SOME REMARKS ON MAESAKA–SEKI–WATANABE'S FORMULA FOR THE MULTIPLE HARMONIC SUMS

SHUJI YAMAMOTO

Abstract. Recently, Maesaka, Seki and Watanabe discovered a surprising equality between multiple harmonic sums and certain Riemann sums which approximate the iterated integral expression of the multiple zeta values. In this paper, we describe the formula corresponding to the multiple zeta-star values and, more generally, to the Schur multiple zeta values of diagonally constant indices. We also discuss the relationship of these formulas with Hoffman's duality identity and an identity due to Kawashima.

1. Introduction

For a finite tuple $\boldsymbol{k} = (k_1, \dots, k_r)$ of positive integers, called an *index*, we define the *multiple harmonic sums*

$$\zeta_{<N}(\boldsymbol{k}) := \sum_{0<m_1<\cdots<m_r<N} \frac{1}{m_1^{k_1} \cdots m_r^{k_r}},$$

where N is any positive integer. By convention, we set $\zeta_{<N}(\varnothing) := 1$ for the empty index $\varnothing$ (i.e., $r = 0$). When the index $\boldsymbol{k}$ is *admissible*, i.e., $k_r \geq 2$ or $\boldsymbol{k} = \varnothing$, the limit

$$\zeta(\boldsymbol{k}) := \lim_{N\to\infty} \zeta_{<N}(\boldsymbol{k}) = \sum_{0<m_1<\cdots<m_r} \frac{1}{m_1^{k_1} \cdots m_r^{k_r}}$$

is called the *multiple zeta value*.

Recently, Maesaka–Seki–Watanabe [7] introduced another kind of finite sum:

$$\zeta^{\flat}_{<N}(\boldsymbol{k}) := \sum_{\substack{0<n_{i1}\leq\cdots\leq n_{ik_i}<N \ (1\leq i\leq r)\\ n_{ik_i}<n_{(i+1)1} \ (1\leq i<r)}} \prod_{i=1}^{r} \frac{1}{(N-n_{i1})n_{i2}\cdots n_{ik_i}}. \tag{1.1}$$

For example, we have

$$\zeta^{\flat}_{<N}(2,1,3) = \sum_{0<n_1\leq n_2<n_3<n_4\leq n_5\leq n_6<N} \frac{1}{(N-n_1)n_2\cdot(N-n_3)\cdot(N-n_4)n_5n_6}.$$

Note that this sum can be written as

$$\zeta^{\flat}_{<N}(2,1,3) = \frac{1}{N^6} \sum_{0<n_1\leq n_2<n_3<n_4\leq n_5\leq n_6<N} \frac{1}{(1-\frac{n_1}{N})\frac{n_2}{N}\cdot(1-\frac{n_3}{N})\cdot(1-\frac{n_4}{N})\frac{n_5}{N}\frac{n_6}{N}},$$

which is a Riemann sum which approximates the well-known iterated integral expression

$$\zeta(2,1,3) = \int_{0<x_1<x_2<x_3<x_4<x_5<x_6<1} \frac{dx_1}{1-x_1}\frac{dx_2}{x_2}\cdot\frac{dx_3}{1-x_3}\cdot\frac{dx_4}{1-x_4}\frac{dx_5}{x_5}\frac{dx_6}{x_6}.$$

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Thus we have

$$\lim_{N\to\infty} \zeta_{<N}(2,1,3) = \zeta(2,1,3) = \lim_{N\to\infty} \zeta^\flat_{<N}(2,1,3),$$

and the same holds for any admissible index. The surprising discovery of Maesaka–Seki–Watanabe [7] is that the equality holds before taking the limit:

Theorem 1.1 ([7, Theorem 1.3])**.** *For any index $\boldsymbol{k}$ and any integer $N > 0$, we have*

$$\zeta_{<N}(\boldsymbol{k}) = \zeta^\flat_{<N}(\boldsymbol{k}). \tag{1.2}$$

In the following, we call this formula (1.2) the *MSW formula*. One may expect various applications and generalizations. In addition to the proof of duality relations given in [7], an application has been given by Seki [9], who provided a new proof of the extended double shuffle relation of multiple zeta values. Hirose, Matsusaka and Seki [1] generalize the MSW formula to the case of multiple polylogarithms.

The purpose of the present article is to show some results related to the MSW formula. First, in §2, we describe the "star version" of it, that is, a formula of the same type for the *multiple star harmonic sum*

$$\zeta^\star_{<N}(\boldsymbol{k}) \coloneqq \sum_{0<m_1\le\cdots\le m_r<N} \frac{1}{m_1^{k_1}\cdots m_r^{k_r}}.$$

The result (Theorem 2.1) is a discrete analogue of the 2-poset integral [11, Corollary 1.3] for the multiple zeta-star value. We will explain how this star version is deduced from the non-star version (1.2) and vice versa.

In §3, we prove a generalization of both non-star and star MSW formulas, namely, the formula for *Schur multiple harmonic sums with diagonally constant indices*. The result is a discretization of the integral expression given by Hirose–Murahara–Onozuka [2]. Our proof is a natural generalization of that of (1.2) given in [7], the so-called "connector method".

In §4, we examine Hoffman's duality identity

$$\zeta^\star_{<N}(\boldsymbol{k}^\vee) = H_{<N}(\boldsymbol{k})$$

in the light of the MSW formula (see §4 for the definition of $\boldsymbol{k}^\vee$ and $H_{<N}$). The combination of this identity and the star MSW formula provides an expression for $H_{<N}(\boldsymbol{k})$. We notice that this expression can be proven directly, applying the computation in §3, and hence a new proof of Hoffman's identity is obtained.

Finally, in §5, we explain the relationship of the star MSW formula with an identity due to Kawashima [6]. We see that Kawashima's identity is, in a sense, equivalent to the star MSW formula.

Notation. For $m, n \in \mathbb{Z}$, we set $[m, n] \coloneqq \{a \in \mathbb{Z} \mid m \le a \le n\}$. We call such a subset of $\mathbb{Z}$ an *interval* in $\mathbb{Z}$, or simply an interval. In particular, the empty set is an interval since it is written as $\emptyset = [m, n]$ with $m > n$. The empty set in a general context is denoted by $\emptyset$, while the empty *index* is denoted by $\varnothing$.

