SOME EXPRESSIONS OF GRAPH ZETAS AND THE TWISTED ALEXANDER POLYNOMIALS, AND THE KNOT VOLUME

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ABSTRACT. In this paper, we consider the zeta function of graphs and the twisted Alexander polynomials of knots. Since they have been developed separately, we explain the relationship between them through the notion of knot graphs. We also provide a new volume presentation of hyperbolic knots using matrix-weighted graphs and Bell polynomials.

1. INTRODUCTION

The zeta function of graphs was defined for regular graphs by Ihara [13]. He showed that its reciprocal is an explicit polynomial. Hashimoto [12] generalized Ihara's result on the zeta function of regular graphs to irregular graphs and showed that its reciprocal is again a polynomial. Bass [2] presented another determinant expression for the Ihara zeta function of irregular graphs using adjacency matrices. After Terras's book [22], Ishikawa, Mitsuhashi, Morita, and Sato have furthered their research into graph zeta functions, obtaining results for various types of graphs (see [14], [18], [20] for example). In recent years, they have started to be applied in several fields, and in this paper we try to lay the foundation of their application to knot theory. In particular, we focus on the Alexander polynomial, which is the most fundamental knot invariant, and its extension, the twisted Alexander polynomial. We will proceed according to Morita's expressions [18].

Since the study of graph zeta and the study of knot theory have been developed separately, there is a wall between them. In order to make it easier for researchers in the respective fields of graph zeta and knot theory to understand, usually omitted terms for graph zeta and knot theory are described as much as possible. In addition, we provide some examples. Moreover, previous researches have often considered the idea of a backtrack of a graph, but since it is not necessary in knot theory, we have reconstructed without this concept. As a result, simpler examples can be used.

As an application of the relation between zeta functions and twisted Alexander polynomials, we provide a volume presentation of hyperbolic knots in the 3-sphere. More precisely, we show that the hyperbolic volume of a knot can be expressed in terms of the traces of the adjacency matrix or the edge matrix of the knot graph through Bell polynomials, which are often used in the study of set partitions. The idea of this volume presentation comes from our previous work on a hyperbolic fibered knot [11].

This paper is organized as follows. In Section 2, we prepare several basic notions about graph zeta functions: Lyndon words, circular products, Foata-Zeilberger's formula for reciprocal characteristic polynomials, edge matrices, and adjacency matrices of digraphs. In Section 3, we review two kinds of weighted zeta functions: one weighted by an element of a commutative $\mathbb{Q}$-algebra, and the other weighted by a matrix whose entries are in a commutative $\mathbb{Q}$-algebra. In particular, we establish the relationship between three kinds of

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matrix-weighted zeta functions. In Section 4, we explain the notion of knot graphs, and study the (twisted) Alexander polynomials of knots from the view point of weighted zeta functions. In the last section, we provide a volume presentation of hyperbolic knots using Bell polynomials.

Throughout this paper, we use the following notation. Let $\mathbb{N}$ be the set of natural numbers. The ring of integers is denoted by $\mathbb{Z}$, and $\mathbb{Q}$ ($\mathbb{C}$ resp.) means the rational (complex resp.) number field. For a finite set X, the number of elements in X is denoted by $\sharp X$. The Kronecker delta is denoted by δ_{xy}, which gives 1 if $x = y$, 0 if $x \neq y$.

2. PRELIMINARIES

2.1. **Words.** According to [18], we review some notations on words. An *alphabet* is a set of noncommutative variables. Let $\mathfrak{A}$ be a set called an alphabet. A finite sequence of elements of $\mathfrak{A}$ is called *words* on $\mathfrak{A}$. The set of all words on $\mathfrak{A}$ is denoted by $\mathfrak{A}^*$. Then $\mathfrak{A}^*$ is a semigroup, the multiplication of which is the concatenation of words. Let $w = a_1a_2\cdots a_\ell \in \mathfrak{A}^*$. The nonnegative integer ℓ is called the *length* of w, denoted by $|w|$. The length of the empty word is defined to be zero. A word w is called *prime* if there is no shorter word $u \in \mathfrak{A}^*$ such that $w = u^k$ for a positive integer k. The *cyclic rearrangement class* of $w = a_1a_2\cdots a_\ell$ is the multiset

$$\{a_1a_2\cdots a_\ell,\ a_2a_3\cdots a_\ell a_1,\ \ldots, a_\ell a_1\cdots a_{\ell-1}\}$$

of ℓ words and we denote it by $\mathrm{Re}(w)$. If w is prime, then any element of $\mathrm{Re}(w)$ has multiplicity 1. Namely, each cyclic rearrangement of w appears just once in $\mathrm{Re}(w)$. If a word u is contained in $\mathrm{Re}(w)$, then we write $u \equiv w$. We can see that $\equiv$ is an equivalence relation on $\mathfrak{A}^*$.

Hereafter, we suppose that an alphabet $\mathfrak{A}$ is finite and totally ordered. We use the total order by $<$. Then $\mathfrak{A}^*$ is also totally ordered by the lexicographical order induced by $<$. We also denote it by $<$. If a word $w \in \mathfrak{A}^*$ is the minimum element in $\mathrm{Re}(w)$, w is called a *Lyndon word* (see e.g. [16]). The set of Lyndon words on $\mathfrak{A}$ is denoted by $\mathrm{Lyn}(\mathfrak{A})$.

Example 2.1. Suppose $\mathfrak{A} = \{1 < 2 < 3\}$. Then 132 is Lyndon, but 321 is not. Moreover, $w = 1212 \notin \mathrm{Lyn}(\mathfrak{A})$ since w is not the minimum element in the multiset $\mathrm{Re}(w) = \{1212,\ 2121,\ 1212,\ 2121\}$. Thus a Lyndon word is necessarily a prime word.

The concept of Lyndon words was crucial in the foundations of free differential calculus [4] and pursued in [21] etc.

Example 2.2. For $\mathfrak{A} = \{1, 2, 3, 4\}$ and $1 < 2 < 3 < 4$, the list of first Lyndon words is as follows. We will use this list in this paper. For example, 1^32 implies 1112.

lengh	Lyndon words
1	$1, 2, 3, 4$
2	$12, 13, 14, 23, 24, 34$
3	$112, 113, 114, 122, 123, 124, 132, 133, 134, 142,$ $143, 144, 223, 224, 233, 234, 243, 244, 334, 344$
4	1^32, 1^33, 1^34, 1122, 1123, 1124, 1132, 1133, 1134, 1142, 1143, 1144, 1213, 1214, 12^3, 1223, 1224, 1232, 1233, 1234, 1242, 1243, 1244, 1314, 1322, 1323, 1324, 1332, 1333, 1334, 1342, 1343, 1344, 1422, 1423, 1424, 1432, 1433, 1434, 1442, 1443, 14^3, 2^33, 2^34, 2233, 2234, 2243, 2244, 2324, 23^3 2334, 2343, 2344, 2433, 2434, 2443, 24^3, 3^34, 3344, 34^3

Although it is not difficult to list all the Lyndon words, it might be difficult to determine whether they are all exhausted. In this case, it is useful to use the Möbius function to display the number of Lyndon words of a particular length. Specifically, the following formula is known:

$$\mathrm{Card}(\mathrm{Lyn}(\mathfrak{A}) \cap \mathfrak{A}^{\ell}) = \frac{1}{\ell} \sum_{\frac{\ell}{d}} \mu(d) [\mathrm{Card}(\mathfrak{A})]^{\frac{\ell}{d}},$$

where μ is the Möbius function. For the details, see [16].

Let R be a commutative ring. Let $\mathrm{Mat}_{\mathfrak{A}}(R)$ denote the set of $n \times n$ matrices $(m_{aa'})_{a,a' \in \mathfrak{A}}$ with $m_{aa'} \in R$ for each a, a', where $n = \sharp\mathfrak{A}$. For $w = a_1 a_2 \cdots a_\ell \in \mathfrak{A}^*$ and $M = (m_{aa'})_{a,a' \in \mathfrak{A}} \in \mathrm{Mat}_{\mathfrak{A}}(R)$, we denote by $\mathrm{circ}_M(w)$ the *circular product*

$$m_{a_1 a_2} m_{a_2 a_3} \cdots m_{a_{\ell-1} a_\ell} m_{a_\ell a_1}$$

of entries in M along w. Let I denote the identity matrix.

Theorem 2.3 ([7]). (1)

$$\det(I - M) = \prod_{l \in \mathrm{Lyn}(\mathfrak{A})} (1 - \mathrm{circ}_M(l)).$$

(2) *Let s be an indeterminate. The reciprocal characteristic polynomial* $1/\det(I - sM)$ *is written by*

$$\frac{1}{\det(I - sM)} = \prod_{l \in \mathrm{Lyn}(\mathfrak{A})} \frac{1}{1 - \mathrm{circ}_M(l) s^{|l|}}.$$

See Examples 2.7 and 3.7 for an interpretation of this theorem. In [7], Foata and Zeilberger provide a short proof of Amitsur's identity.

Theorem 2.4 ([1]). *For square matrices $M_1, M_2, \ldots, M_k$,*

$$\det(I - (M_1 + M_2 + \cdots + M_k)) = \prod_{l \in \mathrm{Lyn}(\mathfrak{A})} \det(I - M_l),$$

where the product runs over all Lyndon words in $\{1, 2, \ldots, k\}$ and $M_l = M_{i_1} M_{i_2} \cdots M_{i_p}$ for $l = i_1 i_2 \cdots i_p$.

2.2. **Graphs.** In this subsection, we define several terminologies and notions on graphs according to [14] and [18].

A *graph* $\Gamma = (V, E)$ is a pair of a set V and a multiset E, where E consists of 2-subsets of V. The elements of V and E are called *vertex* and *edge*, respectively. A graph $\Gamma = (V, E)$ is called *finite* if both V and E are finite sets. If E contains an edge $e = \{u, v\}$ with a multiplicity m, then e is called a *multi edge* with multiplicity m.

A *digraph* $G = (V, A)$ is a pair of a set V and a multiset A, where A consists of ordered pairs $a = (u, v)$ of elements $u, v \in V$. An element of A is called a *directed edge* or an *arc* of G. If $a = (u, v)$ is an arc of G, then u is called the *tail* of a, and v the *head* of a, denoted by $\mathfrak{t}(a)$ and $\mathfrak{h}(a)$ respectively. Since A is a multi-set, it may occur that $\mathfrak{t}(a) = \mathfrak{t}(a')$ and $\mathfrak{h}(a) = \mathfrak{h}(a')$ for distinct $a, a' \in A$. For a loop l, the vertex $n = \mathfrak{t}(l) (= \mathfrak{h}(l))$ is called the *nest* of l. Let $A_{uv} = \{a \in A \mid \mathfrak{t}(a) = u,\ \mathfrak{h}(a) = v\}$. If A contains arcs $a \in A_{uv}$ with a multiplicity m, then a is called a *multi arc* with multiplicity m.

