NOETHERIAN ENVELOPING ALGEBRAS OF SIMPLE GRADED LIE ALGEBRAS

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Abstract. It is shown that if the universal enveloping algebra of a simple $\mathbb{Z}^n$-graded Lie algebra is Noetherian, then the Lie algebra is finite-dimensional.

1. Introduction

Let K be an algebraically closed field of characteristic 0. If a Lie algebra is finite-dimensional, then its enveloping algebra is Noetherian (i.e. Noetherian on the right or on the left, which is equivalent for the enveloping algebras). Whether the converse is true has been asked by many authors, among them R. Amayo and I. Stewart, see [1, Question 27], K. A. Brown, see [3, Question B], J. Dixmier, and V. Latyshev. Besides its intrinsic interest, this is an unavoidable question in the problem of the classification of Noetherian Hopf algebras. S. Sierra and C. Walton stated this question as a Conjecture.

Conjecture 1.1. [10] *The universal enveloping algebra of an infinite-dimensional Lie algebra is not Noetherian.*

Intuitively, since 'large' Lie algebras satisfy the Conjecture, e.g. the enveloping algebra of a free Lie algebra in two generators is not Noetherian, one expects that a counterexample to the Conjecture, if any, should be in some sense 'small'. In this direction, a breakthrough result was obtained in 2013 by Sierra and Walton. Recall that the *Witt algebra* is $W(1) := \operatorname{Der} K[t]$.

Theorem 1.2. [10, Theorem 0.5] *The enveloping algebra of $W(1)$ is not Noetherian.*

This result allows us to conclude that the enveloping algebra of an infinite-dimensional simple $\mathbb{Z}$-graded Lie algebra of finite growth is not Noetherian, by going over the classification of such Lie algebras obtained in [9].

However there are neither classification results for the simple $\mathbb{Z}$-graded Lie algebras of arbitrary growth, nor for the simple $\mathbb{Z}^n$-graded Lie algebras for $n \geq 2$. Nevertheless, the following result will be established.

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Theorem 1.3. *The universal enveloping algebra of an infinite-dimensional simple $\mathbb{Z}^n$-graded Lie algebra is not Noetherian.*

According to our convention below, it is assumed in Theorem 1.3 that *all dimensions of the homogeneous components of the $\mathbb{Z}^n$-graded Lie algebras are finite.*

The proof of Theorem 1.3 is divided into four parts (three of them involve the case $n = 1$). Some parts use concrete results, namely the Theorem of Sierra and Walton and the classification results of the second author [9].

2. Conventions and Preliminaries

2.1. *Conventions about graded vector spaces.*

In the whole paper, we will adopt the following convention. A vector space M endowed with a decomposition $M = \oplus_{\mathbf{m}\in\mathbb{Z}^n} M_{\mathbf{m}}$ will be called a $\mathbb{Z}^n$*-graded* vector space *only if* all homogeneous components $M_{\mathbf{m}}$ are finite-dimensional.

A Lie algebra $\mathcal{L}$ endowed with a $\mathbb{Z}^n$-grading is called a $\mathbb{Z}^n$*-graded Lie algebra* if we have

$$[\mathcal{L}_{\mathbf{n}}, \mathcal{L}_{\mathbf{m}}] \subset \mathcal{L}_{\mathbf{n}+\mathbf{m}} \qquad \text{for any } \mathbf{n}, \mathbf{m} \in \mathbb{Z}^{\mathbf{n}}.$$

A $\mathbb{Z}^n$-graded Lie algebra $\mathcal{L}$ of dimension ≥ 2 without nontrivial proper $\mathbb{Z}^n$-graded ideals is called a *simple $\mathbb{Z}^n$-graded* Lie algebra. For example, $\mathfrak{sl}(2) \otimes K[t, t^{-1}]$ is a simple $\mathbb{Z}$-graded Lie algebra, but it is not simple as a Lie algebra. The definitions of a *$\mathbb{Z}^n$-graded $\mathcal{L}$-module* and a *simple $\mathbb{Z}^n$-graded $\mathcal{L}$-module* are similar.

2.2. *Criteria for Noetherianity of enveloping algebras.*

A *section* of a Lie algebra L is a Lie algebra $\mathfrak{s}$ isomorphic to $\mathfrak{q}/\mathfrak{m}$ for some Lie subalgebra $\mathfrak{q} \subset L$ and some ideal $\mathfrak{m}$ of $\mathfrak{q}$.

The following standard observations are useful, see [10, Lemma 1.7] and [4, Proposition 2.1].

Lemma 2.1. *Let L be a Lie algebra such that $U(L)$ is Noetherian.*

(a) *L satisfies the ascending chain condition on Lie subalgebras.*
(b) *L is finitely presented and $H_k(L)$ is finite-dimensional for any $k \geq 0$.*
(c) *If $\mathfrak{s}$ is a section of L, then $U(\mathfrak{s})$ is also Noetherian.*
(d) *If $\mathfrak{s}$ is an abelian section of L, then* $\dim \mathfrak{s} < \infty$.
(e) *If L is a Lie subalgebra of finite codimension of some Lie algebra L', then $U(L')$ is also Noetherian.* □

2.3. *Examples of enveloping algebras that are not Noetherian.*

Lemma 2.1 allows us to deduce that many Lie algebras satisfy Conjecture 1.1 from Lie algebras that are already known to fulfill it, for instance:

(i) The free Lie algebra Free(Z) on a vector space Z of dimension ≥ 2. Indeed $U(\text{Free}(Z)) \simeq T(Z)$ is not Noetherian.

(ii) [10, Theorem 0.5] The positive Witt algebra W_+. By Lemma 2.1(e), this result is equivalent to the remarkable Theorem 1.2.

See [10], [4] for a list of Lie algebras whose enveloping algebras are not Noetherian by the remarks above. By Lemma 2.1 (c) and (i), another example is a Kac–Moody algebra of indefinite type, cf. [6, Corollary 9.12].

3. Growth of modules over $\mathbb{Z}$-graded Lie algebras

In this section and in the next three, we investigate the Noetherianity condition for $\mathbb{Z}$-graded Lie algebras. The present section involves the questions of finite generation and growth.

Given a $\mathbb{Z}$-graded vector space M and an integer $n \in \mathbb{Z}$, we set

$$M_{\geq n} := \oplus_{k \geq n} M_k.$$

The subspaces $M_{>n}$, $M_{\leq n}$ and $M_{<n}$ are similarly defined.

3.1. *Finite generation.*

Let $\mathcal{L}$ be a $\mathbb{Z}$-graded Lie algebra. We set

$$\mathcal{L}^+ = \mathcal{L}_{>0} \quad \text{and} \quad \mathcal{L}^- = \mathcal{L}_{<0}.$$

Lemma 3.1. *Assume that the Lie algebra $\mathcal{L}$ is finitely generated. Then $\mathcal{L}^+$ and $\mathcal{L}^-$ are finitely generated subalgebras.*

Moreover let M be a finitely generated $\mathbb{Z}$-graded $\mathcal{L}$-module. Then the $\mathcal{L}^+$-module $M_{\geq 0}$ and the $\mathcal{L}^-$-module $M_{\leq 0}$ are finitely generated.