2. MSW formula for multiple star harmonic sums

Recall that, for an index $\boldsymbol{k} = (k_1, \ldots, k_r)$ and a positive integer N, the multiple star harmonic sum is defined by

$$\zeta^\star_{<N}(\boldsymbol{k}) \coloneqq \sum_{0<m_1\le\cdots\le m_r<N} \frac{1}{m_1^{k_1}\cdots m_r^{k_r}}.$$

We also define the ♭-sum

$$\zeta^{\star\flat}_{<N}(\boldsymbol{k}) \coloneqq \sum_{\substack{0<n_{i1}\le\cdots\le n_{ik_i}<N\ (1\le i\le r)\\ n_{(i-1)1}\le n_{ik_i}\ (2\le i\le r)}} \prod_{i=1}^{r} \frac{1}{(N-n_{i1})n_{i2}\cdots n_{ik_i}}. \tag{2.1}$$

For example,

$$\zeta^{\star\flat}_{<N}(2,1,3) = \sum_{n_1\le n_2\le n_3\ge n_4\ge n_5\le n_6} \frac{1}{(N-n_1)n_2n_3\cdot(N-n_4)\cdot(N-n_5)n_6}$$

(here, we omit from the notation the condition $0<n_i<N$ for the running variables n_i). As before, we set $\zeta^{\star}_{<N}(\varnothing)=\zeta^{\star\flat}_{<N}(\varnothing)=1$.

It is well known that the multiple harmonic and star harmonic sums satisfy the "antipode identity": For $r>0$,

$$\sum_{i=0}^{r}(-1)^i\zeta_{<N}(k_1,\dots,k_i)\cdot\zeta^{\star}_{<N}(k_r,\dots,k_{i+1})=0. \tag{2.2}$$

Notice that the ♭-version

$$\sum_{i=0}^{r}(-1)^i\zeta^{\flat}_{<N}(k_1,\dots,k_i)\cdot\zeta^{\star\flat}_{<N}(k_r,\dots,k_{i+1})=0 \tag{2.3}$$

also holds for $r>0$. In fact, for each i, we have

$$\zeta^{\flat}_{<N}(k_1,\dots,k_i)\cdot\zeta^{\star\flat}_{<N}(k_r,\dots,k_{i+1}) = \sum_{\substack{0<n_{j1}\le\cdots\le n_{jk_j}<N\ (1\le j\le r)\\ n_{jk_j}<n_{(j+1)1}\ (1\le j<i)\\ n_{jk_j}\ge n_{(j+1)1}\ (i<j<r)}} \prod_{j=1}^{r}\frac{1}{(N-n_{j1})n_{j2}\cdots n_{jk_j}}.$$

If we decompose the latter sum into two parts according to whether $n_{ik_i}<n_{(i+1)1}$ or $n_{ik_i}\ge n_{(i+1)1}$, the left-hand side of (2.3) becomes a telescopic sum and the equality follows.

The analogue of Theorem 1.1 for the multiple star harmonic sum is the following:

Theorem 2.1. *For any index $\boldsymbol{k}$ and any integer $N>0$, we have*

$$\zeta^{\star}_{<N}(\boldsymbol{k})=\zeta^{\star\flat}_{<N}(\boldsymbol{k}). \tag{2.4}$$

Proof. This follows from Theorem 1.1 by induction on the depth (i.e., the length) r of the index $\boldsymbol{k}$. The formula trivially holds for $\boldsymbol{k}=\varnothing$. When $r>0$, we have two identities

$$\zeta^{\star}_{<N}(k_r,\dots,k_1) = -\sum_{i=1}^{r}(-1)^i\zeta_{<N}(k_1,\dots,k_i)\cdot\zeta^{\star}_{<N}(k_r,\dots,k_{i+1}),$$
$$\zeta^{\star\flat}_{<N}(k_r,\dots,k_1) = -\sum_{i=1}^{r}(-1)^i\zeta^{\flat}_{<N}(k_1,\dots,k_i)\cdot\zeta^{\star\flat}_{<N}(k_r,\dots,k_{i+1})$$

by (2.2) and (2.3), and they are equal by (1.2) and the induction hypothesis. □

Conversely, Theorem 1.1 can be deduced from Theorem 2.1 in the same way. In other words, Theorems 1.1 and 2.1 are equivalent once the identities (2.2) and (2.3) are provided.

3. Schur multiple harmonic sums

Following Nakasuji–Phuksuwan–Yamasaki [8], we define the (skew-)Schur multiple harmonic sum by

$$\zeta_{<N}(\boldsymbol{k}) \coloneqq \sum_{(m_{ij})\in \mathrm{SSYT}_{<N}(D)} \prod_{(i,j)\in D} \frac{1}{m_{ij}^{k_{ij}}}. \tag{3.1}$$

Here $\boldsymbol{k}$ is an index on a skew Young diagram D (that is, $\boldsymbol{k} = (k_{ij})$ is a tuple of positive integers indexed by $(i,j) \in D$), and $\mathrm{SSYT}_{<N}(D)$ denotes the set of semi-standard Young tableaux on D whose entries are positive integers less than N. For example,

$$\zeta_{<N}\left(\begin{array}{|c|c|c|}\hline k_{11} & k_{12} & k_{13} \\ \hline k_{21} & k_{22} & \\ \cline{1-2}\end{array}\right) = \sum_{\substack{m_{11} \le m_{12} \le m_{13} \\ \wedge \quad\ \wedge \\ m_{21} \le m_{22}}} \frac{1}{m_{11}^{k_{11}} m_{12}^{k_{12}} m_{13}^{k_{13}} m_{21}^{k_{21}} m_{22}^{k_{22}}},$$

where we omit the condition $0 < m_{ij} < N$ on running variables m_{ij}. Note that this generalizes both multiple harmonic and star harmonic sums in the sense that

$$\zeta_{<N}\left(\begin{array}{|c|}\hline k_1 \\ \hline \vdots \\ \hline k_r \\ \hline\end{array}\right) = \zeta_{<N}(k_1,\dots,k_r), \qquad \zeta_{<N}\left(\begin{array}{|c|c|c|}\hline k_1 & \cdots & k_r \\ \hline\end{array}\right) = \zeta^\star_{<N}(k_1,\dots,k_r). \tag{3.2}$$

In what follows, we assume that the index $\boldsymbol{k}$ is *diagonally constant*, i.e., k_{ij} depends only on $i-j$. Hirose–Murahara–Onozuka [2] assumed this condition to express the Schur multiple zeta value as the integral associated with a 2-poset. Since our purpose in this section is to provide a finite sum analogue of their integral expression, it is natural to make the same assumption.

