Triangle groups and Hilbert modular varieties

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Abstract

In this paper we study the *Hilbert series* of triangle groups $\Delta(p, q, r)$. The 11 groups in this series are, conjecturally, the only cocompact triangle groups that admit matrix models over totally real fields.

We provide evidence for this conjecture, along with explicit integral models for every group in the Hilbert series. The most remarkable among them, $\Delta(14, 21, 42)$, is the only known triangle group with a split invariant quaternion algebra.

Using this special group, we construct the first example of a *compact* Kobayashi geodesic curve V on a Hilbert modular variety (aside from those that reside on proper Shimura subvarieties). For comparison, there are *no* compact Kobayashi geodesic curves in the moduli space $\mathcal{M}_g$.

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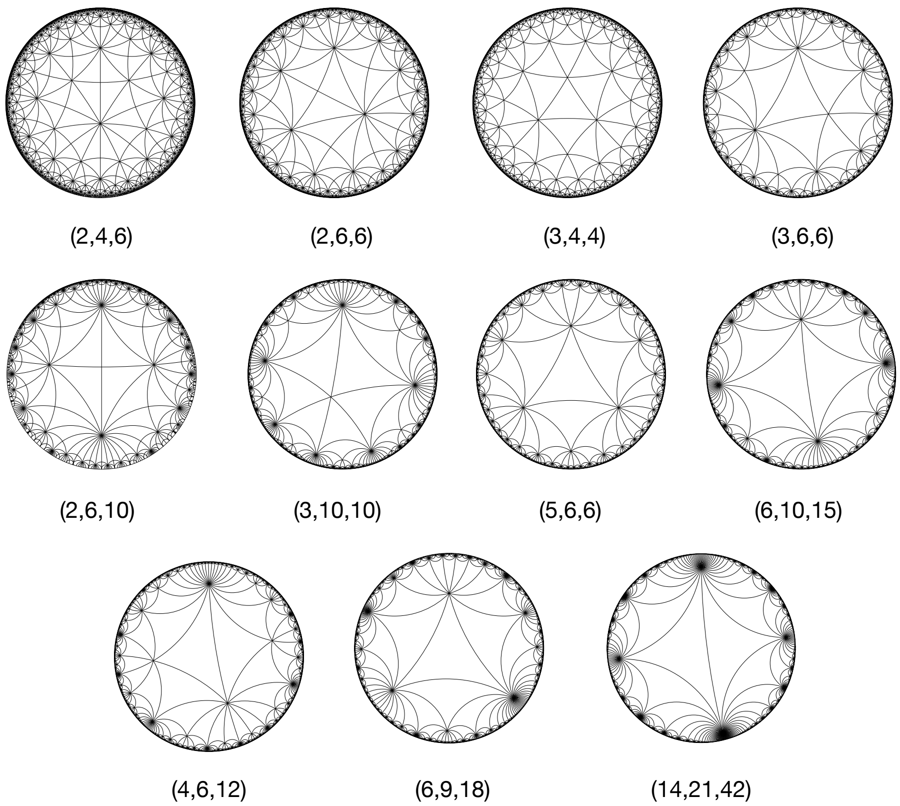

Figure 1. Hyperbolic tilings defined by groups in the Hilbert series.

1 Introduction

This paper gives the first example of a curve on a Hilbert modular variety, $V \subset X_K$, $V \neq X_K$, such that:

1. V is totally geodesic for the Kobayashi metric,
2. V does not lie on a proper Shimura subvariety of X_K, and
3. V is compact (it has no cusps).

The most familiar and commonly studied curves on X_K are the Shimura curves; these satisfy (1) but not (2). Teichmüller curves in moduli space,

$V \subset \mathcal{M}_g$, yield (by passing to the Jacobian) curves satisfying (1) and (2), but they always have cusps. In fact, in [Mc3] we show that every V satisfying (1) and (2) must violate (3) when $\dim X_K = 2$. It is thus natural to wonder if the same is true with $\dim X_K > 2$. We provide a negative answer in Theorem 1.9.

A second goal of this paper is to discuss a remarkable collection of 11 triangle groups acting on $\mathbb{H}$, which we call the *Hilbert series*. Every group Δ in the Hilbert series admits a matrix model over its trace field. The model for $\Delta(14, 21, 42)$ leads to a very explicit example of a curve $V \subset X_K$ satisfying (1), (2) and (3) above (with $\dim X_K = 6$), and hence a proof of Theorem 1.9. See §6 and Figure 5 below for details.

Triangle groups. We begin by defining the Hilbert series, stating its main properties and formulating its conjectural characterizations.

The classical hyperbolic triangle groups

$$\Delta(p, q, r) \subset \mathrm{SL}_2(\mathbb{R})$$

are uniquely determined, up to conjugacy, by triples of positive integers satisfying $1/p + 1/q + 1/r < 1$. They are readily described geometrically as subgroups of index two in the reflection groups associated to the triangles $T(p, q, r) \subset \mathbb{H}$ with internal angles $(\pi/p, \pi/q, \pi/r)$.

Algebraic models. One can also approach triangle groups from an algebraic perspective. Let us say $\Delta = \Delta(p, q, r)$ has a *matrix model* (or simply a *model*) over a subring $A \subset \mathbb{R}$ if it is conjugate to a subgroup of $\mathrm{SL}_2(A)$. Every triangle group has a model over a number field L (see e.g. [CV, eq. (2.7)]). This field necessarily contains the *trace field*

$$K = \mathbb{Q}(\mathrm{tr}\,\Delta) = \mathbb{Q}(\cos \pi/p, \cos \pi/q, \cos \pi/r). \tag{1.1}$$

One can always find a model with $[L : K] = 2$. It is unusual, however, for Δ to admit a model over K itself.

The Hilbert series. In this paper we will study the 11 triangle groups $\Delta(p, q, r)$ in the *Hilbert series*, given by $(p, q, r) =$

$$(2, 4, 6),\quad (2, 6, 6),\quad (3, 4, 4),\quad (3, 6, 6),\quad (2, 6, 10),\quad (3, 10, 10),$$
$$(5, 6, 6),\quad (6, 10, 15)\quad (4, 6, 12),\quad (6, 9, 18),\quad \text{and}\quad (14, 21, 42).$$

See Figure 1. We will show that every group in this series admits a model over its trace field K. In fact, we will show (§3):

Theorem 1.1 *Every group $\Delta(p, q, r)$ in the Hilbert series admits a matrix model over the ring of integers $\mathcal{O}_K$ in its trace field K.*

The most remarkable group in the Hilbert series is $\Delta(14, 21, 42)$. For this group we will also show (§4):

Theorem 1.2 *Every subgroup* Γ *of finite index in* $\Delta(14, 21, 42)$ *admits a matrix model over the ring of integers in* $\mathbb{Q}[\operatorname{Tr}\Gamma]$.

This result will be used to construct an exotic curve on a Hilbert modular variety; see Theorem 1.9 below.

Generators. The proof we offer for Theorems 1.1 and 1.2, provides *explicit matrix generators* for $\Delta(p, q, r)$ in $\mathrm{SL}_2(\mathcal{O}_K)$. This approach makes the proof easy to verify and independently useful for computational work.

Despite the large literature on triangle groups, these algebraically optimized matrices appear to be new; they were found using a general algorithm, described in §2. In the Appendix, we apply the same method to give an integral model for the arithmetic group $\Delta(2, 3, 7)$.

Quaternion algebras. To explain the significance of the Hilbert series more completely, we formulate three conjectures in the language of quaternion algebras.

