{u_0 - F}}{G} \frac{\sum_{l=0}^{n-2} (\omega_X)^{n-1-l} \wedge (\omega_0)^l \wedge \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial} u_0}{(\omega_X)^n} + u_0 = F,$$

where $G = g(u_0 - F)$ for a smooth function $g(x) = \dfrac{n(e^x - 1)}{x}$ of a real variable x. Moreover an equation for $w = \frac{\rho^\lambda}{2\pi} u_0$ translated from the last equation is given by

$$\begin{aligned}
&-\frac{1}{G}\Box_{\omega_0} w - \frac{n e^{u_0 - F}}{G} \frac{\sum_{l=0}^{n-2} (\omega_X)^{n-1-l} \wedge (\omega_0)^l \wedge \sqrt{-1}\partial\overline{\partial} w}{(\omega_X)^n} \\
&+ \frac{\lambda n}{\rho\, G} \left(\frac{(\omega_0)^{n-1}}{(\omega_0)^n} + e^{u_0 - F} \frac{\sum_{l=0}^{n-2} (\omega_X)^{n-1-l} \wedge (\omega_0)^l}{(\omega_X)^n} \right) \wedge \sqrt{-1}(\partial\rho \wedge \overline{\partial} w + \partial w \wedge \overline{\partial}\rho) \\
&- \frac{\rho^\lambda}{G} \left(\Box_{\omega_0} \rho^{-\lambda} + n e^{u_0 - F} \frac{\sum_{l=0}^{n-2} (\omega_X)^{n-1-l} \wedge (\omega_0)^l \wedge \sqrt{-1}\partial\overline{\partial}\rho^{-\lambda}}{(\omega_X)^n} \right) w + 2\pi w = \rho^\lambda F.
\end{aligned}$$

Since all the coefficients of this linear elliptic partial differential equation for w have bounded m-th covariant derivatives for all $m \geq 0$, then the interior Schauder estimates (cf. [20]) can be applied to obtain higher order uniform bounds for w. $\Box$

Remark 4.1. In a local situation of ours, Fu-Hein-Jiang states a complete asymptotics for u_0 near D in [18, Theorem 3.1] such as there exists a constant $c \in \mathbb{R}$ such that

$$u_0 + (n+1)\log\left(1 + \frac{c}{\rho}\right) = O\left(\frac{1}{\rho^m}\right)$$

for any $m \geq 0$. This is a quite stronger result, and this formula directly leads to our conclusion since ρu_0 converges to a constant $-(n+1)c$ as $\rho \to \infty$.

4.2. **Volume growth of Kähler-Einstein metric for Calabi-Yau boundary.** This subsection is devoted to a proof of Theorem 1.2 or Theorem 2.3. Our proof is based on an approximation by the Kähler-Ricci flow (12). However it seems likely that the same result is established more straightforwardly and easily by the ϵ-approximation method, introduced by S.-T. Yau [50], for the complex Monge-Ampère equation (9).

To clarify the growth of the Kähler-Einstein volume form, it suffices from (9) to determine the behavior of $u_0 = u_\alpha(\infty) + (n-1)\log\rho$ near D. We can show that the potential $u_\alpha(t)$ of the Kähler-Ricci flow has the following uniform C^0-estimate:

Proposition 4.2. *There exists a positive constant C such that*

$$\left| u_\alpha(t) - (n-1)\log\left(\alpha e^{-t} + \frac{2}{\rho}\right)\right| \leq C$$

holds on $X \times [0, \infty)$.

As $t \to \infty$, this estimate gives the boundedness of $u_0 = u_\alpha(\infty) + (n-1)\log\rho$ over X and consequently the desired volume growth in Theorem 1.2 or Theorem 2.3.

Remark 4.2. In [11], they make a similar observation on the Kähler-Einstein volume form near an isolated log-canonical singularity in a local but more general situation.

Finally, we provide a proof of Proposition 4.2.