In the many papers on the graph zeta functions, they considered the 'inverse' of a, namely (v, u) for an arc $a = (u, v)$. It is denoted by a^{-1}. They associate to a graph $\Gamma = (V, E)$ a digraph $G(\Gamma)$ which is called the *symmetric digraph* of Γ. The vertices of $G(\Gamma)$ is the same as Γ. If $e = \{u, v\}$ is an edge of Γ, then two arcs $a = (u, v)$ and a^{-1} are associated with e.

The arc set of $G(\Gamma) = (V, A)$ is given by $\{(u, v), (v, u) \mid (u, v) \in E\}$. However, a symmetric digraph is not obtained from a knot by our method, so we basically do not treat such graphs. We do not treat the inverse a^{-1}. Thus, we do not use the notions 'reduce' nor 'backtrack' in this paper. If there are arcs $a = (u, v)$ and (v, u) in G, we denote the arc (v, u) by another symbol (alphabet), e.g. $b = (v, u)$. Then, we may treat some basic digraphs as illustrated in Figure 1.

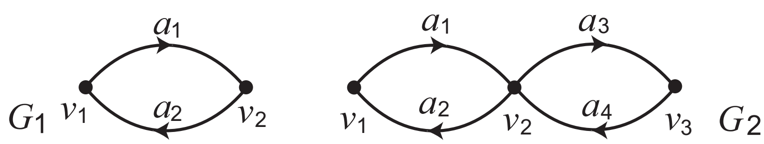

FIGURE 1.

Let $G = (V, A)$ be a digraph. If a sequence $c = (a_1, a_2, \dots, a_\ell)$ satisfies $\mathfrak{h}(a_i) = \mathfrak{t}(a_{i+1})$ for all $i = 1, 2, \dots, \ell - 1$, then c is called a *path* of G, and ℓ is called the *length* of c, denoted by $|c|$. A path $c = (a_1, a_2, \dots, a_\ell)$ is said to be *closed* if $\mathfrak{t}(a_1) = \mathfrak{h}(a_\ell)$. Let $C = C_G$ be the set of closed paths of G, and C_ℓ the set of closed paths of length ℓ. Then we have a disjoint union $C = \sqcup_{\ell \geq 1} C_\ell$. Let $c = (a_1, a_2, \dots, a_\ell) \in C$. The *cyclic rearrangement* $(a_2, a_3, \dots, a_\ell, a_1), (a_3, a_4, \dots, a_1, a_2), \dots, (a_\ell, a_1, \dots, a_{\ell-2}, a_{\ell-1})$ of c is also a closed path of lenght ℓ.

For $c \in C$, we denote by c^k the closed path obtained by making k times iteration to c. Hence, if $c \in C_\ell$, then $c^k \in C_{k\ell}$. A *prime closed path* is a closed path which cannot be written in the form d^k for a shorter $d \in C$. Let $P = P_G$ denote the set of prime closed paths of G. For $c \in C$, there exists $p \in P$ such that $c = p^k$ for some $k \in \mathbb{Z}_{>0}$. Then p is called a *prime section* of c. If p is a prime section of $c \in C$, it is denote by $\pi(c)$. A prime section is not uniquely determined for c, but its cyclic rearrangement class (see the paragraph below for the definition of the class) is uniquely determined. Therefore the length of a prime section is uniquely determined for each c, and we call it the *prime length* of $c \in C$, which is denoted by $\varpi(c)$. Let P_ℓ be the set of prime closed paths of length ℓ. Since the prime length of $c \in C$ is uniquely determined, we have a disjoint union $P = \sqcup_{\ell \geq 1} P_\ell$.

For the digraph G_1 in Figure 1, set $c = (a_1, a_2, a_1, a_2, a_1, a_2)$ and $p = (a_1, a_2)$. Then p is prime, $p = \pi(c)$, $c = p^3$, and $\varpi(c) = |p| = 2$.

Let $c, d \in C$. If d is a cyclic rearrangement of c, we write $c \sim d$. This binary relation $\sim$ is an equivalence relation on C. We call it the *cyclic equivalence*. Let $[C] = [C_G]$ be the quotient $C/\sim$, and $\bar{c}$ the equivalence class with a representative $c \in C$. An element of $[C]$ is called a *cycle* of G. If $c \sim d$ for $c, d \in C$, then c and d have the same length. The length $|\gamma|$ of a cycle $\gamma = \bar{c} \in [C]$ is defined by $|c|$, and it is well-defined. It also shows that $\sim$ is an equivalence relation on C_ℓ, so $[C_\ell] = C_\ell/\sim$ is the set of cycles of length ℓ, and we have $[C] = \sqcup_{\ell \geq 1} [C_\ell]$. A cycle γ, say $\gamma = \bar{c}$, is said to be *prime* if c is prime. The set of prime cycles is denoted by $[P] = [P_G]$. We have $[P] = P/\sim$, since $\sim$ is also an equivalence relation on P. For each $c \in C$, the equivalence class of the prime sections of c is uniquely determined since they are mutually cyclic equivalent. It is called the *prime section* of $\gamma = \bar{c}$, denoted by $\pi(\gamma) \in [P]$. Let $\gamma = \bar{c}$ be a cycle with a representative c. The *prime length* $\varpi(\gamma)$ of γ is defined by $\varpi(\gamma) = \varpi(c)$. It is well-defined since prime sections of c are cyclic equivalent. Let $[P_\ell]$ be the set of prime cycles of length ℓ. Then we have $[P] = \sqcup_{\ell \geq 1} [P_\ell]$.

For the digraph G_1 in Figure 1, set $c = (a_1, a_2, a_1, a_2)$ and $d = (a_2, a_1, a_2, a_1)$. Then $c \sim d, \gamma = \bar{c} = \overline{(a_1, a_2, a_1, a_2)}, \pi(\gamma) = \overline{(a_1, a_2)}$, and $\varpi(\gamma) = \varpi(c) = 2$.

Definition 2.5. Let $G = (V, A)$ be a finite digraph, where A is totally ordered and R is a ring. Let $M = (m_{aa'})_{a,a'\in A} \in \mathrm{Mat}(\sharp A; R)$ be a matrix with the entries which satisfy the condition that $m_{aa'} \neq 0$ implies $\mathfrak{h}(a) = \mathfrak{t}(a')$. A matrix satisfying this condition is called an *edge matrix* of G.

2.3. **Zeta functions.** The zeta function of graphs is known to accept several expressions. According to [18], three of them are called the *Euler product expression* $E(s)$, the *determinant expression* $H(s)$ *of Hashimoto type* (we call it *Hashimoto expression* for short) and the *exponential expression* $Z(s)$. Morita [18] discussed the relationship between them. Then he showed that $H(s)$ can always be reformulated into $E(s)$, and that $E(s)$ can always be reformulated into $Z(s)$. Moreover, he gave conditions from $Z(s)$ to $E(s)$, and $E(s)$ to $H(s)$. In this subsection, we will leave these conditions aside for the time being and present the idea of the Ihara Zeta function, the origin of graph zeta functions, and the related results in a form that is suited to our setting.

The Ihara zeta function for a finite graph is the prototype of combinatorial zeta functions. It is usually defined by the Euler product expression for a kind of finite graph, i.e. a symmetric digraph. Here we define the Ihara zeta function to fit our setting.

Definition 2.6. Let $G = (V, A)$ be a finite digraph, and s an indeterminate. The following formal power series

$$Z_G(s) = \prod_{\gamma\in[P_G]} \frac{1}{1 - s^{|\gamma|}}$$

is called the *Ihara zeta function* of G. It is called an *Euler product expression*.

Let us consider the map

$$\theta\colon A \times A \longrightarrow \{0, 1\} : (a, a') \mapsto \delta_{\mathfrak{h}(a)\mathfrak{t}(a')}.$$

Then we have the edge matrix of G, denoted by $M_G(\theta) = (\theta(a, a'))_{a,a'\in A}$. It is known that the equation

$$Z_G(s) = \frac{1}{\det(I - sM_G(\theta))}$$

holds, and the expression is called *Hashimoto expression*, which is a kind of *determinant expression*. For proof, see the proof of Proposition 3.5, which is given in a more general setting.

Example 2.7. Let G_1 be the digraph illustrated in Figure 1. It has only one prime cycle a_1a_2 whose length is 2. Hence $Z_{G_1}(s) = \dfrac{1}{1 - s^2}$. The edge matrix is $M_{G_1}(\theta) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Then we have $\det(I - sM_{G_1}(\theta)) = 1 - s^2$.

Let G_2 be the digraph illustrated in Figure 1. Because of the infinite number of prime cycles, it can be difficult to calculate $Z_{G_2}(s)$ from the definition. But, using the edge matrix

$$M_{G_2}(\theta) = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix},$$

we can obtain $Z_{G_2}(s) = \dfrac{1}{\det(I - sM_{G_2}(\theta))} = \dfrac{1}{1 - 2s^2}$.

Identify the arc $a_i \in A$ with the alphabet $a_i \in \mathfrak{A}$ in Section 2.1. For $l = a_1 a_2 \cdots a_\ell \in \mathrm{Lyn}(\mathfrak{A})$, we may suppose

$$\mathrm{circ}_M(l) = \theta(a_1, a_2)\theta(a_2, a_3) \cdots \theta(a_\ell, a_1).$$

For l such that $|l| = 2$, $\mathrm{circ}_M(12) = m_{12}m_{21} = 1$, $\mathrm{circ}_M(34) = m_{34}m_{43} = 1$, and $\mathrm{circ}_M(l) = 0\,(l \neq 12,\ 34)$. For l such that $|l| = 4$, $\mathrm{circ}_M(1342) = m_{13}m_{34}m_{42}m_{21} = 1$, and $\mathrm{circ}_M(l) = 0\,(l \neq 1342)$. Similarly, for l such that $|l| = 6$, $\mathrm{circ}_M(121342) = m_{12}m_{21}m_{13}m_{34}m_{42}m_{21} = 1$, $\mathrm{circ}_M(134342) = m_{13}m_{34}m_{43}m_{34}m_{42}m_{21} = 1$, and $\mathrm{circ}_M(l) = 0\,(l \neq 121342,\ 134342)$. The former takes the left cycle twice and the right cycle once, while the latter corresponds to taking the left cycle once and the right cycle twice.