Recall that a quaternion algebra B over a field K of characteristic zero is a central simple algebra of rank 4. We let $\mathrm{SL}(B) \subset B^\times$ denote the group of units of norm one.

The algebra B is *split* if $B \cong \mathrm{M}_2(K)$; otherwise, B is a division algebra. When K is a number field, it is known that the Hasse principle holds:

$$B \text{ is split} \iff B_v = B \otimes_K K_v \text{ is split for every place } v \text{ of } K. \tag{1.2}$$

See e.g. [MR, Theorem 2.7.2]. The algebra B is said to *ramify* at the finitely many places where B_v is not split. We say B splits *over* a field extension L/K if $B \otimes_K L$ is split.

Triangle groups. Now let $\Delta = \Delta(p, q, r) \subset \mathrm{SL}_2(\mathbb{R})$ be a cocompact triangle group with trace field K. Then the associated ring

$$B = \mathbb{Q}[\Delta] \subset \mathrm{M}_2(\mathbb{R})$$

is a quaternion algebra over K. The algebra B is determined, up to isomorphism, by the triple (p, q, r), as we will see explicitly in §2. If B splits over L, then the inclusion

$$\Delta \subset \mathrm{SL}(B \otimes_K L) \cong \mathrm{SL}_2(L)$$

gives a model for Δ over L. The converse also holds. Thus one can regard the inclusion

$$\Delta \subset \mathrm{SL}(B)$$

as a canonical precursor to any matrix model for Δ (including its usual model over $\mathbb{R}$).

Commensurability. Let $\Delta_0 = \langle g^2 : g \in \Delta \rangle$. The *invariant* trace field and quaternion algebra of Δ, defined by

$$K_0 = \mathbb{Q}[\operatorname{tr} \Delta_0] \quad \text{and} \quad B_0 = \mathbb{Q}[\Delta_0],$$

depend only on the *commensurability class* of Δ [MR, Cor. 3.3.5]. We note that

$$\Delta/\Delta_0 \cong H_1(\Delta, \mathbb{Z}/2) \cong (\mathbb{Z}/2)^e,$$

where $e = 2$ if all (p, q, r) are even, $e = 1$ if exactly two are even, and otherwise $e = 0$.

We can now formulate three conjectures concerning all cocompact triangle groups $\Delta = \Delta(p, q, r)$.

Conjecture 1.3 *The quaternion algebra B_v is split at all infinite places v of K $\iff$ Δ belongs to the Hilbert series.*

Conjecture 1.4 *The quaternion algebra B is split $\iff$ Δ belongs to the Hilbert series.*

Conjecture 1.5 *The invariant quaternion algebra B_0 is split $\iff$ Δ is conjugate to* $\Delta(14, 21, 42)$.

The implication $\Longleftarrow$ in each conjecture follows from Theorems 1.1 and 1.2 above. We regard Conjecture 1.3 as the *main* conjecture, since it clearly implies Conjecture 1.4, which in turn implies Conjecture 1.5 (see below).

Added in proof. A proof of Conjecture 1.3 has been announced in [CC].

Totally real models. We now discuss the main conjecture in more detail, and provide evidence in its support.

If B_v is ramified at $v|\infty$, then B_v is isomorphic to Hamilton's quaternions, which can only be split by extending $K_v = \mathbb{R}$ to $\mathbb{C}$. Thus the main conjecture is equivalent to:

Conjecture 1.6 *A cocompact triangle group Δ has a model over a totally real field $\iff$ Δ belongs to the Hilbert series.*

Totally hyperbolic groups. In general, if B is split at s infinite places and ramified at r others, where $r+s=[K:\mathbb{Q}]$, then the embedding

$$\Delta \subset \prod_{v|\infty} \mathrm{SL}(B_v) \cong \mathrm{SU}_2(\mathbb{R})^r \times \mathrm{SL}_2(\mathbb{R})^s$$

gives an isometric action of Δ on $\mathbb{S}^r \times \mathbb{H}^s$, where $\mathbb{S}$ denotes the unit sphere in $\mathbb{R}^3$. We say Δ is *totally hyperbolic* if $r=0$. In these terms, the main conjecture states that:

> *The Hilbert series is the complete list of totally hyperbolic cocompact triangle groups.*

This is [Mc5, Conjecture 1.15].

Evidence. It is straightforward to test if $\Delta(p,q,r)$ is totally hyperbolic. Let $L=\left\{a\in\mathbb{Z}^3 \,:\, \sum a_i = 0 \bmod 2\right\}$, and for $a\in\mathbb{R}^3$, let

$$\|a\|_1 = \sum |a_i| \quad \text{and} \quad \|a-L\|_1 = \inf\{\|a-b\|_1 \,:\, b\in L\}.$$

Then $\Delta(p,q,r)$ is totally hyperbolic $\iff \|ka-L\|_1<1$ for all $k\in(\mathbb{Z}/n)^*$, where $a=(1/p,1/q,1/r)$ and $n=2\,\mathrm{lcm}(p,q,r)$; see [Mc5, Cor. 7.3]. Using this test, we have verified:

Theorem 1.7 *The only totally hyperbolic triangle groups with $p,q,r\le 5000$ are those in the Hilbert series.*

It is also known that the set of cocompact, totally hyperbolic triangle groups is finite [WM, Theorem 4], [Mc5, Cor. 1.9].

We remark that the computation above can be accelerated by choosing a random generator u of $\mathbb{Z}/n$, and then testing ka for $k=u,2u,3u,\ldots \bmod n$ instead of $k=1,2,3\ldots \bmod n$.

Table of invariants. The principal invariants of the groups $\Delta(p,q,r)$ in the Hilbert series are summarized in Table 2. In this table, the second column gives an algebraic number u generating the trace field K. The next two columns give $[K:\mathbb{Q}]$ and $[K_0:\mathbb{Q}]$. Each quaternion algebra B_0/K_0, except the last, is ramified at two finite places of K_0, which lie above two primes $\mathcal{P}$ of $\mathbb{Q}$, listed in the final column. Since the quaternion algebra $B=B_0\otimes_{K_0}K$ splits whenever B_0 splits, this final column shows:

> *Conjecture 1.4 implies 1.5.*

(p,q,r)	$K=\mathbb{Q}(u)$	$\deg K$	$\deg K_0$	$\mathcal{P}$
$(2,4,6)$	$\cos\pi/12$	4	1	$\{2,3\}$
$(2,6,6)$	$\cos\pi/6$	2	1	$\{2,3\}$
$(3,4,4)$	$\cos\pi/4$	2	1	$\{2,3\}$
$(3,6,6)$	$\cos\pi/6$	2	1	$\{2,3\}$
$(2,6,10)$	$\cos\pi/30$	8	2	$\{3,5\}$
$(3,10,10)$	$\cos\pi/10$	4	2	$\{3,5\}$
$(5,6,6)$	$\cos\pi/5+\cos\pi/6$	4	2	$\{3,5\}$
$(6,10,15)$	$\cos\pi/30$	8	4	$\{3,5\}$
$(4,6,12)$	$\cos\pi/12$	4	2	$\{2,3\}$
$(6,9,18)$	$\cos\pi/18$	6	3	$\{2,3\}$
$(14,21,42)$	$\cos\pi/42$	12	6	$\emptyset$

Table 2. The Hilbert series.

We remark that the first four examples in Table 2 are arithmetic and commensurable; the next three are also commensurable; and for the last three examples, $(1/p,1/q,1/r)$ is proportional to $(1,2,3)$.

