Regular Lagrangians are smooth Lagrangians*

Tomohiro Asano†1, Stéphane Guillermou‡2,
Yuichi Ike§3, and Claude Viterbo¶4

1Research Institute for Mathematical Sciences, Kyoto University, Kitashirakawa-Oiwake-Cho, Sakyo-ku, 606-8502, Kyoto, Japan.
2UMR CNRS 6629 du CNRS Laboratoire de Mathématiques Jean LERAY 2 Chemin de la Houssinière, BP 92208, F-44322 NANTES Cedex 3 France
3Institute of Mathematics for Industry, Kyushu University, 744 Motooka, Nishi-ku, Fukuoka-shi, Fukuoka 819-0395, Japan.
4Université Paris-Saclay, CNRS, Laboratoire de Mathématiques d'Orsay, 91405, Orsay, France.

April 8, 2025

To Pierre Schapira for his 80^{th} birthday.

Abstract

We prove that for any element in the γ-completion of the space of smooth compact exact Lagrangian submanifolds of a cotangent bundle, if its γ-support is a smooth Lagrangian submanifold, then the element itself is a smooth Lagrangian. We also prove that if the γ-support of an element in the completion is compact, then it is connected.

1 Introduction

Let M be a C^∞ closed connected manifold. The space $\mathfrak{L}(T^*M)$ of smooth compact exact Lagrangian submanifolds of T^*M carries a distance γ, called the spectral distance (see [Vit92; Oh97; MVZ12; HLS16]). The metric space $(\mathfrak{L}(T^*M), \gamma)$ is not complete, so we consider its completion. Its study was initiated in [Hum08], pursued further in [Vit22b], and has applications to Hamilton–Jacobi equations [Hum08], symplectic homogenization theory [Vit08], and to conformally symplectic dynamics [AHV24].

The elements of the completion $\widehat{\mathfrak{L}}(T^*M)$ are by definition certain equivalence classes of Cauchy sequences with respect to the spectral distance γ. Despite their very abstract nature, they admit a geometric incarnation called the γ*-support*, which was introduced by

*2020 Mathematics Subject Classification: 53D12, 37J11, 35A27
Keywords: γ-support, microlocal sheaf theory, interleaving distance

†`tasano@kurims.kyoto-u.ac.jp`, `tomoh.asano@gmail.com`. Supported by JSPS KAKENHI Grant Number JP24K16920. Also supported by JST, CREST Grant Number JPMJCR24Q1, Japan.

‡`Stephane.Guillermou@univ-nantes.fr`. Supported by ANR COSY (ANR-21-CE40-0002) and Centre Henri Lebesgue (ANR-11-LABX-0020-01).

§`ike@imi.kyushu-u.ac.jp`, `yuichi.ike.1990@gmail.com`. Supported by JSPS KAKENHI Grant Numbers JP21K13801 and JP22H05107. Also supported by JST, CREST Grant Number JPMJCR24Q1, Japan.

¶`Claude.Viterbo@universite-paris-saclay.fr`. Supported by ANR COSY (ANR-21-CE40-0002).

Viterbo in [Vit22b] (as a modification of the support introduced in [Hum08]). It is defined as follows:

Definition 1.1. Let $L_\infty \in \widehat{\mathfrak{L}}(T^*M)$ and $z \in T^*M$. One says that z is in the γ-*support* of L_∞ if for any neighborhood U of z there is $\varphi \in \mathrm{DHam}_c(U)$ such that $\varphi(L_\infty) \neq L_\infty$. Here, $\mathrm{DHam}_c(U)$ denotes the group of Hamiltonian diffeomorphisms compactly supported in U. The set of points in the γ-support of L_∞ is denoted by $\gamma\text{-supp}(L_\infty)$.

For a smooth Lagrangian $L \in \mathfrak{L}(T^*M)$, we easily show $\gamma\text{-supp}(L) = L$. Several questions are of importance for $\gamma\text{-supp}(L_\infty)$. Does $\gamma\text{-supp}(L_\infty)$ characterize L_∞? This is not the case in general (examples can be found in [Vit22b]), but one could still hope it if $\gamma\text{-supp}(L_\infty)$ is small.

Also since γ-supports appear in [AHV24] as higher-dimensional versions of Birkhoff invariant sets, they share some of the properties of the 1-dimensional case. It is proved in loc. cit. that the projection $\pi\colon \gamma\text{-supp}(L_\infty) \to M$ induces an injection in cohomology, but also that the map is not in general surjective. However, is it the case at the H^0 level?

In this note, we give positive answers to the above questions, namely Conjecture 8.2 of [Vit22b] and a question in [AHV24]. That is, we prove, for $L_\infty \in \widehat{\mathfrak{L}}(T^*M)$,

(i) if $\gamma\text{-supp}(L_\infty) = L$ for some $L \in \mathfrak{L}(T^*M)$, then $L_\infty = L$ (see Theorem 5.1),

(ii) if $\gamma\text{-supp}(L_\infty)$ is compact, then it is connected (see Theorem 6.1).

Acknowledgments

The authors are very grateful to Vincent Humilière for many helpful discussions. This paper is the continuation of [AGHIV23]. They also thank Bingyu Zhang for the careful reading of the manuscript. They thank the anonymous referee for the helpful comments.

2 Notations

Throughout this paper, we fix a field $\mathbf{k}$.

Let $\mathcal{L}(T^*M)$ denote the set of compact exact Lagrangian branes, i.e., triples $(L, f_L, \widetilde{G})$, where L is a compact exact Lagrangian submanifold of T^*M, $f_L\colon L \to \mathbb{R}$ is a function satisfying $df_L = \lambda_{|L}$, and $\widetilde{G}$ is a grading of L (see [Sei00; Vit22b]). The action of $\mathbb{R}$ on $\mathcal{L}(T^*M)$ given by $(L, f_L, \widetilde{G}) \mapsto (L, f_L - c, \widetilde{G})$ is denoted by T_c. Let $\mathfrak{L}(T^*M)$ be the set of compact exact Lagrangians, where we do not record primitives or gradings. For L_1, L_2 in $\mathcal{L}(T^*M)$, we define as in [Vit22b] the spectral invariants $c_+(L_1, L_2)$ and $c_-(L_1, L_2)$, and set

$$\mathrm{c}(L_1, L_2) = \max\{c_+(L_1, L_2), 0\} - \min\{c_-(L_1, L_2), 0\}.$$

This defines a distance.[1] For L_1, L_2 in $\mathfrak{L}(T^*M)$, we define the spectral distance between L_1 and L_2 by

$$\gamma(L_1, L_2) = \inf_{c \in \mathbb{R}} \mathrm{c}(L_1, T_c L_2) = c_+(L_1, L_2) - c_-(L_1, L_2).$$

We denote by $\widehat{\mathfrak{L}}(T^*M)$ (resp. $\widehat{\mathcal{L}}(T^*M)$) the completion of $\mathfrak{L}(T^*M)$ (resp. $\mathcal{L}(T^*M)$) with respect to γ (resp. c).

[1] Note that the definition given in [AGHIV23] is not correct, and has to be replaced by the one above. This has been corrected in the published version of [Vit22b].

We denote by $\mathrm{DHam}(T^*M)$ the group of Hamiltonian diffeomorphisms of T^*M (time 1 of an isotopy) and $\mathrm{DHam}_c(T^*M)$ its subgroup made by times 1 of compactly supported isotopies.

We follow the notations of [KS90]. In particular $\mathsf{D}(\mathbf{k}_M)$ is the derived category of sheaves of $\mathbf{k}$-vector spaces on M. An object $F \in \mathsf{D}(\mathbf{k}_M)$ has a microsupport $\mathrm{SS}(F) \subset T^*M$ defined in loc. cit. For $A \subset T^*M$, a closed conic subset, $\mathsf{D}_A(\mathbf{k}_M) := \{F \in \mathsf{D}(\mathbf{k}_M) \mid \mathrm{SS}(F) \subset A\}$ is a triangulated full subcategory of $\mathsf{D}(\mathbf{k}_M)$. We now recall several notions and ideas from [Tam18]. We denote by $(t;\tau)$ the canonical coordinates on $T^*\mathbb{R}$ and we set for short $\{\tau \gtreqless 0\} = T^*M \times \{\tau \gtreqless 0\} \subset T^*(M \times \mathbb{R}_t)$. The Tamarkin category $\mathcal{T}(T^*M)$ is defined as the quotient category $\mathsf{D}(\mathbf{k}_{M\times\mathbb{R}})/\mathsf{D}_{\{\tau\leq 0\}}(\mathbf{k}_{M\times\mathbb{R}})$. The Tamarkin category has a monoidal structure. For $F, F' \in \mathsf{D}(\mathbf{k}_{M\times\mathbb{R}})$ we set $F * F' := \mathrm{R}m_!(q_1^{-1}F \otimes q_2^{-1}F')$, where $q_1, q_2 \colon M \times \mathbb{R}^2 \to M \times \mathbb{R}$ are the projections and m is the addition map $m(x,s,t) = (x, s+t)$. The operation $*$ preserves the left orthogonal ${}^{\perp}\mathsf{D}_{\{\tau\leq 0\}}(\mathbf{k}_{M\times\mathbb{R}})$ and moreover $F \mapsto F * \mathbf{k}_{M\times[0,+\infty[}$ is a projector onto it. This projector induces an equivalence between $\mathcal{T}(T^*M)$ and ${}^{\perp}\mathsf{D}_{\{\tau\leq 0\}}(\mathbf{k}_{M\times\mathbb{R}})$, with which we identify them in what follows. We also set $\mathcal{H}om^*(F, F') := \mathrm{R}q_{1*}\,\mathrm{R}\mathcal{H}om(q_2^{-1}F, m^!F')$ and denote the projection of this $\mathcal{H}om^*$ onto $\mathcal{T}(T^*M)$ by the same symbol. This defines an internal hom $\mathcal{H}om^* \colon \mathcal{T}(T^*M)^{\mathrm{op}} \times \mathcal{T}(T^*M) \to \mathcal{T}(T^*M)$. For $c \in \mathbb{R}$, let $T_c \colon M \times \mathbb{R} \to M \times \mathbb{R}$ be the translation $T_c(x,t) = (x, t{+}c)$. The category $\mathcal{T}(T^*M)$ comes with a family of morphisms of functors $\tau_c \colon \mathrm{id} \to T_{c*}$ for each $c \geq 0$ introduced by Tamarkin. They give rise to an interleaving distance on $\mathcal{T}(T^*M)$ denoted $d_{\mathcal{T}(T^*M)}$ (see [KS18] and [AI20]) defined as follows:

$$d_{\mathcal{T}(T^*M)}(F, F') := \inf \left\{ a+b \;\middle|\; \begin{array}{l} \exists u \colon F \to T_{a*}F',\ \exists v \colon F' \to T_{b*}F, \\ T_{a*}v \circ u = \tau_{a+b}(F),\ T_{b*}u \circ v = \tau_{a+b}(F') \end{array} \right\}.$$

We recall the composition of sheaves. For $F \in \mathsf{D}(\mathbf{k}_{M\times N})$ and $G \in \mathsf{D}(\mathbf{k}_{N\times P})$, set $F \circ G := \mathrm{R}q_{13!}(q_{12}^{-1}F \otimes q_{23}^{-1}G)$, where q_{ij} are the projections from $M \times N \times P$ to the $(i \times j)$ factors. We also consider a mixture of $\circ$ and $*$: for $F \in \mathcal{T}(T^*M \times T^*N)$, $G \in \mathcal{T}(T^*N \times T^*P)$, we set $F \circledast G = \mathrm{R}m_!\mathrm{R}q_{13!}(q_{12}^{-1}F \otimes q_{23}^{-1}G)$ where q_{ij} are projections from $M \times N \times P \times \mathbb{R}^2$ to $M \times N \times \mathbb{R}$, $N \times P \times \mathbb{R}$, $M \times P \times \mathbb{R}^2$ and m the addition map. We set for short $\mathcal{K}^{\circledast}(F) := \mathcal{K} \circledast F$ for $\mathcal{K} \in \mathcal{T}(T^*M^2)$ and $F \in \mathcal{T}(T^*M)$.