Proof. Set $w_\alpha(t) = u_\alpha(t) - (n-1)\log\left(\alpha e^{-t} + \frac{2}{\rho}\right)$. $w_\alpha(t)$ satisfies the equation

$$\dot{w}_\alpha(t) = \log \frac{\left(\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log\left[\hat{\Omega}_{\alpha e^{-t}}\left(\alpha e^{-t} + \frac{2}{\rho}\right)^{n-1}\right] + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}w_\alpha(t)\right)^n}{\left(\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log\left[\hat{\Omega}_{\alpha e^{-t}}\left(\alpha e^{-t} + \frac{2}{\rho}\right)^{n-1}\right]\right)^n} \tag{14}$$

$$- w_\alpha(t) + \frac{\alpha(n-1)e^{-t}}{\alpha e^{-t} + \frac{2}{\rho}} + \log \frac{\left(\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log\left[\hat{\Omega}_{\alpha e^{-t}}\left(\alpha e^{-t} + \frac{2}{\rho}\right)^{n-1}\right]\right)^n}{\hat{\Omega}_0\left(\alpha e^{-t} + \frac{2}{\rho}\right)^{n-1}}.$$

We can see that the last term, denoted by $F(t)$, in this equation is uniformly bounded on $X \times [0, \infty)$. In order to check it, we denote a new Kähler metric $\tilde{\omega}(t) = \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log\left[\hat{\Omega}_{\alpha e^{-t}}\left(\alpha e^{-t} + \frac{2}{\rho}\right)^{n-1}\right]$ and calculate it in detail as

$$\begin{aligned}\tilde{\omega}(t) =&\theta_0 + \left(\alpha e^{-t} + \frac{2}{\rho}\right)\Theta + 2\frac{\sqrt{-1}}{2\pi}\log\rho \wedge \overline{\partial}\log\rho + \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log\left(\alpha e^{-t} + \frac{2}{\rho}\right)^{n-1} \\ =&\left\{\theta_0 + \left(\alpha e^{-t} + \frac{2}{\rho} + \frac{n-1}{\left(\alpha e^{-t} + \frac{2}{\rho}\right)\rho}\frac{2}{\rho}\right)\Theta\right\} \\ &+\left\{2 - \frac{4(n-1)}{\left(\alpha e^{-t} + \frac{2}{\rho}\right)^2\rho^2} + \frac{4(n-1)}{\left(\alpha e^{-t} + \frac{2}{\rho}\right)\rho}\right\}\frac{\sqrt{-1}}{2\pi}\partial\log\rho \wedge \overline{\partial}\log\rho.\end{aligned}$$

Hence it is immediate to observe that $\tilde{\omega}(t)$ is a complete Kähler metric on X of bounded geometry in the sense of Cheng-Yau for any $t \geq 0$. Notice that $\tilde{\omega}(t)$ converge to ω_0 as $t \to \infty$ although ω_0 and $\tilde{\omega}(t)$ at a finite time t have different volume growths. Indeed, we calculate the volume form of $\tilde{\omega}(t)$ at a finite time t as

$$\begin{aligned}\tilde{\omega}(t)^n =&\left\{\theta_0 + \left(\alpha e^{-t} + \frac{2}{\rho} + \frac{n-1}{\left(\alpha e^{-t} + \frac{2}{\rho}\right)\rho}\frac{2}{\rho}\right)\Theta\right\}^n \\ &+n\left\{\theta_0 + \left(\alpha e^{-t} + \frac{2}{\rho} + \frac{n-1}{\left(\alpha e^{-t} + \frac{2}{\rho}\right)\rho}\frac{2}{\rho}\right)\Theta\right\}^{n-1} \\ &\wedge\left(2 - \frac{4(n-1)}{\left(\alpha e^{-t} + \frac{2}{\rho}\right)^2\rho^2} + \frac{4(n-1)}{\left(\alpha e^{-t} + \frac{2}{\rho}\right)\rho}\right)\sqrt{-1}\partial\log\rho \wedge \overline{\partial}\log\rho.\end{aligned}$$