Kobayashi geodesic curves. Next we relate the main conjecture to complex geometry. Recall that any totally real field L of degree d over $\mathbb{Q}$ determines a *Hilbert modular variety*

$$X_L = \mathbb{H}^d/\operatorname{SL}_2(\mathcal{O}_L), \tag{1.3}$$

where $\operatorname{SL}_2(\mathcal{O}_L)$ acts on $\mathbb{H}^d$ via its d distinct embeddings into $\operatorname{SL}_2(\mathbb{R})$.

Let V be a hyperbolic Riemann surface of finite volume, equipped with a holomorphic map to a complex manifold,

$$f : V = \mathbb{H}/\Gamma \to X.$$

By the Schwarz lemma, f is distance non–increasing from the complete hyperbolic metric on V (of constant curvature -4) to the Kobayashi metric on X.

In the rare case that f is a local isometry, we say V is a *Kobayashi geodesic curve* on X (or simply a *geodesic curve*). We also allow V and X to be orbifolds, in which case f must respect the orbifold structure.

The Kobayashi metric on $\mathbb{H}^d$ is the supremum of the hyperbolic metrics on each factor; it descends to give the Kobayashi metric on X_K. As we will see in §6, Conjecture 1.3 is also equivalent to:

Conjecture 1.8 *A finite cover of* $V = \mathbb{H}/\Delta(p,q,r)$ *can be presented as a Kobayashi geodesic curve on a Hilbert modular variety* $\iff$ $\Delta(p,q,r)$ *belongs to the Hilbert series.*

The implication $\Longleftarrow$ is Corollary 6.5 below.

Moduli spaces and Hilbert modular surfaces. We now turn to a question in complex geometry answered by Theorem 1.2 above. This question motivated our investigation of triangle groups.

There are two known cases where a geodesic curve $f : V \to X$ is forced to have a cusp; that is, where V cannot be compact. They occur when:

1. The target X is the *moduli space* $\mathcal{M}_g$ of compact Riemann surfaces of genus $g \geq 2$ [V, Prop. 2.10]; and when
2. The target X is a *Hilbert modular surface* X_L, $\deg(L/\mathbb{Q}) = 2$, and V is not a Shimura curve on X_L [Mc3].

In case (1), the Kobayashi metric on $\mathcal{M}_g$ coincides with the Teichmüller metric, and V is called a *Teichmüller curve* (for a recent survey, see [Mc4]). In case (2), one can normalize so that the lift of f to a map $\widetilde{f} : \mathbb{H} \to \mathbb{H}^2$ has the form

$$\widetilde{f}(z) = (z, f_2(z)),$$

and $f_2 : \mathbb{H} \to \mathbb{H}$ is a hyperbolic isometry if and only if V is a Shimura curve. Most examples of case (1) (all but finitely many in each genus) can be deduced from case (2); see [Mo, Cor. 2.11] and [EFW, Cor. 1.6]).

Higher dimensions. We now address the following question: does a suitable generalization of (2) hold when $\dim X_L > 2$?

It seems likely that the answer is no, already when $\dim X_L = 3$. However, aside from Teichmüller curves and Shimura curves, and their images under Hecke operators, few examples of geodesic curves are known. Nevertheless, using Theorem 1.2, and a construction from [CW] special to triangle groups, in §6 we will show:

Theorem 1.9 *There exists a compact geodesic curve*

$$f : V = \mathbb{H}/\Gamma \to X_L,$$

with $\dim X_L = 6$*, such that* $f(V)$ *is not contained in any proper Shimura subvariety* $S \subset X_L$*.*

For the proof we take Γ to be a suitable subgroup of $\Delta(14, 21, 42)$. The condition on $f(V)$ excludes curves of the form $V \to X_K \to X_L$, where $\dim(X_K) < \dim(X_L)$, as well as compact Shimura curves on X_L.

It would be interesting to find other constructions of totally geodesic curves in Hilbert modular varieties, and determine what happens when $\dim X_L = 3$.

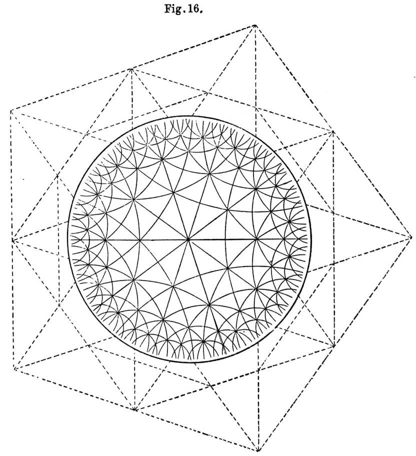

Figure 3. The triangle group $\Delta(2, 4, 5)$ (Schwarz, 1873).

Notes and references. According to Poincaré, cocompact hyperbolic triangle groups first appeared in the work of Schwarz on hypergeometric functions [P, p.168]; see e.g. Figure 3, reproduced from [Sch, p. 240]. Fricke and Klein studied many triangle groups from both a geometric and algebraic point of view in [FK]. For a modern perspective, including the theory of quaternion algebras, see the monographs [MR] and [Voi]. We remark that once formulated, the statement of Theorem 1.2 can also be ratified by less explicit calculations, using Hilbert symbols and the classification of maximal orders.

The groups in the Hilbert series are listed as 'exceptional triples' in [WM, p.362]. This means the existence of matrix models over K is not *ruled out* by their Theorem 2. We are grateful to A. Reid for this reference.

As discussed in [Mc5], one can regard the Hilbert series as a complement to the finite list of arithmetic triangle groups obtained by Takeuchi in [Tak]. In the former case, B_0 is split at all infinite places of K_0; in the latter case, at just one. (The two series overlap when $\deg K_0 = 1$).

More details on geodesic curves, Teichmüller curves and Hilbert modular varieties can be found in [vG], [CW], [Mc1], [Mc2], [Mo], [MV], [BM] and [We]. For an introduction to Shimura varieties, see [Milne].

This paper is a sequel to [Mc5] and [Mc3]. The latter paper includes a discussion of triangle groups with cusps on Hilbert modular surfaces. For recent work on Conjecture 1.3, see [CC].

Acknowledgements. I would like to thank A. Reid and P. Tretkoff (coauthor of [CW]) for many useful conversations.

2 Models for triangle groups

In this section we describe a procedure to produce an explicit matrix model for a triangle group over a given field, or show none exists.

Quaternion algebras. Let B be a quaternion algebra over a number field K. There is a natural K–linear involution $x \mapsto x'$ on B such that $(xy)' = y'x'$ and $x' = x$ if and only if $x \in K$. The (reduced) trace and norm from B to K are defined by $\mathrm{tr}(x) = x + x'$ and $N(x) = xx'$. The trace form

$$\langle x, y\rangle = \frac{1}{2}\,\mathrm{tr}(xy) = \frac{xy + y'x'}{2}$$

is nondegenerate, and satisfies $N(x) = \langle x, x'\rangle$.

We have an orthogonal decomposition

$$B = K \oplus B^0,$$

where $B^0 = \{x \in B \,:\, \mathrm{tr}(x) = 0\}$. Note that $B^0 \cong K^3$ as a vector space over K. The zero set of the norm form restricted to B^0 determines a conic

$$Q(B) \subset \mathbb{P}(B^0) \cong \mathbb{P}^2(K),$$

defined equivalently by $\mathrm{tr}(x^2) = 0$. The group

$$\mathrm{SL}(B) = \{x \in B \,:\, N(x) = 1\} \subset B^\times$$

acts by conjugation on $\mathbb{P}(B^0)$, preserving $Q(B)$.