We begin with preparing the following notation. For a diagonally constant index $\boldsymbol{k}$ on D, we set

$$\begin{aligned} &D_p \coloneqq \{(i,j) \in D \mid i-j = p\}, \\ &p_0 \coloneqq \min\{p \in \mathbb{Z} \mid D_p \neq \emptyset\}, \qquad p_1 \coloneqq \max\{p \in \mathbb{Z} \mid D_p \neq \emptyset\}, \\ &k_p \coloneqq k_{ij} \text{ for any } (i,j) \in D_p. \end{aligned}$$

As a matter of convention, we set $k_p = 1$ for $p \in \mathbb{Z}$ with $D_p = \emptyset$. For each $p \in \mathbb{Z}$, there is a bijection

$$D_p \longrightarrow J_p \coloneqq \{j \in \mathbb{Z} \mid (j+p, j) \in D_p\};\ (i,j) \longmapsto j.$$

Note that J_p is an interval in $\mathbb{Z}$. Moreover, each pair (J_p, J_{p+1}) of intervals must satisfy the following condition.

Definition 3.1. We say a pair (J, J') of intervals in $\mathbb{Z}$ is *consecutive* if $J = [j_0, j_1]$ and $J' = J$, $J \setminus \{j_1\}$, $J \cup \{j_0 - 1\}$ or $J \cup \{j_0 - 1\} \setminus \{j_1\}$. When J is empty, this means that J' is either empty or a singleton $\{j\}$ (in the latter case, we can take $(j_0, j_1) = (j+1, j)$).

For a tuple $\boldsymbol{m} = (m_j) \in [1, N-1]^J$ of integers in $[1, N-1]$ indexed by an interval J, we set $\Pi(\boldsymbol{m}) \coloneqq \prod_{j\in J} m_j$. Then the definition (3.1) of Schur multiple harmonic sum (for a diagonally constant index) is written as

$$\zeta_{<N}(\boldsymbol{k}) = \sum_{\substack{\boldsymbol{m}_p\in[1,N-1]^{J_p} \\ (p_0\le p\le p_1) \\ \boldsymbol{m}_{p_0}\lhd\cdots\lhd\boldsymbol{m}_{p_1}}} \prod_{p=p_0}^{p_1} \frac{1}{\Pi(\boldsymbol{m}_p)^{k_p}},$$

where $\boldsymbol{m}_p$ runs over $[1,N-1]^{J_p}$ for each $p=p_0,\dots,p_1$, satisfying the relation $\boldsymbol{m}_{p_0}\lhd\cdots\lhd\boldsymbol{m}_{p_1}$ defined as follows.

Definition 3.2. Let (J,J') be a consecutive pair of intervals. Let $\boldsymbol{m}=(m_j)_{j\in J}$ and $\boldsymbol{n}=(n_j)_{j\in J'}$ be tuples of integers indexed by J and J' respectively. Then we define the relation

$$\boldsymbol{m}\lhd\boldsymbol{n} \;:\Longleftrightarrow\; \begin{cases} m_j<n_j & \text{if } j\in J \text{ and } j\in J', \\ n_{j-1}\le m_j & \text{if } j\in J \text{ and } j-1\in J'. \end{cases}$$

For later use, we also define

$$\boldsymbol{m}\trianglelefteq\boldsymbol{n} \;:\Longleftrightarrow\; \begin{cases} m_j\le n_j & \text{if } j\in J \text{ and } j\in J', \\ n_{j-1}< m_j & \text{if } j\in J \text{ and } j-1\in J'. \end{cases}$$

Remark 3.3. Notice that the relation $\trianglelefteq$ does *not* mean "$\lhd$ or equal". Nevertheless, we have

$$\boldsymbol{m}\lhd\boldsymbol{n} \iff \boldsymbol{m}\trianglelefteq\boldsymbol{n}-\boldsymbol{1} \tag{3.3}$$

where $\boldsymbol{n}-\boldsymbol{1}=(n_j-1)_{j\in J'}$.

Example 3.4. If $J=J'=\{j\}$ (a singleton), then

$$\boldsymbol{m}\lhd\boldsymbol{n} \iff m_j<n_j, \qquad \boldsymbol{m}\trianglelefteq\boldsymbol{n} \iff m_j\le n_j$$

holds. On the other hand, $J=\{j\}$ and $J'=\{j-1\}$, we have

$$\boldsymbol{m}\lhd\boldsymbol{n} \iff n_{j-1}\le m_j, \qquad \boldsymbol{m}\trianglelefteq\boldsymbol{n} \iff n_{j-1}<m_j.$$

Now we define the $\flat$-sum by using the above notation.

Definition 3.5. For a diagonally constant index $\boldsymbol{k}$ as above and an integer $N>0$, we set

$$\zeta^{\flat}_{<N}(\boldsymbol{k}) \coloneqq \sum_{\substack{\boldsymbol{n}_p^{(l)}\in[1,N-1]^{J_p} \\ \boldsymbol{n}_p^{(l)}\trianglelefteq\boldsymbol{n}_p^{(l+1)}\ (1\le l<k_p) \\ \boldsymbol{n}_p^{(k_p)}\lhd\boldsymbol{n}_{p+1}^{(1)}\ (p_0\le p<p_1)}} \prod_{p=p_0}^{p_1} \frac{1}{\Pi(\boldsymbol{N}-\boldsymbol{n}_p^{(1)})\Pi(\boldsymbol{n}_p^{(2)})\cdots\Pi(\boldsymbol{n}_p^{(k_p)})},$$

where $\boldsymbol{n}_p^{(l)}$ runs over $[1,N-1]^{J_p}$ for each $p=p_0,\dots,p_1$ and $l=1,\dots,k_p$, satisfying the indicated relatioins. Moreover, for any tuple $\boldsymbol{n}=(n_j)_{j\in J}\in[1,N-1]^J$, we set $\boldsymbol{N}-\boldsymbol{n}\coloneqq(N-n_j)_{j\in J}$.