We put an analytic structure on M and define $\mathcal{T}_{\mathrm{lc}}(T^*M)$ as the subcategory of $\mathcal{T}(T^*M)$ made by objects that are limits (for the interleaving distance) of constructible sheaves. We remark that for a submanifold N of M, the pull-back to $N \times \mathbb{R}$ commutes with T_{c*} and τ_c. It follows that the pull-back is a contraction and hence sends $\mathcal{T}_{\mathrm{lc}}(T^*M)$ to $\mathcal{T}_{\mathrm{lc}}(T^*N)$.

For an object $F \in \mathcal{T}(T^*M)$, we define its reduced microsupport $\mathrm{RS}(F) \subset T^*M$ by

$$\mathrm{RS}(F) := \overline{\rho_t(\mathrm{SS}(F) \cap \{\tau > 0\})},$$

where $\rho_t \colon \{\tau > 0\} \to T^*M, (x,t;\xi,\tau) \mapsto (x;\xi/\tau)$. For a closed subset $A \subset T^*M$, we let $\mathcal{T}_A(T^*M)$ be the full subcategory of $\mathcal{T}(T^*M)$ consisting of the F with $\mathrm{RS}(F) \subset A$. We also set $\mathcal{T}_{\mathrm{lc},A}(T^*M) = \mathcal{T}_A(T^*M) \cap \mathcal{T}_{\mathrm{lc}}(T^*M)$.

3 Preliminaries

We recall that we have a quantization map for Hamiltonian isotopies $Q \colon \mathrm{DHam}_c(T^*M) \to \mathcal{T}(T^*M^2)$ introduced in [GKS12]. It is defined so that $\mathrm{RS}(Q(\varphi))$ is the graph of φ. For $\varphi \in \mathrm{DHam}_c(T^*M)$ and $\mathcal{K}_\varphi = Q(\varphi)$, the action of $\mathcal{K}_\varphi$ on $\mathcal{T}(T^*M)$, $F \mapsto \mathcal{K}_\varphi^{\circledast}(F) = \mathcal{K}_\varphi \circledast F$, is an auto-equivalence of category and we have $\mathrm{RS}(\mathcal{K}_\varphi^{\circledast}(F)) = \varphi(\mathrm{RS}(F))$. The category

$\mathcal{T}_{\rm lc}(T^*M^2)$ is not a group but it comes with the operation $\circledast$ which is associative and has $\mathbf{k}_{\Delta_M\times[0,+\infty[}$ as a unit element. Then Q respects the operations on $\mathrm{DHam}_c(T^*M)$ and $\mathcal{T}_{\rm lc}(T^*M^2)$: $Q(\varphi\circ\psi)\simeq Q(\varphi)\circledast Q(\psi)$.

We also have a quantization map for smooth compact exact Lagrangians, denoted by the same letter, $Q\colon \mathcal{L}(T^*M)\to\mathcal{T}(T^*M)$ defined more recently in [Gui23; Vit19], constructed so that $\mathrm{RS}(Q(L))=L$ for any $L\in\mathcal{L}(T^*M)$. This functor is an isometric embedding for the spectral and interleaving distances respectively (see [GV24, prop 6.3]): for $L_1,L_2\in\mathcal{L}(T^*M)$,

$$d_{\mathcal{T}(T^*M)}(Q(L_1),Q(L_2))=\gamma(L_1,L_2).$$

Since the map Q is an isometry, it extends to the completion[2] as an isometric embedding $\widehat{Q}\colon\widehat{\mathcal{L}}(T^*M)\to\mathcal{T}(T^*M)$ defined in [GV24]. We notice that $\widehat{Q}(T_c(\widetilde{L}_\infty))\simeq T_{c*}\widehat{Q}(\widetilde{L}_\infty)$. The main result of [AGHIV23] is the following connection between microsupport, γ-support and quantization:

$$\mathrm{RS}(\widehat{Q}(\widetilde{L}_\infty))=\gamma\text{-}\mathrm{supp}(\widetilde{L}_\infty)\quad\text{for any } \widetilde{L}_\infty\in\widehat{\mathcal{L}}(T^*M).$$

An approximation argument is missing in [GV24], which we shall now provide.

Proposition 3.1. *For any $\widetilde{L}_\infty\in\widehat{\mathcal{L}}(T^*M)$, one has $\widehat{Q}(\widetilde{L}_\infty)\in\mathcal{T}_{\rm lc}(T^*M)$.*

Proof. According to [CE12], Corollary 6.25, an element $\widetilde{L}\in\mathcal{L}(T^*M)$, is C^k-approximated for any $k\geq 1$ by analytic Lagrangians $\widetilde{L}_i$. We thus find that $\widetilde{L}=C^k-\lim\widetilde{L}_i$ hence $\widetilde{L}_i=\varphi_i(\widetilde{L})$ where φ_i is generated by a C^k-small Hamiltonian. According to Lemma 3.2 the distance between $\widetilde{L}$ and $\widetilde{L}_i$ can then be chosen arbitrarily small. As a result $\widetilde{L}$ is a γ-limit of analytic Lagrangians. According to [KS90] Theorem 8.4.2, the $Q(\widetilde{L}_i)$ are constructible hence their limit $Q(\widetilde{L})$ is in $\mathcal{T}_{\rm lc}(T^*M)$. Since $\widetilde{L}_\infty$ can be written as a Cauchy sequence of elements of $\mathcal{L}(T^*M)$, the claim follows. □

Lemma 3.2. *Let $h\colon (T^*(M\times\mathbb{R})\setminus 0_{M\times\mathbb{R}})\times I\to\mathbb{R}$ be a homogeneous Hamiltonian function and ϕ be the associated homogeneous Hamiltonian isotopy. Let $K\in\mathsf{D}((M\times\mathbb{R})^2\times I)$ be the sheaf associated with ϕ constructed in [GKS12]. Then, for any $F\in\mathcal{T}(T^*M)$*

$$d_{\mathcal{T}(T^*M)}(F,K_1\circ F)\leq 4\int_0^1\max|h_s(x,t;\xi,1)|\,ds.$$

Proof. First note that we have

$$\begin{aligned}\mathrm{SS}(K)\subset\big\{&(\phi_s(x,t;\xi,\tau),(x,t;-\xi,-\tau),(s,-h_s(\phi_s(x,t;\xi,\tau))))\mid\\ &(x,t;\xi,\tau)\in T^*(M\times\mathbb{R})\setminus 0_{M\times\mathbb{R}},s\in I\big\},\end{aligned}$$

which implies

$$\mathrm{SS}(K\circ F)\subset T^*M\times\{(t,s;\tau,\sigma)\mid\tau\geq 0,-\max h_s(x,t;\xi,\tau)\leq\sigma\leq-\min h_s(x,t;\xi,\tau)\}.$$

Since h is homogeneous, we get $h_s(x,t;\xi,\tau)=\tau h_s(x,t;\xi/\tau,1)$ for $\tau>0$. Thus, we can apply the same proof in Theorem 4.16 [AI20] to get

$$d_{\mathcal{T}(T^*M)}(F,K_1\circ F)\leq 2\int_0^1(\max h_s(x,t;\xi,1)-\min h_s(x,t;\xi,1))\,ds.$$

Here, note that the distance $d_{\mathcal{T}(T^*M)}$ is slightly different from the distance $d_{\mathcal{D}(M)}$ in [AI20] and we have $d_{\mathcal{T}(T^*M)}\leq 2d_{\mathcal{D}(M)}$. The right-hand side of the inequality is bounded above by the desired integral. □

[2]Note that the image is complete, but the map is not onto.

Let $L \in \mathfrak{L}(T^*M)$ and $L_\infty \in \widehat{\mathfrak{L}}(T^*M)$. We assume that γ-supp$(L_\infty) = L$ and we want to prove that $L_\infty \in \mathfrak{L}(T^*M)$ and $L_\infty = L$. Let $\widetilde{L} = (L, f_L, \widetilde{G}) \in \mathcal{L}(T^*M)$ be a lift of L and let $\widetilde{L}_\infty \in \widehat{\mathcal{L}}(T^*M)$ be a lift of L_∞. In view of [AGHIV23] our assumption means that the sheaf $F_{\widetilde{L}_\infty} = \widehat{Q}(\widetilde{L}_\infty) \in \mathcal{T}(T^*M)$ satisfies $\mathrm{RS}(F_{\widetilde{L}_\infty}) = L$. To prove that $L_\infty = L$ it is enough to see that F_{L_∞} is isomorphic to $F_L = Q(\widetilde{L})$, up to translation (in t) and shift (in grading). To this end, we shall characterize the objects F of $\mathcal{T}_{\mathrm{lc},L}(T^*M)$ with $\mathrm{SS}(F) = T_c(\Lambda)$ for some $c \in \mathbb{R}$, where $\Lambda \subset T^*(M \times \mathbb{R})$ is the cone over a Legendrian lift of L and T_c also denotes the translation on $T^*(M \times \mathbb{R})$ by c (see Definition 4.1, Lemma 4.2 and Proposition 4.11). Explicitly

$$\Lambda = \{(x, \tau p, -f_L(x, p), \tau) \mid \tau > 0, (x, p) \in L\}.$$

Hence Λ is a conic Lagrangian submanifold of $T^*(M \times \mathbb{R})$ contained in $\{\tau > 0\}$. Note that the coisotropic submanifold $\rho_t^{-1}(L)$ is foliated by the translates of Λ: $\rho_t^{-1}(L) = \bigsqcup_{c \in \mathbb{R}} T_c(\Lambda)$. It is not too difficult to see that any closed conic coisotropic subset of $\rho_t^{-1}(L)$ is a union of translates of Λ. Hence for any non zero $F \in \mathcal{T}_L(T^*M)$, $\mathrm{SS}(F)$ contains at least $T_c(\Lambda)$ for some $c \in \mathbb{R}$. However we shall not use these facts.