Here is an interpretation of Theorem 2.3. We have the formal power series:

$$\begin{aligned} Z_{G_2}(s) &= \frac{1}{\det(I - sM_{G_2}(\theta))} \\ &= \frac{1}{1 - 2s^2} \\ &= 1 + 2s^2 + 4s^4 + 8s^6 + 16s^8 + 32s^{10} + O\left(s^{11}\right). \end{aligned} \tag{2.1}$$

On the other hand, we have:

$$\begin{aligned} \prod_{l \in \mathrm{Lyn}(\mathfrak{A}), |l| \leq 2} \frac{1}{1 - \mathrm{circ}_M(l) s^{|l|}} &= \frac{1}{(1 - s^2)^2} \\ &= 1 + 2s^2 + 3s^4 + 4s^6 + 5s^8 + 6s^{10} + O\left(s^{11}\right), \end{aligned}$$

where the first two terms match the formula (2.1),

$$\begin{aligned} \prod_{l \in \mathrm{Lyn}(\mathfrak{A}), |l| \leq 4} \frac{1}{1 - \mathrm{circ}_M(l) s^{|l|}} &= \frac{1}{(1 - s^2)^2 (1 - s^4)} \\ &= 1 + 2s^2 + 4s^4 + 6s^6 + 9s^8 + 12s^{10} + O\left(s^{11}\right), \end{aligned}$$

where the first three terms match the equation (2.1),

$$\begin{aligned} \prod_{l \in \mathrm{Lyn}(\mathfrak{A}), |l| \leq 6} \frac{1}{1 - \mathrm{circ}_M(l) s^{|l|}} &= \frac{1}{(1 - s^2)^2 (1 - s^4)(1 - s^6)^2} \\ &= 1 + 2s^2 + 4s^4 + 8s^6 + 13s^8 + 20s^{10} + O\left(s^{11}\right), \end{aligned}$$

where the first four terms match the formula (2.1). Theorem 2.3 illustrates this asymptotic behavior. Thus it can be viewed as the Euler product expression for $1/\det(I - sM)$ since the set $\mathrm{Lyn}(\mathfrak{A})$ gives the primes in $\mathfrak{A}^*$.

As we observed in Example 2.7, from the definition we have the following:

$$\{\text{Prime cycles in } G\} \overset{1:1}{\longleftrightarrow} \{l \in \mathrm{Lyn}(\mathfrak{A}) \mid \mathrm{circ}_M(l) \neq 0\}. \tag{2.2}$$

Suppose M is an edge matrix of a digraph $G = (V, A)$. By identifying a path $c = (a_1, a_2, \ldots, a_\ell)$ of G with a word $a_1 a_2 \cdots a_\ell \in \mathfrak{A}^*$, one can consider the circular product $\mathrm{circ}_M(c) = m_{a_1 a_2} m_{a_2 a_3} \cdots m_{a_\ell a_1}$. The condition $\mathrm{circ}_M(c) \neq 0$ implies that the path c is closed.

According to [18], there is one more expression of the zeta function, which is called the *exponential expression*. Let N_ℓ be the number of closed paths of length ℓ in a digraph G. Then, it is known that

$$Z_G(s) = \exp\left(\sum_{\ell \geq 1} \frac{N_\ell}{\ell} s^\ell\right).$$

We should note the difference between cycles and closed paths. For proof, see the proof of Proposition 3.5, which is given in a more general setting.

Example 2.8. Let G_2 be the digraph illustrated in Figure 1. By the calculations in Example 2.7, we have:

$$\log Z_{G_2}(s) = \log \frac{1}{1-2s^2} = 2s^2 + 2s^4 + \frac{8s^6}{3} + 4s^8 + \frac{32s^{10}}{5} + O\left(s^{11}\right).$$

Thus we have: $N_1 = 0,\ N_2 = 4,\ N_3 = 0,\ N_4 = 8,\ N_5 = 0,\ N_6 = 16$. For example, N_6 consists of $\{\{(12)^3, (21)^3\}, \{(34)^3, (43)^3\}, \{121342, 213421, 134212, 342121, 421213, 212134\}, \{134342, 343421, 434213, 342134, 421343, 213434\}\}$.

Proposition 2.9.

$$N_\ell = \mathrm{tr}\,(M_G(\theta))^\ell.$$

Proof. This proposition can be proved using the following two facts in linear algebra: $\log(I + X) = \sum_{k=1}^{\infty}(-1)^{k-1}\frac{1}{k}X^k$ and $\mathrm{tr}\,\log X = \log\det X$ for an invertible matrix X. Confirm it and calculate for the case of the digraph G_2 in Example 2.7.

In fact, we have the following equations:

$$\begin{aligned}\log\det(I - sM_G(\theta))^{-1} &= -\log\det(I - sM_G(\theta)) = -\mathrm{tr}\sum_{\ell=1}^{\infty}(-1)^{\ell-1}\frac{1}{\ell}(-sM_G(\theta))^\ell \\ &= -\mathrm{tr}\sum_{\ell=1}^{\infty}(-1)^{2\ell-1}\frac{1}{\ell}(M_G(\theta))^\ell s^\ell = \sum_{\ell=1}^{\infty}\frac{\mathrm{tr}\,(M_G(\theta))^\ell}{\ell}s^\ell.\end{aligned}$$

Hence, we obtain $N_\ell = \mathrm{tr}\,(M_G(\theta))^\ell$. □

The next lemma plays an important role in our setting. See Section 5 in [14] for the precise proof.

Lemma 2.10. *Let P be an $m \times n$ matrix and Q an $n \times m$ matrix. Then* $\det(I - PQ) = \det(I - QP)$.

Proof. If P is a regular matrix, we have $\det(I - PQ) = \det P^{-1}\det(I - PQ)\det P = \det(P^{-1}IP - P^{-1}PQP) = \det(I - QP)$.

In general, if A is a regular matrix of the size $m \times m$ and D is a square matrix, then $\det\begin{pmatrix} A & P \\ Q & D\end{pmatrix} = \det A \cdot \det(D - QA^{-1}P)$. Similarly, if D is a regular matrix of the size $n \times n$ and A is a square matrix, then $\det\begin{pmatrix} A & P \\ Q & D\end{pmatrix} = \det D \cdot \det(A - PD^{-1}Q)$. Suppose both A and D are the identity matrices, we have the conclusion. □

Definition 2.11. The matrix $A_G = (a_{uv})_{u,v\in V}$ with entries $a_{uv} = \sharp A_{uv}$ is called the *adjacency matrix* of a digraph G.

Example 2.12. For the digraphs G_1 and G_2 in Figure 1, $A_{G_1} = \begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}$, $A_{G_2} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0\end{pmatrix}$.

Proposition 2.13. *For a finite digraph $G = (V, A)$, we have*

$$Z_G(s) = \frac{1}{\det(I - sA_G)}.$$

Proof. We have only to prove $\det(I - sA_G) = \det(I - sM_G(\theta))$ where $M_G(\theta)$ is the edge matrix for G. Let $H = (h_{av})_{a\in A, v\in V}$ and $J = (j_{ua'})_{u\in V, a'\in A}$ denote the matrices with entries $h_{av} = \delta_{\mathfrak{h}(a)v}$ and $j_{ua'} = \delta_{u\mathfrak{t}(a')}$ respectively. One can see that $M_G(\theta) = HJ$ and $A_G = JH$. Then we have the conclusion by Lemma 2.10. □

3. Weighted zeta functions

A natural extension of the definition of the Ihara zeta function is a zeta function for weighted graphs with positive natural numbers attached to the arcs, e.g. see Section 6 in [22]. However, when considering applications to knot theory, a more general setting is required. In this section, following [18] and [14], we provide a setting that anticipates applications to the (twisted) Alexander polynomials.

3.1. **A weighted graph.** Let $G = (V, A)$ be a digraph. Let R be a commutative $\mathbb{Q}$-algebra, $\omega\colon A \to R$ be a map. Then ω is called a *weight*, and the pair (G, ω) is called a *weighted graph*. Suppose $A = \mathfrak{A}$ and let W be the diagonal matrix $W = (w_{aa'})_{a,a'\in\mathfrak{A}}$ such that $w_{aa'} = \omega(a)\delta_{aa'}$. Then, we call W the *weight matrix* for (G, ω). Let us consider the following three maps:

$$\theta_0\colon A \times A \to \{0,1\} : (a, a') \mapsto \delta_{\mathfrak{h}(a)\mathfrak{t}(a')}$$
$$\theta_1\colon A \times A \to R : (a, a') \mapsto \omega(a)\delta_{\mathfrak{h}(a)\mathfrak{t}(a')}$$
$$\theta_2\colon A \times A \to R : (a, a') \mapsto \omega(a')\delta_{\mathfrak{h}(a)\mathfrak{t}(a')}.$$

Then, we have the edge matrix $M_G(\theta_0)$ for the digraph G and the edge matrix $M_G(\theta_i)(i = 1, 2)$ for the weighted graph (G, ω).

Lemma 3.1. $\det(I - M_G(\theta_1)) = \det(I - M_G(\theta_2))$.

Proof. We have $\det(I - M_G(\theta_1)) = \det(I - M_G(\theta_0)W) = \det(I - WM_G(\theta_0)) = \det(I - M_G(\theta_2))$ by Lemma 2.10. □

It is not difficult to apply the arguments on θ in Subsection 2.3 to this θ_2. According to [18], we use the *edge matrix* $M_G(\omega) = M_G(\theta_2)$ in this paper. Theorem 2.3 holds for the edge matrix $M_G(\omega)$.

Definition 3.2. Let s be an indeterminate. The formal power series $H_G(s; \omega)$ is defined as follows:

$$H_G(s;\omega) = \frac{1}{\det(I - sM_G(\omega))}.$$

Example 3.3. Let G_1 and G_2 be digraphs as illustrated in Figure 1. We give weights ω_1 (ω_2 resp.) for G_1 (G_2 resp.) such that $\omega_1(a_1) = t, \omega_1(a_2) = 1 - t$ ($\omega_2(a_1) = t, \omega_2(a_2) = 1 - t^{-1}, \omega_2(a_3) = t^{-1}, \omega_2(a_4) = 1 - t$, resp.).

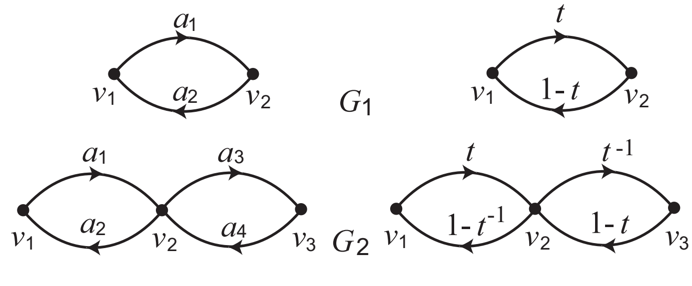

FIGURE 2.

Then we have

$$M_{G_1}(\omega_1) = \begin{pmatrix} 0 & 1-t \\ t & 0 \end{pmatrix}, \quad M_{G_2}(\omega_2) = \begin{pmatrix} 0 & 1-t^{-1} & t^{-1} & 0 \\ t & 0 & 0 & 0 \\ 0 & 0 & 0 & 1-t \\ 0 & 1-t^{-1} & t^{-1} & 0 \end{pmatrix},$$

$\det(I - sM_{G_1}(\omega_1)) = 1 + (-t + t^2)s^2$, and $\det(I - sM_{G_2}(\omega_2)) = 1 + (-t^{-1} + 2 - t)s^2$.