Proof. By hypothesis, there is an integer $d > 0$ such that $\oplus_{-d \leq k \leq d} \mathcal{L}_k$ generates $\mathcal{L}$. By Lemma 18 of [8], $\oplus_{1 \leq k \leq d} \mathcal{L}_k$ generates $\mathcal{L}^+$ and $\oplus_{-d \leq k \leq -1} \mathcal{L}_k$ generates $\mathcal{L}^-$, which proves the first assertion.

Let S be a finite set of generators of M. There is an integer e such that S lies in $M_{\leq e}$. Since $M_{\leq e}$ is a $\mathcal{L}_{\leq 0}$-module, we have $M = U(\mathcal{L}^+).M_{\leq e}$. Since in addition $\mathcal{L}^+$ is generated by $\oplus_{1 \leq k \leq d} \mathcal{L}_k$, we have

$$M_n = \sum_{1 \leq k \leq d} \mathcal{L}_k.M_{n-k},$$

for any $n > e$. It follows easily that the $\mathcal{L}^+$-module $M_{\geq 0}$ is finitely generated. The proof of the finite generation of the $\mathcal{L}^-$-module $M_{\leq 0}$ is identical. □

3.2. *Finite and intermediate growth.*

A $\mathbb{Z}$-graded vector space M is called of *finite growth* if the function

$$n \mapsto \dim M_n$$

is bounded by a polynomial. It is called of *intermediate growth* if both limits

$$\limsup \frac{\log^+(\dim M_n)}{n} \qquad \text{and} \qquad \limsup \frac{\log^+(\dim M_{-n})}{n}$$

are zero, where the function $\log^+$ is defined by $\log^+(x) = \log(x)$ if $x \geq 1$ and $\log^+(x) = 0$ otherwise. The formal series

$$\chi_M^{\pm}(z) := \sum_{n\geq 0} \dim\, M_{\pm n}\, z^n$$

are called the *two generating series* of M. Equivalently, M has intermediate growth iff both series $\chi_M^+(z)$ and $\chi_M^-(z)$ are convergent for $|z| < 1$.

Assume now that $M = \oplus_{n\geq 1} M_n$ is a positively graded vector space. Then the symmetric algebra $S(M)$ is a nonnegatively graded vector space. The following lemma is well-known.

Lemma 3.2. *Assume that the positively graded vector space M has intermediate growth. Then $S(M)$ also has intermediate growth.*

Proof. For any integer $n \geq 1$, set $a_n = \dim\, M_n$. We have

$$\chi_M^+(z) = \sum_{n\geq 1} a_n z^n, \qquad \chi_{S(M)}^+(z) = \prod_{n\geq 1} \frac{1}{(1-z^n)^{a_n}}.$$

If M is finite-dimensional, $S(M)$ has finite growth. Otherwise, the lemma follows because these series have the same radius of convergence. □

3.3. *Growth of $\mathbb{Z}$-graded $\mathcal{L}$-modules.*

Let $\mathcal{L}$ be a $\mathbb{Z}$-graded Lie algebra and let M be a $\mathbb{Z}$-graded $\mathcal{L}$-module. For any n, let M_n^{int} be the subspace of all $m \in M_n$ such that $U(\mathcal{L}^+).m$ has intermediate growth. Set $M^{\mathrm{int}} = \oplus_{n\in\mathbb{Z}} M_n^{\mathrm{int}}$.

Lemma 3.3. *The subspace M^{int} is a $\mathcal{L}$-submodule.*

Proof. Since M^{int} is clearly a $\mathcal{L}^+$-module, it is enough to show that for any homogeneous elements $u \in \mathcal{L}$ of degree $d \leq 0$ and $v \in M^{\mathrm{int}}$, $u.v$ belongs to M^{int}. First note that

$$U(\mathcal{L}^+)u \subset U(\mathcal{L}^+)\mathcal{L}_{\geq d} = \mathcal{L}_{\geq d} U(\mathcal{L}^+) = U(\mathcal{L}^+) \oplus \oplus_{d\leq k\leq 0}\, \mathcal{L}_k U(\mathcal{L}^+).$$

Therefore we have

$$U(\mathcal{L}^+)u.v \subset U(\mathcal{L}^+).v + \sum_{d\leq k\leq 0} \mathcal{L}_k U(\mathcal{L}^+).v.$$

Thus $U(\mathcal{L}^+)u.v$ has intermediate growth, i.e. $u.v$ belongs to M^{int}. □

Lemma 3.4. *Let $\mathcal{L}$ be a finitely generated $\mathbb{Z}$-graded Lie algebra and let M be a simple $\mathbb{Z}$-graded $\mathcal{L}$-module. Assume that, for some homogeneous $v \in M \setminus 0$, the vector space $\mathcal{L}.v$ has intermediate growth. Then M has intermediate growth.*

Proof. Let $\mathcal{K}^+ = \{x \in \mathcal{L}^+ \mid x.v = 0\}$. As a graded space, the $\mathcal{L}^+$-module $\operatorname{Ind}_{\mathcal{K}^+}^{\mathcal{L}^+} Kv$ is isomorphic to $S(\mathcal{L}^+/\mathcal{K}^+)$. By Lemma 3.2, $\operatorname{Ind}_{\mathcal{K}^+}^{\mathcal{L}^+} Kv$ has intermediate growth. Thus $U(\mathcal{L}^+).v$, a quotient of $\operatorname{Ind}_{\mathcal{K}^+}^{\mathcal{L}^+} Kv$, has intermediate growth too.

Since M is simple, from Lemma 3.3 we infer that any cyclic $U(\mathcal{L}^+)$-submodule of M has intermediate growth. Now the $\mathcal{L}^+$-module $M_{\geq 0}$ is finitely generated by Lemma 3.1, hence $M_{\geq 0}$ has intermediate growth. Similarly $M_{\leq 0}$ has intermediate growth; therefore M has intermediate growth. □

4. Rank one Lie algebras of class $\mathscr{V}$

We define first the general notions of roots and rank of a $\mathbb{Z}$-graded Lie algebra $\mathcal{L}$. Then we split the proof that $U(\mathcal{L})$ is not Noetherian into three parts: Lie algebras of class $\mathscr{V}$ are treated in this section; the next section 5 is devoted to class $\mathscr{S}$; the last section 6 deals with the Lie algebras of rank ≥ 2.