4 Cohomologically chordless sheaves

The main result we want to prove, Theorem 5.1, is about the space $\widehat{\mathcal{L}}(T^*M)$ and its statement is independent of sheaves. However, our proof starts by embedding this space in the category of sheaves via the functor Q. This embedding Q is far from being essentially surjective. We do not try to characterize its image, but we give here a useful property, *cohomologically chordless*, shared by the sheaves in its image. This property is a cohomological consequence of the following geometric property: if $F = Q(\widetilde{L})$ for some smooth Lagrangian brane $\widetilde{L} = (L, f_L, \widetilde{G})$, then the reduction map $\mathrm{SS}(F) \cap ST^*(M \times \mathbb{R}) \to T^*M$ is an embedding with image L. In other words, the Legendrian $\mathrm{SS}(F) \cap ST^*(M \times \mathbb{R})$ has no Reeb chords. Unfortunately, this geometric property is not necessarily preserved by taking limits since a γ-support may have double points (see Ex. 6.22 in [Vit22b]). However, the geometric property easily implies the following (already used in [Gui23, chapter XII.4]), which is stable by completion: $\mathrm{RHom}(F, T_{c*}F)$ is constant when c runs over $\mathbb{R}_{>0}$ or over $\mathbb{R}_{<0}$ (and in the latter case it is zero). Our Definition 4.1 below only retains the case $\mathbb{R}_{<0}$ but gives a slightly stronger version.

As already mentioned, even for a cohomological chordless sheaf F, the map $\mathrm{SS}(F) \cap ST^*(M \times \mathbb{R}) \to \mathrm{RS}(F)$ may not be injective. However we can give a sheafy statement analog to our main theorem: if F is cohomological chordless and $\mathrm{RS}(F)$ is a smooth exact Lagrangian submanifold, then $\mathrm{SS}(F) \cap ST^*(M \times \mathbb{R}) \to \mathrm{RS}(F)$ is a bijection (see Proposition 4.11 below for a more precise statement).

We denote by $q\colon M \times \mathbb{R} \to M$ the projection.

Definition 4.1. Let $F \in \mathcal{T}(T^*M)$. We say that F is *cohomologically chordless* if

$$\mathrm{RHom}(F \otimes q^{-1}G, T_{c*}F) \simeq 0$$

for all $c < 0$ and all locally constant $G \in \mathsf{D}(\mathbf{k}_M)$ (we say that an object of $\mathsf{D}(\mathbf{k}_M)$ is locally constant if its cohomology sheaves are locally constant).

Before proving Proposition 4.11 we give several results about cohomologically chordless sheaves.

Lemma 4.2. *Let $F \in \mathcal{T}_L(T^*M)$ with $\mathrm{SS}(F) = T_{c_0}(\Lambda)$ for some $c_0 \in \mathbb{R}$ and $F|_{M\times\{t\}} \simeq 0$ for $t \ll 0$. Then F is cohomologically chordless.*

Proof. This is already done in [Gui23, Lemma 12.4.4], but we sketch the proof for the convenience of the reader. First the microsupports of $F \otimes q^{-1}G$ and $T_{c*}F$ do not meet when c runs over $]-\infty, 0[$, hence $\mathrm{RHom}(F \otimes q^{-1}G, T_{c*}F)$ is independent of $c < 0$ by a variation on the Morse theorem for sheaves [KS90, Corollary 5.4.19] (see [Nad16] or [Gui23, Corollary 1.2.17]). We choose a such that $\Lambda \subset T^*(M \times]-a, a[)$. For $c < -2a$ we obtain that $T_{c*}F$ is locally constant on $\mathrm{supp}(F \otimes q^{-1}G)$, say $T_{c*}F \simeq q^{-1}G' \simeq q^! G'[-1]$ there. Then $\mathrm{RHom}(F \otimes q^{-1}G, T_{c*}F)$ is isomorphic to $\mathrm{RHom}(F \otimes q^{-1}G, q^! G'[-1])$. Using the adjunction $(\mathrm{R}q_!, q^!)$ and the projection formula $\mathrm{R}q_!(F \otimes q^{-1}G) \simeq \mathrm{R}q_! F \otimes G$, it is then enough to check that $\mathrm{R}q_! F \simeq 0$. This can be proved stalkwise: $(\mathrm{R}q_! F)_x \simeq \mathrm{R}\Gamma_c(\{x\} \times \mathbb{R}; F|_{\{x\}\times\mathbb{R}})$ and the vanishing follows again from the Morse theorem for sheaves since $\mathrm{SS}(F|_{\{x\}\times\mathbb{R}}) \subset \{\tau \geq 0\}$ and $F|_{\{x\}\times\mathbb{R}}$ vanishes near $-\infty$. □

Lemma 4.3. *Let $(F_i)_{i\in\mathbb{N}}$, be a convergent sequence in $\mathcal{T}(T^*M)$ and set $F = \lim_i F_i$ (the limit being for the distance $d_{\mathcal{T}(T^*M)}$). We assume that F_i is cohomologically chordless for each $i \in \mathbb{N}$. Then F is cohomologically chordless.*

Proof. By [GV24, Proposition 6.25] (or [AI24, Theorem 4.3]), up to taking a subsequence, there exist a sequence of positive numbers $(\varepsilon_i)_{i\in\mathbb{N}}$ converging to 0 and morphisms

$$f_i \colon T_{-\varepsilon_i *}F_i \to T_{-\varepsilon_{i+1}*}F_{i+1}, \quad u_i \colon T_{-\varepsilon_i *}F_i \to F \tag{4.1}$$

such that $u_{i+1} \circ f_i = u_i$, for all n, and the morphism $\mathrm{hocolim}\, T_{-\varepsilon_i *}F_i \to F$ induced by the u_i's is an isomorphism, where hocolim is the sequential homotopy colimit described in [BN93] (see also [KS06, Notation 10.5.10]). The same proposition holds with homotopy limits instead of homotopy colimits and we can write in the same way (taking a subsequence again) $F \xrightarrow{\sim} \mathrm{holim}\, T_{\eta_j *}F_j$ for some other sequence $(\eta_i)_{i\in\mathbb{N}}$.

Since the tensor product commutes with direct sums, it also commutes with homotopy colimits and we have, for any $G \in \mathsf{D}(\mathbf{k}_M)$, $F \otimes q^{-1}G \simeq \mathrm{hocolim}(T_{-\varepsilon_i *}F_i \otimes q^{-1}G)$. Recall that the category of sheaves on a topological space X is a Grothendieck category, so we may apply Lemma 4.4 and infer that $\mathrm{RHom}(F \otimes q^{-1}G, T_{c*}F)$ is a homotopy limit of $E_i = \mathrm{RHom}(T_{-\varepsilon_i *}F_i \otimes q^{-1}G, T_{(\eta_i+c)*}F)$. For a given $c < 0$ and for i big enough we have $\varepsilon_i + \eta_i + c < 0$ and then $E_i \simeq 0$. It follows that $\mathrm{RHom}(F \otimes q^{-1}G, T_{c*}F)$ vanishes. □

Lemma 4.4. *Let $\mathcal{C}$ be a Grothendieck category. Let (A_i, f_i), $i \in \mathbb{N}$, be an inductive system in $\mathsf{D}(\mathcal{C})$, with homotopy colimit A, and let (B_j, g_j), $j \in \mathbb{N}$, be a projective system, with homotopy limit B. Then $\mathrm{RHom}(A, B)$ is a homotopy limit of the system $(\mathrm{RHom}(A_i, B_i), h_i)$ where h_i is the morphism induced by composition with f_i, g_i.*

Proof. According to [Hov01], Theorem 2.2, the category $\mathrm{Ch}(\mathcal{C})$ of chain complexes on $\mathcal{C}$ is a model category having homotopical category $\mathsf{D}(\mathcal{C})$. We denote by $\mathbb{V}_{\mathbf{k}}$ the category of $\mathbf{k}$-vector spaces.

We apply results of [CS02] where homotopy (co)limits are defined for categories with weak equivalences. If $\mathcal{A}$ is such a category and I is a small category, we have a functor $\mathrm{holim}'_I \colon \mathrm{Fun}(I, \mathcal{A}) \to \mathrm{Ho}(\mathcal{A})$ (in particular $\mathrm{holim}'_I \colon \mathrm{Fun}(I, \mathrm{Ch}(\mathcal{C})) \to \mathrm{Ho}(\mathrm{Ch}(\mathcal{C})) = \mathsf{D}(\mathcal{C})$. In the proof of Lemma 4.3 the notation $\mathrm{holim}_I F$ applies to $F \in \mathrm{Fun}(I, \mathsf{D}(\mathcal{C}))$ — this is not a functor: $\mathrm{holim}_I F$ is well-defined up to a non-unique isomorphism. We use the notation $\mathrm{holim}'_I F$ to avoid confusion (but this is denoted by holim_I in [CS02]). We have $\mathrm{holim}'_I F \simeq \mathrm{holim}_I Q \circ F$ where $Q \colon \mathcal{A} \to \mathrm{Ho}(\mathcal{A})$ is the quotient.