Again, this generalizes both (1.1) and (2.1), that is, we have

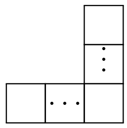
$$\zeta^{\flat}_{<N}\left(\begin{array}{|c|}\hline k_1 \\ \hline \vdots \\ \hline k_r \\ \hline\end{array}\right) = \zeta^{\flat}_{<N}(k_1,\dots,k_r), \qquad \zeta^{\flat}_{<N}\left(\begin{array}{|c|c|c|}\hline k_1 & \cdots & k_r \\ \hline\end{array}\right) = \zeta^{\star\flat}_{<N}(k_1,\dots,k_r). \tag{3.4}$$

Now we state the main theorem of this section.

Theorem 3.6. *For any diagonally constant index $\boldsymbol{k}$ and any integer $N > 0$, we have*

$$\zeta_{<N}(\boldsymbol{k}) = \zeta^{\flat}_{<N}(\boldsymbol{k}). \tag{3.5}$$

Remark 3.7. In view of identities (3.2) and (3.4), Theorem 3.6 includes both Theorems 1.1 and 2.1. More generally, if the diagram D is of anti-hook type, i.e., D has the shape



then the formula (3.5) is the finite sum analogue of the integral-series identity [4, Theorem 4.1].

We prove Theorem 3.6 by the method of *connected sums*. Though this proof may look quite complicated, it is indeed a natural extension of the proof of Theorem 1.1 given in [7]. The new ingredient is a trick using certain determinants. It is worthy of attention that a similar use of determinants appears in [2].

For $0 \le m, n \le N-1$, set

$$C_N(m,n) := \binom{n}{m} \Big/ \binom{N-1}{m}.$$

By definition, we have

$$\begin{aligned} C_N(0,n) = 1 = C_N(m, N-1) \quad & \text{for } 0 \le m, n \le N-1, \\ C_N(m,n) = 0 \quad & \text{for } 0 \le n < m \le N-1. \end{aligned}$$

Lemma 3.8. (1) *For $0 < m < N$ and $0 \le n \le n' < N$,*

$$\frac{1}{m}\big(C_N(m,n') - C_N(m,n)\big) = \sum_{b=n+1}^{n'} C_N(m,b)\frac{1}{b}.$$

(2) *For $0 \le m \le m' < N$ and $0 < n < N$,*

$$\sum_{a=m+1}^{m'} C_N(a,n)\frac{1}{n} = \big(C_N(m,n-1) - C_N(m',n-1)\big)\frac{1}{N-n}.$$

Proof. These are immediate consequences of the identities

$$\frac{1}{m}\big(C_N(m,b) - C_N(m,b-1)\big) = C_N(m,b)\frac{1}{b}$$

and

$$C_N(a,n)\frac{1}{n} = \big(C_N(a-1,n-1) - C_N(a,n-1)\big)\frac{1}{N-n},$$

respectively. □

Remark 3.9. The relation (1) of the above lemma is the same as (4.1) of [7, Lemma 4.1], though their symbol $C_N(n,m)$ corresponds to our $C_{N+1}(n,m)$. We also note that (4.2) of [7, Lemma 4.1] is obtained by combining (1) and (2) of our lemma.

Next we extend Lemma 3.8 to certain determinants.

Definition 3.10. Let (J, J') be a pair of consecutive intervals, and $\boldsymbol{m} = (m_j)_{j\in J}$ and $\boldsymbol{n} = (n_{j'})_{j'\in J'}$ be tuples of elements in $[1, N-1]$ indexed by them. We set $\tilde{J} = J \cup J'$ and define the symbols $\tilde{m}_j$ and $\tilde{n}_j$ for $j \in \tilde{J}$ by

$$\tilde{m}_j = \begin{cases} m_j & (j \in J), \\ 0 & (j \notin J), \end{cases} \qquad \tilde{n}_j = \begin{cases} n_j & (j \in J'), \\ N-1 & (j \notin J'). \end{cases}$$

Then we define

$$D_N(\boldsymbol{m}, \boldsymbol{n}) := \det\bigl(C_N(\tilde{m}_{j_1}, \tilde{n}_{j_2})\bigr)_{j_1, j_2 \in \tilde{J}}.$$

Lemma 3.11. *When J or J' is empty, one has $D_N(\boldsymbol{m}, \boldsymbol{n}) = 1$.*

Proof. This is trivial when $J = J' = \emptyset$. If J is empty and J' is not, then J' has to be a singleton. Hence $\boldsymbol{n}$ is represented by a single element $n \in [1, N-1]$ and it holds that

$$D_N(\boldsymbol{m}, \boldsymbol{n}) = C_N(0, n) = 1.$$

Similarly, if J' is empty and J is a singleton, then $\boldsymbol{m}$ is represented by $m \in [1, N-1]$ and it holds that

$$D_N(\boldsymbol{m}, \boldsymbol{n}) = C_N(m, N-1) = 1. \qquad \square$$

We say a tuple $\boldsymbol{m} = (m_j)_{j\in J}$ of integers indexed by an interval J is *non-decreasing* if $m_j \le m_{j+1}$ holds whenever $j, j+1 \in J$.

Lemma 3.12. *Let (J, J') be a consecutive pair of intervals.*

(1) *If $\boldsymbol{m} \in [1, N-1]^J$ and $\boldsymbol{n} \in [0, N-1]^{J'}$ are non-decreasing, we have*

$$\frac{1}{\Pi(\boldsymbol{m})} D_N(\boldsymbol{m}, \boldsymbol{n}) = \sum_{\substack{\boldsymbol{b}\in[1,N-1]^J \\ \boldsymbol{b} \trianglelefteq \boldsymbol{n}}} D_N(\boldsymbol{m}, \boldsymbol{b}) \frac{1}{\Pi(\boldsymbol{b})}. \tag{3.6}$$

(2) *If $\boldsymbol{m} \in [0, N-1]^J$ and $\boldsymbol{n} \in [1, N-1]^{J'}$ are non-decreasing, we have*

$$\sum_{\substack{\boldsymbol{a}\in[1,N-1]^{J'} \\ \boldsymbol{m} \triangleleft \boldsymbol{a}}} D_N(\boldsymbol{a}, \boldsymbol{n}) \frac{1}{\Pi(\boldsymbol{n})} = D_N(\boldsymbol{m}, \boldsymbol{n} - \boldsymbol{1}) \frac{1}{\Pi(\boldsymbol{N} - \boldsymbol{n})}. \tag{3.7}$$

Proof. First let us show (3.6). By shifting the numbering, we assume $J = [1, \bar{\jmath}]$. Put $C_{j_1 j_2} = C_N(\tilde{m}_{j_1}, \tilde{n}_{j_2})$ to simplify the notation. The proof is divided into two cases: (a) $0 \notin J'$, (b) $0 \in J'$.

