# LOW-LYING ZEROS OF SYMMETRIC POWER $L$-FUNCTIONS WEIGHTED BY SYMMETRIC SQUARE $L$-VALUES

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ABSTRACT. Given a totally real number field $F$ and its adèle ring $\mathbb{A}_F$, let $\pi$ vary in the set of irreducible cuspidal automorphic representations of $\mathrm{PGL}_2(\mathbb{A}_F)$ corresponding to primitive Hilbert modular forms of a fixed weight. We determine the symmetry type of the one-level density of low-lying zeros of the symmetric power $L$-functions $L(s, \mathrm{Sym}^r(\pi))$ weighted by special values of the symmetric square $L$-functions $L(\frac{z+1}{2}, \mathrm{Sym}^2(\pi))$ at $z \in [0,1]$ in the level aspect. If $0 < z \leqslant 1$, our weighted density in the level aspect has the same symmetry type as Ricotta and Royer's density of low-lying zeros of symmetric power $L$-functions for $F = \mathbb{Q}$ with harmonic weight. Hence our result is regarded as a $z$-interpolation of Ricotta and Royer's result. If $z = 0$, the density of low-lying zeros weighted by central values is of a different type only when $r = 2$.

## 1. INTRODUCTION

Studying zeros of $L$-functions is one of principal problems in number theory as it was originated by the Riemann hypothesis. However, as of now, the investigation of the non-trivial zeros of an individual $L$-function is still far from completion. Instead of individual $L$-functions, a family of $L$-functions is more tractable. Concerning zeros of $L$-functions in a family, Katz and Sarnak [26], [27] conjectured that the distribution of low-lying zeros of a family of $L$-functions should have a density function with symmetry type arising in random matrix theory. According to their conjecture, it is expected that the following five density functions should describe the density of low-lying zeros of $L$-functions in a family:

$$W(\mathrm{Sp})(x) = 1 - \frac{\sin 2\pi x}{2\pi x},$$
$$W(\mathrm{O})(x) = 1 + \frac{1}{2}\delta_0(x),$$
$$W(\mathrm{SO(even)})(x) = 1 + \frac{\sin 2\pi x}{2\pi x},$$
$$W(\mathrm{SO(odd)})(x) = 1 - \frac{\sin 2\pi x}{2\pi x} + \delta_0(x),$$
$$W(\mathrm{U})(x) = 1,$$

where $\delta_0$ is the Dirac delta distribution supported at 0. The conjecture was derived from their works [26] and [27] on function field cases, which study statistics of zeros for a geometric family of zeta functions for a function field over a finite field.

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Later, Iwaniec, Luo and Sarnak [25] studied densities of low-lying zeros of the standard $L$-functions and those of symmetric square $L$-functions associated to holomorphic elliptic cusp forms both in the weight aspect and in the level aspect, assuming GRH for the relevant $L$-functions. Inspired by their study, densities of low-lying zeros of families of automorphic $L$-functions have been investigated in several settings such as Hilbert modular forms ([34]), Siegel modular forms of degree 2 ([30], [31]) and Hecke-Maass forms ([1], [2], [22], [35], [37], [43]).

As of now, the broadest setting for low-lying zeros of automorphic $L$-functions was investigated by Shin and Templier [47]. They treated a general family of automorphic $L$-functions $L(s, \pi, r)$ and obtained the one-level density of low-lying zeros in the family both in the level aspect and in the weight aspect. In their work, $\pi$ varies in discrete automorphic representations of $G(\mathbb{A}_F)$, where $G$ is a connected reductive group over a number field $F$. Furthermore they assumed that $G$ admits discrete series representations at all archimedean places, and that $r : {}^L G \to \mathrm{GL}_d(\mathbb{C})$ is an irreducible $L$-morphism satisfying the hypothesis on the Langlands functoriality principle in their work. Before their study, Güloğlu [23] and Ricotta and Royer [44] had considered the functorial lifting corresponding to the symmetric tensor representation $\mathrm{Sym}^r : {}^L\mathrm{PGL}_2 \to \mathrm{GL}_{r+1}(\mathbb{C})$ for $r \in \mathbb{N}$ and gave densities of low-lying zeros of the symmetric power $L$-functions $L(s, \mathrm{Sym}^r(f))$ attached to holomorphic elliptic cusp forms $f$ in the weight aspect [23] with GRH and in the level aspect [44] without GRH, respectively, under the hypothesis on analytic properties of $L(s, \mathrm{Sym}^r(f))$.

Recently, Knightly and Reno [32] studied the density of low-lying zeros of the standard $L$-functions $L(s, f)$ attached to holomorphic elliptic cusp forms $f$ weighted by central values $L(1/2, f)$, and found the change of symmetry type of the density from $W(\mathrm{O})$ in the usual setting to $W(\mathrm{Sp})$ in the weighted setting. Their study was inspired by the works of Kowalski, Saha and Tsimerman [33] and of Dickson [13] on the asymptotic formula of the average of spinor $L$-functions $L(s, f, \mathrm{Spin})$ attached to holomorphic Siegel cusp forms $f$ of degree 2 weighted by the squares of the absolute values of the Fourier coefficients of $f$ at the unit matrix. In [33] and [13], they observed a phenomenon of changing the symmetry type from $W(\mathrm{O})$ to $W(\mathrm{Sp})$. They considered the phenomenon as weak evidence toward Böcherer's conjecture which was not proved at that time (now proved by Furusawa and Morimoto [19, Theorem 2]). Indeed, the Fourier coefficient of each $f$ in their weight factor is a Bessel period of $f$, and Böcherer's conjecture shows that the square of the absolute value of the Bessel period of $f$ is identical to $L(1/2, f, \mathrm{Spin})L(1/2, f \otimes \chi_{-4}, \mathrm{Spin})$, where $\chi_{-4}$ is the quadratic Dirichlet character modulo 4.

