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On capitulations and pseudo-null submodules in certain $\mathbb{Z}_p^d$-extensions

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Abstract. Let p be a prime number. By a result of Ozaki, the capitulations of ideals in $\mathbb{Z}_p$-extensions and the finite submodules of Iwasawa modules are closely related. In this article, we discuss this relationship in $\mathbb{Z}_p^d$-extensions.

1. Introduction

Let p be a fixed prime number and $k/\mathbb{Q}$ a fixed finite extension, where denote by $\mathbb{Q}$ the field of rational numbers. For a number field F, let A_F be the p-part of the ideal class group of F. Let $\mathbb{Z}_p$ be the ring of p-adic integers. Let k_∞/k be a $\mathbb{Z}_p$-extension and k_n its n-th layer for each non-negative integer n, namely, k_n is the unique intermediate field of k_∞/k such that $[k_n : k] = p^n$. Let $X_{k_\infty} = \varprojlim_n A_{k_n}$, the projective limit is taken with respect to norm maps. The module X_{k_∞} is also defined to be the Galois group $\mathrm{Gal}(L_{k_\infty}/k_\infty)$ of the maximal unramified abelian pro-p extension L_{k_∞}/k_∞. We then have natural projection maps $X_{k_\infty} \to A_{k_n}$ for all $n \geq 0$. Let $A_{k_\infty} = \varinjlim_n A_{k_n}$, the inductive limit is taken with respect to lifting maps. We then have lifting maps $A_{k_n} \to A_{k_\infty}$ for all $n \geq 0$. It is well known that X_{k_∞} is a module over the completed group ring $\mathbb{Z}_p[\![\mathrm{Gal}(k_\infty/k)]\!]$. Let $X^0_{k_\infty}$ be the maximal finite submodule of X_{k_∞}. Then Ozaki obtained the following.

Theorem 1.1 (Ozaki [**15**]). *Suppose that k_∞/k is totally ramified at all ramified primes. Then we have $\mathrm{Ker}(A_{k_n} \to A_{k_\infty}) = \mathrm{Im}(X^0_{k_\infty} \to A_{k_n})$ for all $n \geq 0$. In particular, $X^0_{k_\infty} \neq 0$ if and only if $\mathrm{Ker}(A_{k_n} \to A_{k_\infty}) \neq 0$ for some $n \geq 0$.*

For the cyclotomic $\mathbb{Z}_p$-extensions k_∞/k of totally real fields k, the non-triviality of $X^0_{k_\infty}$ is studied as a weak form of Greenberg's conjecture (for Greenberg's conjecture see [**5**], and for a weak form of Greenberg's conjecture see [**13**], [**14**]). In this article, we discuss the relationship between kernels of lifting maps and pseudo-null submodules in $\mathbb{Z}_p^d$-extensions.

For a positive integer d, an algebraic extension K/k is called a $\mathbb{Z}_p^d$-extension if K/k is a Galois extension and $\mathrm{Gal}(K/k) \simeq \mathbb{Z}_p^d$ as topological groups. The composite field $\tilde{k}$ of all $\mathbb{Z}_p$-extensions of k is a $\mathbb{Z}_p^d$-extension for some $d > 0$. Let K/k be a $\mathbb{Z}_p^d$-extension. Let $X_K = \varprojlim_{k \subseteq k' \subseteq K, [k':k] < \infty} A_{k'}$, the projective limit is taken with respect to norm maps. The