We will apply Section 31.5 from [CS02] which states that if $F\colon I\times J\to\mathcal{C}$ is a functor to a model category, then $\mathrm{holim}'_{I\times J}F\simeq\mathrm{holim}_I\,\mathrm{holim}'_J F\simeq\mathrm{holim}_J\,\mathrm{holim}'_I F$. In our case $I=J=\mathbb{N}^{\mathrm{op}}$. We first lift the diagram $i\mapsto A_i$ to a similar diagram in the set of chain complexes on $\mathcal{C}$. We shall use the same notation for the lift. We do the same for $j\mapsto B_j$ and we may further impose that each B_j is a complex of injectives. We then define a functor $F\colon(\mathbb{N}^{\mathrm{op}})^2\to\mathrm{Ch}(\mathbb{V}_{\mathbf{k}})$ by $F(i,j)=\mathrm{Hom}(A_i,B_j)$. Since the B_j's are injective, we have $\mathrm{holim}'_i F(i,j)\simeq\mathrm{RHom}(\mathrm{hocolim}'_i A_i,B_j)\simeq\mathrm{RHom}(A,B_j)$ for each j. From the definition of holim we also have $\mathrm{holim}_j\,\mathrm{RHom}(A,B_j)\simeq\mathrm{RHom}(A,\mathrm{holim}'_j B_j)$. Hence

$$\mathrm{RHom}(A,B)\simeq\mathrm{holim}'_{(i,j)\in(\mathbb{N}^{\mathrm{op}})^2}\mathrm{Hom}(A_i,B_j).$$

According to 31.6 (loc. cit.) for $F\colon I\to\mathcal{C}$ a functor in a model category and $f\colon J\to I$ an initial functor, the map

$$\mathrm{holim}'_I F\to\mathrm{holim}'_J f^*F$$

is a weak equivalence. Using the fact that the inclusion of the diagonal $\mathbb{N}^{\mathrm{op}}$ in $(\mathbb{N}^{\mathrm{op}})^2$ is initial we get

$$\mathrm{RHom}(A,B)\simeq\mathrm{holim}'_{i\in\mathbb{N}^{\mathrm{op}}}\mathrm{Hom}(A_i,B_i)\simeq\mathrm{holim}_{i\in\mathbb{N}^{\mathrm{op}}}\mathrm{RHom}(A_i,B_i).$$

This concludes the proof. □

We now prove that, if $F\in\mathcal{T}_{\mathrm{lc},0_M}(T^*M)$ is cohomologically chordless, then $\mathrm{SS}(F)=0_M\times(\{c_0\}\times]0,\infty[)$ for some $c_0\in\mathbb{R}$ (Proposition 4.6 below).

We first recall a microlocal characterization of the inverse image of sheaves by a projection with contractible fibers.

Lemma 4.5. *Let N be a manifold and let I be an open interval (or more generally a contractible manifold). Let $p\colon N\times I\to N$ be the projection and let $i_a\colon N\times\{a\}\to N\times I$ be the inclusion, for $a\in I$. Then $p^{-1}\colon\mathsf{D}(\mathbf{k}_N)\to\mathsf{D}_{T^*N\times 0_I}(\mathbf{k}_{N\times I})$ is an equivalence of categories, with inverses $\mathrm{R}p_*$ and i_a^{-1}, $a\in I$. Moreover, in the case $N=\mathbb{R}$, these functors induce equivalences $\mathcal{T}(\mathrm{pt})\simeq\mathcal{T}_{0_I}(T^*I)$ and $\mathcal{T}_{\mathrm{lc}}(\mathrm{pt})\simeq\mathcal{T}_{\mathrm{lc},0_I}(T^*I)$.*

Proof. Proposition 2.7.8 of [KS90] says that p^{-1} and $\mathrm{R}p_*$ give equivalences between $\mathsf{D}(\mathbf{k}_N)$ and $\mathsf{D}_p(\mathbf{k}_{N\times I})$, where the latter category is the subcategory of $\mathsf{D}(\mathbf{k}_{N\times I})$ whose objects restrict to constant sheaves on the fibers. Now Proposition 5.4.5 of [KS90] says that $\mathsf{D}_p(\mathbf{k}_{N\times I})$ coincides with $\mathsf{D}_{T^*N\times 0_I}(\mathbf{k}_{N\times I})$. Since $i_a^{-1}\circ p^{-1}\simeq\mathrm{id}_{\mathsf{D}(\mathbf{k}_N)}$, we deduce that i_a^{-1} is also an inverse to p^{-1}.

The functors p^{-1} and i_a^{-1} commute with $-\ast\mathbf{k}_{[0,+\infty[}$ and we deduce $\mathcal{T}(\mathrm{pt})\simeq\mathcal{T}_{0_I}(T^*I)$. Moreover they send constructible sheaves to constructible sheaves and are 1-Lipschitz with respect to the interleaving distance. Hence they also induce the last equivalence of the lemma. □

Proposition 4.6. *Let $F\in\mathcal{T}_{\mathrm{lc},0_M}(T^*M)$ such that F is cohomologically chordless. Then there exists c_0 and a locally constant sheaf G_0 on M such that $F\simeq G_0\boxtimes\mathbf{k}_{[c_0,+\infty[}$.*

Proof. (i) For $c\in\mathbb{R}$ we set $G'_c=\mathrm{R}q_*\,\mathrm{R}\mathcal{H}om(F,T_{c*}F)$. By Lemma 4.5, for any open ball $B\subset M$, we have $F|_{B\times\mathbb{R}}\simeq p^{-1}F'$ for some $F'\in\mathcal{T}_{\mathrm{lc}}(\mathrm{pt})$, where $p\colon B\times\mathbb{R}\to\mathbb{R}$ is the projection. It follows that $\mathrm{R}\mathcal{H}om(F,T_{c*}F)|_{B\times\mathbb{R}}\simeq p^{-1}\,\mathrm{R}\mathcal{H}om(F',T_{c*}F')$. By base change we deduce that $G'_c|_B$ is constant. Hence G'_c is locally constant. We also have the adjunction isomorphism

$$\begin{aligned}\mathrm{RHom}(F\otimes q^{-1}G'_c,T_{c*}F)&\simeq\mathrm{RHom}(q^{-1}G'_c,\mathrm{R}\mathcal{H}om(F,T_{c*}F))\\&\simeq\mathrm{RHom}(G'_c,\mathrm{R}q_*\,\mathrm{R}\mathcal{H}om(F,T_{c*}F))=\mathrm{RHom}(G'_c,G'_c).\end{aligned}$$

Since F is cohomologically chordless, it follows that $G'_c \simeq 0$ for any $c < 0$.

(ii) Let $x \in M$ be given and let B be a small ball around x. With the same notations as in (i) we have $\mathrm{R}\mathcal{H}om(F, T_{c*}F)|_{B\times\mathbb{R}} \simeq p^{-1}\,\mathrm{R}\mathcal{H}om(F', T_{c*}F')$ and the base change formula gives $\mathrm{R}\Gamma(B; G'_c) \simeq \mathrm{RHom}(F', T_{c*}F')$. For $c < 0$ we thus obtain $\mathrm{RHom}(F', T_{c*}F') \simeq 0$. Let us check that this implies $F' \simeq E \otimes \mathbf{k}_{[c_0,+\infty[}$ for some constant sheaf E on $\mathbb{R}$ and some $c_0 \in \mathbb{R}$.

By [GV24, Corollary B.12] we have a decomposition $F' \simeq \bigoplus_{j\in\mathcal{I}} \mathbf{k}_{[a_j,b_j[}[d_j]$, where $\mathcal{I}$ is a countable set and $a_j \in \mathbb{R}$, $b_j \in \mathbb{R}\cup\{+\infty\}$, $d_j \in \mathbb{Z}$. If F' is not of the form $E\otimes\mathbf{k}_{[c_0,+\infty[}$, then there exists n with $b_n \neq +\infty$ or there exist n, m with $a_n \neq a_m$ (say $a_n < a_m$). In the first case we write $F' \simeq \mathbf{k}_{[a_n,b_n[}[d_n] \oplus F''$ and see that $H(c) := \mathrm{RHom}(\mathbf{k}_{[a_n,b_n[}, \mathbf{k}_{[c+a_n,c+b_n[})$ is a direct summand of $\mathrm{RHom}(F', T_{c*}F')$. By Lemma 4.7 below $H(c) \simeq \mathbf{k}[-1]$ for $a_n - b_n < c < 0$. The second case is similar, with the use of the fact that $\mathrm{Hom}(\mathbf{k}_{[a_n,+\infty[}, \mathbf{k}_{[c+a_m,+\infty[}) \simeq \mathbf{k}$ for $a_n - a_m < c$. In both cases we have $\mathrm{RHom}(F', T_{c*}F') \neq 0$ and get a contradiction. Hence $F' \simeq E \otimes \mathbf{k}_{[c_0,+\infty[}$ for some constant sheaf E and $c_0 \in \mathbb{R}$, as claimed.

(iii) Summing up, we have for any $x \in M$ and ball B around x an isomorphism $F|_{B\times\mathbb{R}} \simeq p^{-1}(E \otimes \mathbf{k}_{[c_0,+\infty[})$ where $p\colon B \times \mathbb{R} \to \mathbb{R}$ is the projection, $E \in \mathsf{D}(\mathbf{k})$ and $c_0 \in \mathbb{R}$. Since M is connected, c_0 does not depend on x. It follows that F is supported on $M \times [c_0, +\infty[$, hence $\mathrm{R}\mathcal{H}om(\mathbf{k}_{M\times[c_0,+\infty[}, F) \xrightarrow{\sim} F$.

Let us set $G_0 = \mathrm{R}q_*F$. The image of id_{G_0} by the adjunction isomorphisms

$$\begin{aligned}\mathrm{Hom}(G_0, \mathrm{R}q_*F) &\simeq \mathrm{Hom}(q^{-1}G_0, F)\\ &\simeq \mathrm{Hom}(q^{-1}G_0, \mathrm{R}\mathcal{H}om(\mathbf{k}_{M\times[c_0,+\infty[}, F)) \simeq \mathrm{Hom}(q^{-1}G_0 \otimes \mathbf{k}_{M\times[c_0,+\infty[}, F)\end{aligned}$$

gives a morphism $u\colon q^{-1}G_0 \otimes \mathbf{k}_{M\times[c_0,+\infty[} \to F$. By (ii) it is locally an isomorphism, hence it is an isomorphism. □

Lemma 4.7. *Let $a, c \in \mathbb{R}$ and $b, d \in \mathbb{R} \cup \{+\infty\}$ with $a < b$, $c < d$. We have*

$$\begin{aligned}\mathrm{R}\mathcal{H}om(\mathbf{k}_{[a,b[}, \mathbf{k}_{[c,d[}) &\simeq \mathrm{R}\mathcal{H}om(\mathbf{k}_{[a,b[\cap]c,d]}, \mathbf{k}_{\mathbb{R}})\\ &\simeq \begin{cases}\mathbf{k}_{[c,b]} & \text{if } a \le c < b \le d,\\ \mathbf{k}_{]a,d[} & \text{if } c < a < d < b,\\ \mathbf{k}_{\{a\}}[-1] & \text{if } a = d,\\ \mathbf{k}_I & \text{else, where } I \text{ is half closed or empty}\end{cases}\end{aligned}$$

and in particular

$$\mathrm{RHom}(\mathbf{k}_{[a,b[}, \mathbf{k}_{[c,d[}) \simeq \begin{cases}\mathbf{k} & \text{if } a \le c < b \le d,\\ \mathbf{k}[-1] & \text{if } c < a \le d < b,\\ 0 & \text{else.}\end{cases}$$

Proof. For an interval I with non empty interior let us write $I^* = (\overline{I}\setminus I)\cup\mathrm{Int}(I)$ (in words, we turn closed ends into open ones and conversely). Then $\mathrm{R}\mathcal{H}om(\mathbf{k}_I, \mathbf{k}_{\mathbb{R}}) \simeq \mathbf{k}_{I^*}$. In particular $\mathrm{R}\mathcal{H}om(\mathbf{k}_{[a,b[}, \mathbf{k}_{[c,d[}) \simeq \mathrm{R}\mathcal{H}om(\mathbf{k}_{[a,b[}, \mathrm{R}\mathcal{H}om(\mathbf{k}_{]c,d]}, \mathbf{k}_{\mathbb{R}})) \simeq \mathrm{R}\mathcal{H}om(\mathbf{k}_{[a,b[} \otimes \mathbf{k}_{]c,d]}, \mathbf{k}_{\mathbb{R}})$, which gives the first isomorphism. The second one follows by a case by case check, together with the additional isomorphism $\mathrm{R}\mathcal{H}om(\mathbf{k}_{\{a\}}, \mathbf{k}_{\mathbb{R}}) \simeq \mathbf{k}_{\{a\}}[-1]$. The last assertion is obtained by taking global sections. □

We now check that $\mathrm{DHam}_c(T^*M)$ and its completion preserve cohomologically chordless sheaves.