4.1. *Roots and rank.*

Let $\mathcal{L} = \oplus_{n\in\mathbb{Z}}\mathcal{L}_n$ be a $\mathbb{Z}$-graded Lie algebra. We fix, once and for all, a Cartan subalgebra $\mathfrak{h}$ of $\mathcal{L}_0$, i.e., $\mathfrak{h}$ is a nilpotent self-normalizing subalgebra of $\mathcal{L}_0$ [2]. For any $\alpha \in \mathfrak{h}^*$ and any $n \in \mathbb{Z}$, we set

$$\mathcal{L}_n^\alpha = \{x \in \mathcal{L}_n \mid (\operatorname{ad}(h) - \alpha(h))^N(x) = 0 \quad \forall h \in \mathfrak{h} \text{ and } N \gg 0\}.$$

Also, $\mathcal{L}^{\widetilde{\alpha}} := \mathcal{L}_n^\alpha$ for $\widetilde{\alpha} = (\alpha, n) \in \mathfrak{h}^* \times \mathbb{Z}$. The *set of roots* of $\mathcal{L}$ is the set $\Delta := \{\widetilde{\alpha} \mid \mathcal{L}^{\widetilde{\alpha}} \neq 0\}$ (with our nonstandard definition, $(0, 0)$ is a root whenever $\mathcal{L}_0 \neq 0$). Therefore

$$\mathcal{L} = \oplus_{\widetilde{\alpha}\in\Delta}\, \mathcal{L}^{\widetilde{\alpha}}$$

is the generalized root space decomposition of $\mathcal{L}$.

A root $\widetilde{\alpha} = (\alpha, n)$ is called *real* if $\alpha \neq 0$ and *imaginary* otherwise. Let Δ_{re}, respectively Δ_{im}, be the set of real, respectively imaginary, roots.

The *root lattice* is the subgroup $Q \subset \mathfrak{h}^* \times \mathbb{Z}$ generated by Δ. By definition the *rank* of $\mathcal{L}$ is the rank of Q.

Remark 4.1. It is proved in [2] that any two Cartan subalgebras of $\mathcal{L}_0$ are conjugated by an automorphism of $\mathcal{L}_0$. In fact the proof of [2] shows that they are conjugated by a degree-preserving automorphism of $\mathcal{L}$. Therefore the root lattice is independent of the choice of a Cartan subalgebra of $\mathcal{L}_0$. Since we do not need this fact, we will not provide more details.

4.2. *Rank one Lie algebras.*

Let $\mathcal{L} = \oplus_{n\in\mathbb{Z}}\mathcal{L}_n$ be a $\mathbb{Z}$-graded Lie algebra of rank one. Assume that $\mathcal{L} \neq \mathcal{L}_0$. Then $\mathcal{L}_0 = \mathfrak{h}$ is a nilpotent Lie algebra and there exists $\widetilde{\alpha} = (\alpha, 1) \in \mathcal{L}_0^* \times \mathbb{Z}$ such that Δ lies in $\mathbb{Z}.\widetilde{\alpha}$. We keep the terminology of [9]. When $\alpha = 0$ or, equivalently, when the set of real roots is void, we say that $\mathcal{L}$ belongs to the class $\mathscr{V}$ (for the class $\mathscr{V}$, the Lie algebra $\mathcal{L}_0$ could be 0).

Otherwise, we say that $\mathcal{L}$ belongs to the class $\mathscr{S}$. Here the letter $\mathscr{S}$ stands for string, because, roughly speaking, all real roots are on a "string".

4.3. *Rank one Lie algebras of class $\mathscr{V}$.*

This case follows easily from the next result.

Lemma 4.2. [7, Lemma 22] *Let $\mathcal{L}$ be a $\mathbb{Z}$-graded Lie algebra of class $\mathscr{V}$. If $\mathcal{L} = [\mathcal{L}, \mathcal{L}]$, then $\mathcal{L}$ is not finitely generated.* □

Corollary 4.3. *Let $\mathcal{L}$ be a simple $\mathbb{Z}$-graded Lie algebra of class $\mathscr{V}$. Then $U(\mathcal{L})$ is not Noetherian.*

Proof. Immediate from Lemmas 2.1(b) and 4.2. □

5. Rank one Lie algebras of class $\mathscr{S}$

The case of Lie algebras of class $\mathscr{S}$ is more difficult than the previous one. Recall that a $\mathbb{Z}$-graded Lie algebra $\mathcal{L}$ belongs to $\mathscr{S}$ if $\mathcal{L} \neq \mathcal{L}_0$, $\mathcal{L}_0$ is nilpotent and there exists a nonzero $\alpha \in \mathcal{L}_0^*$ such that $\mathcal{L}_n = \mathcal{L}_n^{n\alpha}$ for any $n \in \mathbb{Z}$.

The main step is Theorem 5.8, which is implicit in [9]. Navigating through chapters 7 and 8 of *loc. cit.* is not easy. Thus for the sake of the reader, we rewrite parts of those in a convenient way.

We need the following definition. For $n \neq 0$, let $\mathcal{L}\{n\}$ be the Lie algebra $\mathcal{L}$ endowed with a grading rescaled by a factor of n, i.e. we have

$$\mathcal{L}\{n\}_{nk} = \mathcal{L}_k, \quad k \in \mathbb{Z}, \qquad\qquad \mathcal{L}\{n\}_m = 0 \text{ if } n \not| m.$$

The $\mathbb{Z}$-graded Lie algebra $\mathcal{L}\{n\}$, again in class $\mathscr{S}$, is called a *rescaling of* $\mathcal{L}$.

5.1. *Local Lie algebras.*

Let P be the set of pairs of integers (i, j) with $i, j, i+j \in \{-1, 0, 1\}$. Following [5], see [6, Exercise 1.8, p. 13], a *local Lie algebra* is a graded vector space

$$G = G_{-1} \oplus G_0 \oplus G_1$$

endowed with a degree preserving bracket $[\,,\,]$ which is defined only on $\cup_{(i,j)\in P} G_i \times G_j$ and which satisfies the Jacobi identity whenever it makes sense. Equivalently, this means that G_0 is a Lie algebra, G_1 and G_{-1} are G_0-modules and the bracket $[,] : G_{-1} \times G_1 \to G_0$ is G_0-equivariant.

The notions of morphisms between local Lie algebras, local Lie subalgebras and local ideals are defined in an evident way. Analogously a local Lie algebra S is a called a *section* of G if S is isomorphic to H/K for some local subalgebra $H \subset G$ and some local ideal K of H.