(a) In this case, $\tilde{J} = J = [1, \bar{\jmath}]$. If we set $\tilde{n}_0 := 0$, then we have

$$\begin{aligned} \frac{1}{\Pi(\boldsymbol{m})} D_N(\boldsymbol{m}, \boldsymbol{n}) &= \det\left(\frac{1}{m_{j_1}} C_{j_1 j_2}\right)_{j_1, j_2 \in J} \\ &= \det\left(\frac{1}{m_{j_1}} \bigl(C_{j_1 j_2} - C_{j_1 (j_2 - 1)}\bigr)\right)_{j_1, j_2 \in J} \end{aligned} \tag{3.8}$$

since $C_N(m, 0) = 0$ for any $m > 0$. By Lemma 3.8 (1), this is

$$\begin{aligned} &= \det\left(\sum_{\tilde{n}_{j_2 - 1} < b_{j_2} \le \tilde{n}_{j_2}} C_N(m_{j_1}, b_{j_2}) \frac{1}{b_{j_2}}\right)_{j_1, j_2 \in J} \\ &= \sum_{\substack{\boldsymbol{b}\in[1,N-1]^J \\ \boldsymbol{b} \trianglelefteq \boldsymbol{n}}} D_N(\boldsymbol{m}, \boldsymbol{b}) \frac{1}{\Pi(\boldsymbol{b})} \end{aligned}$$

(note that the last equality holds regardless whether $J' = J$ or $J' \subsetneq J$, since in the latter case, the extra condition $b_{\bar{\jmath}} \leq \tilde{n}_{\bar{\jmath}} = N-1$ holds automatically). Thus we have shown (3.6) in the case (a).

(b) In this case, $\tilde{J} = [0, \bar{\jmath}]$. Since $C_N(0, n) = 1$ for any n, we have

$$\begin{aligned}
\frac{1}{\Pi(\boldsymbol{m})} D_N(\boldsymbol{m}, \boldsymbol{n}) &= \frac{1}{\Pi(\boldsymbol{m})} \begin{vmatrix} 1 & 1 & \cdots & 1 \\ C_{10} & C_{11} & \cdots & C_{1\bar{\jmath}} \\ \vdots & \vdots & & \vdots \\ C_{\bar{\jmath}0} & C_{\bar{\jmath}1} & \cdots & C_{\bar{\jmath}\bar{\jmath}} \end{vmatrix} \\
&= \frac{1}{\Pi(\boldsymbol{m})} \begin{vmatrix} 1 & 0 & \cdots & 0 \\ C_{10} & C_{11} - C_{10} & \cdots & C_{1\bar{\jmath}} - C_{1(\bar{\jmath}-1)} \\ \vdots & \vdots & & \vdots \\ C_{\bar{\jmath}0} & C_{\bar{\jmath}1} - C_{\bar{\jmath}0} & \cdots & C_{\bar{\jmath}\bar{\jmath}} - C_{\bar{\jmath}(\bar{\jmath}-1)} \end{vmatrix} \\
&= \det\left(\frac{1}{m_{j_1}} \left(C_{j_1 j_2} - C_{j_1(j_2-1)} \right) \right)_{j_1, j_2 \in J}
\end{aligned}$$

which is the expression in (3.8) and the rest of the proof is the same.

Next we show (3.7). By renumbering if necessary, we assume $J' = [1, \bar{\jmath}]$. Put $C'_{j_1 j_2} = C_N(\tilde{m}_{j_1}, \tilde{n}_{j_2} - 1)$. Again we consider two cases: (a) $\bar{\jmath}+1 \notin J$ or (b) $\bar{\jmath}+1 \in J$.

(a) In this case, $\tilde{J} = J' = [1, \bar{\jmath}]$. If we set $\tilde{m}_{\bar{\jmath}+1} \coloneqq N-1$, we have

$$\begin{aligned}
D_N(\boldsymbol{m}, \boldsymbol{n}-\boldsymbol{1}) \frac{1}{\Pi(\boldsymbol{N}-\boldsymbol{n})} &= \det\left(C'_{j_1 j_2} \frac{1}{N - n_{j_2}} \right)_{j_1, j_2 \in J'} \\
&= \det\left(\left(C'_{j_1 j_2} - C'_{(j_1+1) j_2} \right) \frac{1}{N - n_{j_2}} \right)_{j_1, j_2 \in J'}
\end{aligned} \tag{3.9}$$

since $C_N(N-1, n-1) = 0$ for any $n \in [1, N-1]$. By Lemma 3.8 (2), this is

$$\begin{aligned}
&= \det\left(\sum_{\tilde{m}_{j_1} < a_{j_1} \leq \tilde{m}_{j_1+1}} C_N(a_{j_1}, n_{j_2}) \frac{1}{n_{j_2}} \right)_{j_1, j_2 \in J'} \\
&= \sum_{\substack{\boldsymbol{a} \in [1, N-1]^{J'} \\ \boldsymbol{m} \lhd \boldsymbol{a}}} D_N(\boldsymbol{a}, \boldsymbol{n}) \frac{1}{\Pi(\boldsymbol{n})}
\end{aligned}$$

(again, the last equality holds regardless whether $J = J'$ or $J \subsetneq J'$). Thus we have shown (3.7) in the case (a).