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module X_K is also defined to be the Galois group of the maximal unramified abelian pro-p extension L_K/K. Then the completed group ring $\mathbb{Z}_p[\![\mathrm{Gal}(K/k)]\!]$ acts on X_K. Then it is known that that X_K is a finitely generated torsion $\mathbb{Z}_p[\![\mathrm{Gal}(K/k)]\!]$-module, see Lemma 2.2 of the below. Let X_K^0 be the maximal pseudo-null $\mathbb{Z}_p[\![\mathrm{Gal}(K/k)]\!]$-submodule of X_K, here, a $\mathbb{Z}_p[\![\mathrm{Gal}(K/k)]\!]$-module is called pseudo-null if the annihilator ideal is not contained in any height 1 prime ideals. When $K = \tilde{k}$, the non-triviality of $X_{\tilde{k}}^0$ is studied as a weak form of Greenberg's generalized conjecture (for Greenberg's generalized conjecture see [**7**], and for a weak form of Greenberg's generalized conjecture see [**18**], [**14**] and [**11**]). Let $A_K = \varinjlim_{k\subseteq k'\subseteq K,[k':k]<\infty} A_{k'}$, the inductive limit is taken with respect to lifting maps. Let $A_{k'} \to A_K$ be the lifting map. In this article, we mainly discuss by putting the following assumption on $\mathbb{Z}_p^d$-extensions:

CONDITION A. The prime number p does not split in $k/\mathbb{Q}$ and K/k is totally ramified at the unique prime of k lying above p.

The results of this article are as follows.

THEOREM 1.2. *Let K/k be a $\mathbb{Z}_p^d$-extension. Suppose that the condition A holds and that $A_k \simeq \mathbb{Z}/p^c$ for some $c \in \mathbb{Z}_{>0}$. If there is an intermediate field $k \subseteq k' \subseteq K$ with $[k' : k] < \infty$ such that $\mathrm{Ker}(A_{k'} \to A_K) \neq 0$, then $X_K^0 \neq 0$.*

We must mention here that, by Iwasawa's result [**8**], under the condition A, if $A_k = 0$ then $X_K = 0$.

THEOREM 1.3. *Let K/k be a $\mathbb{Z}_p^d$-extension. Suppose that the condition A holds. If $X_K^0 \neq 0$, then there is an intermediate field $k \subseteq k' \subseteq K$ with $[k' : k] < \infty$ such that $\mathrm{Ker}(A_{k'} \to A_K) \neq 0$.*

COROLLARY 1.1. *Let K/k be a $\mathbb{Z}_p^d$-extension. Suppose that the condition A holds and that $A_k \simeq \mathbb{Z}/p^c$ for some $c \in \mathbb{Z}_{>0}$. Then $X_K^0 \neq 0$ if and only if there is an intermediate field $k \subseteq k' \subseteq K$ with $[k' : k] < \infty$ such that $\mathrm{Ker}(A_{k'} \to A_K) \neq 0$.*

There have been some related earlier studies, we introduce here two of them.

THEOREM 1.4 (PROPOSITION 5.B OF MINARDI [**10**]). *Let K/k be a $\mathbb{Z}_p^d$-extension. Suppose that the condition A holds. Then $X_K = X_K^0$ if and only if there is a sub-$\mathbb{Z}_p$-extension F_∞/F of K/k with $[F : k] < \infty$ such that $A_F = \mathrm{Ker}(A_F \to A_{F_\infty})$.*

THEOREM 1.5 (LAI AND TAN [**9**]). *Let K/k be a $\mathbb{Z}_p^d$-extension. Then we have $\varprojlim_{k\subseteq k'\subseteq K,[k':k]<\infty} \mathrm{Ker}(A_{k'} \to A_K) \subseteq X_K^0$.*

We set here some notations. For a profinite group G, let $\Lambda_G = \mathbb{Z}_p[\![G]\!]$ be the completed group ring of G with coefficients in $\mathbb{Z}_p$. In the rest of this section, let $G \simeq \mathbb{Z}_p^d$. It is known as Serre's isomorphism that Λ_G is isomorphic to the formal power series ring in d-variables with coefficients in $\mathbb{Z}_p$. Hence Λ_G is a noetherian, integrally closed, complete, regular local ring. By Auslander–Buchsbaum's theorem [**1**], Λ_G and $\Lambda_G/p\Lambda_G$

are UFDs. A finitely generated Λ_G-module M is called pseudo-null if the annihilator ideal of M over Λ_G is not contained in any height 1 prime ideals of Λ_G. When $d = 1$, it is known that M is pseudo-null if and only if is finite. For a topological group $\mathfrak{G}$ and a topological $\mathfrak{G}$-module M, let $M^{\mathfrak{G}}$ and $M_{\mathfrak{G}}$ be the $\mathfrak{G}$-invariant submodule and the $\mathfrak{G}$-coinvariant module of M. For an algebraic extension $F/\mathbb{Q}$ not necessary finite, let L_F/F be the maximal unramified abelian pro-p extension and X_F its Galois group. Let A_F be the p-part of the ideal class group of F. If $[F : \mathbb{Q}] < \infty$, $X_F \simeq A_F$ by unramified class field theory.