One can think of $\mathrm{SL}(B)$ and $Q(B)$ as potentially twisted forms of $\mathrm{SL}_2(K)$ and $\mathbb{P}^1(K)$. When $B = \mathrm{M}_2(K)$, the norm and trace on B agree with the usual trace and determinant on $\mathrm{M}_2(K)$.

Matrices for *B*. It is useful to be able to construct an explicit isomorphism $B \cong \mathrm{M}_2(K)$ when one exists. To this end, we recall that the following three statements are equivalent:

1. The conic $Q(B)$ has a point over K.

2. There exists a $u \neq 0$ in B such that $u^2 = 0$.

3. The algebra B is split.

The proof is constructive. Suppose $[u] \in \mathbb{P}B^0$ represents a point on $Q(B)$. Then $\operatorname{tr}(u) = N(u) = 0$, and hence $u \neq 0$ but $u^2 = -uu' = -N(u) = 0$. It follows that $\dim(Bu) \leq 2$. Since the trace form is nondegenerate, $\operatorname{tr}(xu) = 1$ for some $xu \in Bu$; thus

$$Bu = Ku \oplus Kxu \cong K^2.$$

Now we are done: the left action of B on Bu gives a map of K–algebras,

$$\phi : B \to \operatorname{End}(Bu) \cong \mathrm{M}_2(K),$$

and since B is simple, ϕ is an isomorphism. Thus (1) $\implies$ (2) $\implies$ (3), and the implication (3) $\implies$ (1) is immediate.

The regular representation. Now consider a nonelementary Fuchsian group $\Gamma \subset \mathrm{SL}_2(\mathbb{R})$, generated by three elements a, b and c satisfying

$$(\operatorname{tr} a, \operatorname{tr} b, \operatorname{tr} c) = (\alpha, \beta, \gamma) \quad \text{and} \quad abc = -I.$$

Let $K = \mathbb{Q}(\alpha, \beta, \gamma)$ and let $B = \mathbb{Q}[\Gamma]$. Since Γ is nonelementary, the matrices $\{I, a, b, c\}$ form a basis for $\mathrm{M}_2(\mathbb{R})$ over $\mathbb{R}$, and hence a basis for the quaternion algebra B over K. With respect to this basis, the left regular representation gives an embedding of algebras,

$$\psi : B \to \mathrm{M}_4(K).$$

Under this embedding, $N(a) = \det(a)$ and $1 = I$.

To compute the matrix $\psi(a)$, we need to express a^2, ab and ac in terms of the basis above. The first two products are immediate: since $a + a' = \alpha$ and $aa' = \det(a) = 1$, we have

$$a^2 = \alpha a - 1,$$

and from $abc = -1$ we get $ab = -c^{-1} = c - \gamma$.

To compute ac, first let $a_0 = a - \alpha/2$ denote the projection of a to B^0, and similarly for b_0 and c_0. Then

$$a_0 c_0 + c_0 a_0 = \operatorname{tr}(a_0 c_0),$$

because $x' = -x$ for all $x \in B^0$. Expanding this expression, we find

$$ac = \gamma a + \alpha c - b - \alpha\gamma,$$

and hence

$$\psi(a) = \begin{pmatrix} 0 & -1 & -\gamma & -\alpha\gamma \\ 1 & \alpha & 0 & \gamma \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & \alpha \end{pmatrix}.$$

Similar reasoning leads to matrices for $\psi(b)$ and $\psi(c)$, also expressed in terms of (α, β, γ).

Models for triangle groups. With the regular representation of B in hand, an explicit model for $\Delta(p, q, r)$ over its trace field K — if it exists — can be computed as follows.

1. Construct $B \subset \mathrm{M}_4(K)$ as above, using

$$(\operatorname{tr} a, \operatorname{tr} b, \operatorname{tr} c) = 2(\cos \pi/p, \cos \pi/q, \cos \pi/r).$$

2. Compute the homogeneous polynomial

$$P(x, y, z) = \operatorname{tr}((xa_0 + yb_0 + zc_0)^2)$$

defining the conic $Q(B) \subset \mathbb{P}B^0$. (Note that the trace of u in B is half the trace of $\psi(u)$ in $\mathrm{M}_4(K)$.)

3. Find a nonzero solution to the homogeneous equation $P(x, y, z) = 0$ over K, to obtain a K–rational point $p \in Q(B)$. (This is a standard problem in computer algebra.)

4. Let $[u] \in \mathbb{P}(B^0)$ represent p. Then $u^2 = 0$.

5. Choose an isomorphism $Bu \cong K^2$, and compute the representation

$$\phi : B \to \operatorname{End}(Bu) \cong \mathrm{M}_2(K).$$

6. The restriction of ϕ to $\Gamma \subset \mathrm{SL}(B)$ then gives the desired model

$$\phi : \Gamma \to \mathrm{SL}_2(K).$$

Obstructions. Of course B may not split over K. In this case one can obtain, by the same procedure, a model for $\Delta(p, q, r)$ over $L \supset K$, whenever $P(x, y, z) = 0$ has a nontrivial solution over L.

The obstruction to splitting can be found by first computing the Hilbert symbol for B, given in [MR, §3.6]; and then computing the ramified primes of B. The last column of Table 2 was obtained in this way.

3 Generators for the Hilbert series

In this section we give a concise proof of Theorem 1.1, constructed using the method of §2. (Some care was required to obtain models over $\mathcal{O}_K$.)

Proof of Theorem 1.1. For each group $\Delta(p,q,r)$ in the Hilbert series, we give below an algebraic integer t, and a pair of matrices $a, b \in \mathrm{SL}_2(\mathbb{Z}[t])$, such that $\mathcal{O}_K = \mathbb{Z}[t]$ and

$$\Delta(p,q,r) = \langle a, b \rangle. \tag{3.1}$$

Thus $\langle a, b\rangle$ gives the desired model of $\Delta(p,q,r)$ over $\mathcal{O}_K$. To verify (3.1), the reader need only check that

$$(\operatorname{tr} a, \operatorname{tr} b, \operatorname{tr} c) = 2(\cos \pi/p, \cos \pi/q, \cos \pi/r),$$

where $abc = -I$. We have included c where space permits. In the special case of $\Delta(5,6,6)$, $\mathcal{O}_K = \mathbb{Z}[s,t]$ requires two generators.

Here are the required generators for each group.