Let $c = (a_1, a_2, \ldots, a_\ell)$ be a path of G. We define the *weight* $\omega(c)$ for the path c by $\omega(c) = \omega(a_1)\omega(a_2)\cdots\omega(a_\ell)$. If c is closed, i.e. if $c \in C_\ell$, then we can define the weight of a cycle $\gamma = \bar{c}$ by $\omega(\gamma) = \omega(c)$, which is well-defined. For each positive integer $\ell \geq 1$, we define

$$N_\ell(\omega) = \sum_{c \in C_\ell} \omega(c). \tag{3.1}$$

We note that $N_\ell(\omega) = \sum_{\substack{\gamma \in [C_\ell] \\ \varpi(\gamma) | \ell}} \varpi(\gamma)\omega(\pi(\gamma))^{\ell/\varpi(\gamma)}$ holds.

Definition 3.4. Let s be an indeterminate. The formal power series $Z_G(s;\omega)$ and $E_G(s;\omega)$ are defined as follows:

$$Z_G(s;\omega) = \exp\left(\sum_{\ell \geq 1} \frac{N_\ell(\omega)}{\ell} s^\ell\right),$$

$$E_G(s;\omega) = \prod_{\gamma \in [P_G]} \frac{1}{1 - \omega(\gamma)s^{|\gamma|}}.$$

Proposition 3.5.

$$H_G(s;\omega) = Z_G(s;\omega) = E_G(s;\omega).$$

Proof. The following proof is essentially the same as the proof of Theorems 12 and 13 in [18].

Taking the logarithm of $E_G(s;\omega)$ and using the relation $\log(1+x) = \sum_{k\geq 1}(-1)^{k-1}\frac{1}{k}x^k$, we have

$$\log E_G(s;\omega) = \log \prod_{\gamma \in [P_G]} \frac{1}{1 - \omega(\gamma)s^{|\gamma|}} = \sum_{\gamma \in [P_G]} \sum_{k \geq 1} \frac{1}{k}\omega(\gamma)^k s^{k|\gamma|}.$$

Setting $\ell = k|\gamma|$, we see that $\omega(\gamma)^k = \omega(\gamma^k) = \omega(c)$ holds for $c \in C_\ell$ and $\gamma \in P_G$ satisfying $c = \gamma^k$. Then, the last term equals

$$\begin{aligned}\sum_{\gamma \in [P_G]} \sum_{k \geq 1} \frac{1}{k}\omega(\gamma)^k s^{k|\gamma|} &= \sum_{\gamma \in P_G} \sum_{k \geq 1} \frac{1}{|\gamma|}\left(\frac{1}{k}\omega(\gamma^k)\right) s^{k|\gamma|} = \sum_{\ell \geq 1} \sum_{c \in C_\ell} \frac{1}{\ell}\omega(c)s^\ell \\ &= \sum_{\ell \geq 1} \frac{1}{\ell} \sum_{c \in C_\ell} \omega(c)s^\ell = \sum_{\ell \geq 1} \frac{N_\ell(\omega)}{\ell} s^\ell.\end{aligned}$$

Thus $E_G(s;\omega) = Z_G(s;\omega)$ holds true.

Next we show that $E_G(s;\omega) = H_G(s;\omega)$. Suppose $\gamma = \overline{(a_1, a_2, \ldots, a_\ell)}$ is a prime cycle in a digraph $G = (V, A)$. By the correspondence (2.2), there exists only one Lyndon word $l = a_{i_1}a_{i_2}\cdots a_{i_\ell} \in \mathrm{Lyn}(A)$ which is a cyclic rearrangement of $a_1a_2\cdots a_\ell$. By the definitions of the edge matrix $M_G(\omega) = M_G(\theta_2) = (m_{aa'})_{a,a' \in A}$ and the circular product $\mathrm{circ}_{M_G(\omega)}(l)$, we obtain

$$\begin{aligned}\omega(\gamma) &= \omega(a_1)\omega(a_2)\cdots\omega(a_\ell) = \omega(a_{i_1})\omega(a_{i_2})\cdots\omega(a_{i_\ell}) \\ &= m_{a_{i_\ell}a_{i_1}} m_{a_{i_1}a_{i_2}} \cdots m_{a_{i_{\ell-1}}a_{i_\ell}} \\ &= \mathrm{circ}_{M_G(\omega)}(l).\end{aligned}$$

Hence, we have

$$\begin{aligned}E_G(s;\omega) &= \prod_{\gamma \in [P_G]} \frac{1}{1-\omega(\gamma)s^{|\gamma|}} \\ &= \prod_{l \in \mathrm{Lyn}(\mathfrak{A})} \frac{1}{1-\mathrm{circ}_{M_G(\omega)}(l)s^{|l|}} \\ &= \frac{1}{\det(I - sM_G(\omega))} = H_G(s;\omega)\end{aligned}$$

by Theorem 2.3. This completes the proof of Proposition 3.5. □

By the same argument as in the proof of Proposition 2.9, we have:

Proposition 3.6.

$$N_\ell(\omega) = \mathrm{tr}\,(M_G(\omega))^\ell.$$

Example 3.7. Let G_2 be the digraph with weight ω_2 as illustrated in Figure 2. Then, using Example 3.3, we have:

$$\begin{aligned}H_{G_2}(s;\omega_2) &= \frac{1}{\det(I - sM_{G_2}(\omega_2))} = \frac{1}{1+(-t^{-1}+2-t)s^2} = \frac{1}{1-\frac{(t-1)^2}{t}s^2} \\ &= 1 + \frac{(t-1)^2}{t}s^2 + \frac{(t-1)^4}{t^2}s^4 + \frac{(t-1)^6}{t^3}s^6 \\ &\quad + \frac{(t-1)^8}{t^4}s^8 + \frac{(t-1)^{10}}{t^5}s^{10} + O\left(s^{11}\right).\end{aligned} \tag{3.2}$$

On the other hand, we have:

$$\begin{aligned}\prod_{\gamma \in [P_2]} \frac{1}{1-\omega(\gamma)s^{|\gamma|}} &= \frac{1}{1-t(1-t^{-1})s^2} \cdot \frac{1}{1-t^{-1}(1-t)s^2} \\ &= 1 + \frac{(t-1)^2}{t}s^2 + \frac{(t-1)^2\left(t^2-t+1\right)}{t^2}s^4 + \frac{(t-1)^4\left(t^2+1\right)}{t^3}s^6 \\ &\quad + \frac{(t-1)^4\left(t^4-t^3+t^2-t+1\right)}{t^4}s^8 + \frac{(t-1)^6\left(t^4+t^2+1\right)}{t^5}s^{10} + O\left(s^{11}\right),\end{aligned}$$

where the first two terms match the formula (3.2),

$$\begin{aligned}\prod_{\gamma \in [P_2]\cup[P_4]} \frac{1}{1-\omega(\gamma)s^{|\gamma|}} &= 1 + \frac{(t-1)^2}{t}s^2 + \frac{(t-1)^4}{t^2}s^4 + \frac{(t-1)^4\left(t^2-t+1\right)}{t^3}s^6 \\ &\quad + \frac{(t-1)^4\left(t^2-t+1\right)^2}{t^4}s^8 + \frac{(t-1)^6\left(t^4-t^3+2t^2-t+1\right)}{t^5}s^{10} + O\left(s^{11}\right),\end{aligned}$$

where the first three terms match the equation (3.2),

$$\prod_{\gamma \in [P_2]\cup[P_4]\cup[P_6]} \frac{1}{1-\omega(\gamma)s^{|\gamma|}} = 1 + \frac{(t-1)^2}{t}s^2 + \frac{(t-1)^4}{t^2}s^4 + \frac{(t-1)^6}{t^3}s^6$$

$$+\frac{(t-1)^4\left(t^4-3t^3+5t^2-3t+1\right)}{t^4}s^8$$
$$+\frac{(t-1)^6\left(t^4-2t^3+4t^2-2t+1\right)}{t^5}s^{10}+O\left(s^{11}\right),$$

where the first four terms match the equation (3.2). Moreover, Proposition 3.5 implies that

$$\log Z_{G_2}(s;\omega)=\frac{(t-1)^2}{t}s^2+\frac{(t-1)^4}{2t^2}s^4+\frac{(t-1)^6}{3t^3}s^6+\frac{(t-1)^8}{4t^4}s^8$$
$$+\frac{(t-1)^{10}}{5t^5}s^{10}+O\left(s^{11}\right).$$

On the other hand, by a direct calculation of matrices, we have

$$\operatorname{tr}\left(M_G(\omega_2)\right)^2=\frac{2(t-1)^2}{t},\ \operatorname{tr}\left(M_{G_2}(\omega_2)\right)^3=0,\ \operatorname{tr}\left(M_{G_2}(\omega_2)\right)^4=\frac{2(t-1)^4}{t^2},$$
$$\operatorname{tr}\left(M_{G_2}(\omega_2)\right)^5=0,\ \operatorname{tr}\left(M_{G_2}(\omega_2)\right)^6=\frac{2(t-1)^6}{t^3},\ \operatorname{tr}\left(M_{G_2}(\omega_2)\right)^7=0,$$
$$\operatorname{tr}\left(M_{G_2}(\omega_2)\right)^8=\frac{2(t-1)^8}{t^4},\ \operatorname{tr}\left(M_{G_2}(\omega_2)\right)^9=0,\ \operatorname{tr}\left(M_{G_2}(\omega_2)\right)^{10}=\frac{2(t-1)^{10}}{t^5}.$$

Definition 3.8. Let (G,ω) be a weighted graph where $G=(V,A)$ and $\omega\colon A\to R$ is a weight. The *weighted adjacency matrix* $A_G(\omega)$ is defined as follows:

(i) $A_G(\omega)=(a_{uv})_{u,v\in V}$;
(ii) $a_{uv}=\sum_{a\in A_{uv}}\omega(a)$.

Proposition 3.9.

$$H_G(s;\omega)=\frac{1}{\det(I-sA_G(\omega))}.$$

Proof. The idea of the proof is the same as that of Proposition 2.13. We have only to prove $\det(I-sA_G(\omega))=\det(I-sM_G(\omega))$ where $M_G(\omega)$ is the edge matrix for G. Let $H=(h_{av})_{a\in A,v\in V}$ and $J=(j_{ua'})_{u\in V,a'\in A}$ denote the matrices with entries $h_{av}=\delta_{\mathfrak{h}(a)v}$ and $j_{ua'}=\omega(a')\delta_{u\mathfrak{t}(a')}$ respectively. One can see that $M_G(\omega)=HJ$ and $A_G(\omega)=JH$. Then we have the conclusion by Lemma 2.10. □

Example 3.10. Let G_1 and G_2 be weighted digraphs as illustrated in Figure 2. Then, we obtain

$$A_{G_1}(\omega_1)=\begin{pmatrix}0 & t\\ 1-t & 0\end{pmatrix}\quad\text{and}\quad A_{G_2}(\omega_2)=\begin{pmatrix}0 & t & 0\\ 1-t^{-1} & 0 & t^{-1}\\ 0 & 1-t & 0\end{pmatrix}.$$

Hence, $\det(I-sA_{G_1}(\omega_1))=1+(-t+t^2)s^2$ and $\det(I-sA_{G_2}(\omega_2))=1+(-t^{-1}+2-t)s^2$. Comparing with Example 3.3, we see that the equalities $\det(I-sM_{G_i}(\omega_i))=\det(I-sA_{G_i}(\omega_i))$ $(i=1,2)$ hold.