2. Preliminaries

LEMMA 2.1. *Let A be a UFD and I an ideal of A. The following three statements are equivalent.*
(1) *The ideal I is not contained in any height 1 prime ideals of A.*
(2) *There are $f, g \in I$ such that f and g are relatively prime.*
(3) *For all $0 \neq f \in A$ there is $g \in I$ such that f and g are relatively prime.*

PROOF. (3) $\Rightarrow$ (2) : Trivial. (2) $\Rightarrow$ (1) : Let $f, g \in I$ and suppose that f and g are relatively prime. Then there is no prime element $q \in A$ such that both of f and g are divided by q. Since $(f, g) \subseteq I$, I is not contained in any height 1 prime ideals. (1) $\Rightarrow$ (3) : The following proof is written in Lemma 4.3 of [**10**]. Suppose that I is not contained in any height 1 prime ideals of A. Let s be the number of pairwise non-associated prime factors of f. We prove by using induction on s. Let $s = 1$. Then $f = uq_1^m$ for a unit u and an integer m. Since I is not contained in any height 1 prime ideals, it follows that $I \not\subseteq (q_1)$, and hence there is $g \in I$ such that f and g are relatively prime. Suppose that $s > 1$. Let $f = f_1 f_2$ be a decomposition of f by non units f_1, f_2 such that f_1 and f_2 are relatively prime. By the assumption of our induction, there are $g_1, g_2 \in I$ such that each of two pairs of elements f_1, g_1 and f_2, g_2 are relatively prime respectively. Put $g = g_2 f_1 + g_1 f_2 \in I$. Then f and g are relatively prime. □

LEMMA 2.2. *Let K/k be a $\mathbb{Z}_p^d$-extension and $G = \mathrm{Gal}(K/k)$. Then X_K is a finitely generated torsion Λ_G-modules.*

PROOF. This lemma was shown by Greenberg [**4**] for the extension $\tilde{k}/k$, and some authors mentioned that the statement of Lemma 2.2 also holds true for arbitrary $\mathbb{Z}_p^d$-extensions. For readers, we prefer to provide the proof here. Greenberg's proof depends on the existence of the cyclotomic $\mathbb{Z}_p$-extensions. The cyclotomic $\mathbb{Z}_p$-extension of a number field is ramified at all primes lying above p, and this property is a keystone to prove the statement of Lemma 2.2. Let K/k be a $\mathbb{Z}_p^d$-extension with $d \geq 1$ and S the set of all primes of k which ramify in K/k. Here it suffices to show that there is a $\mathbb{Z}_p$-extension k_∞/k such that $k_\infty \subseteq K$ and that k_∞/k is ramified at all primes in S. Such a $\mathbb{Z}_p$-extension k_∞/k plays the role of the cyclotomic $\mathbb{Z}_p$-extensions in Greenberg's proof, see Theorem 1 of [**4**]. We shall prove by induction on $d \geq 1$. The case that $d = 1$ is trivial. Let $d > 1$ and assume that the statement of the lemma holds true for all $\mathbb{Z}_p^r$-extensions K_0/k which are exactly ramified at all primes in S with $r < d$. Let $I_v \subseteq \mathrm{Gal}(K/k)$ be the inertia subgroup at $v \in S$. Put $S_1 = \{w \in S \mid I_w \simeq \mathbb{Z}_p\}$. Since $d > 1$ and S_1 is a finite set, we can choose $x \in \mathrm{Gal}(K/k)$ such that $x \notin \mathrm{Gal}(K/k)^p$ and that $\overline{\langle x \rangle} \cap I_w = 0$ for all $w \in S_1$. Let K_0 be the fixed field of $\overline{\langle x \rangle}$ in K. Then K_0/k is a $\mathbb{Z}_p^{d-1}$-extension. Let $u \in S \setminus S_1$. It follows that the $\mathbb{Z}_p$-rank of I_u is greater than 1. Then $I_u\overline{\langle x \rangle}/\overline{\langle x \rangle}$ has $\mathbb{Z}_p$-rank at least 1. Thus the inertia subgroup in K_0/k at each $v \in S$ is not trivial, and hence K_0/k is ramified at all primes in S. By the assumption of our induction, there is a $\mathbb{Z}_p$-extension k_∞/k such that $k_\infty \subseteq K_0$ and that k_∞/k is ramified at all primes in S. This completes the proof. This proof is a generalization of Lemma 5 of [**3**]. □