(b) In this case, $\tilde{J} = [1, \bar{\jmath}+1]$. Since $C_N(m, N-1) = 1$ for any $m \in [0, N-1]$, we have

$$D_N(\boldsymbol{m}, \boldsymbol{n}-\boldsymbol{1}) \frac{1}{\Pi(\boldsymbol{N}-\boldsymbol{n})} = \begin{vmatrix} C'_{11} & \cdots & C'_{1\bar{\jmath}} & 1 \\ \vdots & & \vdots & \vdots \\ C'_{\bar{\jmath}1} & \cdots & C'_{\bar{\jmath}\bar{\jmath}} & 1 \\ C'_{(\bar{\jmath}+1)1} & \cdots & C'_{(\bar{\jmath}+1)\bar{\jmath}} & 1 \end{vmatrix} \frac{1}{\Pi(\boldsymbol{N}-\boldsymbol{n})}$$
$$= \begin{vmatrix} C'_{11} - C'_{21} & \cdots & C'_{1\bar{\jmath}} - C'_{2\bar{\jmath}} & 0 \\ \vdots & & \vdots & \vdots \\ C'_{\bar{\jmath}1} - C'_{(\bar{\jmath}+1)1} & \cdots & C'_{\bar{\jmath}\bar{\jmath}} - C'_{(\bar{\jmath}+1)\bar{\jmath}} & 0 \\ C'_{(\bar{\jmath}+1)1} & \cdots & C'_{(\bar{\jmath}+1)\bar{\jmath}} & 1 \end{vmatrix} \frac{1}{\Pi(\boldsymbol{N}-\boldsymbol{n})}$$
$$= \det\left((C'_{j_1 j_2} - C'_{(j_1+1)j_2}) \frac{1}{N - n_{j_2}} \right)_{j_1, j_2 \in J'},$$

which is the expression in (3.9), and the rest of the proof is the same.

The proof is complete. □

Now let us introduce the following connected sum

$$Z(\boldsymbol{k}; q) := \sum_{\substack{\boldsymbol{m}_p \ (p_0 \le p \le q) \\ \boldsymbol{n}_p^{(l)} \ (q < p \le p_1)}} \left(\prod_{p=p_0}^{q} \frac{1}{\Pi(\boldsymbol{m}_p)^{k_p}} \right) \cdot D_N(\boldsymbol{m}_q, \boldsymbol{n}_{q+1}^{(1)} - \boldsymbol{1})$$
$$\cdot \left(\prod_{p=q+1}^{p_1} \frac{1}{\Pi(\boldsymbol{N} - \boldsymbol{n}_p^{(1)}) \Pi(\boldsymbol{n}_p^{(2)}) \cdots \Pi(\boldsymbol{n}_p^{(k_p)})} \right)$$

for $q = p_0 - 1, p_0, \ldots, p_1$. Here $\boldsymbol{m}_p \in [1, N-1]^{J_p}$ for $p_0 \le p \le q$ run satisfying

$$\boldsymbol{m}_{p_0} \lhd \cdots \lhd \boldsymbol{m}_q,$$

and $\boldsymbol{n}_p^{(l)} \in [1, N-1]^{J_p}$ for $q < p \le p_1$ and $1 \le l \le k_p$ run satisfying

$$\boldsymbol{n}_p^{(l)} \unlhd \boldsymbol{n}_p^{(l+1)} \ \text{ for } 1 \le l < k_p \text{ and } \ \boldsymbol{n}_p^{(k_p)} \lhd \boldsymbol{n}_{p+1}^{(1)} \text{ for } q < p < p_1.$$

We also set $\boldsymbol{m}_{p_0-1}$ and $\boldsymbol{n}_{p_1+1}^{(1)}$ to be the empty tuple, which is compatible with the fact that J_{p_0-1} and J_{p_1+1} are the empty set.

Proof of Theorem 3.6. By using Lemma 3.11, one can check that

$$\zeta_{<N}(\boldsymbol{k}) = Z(\boldsymbol{k}; p_1), \qquad \zeta^{\flat}_{<N}(\boldsymbol{k}) = Z(\boldsymbol{k}; p_0 - 1).$$

On the other hand, for $p_0 \le q \le p_1$, one shows the equality $Z(\boldsymbol{k}; q) = Z(\boldsymbol{k}; q-1)$ by applying (3.6) k_q times and then (3.7) once (the equivalence (3.3) is also used at the first application of (3.6)). Thus we have

$$\zeta_{<N}(\boldsymbol{k}) = Z(\boldsymbol{k}; p_1) = Z(\boldsymbol{k}; p_1 - 1) = \cdots = Z(\boldsymbol{k}; p_0 - 1) = \zeta^{\flat}_{<N}(\boldsymbol{k}),$$

as desired. □

4. Hoffman's duality identity

For a non-empty index (in the usual sense) $\boldsymbol{k} = (k_1, \dots, k_r)$, we define the *multiple sum of Hoffman's type* by

$$H_{<N}(\boldsymbol{k}) = \sum_{1 \le m_1 \le \cdots \le m_r < N} \frac{(-1)^{m_r - 1}}{m_1^{k_1} \cdots m_r^{k_r}} \binom{N-1}{m_r}.$$

On the other hand, the *Hoffman dual* of the index $\boldsymbol{k}$ is defined by

$$\boldsymbol{k}^\vee = (\underbrace{1, \dots, 1}_{k_1} + \underbrace{1, \dots, 1}_{k_2} + \cdots + \underbrace{1, \dots, 1}_{k_r}).$$

In other words, $\boldsymbol{k}^\vee$ is obtained from

$$\boldsymbol{k} = (\underbrace{1 + \cdots + 1}_{k_1}, \underbrace{1 + \cdots + 1}_{k_2}, \dots, \underbrace{1 + \cdots + 1}_{k_r})$$

by changing plus symbols to commas and vice versa.

Theorem 4.1 (Hoffman's duality identity)**.** *For any non-empty index $\boldsymbol{k} = (k_1, \dots, k_r)$ and any integer $N > 0$, we have*

$$\zeta^\star_{<N}(\boldsymbol{k}^\vee) = H_{<N}(\boldsymbol{k}). \tag{4.1}$$

This theorem was proved by Hoffman [3] and independently by Kawashima [5]. Some different proofs are also known, e.g., one based on the integral expression [11] and one by the connected method [10].