Given a $\mathbb{Z}$-graded Lie algebra $\mathcal{L}$, its *local part*

$$\mathcal{L}_{loc} := \mathcal{L}_{-1} \oplus \mathcal{L}_0 \oplus \mathcal{L}_1$$

is evidently a local Lie algebra. Conversely, given a local Lie algebra G there are $\mathbb{Z}$-graded Lie algebras whose local part is G. One of them, denoted by $\mathcal{L}_{\max}(G)$, is defined as follows. As a vector space we have

$$\mathcal{L}_{\max}(G) = \mathrm{Free}(G_{-1}) \oplus G_0 \oplus \mathrm{Free}(G_1)$$

where $\mathrm{Free}(G_{\pm1})$ denotes the free Lie algebra on the vector space $G_{\pm1}$. Then the local Lie bracket and the $\mathbb{Z}$-grading extend uniquely to $\mathcal{L}_{\max}(G)$ [5]. Indeed the functor $G \to \mathcal{L}_{\max}(G)$ is the left adjoint of the functor $\mathcal{L} \to \mathcal{L}_{loc}$ [8]. Let $\mathcal{I}$ be the largest graded ideal of $\mathcal{L}_{\max}(G)$ such that $\mathcal{I} \cap G = 0$ and set

$$\mathcal{L}_{\min}(G) = \mathcal{L}_{\max}(G)/\mathcal{I}.$$

Notice that, if $\mathcal{L}$ is a Lie $\mathbb{Z}$-graded Lie algebra which is generated by its local part G, then there are natural epimorphisms

$$\mathcal{L}_{\max}(G) \twoheadrightarrow \mathcal{L} \qquad \text{and} \qquad \mathcal{L} \twoheadrightarrow \mathcal{L}_{\min}(G),$$

so $\mathcal{L}$ is between the Lie algebras $\mathcal{L}_{\max}(G)$ and $\mathcal{L}_{\min}(G)$. We conclude:

Lemma 5.1. *Let G be a local Lie algebra and let $\mathcal{L}$ be a $\mathbb{Z}$-graded Lie algebra. If G is a section of $\mathcal{L}_{loc}$, then $\mathcal{L}_{\min}(G)$ is a section of $\mathcal{L}$.* □

5.2. *Four basic simple Lie algebras of class $\mathscr{S}$.*

We start recalling the definitions of some Lie algebras of class $\mathscr{S}$.

- The *centerless Virasoro algebra* is $W = \mathrm{Der}\, K[t, t^{-1}]$. It has a natural grading, relative to which the element $L_n := t^{n+1}\frac{\mathrm{d}}{\mathrm{d}t}$ is homogeneous of degree n. We have $\mathfrak{h} = K.L_0$.
- The *Witt algebra* is $W(1) = \mathrm{Der}\, K[t]$; it is a graded subalgebra of W.
- The Lie algebra $\mathfrak{sl}(2)$; it is the Lie subalgebra of W with basis $\{L_{-1}, L_0, L_1\}$.
- The contragredient Lie algebra $G(\begin{smallmatrix}2&2\\2&2\end{smallmatrix})$. It is generated by five elements h, e_1, e_2, f_1, f_2 and defined by the following relations

$$[h, e_i] = 2e_i, \qquad [h, f_i] = -2f_i, \qquad [e_i, f_j] = \delta_{i,j}\, h, \tag{1}$$

  for any $i, j \in \{1, 2\}$, where, as usual, $\delta_{i,j}$ is the Kronecker symbol. It has a $\mathbb{Z}$-grading relative to which the e_i's have degree one, h has degree zero and the f_i's have degree -1.

These four $\mathbb{Z}$-graded Lie algebras and their rescalings are simple Lie algebras of class $\mathscr{S}$ (for the simplicity of $G(\begin{smallmatrix}2&2\\2&2\end{smallmatrix})$ see [5]). They play a central role in what follows.

5.3. *Non-Abelian free subalgebras of $\mathbb{Z}$-graded Lie algebras.*

Let $G(\begin{smallmatrix}2&2\\2&2\end{smallmatrix})_{loc}$ be the local part of the Lie algebra $G(\begin{smallmatrix}2&2\\2&2\end{smallmatrix})$. Since $G(\begin{smallmatrix}2&2\\2&2\end{smallmatrix})$ is generated by its local part and is defined by local relations, we have

$$G(\begin{smallmatrix}2&2\\2&2\end{smallmatrix}) = \mathcal{L}_{\max}(G(\begin{smallmatrix}2&2\\2&2\end{smallmatrix})_{loc}).$$

Lemma 5.2. *We have* $G(\begin{smallmatrix}2&2\\2&2\end{smallmatrix}) = \mathcal{L}_{\min}(G(\begin{smallmatrix}2&2\\2&2\end{smallmatrix})_{loc})$.

Proof. This follows because the Lie algebra $G(\begin{smallmatrix}2&2\\2&2\end{smallmatrix})$ is simple [5]. □

Lemma 5.3. *Let $\mathcal{L}$ be a $\mathbb{Z}$-graded Lie algebra. If $G(\begin{smallmatrix}2&2\\2&2\end{smallmatrix})_{loc}$ is a section of $\mathcal{L}_{loc}$, the Lie algebra $\mathcal{L}$ contains a non-Abelian free Lie subalgebra.*

Proof. By Lemmas 5.1 and 5.2, $G(\begin{smallmatrix}2&2\\2&2\end{smallmatrix})$ is a section of $\mathcal{L}$. The Lie subalgebra of $G(\begin{smallmatrix}2&2\\2&2\end{smallmatrix})$ generated by e_1 and e_2 is free of rank two. Hence $\mathcal{L}$ admits a non-Abelian free section, which can be lifted to a Lie subalgebra of $\mathcal{L}$. □

5.4. *A simple criterion for a section isomorphic to* $G(\begin{smallmatrix}2&2\\2&2\end{smallmatrix})$.

Let $\mathcal{L}$ be a $\mathbb{Z}$-graded Lie algebra of class $\mathscr{S}$ with $\alpha \in \mathcal{L}_0^*$ as above. In this subsection and in the next two, we do not assume that $\mathcal{L}$ is simple as a $\mathbb{Z}$-graded algebra. We will describe criteria for the existence of a section of $\mathcal{L}$ isomorphic to $G(\begin{smallmatrix}2&2\\2&2\end{smallmatrix})$.

For $n \neq 0$, let $B_n : \mathcal{L}_{-n} \times \mathcal{L}_n \to K$ be the bilinear map

$$B_n : (x, y) \in \mathcal{L}_{-n} \times \mathcal{L}_n \mapsto \alpha([x, y]).$$

Let $\mathcal{K}_n$ and $\mathcal{K}_{-n}$ be its right kernel and its left kernel. Since $\alpha([\mathcal{L}_0, \mathcal{L}_0]) = 0$, the bilinear map B_n is $\mathcal{L}_0$-equivariant. Also $\mathcal{K}_0 := \operatorname{Ker} \alpha$ is an ideal of $\mathcal{L}_0$.

Given a finite-dimensional $\mathcal{L}_0$-module M, its *cosocle* is its maximal semi-simple quotient. It is denoted by Cosoc M. For any $\beta \in (\mathcal{L}_0/[\mathcal{L}_0, \mathcal{L}_0])^*$, let K_β be the one-dimensional $\mathcal{L}_0$-module on which any $h \in \mathcal{L}_0$ acts by multiplication by $\beta(h)$. Obviously, Cosoc $\mathcal{L}_n/\mathcal{K}_n$ is direct sum of copies of $K_{n\alpha}$.

Lemma 5.4. *Assume that* Cosoc $\mathcal{L}_n/\mathcal{K}_n$ *has dimension* ≥ 2 *for some* $n > 0$. *Then, up to a rescaling,* $G(\begin{smallmatrix}2&2\\2&2\end{smallmatrix})$ *is a section of* $\mathcal{L}$.