Lemma 2.3. *Let K/k be a $\mathbb{Z}_p^d$-extension. Suppose that the condition A holds. For each intermediate field F of K/k, we have $X_F \simeq (X_K)_{\mathrm{Gal}(K/F)}$.*

Proof. It follows that K/F is a $\mathbb{Z}_p^r$-extension for some $r \leq d$. Let $\mathrm{Gal}(K/F) = \overline{\langle \sigma_1, \cdots, \sigma_r \rangle}$. One can see that $(X_K)_{\mathrm{Gal}(K/F)} = X_K/(\sigma_1 - 1, \cdots, \sigma_r - 1)X_K$. Let K_i be the fixed field of $\overline{\langle \sigma_{i+1}, \cdots, \sigma_r \rangle}$ for $0 \leq i \leq r-1$. Then we have a tower of fields $F = K_0 \subseteq K_1 \subseteq \cdots \subseteq K_r = K$. By the condition A, the extension K/k is totally ramified at the unique prime of k lying above p, and hence extensions K_i/K_{i-1} are also totally ramified at the unique prime of K_{i-1} lying above p for all i with $1 \leq i \leq r$. Let L_i be the maximal subfield of L_{K_i} which is abelian over K_{i-1}. It holds that $\mathrm{Gal}(L_i/K_i) \simeq (X_{K_i})_{\mathrm{Gal}(K_i/K_{i-1})} = X_{K_i}/(\sigma_i - 1)X_{K_i}$. Let I_i be the inertia subgroup in L_i/K_{i-1} of the unique prime of K_{i-1} lying above p. It then holds that $I_i = \mathrm{Gal}(L_i/L_{K_{i-1}})$. Since L_i/K_i is unramified, we have $I_i \cap \mathrm{Gal}(L_i/K_i) = 1$, and hence $L_i = K_i L_{K_{i-1}}$ holds. By the definition of L_i it follows that $L_{K_{i-1}} \cap K_i = K_{i-1}$, and hence we have $X_{K_{i-1}} \simeq \mathrm{Gal}(L_i/K_i) \simeq X_{K_i}/(\sigma_i - 1)X_{K_i}$ for all i. Thus it holds that $X_F \simeq X_K/(\sigma_1 - 1, \cdots, \sigma_r - 1)X_K = (X_K)_{\mathrm{Gal}(K/F)}$. □

Lemma 2.4. *Let $\Gamma \simeq \mathbb{Z}_p$ and M a finitely generated torsion Λ_Γ-module. Then M has no non-trivial finite submodules if and only if there is an exact sequence $0 \to \Lambda_\Gamma^{\oplus r} \to \Lambda_\Gamma^{\oplus r} \to M \to 0$ for some $r \in \mathbb{Z}_{>0}$.*

Proof. See Proposition 2.1 of [**17**]. □

Lemma 2.5. *Let $G \simeq \mathbb{Z}_p^d$ with $d > 0$. Let N be a finitely generated torsion Λ_G-module. Assume that N has an annihilator $\Phi \in \Lambda_G$ such that $\Phi \not\equiv 0 \bmod p\Lambda_G$. Then G contains at least one subgroup H such that $G/H \simeq \mathbb{Z}_p$ with the property that N is finitely generated over Λ_H.*