3.2. **The matrix-weighted graph.** In this subsection, we state a matrix-weighted zeta function of a digraph $G=(V,A)$. It was introduced in [24] and [20], and has been developed in [17] and [6]. Our setting here is different from them, so we introduce the one that suits our setting.

Suppose $V=\{v_1,v_2,\dots,v_m\}$ and $(n_1,n_2,\dots,n_m)\in\mathbb{N}^m$. Set $n_{v_i}=n_i\,(1\leq i\leq m)$. Then, for each $a\in A_{v_iv_j}$, let $\Omega(a)$ be an $n_i\times n_j$ matrix whose entries are in a commutative

$\mathbb{Q}$-algebra. The set $\{\Omega(a) \mid a \in A\}$ is called the *matrix-weight* of a digraph G. For a closed path $c = (a_{i_1}, a_{i_2}, \ldots, a_{i_\ell})$, set

$$\Omega(c) = \Omega(a_{i_1})\Omega(a_{i_2}) \cdots \Omega(a_{i_\ell}).$$

The weights obtained by cyclic rearrangement are different, but the following definition is well-defined according to Lemma 2.10.

Definition 3.11.

$$E_G(s;\Omega) = \prod_{\gamma \in [P_G]} \frac{1}{\det(I - s^{|\gamma|}\Omega(\gamma))}.$$

By the same idea as in Subsection 3.1, we can have the map

$$\Theta\colon A \times A \to \mathrm{Mat}(R) : (a, a') \mapsto \delta_{\mathfrak{h}(a)\mathfrak{t}(a')}\Omega(a'),$$

and the *edge matrix* $M_G(\Theta)$ whose (a, a')-entry is the $n'_i \times n'_j$ block matrix for $a' \in A_{v'_i v'_j}$. Similarly, we have the *adjacency matrix* $A_G(\Omega) = \left(\sum_{a \in A_{uv}} \Omega(a)\right)_{u,v \in V}$. As in Subsection 3.1, we use the notation $M_G(\Omega) = M_G(\Theta)$.

Example 3.12. Let G_2 be the digraph as in Figure 1. Then we have:

$$M_{G_2}(\Omega) = \begin{pmatrix} O & \Omega(a_2) & \Omega(a_3) & O \\ \Omega(a_1) & O & O & O \\ O & O & O & \Omega(a_4) \\ O & \Omega(a_2) & \Omega(a_3) & O \end{pmatrix}, A_{G_2}(\Omega) = \begin{pmatrix} O & \Omega(a_1) & O \\ \Omega(a_2) & O & \Omega(a_3) \\ O & \Omega(a_4) & O \end{pmatrix},$$

where O denotes the zero matrix.

Definition 3.13. Let s be an indeterminate. The formal power series $H_G(s;\Omega)$ is defined as follows:

$$H_G(s;\Omega) = \frac{1}{\det(I - sM_G(\Omega))}.$$

Example 3.14. Let G_2 be the digraph as in Figure 1 with the following matrix-weight:

$$\Omega(a_1) = \begin{pmatrix} 0 & \frac{1}{2}(1-\sqrt{3}i)t \\ -\frac{1}{2}(1+\sqrt{3}i)t & 2t \end{pmatrix}, \Omega(a_2) = \begin{pmatrix} -\frac{1}{t} + \frac{1}{2}(1+\sqrt{3}i) & -1 \\ \frac{1}{2t}(-1+\sqrt{3}i) + \frac{1}{2}(1-\sqrt{3}i) & -\frac{1}{t}+1 \end{pmatrix},$$

$$\Omega(a_3) = \begin{pmatrix} \frac{1}{t} & 0 \\ \frac{1}{2t}(1-\sqrt{3}i) & \frac{1}{t} \end{pmatrix}, \Omega(a_4) = \begin{pmatrix} 1 - \frac{1}{2}(1+\sqrt{3}i)t & t \\ \frac{1}{2}(1+\sqrt{3}i)t & 1 + \frac{1}{2}(-3+\sqrt{3}i)t \end{pmatrix}.$$

Then, $H_{G_2}(s;\Omega)^{-1} = \det(I - sM_{G_2}(\Omega)) = 1 + \left(-2t + 3 - \dfrac{2}{t}\right)s^2 + \dfrac{(t-1)^4}{t^2}s^4$ holds.

Recently, the following amount were introduced in [6] and [17]. This is a generalization of (3.1). We set

$$N_\ell(\Omega) = \sum_{c \in C_\ell} \operatorname{tr}\Omega(c).$$

Here, if a closed path $d \in C_\ell$ is a cyclic rearrangement of $c \in C_\ell$, then it holds $\operatorname{tr}\Omega(d) = \operatorname{tr}\Omega(c)$.

Definition 3.15. Let s be an indeterminate. The formal power series $Z_G(s;\Omega)$ is defined as follows:

$$Z_G(s;\Omega) = \exp\left(\sum_{\ell \geq 1} \frac{N_\ell(\Omega)}{\ell} s^\ell\right).$$

Theorem 3.16.

$$H_G(s;\Omega) = Z_G(s;\Omega) = E_G(s;\Omega).$$

Proof. First, we show $E_G(s;\Omega) = Z_G(s;\Omega)$. This proof is a hybrid of the proofs of Propositions 2.9 and 3.5.

Taking the logarithm of $E_G(s;\Omega)$, we have:

$$\begin{aligned}\log E_G(s;\Omega) &= \log \prod_{\gamma\in[P_G]} \frac{1}{\det(I - s^{|\gamma|}\Omega(\gamma))} = \sum_{\gamma\in[P_G]} \log \frac{1}{\det(I - s^{|\gamma|}\Omega(\gamma))} \\ &= -\sum_{\gamma\in[P_G]} \log\det(I - s^{|\gamma|}\Omega(\gamma)) = -\sum_{\gamma\in[P_G]} \mathrm{tr}\ \log(I - s^{|\gamma|}\Omega(\gamma)) \\ &= -\sum_{\gamma\in[P_G]} \mathrm{tr} \sum_{k\geq 1} (-1)^{k-1}\frac{1}{k}(-s^{|\gamma|}\Omega(\gamma))^k = \sum_{\gamma\in[P_G]}\sum_{k\geq 1}\frac{1}{k}s^{k|\gamma|}\mathrm{tr}\,(\Omega(\gamma))^k.\end{aligned}$$

Set $\ell = k|\gamma|$. Then, as in the proof of Proposition 3.5, the last term is equal to

$$\sum_{\ell\geq 1}\frac{1}{\ell}\sum_{c\in C_\ell} s^\ell \mathrm{tr}\,\Omega(c) = \sum_{\ell\geq 1}\frac{N_\ell(\Omega)}{\ell}s^\ell.$$

This means $E_G(s;\Omega) = Z_G(s;\Omega)$.

Next, we show $E_G(s;\Omega) = H_G(s;\Omega)$. Let $M_G(\Omega)$ be the edge matrix of the digraph $G = (V, A)$ with matrix-weights. Its (a, a')-entry consists of the $n'_i \times n'_j$ block matrix for $a' \in A_{v'_i v'_j}$. Let B_j be the matrix whose a'-column is the same as that of $M_G(\Omega)$ and other columns are all O. (Refer to the next example.) Then $M_G(\Omega) = B_1 + B_2 + \cdots + B_{\sharp A}$ and

$$\prod_{\gamma\in[P_G]} \det(I - s^{|\gamma|}\Omega(\gamma)) = \prod_{l\in\mathrm{Lyn}(\mathfrak{A})} \det(I - s^{|l|}B_l)$$

where $B_l = B_{i_1}B_{i_2}\cdots B_{i_p}$ for a Lyndon word $l = i_1 i_2 \cdots i_p$. By Theorem 2.4, the righthand side of the equation is equal to $\det(I - sM_G(\Omega))$. This completes the proof of Theorem 3.16. □

Example 3.17. We use the data in Examples 3.12 and 3.14. We denote the edge matrix of the digraph G_2 by $M_{G_2}(\Omega) = B_1 + B_2 + B_3 + B_4$, where

$$B_1 = \begin{pmatrix} O & O & O & O \\ \Omega(a_1) & O & O & O \\ O & O & O & O \\ O & O & O & O \end{pmatrix}, \quad B_2 = \begin{pmatrix} O & \Omega(a_2) & O & O \\ O & O & O & O \\ O & O & O & O \\ O & \Omega(a_2) & O & O \end{pmatrix},$$

$$B_3 = \begin{pmatrix} O & O & \Omega(a_3) & O \\ O & O & O & O \\ O & O & O & O \\ O & O & \Omega(a_3) & O \end{pmatrix} \quad \text{and} \quad B_4 = \begin{pmatrix} O & O & O & O \\ O & O & O & O \\ O & O & O & \Omega(a_4) \\ O & O & O & O \end{pmatrix}.$$

As in Example 3.14, $\det(I - sM_{G_2}(\Omega)) = 1 + \left(-2t + 3 - \frac{2}{t}\right)s^2 + \frac{(t-1)^4}{t^2}s^4$. Set

$$f(k) = \det(I - sM_{G_2}(\Omega)) - \prod_{l\in\mathrm{Lyn}(\mathfrak{A}), |l|\leq k} \det(I - s^{|l|}B_l).$$

Note that if $|l| = 2$, $B_l = B_1B_2$ or B_3B_4. If $|l| = 4$, $B_l = B_1B_3B_4B_2$. Then the coefficient of $s^i (0 \leq i \leq 3)$ of $f(2)$ is 0. Similarly, we can confirm that the coefficient of $s^i (0 \leq i \leq 5)$ of $f(4)$ is 0, the coefficient of $s^i (0 \leq i \leq 7)$ of $f(6)$ is 0, and so on. Theorem 2.4 assures that $\lim_{k\to\infty} f(k) = 0$.