Proof. Since we did not assume that $\mathcal{L}$ is simple, we can assume that $n = 1$. By assumption, there is a $\mathcal{L}_0$-module $\mathcal{I}_1$ with $\mathcal{K}_1 \subset \mathcal{I}_1 \subset \mathcal{L}_1$ such that $\mathcal{L}_1/\mathcal{I}_1$ is isomorphic to $K_\alpha \oplus K_\alpha$. Let $\mathcal{L}'_{-1} \subset \mathcal{L}_{-1}$ be the orthogonal of $\mathcal{I}_1$ with respect to the bilinear map B_1.

By definition $\mathcal{L}'_{-1}$ contains $\mathcal{K}_{-1}$ and the $\mathcal{L}_0$-module $\mathcal{L}'_{-1}/\mathcal{K}_{-1}$ is isomorphic to $K_{-\alpha} \oplus K_{-\alpha}$. It follows that $\mathcal{I} := \mathcal{K}_{-1} \oplus \mathcal{K}_0 \oplus \mathcal{I}_1$ is a local ideal of the local Lie algebra $\mathcal{G} := \mathcal{L}'_{-1} \oplus \mathcal{L}_0 \oplus \mathcal{L}_1$. Since $\mathcal{G}/\mathcal{I}$ is clearly isomorphic to the local part of $G(\begin{smallmatrix}2&2\\2&2\end{smallmatrix})$, it follows from Lemmas 5.1 and 5.2 that $G(\begin{smallmatrix}2&2\\2&2\end{smallmatrix})$ is a section of $\mathcal{L}$. □

5.5. *Cyclic modules.*

Let $\mathfrak{g}$ be a Lie algebra and let M be a finite-dimensional $\mathfrak{g}$-module.

Lemma 5.5. *There is a polynomial P such that the dimension of any cyclic $U(\mathfrak{g})$-module in $M^{\otimes n}$ is bounded by $P(n)$, for any $n \geq 1$.*

Proof. For any positive integer n, let I_n be the annihilator of the $U(\mathfrak{g})$-module $M^{\otimes n}$. Since the coproduct is cocommutative, $U(\mathfrak{g})/I_n$ embeds into $H^0(S_n, \mathrm{End}(M)^{\otimes n})$, where the symmetric group S_n acts on $\mathrm{End}(M)^{\otimes n}$ by permutation of the factors. Thus for any cyclic $U(\mathfrak{g})$-submodule C of $M^{\otimes n}$, the inequality

$$\dim C \leq \dim U(\mathfrak{g})/I_n \leq \dim S^n(\mathrm{End}(M)) = \binom{n+(\dim M)^2-1}{n}$$

is a polynomial bound of degree $(\dim V)^2 - 1$. □

5.6. *An improved criterion for a section isomorphic to $G\binom{2\,2}{2\,2}$.*

Using the notation of the previous sections, we show that the conclusion of Lemma 5.4 holds with a weaker hypothesis.

Lemma 5.6. *Assume that $\mathcal{L}_n/\mathcal{K}_n$ has dimension ≥ 2 for some $n > 0$. Then, up to a rescaling, $G\binom{2\,2}{2\,2}$ is a section of $\mathcal{L}$.*

Proof. We can assume that $n = 1$ and that $\mathcal{L}$ is generated by its local part. This implies that $\mathcal{L}^+$ is generated by $\mathcal{L}_1$.

By Lemma 5.5, the dimensions of the cyclic $\mathcal{L}_0$-modules in $\mathcal{L}_1^{\otimes n}$ are bounded by a polynomial on n. Obviously, the same property holds for its quotient $\mathcal{L}_n/\mathcal{K}_n$. However, by Lemma 7.9 of [9], the function $n \mapsto \mathrm{rk}\ B_n = \dim \mathcal{L}_n/\mathcal{K}_n$ has infinite growth (i.e. it is not bounded by a polynomial). Hence, when n goes to ∞, the minimal number of generators of the $\mathcal{L}_0$-module $\mathcal{L}_n/\mathcal{K}_n$ is arbitrarily large. Thus the function $n \mapsto \dim\ \mathrm{Cosoc}(\mathcal{L}_n/\mathcal{K}_n)$ is unbounded.

Therefore, for some n, we have

$$\dim\ \mathrm{Cosoc}(\mathcal{L}_n/\mathcal{K}_n) \geq 2.$$

Thus by Lemma 5.4, $G\binom{2\,2}{2\,2}$ is a section of $\mathcal{L}$. □

5.7. *The dichotomy for the class $\mathscr{S}$.*

From now on, we assume that the Lie algebra $\mathcal{L}$ of class $\mathscr{S}$ is simple. It is implicitly proved in [9, Chapter 8] that $\mathcal{L}$ is isomorphic to W or $W(1)$, under the hypothesis

($\mathcal{H}_1$) all bilinear forms B_n have rank ≤ 1.

Unfortunately, the explicit hypothesis used in [9, Chapter 8] is

($\mathcal{H}_2$) the Lie algebra $\mathcal{L}$ has intermediate growth.

It would be long to go into the details of *loc. cit.* to explain why ($\mathcal{H}_1$) can be used instead of ($\mathcal{H}_2$). Here we can assume that $\mathcal{L}$ is finitely generated. Under this additional hypothesis, the next lemma gives an easy explanation.

Lemma 5.7. *If $\mathcal{L}$ is finitely generated, then $(\mathcal{H}_1)$ implies $(\mathcal{H}_2)$.*

Proof. Let $M := \oplus_{n\in\mathbb{Z}} \mathcal{L}_n^*$ be the graded dual of the adjoint module. The hypothesis $(\mathcal{H}_1)$ means that the $\mathbb{Z}$-graded space $\mathcal{L}.\alpha$ has homogenous components of dimension ≤ 1. By Lemma 3.4, we see that M has intermediate growth, i.e. $(\mathcal{H}_2)$ holds. □

The following result is implicitly proved in [9], even without the hypothesis of finite generation.

Theorem 5.8. *Let $\mathcal{L}$ be a simple $\mathbb{Z}$-graded Lie algebra of class $\mathscr{S}$. Assume that $\mathcal{L}$ is finitely generated. Then*

(i) *either $\mathcal{L}$ is isomorphic to $\mathfrak{sl}(2)$, $W(1)$ or W,*
(ii) *or $\mathcal{L}$ contains a nonabelian free Lie algebra.*

Proof. (i) First assume that the bilinear form B_n has rank ≤ 1 for any n. By Lemma 5.7, $\mathcal{L}$ has intermediate growth. Thus it follows from Proposition 8.9 of [9] that $\mathcal{L}$ is isomorphic to $\mathfrak{sl}(2)$, $W(1)$ or W.