Proof. See Lemma 2 of [**6**]. □

Lemma 2.6. *Let $d \geq 3$ and $G \simeq \mathbb{Z}_p^d$. Let H be a subgroup of G such that $G/H \simeq \mathbb{Z}_p$. Let N be a finitely generated pseudo-null Λ_G-module. Suppose that N is finitely generated over Λ_H. Then for all but finitely many subgroups V of H with $H/V \simeq \mathbb{Z}_p^{d-2}$, N_V is a pseudo-null $\Lambda_{G/V}$-module.*

Proof. This lemma is shown in [**10**] as a Corollary of Proposition 4.C. Here, we give a somewhat simpler proof. Let H be a subgroup of G such that N is finitely generated over Λ_H with $G/H \simeq \mathbb{Z}_p$. Let $\tau \in G$ be an element such that $G = H \times \overline{\langle \tau \rangle}$. Put $T = \tau - 1$, and we shall identify by Serre's isomorphism Λ_G with $\Lambda_H[\![T]\!]$, the formal power series ring in the variable T with coefficients in Λ_H. Hence all Λ_G-module can be regarded as $\Lambda_H[\![T]\!]$-modules. Since N is finitely generated over Λ_H, by Cayley–Hamilton's theorem, there is a monic polynomial $f \in \Lambda_H[T]$ such that f annihilates N. By the Weierstass preparation theorem, we may assume that f is a distinguished polynomial of degree greater than 0, see Definition 2 and Proposition 6 in Section 3 of Chapter 7 of [**2**]. Since N is pseudo-null, by Lemma 2.1, there is an annihilator $g \in \Lambda_G = \Lambda_H[\![T]\!]$ of N such that f and g are relatively prime. If we need, by adding f to g and by the Weierstrass preparation theorem, we may assume that g is also a distinguished polynomial in $\Lambda_H[T]$. By Proposition 7 of Section 3 of Chapter 7 of [**2**], f and g are relatively prime in Λ_G if and only if are relatively prime in $\Lambda_H[T]$. Hence there are polynomials A and B of $Q_{\Lambda_H}[T]$ such that $Af + Bg = 1$, here Q_{Λ_H} denotes the field of fractions of Λ_H. Choose an element $\alpha \in \Lambda_H$ such that $\alpha A, \alpha B \in \Lambda_H[T]$, hence it holds that $\alpha A f + \alpha B g = \alpha$. Let $\sigma \in H \setminus H^p$. By the choice of f and g, we have $f, g \not\equiv 0 \bmod (\sigma - 1)\Lambda_G$. Since Λ_G is a UFD and $(\sigma - 1)\Lambda_G$ is a prime ideal of Λ_G, there are only finitely many prime ideals of the form $(\sigma - 1)\Lambda_G$ so that $\alpha \equiv 0 \bmod (\sigma - 1)\Lambda_G$. Let $V = \overline{\langle\sigma\rangle}$ with the property that $\alpha \not\equiv 0 \bmod (\sigma - 1)\Lambda_G$. For each $h \in \Lambda_G$, let h_V be the image of h with respect to the map $\Lambda_G \to \Lambda_{G/V} = \Lambda_{H/V}[\![T]\!]$. Thus it holds that $(\alpha A)_V f_V + (\alpha B)_V g_V = \alpha_V \neq 0$. This implies that f_V and g_V are relatively prime in $\Lambda_{H/V}[T]$. Further $f_V, g_V \neq 0$ and both of f_V, g_V annihilate N_V. Therefore, N_V is a pseudo-null $\Lambda_{H/V}[\![T]\!] = \Lambda_{G/V}$-module. □

LEMMA 2.7. *Let K/k be a $\mathbb{Z}_p^d$-extension. Suppose that the condition A holds. Let $1 \neq \sigma \in G = \mathrm{Gal}(K/k)$. Then a generator of the characteristic ideal of X_K over Λ_G and $\sigma - 1$ are relatively prime.*