Since $\theta_0 + \left(\alpha e^{-t} + \frac{2}{\rho} + \frac{n-1}{(\alpha e^{-t}+\frac{2}{\rho})\rho}\frac{2}{\rho}\right)\Theta$ is strictly positive near D, the first term in this formula is bounded on X and the second term is of the same form with $\hat{\Omega}_0$, that is, the power of ρ is 2. From these growth properties, we know that the growth of $\tilde{\omega}(t)^n$ is actually the same with $\hat{\Omega}_0$, that is, the power of ρ is 2. We further proceed to calculate $F(t)$ as

$$F(t) = \log\left[\frac{\|\sigma\|^2\rho^2}{\Omega\left(\alpha e^{-t}+\dfrac{2}{\rho}\right)^{n-1}}\left\{\theta_0+\left(\alpha e^{-t}+\frac{2}{\rho}+\frac{n-1}{\left(\alpha e^{-t}+\dfrac{2}{\rho}\right)\rho}\frac{2}{\rho}\right)\Theta\right\}^n\right.$$
$$+n\left\{\frac{\theta_0}{\alpha e^{-t}+\dfrac{2}{\rho}}+\left(1+\frac{2(n-1)}{\left(\alpha e^{-t}+\dfrac{2}{\rho}\right)^2\rho^2}\right)\Theta\right\}^{n-1}$$
$$\left.\wedge\left(2-\frac{4(n-1)}{\left(\alpha e^{-t}+\dfrac{2}{\rho}\right)^2\rho^2}+\frac{4(n-1)}{\left(\alpha e^{-t}+\dfrac{2}{\rho}\right)\rho}\right)\frac{\|\sigma\|^2\rho^2\sqrt{-1}\partial\log\rho\wedge\overline{\partial}\log\rho}{\Omega}\right].$$

A crucial point is that the second term in this description has $1+\frac{2(n-1)}{\left(\alpha e^{-t}+\frac{2}{\rho}\right)^2\rho^2}$ as a coefficient of Θ, which is away from 0 uniformly in t. So we can see from this reason that $F(t)$ is uniformly bounded on $X\times[0,\infty)$. Moreover, it follows from (14) and the inequality in [28, page 408] that

(15)
$$\left(\frac{\partial}{\partial t}-\Box_{2\pi\tilde{\omega}(t)}\right)w_\alpha(t)\le -w_\alpha(t)+\frac{\alpha(n-1)e^{-t}}{\alpha e^{-t}+\dfrac{2}{\rho}}+F(t)\le\left(\frac{\partial}{\partial t}-\Box_{2\pi\omega_X(t)}\right)w_\alpha(t)$$

holds. Therefore combined with the fact that $w_\alpha(0) = -(n-1)\log\left(\alpha+\frac{2}{\rho}\right)$ is bounded on X, the parabolic maximum principle (cf. [38, Lemma 4.7]) is applied for (15) to derive the uniform boundedness of $w_\alpha(t)$. This completes the proof. □

References

[1] A. Ash, D. Mumford, M. Rapoport, and Y.-S. Tai, *Smooth compactifications of locally symmetric varieties*, 2nd ed., Cambridge Mathematical Library, Cambridge University Press, Cambridge, 2010. With the collaboration of Peter Scholze.

[2] H. Auvray, *Asymptotic properties of extremal Kähler metrics of Poincaré type*, Proc. Lond. Math. Soc. (3) **115** (2017), no. 4, 813–853.

[3] S. Bando, *Einstein Kähler metrics of negative Ricci curvature on open Kähler manifolds*, Kähler metric and moduli spaces, Adv. Stud. Pure Math., vol. 18, Academic Press, Boston, MA, 1990, pp. 105–136.

[4] H. Bass, M. Lazard, and J.-P. Serre, *Sous-groupes d'indice fini dans* $\mathbf{SL}(n, \mathbf{Z})$, Bull. Amer. Math. Soc. **70** (1964), 385–392 (French).

[5] H. Bass, J. Milnor, and J.-P. Serre, *Solution of the congruence subgroup problem for* SL_n $(n\ge 3)$ *and* Sp_{2n} $(n\ge 2)$, Inst. Hautes Études Sci. Publ. Math. **33** (1967), 59–137.

[6] S. Boucksom, *On the volume of a line bundle*, Internat. J. Math. **13** (2002), no. 10, 1043–1063.

[7] H. D. Cao, *Deformation of Kähler metrics to Kähler-Einstein metrics on compact Kähler manifolds*, Invent. Math. **81** (1985), no. 2, 359–372.

[8] J. Carlson and P. Griffiths, *A defect relation for equidimensional holomorphic mappings between algebraic varieties*, Ann. of Math. (2) **95** (1972), 557–584.

[9] A. Chau, *Convergence of the Kähler-Ricci flow on noncompact Kähler manifolds*, J. Differential Geom. **66** (2004), no. 2, 211–232.