(ii) Otherwise, the bilinear form has rank ≥ 2 for some n. By Lemma 5.6 the Lie algebra $G\binom{2\,2}{2\,2}$ is a section of $\mathcal{L}$. By Lemma 5.3, the Lie algebra $\mathcal{L}$ contains a nonabelian free Lie algebra. □

Corollary 5.9. *Let $\mathcal{L}$ be a simple $\mathbb{Z}$-graded Lie algebra of class $\mathscr{S}$. Then $U(\mathcal{L})$ is not Noetherian, except if $\mathcal{L}$ is isomorphic to $\mathfrak{sl}(2)$.*

Proof. Assume that $\mathcal{L}$ is infinite-dimensional. By Lemma 2.1(b), we can assume that $\mathcal{L}$ is finitely generated. By Theorem 5.8, $\mathcal{L}$ contains a subalgebra isomorphic to the Witt algebra $W(1)$ or a nonabelian free subalgebra. In the first case, Theorem 1.2 implies that $U(\mathcal{L})$ is not Noetherian. In the second case, we already observed that the enveloping algebra of a nonabelian free Lie algebra is not Noetherian, so neither is $U(\mathcal{L})$. □

6. $\mathbb{Z}$-graded Lie algebras of rank ≥ 2

In this section we investigate the Noetherianity condition for $\mathbb{Z}$-graded Lie algebras of rank ≥ 2.

6.1. *Weakly $\mathbb{Z}$-graded Lie algebras.*

We will encounter Lie algebras $\mathcal{M}$ with a decomposition $\mathcal{M} = \oplus\mathcal{M}_n$ satisfying $[\mathcal{M}_n, \mathcal{M}_m] \subset \mathcal{M}_{n+m}$ where the homogeneous components could be of infinite dimension; we shall call them *weakly $\mathbb{Z}$-graded* Lie algebras.

Lemma 6.1. *Let $\mathcal{L} = \oplus_{n\in\mathbb{Z}} \mathcal{L}_n$ be a weakly $\mathbb{Z}$-graded Lie algebra such that $U(\mathcal{L})$ is Noetherian. Then $\mathcal{L}_n$ is finite-dimensional for any $n \in \mathbb{Z} \setminus 0$.*

Moreover if $\mathcal{L}$ is simple and $\mathcal{L} \neq \mathcal{L}_0$, then $\mathcal{L}_0$ is also finite-dimensional.

Proof. As before, set $\mathcal{L}^+ = \mathcal{L}_{>0}$ and $\mathcal{L}^- = \mathcal{L}_{<0}$.

By Lemma 2.1, $U(\mathcal{L}^+)$ is Noetherian, hence $\mathcal{L}^+$ is finitely generated. It follows easily that all homogeneous components of $\mathcal{L}^+$ are finite-dimensional.

Similarly, all homogenous components of $\mathcal{L}^-$ are finite-dimensional, which proves the first assertion.

Assume now that $\mathcal{L}$ is simple and $\mathcal{L} \neq \mathcal{L}_0$. Since $\mathcal{L}^+$ and $\mathcal{L}^-$ are finitely generated, there is an integer $d > 0$ such that $\oplus_{1\leq i\leq d} \mathcal{L}_i$ generates $\mathcal{L}^+$ and $\oplus_{1\leq i\leq d} \mathcal{L}_{-i}$ generates $\mathcal{L}^-$. Set

$$M = \oplus_{1\leq i\leq d} \mathcal{L}_i \bigoplus \oplus_{1\leq i\leq d} \mathcal{L}_{-i}$$

and let $\mathcal{K}$ be the annhilator in $\mathcal{L}_0$ of the $\mathcal{L}_0$-module M. Since M is a finite-dimensional $\mathcal{L}_0$-module, $\mathcal{L}_0/\mathcal{K} = 0$ is finite-dimensional. We have

$$[\mathcal{K}, \mathcal{L}^{\pm}] = 0, [\mathcal{K}, \mathcal{L}_0] \subset \mathcal{L}_0, \text{ and } \mathcal{K} \subset \mathcal{L}_0. \tag{2}$$

Hence $\mathcal{K}$ is a proper ideal of $\mathcal{L}$. Since $\mathcal{L}$ is simple, the ideal $\mathcal{K}$ is trivial. It follows that $\mathcal{L}_0$ is finite-dimensional. □

6.2. *The hypothesis* $(\mathcal{H})$.

Let $\mathcal{L}$ be a $\mathbb{Z}$-graded Lie algebra. Consider the following hypothesis

$(\mathcal{H})$ There exist $\widetilde{\alpha}, \widetilde{\beta} \in Q$, $\widetilde{\beta} \notin \mathbb{Q}.\widetilde{\alpha}$, such that $(\widetilde{\beta} + \mathbb{Z}.\widetilde{\alpha}) \cap \Delta$ is infinite.

Lemma 6.2. *If $\mathcal{L}$ satisfies the hypothesis $(\mathcal{H})$, then $U(\mathcal{L})$ is not Noetherian.*

Proof. For any integer $k \geq 1$, set $\Delta(k) = (k.\widetilde{\beta} + \mathbb{Z}.\widetilde{\alpha}) \cap \Delta$ and

$$\mathcal{M}_k = \oplus_{\widetilde{\gamma}\in\Delta(k)} \mathcal{L}^{\widetilde{\gamma}}.$$

Since we have $[\mathcal{M}_k, \mathcal{M}_l] \subset \mathcal{M}_{k+l}$ for any $k, l \geq 1$, the vector space

$$\mathcal{M} := \oplus_{k\geq 1} \mathcal{M}_k$$

is a weakly $\mathbb{Z}$-graded Lie algebra. Since $\widetilde{\alpha}$ and $\widetilde{\beta}$ are linearly independent over $\mathbb{Q}$, the sets $\Delta(k)$ are pairwise disjoint. Hence $\mathcal{M}$ is a Lie subalgebra of $\mathcal{L}$. Since $\mathcal{M}_1$ is infinite-dimensional, by Lemma 6.1 $U(\mathcal{M})$ is not Noetherian and by Lemma 2.1 $U(\mathcal{L})$ is not Noetherian. □

6.3. *Constructions of ideals in Lie algebras.*

The next two lemmas show that certain subspaces of a Lie algebra are indeed ideals. Results of this kind are useful in the study of simple Lie algebras.

Let $\mathcal{L}$ be a Lie algebra.

Lemma 6.3. [7, Lemma 6] *Let $\mathcal{A}$ and $\mathcal{B}$ be linear subspaces of $\mathcal{L}$ such that $\mathcal{L} = \mathcal{A} + \mathcal{B}$ and $[\mathcal{A}, \mathcal{B}] \subset \mathcal{B}$. Then $\mathcal{B} + [\mathcal{B}, \mathcal{B}]$ is an ideal of $\mathcal{L}$.* □

Let L be a linear subspace of $\mathcal{L}$. We say that an element $x \in \mathcal{L}$ is *locally L-nilpotent* if we have $\mathrm{Ad}(L)^{1+n}(x) = 0$ for $n \gg 0$. Let $\mathcal{N}$ be the space of all locally L-nilpotent elements and set $\mathcal{I} := \cap_{N\geq 0} \mathrm{Ad}(L)^N(\mathcal{L})$.