By a similar argument to the proof of Proposition 3.9, we have:

Proposition 3.18.

$$H_G(s;\Omega) = \frac{1}{\det(I - sA_G(\Omega))}.$$

Proposition 3.19.

$$N_\ell(\Omega) = \mathrm{tr}\,(M_G(\Omega))^\ell = \mathrm{tr}\,(A_G(\Omega))^\ell.$$

Proof. A similar argument to the proofs of Propositions 2.9 and 3.9 yields this proposition. □

Example 3.20. Let G_2 be the digraph illustrated in Figure 1, and we use the data in Examples 2.8 and 3.14. We have $\mathrm{tr}\,(M_{G_2}(\Omega))^6 = \mathrm{tr}\,(A_{G_2}(\Omega))^6 = \frac{4}{t^3} - \frac{6}{t^2} + 6 - 6t^2 + 4t^3$ by the direct calculation. On the other hand,

$$\begin{aligned}
\mathrm{tr}\,\Omega((a_1a_2)^3) &= \frac{4t^3 - 3\left(1 - 3\sqrt{3}i\right)t^2 - 12\left(1 + \sqrt{3}i\right)t + 3\left(3 + \sqrt{3}i\right)}{2},\\
\mathrm{tr}\,\Omega((a_3a_4)^3) &= \frac{3\left(3 - \sqrt{3}i\right)t^3 - 12\left(1 - \sqrt{3}i\right)t^2 - 3\left(1 + 3\sqrt{3}i\right)t + 4}{2t^3},\\
\mathrm{tr}\,\Omega(a_1a_2a_1a_3a_4a_2) &= -\frac{\left(1 + 3\sqrt{3}i\right)t^3 - 8t^2 + 2\left(1 - 4\sqrt{3}i\right)t + 4\left(1 + \sqrt{3}i\right)}{2t},\\
\mathrm{tr}\,\Omega(a_1a_3a_4a_3a_4a_2) &= -\frac{4\left(1 - \sqrt{3}i\right)t^3 + 2\left(1 + 4\sqrt{3}i\right)t^2 - 8t + 1 - 3\sqrt{3}i}{2t^2}.
\end{aligned}$$

Thus, we have:

$$\begin{aligned}
N_6(\Omega) &= \sum_{c\in C_6} \mathrm{tr}\,\Omega(c)\\
&= 2\mathrm{tr}\,\Omega((a_1a_2)^3) + 2\mathrm{tr}\,\Omega((a_3a_4)^3) + 6\mathrm{tr}\,\Omega(a_1a_2a_1a_3a_4a_2) + 6\mathrm{tr}\,\Omega(a_1a_3a_4a_3a_4a_2)\\
&= \frac{4}{t^3} - \frac{6}{t^2} + 6 - 6t^2 + 4t^3.
\end{aligned}$$

Accordingly, we see that the equalities $N_6(\Omega) = \mathrm{tr}\,(M_{G_2}(\Omega))^6 = \mathrm{tr}\,(A_{G_2}(\Omega))^6$ hold for the digraph G_2.

4. The Alexander polynomial

There has been some work on (twisted) Alexander polynomials and graphs, but here we summarize the previous work in a way that makes it applicable to future applications.

4.1. **The knot graph.** The idea of the knot graph can be found in [15]. It was later formalized by [8] and named arc graph. In [10], the first author of the present paper formulated the idea of Lin and Wang [15] without knowing [8] and named it knot graph, but it is essentially the same as the one formulated by [8]. In this paper, we adopt a hybrid of the two and call it *knot graph*.

We consider a diagram of an oriented knot K with no kink, together with a base point decorated with $*$. A knot graph G_K is constructed by the following steps (see Figure 3). We denote by K again a diagram of K.

(i) Decompose K into overpaths $k_1, k_2, \ldots, k_m$ along the orientation of K from $*$.
(ii) A vertex v_i of $\tilde{G}_K$ corresponds to the arc k_i.
(iii) An edge of $\tilde{G}_K$ corresponds to a transition of K one to one: when we walk along K,

- if we go under at a crossing along the orientation of K, we draw a corresponding blue arc (depicted with a small circle on it);
- if we jump up at a crossing along the orientation of K, we draw a corresponding red arc.

Then $\tilde{G}_K$ has vertices $v_1, \ldots, v_m$ and m blue arcs (v_i, v_{i+1}) (mod m) and m red arcs (v_i, v_j).

(iv) Each vertex is labelled with the sign corresponding to that of the crossing.

(v) The *knot graph* G_K is obtained from $\tilde{G}_K$ by deleting the vertex v_m and arcs associated with v_m.

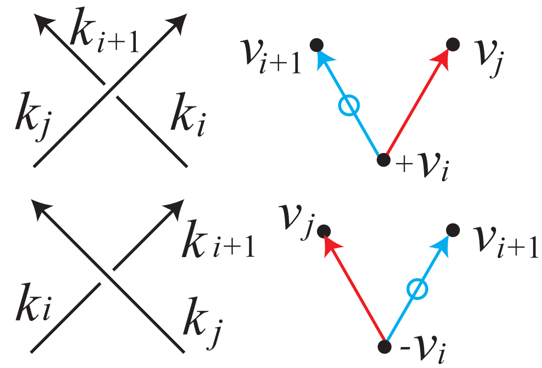

FIGURE 3.

Example 4.1. We put the cases of the trefoil and the figure-eight knot in Figure 4.

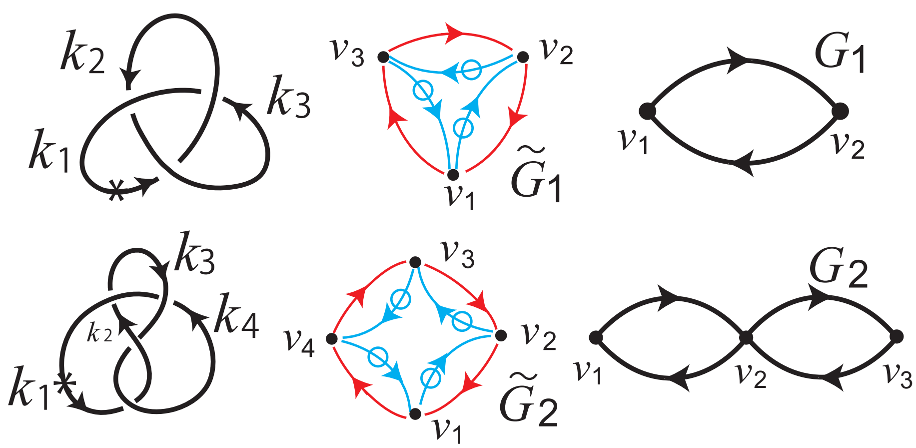

FIGURE 4.

Remark 4.2. (1) The assumption of no kink assures that a knot graph does not have a loop and multi arcs.

(2) Each vertex of $\tilde{G}_K$ has two arcs whose tails are the vertex.

4.2. **The classical Alexander polynomial.** The fundamental group of a knot complement has a Wirtinger presentation. It can be obtained as follows. Let K be an oriented knot and we also denote by K its diagram. We decompose K into overpaths $k_1, k_2, \ldots, k_m$ along the orientation of K. We denote by $\mathfrak{c}_i$ the crossing which corresponds to the terminal of the path k_i

$(i = 1, 2, \ldots, m)$. Let $\nu(K)$ be an open tubular neighborhood of K and $E_K = S^3 \setminus \nu(K)$ the exterior of K in the 3-sphere S^3. We take a generator x_i of $\pi_1(E_K)$ associated with the over-path k_i. If sign$(\mathfrak{c}_i) = +1$, then we take the relator $r_i\colon\ x_i x_j x_{i+1}^{-1} x_j^{-1}$. If sign$(\mathfrak{c}_i) = -1$, then we take the relator $r_i\colon\ x_i x_j^{-1} x_{i+1}^{-1} x_j$. See Figure 5. It is known that the relator r_m can be obtained from others, so we have a presentation of $\pi_1(E_K)$: $\langle x_1, x_2, \ldots, x_m \mid r_1, r_2, \ldots, r_{m-1}\rangle$, which is called a *Wirtinger presentation*.

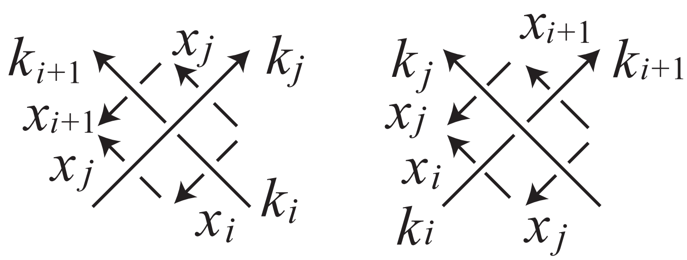

FIGURE 5.

Example 4.3. Let K be the figure-eight knot as illustrated in Figure 4. Then we may have the Wirtinger presentation of $\pi_1(E_K)$:

$$\langle x_1, x_2, x_3, x_4 \mid r_1 = x_1 x_4 x_2^{-1} x_4^{-1}, r_2 = x_2 x_1^{-1} x_3^{-1} x_1, r_3 = x_3 x_2 x_4^{-1} x_2^{-1}\rangle.$$

We denote by F_m the free group $\langle x_1, x_2, \ldots, x_m\rangle$ of degree m, and let $\mathbb{Z}[F_m]$ be the group ring associated with F_m. Let $\frac{\partial}{\partial x_j} : \mathbb{Z}[F_m] \to \mathbb{Z}[F_m]\,(j = 1, 2, \ldots, m)$ be the free differential. It is characterized by the following properties:

(i) $\dfrac{\partial}{\partial x_j}$ is linear on $\mathbb{Z}$

(ii) For any i and j, $\dfrac{\partial x_i}{\partial x_j} = \delta_{ij}$,

(iii) For any $y,\ y' \in F_m$, $\dfrac{\partial}{\partial x_j}(yy') = \dfrac{\partial y}{\partial x_j} + y\dfrac{\partial y'}{\partial x_j}$.

Then we can have $\dfrac{\partial 1}{\partial x_j} = 0$ and $\dfrac{\partial y^{-1}}{\partial x_j} = -y^{-1}\dfrac{\partial y}{\partial x_j}$. Moreover, let

$$\alpha\colon \pi_1(E_K) \to H_1(E_K; \mathbb{Z}) \cong \mathbb{Z} = \langle t\rangle$$

be the abelianization homomorphism, which is given by $\alpha(x_i) = t$. This map α naturally induces the ring homomorphism $\tilde{\alpha}$ from $\mathbb{Z}[\pi_1(E_K)]$ to $\mathbb{Z}[t^{\pm 1}]$.

For a Wirtinger presentation $\langle x_1, x_2, \ldots, x_m \mid r_1, r_2, \ldots, r_{m-1}\rangle$ of $\pi_1(E_K)$, we denote by $\phi\colon \mathbb{Z}[F_m] \to \mathbb{Z}[t^{\pm 1}]$ the composition of the surjection $\mathbb{Z}[F_m] \to \mathbb{Z}[\pi_1(E_K)]$ induced naturally and the map $\mathbb{Z}[\pi_1(E_K)] \to \mathbb{Z}[t^{\pm 1}]$ given by $\tilde{\alpha}$.