One can combine this identity (4.1) with the MSW formula (2.4) for $\zeta^\star_{<N}(\boldsymbol{k}^\vee)$, and make the "change of variables" $n_{ij} \leftrightarrow N - n_{ij}$. Then one obtains the following MSW-like expression of the multiple sum of Hoffman's type:

Proposition 4.2. *We have*

$$H_{<N}(\boldsymbol{k}) = H^\flat_{<N}(\boldsymbol{k}), \tag{4.2}$$

where

$$H^\flat_{<N}(\boldsymbol{k}) := \sum_{\substack{0 < n_{i1} \le \cdots \le n_{ik_i} < N \ (1 \le i \le r) \\ n_{(i-1)1} \le n_{ik_i} \ (2 \le i \le r)}} \frac{1}{n_{r1} \cdots n_{rk_r}} \prod_{i=1}^{r-1} \frac{1}{(N - n_{i1}) n_{i2} \cdots n_{ik_i}}.$$

The relation of the identities (2.4), (4.1) and (4.2) is summarized as follows:

$$\begin{tikzcd}
\zeta^\star_{<N}(\boldsymbol{k}^\vee) \arrow[r, equal, "(2.4)"] \arrow[d, equal, "(4.1)"'] & \zeta^{\star\flat}_{<N}(\boldsymbol{k}^\vee) \arrow[d, equal, "n_{ij} \leftrightarrow N - n_{ij}"] \\
H_{<N}(\boldsymbol{k}) \arrow[r, equal, "(4.2)"] & H^\flat_{<N}(\boldsymbol{k})
\end{tikzcd}$$

It is worth noticing that one can also prove (4.2) directly, because one then obtains yet another proof of Hoffman's identity (4.1). For this, one starts from showing that

$$\begin{aligned}
\sum_{m_1=1}^{m_2}\frac{1}{m_1^{k_1}} &= \sum_{1\le m_1\le m_2}\frac{1}{m_1^{k_1}}\big(C_N(m_1,N-1)-C_N(m_1,0)\big)\\
&\overset{(1)}{=} \sum_{\substack{1\le m_1\le m_2\\ 1\le n_{1k_1}<N}}\frac{1}{m_1^{k_1-1}}\big(C_N(m_1,n_{1k_1})-C_N(m_1,0)\big)\frac{1}{n_{1k_1}}\\
&\overset{(1)}{=} \sum_{\substack{1\le m_1\le m_2\\ 1\le n_{1(k_1-1)}\le n_{1k_1}<N}}\frac{1}{m_1^{k_1-2}}\big(C_N(m_1,n_{1(k_1-1)})-C_N(m_1,0)\big)\frac{1}{n_{1(k_1-1)}n_{1k_1}}\\
&\overset{(1)}{=}\cdots\overset{(1)}{=} \sum_{\substack{1\le m_1\le m_2\\ 1\le n_{11}\le\cdots\le n_{1k_1}<N}}\big(C_N(m_1,n_{11})-C_N(m_1,0)\big)\frac{1}{n_{11}\cdots n_{1k_1}}\\
&\overset{(2)}{=} \sum_{1\le n_{11}\le\cdots\le n_{1k_1}<N}\big(C_N(0,n_{11}-1)-C_N(m_2,n_{11}-1)\big)\frac{1}{(N-n_{11})n_{12}\cdots n_{1k_1}}\\
&= \sum_{1\le n_{11}\le\cdots\le n_{1k_1}<N}\big(C_N(m_2,N-1)-C_N(m_2,n_{11}-1)\big)\frac{1}{(N-n_{11})n_{12}\cdots n_{1k_1}}.
\end{aligned}$$

Here one uses Lemma 3.8 (1) and (2) as indicated. Note also that $C_N(m_1,N-1)=1$, $C_N(m_1,0)=0$ and $C_N(0,n_{11}-1)=1=C_N(m_2,N-1)$. By repeating such transformations, one arrives at

$$\begin{aligned}
H_{<N}(\boldsymbol{k}) = \sum_{\substack{0<n_{i1}\le\cdots\le n_{ik_i}<N\ (1\le i\le r)\\ n_{(i-1)1}\le n_{ik_i}\ (2\le i\le r)}}\sum_{m_r=1}^{N-1}(-1)^{m_r-1}\binom{N-1}{m_r}C_N(m_r,n_{r1})&\\
\cdot\frac{1}{n_{r1}\cdots n_{rk_r}}\prod_{i=1}^{r-1}\frac{1}{(N-n_{i1})n_{i2}\cdots n_{ik_i}}&.
\end{aligned}$$

Then, since

$$\sum_{m_r=1}^{N-1}(-1)^{m_r-1}\binom{N-1}{m_r}C_N(m_r,n_{r1}) = \sum_{m_r=1}^{n_{r1}}(-1)^{m_r-1}\binom{n_{r1}}{m_r}=1$$

by the binomial theorem, the equality (4.2) follows.

Remark 4.3. Beginning with $\zeta^\star_{<N}(\boldsymbol{k})$ instead of $H_{<N}(\boldsymbol{k})$ and following the same computation, one arrives at $\zeta^{\star\flat}_{<N}(\boldsymbol{k})$ (in this case, the final step using the binomial theorem is not necessary). This direct proof of Theorem 2.1 is just a specialization of the proof of Theorem 3.6 given in the previous section.

5. Relationship with Kawashima's identity

In this section, we relate Theorem 2.1 to an identity due to Kawashima. For a nonempty index $\boldsymbol{k}=(k_1,\dots,k_r)$ and a complex variable z with $\Re(z)>-1$, consider the nested series

$$G_{\boldsymbol{k}}(z) := \sum_{\substack{0<m_{11}\le\cdots\le m_{1k_1}\\ <m_{21}\le\cdots\le m_{2k_2}\\ \vdots \\ <m_{r1}\le\cdots\le m_{rk_r}}} \prod_{i=1}^{r-1} \frac{1}{(m_{i1}+z)\cdots(m_{i(k_i-1)}+z)m_{ik_i}} \cdot \frac{1}{(m_{r1}+z)\cdots(m_{r(k_r-1)}+z)}\left(\frac{1}{m_{rk_r}}-\frac{1}{m_{rk_r}+z}\right).$$

Here m_{ij} runs over all positive integers satisfying the indicated inequalities, hence $G_{\boldsymbol{k}}(z)$ is an *infinite* series. For example,

$$G_{2,1,2}(z) = \sum_{0<m_1\le m_2<m_3<m_4\le m_5} \frac{1}{(m_1+z)m_2\,m_3(m_4+z)}\left(\frac{1}{m_5}-\frac{1}{m_5+z}\right).$$

This definition differs from Kawashima's original one in that we take the Hoffman dual of the index. In our notation, Kawashima's identity is stated as follows.