PROOF. Because we study on a generator of the characteristic ideal of X_K, we may assume that $X_K^0 = 0$. By the structure theorem of finitely generated torsion Λ_G-modules, see for example (5.1.7) Proposition of [**12**], there is an exact sequence $0 \to X_K \to E \to Z \to 0$ of Λ_G-modules, here, we denote by E an elementally finitely generated torsion Λ_G-module and by Z a pseudo-null Λ_G-module. Let $\{\sigma_1, \sigma_2, \cdots, \sigma_d\}$ be a basis of G over $\mathbb{Z}_p$ such that $\sigma = \sigma_1^{p^a}$ with some non-negative integer a. Then one can see that $Z/(\sigma-1)Z$ is a finitely generated torsion Λ_H-module, where we let $H = \overline{\langle\sigma_2, \cdots, \sigma_d\rangle}$. Indeed, since $Z/(\sigma - 1)Z$ is also pseudo-null over Λ_G, there is an annihilator u of $Z/(\sigma - 1)Z$ over Λ_G such that u and $\sigma - 1$ are relatively prime. By the Weierstrass preparation theorem, if u is not reduced in $\Lambda_G = \Lambda_H[\![\overline{\sigma_1}]\!]$, by adding $\sigma - 1$ to u, we may assume that u is a distinguished polynomial in $\Lambda_H[\sigma_1 - 1]$. Since $\sigma - 1$ and u are relatively prime in $Q_{\Lambda_H}[\sigma_1 - 1]$, there are f and g of $Q_{\Lambda_H}[\sigma_1 - 1]$ such that $f(\sigma - 1) + gu = 1$. Thus there is $0 \neq v \in \Lambda_H$ such that $v = (vf)(\sigma - 1) + (vg)u$. This shows that $Z/(\sigma - 1)Z$ is torsion over Λ_H.

Let M be the fixed field of $\overline{\langle\sigma\rangle}$. Then we have a decomposition $\mathrm{Gal}(M/k) \simeq \mathbb{Z}/p^a \times \overline{\langle\sigma_2, \cdots, \sigma_d\rangle}$. Let F be the fixed field of $H = \overline{\langle\sigma_2, \cdots, \sigma_d\rangle}$ in M. Then $[F : k] < \infty$ and M/F is a $\mathbb{Z}_p^{d-1}$-extension. It is known by Lemma 2.2 that the module X_M is finitely generated and torsion over Λ_H. By Lemma 2.3, one can see that $X_M \simeq (X_K)_{\mathrm{Gal}(K/M)} = X_K/(\sigma-1)X_K$. Then we have an exact sequence $X_M \to E/(\sigma-1)E \to Z/(\sigma-1)Z \to 0$. Let $f \in \Lambda_G$ be a generator of the characteristic ideal of X_K over Λ_G. Now, suppose that f and $\sigma - 1$ are not relatively prime. Let q be a common prime factor of f and $\sigma - 1$.

Then E is a module of the form

$$E = \Lambda_G/q^e\Lambda_G \oplus \bigoplus_{i=1}^{s} \Lambda_G/q_i^{e_i}\Lambda_G,$$

where $q_1, \cdots, q_s \in \Lambda_G$ denote prime elements of Λ_G, and $e, e_1, \cdots, e_s$ are positive integers. As we checked, $Z/(\sigma-1)Z$ is torsion over Λ_H. Since

$$\Lambda_G/(q, \sigma-1) = \Lambda_G/q\Lambda_G = (\mathbb{Z}_p[\![\overline{\langle\sigma_1\rangle}]\!]/q\mathbb{Z}_p[\![\overline{\langle\sigma_1\rangle}]\!])[\![H]\!] \supseteq \Lambda_H,$$

$\Lambda_G/(q, \sigma-1)$ is not torsion over Λ_H, and $\Lambda_G/(q^e, \sigma-1)$ is also not torsion since there is a surjective morphism $\Lambda_G/(q^e, \sigma-1) \to \Lambda_G/(q, \sigma-1)$. This contradicts to the fact that $X_M \simeq X_K/(\sigma-1)X_K$ is torsion over Λ_H. Therefore there are no common prime factors of f and $\sigma - 1$. □