Let us consider the crossings in Figure 5. In case of sign$(\mathfrak{c}_i) = +1$, for $r_i = x_i x_j x_{i+1}^{-1} x_j^{-1}$, we have:

$$\frac{\partial r_i}{\partial x_i} = 1 \overset{\phi}{\longmapsto} 1,\ \frac{\partial r_i}{\partial x_{i+1}} = -x_i x_j x_{i+1}^{-1} \overset{\phi}{\longmapsto} -t,$$
$$\frac{\partial r_i}{\partial x_j} = x_i - x_i x_j x_{i+1}^{-1} x_j^{-1} \overset{\phi}{\longmapsto} t - 1,$$

because $r_i = 1$ holds in $\mathbb{Z}[\pi_1(E_K)]$. Similarly, in case of $\operatorname{sign}(\mathfrak{c}_i) = -1$, for $r_i = x_i x_j^{-1} x_{i+1}^{-1} x_j$, we have:

$$\frac{\partial r_i}{\partial x_i} = 1 \overset{\phi}{\longmapsto} 1,\ \frac{\partial r_i}{\partial x_{i+1}} = -x_i x_j^{-1} x_{i+1}^{-1} \overset{\phi}{\longmapsto} -t^{-1},$$
$$\frac{\partial r_i}{\partial x_j} = -x_i x_j^{-1} + x_i x_j^{-1} x_{i+1}^{-1} \overset{\phi}{\longmapsto} -1 + t^{-1}.$$

Definition 4.4. The *Alexander polynomial* $\Delta_K(t)$ of a knot K is defined as follows:

$$\Delta_K(t) = \det\left(\phi\left(\frac{\partial r_i}{\partial x_j}\right)\right)_{1\le i,j\le m-1}.$$

Omitting the m-th column of the matrix $\left(\phi\left(\frac{\partial r_i}{\partial x_j}\right)\right)_{\substack{1\le i\le m-1\\ 1\le j\le m}}$ corresponds to deleting the vertex v_m and its associated edges in $\tilde{G}_K$. The Alexander polynomial is an invariant of a knot up to multiplication by $\pm t^k$ ($k \in \mathbb{Z}$).

Example 4.5. Let K be the figure-eight knot as illustrated in Figure 4, and we have the Wirtinger presentation of $\pi_1(E_K)$ as in Example 4.3. A direct calculation shows that

$$\left(\phi\left(\frac{\partial r_i}{\partial x_j}\right)\right)_{1\le i,j\le 3} = \begin{pmatrix} 1 & -t & 0 \\ -1+t^{-1} & 1 & -t^{-1} \\ 0 & t-1 & 1 \end{pmatrix}.$$

Further, we obtain $\Delta_K(t) = -\dfrac{1}{t} + 3 - t$. Note that $\left(\phi\left(\dfrac{\partial r_i}{\partial x_j}\right)\right)_{1\le i,j\le 3} = I - A_{G_2}(\omega)$ holds, where $A_{G_2}(\omega)$ is the weighted adjacency matrix in Example 3.10.

Let $G_K = (V, A)$ be a knot graph of a knot K and recall that G_K does not have a loop and multi arcs (see Remark 4.2). We define the *Alexander weight* as follows ([10, 15, 8]).

Definition 4.6. Let $\omega\colon A \to R = \mathbb{Q}[t^{\pm 1}]$ be

$$\omega(a) = \omega(v_i, v_j) = \begin{cases} t^{\operatorname{sign}(v_i)} & \text{if } j = i+1, \\ 1 - t^{\operatorname{sign}(v_i)} & \text{otherwise.} \end{cases}$$

Then we call the map ω the *Alexander weight*. See Figure 6.

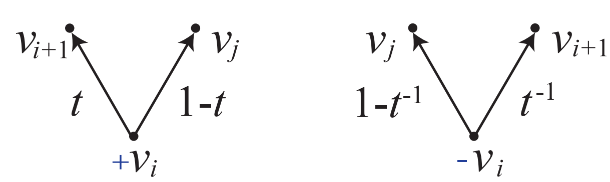

FIGURE 6.

The first equality of the following theorem is proven in [15] and [8].

Theorem 4.7 (Theorem 4.3 in [15], Theorem 1 in [8])**.** *Let ω be the Alexander weight for a knot graph G_K. Then we have:*

$$\Delta_K(t) = \det(I - A_{G_K}(\omega)) = \det(I - M_{G_K}(\omega)).$$

Proof. By the definition of the Alexander weight, we see that

$$A_{G_K}(\omega) = I - \left(\phi\left(\frac{\partial r_i}{\partial x_j}\right)\right)_{1\le i,j\le m-1}$$

holds. Hence the assertion follows from the proof of Proposition 3.9 immediately. □

Example 4.8. The knot graphs for the trefoil knot and the figure-eight knot are given in Example 4.1. In Examples 3.3 and 3.10, the Alexander weights for these knots and the Alexander polynomials are provided, if we substitute $s = 1$ for $\det(I - sM_{G_i}(\omega_i))$ and $\det(I - sA_{G_i}(\omega_i))$.

4.3. **The twisted Alexander polynomial.** We use the notation from the previous subsection and Wada's original paper [23].

Let $\rho\colon \pi_1(E_K) \to \mathrm{SL}(n;\mathbb{C})$ be a linear representation. This map naturally induces the ring homomorphism $\tilde\rho$ from $\mathbb{Z}[\pi_1(E_K)]$ to $\mathrm{Mat}(n;\mathbb{C})$. Then, $\tilde\rho\otimes\tilde\alpha$ defines a ring homomorphism $\mathbb{Z}[\pi_1(E_K)] \to \mathrm{Mat}(n;\mathbb{C}[t^{\pm1}])$. As in Subsection 4.2, we denote by $\Phi\colon \mathbb{Z}[F_m] \to \mathrm{Mat}(n;\mathbb{C}[t^{\pm1}])$ the composition of the surjection $\mathbb{Z}[F_m] \to \mathbb{Z}[\pi_1(E_K)]$ induced naturally and the map $\mathbb{Z}[\pi_1(E_K)] \to \mathrm{Mat}(n;\mathbb{C}[t^{\pm1}])$ given by $\tilde\rho\otimes\tilde\alpha$.

Let us consider the crossings in Figure 5 in the same way as for the Alexander polynomials. Set $X_i = \rho(x_i)$. In case of $\mathrm{sign}(\mathfrak{c}_i) = +1$, for $r_i = x_i x_j x_{i+1}^{-1} x_j^{-1}$, we have:

$$\Phi\left(\frac{\partial r_i}{\partial x_i}\right) = I,\ \Phi\left(\frac{\partial r_i}{\partial x_{i+1}}\right) = -tX_j,\ \Phi\left(\frac{\partial r_i}{\partial x_j}\right) = tX_i - I.$$

Similarly, in case of $\mathrm{sign}(\mathfrak{c}_i) = -1$, for $r_i = x_i x_j^{-1} x_{i+1}^{-1} x_j$, we have:

$$\Phi\left(\frac{\partial r_i}{\partial x_i}\right) = I,\ \Phi\left(\frac{\partial r_i}{\partial x_{i+1}}\right) = -t^{-1}X_j^{-1},\ \Phi\left(\frac{\partial r_i}{\partial x_j}\right) = -X_iX_j^{-1} + t^{-1}X_j^{-1}.$$

The twisted Alexander polynomial $\Delta_{K,\rho}(t)$ is defined as follows:

Definition 4.9.

$$\Delta_{K,\rho}(t) = \frac{\det\left(\Phi\left(\frac{\partial r_i}{\partial x_j}\right)\right)_{1\le i,j\le m-1}}{\det\Phi(x_m - 1)}.$$

The twisted Alexander polynomial is well-defined up to multiplication by $t^k (k\in\mathbb{Z})$ if n is even and by $\pm t^k$ if n is odd. For simplicity, we denote the numerator (denominator resp.) of the twisted Alexander polynomial by $\Delta^1_{K,\rho}(t)$ ($\Delta^0_{K,\rho}(t)$ resp.).

According to [10], we define the following weight. See Figure 7.

Definition 4.10. The *twisted Alexander weight* is defined as follows:

$$\Omega(a) = \Omega(v_i, v_j) = \begin{cases} t^{\mathrm{sign}(v_i)} X_j^{\mathrm{sign}(v_i)} & \text{if } j = i+1, \\ I - tX_i & \text{if } j \ne i+1 \text{ and } \mathrm{sign}(v_i) = +1, \\ X_iX_j^{-1} - \dfrac{1}{t}X_j^{-1} & \text{if } j \ne i+1 \text{ and } \mathrm{sign}(v_i) = -1. \end{cases}$$

Theorem 4.11 ([10]). *Let Ω be the twisted Alexander weight for a knot graph G_K. Then we have:*

$$\Delta^1_{K,\rho}(t) = \det(I - A_{G_K}(\Omega)) = \det(I - M_{G_K}(\Omega)).$$

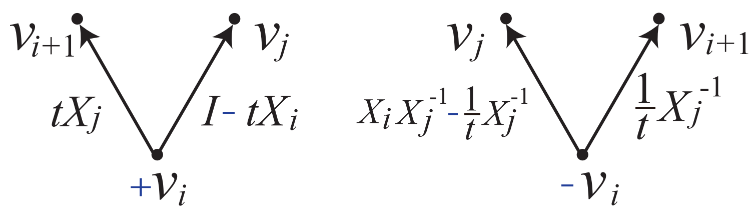

FIGURE 7.

Proof. By the definition of the twisted Alexander weight, we see that

$$A_{G_K}(\Omega) = I - \left(\Phi \left(\frac{\partial r_i}{\partial x_j} \right) \right)_{1 \le i,j \le m-1}$$

holds. Hence we obtain the desired formula using Proposition 3.18. □

Example 4.12. We use the data in Examples 3.14, 4.1 and 4.3. Let K be the figure-eight knot as illustrated in Figure 4 and the presentation of $\pi_1(E_K)$ as in Example 4.3. Set

$$X_1 = \begin{pmatrix} 1 & 0 \\ \frac{-1}{2}(1-\sqrt{3}i) & 1 \end{pmatrix}, X_2 = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, X_3 = \begin{pmatrix} \frac{1}{2}(1+\sqrt{3}i) & -1 \\ \frac{-1}{2}(1+\sqrt{3}i) & \frac{1}{2}(3-\sqrt{3}i) \end{pmatrix},$$

$$\text{and} \quad X_4 = \begin{pmatrix} 0 & \frac{1}{2}(1-\sqrt{3}i) \\ \frac{-1}{2}(1+\sqrt{3}i) & 2 \end{pmatrix}.$$

Define a map $\rho\colon \pi_1(E_K) \to \mathrm{SL}(2;\mathbb{C})$ by $\rho(x_i) = X_i$. Then one can confirm that ρ satisfies the relators $r_i \, (1 \le i \le 3)$, so it is a representation of $\pi_1(E_K)$. Moreover, we can confirm that $\Omega(a_1) = tX_4, \Omega(a_2) = X_2X_1^{-1} - \frac{1}{t}X_1^{-1}, \Omega(a_3) = \frac{1}{t}X_1^{-1}$, and $\Omega(a_4) = I - tX_3$. Hence $\Delta^1_{K,\rho}(t) = \dfrac{(t-1)^2(t^2-4t+1)}{t^2} = H_{G_2}(1;\Omega)^{-1}$. Since $\Delta^0_{K,\rho}(t) = \det \Phi(x_4 - 1) = (t-1)^2$, we have $\Delta_{K,\rho}(t) \doteq t^2 - 4t + 1$.