[10] S. Y. Cheng and S. T. Yau, *On the existence of a complete Kähler metric on noncompact complex manifolds and the regularity of Fefferman's equation*, Comm. Pure Appl. Math. **33** (1980), no. 4, 507–544.

[11] V. Datar, X. Fu, and J. Song, *Kähler-Einstein metrics near an isolated log-canonical singularity*, arXiv:2106.05486 to appear in J. Reine Angew. Math.

[12] J.-P. Demailly, *Analytic methods in algebraic geometry*, Surveys of Modern Mathematics, vol. 1, International Press, Somerville, MA; Higher Education Press, Beijing, 2012.

[13] G. Di Cerbo and L. F. Di Cerbo, *Positivity in Kähler-Einstein theory*, Math. Proc. Cambridge Philos. Soc. **159** (2015), no. 2, 321–338.

[14] ———, *Effective results for complex hyperbolic manifolds*, J. Lond. Math. Soc. (2) **91** (2015), no. 1, 89–104.

[15] P. Eyssidieux, V. Guedj, and A. Zeriahi, *Singular Kähler-Einstein metrics*, J. Amer. Math. Soc. **22** (2009), no. 3, 607–639.

[16] H. Fang and X. Fu, *On the construction of a complete Kähler-Einstein metric with negative scalar curvature near an isolated log-canonical singularity*, Proc. Amer. Math. Soc. **149** (2021), no. 9, 3965–3976.

[17] E. Freitag, *Siegelsche Modulfunktionen*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 254, Springer-Verlag, Berlin, 1983 (German).

[18] X. Fu, H.-J. Hein, and X. Jiang, *Asymptotics of Kähler-Einstein metrics on complex hyperbolic cusps*, arXiv:2108.13390.

[19] P. Gao, S.-T. Yau, and W. Zhou, *Nonexistence for complete Kähler-Einstein metrics on some noncompact manifolds*, Math. Ann. **369** (2017), no. 3-4, 1271–1282.

[20] D. Gilbarg and N. S. Trudinger, *Elliptic partial differential equations of second order*, Classics in Mathematics, Springer-Verlag, Berlin, 2001. Reprint of the 1998 edition.

[21] H. Grauert, *Über Modifikationen und exzeptionelle analytische Mengen*, Math. Ann. **146** (1962), 331–368 (German).

[22] J. W. Hoffman and S. H. Weintraub, *Cohomology of the boundary of Siegel modular varieties of degree two, with applications*, Fund. Math. **178** (2003), no. 1, 1–47.

[23] F. Holland, *Another proof of Hadamard's determinantal inequality*, Irish Math. Soc. Bull. **59** (2007), 61–64.

[24] X. Jiang and Y. Shi, *Asymptotic expansions of complete Kähler-Einstein metrics with finite volume on quasi-projective manifolds*, Sci. China Math. **65** (2022), no. 9, 1953–1974.

[25] Y. Kawamata, *Minimal models and the Kodaira dimension of algebraic fiber spaces*, J. Reine Angew. Math. **363** (1985), 1–46.

[26] S. Kikuta, *The limits on boundary of orbifold Kähler-Einstein metrics and Kähler-Ricci flows over quasi-projective manifolds*, Math. Ann. **361** (2015), no. 1-2, 477–510.

[27] ———, *Boundary behavior of Kähler-Einstein metric of negative Ricci curvature over quasi-projective manifolds with boundaries of general type*, Kodai Math. J. **44** (2021), no. 1, 81–114.

[28] R. Kobayashi, *Kähler-Einstein metric on an open algebraic manifold*, Osaka J. Math. **21** (1984), no. 2, 399–418.

[29] ———, *Einstein-Kaehler metrics on open algebraic surfaces of general type*, Tohoku Math. J. (2) **37** (1985), no. 1, 43–77.

[30] O. A. Ladyženskaja, V. A. Solonnikov, and N. N. Ural′ceva, *Linear and quasilinear equations of parabolic type*, Translated from the Russian by S. Smith. Translations of Mathematical Monographs, Vol. 23, American Mathematical Society, Providence, R.I., 1968 (Russian).

[31] G. M. Lieberman, *Second order parabolic differential equations*, World Scientific Publishing Co. Inc., River Edge, NJ, 1996.