$\boldsymbol{\Delta(2,4,6)}$: $t = 2\cos \pi/12 = (1+\sqrt{3})/\sqrt{2})$; $a, b =$

$$\begin{pmatrix} -t^3+5t+2 & 2t^3+t^2-6t-2 \\ -2t^3-t^2+6t+2 & t^3-5t-2 \end{pmatrix}, \begin{pmatrix} 2t^3+t^2-6t-2 & -t^3+5t+1 \\ t^3-5t-2 & -t^3-t^2+3t+2 \end{pmatrix}.$$

$\boldsymbol{\Delta(2,6,6)}$: $t = 2\cos \pi/6 = \sqrt{3}$;

$$a = \begin{pmatrix} 2-t & t-1 \\ 2-2t & t-2 \end{pmatrix}, \; b = \begin{pmatrix} t-1 & 1 \\ t-2 & 1 \end{pmatrix}, \; c = \begin{pmatrix} t & 1 \\ -1 & 0 \end{pmatrix}.$$

$\boldsymbol{\Delta(3,4,4)}$: $t = 2\cos \pi/4 = \sqrt{2}$;

$$a = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}, \; b = \begin{pmatrix} -1 & -t \\ t+1 & t+1 \end{pmatrix}, \; c = \begin{pmatrix} -1 & -t-1 \\ t & t+1 \end{pmatrix}.$$

$\boldsymbol{\Delta(3,6,6)}$: $t = 2\cos \pi/6 = \sqrt{3}$;

$$a = \begin{pmatrix} 2-t & t \\ 3-2t & t-1 \end{pmatrix}, \; b = \begin{pmatrix} t & t+2 \\ t-2 & 0 \end{pmatrix}, \; c = \begin{pmatrix} t & 1 \\ -1 & 0 \end{pmatrix}.$$

$\boldsymbol{\Delta(2,6,10)}$: $t = 2\cos \pi/30$;

$$a = \begin{pmatrix} \left(t^2-2\right)\left(t^2-1\right)^2 & 2t^5-7t^3+5t \\ -\left(2t^5-7t^3+5t\right) & -\left(t^2-2\right)\left(t^2-1\right)^2 \end{pmatrix},$$

$$b = \begin{pmatrix} 2t^5 - 7t^3 + 5t & 2t^4 - 4t^2 \\ -\left(t^2-2\right)\left(t^2-1\right)^2 & -t^5 + 2t^3 \end{pmatrix}.$$

$\Delta(3, 10, 10)$: $t = 2\cos\pi/10$;

$$a = \begin{pmatrix} 3t^3 - 5t^2 - 4t + 6 & -2t^3 + 5t^2 + t - 7 \\ 3t^3 - 6t^2 - 3t + 8 & -3t^3 + 5t^2 + 4t - 5 \end{pmatrix},\ b = \begin{pmatrix} 0 & -1 \\ 1 & t \end{pmatrix}.$$

$\Delta(5, 6, 6)$: $s = 2\cos\pi/5 = (1+\sqrt{5})/2,\ t = 2\cos\pi/6 = \sqrt{3}$;

$$a = \begin{pmatrix} 0 & -1 \\ 1 & s \end{pmatrix},\ b = \begin{pmatrix} -st + 2s + 1 & -st + s - t + 1 \\ st - 2s + t - 1 & st - 2s + t - 1 \end{pmatrix}.$$

$\Delta(6, 10, 15)$: $t = 2\cos\pi/30$;

$$a = \begin{pmatrix} 0 & -1 \\ 1 & t^5 - 5t^3 + 5t \end{pmatrix},\ b =$$

$$\begin{pmatrix} -4t^7 + t^6 + 27t^5 - 6t^4 - 51t^3 + 10t^2 + 25t - 2 & t^7 - 2t^6 - 6t^5 + 14t^4 + 9t^3 - 29t^2 - 2t + 15 \\ -2t^7 - t^6 + 12t^5 + 6t^4 - 18t^3 - 5t^2 + 4t - 5 & 4t^7 - t^6 - 27t^5 + 6t^4 + 52t^3 - 10t^2 - 28t + 2 \end{pmatrix}.$$

$\Delta(4, 6, 12)$: $t = 2\cos\pi/12$;

$$a = \begin{pmatrix} t^3 - t^2 - 2t + 1 & -t^3 + 2t^2 + t - 2 \\ 1 - t^2 & t^2 - t - 1 \end{pmatrix},\ b = \begin{pmatrix} -t^3 + 2t^2 + t - 2 & 2t^3 - 2t^2 - 3t \\ -t^3 + t^2 + 2t - 1 & t^3 - t^2 - t \end{pmatrix}.$$

$\Delta(6, 9, 18)$: $t = 2\cos\pi/18$;

$$a = \begin{pmatrix} 3t^5 - 19t^3 + 5t^2 + 24t - 6 & t^5 - 4t^4 - 4t^3 + 19t^2 + 3t - 22 \\ -4t^5 + 4t^4 + 17t^3 - 14t^2 - 15t + 16 & -3t^5 + 20t^3 - 5t^2 - 27t + 6 \end{pmatrix},$$

$$b = \begin{pmatrix} -t^5 + 6t^3 - 2t^2 - 6t + 5 & -2t^5 + t^4 + 11t^3 - 4t^2 - 14t + 3 \\ t^5 - 2t^4 - 5t^3 + 8t^2 + 8t - 6 & t^5 - 6t^3 + 3t^2 + 6t - 7 \end{pmatrix}.$$

$\Delta(14, 21, 42)$: $t = 2\cos\pi/42$; $a =$

$$\begin{pmatrix} -3t^{11} + 33t^9 - 131t^7 + 227t^5 - 165t^3 + 36t & 3t^{11} - 32t^9 + 123t^7 - 206t^5 + 144t^3 - 31t \\ -3t^{11} + 33t^9 - 132t^7 + 234t^5 - 179t^3 + 45t & 3t^{11} - 33t^9 + 131t^7 - 227t^5 + 166t^3 - 39t \end{pmatrix},$$

$$b = \begin{pmatrix} 0 & -1 \\ 1 & t^2 - 2 \end{pmatrix}.$$

∎

4 The $(14, 21, 42)$ triangle group

In this section we will show:

Theorem 4.1 *The unique subgroup* Δ_0 *of index two in* $\Delta(14, 21, 42)$ *admits a model over the ring of integers in its trace field.*

We then deduce Theorem 1.2.

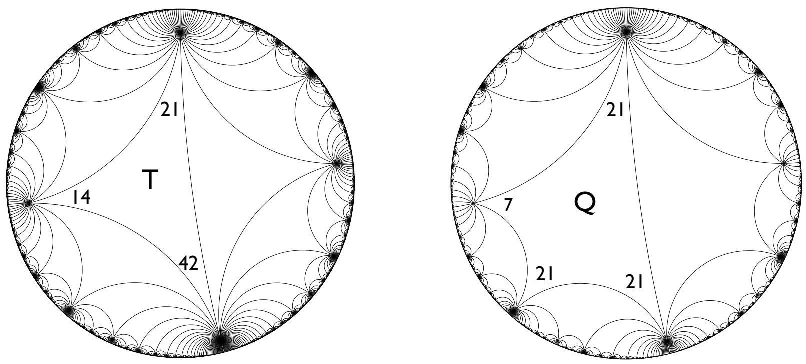

Figure 4. The triangle group $\Delta(14, 21, 42)$ and its quadrilateral subgroup Δ_0.

From triangles to quadrilaterals. Let Δ_0 be the unique subgroup of index two in $\Delta = \Delta(14, 21, 42)$. It is readily verified that these groups have trace fields $K_0 = \mathbb{Q}(\cos \pi/21)$ and $K = \mathbb{Q}(\cos \pi/42)$ respectively. In fact, $\Delta_0 = \langle g^2 \, : \, g \in \Delta \rangle$; thus K_0 is the *invariant* trace field of Δ.

Just as Δ has index two in the reflection group for the triangle $T = T(14, 21, 42)$, Δ_0 has index two in the reflection group for the symmetric quadrilateral $Q(7, 21, 21, 21)$ built from two copies of T; see Figure 4. To describe Δ_0 explicitly, let (a, b, c) be the generators for $\Delta(14, 21, 42)$ given in §3. These generators satisfy $a^{14} = b^{21} = c^{42} = abc = -I$. Then

$$(A, B, C, D) = (a^2, b, c^2, c^{-1}bc)$$

are generators for Δ_0, satisfying

$$A^7 = B^{21} = C^{21} = D^{21} = -ABCD = -I.$$

There is one generator for each vertex of Q.