Proposition 5.1 ([6, Proposition 3.2]). *For any non-empty index $\boldsymbol{k}$ and any integer $N>0$,*

$$G_{\overleftarrow{\boldsymbol{k}}}(N-1) = \zeta^{\star}_{<N}(\boldsymbol{k}) \tag{5.1}$$

holds, where $\overleftarrow{\boldsymbol{k}} := (k_r,\dots,k_1)$ for $\boldsymbol{k}=(k_1,\dots,k_r)$.

Remark 5.2. The main result of the article [6] is the identity $G_{\overleftarrow{\boldsymbol{k}}}(z) = F_{\boldsymbol{k}}(z)$, where $F_{\boldsymbol{k}}(z)$ is a function (called the *Kawashima function*) characterized as the Newton series that interpolates multiple star harmonic sums: $F_{\boldsymbol{k}}(N-1) = \zeta^{\star}_{<N}(\boldsymbol{k})$. Thus the above Proposition 5.1 states that $G_{\overleftarrow{\boldsymbol{k}}}(N-1) = F_{\boldsymbol{k}}(N-1)$ holds for any integer $N>0$. This is an important step in Kawashima's proof of the identity $G_{\overleftarrow{\boldsymbol{k}}}(z) = F_{\boldsymbol{k}}(z)$ as functions.

Now let us combine Proposition 5.1 with Theorem 2.1. The result is the following:

Proposition 5.3. *For any non-empty index $\boldsymbol{k}$ and any integer $N>0$,*

$$G_{\overleftarrow{\boldsymbol{k}}}(N-1) = \zeta^{\star\flat}_{<N}(\boldsymbol{k}) \tag{5.2}$$

holds.

There is a triangle

$$\begin{tikzcd} \zeta^{\star}_{<N}(\boldsymbol{k}) \arrow[rr, equal, "(2.4)"] \arrow[dr, equal, "(5.1)"'] & & \zeta^{\star\flat}_{<N}(\boldsymbol{k}) \arrow[dl, equal, "(5.2)"] \\ & G_{\overleftarrow{\boldsymbol{k}}}(N-1) & \end{tikzcd}$$

of three quantities, and we have deduced the equality (5.2) from the other two. On the other hand, (5.2) can be shown directly, by using the transformations

$$\begin{aligned} &\frac{1}{m'+N-1}\sum_{m=m'}^{\infty}\left(\frac{1}{m+N-1-n''}-\frac{1}{m+N-n'}\right) \\ &= \frac{1}{m'+N-1}\sum_{n=n'}^{n''}\frac{1}{m'+N-1-n} = \sum_{n=n'}^{n''}\left(\frac{1}{m'+N-1-n}-\frac{1}{m'+N-1}\right)\frac{1}{n} \end{aligned} \tag{5.3}$$

and

$$\begin{aligned}&\frac{1}{m'}\sum_{m=m'+1}^{\infty}\left(\frac{1}{m+N-1-n''}-\frac{1}{m+N-n'}\right)\\ &=\frac{1}{m'}\sum_{n=n'}^{n''}\frac{1}{m'+N-n}=\sum_{n=n'}^{n''}\left(\frac{1}{m'}-\frac{1}{m'+N-n}\right)\frac{1}{N-n}\end{aligned}\tag{5.4}$$

repeatedly. For instance, starting from

$$G_{2,2}(N-1)=\sum_{0<m_1\le m_2<m_3\le m_4}\frac{1}{(m_1+N-1)m_2(m_3+N-1)}\left(\frac{1}{m_4}-\frac{1}{m_4+N-1}\right),$$

one proceeds

$$\begin{aligned}&\frac{1}{m_3+N-1}\sum_{m_4=m_3}^{\infty}\left(\frac{1}{m_4}-\frac{1}{m_4+N-1}\right)\\ &\qquad\overset{(5.3)}{=}\sum_{n_4=1}^{N-1}\left(\frac{1}{m_3+N-1-n_4}-\frac{1}{m_3+N-1}\right)\frac{1}{n_4},\\ &\frac{1}{m_2}\sum_{m_3=m_2+1}^{\infty}\left(\frac{1}{m_3+N-1-n_4}-\frac{1}{m_3+N-1}\right)\\ &\qquad\overset{(5.4)}{=}\sum_{n_3=1}^{n_4}\left(\frac{1}{m_2}-\frac{1}{m_2+N-n_3}\right)\frac{1}{N-n_3},\\ &\frac{1}{m_1+N-1}\sum_{m_2=m_1}^{\infty}\left(\frac{1}{m_2}-\frac{1}{m_2+N-n_3}\right)\\ &\qquad\overset{(5.3)}{=}\sum_{n_2=n_3}^{N-1}\left(\frac{1}{m_1+N-1-n_2}-\frac{1}{m_1+N-1}\right)\frac{1}{n_2},\end{aligned}$$

and ends with a telescopic sum

$$\sum_{m_1=1}^{\infty}\left(\frac{1}{m_1+N-1-n_2}-\frac{1}{m_1+N-1}\right)=\sum_{n_1=1}^{n_2}\frac{1}{N-n_1}.$$

The result is

$$G_{2,2}(N-1)=\sum_{n_4=1}^{N-1}\sum_{n_3=1}^{n_4}\sum_{n_2=n_3}^{N-1}\sum_{n_1=1}^{n_2}\frac{1}{(N-n_1)n_2(N-n_3)n_4}=\zeta^{\star\flat}_{<N}(2,2).$$

Thus we obtain a new proof of Proposition 5.1 through Theorem 2.1 and Proposition 5.3. Conversely, Propositions 5.1 and 5.3 implies Theorem 2.1, and hence Theorem 1.1. This could be another possible way of finding the MSW formula!

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Interdisciplinary Faculty of Science and Engineering, Shimane University, 1060 Nishi-Kawatsu, Matsue, 690-8504, Japan

Email address: yamashu@riko.shimane-u.ac.jp