Lemma 6.4. *The subspace $\mathcal{N}$ is a Lie subalgebra and $\mathcal{I}$ is a $\mathcal{N}$-submodule. Consequently, if $\mathcal{N} = \mathcal{L}$, the subspace $\mathcal{I}$ is an ideal.*

Proof. Let $x, y \in \mathcal{L}$. For any $N \geq 0$, we have

(i) $\mathrm{Ad}(L)^N([x, y]) \subset \sum_{0\leq k\leq N}([\mathrm{Ad}(L)^k(x), \mathrm{Ad}(L)^{N-k}(y)])$,

(ii) $[x, \mathrm{Ad}(L)^N(y)] \subset \sum_{0\leq k\leq N} \mathrm{Ad}(L)^{N-k}([\mathrm{Ad}(L)^k(x), y])$.

The first identity shows that $[\mathcal{N},\mathcal{N}] \subset \mathcal{N}$, i.e. $\mathcal{N}$ is a Lie subalgebra. The second identity shows that $[\mathcal{N},\mathcal{I}] \subset \mathcal{I}$, i.e. $\mathcal{I}$ is a $\mathcal{N}$-submodule. □

6.4. *A dichotomy for the $\mathbb{Z}$-graded Lie algebras of rank ≥ 2.*

Let $\mathcal{L}$ be a simple $\mathbb{Z}$-graded Lie algebra of rank ≥ 2. We now define two hypothetical properties, and show that any such $\mathcal{L}$ satisfies one of them. By the end of the section it will be clear that these properties are mutually exclusive.

To start with, we define the notion of a string. Let $\widetilde{\alpha} \in Q$ and $\widetilde{\beta} \in \Delta$. There are $a, b \in \mathbb{Z} \cup \{\pm\infty\}$ with $a < 0 < b$ such that

(i) $\widetilde{\beta} + k\widetilde{\alpha}$ belongs to Δ for any $k \in]a, b[$, but

(ii) neither $\widetilde{\beta} + a\widetilde{\alpha}$ nor $\widetilde{\beta} + b\widetilde{\alpha}$ belongs to Δ.

The set $\{\widetilde{\beta} + k\widetilde{\alpha} \mid k \in]a, b[\}$ is called the *$\widetilde{\alpha}$-string through $\widetilde{\beta}$.*

The first hypothetical property $(\mathcal{H}_{\mathrm{re}})$ is the following:

$(\mathcal{H}_{\mathrm{re}})$ There exist $\widetilde{\alpha} \in \Delta_{\mathrm{re}}, \quad \widetilde{\beta} \in \Delta, \quad \widetilde{\beta} \notin \mathbb{Q}.\widetilde{\alpha}$, such that the $\widetilde{\alpha}$-string through $\widetilde{\beta}$ is infinite.

The hypothesis $(\mathcal{H}_{\mathrm{re}})$ is obviously stronger than $(\mathcal{H})$.

The second hypothetical property is the notion of weak integrability. Following [9], we say that $\mathcal{L}$ is *weakly integrable* if, for any $\widetilde{\alpha} \in \Delta_{\mathrm{re}}$, we have

$$\bigcap_{n\geq 0} \mathrm{Ad}(\mathcal{L}^{\widetilde{\alpha}})^n(\mathcal{L}) = 0.$$

Lemma 6.5. *Let $\mathcal{L}$ be a simple $\mathbb{Z}$-graded algebra of rank ≥ 2. Then either*

(a) *$\mathcal{L}$ satisfies the hypothesis $(\mathcal{H}_{\mathrm{re}})$, or*

(b) *$\mathcal{L}$ is weakly integrable.*

Proof. Assuming that $\mathcal{L}$ does not satisfy $(\mathcal{H}_{\mathrm{re}})$, we will prove that $\mathcal{L}$ is weakly integrable. Let $\widetilde{\alpha} \in \Delta_{\mathrm{re}}$, let $\mathcal{N}$ be the space of locally $\mathcal{L}^{\widetilde{\alpha}}$-nilpotent elements and set

$$\mathcal{I} = \cap_{N\geq 0} \mathrm{Ad}(\mathcal{L}^{\widetilde{\alpha}})^N(\mathcal{L}) \qquad \mathcal{A} = \oplus_{\widetilde{\beta}\in\mathbb{Q}.\widetilde{\alpha}} \mathcal{L}^{\widetilde{\beta}} \qquad \text{and} \qquad \mathcal{B} = \bigoplus_{\widetilde{\beta}\notin\mathbb{Q}.\widetilde{\alpha}} \mathcal{L}^{\widetilde{\beta}}.$$

First, we prove that $\mathcal{L} = \mathcal{N}$. For any $\widetilde{\beta} \notin \mathbb{Q}.\widetilde{\alpha}$, there is an integer $N > 0$ such that $\mathcal{L}^{\widetilde{\beta}+N\widetilde{\alpha}} = 0$. It follows that $\mathrm{Ad}^N(\mathcal{L}^{\widetilde{\alpha}})(\mathcal{L}^{\widetilde{\beta}}) = 0$. Therefore $\mathcal{N}$ contains $\mathcal{L}^{\widetilde{\beta}}$ for any $\widetilde{\beta} \notin \mathbb{Q}.\widetilde{\alpha}$, i.e. $\mathcal{N}$ contains $\mathcal{B}$. By Lemma 6.4, $\mathcal{N}$ contains $\mathcal{B} + [\mathcal{B},\mathcal{B}]$. Since $[\mathcal{A},\mathcal{B}] \subset \mathcal{B}$ and $\mathcal{L} = \mathcal{A} + \mathcal{B}$, the space $\mathcal{B} + [\mathcal{B},\mathcal{B}]$ is an ideal by Lemma 6.3. By simplicity of $\mathcal{L}$, we deduce that $\mathcal{B} + [\mathcal{B},\mathcal{B}] = \mathcal{L}$, and therefore we have $\mathcal{N} = \mathcal{L}$.

Next, we prove that $\mathcal{I} = 0$. Let $\widetilde{\beta}$ be a root which is not proportional to $\widetilde{\alpha}$. There is an integer $N > 0$ such that $\mathcal{L}^{\widetilde{\beta}-N\widetilde{\alpha}} = 0$. Hence $\mathcal{L}^{\widetilde{\beta}}$ is not contained in $\mathrm{Ad}^N(\mathcal{L}^{\widetilde{\alpha}})(\mathcal{L})$. It follows that $\mathcal{I}$ does not contain $\mathcal{L}^{\widetilde{\beta}}$. However, by Lemma 6.4, $\mathcal{I}$ is an ideal of $\mathcal{L}$. Hence we have $\mathcal{I} = 0$. In other words, $\mathcal{L}$ is weakly integrable. □

6.5. *Non-Noetherianity for $\mathbb{Z}$-graded Lie algebras of rank ≥ 2.*

Corollary 6.6. *Let $\mathcal{L}$ be a simple $\mathbb{Z}$-graded algebra of rank ≥ 2. If $U(\mathcal{L})$ is Noetherian, then $\mathcal{L}$ is finite-dimensional.*

Proof. By Lemma 6.5, $\mathcal{L}$ satisfies the hypothesis $(\mathcal{H}_{\mathrm{re}})$ or $\mathcal{L}$ is weakly integrable. In the first case, $U(\mathcal{L})$ is not Noetherian by Lemma 6.2.