Proof of Theorem 4.1. It can now be checked directly, from the explicit matrices a, b, c given in §3, that we have $A, B, C, D \in \mathrm{SL}_2(\mathcal{O}_{K_0})$. ■

Matrices. Since the matrices for Δ_0 are simpler than those for Δ, we also give them explicitly: setting $t = 2\cos \pi/21 \in \mathcal{O}_{K_0}$, we have:

$$A = \begin{pmatrix} t^5 - 4t^3 - t^2 + 3t & -t^5 + t^4 + 6t^3 - 3t^2 - 10t \\ t^3 - 1 & -t^5 + 5t^3 + t^2 - 6t \end{pmatrix}, \ B = \begin{pmatrix} 0 & -1 \\ 1 & t \end{pmatrix},$$

$$C = \begin{pmatrix} t^5 + t^4 - 4t^3 - 4t^2 + 2t & t^4 + t^3 - 4t^2 - 5t \\ -t^4 - t^3 + 4t^2 + 5t - 1 & -t^5 - t^4 + 4t^3 + 4t^2 - t \end{pmatrix}, \quad \text{and}$$

$$D = \begin{pmatrix} -t^5 + 2t^4 + 7t^3 - 6t^2 - 13t - 1 & 2t^5 + t^4 - 9t^3 - 5t^2 + 6t \\ -t^5 + 5t^3 + 2t^2 - 5t + 1 & t^5 - 2t^4 - 7t^3 + 6t^2 + 14t + 1 \end{pmatrix}.$$

Galois theory. To prove Theorem 1.2, we must generalize Theorem 4.1 to all subgroups of finite index in $\Delta = \Delta(14, 21, 42)$.

For this purpose, note that the trace field $K = \mathbb{Q}(\cos \pi/42)$ of Δ is a quadratic extension of K_0, with Galois group $G = \mathrm{Gal}(K/K_0) \cong \mathbb{Z}/2$. Let $g \mapsto g'$ denote the action of G on Δ. We claim that

$$\Delta^G = \Delta_0.$$

Indeed, noting that only odd powers of t appear in the expression for a in §3, and only even powers in the expression for b, we find that $a' = -a$, $b' = b$ and $c' = -c$, and hence A, B, C and D are all fixed by G. This shows $\Delta_0 \subset \Delta^G$, and equality must hold since $\Delta^G \neq \Delta$.

Lemma 4.2 *Let $\Gamma \subset \Delta(14, 21, 42)$ be a subgroup of finite index. Then the trace field of Γ is $K_0 \iff \Gamma \subset \Delta_0$.*

Proof. Choose elements $a_1, a_2, a_3, a_4 \in \Delta_0 \cap \Gamma$ that form a basis for $\mathrm{M}_2(K_0)$ over K_0. Suppose the trace field of Γ is K_0. Then for any $g \in \Gamma$, we have

$$\mathrm{tr}(a_i g) = \mathrm{tr}(a_i g)' = \mathrm{tr}(a_i g')$$

for all i. But the trace pairing is nondegenerate, so $g' = g$, and thus $\Gamma \subset \Delta^G = \Delta_0$. The converse is immediate. ∎

Proof of Theorem 1.2. Since $[K : K_0] = 2$, the trace field of any subgroup of finite index $\Gamma \subset \Delta$ is either K or K_0. In the first case, Γ has a model over $\mathcal{O}_K$ because Δ does; and in the second case, it has a model over $\mathcal{O}_{K_0}$ because $\Gamma \subset \Delta_0$. ∎

5 Zariski density

Let L be a totally real field of degree d over $\mathbb{Q}$, and let $\mathrm{SL}_2(L)$ be embedded in $\mathrm{SL}_2(\mathbb{R})^d$ using the d distinct real places of L. In this section we will show:

Theorem 5.1 *Let Γ be a subgroup of $\mathrm{SL}_2(L) \subset \mathrm{SL}_2(\mathbb{R})^d$. Then Γ is Zariski dense in $\mathrm{SL}_2(\mathbb{R})^d \iff$*

1. *The invariant trace field of Γ is L; and*
2. *The group Γ is not virtually solvable.*

This result will be used, in the proof of Theorem 1.9, to show that the curve $V \subset X_L$ does not lie on a proper Shimura subvariety.

Remarks.

1. The Zariski closure of $\Gamma \subset \mathrm{SL}_2(\mathbb{R})^d$ coincides with the smallest Lie group H such that
$$\Gamma \subset H \subset \mathrm{SL}_2(\mathbb{R})^d \quad \text{and} \quad |H/H^0| < \infty;$$
cf. [Bor, I.1]. More formally, Theorem 5.1 concerns the Zariski density of Γ in the $\mathbb{Q}$–algebraic group $T = \mathrm{Res}_{L/\mathbb{Q}} \mathrm{SL}_2$, which satisfies $T(\mathbb{Q}) = \mathrm{SL}_2(L)$ and $T(\mathbb{R}) \cong \mathrm{SL}_2(\mathbb{R})^d$.

2. It is critical in Theorem 5.1 that L is the *invariant* trace field of Γ. For example, the trace field of $\Gamma = \Delta(3,4,4)$ is $K = \mathbb{Q}(\sqrt{3})$, while its invariant trace field is $K_0 = \mathbb{Q}$. If we embed Γ into $\mathrm{SL}_2(K)$ using the generators (a,b,c) given in §2, then their Galois conjugates satisfy
$$a' = hah^{-1}, b' = -hbh^{-1} \quad \text{and} \quad c' = -hch^{-1},$$
where $h = \left(\begin{smallmatrix} -1 & 2 \\ 2 & 1 \end{smallmatrix}\right)$. Hence Γ is not Zariski dense in $\mathrm{SL}_2(\mathbb{R})^2$; it lies in the Lie subgroup H defined by $g_2 = \pm h g_1 h^{-1}$. A similar statement holds for all groups in the Hilbert series.

Background on $\mathbf{SL_2}(\mathbb{R})$. A group Γ is *virtually solvable* if it contains a solvable subgroup of finite index. We note that:

$\Gamma \subset \mathrm{SL}_2(\mathbb{R})$ *is Zariski dense* $\iff$ Γ *is not virtually solvable,* since H^0 is solvable for every proper algebraic subgroup $H \subset \mathrm{SL}_2(\mathbb{R})$.