Lemma 4.8. *Let $\varphi \in \mathrm{DHam}_c(T^*M)$ and $\mathcal{K}_\varphi = Q(\varphi)$. Let $F \in \mathcal{T}(T^*M)$ be cohomologically chordless. Then $\mathcal{K}_\varphi^{\circledast}(F)$ is cohomologically chordless.*

Proof. Since $\mathcal{K}_\varphi^{\circledast}$ is an equivalence, we have

$$\mathrm{RHom}(\mathcal{K}_\varphi^{\circledast}(F \otimes q^{-1}G), \mathcal{K}_\varphi^{\circledast}(T_{c*}F)) \simeq \mathrm{RHom}(F \otimes q^{-1}G, T_{c*}F).$$

Hence it is enough to check that $\mathcal{K}_\varphi^{\circledast}$ commutes with T_{c*}, which is clear by the definition of $\circledast$, and that

$$\mathcal{K}_\varphi^{\circledast}(F \otimes q^{-1}G) \simeq \mathcal{K}_\varphi^{\circledast}(F) \otimes q^{-1}G.$$

Since φ is the time 1 of some isotopy, both sides of this isomorphism are restrictions at time 1 of sheaves in $\mathcal{T}_A(T^*(M \times \mathbb{R}))$, where $A \subset T^*(M \times \mathbb{R})$ is given by $A = \{(x, \xi, s, \sigma) \mid \sigma = h(x, \xi, s)\}$, with h the Hamiltonian function of φ. Both sheaves coincide at time 0 and the result follows from a uniqueness property in this situation (see for example Corollary 2.1.5 in [Gui23]). ☐

We equip $\mathrm{DHam}_c(T^*M)$ with the sheaf-theoretic spectral metric γ^s defined as

$$\gamma^s(\varphi, \varphi') = d_{\mathcal{T}(T^*M^2)}(\mathcal{K}_\varphi, \mathcal{K}_{\varphi'}).$$

Denote by $\widehat{\mathrm{DHam}}_c(T^*M)$ the completion of $\mathrm{DHam}_c(T^*M)$ with respect to γ^s. By the completeness of $\mathcal{T}(T^*M^2)$ with respect to $d_{\mathcal{T}(T^*M^2)}$ [AI24; GV24], we can extend the map Q as

$$\widehat{Q} \colon \widehat{\mathrm{DHam}}_c(T^*M) \to \mathcal{T}_{\mathrm{lc}}(T^*M^2).$$

As a completion of a group, $\widehat{\mathrm{DHam}}_c(T^*M)$ is a group and the formula $Q(\varphi \circ \psi) \simeq Q(\varphi) \circledast Q(\psi)$ given at the beginning of §3 extends to $\widehat{Q}$.

Lemma 4.9. *Let $\varphi_\infty \in \widehat{\mathrm{DHam}}_c(T^*M)$ and $\mathcal{K}_{\varphi_\infty} = \widehat{Q}(\varphi_\infty)$. Then $\mathcal{K}_{\varphi_\infty}^{\circledast} \colon \mathcal{T}(T^*M) \to \mathcal{T}(T^*M)$, $F \mapsto \mathcal{K}_{\varphi_\infty}^{\circledast}(F) := \mathcal{K}_{\varphi_\infty} \circledast F$ is an equivalence of categories. Moreover, if $F \in \mathcal{T}(T^*M)$ is cohomologically chordless, so is $\mathcal{K}_{\varphi_\infty}^{\circledast}(F)$.*

Proof. We find that $\mathcal{K}_{\varphi_\infty}$ has an inverse with respect to $\circledast$ given by $\mathcal{K}_{\varphi_\infty}^{-1} = \widehat{Q}(\varphi_\infty^{-1})$. The first assertion is then clear. Writing φ_∞ as a limit of a Cauchy sequence $(\varphi_n)_n$ of $\mathrm{DHam}_c(T^*M)$, the sequence $\mathcal{K}_{\varphi_n}^{\circledast}(F)$ converges to $\mathcal{K}_{\varphi_\infty}^{\circledast}(F)$. Hence the second assertion follows from Lemmas 4.3 and 4.8. ☐

Now we extend Proposition 4.6 to the case of a general exact Lagrangian L. To reduce the problem to Proposition 4.6 we shall use a result Arnaud, Humilière, and Viterbo [AHV24].

Theorem 4.10 (cf. [AHV24])**.** *Let $L \in \mathfrak{L}(T^*M)$ be a compact exact Lagrangian submanifold of T^*M. Then, there exists $\varphi_\infty \in \widehat{\mathrm{DHam}}_c(T^*M)$ such that $\varphi_\infty(L) = 0_M$, where both sides should be understood as elements in $\widehat{\mathfrak{L}}(T^*M)$. Moreover, the functor $\mathcal{K}_{\varphi_\infty}^{\circledast} \colon \mathcal{T}(T^*M^2) \to \mathcal{T}(T^*M^2)$ sends $\mathcal{T}_L(T^*M)$ to $\mathcal{T}_{0_M}(T^*M)$.*

The theorem is proved in [AHV24] for the completion of $\mathrm{DHam}_c(T^*M)$ with respect to the usual spectral metric γ. In Appendix A, we give a proof for the sheaf-theoretic spectral metric γ^s. Note that if $\mathbf{k}$ is of characteristic 2 or M is spin, then γ^s coincides with the usual spectral metric γ (see [GV24]).

Proposition 4.11. *Let $\widetilde{L} \in \mathcal{L}(T^*M)$ be a lift of $L \in \mathfrak{L}(T^*M)$ and set $F_{\widetilde{L}} = Q(\widetilde{L})$. Let $\varphi_\infty \in \widehat{\mathrm{DHam}}_c(T^*M)$ be given by Theorem 4.10 and set $\mathcal{K}_{\varphi_\infty} = \widehat{Q}(\varphi_\infty) \in \mathcal{T}_{\mathrm{lc}}(T^*M^2)$.*

(i) *There exist a locally constant sheaf G_0 on M, of rank 1, and $c_0 \in \mathbb{R}$ such that* $\mathcal{K}^{\circledast}_{\varphi_\infty}(F_{\widetilde{L}}) \simeq G_0 \boxtimes \mathbf{k}_{[c_0,+\infty[}$.

(ii) *Let $F \in \mathcal{T}_{\mathrm{lc},L}(T^*M)$ such that F is cohomologically chordless. Then there exists c_1 and a locally constant sheaf G_1 on M such that $F \simeq T_{c_1 *}(F_{\widetilde{L}} \otimes q^{-1}G_1)$.*

Proof. Let F be as in (ii). By Theorem 4.10 $\mathcal{K}^{\circledast}_{\varphi_\infty}(F) \in \mathcal{T}_{\mathrm{lc},0_M}(T^*M)$, and by Lemma 4.9 it is cohomologically chordless. By Proposition 4.6, we deduce $\mathcal{K}^{\circledast}_{\varphi_\infty}(F) \simeq G_F \boxtimes \mathbf{k}_{[c_F,+\infty[}$ for some $c_F \in \mathbb{R}$ and some locally constant sheaf G_F on M.

By Lemma 4.2 the sheaf $F_{\widetilde{L}}$ satisfies the property in (ii). In particular, we have (i) with $c_0 = c_{F_{\widetilde{L}}}$ and $G_0 = G_{F_{\widetilde{L}}}$, but we have to check that G_0 is of rank 1. We set $F_1 = (\mathcal{K}^{-1}_{\varphi_\infty})^{\circledast}(\mathbf{k}_{[c_0,+\infty[})$. Then $\mathcal{K}^{\circledast}_{\varphi_\infty}(F_1 \otimes q^{-1}G_0) \simeq \mathcal{K}^{\circledast}_{\varphi_\infty}(F_{\widetilde{L}})$. Hence $F_1 \otimes q^{-1}G_0 \simeq F_{\widetilde{L}}$. Restricting to $M \times \{t\}$ for $t \gg 0$ we obtain $(F_1|_{M\times\{t\}}) \otimes G_0 \simeq F_{\widetilde{L}}|_{M\times\{t\}} \simeq \mathbf{k}_M$, which implies that G_0 is of rank 1. This proves (i).

Now we come back to F as in (ii). We have

$$\mathcal{K}^{\circledast}_{\varphi_\infty}(F) \simeq G_F \boxtimes \mathbf{k}_{[c_F,+\infty[} \simeq T_{c_1 *}(G_1 \otimes (G_0 \boxtimes \mathbf{k}_{[c_0,+\infty[})),$$

where $c_1 = c_F - c_0$ and $G_1 = G_F \otimes G_0^{-1}$. Hence $\mathcal{K}^{\circledast}_{\varphi_\infty}(F) \simeq \mathcal{K}^{\circledast}_{\varphi_\infty}(T_{c_1 *}(F_{\widetilde{L}} \otimes q^{-1}G_1))$ and (ii) follows. □

5 Regular Lagrangians are smooth Lagrangians

We now prove our first main result, as a corollary of the characterization of cohomologically chordless sheaves obtained in Proposition 4.11.