In the latter case, $\mathcal{L}$ is isomorphic to an affine Lie algebra or it has finite dimension by [9, Theorem 4]. But if $\mathcal{L}$ is an affine Lie algebra, then it has an infinite-dimensional abelian subalgebra, hence $U(\mathcal{L})$ is not Noetherian. □

7. Proof of the Main Result

7.1. *The endomorphisms of simple $\mathbb{Z}^n$-graded modules.*

Let $\mathcal{L}$ be a $\mathbb{Z}^n$-graded Lie algebra and let M be a simple $\mathbb{Z}^n$-graded module.

Lemma 7.1. *If M is not simple (as a non-graded module), then there exists $\theta \in \mathrm{End}_{\mathcal{L}}(M)$ which is invertible and homogeneous of degree $\mathbf{p}$ for some $\mathbf{p} \in \mathbb{Z}^n \backslash 0$.*

Proof. Any $v \in M$ decomposes as $v = \sum_{\mathbf{m}} v_{\mathbf{m}}$ where $v_{\mathbf{m}} \in M_{\mathbf{m}}$. By definition the *support* of v is the set

$$\mathrm{supp}(v) := \{\mathbf{n} \in \mathbb{Z}^n \mid v_{\mathbf{n}} \neq 0\}.$$

Assume that M is not simple. Let $v \in M \setminus 0$ be the generator of a proper submodule with a support of minimal cardinality. Since M is simple as a $\mathbb{Z}^n$-graded module, v is not homogenous. Hence $\mathrm{supp}(v)$ contains distinct elements $\mathbf{m}, \mathbf{n}$.

We claim that $\mathrm{Ann}(v_{\mathbf{n}}) \subset \mathrm{Ann}(v_{\mathbf{m}})$, where $\mathrm{Ann}(m)$ denotes the annihilator of m in $U(\mathcal{L})$, for any $m \in M$. Since $\mathrm{Ann}(v_{\mathbf{n}})$ is $\mathbb{Z}^n$-graded, it is enough to show that any homogenous element $u \in \mathrm{Ann}(v_{\mathbf{n}})$ belongs to $\mathrm{Ann}(v_{\mathbf{m}})$. Since $u.v_{\mathbf{n}} = 0$, the support $u.v$ lies in $(\mathbf{d} + \mathrm{supp}(v)) \setminus \{\mathbf{d} + \mathbf{n}\}$, where $\mathbf{d}$ is the degree of u. By minimality of the cardinality of $\mathrm{supp}(v)$, we deduce that $u.v = 0$ which proves the claim.

Hence there exists $\theta \in \mathrm{End}_{\mathcal{L}}(M)$ mapping $v_{\mathbf{n}}$ to $v_{\mathbf{m}}$. Clearly, θ is homogeneous of degree $\mathbf{p} = \mathbf{m} - \mathbf{n} \in \mathbb{Z}^n$. Since $\mathrm{Ker}\,\theta$ and $\mathrm{Im}\,\theta$ are graded submodules, $\mathrm{Ker}\,\theta = 0$ and $\mathrm{Im}\,\theta = M$, hence θ is invertible. □

7.2. *Simple $\mathbb{Z}^n$-graded Lie algebras which are not simple.*

Let $\mathcal{L}$ be a $\mathbb{Z}^n$-graded Lie algebra. The algebra of endomorphisms of the adjoint module is called the *centroid* of $\mathcal{L}$.

Lemma 7.2. *If the simple $\mathbb{Z}^n$-graded Lie algebra $\mathcal{L}$ is not simple as a Lie algebra, then it contains an infinite-dimensional abelian subalgebra.*

Proof. By Lemma 7.1, there is an element $\theta \neq 0$ in the centroid which is homogeneous of degree $\mathbf{m} \in \mathbb{Z}^n \setminus 0$. Let $0 \neq x \in \mathcal{L}$ be an homogeneous element. Let $\mathfrak{m}$ be the linear span of $\{\theta^p(x) : p \in \mathbb{Z}\}$. For $p, q \in \mathbb{Z}$, we have

$$[\theta^p(x), \theta^q(x)] = \theta^{p+q}([x, x]) = 0,$$

hence $\mathfrak{m}$ is a abelian subalgebra. Moreover the elements $\theta^p(x)$ are nonzero elements of different degrees, hence $\mathfrak{m}$ is infinite-dimensional. $\square$

Corollary 7.3. *Assume that the simple $\mathbb{Z}^n$-graded Lie algebra $\mathcal{L}$ is not simple as a Lie algebra. Then $U(\mathcal{L})$ is not Noetherian.*

Proof. This is a consequence of Lemmas 7.2 and 2.1 (d). $\square$

7.3. *Proof of the main result.*

We can now prove the main Theorem of this article.

Theorem 1.3. *Let $\mathcal{L}$ be a simple $\mathbb{Z}^n$-graded Lie algebra of infinite dimension. Its enveloping algebra $U(\mathcal{L})$ is not Noetherian.*

Proof. We can assume that $\mathcal{L}$ is simple as a Lie algebra, otherwise $U(\mathcal{L})$ is not Noetherian by Corollary 7.3.

There exists $\mathbf{m} = (m_1, \ldots, m_n) \neq 0$ such that $\mathcal{L}_{\mathbf{m}} \neq 0$. Without loss of generality, we can assume that $m_1 \neq 0$. Define the weakly $\mathbb{Z}$-graded Lie algebra $\mathcal{L}'$ (which is $\mathcal{L}$ as Lie algebra) by the requirement that

$$\mathcal{L}'_m = \bigoplus_{(m_2,\ldots,m_n) \in \mathbb{Z}^{n-1}} \mathcal{L}_{(m,m_2,\ldots,m_n)}.$$

We can assume that all homogeneous components of $\mathcal{L}'$ are finite-dimensional, otherwise $U(\mathcal{L})$ is not Noetherian by Lemma 6.1.

Therefore $\mathcal{L}'$ is a simple $\mathbb{Z}$-graded Lie algebra. If $\mathcal{L}'$ has rank one, $U(\mathcal{L})$ is not Noetherian by Corollaries 4.3 and 5.9. Otherwise $\mathcal{L}'$ has rank ≥ 2 and $U(\mathcal{L})$ is not Noetherian by Corollary 6.6. $\square$

Remark 7.4. Let $\mathcal{L}$ be a $\mathbb{Z}^n$-graded Lie algebra. If $\mathcal{L}$ has a simple infinite-dimensional graded section, then Theorem 1.3 implies that $U(\mathcal{L})$ is not Noetherian. In other words, if $U(\mathcal{L})$ is Noetherian, then any simple graded section has finite dimension, in particular any maximal graded ideal has finite codimension.

Remark 7.5. In the theorem, we had assumed that all homogenous components of $\mathcal{L}$ are finite-dimensional. In fact we can replace this condition by the weaker hypothesis that $\mathcal{L} \neq \mathcal{L}_0$, see Lemma 6.1.

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