Lemma 2.8. *Let K/k be a $\mathbb{Z}_p^d$-extension. Suppose that the condition A holds. Let $1 \neq \sigma \in \mathrm{Gal}(K/k)$. Then we have $(X_K/X_K^0)^{\overline{\langle\sigma\rangle}} = 0$.*

Proof. By Lemma 2.7, a generator of the characteristic ideal of X_K and $\sigma - 1$ are relatively prime. Since X_K/X_K^0 has no non-trivial pseudo-null submodules, we have $(X_K/X_K^0)^{\overline{\langle\sigma\rangle}} = 0$. □

Lemma 2.9. *Let N be a non-trivial pseudo-null Λ_U-module, where $U \simeq \mathbb{Z}_p^2$. Assume that N_U is finite. Then there exists at least one subgroup V of U such that $U/V \simeq \mathbb{Z}_p$ with the property that N_V contains a non-trivial finite $\Lambda_{U/V}$-submodule.*

Proof. See Lemma 5 of [**6**]. □

3. Proof of Theorem 1.2

Let K/k be a $\mathbb{Z}_p^d$-extension and suppose that the condition A holds. Suppose also that $A_k \simeq \mathbb{Z}/p^c$ for some $c > 0$, and $\mathrm{Ker}(A_{k'} \to A_K) \neq 0$ for some $k \subseteq k' \subseteq K$ with $[k' : k] < \infty$. There is a finite extension k_1'/k' with $k_1' \subseteq K$ such that $\mathrm{Ker}(A_{k'} \to A_{k_1'}) \neq 0$. Then one can find a finite cyclic extension F'/F such that $k' \subseteq F \subseteq F' \subseteq k_1'$ and that $\mathrm{Ker}(A_F \to A_{F'}) \neq 0$. Since K/F is a $\mathbb{Z}_p^d$-extension, there is a $\mathbb{Z}_p$-extension F_∞/F such that $F' \subseteq F_\infty \subseteq K$ and that $\mathrm{Ker}(A_F \to A_{F_\infty}) \neq 0$. By Theorem 1.1, we have $X_{F_\infty}^0 \neq 0$. Let $G = \mathrm{Gal}(K/k)$, $H = \mathrm{Gal}(K/F_\infty)$ and $\Gamma = \mathrm{Gal}(F_\infty/F)$. By Nakayama's lemma and Lemma 2.3, since $A_k \simeq (X_K)_G$, X_K is cyclic over Λ_G. Let $0 \to I \to \Lambda_G \to X_K \to 0$ be an exact sequence of Λ_G-modules with an ideal I of Λ_G. Since Λ_G is noetherian, I is finitely generated. Put $I = (h_1, \cdots, h_s)$ for some elements $h_1, \cdots, h_s \in \Lambda_G$. Let h be a greatest common divisor of $h_1, \cdots, h_s$. Suppose that $X_K^0 = 0$. Let $I_0 = (h_1/h, \cdots, h_s/h)$. It holds that $h\Lambda_G/I \simeq \Lambda_G/I_0$. Since elements $h_1/h, \cdots, h_s/h$ have no non-trivial common divisor, I_0 is not contained in any height 1 prime ideals of Λ_G. Hence $\Lambda_G/I_0 \simeq h\Lambda_G/I$ is a pseudo-null Λ_G-module. Since $X_K \simeq \Lambda_G/I$ has no non-trivial pseudo-null submodules, we have $I = h\Lambda_G$. Thus there is an exact sequence $0 \to \Lambda_G \to \Lambda_G \to X_K \to 0$. Since $(\Lambda_G)_H \simeq \Lambda_{G/H}$, we have an exact sequence $\Lambda_{G/H} \to \Lambda_{G/H} \to X_{F_\infty} \to 0$. By the definitions of G and H, we have $(G/H)/\Gamma = \mathrm{Gal}(F/k)$. Since $(\Lambda_{G/H})_\Gamma \simeq \mathbb{Z}_p^{\oplus[F:k]}$ and $\Lambda_{G/H}$ is torsion free over $\mathbb{Z}_p$, it holds that $\Lambda_{G/H} \simeq \Lambda_\Gamma^{\oplus[F:k]}$ as Λ_Γ-modules. From the fact that X_{F_∞} is a torsion Λ_Γ-module, the kernel of $\Lambda_{G/H} \to \Lambda_{G/H}$ is a submodule of a free Λ_Γ-module and is of rank 0, and hence is trivial. Therefore we have an exact sequence $0 \to \Lambda_\Gamma^{\oplus[F:k]} \to \Lambda_\Gamma^{\oplus[F:k]} \to X_{F_\infty} \to 0$. This implies that $X_{F_\infty}^0 = 0$ by Lemma 2.4. This contradicts to the fact that $X_{F_\infty}^0 \neq 0$. Thus we have $X_K^0 \neq 0$. □