Theorem 5.1. *Let $L_\infty \in \widehat{\mathfrak{L}}(T^*M)$. We assume that $L = \gamma\text{-supp}(L_\infty)$ is a compact exact Lagrangian submanifold of T^*M. Then $L_\infty = L$ in $\mathfrak{L}(T^*M)$.*

Proof. We lift L_∞ to $\widetilde{L}_\infty \in \widehat{\mathcal{L}}(T^*M)$ and set $F_\infty = \widehat{Q}(\widetilde{L}_\infty)$. By definition, $\widetilde{L}_\infty$ is the equivalence class of a Cauchy sequence $(L_n)_n$ in $\mathcal{L}(T^*M)$ and, by [GV24] or [AI24], the sequence of associated sheaves F_{L_n} converges in $\mathcal{T}(T^*M)$ to F_∞. By [AGHIV23] we know that $\mathrm{RS}(F_\infty) = L$. We lift L into $\widetilde{L} \in \mathcal{L}(T^*M)$ and let $F_{\widetilde{L}}$ be the associated sheaf. By Lemma 4.3 and Proposition 4.11 we have $F_\infty \simeq T_{c_1 *}(F_{\widetilde{L}} \otimes q^{-1}G_1)$ for some $c_1 \in \mathbb{R}$ and some locally constant sheaf G_1 on M. Let $x \in M$ be given. The sequence $F_{L_n}|_{\{x\}\times\mathbb{R}}$ converges to $F_\infty|_{\{x\}\times\mathbb{R}}$ and $\mathrm{R}\Gamma(\{x\} \times \mathbb{R}; F_{L_n}|_{\{x\}\times\mathbb{R}}) \simeq \mathbf{k}$ for all n. Hence $\mathrm{R}\Gamma(\{x\} \times \mathbb{R}; F_\infty|_{\{x\}\times\mathbb{R}}) \simeq \mathbf{k}$. It follows that $(G_1)_x \simeq \mathbf{k}$. The same kind of argument shows that $\mathrm{R}\Gamma(M; G_1) \simeq \mathrm{R}\Gamma(M \times \mathbb{R}; F_\infty) \simeq \mathbf{k}$. Hence $G_1 \simeq \mathbf{k}_M$ and $F_\infty \simeq T_{c_1 *}F_{\widetilde{L}}$, which implies $\widetilde{L}_\infty = T_{c_1}(\widetilde{L})$, hence $L_\infty = L$. □

Remark 5.2. Note that for $L = 0_M$ in T^*M and for a manifold M satisfying a certain condition (denoted by $(\star)$ in [Vit22a]), this theorem follows from Theorem 8.6 of [Vit22b](version 2) as a consequence of Theorem 6.3 in [Vit22a]. This was removed from the published version of [Vit22b] and included in [Vit22a].

6 Compact γ-supports are connected

Our second main result answers a question in [AHV24].

Theorem 6.1. *Let $L_\infty \in \widehat{\mathfrak{L}}(T^*M)$ and assume that $\gamma\text{-supp}(L_\infty)$ is compact. Then $\gamma\text{-supp}(L_\infty)$ is connected.*

Lemma 6.2. *For any* $\widetilde{L}_\infty \in \widehat{\mathcal{L}}(T^*M)$*, one has* $\mathrm{End}_{\mathcal{T}(T^*M)}(F_{\widetilde{L}_\infty}) \simeq \mathbf{k}$.

Proof. Write $\widetilde{L}_\infty$ as the equivalence class of a Cauchy sequence $(\widetilde{L}_n)_n$, where $\widetilde{L}_n \in \mathcal{L}(T^*M)$. Then, we find that $\mathrm{R}q_* \mathcal{H}om^*(F_{\widetilde{L}_n}, F_{\widetilde{L}_n}) \simeq \mathrm{R}\Gamma(M; \mathbf{k}_M) \otimes \mathbf{k}_{[0,+\infty[}$ for any n by [Vit19, Proposition 9.11]. Since $\gamma(\widetilde{L}_n, \widetilde{L}_\infty) \to 0$ and $\mathcal{H}om^*$ is continuous for the interleaving distance, we find that $\mathrm{R}q_* \mathcal{H}om^*(F_{\widetilde{L}_n}, F_{\widetilde{L}_n})$ converges to $\mathrm{R}q_* \mathcal{H}om^*(F_{\widetilde{L}_\infty}, F_{\widetilde{L}_\infty})$. This implies that

$$d_{\mathcal{T}(\mathrm{pt})}(\mathrm{R}q_* \mathcal{H}om^*(F_{\widetilde{L}_\infty}, F_{\widetilde{L}_\infty}), \mathrm{R}\Gamma(M; \mathbf{k}_M) \otimes \mathbf{k}_{[0,+\infty[}) = 0,$$

from which we deduce $\mathrm{R}q_* \mathcal{H}om^*(F_{\widetilde{L}_\infty}, F_{\widetilde{L}_\infty}) \simeq \mathrm{R}\Gamma(M; \mathbf{k}_M) \otimes \mathbf{k}_{[0,+\infty[}$ by [GV24, Proposition B.8]. Thus, we obtain

$$\mathrm{Hom}(F_{\widetilde{L}_\infty}, F_{\widetilde{L}_\infty}) \simeq H^0 \,\mathrm{RHom}(\mathbf{k}_{[0,+\infty[}, \mathrm{R}q_* \mathcal{H}om^*(F_{\widetilde{L}_\infty}, F_{\widetilde{L}_\infty})) \simeq \mathbf{k},$$

which proves the lemma. □

Our next Lemma is a variant of microlocal cut-off lemma. A cut-off functor associated with an open subset Ω of a cotangent bundle sends a sheaf F to a sheaf F' such that $\mathrm{SS}(F') \subset \overline{\Omega}$ and $\mathrm{SS}(F') \cap \Omega = \mathrm{SS}(F)$. Such functors were first introduced in [KS90] for special cases of Ω and more recently in [DAg96; Chi17; Zha24; Zha23; Kuo23; KSZ23; Zha25]. In general $\mathrm{SS}(F') \cap \partial\Omega$ will not be bounded by $\mathrm{SS}(F) \cap \partial\Omega$. However, when $\mathrm{SS}(F) \cap \partial\Omega$ is empty, we check that it holds true.

Lemma 6.3. *Let* U *be an open subset of* T^*M *and* $F \in \mathcal{T}(T^*M)$*. Assume that* $\mathrm{RS}(F) \cap U$ *is compact and* $\mathrm{RS}(F) \cap \partial U = \varnothing$*, where* ∂U *is the topological boundary defined as* $\overline{U} \setminus U$*. Then one has an exact triangle*

$$P(U) \circledast F \to F \to Q(U) \circledast F \xrightarrow{+1}$$

with $\mathrm{RS}(P(U) \circledast F) = \mathrm{RS}(F) \cap U$ *and* $\mathrm{RS}(Q(U) \circledast F) = \mathrm{RS}(F) \cap (T^*M \setminus \overline{U})$*. Here* $P(U), Q(U) \colon \mathcal{T}(T^*M) \to \mathcal{T}(T^*M)$ *are the microlocal projectors associated with* U *(see [Chi17; Zha24; Zha23]).*

Proof. Set $A_1 := \mathrm{RS}(F) \cap U$ and $A_2 := \mathrm{RS}(F) \setminus A_1$. To show $P(U) \circledast F \in \mathcal{T}_{A_1}(T^*M)$ and $Q(U) \circledast F \in \mathcal{T}_{A_2}(T^*M)$, we use (a version of) Kuo's description of projectors by microlocal wrapping [Kuo23; KSZ23].

Let $C_c^\infty(U)$ be the poset of compactly supported smooth functions on U and $H_\bullet \colon \mathbb{N} \to C_c^\infty(U)$ be a final functor satisfying $H_n \equiv n$ on a neighborhood of A_1 for each n. The microlocal projector $Q(U)$ is described as (see after (4.1) for the notation hocolim)

$$Q(U) \simeq \operatorname*{hocolim}_{n \in \mathbb{N}} \mathcal{K}_{H_n}.$$

Since hocolim commutes with $\circledast$, $Q(U) \circledast F \simeq \mathrm{hocolim}_n(\mathcal{K}_{H_n} \circledast F)$. Since dH_n vanishes on a neighborhood of $\mathrm{RS}(F)$, $\mathrm{RS}(\mathcal{K}_{H_n} \circledast F) = \mathrm{RS}(F) = A_1 \cup A_2$. Hence $\mathrm{RS}(\mathrm{hocolim}_n(\mathcal{K}_{H_n} \circledast F)) \subset A_1 \cup A_2$. On the other hand, $\mathrm{RS}(Q(U) \circledast F) \subset T^*M \setminus U$ by a formal property of the projector $Q(U)$ and hence,

$$\mathrm{RS}(Q(U) \circledast F) \subset (T^*M \setminus U) \cap (A_1 \cup A_2) = A_2.$$

Since the morphism $F \to Q(U) \circledast F$ is an isomorphism on $T^*M \setminus \overline{U}$, $(P(U) \circledast F) \cap (T^*M \setminus \overline{U}) = \varnothing$. The triangle inequality for microsupports shows $\mathrm{RS}(P(U) \circledast F) = A_1$. □

The next lemma is a wide generalization of [Gui23, Proposition 3.3.2], where the result is local and the sets A_1, A_2 are supposed "unknotted".

Lemma 6.4. *Let $F \in \mathcal{T}(T^*M)$ and assume that $\mathrm{RS}(F)$ is decomposed into two compact disjoint subsets A_1 and A_2. Then there exist $F_1, F_2 \in \mathcal{T}(T^*M)$ such that $\mathrm{RS}(F_i) = A_i$ and $F \simeq F_1 \oplus F_2$.*

Proof. Take an open neighborhood U of A_1 such that $\overline{U} \cap A_2 = \varnothing$. Applying Lemma 6.3, we have an exact triangle in $\mathcal{T}(T^*M)$

$$P(U) \circledast F \to F \to Q(U) \circledast F \xrightarrow{+1}$$

with $P(U) \circledast F \in \mathcal{T}_{A_1}(T^*M)$ and $Q(U) \circledast F \in \mathcal{T}_{A_2}(T^*M)$. Set $F_1 := P(U) \circledast F$ and $F_2 := Q(U) \circledast F$. Since $A_1 \cap A_2 = \varnothing$ and each A_i is compact, by Tamarkin's separation theorem we have $\mathrm{Hom}_{\mathcal{T}(T^*M)}(F_2, F_1[1]) = 0$. Then by the above exact triangle, $F \simeq F_1 \oplus F_2$. $\square$

Proof of Theorem 6.1. Suppose that γ-supp(L_∞) is decomposed into two non-empty compact disjoint subsets A_1 and A_2. Let $\widetilde{L}_\infty$ be a lift of L_∞. Set $F_{\widetilde{L}_\infty} = Q(\widetilde{L}_\infty) \in \mathcal{T}(T^*M)$. By a result of [AGHIV23], we have γ-supp$(L_\infty) = \mathrm{RS}(F_{\widetilde{L}_\infty})$. By Lemma 6.4, there exist $F_i \in \mathcal{T}_{A_i}(T^*M)$ such that $F_{\widetilde{L}_\infty} \simeq F_1 \oplus F_2$. Since $F_{\widetilde{L}_\infty}$ is indecomposable by Lemma 6.2, either F_1 or F_2 is zero. This is a contradiction. $\square$

Remarks 6.5. (i) The connectedness does not hold for elements in $\widehat{\mathfrak{L}}(T^*M)$ with non-compact γ-support. Indeed, consider the situation in Figure 6.1. Since f_j C^0-converges to f, $L_j = \mathrm{graph}(df_j)$ γ-converges to some L_∞ in $\widehat{\mathfrak{L}}(T^*S^1)$. Clearly the γ-support of L_∞ is the union of the two connected components represented in Figure 6.1(b).