It is well–known that the adjoint action of $\mathrm{SL}_2(\mathbb{R})$ on its Lie algebra is irreducible, and similarly for any Zariski–dense subgroup Γ. In fact its action generates the full matrix algebra; we have:

$$\mathbb{R}[\mathrm{Ad}\,\mathrm{SL}_2(\mathbb{R})] = \mathbb{R}[\mathrm{Ad}\,\Gamma] = \mathrm{End}(\mathrm{sl}_2(\mathbb{R})) \cong \mathrm{M}_3(\mathbb{R}). \tag{5.1}$$

If the eigenvalues of $g \in \mathrm{SL}_2(\mathbb{R})$ are (μ, μ^{-1}), then those of $\mathrm{Ad}\, g$ are $(\mu^2, 1, \mu^{-2})$, and hence $\mathrm{tr}\,\mathrm{Ad}\, g = \mathrm{tr}(g^2) - 1$. This shows the adjoint trace field and the invariant trace field coincide: we have

$$\mathbb{Q}(\mathrm{tr}\,\mathrm{Ad}\,\Gamma) = \mathbb{Q}(\mathrm{tr}(g^2)\ :\ g \in \Gamma). \tag{5.2}$$

Notation. As above, let L be a totally real field with $d = [L : \mathbb{Q}]$. Choose an ordering for the d distinct embeddings $L \to \mathbb{R}$, and let

$$\lambda \mapsto (\lambda_1, \ldots, \lambda_d) \quad \text{and} \quad g \mapsto (g_1, \ldots, g_d)$$

denote the corresponding embeddings

$$L \to \mathbb{R}^d \quad \text{and} \quad \mathrm{SL}_2(L) \to G = \mathrm{SL}_2(\mathbb{R})^d.$$

For clarity, we will also write $G = \prod_1^d G_i$, and identity each factor $G_i \cong \mathrm{SL}_2(\mathbb{R})$ with the corresponding subgroup of G. Then the Lie algebra of G over $\mathbb{R}$ is given by

$$\mathrm{Lie}(G) = \bigoplus_1^d \mathrm{Lie}(G_i).$$

Let $\pi_i : \mathrm{Lie}(G) \to \mathrm{Lie}(G_i)$ denote the natural projection to each summand.

Proof of Theorem 5.1. Assume that Γ is not virtually solvable and its invariant trace field is L. Let $H \subset G$ be a Lie subgroup with finitely many components such that $\Gamma \subset H$. Note that the algebra generated by the adjoint representation of Γ,

$$A = \mathbb{R}[\mathrm{Ad}\,\Gamma] \subset \mathrm{End}(\mathrm{Lie}(G)),$$

preserves each summand $\mathrm{Lie}(G_i)$, as well as $\mathrm{Lie}(H)$. Since Γ is not virtually solvable, its projection to G_i is Zariski dense, and hence

$$A|\,\mathrm{Lie}(G_i) = \mathrm{End}(\mathrm{Lie}(G_i)) \tag{5.3}$$

by equation (5.1). Consequently $\pi_i(\mathrm{Lie}(H)) = \mathrm{Lie}(G_i)$ for all i, since the image is a nonzero module for the $A|\,\mathrm{Lie}(G_i)$.

To show that Γ is Zariski dense in G, it suffices to show that $\pi_i \in A$ for all i, for then we have $\bigoplus \pi_i(\mathrm{Lie}(H)) = \bigoplus \mathrm{Lie}(G_i) = \mathrm{Lie}(G) \subset \mathrm{Lie}(H)$ and hence $H = G$.

By assumption L is the invariant trace field of Γ, and hence also the trace field of $\mathrm{Ad}\,\Gamma$, by equation (5.2). Now for any $g \in G$, the eigenvalues of $\mathrm{Ad}\,g$ have the form $(\mu, 1, \mu^{-1})$, where

$$\lambda = \mu + \mu^{-1} = (\mathrm{tr}\,\mathrm{Ad}\,g) - 1 \in L.$$

Thus the eigenvalues of

$$S = \mathrm{Ad}\,g + \mathrm{Ad}\,g^{-1}$$

are $(2, \lambda, \lambda)$. Since $L = \mathbb{Q}(\mathrm{tr}\,\mathrm{Ad}\,\Gamma)$, for any $j \neq i$ we can choose g such that $\lambda_j \neq \lambda_i$. Since $\mathrm{Ad}\,g$ has distinct eigenvalues, S is diagonalizable, and hence

$$U = (S - 2)(S - \lambda_j)$$

satisfies $U|\,\mathrm{Lie}(G_j) = 0$, but $U|\,\mathrm{Lie}(G_i) \neq 0$.

In view of (5.1), the two–sided ideal $J \subset A$ generated by U satisfies $J|\,\mathrm{Lie}(G_i) = \mathrm{End}(\mathrm{Lie}\,G_i)$. Thus J contains an element T_{ij} that acts by multiplication by 0 on $\mathrm{Lie}(G_j)$ and by 1 on $\mathrm{Lie}(G_i)$. Therefore $\pi_i = \prod_{i \neq j} T_{ij} \in A$, and hence $H = G$, completing the proof in one direction.

For the converse, suppose that (i) the invariant trace field of Γ is a proper subfield L_0 of L, or (ii) Γ is virtual solvable. In case (i), there exist a pair of distinct indices such that $\lambda_i = \lambda_j$ for all $\lambda \in L_0$, and thus Γ is contained the proper subvariety of G defined by $\mathrm{Tr}\,\mathrm{Ad}\,g_i = \mathrm{Tr}\,\mathrm{Ad}\,g_j$. In the second case, the projection of Γ to G_1 is not Zariski dense. So in either case, Γ is not Zariski dense in G. ■

For a related argument, see [PR, Lemma 5.7].

6 Curves on Hilbert modular varieties

In this section we prove Theorem 1.9, and more generally explain the close relationship between the Hilbert series and Hilbert modular varieties.

Modular embeddings. We begin by reviewing a construction from [CW], adapted to the case at hand. Let $L \subset \mathbb{R}$ be a totally real number field of degree d over $\mathbb{Q}$. As in the previous section, we choose an ordering for the real places of L, and denote the corresponding maps $L \to \mathbb{R}^d$ and $\mathrm{GL}_2(L) \to \mathrm{GL}_2(\mathbb{R})^d$ by $\lambda \mapsto (\lambda_i)$ and $g \mapsto (g_i)$.

In addition to the traditional modular variety $X_L = \mathbb{H}^d / \mathrm{SL}_2(\mathcal{O}_L)$, it is useful to consider the disconnected variety

$$Y_L = (\pm\mathbb{H})^d / \mathrm{SL}_2(\mathcal{O}_L).$$

We say two complex manifolds (or orbifolds) are commensurable if they have isomorphic finite covers.

Proposition 6.1 *Each component of Y_L is commensurable to X_L.*

Proof. Let $g = \left(\begin{smallmatrix} \lambda & 0 \\ 0 & 1 \end{smallmatrix}\right)$ where $\lambda \neq 0$ in L. Then conjugation by g sends $\mathrm{SL}_2(\mathcal{O}_L)$ to a commensurable subgroup of $\mathrm{SL}_2(L)$. On the other hand, as a Möbius transformation, g_i sends $\mathbb{H}$ to $-\mathbb{H}$ whenever $\lambda_i < 0$. Since every possible sequence of signs can be achieved for some $\lambda \in L$, all components of Y_L are commensurable, including the one isomorphic to X_L. ■

Corollary 6.2 *If V is a geodesic curve on Y_L, then a finite cover of V can be presented as a geodesic curve on X_L.*

Theorem 6.3 (Cohen and Wolfart) *Let $\Gamma \subset \Delta(p, q, r)$ be a subgroup of finite index in a triangle group. Suppose that Γ has a model over $\mathcal{O}_L$. Then there exists a holomorphic map*

$$f : V = \mathbb{H}/\Gamma \to (\pm\mathbb{H})^d / \mathrm{SL}_2(\mathcal{O}_L),$$

presenting V as a geodesic curve on Y_L.

The map f is referred to as a *modular embedding* in [CW].

Sketch of the proof. Assume for simplicity that $\Gamma = \Delta(p, q, r)$. Since Γ has a model over $\mathcal{O}_L$, we may also assume that

$$\Delta(p, q, r) \subset \mathrm{SL}_2(\mathcal{O}_L) \subset \mathrm{SL}_2(\mathbb{R}),$$

where the embedding $L \to \mathbb{R}$ is given by $\lambda \mapsto \lambda_1$.