4. Proof of Theorem 1.3

Let K/k be a $\mathbb{Z}_p^d$-extension. Suppose that the condition A holds, and that $X_K^0 \neq 0$. Let $G = \mathrm{Gal}(K/k)$. First, we suppose that $d \geq 3$. Since X_K^0 is pseudo-null, there is an annihilator $\Phi \in \Lambda_G$ of X_K^0 such that $\Phi \not\equiv 0 \bmod p\Lambda_G$. By Lemma 2.5, there is a subgroup H of G such that $G/H \simeq \mathbb{Z}_p$ and that X_K^0 is finitely generated over Λ_H. By Nakayama's lemma and Lemma 2.6, there is $\sigma \in H \setminus H^p$ such that $X_K^0/(\sigma-1)X_K^0$ is a non-trivial pseudo-null $\Lambda_{G/\overline{\langle\sigma\rangle}}$-module. Let $K^{\overline{\langle\sigma\rangle}}$ be the fixed field of σ in K. By Lemma 2.8, $X_K^0/(\sigma-1)X_K^0 \to X_K/(\sigma-1)X_K \simeq X_{K^{\overline{\langle\sigma\rangle}}}$ is injective. Hence we have $X_{K^{\overline{\langle\sigma\rangle}}}^0 \neq 0$. Let K^H be the fixed field of H. Then one sees that K^H/k is a $\mathbb{Z}_p$-extension. By doing the same arguments, we can find a $\mathbb{Z}_p^2$-extension L/k such that $X_L^0 \neq 0$ and $K^H \subseteq L$. Next we consider the $\mathbb{Z}_p^2$-extension L/k. Put $U = \mathrm{Gal}(L/k)$, and let V_0 be a closed subgroup of U such that $U/V_0 \simeq \mathbb{Z}_p$, and $E \subseteq L$ the fixed field of V_0. Let γ be a topological generator of $\mathrm{Gal}(E/k) = U/V_0$. From the condition A, it holds that $A_k \simeq X_E/(\gamma-1)X_E$, and hence $X_E/(\gamma-1)X_E$ is finite. This shows that a generator of the characteristic ideal of X_E over Λ_{U/V_0} and $\gamma - 1$ are relatively prime. By Lemma 2.8, the map $(X_L^0)_{V_0} \to (X_L)_{V_0} \simeq X_E$ is injective. Hence a generator of the characteristic ideal of $(X_L^0)_{V_0}$ and $\gamma-1$ are also relatively prime. This shows that $(X_L^0)_U = (X_L^0)_{V_0}/(\gamma-1)(X_L^0)_{V_0}$ is finite. By Lemma 2.9, there is a subgroup $V \subseteq U$ such that $U/V \simeq \mathbb{Z}_p$ with the property that $(X_L^0)_V$ has a non-trivial finite submodule. Let k_∞ be the fixed field of V in L. Then it follows that $X_{k_\infty}^0 \neq 0$. By Theorem 1.1, $\mathrm{Ker}(A_{k_n} \to A_{k_\infty}) \neq 0$ for some $n \geq 0$. Since $\mathrm{Ker}(A_{k_n} \to A_{k_\infty}) \subseteq \mathrm{Ker}(A_{k_n} \to A_K)$, this completes the proof. □

Remark. In fact, from Theorem 2 of **[16]**, one can further see that $(X_L^0)_V$ is finite for all but finite subgroups V of U such that $U/V \simeq \mathbb{Z}_p$.

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