Using the results from the previous section, we can obtain some expressions for (the numerator of) the twisted Alexander polynomial in terms of three kinds of matrix-weighted zeta functions evaluated at $s = 1$.

Corollary 4.13. *For the twisted Alexander weight Ω of a knot graph G_K, we have*

$$\Delta^1_{K,\rho}(t) = H_{G_K}(1;\Omega)^{-1} = Z_{G_K}(1;\Omega)^{-1} = E_{G_K}(1;\Omega)^{-1}.$$

In the next section, as an application of Theorem 4.11, we provide a volume presentation of a hyperbolic knot in S^3 using Bell polynomials.

5. A VOLUME PRESENTATION OF A HYPERBOLIC KNOT

In our previous paper [11], we provided a volume presentation of a hyperbolic fibered knot, whose complement in S^3 admits the complete hyperbolic metric of finite volume and fibers over the circle. More precisely, we showed that the hyperbolic volume of a fibered knot can be expressed in terms of the traces of powers of a *monodromy matrix* through Bell polynomials. In this section, we exhibit a new volume formula for a hyperbolic knot (not necessarily fibered) using graph data and Bell polynomials.

5.1. **Bell polynomial.** Bell polynomials are often used in the study of set partitions. See [5]. Note that Wolfram Mathematica mounts Bell polynomials.

The *partial Bell polynomials* are the polynomials $B_{k,j} = B_{k,j}(x_1, x_2, \ldots, x_{k-j+1})$ in an infinite number of variables $x_1, x_2, \ldots$, defined by the formal double series expression:

$$\exp\left(u\sum_{m\geq 1} x_m \frac{t^m}{m!}\right) = \sum_{k\geq j\geq 0} B_{k,j}\frac{t^k}{k!}u^j = 1 + \sum_{k\geq 1}\frac{t^k}{k!}\left(\sum_{j=1}^{k} u^j B_{k,j}(x_1, x_2, \ldots)\right).$$

The *complete Bell polynomials* B_k are defined by

$$B_k = B_k(x_1, x_2, \ldots, x_k) = \sum_{j=1}^{k} B_{k,j}, \quad B_0 = 1.$$

There is a list of the values of the partial Bell polynomials $B_{k,j}$ for the small numbers k, j on pages 307 in [5]. For the reader's convenience, we exhibit some of them:

$$B_{1,1} = x_1,\ B_{2,1} = x_2,\ B_{2,2} = x_1^2,\ B_{3,1} = x_3,\ B_{3,2} = 3x_1x_2,\ B_{3,3} = x_1^3,$$

$$B_{4,1} = x_4,\ B_{4,2} = 4x_1x_3 + 3x_2^2,\ B_{4,3} = 6x_1^2x_2,\ B_{4,4} = x_1^4,$$

$$B_{5,1} = x_5,\ B_{5,2} = 5x_1x_4 + 10x_2x_3,\ B_{5,3} = 10x_1^2x_3 + 15x_1x_2^2,\ B_{5,4} = 10x_1^3x_2,\ B_{5,5} = x_1^5.$$

Let A be a $d \times d$ matrix and $\varphi_A(\lambda)$ be the characteristic polynomial of A with coefficients $p_1, p_2, \ldots, p_d$:

$$\varphi_A(\lambda) = \lambda^d - p_1\lambda^{d-1} - p_2\lambda^{d-2} - \cdots - p_d.$$

Suppose that $\lambda_1, \lambda_2, \ldots, \lambda_d$ are the eigenvalues of A, i.e. $\varphi_A(\lambda) = (\lambda-\lambda_1)(\lambda-\lambda_2)\cdots(\lambda-\lambda_d)$. Then, A^k has the eigenvalues $\lambda_1^k, \lambda_2^k, \ldots, \lambda_d^k$ and the trace $\operatorname{tr} A^k = \sum_{j=1}^{d}\lambda_j^k$ $(k = 1, 2, \ldots)$. We denote $\operatorname{tr} A^k$ by q_k for simplicity.

The traces q_k $(k = 1, 2, \ldots, d)$ have a relation to the coefficients p_k by Newton's identities:

$$kp_k = q_k - p_1q_{k-1} - \cdots - p_{k-1}q_1 \quad (k = 1, 2, \ldots, d).$$

The coefficient p_k can be expressed as follows.

Lemma 5.1 (Lemma 3.3 in [11])**.** *Let $B_k(x_1, x_2, \ldots, x_k)$ be the complete Bell polynomial. Then, we have*

$$\begin{aligned} p_k &= -\frac{1}{k!}B_k\big(-q_1, -1!q_2, -2!q_3, \ldots, -(k-1)!q_k\big) \\ &= -\sum_{m_1+2m_2+\cdots+km_k=k}\prod_{j=1}^{k}\frac{(-q_j)^{m_j}}{m_j!\,j^{m_j}} \end{aligned}$$

where $m_1 \geq 0, \ldots, m_k \geq 0$.

5.2. **Hyperbolic volume.** In this subsection, we review the volume formula of a hyperbolic knot using the higher-dimensional twisted Alexander polynomials according to [3] (see also [9]). We also provide a new volume formula using Bell polynomials.

An orientable hyperbolic 3-manifold has a natural representation of its fundamental group into $\mathrm{PSL}(2;\mathbb{C})$, which is unique up to conjugation. We call it the *holonomy representation* of the 3-manifold. It is known that the holonomy representation lifts $\mathrm{SL}(2;\mathbb{C})$, and a lift is unique up to multiplication with a representation into the center of $\mathrm{SL}(2;\mathbb{C})$. In this paper, we consider the lift such that the trace of the image of a meridian of a hyperbolic knot is equal to 2. Of course, another choice of lift works as well.

For $n \geq 2$, let us recall that, up to conjugation, the unique n-dimensional irreducible representation of $\mathrm{SL}(2;\mathbb{C})$ is the $(n-1)$-th symmetric power. We denote it by $\sigma_n\colon \mathrm{SL}(2;\mathbb{C}) \to \mathrm{SL}(n;\mathbb{C})$. Let K be a hyperbolic knot in S^3, namely, the interior of E_K admits the complete hyperbolic metric of finite volume. For our lift of the holonomy representation, we consider the composition with σ_n, and denote this representation by $\rho_n\colon \pi_1(E_K) \to \mathrm{SL}(n;\mathbb{C})$. Here we note that ρ_2 is conjugate to our lift of the holonomy representation.

The next theorem is a volume formula of a hyperbolic knot using the twisted Alexander polynomials. The set of unit complex numbers is denoted by $\mathbb{S}^1 = \{\zeta \in \mathbb{C} \mid |\zeta| = 1\}$.

Theorem 5.2 ([3, Theorem 1.1])**.** *For a hyperbolic knot K in S^3 and for any $\zeta \neq 1 \in \mathbb{S}^1$,*

$$\frac{1}{4\pi}\mathrm{Vol}(S^3 \setminus K) = \lim_{n\to\infty} \frac{1}{n^2} \log |\Delta_{K,\rho_n}(\zeta)|$$

holds.

Let Ω_n be the twisted Alexander weight for the knot graph G_K of a hyperbolic knot K, which corresponds to the representation $\rho_n\colon \pi_1(E_K) \to \mathrm{SL}(n;\mathbb{C})$. Further, we set $q_k(t) = \mathrm{tr}\,(A_{G_K}(\Omega_n))^k \in \mathbb{C}[t^{\pm 1}]$ $(k = 1, 2, \ldots, d)$ and

$$\vec{q}(t) = \big(-q_1(t), -1!q_2(t), -2!q_3(t), \ldots, -(d-1)!q_d(t)\big).$$

Then, Proposition 3.19 and the definition of $N_\ell(\Omega_n)$ imply that we can compute $\vec{q}(t)$ by the sum of the traces of matrices over all closed paths with fixed length on the knot graph G_K.

Now, we provide a new volume presentation of a hyperbolic knot.

Theorem 5.3. *For any $\zeta \neq 1 \in \mathbb{S}^1$, we have*

$$\frac{1}{4\pi}\mathrm{Vol}(S^3 \setminus K) = \lim_{n\to\infty} \frac{1}{n^2} \log \left| \sum_{k=0}^{mn} \frac{B_k(\vec{q}(\zeta))}{k!} \right|,$$

where m is the number of vertices in the knot graph G_K.

Proof. Using Theorem 4.11, we see that the numerator of the twisted Alexander polynomial can be written as $\Delta^1_{K,\rho_n}(t) = \det(I - A_{G_K}(\Omega_n)) = \det(sI - A_{G_K}(\Omega_n))|_{s=1}$. On the other hand, by Lemma 5.1

$$\det(sI - A_{G_K}(\Omega_n)) = s^d - p_1 s^{d-1} - p_2 s^{d-2} - \cdots - p_d,$$

where $d = mn$ and $p_k = -\frac{1}{k!}B_k(\vec{q}(t))$ $(k = 1, 2, \ldots, d)$. Thus, we have

$$\Delta^1_{K,\rho_n}(t) = 1 + \sum_{k=1}^{d} \frac{B_k(\vec{q}(t))}{k!} = \sum_{k=0}^{d} \frac{B_k(\vec{q}(t))}{k!}.$$

As for the denominator of the twisted Alexander polynomial, we obtain $\Delta^0_{K,\rho_n}(t) = (t-1)^n$ (see Lemmas 2.6 and 2.7 in [11]). Hence, using Theorem 5.2 for $\zeta \neq 1 \in \mathbb{S}^1$, we have

$$\begin{aligned}\frac{1}{4\pi}\mathrm{Vol}(S^3 \setminus K) &= \lim_{n\to\infty} \frac{1}{n^2} \log |\Delta_{K,\rho_n}(\zeta)| = \lim_{n\to\infty} \frac{1}{n^2} \log \left| \frac{\sum_{k=0}^{d} \frac{B_k(\vec{q}(\zeta))}{k!}}{(\zeta - 1)^n} \right| \\ &= \lim_{n\to\infty} \frac{1}{n^2} \left(\log \left| \sum_{k=0}^{d} \frac{B_k(\vec{q}(\zeta))}{k!} \right| - n \log |\zeta - 1| \right) \\ &= \lim_{n\to\infty} \frac{1}{n^2} \log \left| \sum_{k=0}^{d} \frac{B_k(\vec{q}(\zeta))}{k!} \right|.\end{aligned}$$

This completes the proof of Theorem 5.3. $\square$

We remark here that the hyperbolic volume of K does not depend on the denominator of the twisted Alexander polynomial evaluated at $t = \zeta$. We can also use the trace of the edge matrix $M_{G_K}(\Omega_n)$ to provide a similar expression for the hyperbolic volume.

Wolfram Mathematica was used for the calculations in this paper. We became aware of the existence of the literature [19] after completing this paper.

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