For each $i = 1, 2, \ldots, d$, let $T_i \subset \mathbb{H}$ denote the hyperbolic triangle whose vertices are the fixed points of (a_i, b_i, c_i). Assume the orientations of T_1 and T_i agree. Then, by the Riemann mapping theorem, there is a unique conformal map

$$f_i : T_1 \to T_i,$$

respecting vertices. This map can be analytically continued, by Schwarz reflection through the sides of T_1 and T_i, to a conformal map $f_i : \mathbb{H} \to \mathbb{H}$ satisfying

$$f_i(g_1 \cdot z) = g_i \cdot f_i(z) \tag{6.1}$$

for all $g \in \Gamma$ and $z \in \mathbb{H}$. If the orientations of T_1 and T_i do *not* agree, we construct $f_i : \mathbb{H} \to -\mathbb{H}$ using the analogous Riemann mapping from T_1 to the complex conjugate of T_i. This map also satisfies (6.1).

Let $F : \mathbb{H} \to (\pm\mathbb{H})^d$ be the map defined by

$$F(z) = (z, f_2(z), \ldots, f_d(z)).$$

This map is an isometry from the hyperbolic metric to the Kobayashi metric, since it is the identity on the first coordinate, and by virtue of equation (6.1), it descends to give the desired map $f : V \to Y_L$. ∎

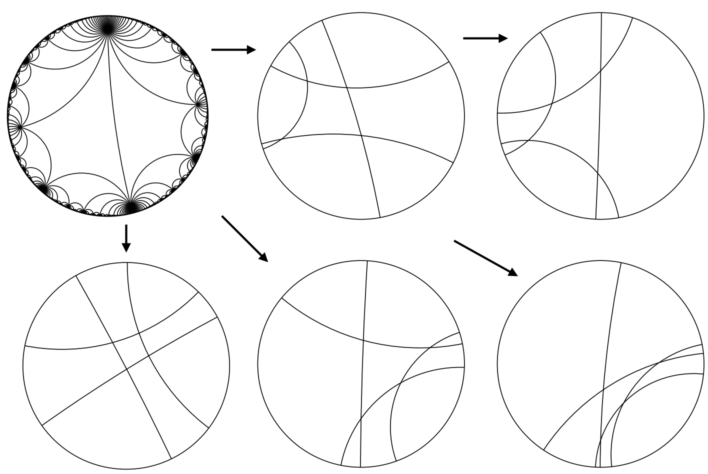

Figure 5. Constructing an equivariant map $\widetilde{f_0} : \mathbb{H} \to \mathbb{H}^6$ for $\Delta_0(14, 21, 42)$.

Proof of Theorem 1.9. Let $L = K_0$ be the trace field of Δ_0, the unique subgroup of index two in $\Delta(14, 21, 42)$. By Theorem 1.2, Δ_0 has a model over $\mathcal{O}_L$; thus Theorem 6.3 provides us with a holomorphic local isometry

$$f_0 : \mathbb{H}/\Delta_0 \to Y_L.$$

The components of the map $\widetilde{f_0}$ are illustrated in Figure 5. By Corollary 6.2, we can pass to a subgroup Γ of finite index in Δ_0 to obtain a geodesic curve

$$f : V = \mathbb{H}/\Gamma \to X_L.$$

Now suppose $f(V)$ is contained in a Shimura subvariety $S \subset X_L$. Then the universal cover $\widetilde{S}$ of S in $\mathbb{H}^d$ is stabilized by Γ, and hence also by its Zariski closure. But Γ is Zariski dense in $\mathrm{SL}_2(\mathbb{R})^d$ by Theorem 5.1, so $\widetilde{S} = \mathbb{H}^d$ and $S = X_L$.

(The conclusion $S = X_L$ also follows from formal properties of morphisms between Shimura varieties; see [Milne, §5.15].) ∎

Remark. In [CW, pp.95–97], it is assumed that all the triangles $T_i \subset \mathbb{H}$ have the same orientation, and that $\Delta(p,q,r)$ has a model over its trace field K; the formulation of Theorem 6.3 corrects these points (see also [Ri]).

Further consequences. We conclude with three more Corollaries of Theorem 6.3.

Corollary 6.4 *A cocompact triangle group Δ has a model over a totally real field $\iff$ $V = \mathbb{H}/\Delta$ has a finite cover that can be realized as a geodesic curve on a Hilbert modular variety.*

Proof. Suppose we have a model $\Delta \subset \mathrm{SL}_2(L)$, where L is totally real, and let $B \cong \mathrm{M}_2(L)$ be the corresponding quaternion algebra. Then $\mathcal{O}_\Delta = \mathbb{Z}[\Delta]$ is an order in B, as can be seen from the fact that the representation ψ of §2 sends Δ into $\mathrm{M}_4(\mathcal{O}_L)$. Since any two orders in B are commensurable, there is a subgroup Γ of finite index in Δ such that

$$\Gamma \subset \mathcal{O}_\Delta \cap \mathrm{M}_2(\mathcal{O}_L),$$

and thus Γ has a model over $\mathcal{O}_L$. By Theorem 6.3, the finite cover $\mathbb{H}/\Gamma$ of V can be realized as a geodesic curve on Y_L, and by Corollary 6.2, a further finite cover can be realized on X_L.

For the converse, suppose a finite cover of V can be realized as a totally geodesic curve on X_L. Then a subgroup of finite index in Δ has a model over L, so the invariant quaternion algebra B_0 for Δ splits over L. It follows that Δ itself has a model over the (totally real) compositum of L and its trace field K. ∎

Corollary 6.5 *Let $\Delta = \Delta(p,q,r)$ be a triangle group in the Hilbert series. Then a finite cover of $V = \mathbb{H}/\Delta$ can be presented as a geodesic curve on X_K, where K is trace field of Δ.*

Corollary 6.6 *Conjectures 1.3 and 1.8 are equivalent.*

These last two results follow from Corollary 6.4, using Theorem 1.1 and the equivalence of Conjectures 1.3 and 1.6.

A Appendix: The group $\Delta(2,3,7)$

In this Appendix we treat one more triangle group, to illustrate the fact that the method in §2 is rather general. We choose the Fuchsian group of smallest covolume, namely the famous arithmetic group $\Delta(2,3,7)$, which appears in Fricke's paper [Fr] and figures in the Hurwitz bound $\mathrm{Aut}(X) \le 84(g-1)$.

This triangle group admits no model over its trace field $K = \mathbb{Q}(\cos 2\pi/7)$. Instead, we give the following model over the ring of integers $\mathcal{O}_L$, where $L = \mathbb{Q}(t)$ and $t = \sqrt{2\cos 2\pi/7}$:

$$a = \begin{pmatrix} t^5+t^4+t^2-2t-1 & 2 \\ t^5+t^3-t^2-2t-1 & -t^5-t^4-t^2+2t+1 \end{pmatrix},$$

$$b = \begin{pmatrix} -t^5-t^3+t^2+2t+1 & t^5+t^4+t^2-2t-1 \\ -t^2+t-1 & t^5+t^3-t^2-2t \end{pmatrix}, \quad \text{and}$$

$$c = \begin{pmatrix} t^5+t^4+t^2-2t-1 & 1 \\ t^5+t^3-t^2-2t & -t^5+2t \end{pmatrix}.$$

As expected, L is not totally real, since both Galois conjugates of $\cos(2\pi/7)$ are negative. For a more complete discussion of $\Delta(2,3,7)$, and its relationship to the Klein quartic and other matters, see [El].

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