(ii) One can construct examples of sequences F_i of elements in $\mathcal{T}(T^*M)$ such that $\mathrm{RS}(F_i)$ remain in a fixed compact set, are connected, F_i converges to F for $d_{\mathcal{T}(T^*M)}$, but $\mathrm{RS}(F)$ is not connected. As a result F is not in the image of $\widehat{Q}$.

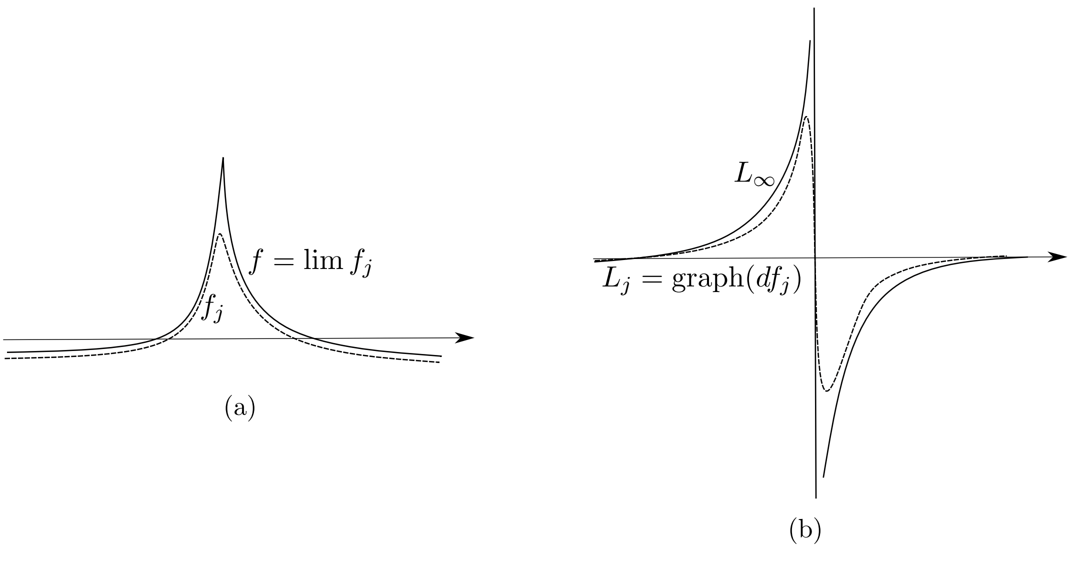

Figure 6.1: An element $L_\infty = \gamma\text{-}\lim(L_j)$ in $\widehat{\mathfrak{L}}(T^*S^1)$ with non-compact and disconnected γ-support. Figure (b) is the differential of Figure (a), the f_j, L_j correspond to the dashed curves, f, L_∞ to the solid curves.

A The weak nearby Lagrangian conjecture for the sheaf-theoretic spectral metric

In this appendix, we prove a sheaf-theoretic version of a result of Arnaud, Humilière, and Viterbo [AHV24]. The following theorem is for the sheaf-theoretic spectral metric γ^s.

Theorem A.1 (cf. [AHV24])**.** *Let $L \in \mathfrak{L}(T^*M)$ be a compact exact Lagrangian submanifold of T^*M. Then, there exists $\varphi_\infty \in \widehat{\mathrm{DHam}_c}(T^*M)$ such that $\varphi_\infty(L) = 0_M$, where both sides should be understood as elements in $\widehat{\mathfrak{L}}(T^*M)$. Moreover, the functor $\mathcal{K}^{\circledast}_{\varphi_\infty} \colon \mathcal{T}(T^*M^2) \to \mathcal{T}(T^*M^2)$ sends $\mathcal{T}_L(T^*M)$ to $\mathcal{T}_{0_M}(T^*M)$.*

The zero-section 0_M is the fixed point set of the canonical Liouville flow. We set ψ_0^s to be the time-s map of the canonical Liouville flow and set $\psi_0 := \psi_0^1$. The map ψ_0^s is the multiplication by e^s on each fiber. By [AHV24, Proposition 7.3], we see that L is also the fixed point set of the Liouville flow of another Liouville 1-form which coincides with the canonical Liouville form outside a compact subset. Set ψ_1 to be the time-1 map of this latter Liouville flow.

We will find φ_∞ as a fixed point of a contraction on $\widehat{\mathrm{DHam}_c}(T^*M)$. To construct the contraction, we first note the following.

Lemma A.2. *Let $\psi \colon T^*M \to T^*M$ be a diffeomorphism such that $\psi^*\lambda = a\lambda$ for some $a > 0$. Moreover, let $H \colon T^*M \times I \to \mathbb{R}$ be a compactly supported function and set $\tilde{H} := a^{-1} H \circ \psi$. Then, $\psi^{-1} \circ \phi_s^H \circ \psi = \phi_s^{\tilde{H}}$ and $\lambda(X_{\tilde{H}_s}) = a^{-1}\lambda(X_{H_s}) \circ \psi$ for $s \in I$.*

Proof. For $s \in I$, we have

$$\begin{aligned} \iota_{X_{\tilde{H}_s}}\omega &= -d(a^{-1}H_s \circ \psi) \\ &= -a^{-1}\psi^* dH_s \\ &= a^{-1}\psi^*(\iota_{X_{H_s}}\omega) \\ &= a^{-1}\iota_{\psi^* X_{H_s}}(\psi^*\omega) \\ &= \iota_{\psi^* X_{H_s}}\omega, \end{aligned}$$

where $\psi^* X := (d\psi)^{-1} X \circ \psi$. This shows that $X_{\tilde{H}_s} = \psi^* X_{H_s}$, which implies $\phi_s^H \circ \psi = \psi \circ \phi_s^{\tilde{H}}$. Moreover, since $\lambda = a^{-1}\psi^*\lambda$, we have

$$\iota_{X_{\tilde{H}_s}}\lambda = \iota_{\psi^* X_{H_s}}\lambda = a^{-1}\iota_{\psi^* X_{H_s}}(\psi^*\lambda) = a^{-1}\psi^*\iota_{X_{H_s}}\lambda,$$

which proves the second equality. □

Lemma A.3. *For a compactly supported function $H \colon T^*M \times I \to \mathbb{R}$, let $K_{\phi^H} \in \mathsf{D}(M^2 \times I \times \mathbb{R})$ be the sheaf quantization of the Hamiltonian isotopy ϕ^H. Then, for such a function H, one has $K_{\psi_0^{-1} \circ \phi^H \circ \psi_0} \simeq f_* K_{\phi^H}$, where f is defined as*

$$f \colon M^2 \times \mathbb{R} \times I \to M^2 \times \mathbb{R} \times I, \quad (x_1, x_2, t, s) \mapsto (x_1, x_2, e^{-1}t, s).$$

Proof. Set $\tilde{H} := e^{-1} H \circ \psi_0$. We also define

$$u_{H,s}(p) := \int_0^s (H_{s'} - \lambda(X_{H_{s'}}))(\phi_{s'}^H(p))\, ds'$$

for $H\colon T^*M \times I \to \mathbb{R}$ and $s \in I$. Then, by Lemma A.2, we have

$$\begin{aligned} u_{\tilde{H},s}(p) &= \int_0^s (\tilde{H}_{s'} - \lambda(X_{\tilde{H}_{s'}}))(\psi_0^{-1}\phi_{s'}^H\psi_0(p))\, ds' \\ &= e^{-1}\int_0^s (H_{s'} - \lambda(X_{H_{s'}}))(\phi_{s'}^H(\psi_0(p)))\, ds' \\ &= e^{-1}u_{H,s}(\psi_0(p)). \end{aligned}$$

We shall estimate the microsupports of $K_{\psi_0^{-1}\circ\phi^H\circ\psi_0}$ and $f_*K_{\phi^H}$. On the one hand, we have

$$\begin{aligned} &\mathring{\mathrm{SS}}(K_{\psi_0^{-1}\circ\phi^H\circ\psi_0}) \\ &= \left\{((x';\xi'),(x;-\xi),(u_{\tilde{H},s}(x;\xi/\tau),\tau),(s,-\tau\tilde{H}_s(\phi_s^{\tilde{H}}(x;\xi/\tau))))\ \middle|\ (x';\xi'/\tau) = \phi_s^{\tilde{H}}(x;\xi/\tau)\right\} \\ &= \begin{array}{l}\{((x';\xi'),(x;-\xi),(e^{-1}u_{H,s}(x;e\xi/\tau),\tau),(s,-\tau e^{-1}H_s(\phi_s^H(x;e\xi/\tau))))| \\ \quad (x';e\xi'/\tau) = \phi_s^H(x;e\xi/\tau)\}\,.\end{array} \end{aligned}$$

On the other hand, we have

$$\begin{aligned} &\mathring{\mathrm{SS}}(f_*K_{\phi^H}) \\ &= \begin{array}{l}\{((x';\xi'),(x;-\xi),(e^{-1}u_{H,s}(x;\xi/\tau),e\tau),(s,-\tau H_s(\phi_s^H(x;\xi/\tau))))| \\ \quad (x';\xi'/\tau) = \phi_s^H(x;\xi/\tau)\}\end{array} \\ &= \begin{array}{l}\{((x';\xi'),(x;-\xi),(e^{-1}u_{H,s}(x;e\xi/\tilde{\tau}),\tilde{\tau}),(s,-\tilde{\tau}e^{-1}H_s(\phi_s^H(x;e\xi/\tilde{\tau}))))| \\ \quad (x';e\xi'/\tilde{\tau}) = \phi_s^H(x;e\xi/\tilde{\tau})\}\,.\end{array} \end{aligned}$$