[32] J. Lott and Z. Zhang, *Ricci flow on quasi-projective manifolds*, Duke Math. J. **156** (2011), no. 1, 87–123.

[33] N. Mok, *Projective algebraicity of minimal compactifications of complex-hyperbolic space forms of finite volume*, Perspectives in analysis, geometry, and topology, Progr. Math., vol. 296, Birkhäuser/Springer, New York, 2012, pp. 331–354.

[34] D. Mumford, *Hirzebruch's proportionality theorem in the noncompact case*, Invent. Math. **42** (1977), 239–272.

[35] Y. Namikawa, *Toroidal compactification of Siegel spaces*, Lecture Notes in Mathematics, vol. 812, Springer, Berlin, 1980.

[36] F. Rochon and Z. Zhang, *Asymptotics of complete Kähler metrics of finite volume on quasiprojective manifolds*, Adv. Math. **231** (2012), no. 5, 2892–2952.

[37] G. Schumacher, *Asymptotics of Kähler-Einstein metrics on quasi-projective manifolds and an extension theorem on holomorphic maps*, Math. Ann. **311** (1998), no. 4, 631–645.

[38] W.-X. Shi, *Ricci flow and the uniformization on complete noncompact Kähler manifolds*, J. Differential Geom. **45** (1997), no. 1, 94–220.

[39] T. Shioda, *On elliptic modular surfaces*, J. Math. Soc. Japan **24** (1972), 20–59.

[40] J. Song and G. Tian, *Canonical measures and Kähler-Ricci flow*, J. Amer. Math. Soc. **25** (2012), no. 2, 303–353.

[41] J. Song and B. Weinkove, *An introduction to the Kähler-Ricci flow*, An introduction to the Kähler-Ricci flow, Lecture Notes in Math., vol. 2086, Springer, Cham, 2013, pp. 89–188.

[42] G. Tian and S.-T. Yau, *Existence of Kähler-Einstein metrics on complete Kähler manifolds and their applications to algebraic geometry*, Mathematical aspects of string theory (San Diego, Calif., 1986), Adv. Ser. Math. Phys., vol. 1, World Sci. Publishing, Singapore, 1987, pp. 574–628.

[43] G. Tian and Z. Zhang, *On the Kähler-Ricci flow on projective manifolds of general type*, Chinese Ann. Math. Ser. B **27** (2006), no. 2, 179–192.

[44] H. Tsuji, *Existence and degeneration of Kähler-Einstein metrics on minimal algebraic varieties of general type*, Math. Ann. **281** (1988), no. 1, 123–133.

[45] ———, *A characterization of ball quotients with smooth boundary*, Duke Math. J. **57** (1988), no. 2, 537–553.

[46] ———, *Canonical measures and the dynamical systems of Bergman kernels*, arXiv:0805.1829.

[47] W. Wang, *On the smooth compactification of Siegel spaces*, J. Differential Geom. **38** (1993), no. 2, 351–386.

[48] D. Wu, *Higher canonical asymptotics of Kähler-Einstein metrics on quasi-projective manifolds*, Comm. Anal. Geom. **14** (2006), no. 4, 795–845.

[49] ———, *Kähler-Einstein metrics of negative Ricci curvature on general quasi-projective manifolds*, Comm. Anal. Geom. **16** (2008), no. 2, 395–435.

[50] S. T. Yau, *On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation. I*, Comm. Pure Appl. Math. **31** (1978), no. 3, 339–411.

[51] ———, *A splitting theorem and algebraic geometric characterization of locally Hermitian symmetric spaces*, Commun. Anal. Geom. **1** (1993), no. 3, 473?486.

[52] S.-T. Yau and Y. Zhang, *Hodge bundles on smooth compactifications of Siegel varieties and applications*, ICCM Not. **7** (2019), no. 2, 1–18.

[53] ———, *The geometry on smooth toroidal compactifications of Siegel varieties*, Amer. J. Math. **136** (2014), no. 4, 859–941.

[54] Y. Zhang, *Mumford compactification of Siegel variety from the viewpoint of Kähler-Einstein metric*, Proceedings of the Sixth International Congress of Chinese Mathematicians. Vol. I, Adv. Lect. Math. (ALM), vol. 36, Int. Press, Somerville, MA, 2017, pp. 541–555.

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