Since $(f_*K_{\phi^H})|_{s=0} \simeq \mathbf{k}_{\Delta\times\{0\}}$, by the uniqueness of the sheaf quantization ([GKS12]), we conclude. □

Proof of Theorem A.1. By Lemma A.3, for $\varphi \in \mathrm{DHam}_c(T^*M)$, we have $\mathcal{K}_{\psi_0^{-1}\circ\varphi\circ\psi_0} = f_*\mathcal{K}_\varphi$, where $\mathcal{K}_\varphi = Q(\varphi)$ and f is defined, by abuse of notation, as

$$f\colon M^2 \times \mathbb{R} \to M^2 \times \mathbb{R}, \quad (x_1,x_2,t) \mapsto (x_1,x_2,e^{-1}t).$$

This implies, setting $h = \psi_0^{-1}\circ\psi_1 \in \mathrm{DHam}_c(T^*M)$, that the map

$$T\colon \mathrm{DHam}_c(T^*M) \to \mathrm{DHam}_c(T^*M), \quad \varphi \mapsto \psi_0^{-1}\circ\varphi\circ\psi_1 = \psi_0^{-1}\circ\varphi\circ\psi_0\circ h$$

is a contraction. Note that $h = \psi_0^{-1}\circ\psi_1 \in \mathrm{DHam}_c(T^*M)$ is proved in the proof of [AHV24, Theorem 7.4]. Indeed, we have

$$\begin{aligned} d_{\mathcal{T}(T^*M^2)}(\mathcal{K}_{T\varphi},\mathcal{K}_{T\varphi'}) &= d_{\mathcal{T}(T^*M^2)}(\mathcal{K}_{\psi_0^{-1}\circ\varphi\circ\psi_0\circ h},\mathcal{K}_{\psi_0^{-1}\circ\varphi'\circ\psi_0\circ h}) \\ &= d_{\mathcal{T}(T^*M^2)}(\mathcal{K}_{\psi_0^{-1}\circ\varphi\circ\psi_0}\circledast\mathcal{K}_h,\mathcal{K}_{\psi_0^{-1}\circ\varphi'\circ\psi_0}\circledast\mathcal{K}_h) \\ &= d_{\mathcal{T}(T^*M^2)}(\mathcal{K}_{\psi_0^{-1}\circ\varphi\circ\psi_0},\mathcal{K}_{\psi_0^{-1}\circ\varphi'\circ\psi_0}) \\ &= d_{\mathcal{T}(T^*M^2)}(f_*\mathcal{K}_\varphi,f_*\mathcal{K}_{\varphi'}) \\ &= e^{-1}d_{\mathcal{T}(T^*M^2)}(\mathcal{K}_\varphi,\mathcal{K}_{\varphi'}), \end{aligned}$$

where the last equality follows from the fact that an (a,b)-isomorphism for $(\mathcal{K}_\varphi,\mathcal{K}_{\varphi'})$ gives an $(e^{-1}a,e^{-1}b)$-isomorphism for $(f_*\mathcal{K}_\varphi,f_*\mathcal{K}_{\varphi'})$ and vice versa.

Hence, the map T extends to the completion as a contraction, which we will also denote by $T\colon \widehat{\mathrm{DHam}_c}(T^*M) \to \widehat{\mathrm{DHam}_c}(T^*M)$. Therefore, there exists a unique fixed point $\varphi_\infty \in \widehat{\mathrm{DHam}_c}(T^*M)$ so that $\psi_0^{-1}\varphi_\infty = \varphi_\infty\psi_1^{-1}$, where both sides should be understood as actions on $\widehat{\mathfrak{L}}(T^*M)$. Since $L \in \widehat{\mathfrak{L}}(T^*M)$ is the unique fixed point of the action of ψ_1^{-1} and $0_M \in \widehat{\mathfrak{L}}(T^*M)$ is the unique fixed point of the action of ψ_0^{-1}, we find $\varphi_\infty(L) = 0_M$.

By its construction, the element $\varphi_\infty \in \widehat{\mathrm{DHam}_c}(T^*M)$ is represented as the sequence $(\varphi_n)_{n\in\mathbb{N}}$ with $\varphi_n := \psi_0^{-n} \circ \psi_1^n$. Thus, we obtain

$$\liminf_n \varphi_n(L) = \liminf_n \psi_0^{-n} \circ \psi_1^n(L) = \liminf_n \psi_0^{-n}(L) = 0_M.$$

Since taking the microsupport is "continuous" by [GV24, Proposition 6.26], for any $F \in \mathcal{T}_L(T^*M)$, we have

$$\mathrm{RS}(K^{\circledast}_{\varphi_\infty}(F)) \subset \liminf_n \varphi_n(L) = 0_M,$$

which proves the result. □

References

[AHV24] M.-C. Arnaud, V. Humilière, and C. Viterbo. *Higher Dimensional Birkhoff attractors*. 2024. arXiv: 2404.00804 [math.SG].

[AGHIV23] T. Asano, S. Guillermou, V. Humilière, Y. Ike, and C. Viterbo. "The γ-support as a micro-support". *Comptes Rendus. Mathématique* 361 (2023), pp. 1333–1340.

[AI20] T. Asano and Y. Ike. "Persistence-like distance on Tamarkin's category and symplectic displacement energy". *J. Symplectic Geom.* 18.3 (2020), pp. 613–649.

[AI24] T. Asano and Y. Ike. "Completeness of derived interleaving distances and sheaf quantization of non-smooth objects". *Math. Ann.* (2024).

[BN93] M. Bökstedt and A. Neeman. "Homotopy limits in triangulated categories". *Compositio Math.* 86.2 (1993), pp. 209–234.

[CS02] W. Chacholski and J. Scherer. "Homotopy theory of diagrams". *Mem. Amer. Math. Soc.* 155.736 (2002).

[Chi17] S.-F. Chiu. "Nonsqueezing property of contact balls". *Duke Math. J.* 166.4 (2017), pp. 605–655.

[CE12] K. Cieliebak and Y. Eliashberg. *From Stein to Weinstein and back. Symplectic Geometry and Affine complex manifolds.* Vol. 59. Colloquium Publications. AMS, Providence RI, 2012.

[DAg96] A. D'Agnolo. "On the microlocal cut-off of Sheaves". *Journal of the Juliusz Schauder Center* 8 (1996), pp. 161–167.

[Gui23] S. Guillermou. "Sheaves and symplectic geometry of cotangent bundles". *Astérisque* 440 (2023).

[GV24] S. Guillermou and C. Viterbo. "The Singular Support of Sheaves is γ-Coisotropic". *Geom. Funct. Anal.* 34 (2024), pp. 1052–1113.

[GKS12] S. Guillermou, M. Kashiwara, and P. Schapira. "Sheaf quantization of Hamiltonian isotopies and applications to non-displaceability problems". *Duke Math. J. 161, no. 2 (2012), 201-245* 161.2 (2012), pp. 201–245.

[Hov01] M. Hovey. "Model category structures on chain complexes of sheaves". *Trans. Amer. Math. Soc.* 353.6 (2001), pp. 2441–2457.

[Hum08] V. Humilière. "On some completions of the space of Hamiltonian maps." *Bulletin de la Soc. Math. de France* 136.3 (2008), pp. 373–404.

[HLS16] V. Humilière, R. Leclercq, and S. Seyfaddini. "Reduction of symplectic homeomorphisms". *Ann. Sci. Éc. Norm. Supér. (4)* 49.3 (2016), pp. 633–668.

[KS90] M. Kashiwara and P. Schapira. *Sheaves on manifolds.* Vol. 292. Grundlehren der Math. Wissenschaften. Springer-Verlag, 1990.

[KS06] M. Kashiwara and P. Schapira. *Categories and Sheaves.* Vol. 332. Grundlehren der Mathematische Wissenschaften. Springer-Verlag, 2006.

[KS18] M. Kashiwara and P. Schapira. "Persistent homology and microlocal sheaf theory". *J. Appl. Comput. Topol.* 2.1-2 (2018), pp. 83–113.

[Kuo23] C. Kuo. "Wrapped sheaves". *Adv. Math.* 415 (2023), Paper No. 108882, 71.

[KSZ23] C. Kuo, V. Shende, and B. Zhang. *On the Hochschild cohomology of Tamarkin categories.* 2023. arXiv: `2312.11447 [math.SG]`.

[MVZ12] A. Monzner, N. Vichery, and F. Zapolsky. "Partial quasimorphisms and quasistates on cotangent bundles, and symplectic homogenization". *J. Mod. Dyn.* 6.2 (2012), pp. 205–249.

[Nad16] D. Nadler. *Non-characteristic expansions of Legendrian singularities.* 2016. arXiv: `1507.01513 [math.SG]`.

[Oh97] Y.-G. Oh. "Symplectic topology as the geometry of action functional. I. Relative Floer theory on the cotangent bundle". *J. Differential Geom.* 46.3 (1997), pp. 499–577.

[Sei00] P. Seidel. "Graded Lagrangian submanifolds". *Bulletin de la S. M. F.* 128.1 (2000), pp. 103–149.

[Tam18] D. Tamarkin. "Microlocal Condition for Non-displaceability". In: *Algebraic and Analytic Microlocal Analysis.* Ed. by M. Hitrik, D. Tamarkin, B. Tsygan, and S. Zelditch. Cham: Springer International Publishing, 2018, pp. 99–223.

[Vit92] C. Viterbo. "Symplectic topology as the geometry of generating functions". *Math. Ann.* 292.4 (1992), pp. 685–710.

[Vit08] C. Viterbo. *Symplectic Homogenization.* 2008. arXiv: `0801.0206 [math.SG]`.

[Vit19] C. Viterbo. *Sheaf Quantization of Lagrangians and Floer cohomology.* 2019. arXiv: `1901.09440 [math.SG]`.

[Vit22a] C. Viterbo. *Inverse reduction inequalities for spectral numbers and applications.* 2022. arXiv: `2203.13172 [math.SG]`.

[Vit22b] C. Viterbo. *On the supports in the Humilière completion and γ-coisotropic sets.* 2022. arXiv: `2204.04133 [math.SG]`.

[Zha23] B. Zhang. *Idempotence of microlocal kernels and S^1-equivariant Chiu-Tamarkin invariant.* 2023. arXiv: `2306.12316 [math.SG]`.

[Zha24] B. Zhang. "Capacities from the Chiu-Tamarkin complex". *J. Symplectic Geom.* 22.3 (2024), pp. 441–524.

[Zha25] B. Zhang. "Non-linear microlocal cut-off functors". *Rend. Sem. Mat. Univ. Padova* (2025). published online first.