In this article, in order to observe phenomena of changes of densities by special $L$-values in other settings, we consider low-lying zeros of the symmetric power $L$-functions $L(s, \mathrm{Sym}^r(\pi))$ associated to cuspidal representations $\pi$ of $\mathrm{PGL}_2(\mathbb{A}_F)$ for a given totally real number field $F$. Our weight factors are special values of symmetric square $L$-functions $L(\frac{z+1}{2}, \mathrm{Sym}^2(\pi))$ at each $z \in [0, 1]$.

### 1.1. **Density of low-lying zeros weighted by symmetric square $L$-functions.**

To explain our result in some detail, we prepare notation. Let $F$ be a totally real number field with $d_F = [F : \mathbb{Q}] < \infty$ and $\mathbb{A} = \mathbb{A}_F$ the adèle ring of $F$. Let $\Sigma_\infty$ denote the set of the archimedean places of $F$. Let $l = (l_v)_{v \in \Sigma_\infty}$ be a family of positive even integers and let $\mathfrak{q}$ be a non-zero prime ideal of the integer ring $\mathfrak{o}$ of $F$. We denote by $\Pi_{\mathrm{cus}}^*(l,\mathfrak{q})$ the set of all irreducible cuspidal automorphic representations $\pi=\otimes_v'\pi_v$ of $\mathrm{PGL}_2(\mathbb{A})$ such that the conductor of $\pi$ equals $\mathfrak{q}$ and $\pi_v$ for each $v\in\Sigma_\infty$ is isomorphic to the discrete series representation of $\mathrm{PGL}_2(\mathbb{R})$ with minimal $\mathrm{O}(2)$-type $l_v$. For any $r\in\mathbb{N}$, we treat in this article the symmetric power $L$-functions $L(s,\mathrm{Sym}^r(\pi))$ for $\pi\in\Pi_{\mathrm{cus}}^*(l,\mathfrak{q})$ explained in §3.1 (see also [12] and [44]), and consider the hypothesis $\mathrm{Nice}(\pi,r)$ that

- $L(s,\mathrm{Sym}^r(\pi))$ is continued to an entire function on $\mathbb{C}$ of order 1, and
- $L(s,\mathrm{Sym}^r(\pi))$ satisfies the functional equation
$$L(s,\mathrm{Sym}^r(\pi))=\epsilon_{\pi,r}(D_F^{r+1}\mathrm{N}(\mathfrak{q})^r)^{1/2-s}L(1-s,\mathrm{Sym}^r(\pi)),$$
where $D_F$ is the absolute value of the discriminant of $F/\mathbb{Q}$, $\mathrm{N}(\mathfrak{q})$ is the absolute norm of $\mathfrak{q}$, and $\epsilon_{\pi,r}\in\{\pm1\}$.

This hypothesis is expected to be true in the viewpoint of the Langlands functoriality principle. Recently, $\mathrm{Nice}(\pi,r)$ was proved by Newton and Thorne [40] for all $r\in\mathbb{N}$, all non-zero prime ideals $\mathfrak{q}$ and all $\pi\in\Pi_{\mathrm{cus}}^*(l,\mathfrak{q})$ if we restrict our case to elliptic modular forms ($F=\mathbb{Q}$). Moreover they announced a preprint [42] on the symmetric power functoriality of Hilbert modular forms. Thus we strongly believe that the hypothesis $\mathrm{Nice}(\pi,r)$ is no longer needed. For known results concerning $\mathrm{Nice}(\pi,r)$, we refer to §3.1 in details.

Throughout this article, we fix $F$ and $l$, and assume $\mathrm{Nice}(\pi,r)$ for all non-zero prime ideals $\mathfrak{q}$ and all $\pi\in\Pi_{\mathrm{cus}}^*(l,\mathfrak{q})$. In what follows, we consider the distribution of low-lying zeros of $L(s,\mathrm{Sym}^r(\pi))$ for $\pi\in\Pi_{\mathrm{cus}}^*(l,\mathfrak{q})$ with $\mathrm{N}(\mathfrak{q})\to\infty$ without assuming GRH for any of the $L$-functions. The one-level density of such low-lying zeros is defined as

$$D(\mathrm{Sym}^r(\pi),\phi)=\sum_{\rho=1/2+i\gamma}\phi\left(\frac{\log Q(\mathrm{Sym}^r(\pi))}{2\pi}\gamma\right) \tag{1.1}$$

for any Paley-Wiener functions $\phi$, where $\rho=1/2+i\gamma$ ($\gamma\in\mathbb{C}$) runs over all zeros of $L(s,\mathrm{Sym}^r(\pi))$ counted with multiplicity, and $Q(\mathrm{Sym}^r(\pi))$ is the analytic conductor of $\mathrm{Sym}^r(\pi)$. Here a Paley-Wiener function is given by a Schwartz function $\phi$ on $\mathbb{R}$ such that the Fourier transform
$$\hat{\phi}(\xi)=\int_{\mathbb{R}}\phi(x)e^{-2\pi i\xi x}dx$$
of $\phi$ has a compact support. By the compactness, $\phi$ is extended to an entire function on $\mathbb{C}$. By the symmetry of zeros of $L(s,\mathrm{Sym}^r(\pi))$, we may assume that $\phi$ is even.

Before stating our results, we recall Ricotta and Royer's result [44] on elliptic modular forms. When $F=\mathbb{Q}$, the set $\Pi_{\mathrm{cus}}^*(l,q\mathbb{Z})$ for an even positive integer $l$ and a prime number $q$ is identified with the set $H_l^*(q)$ of normalized new Hecke eigenforms in the space $S_l(\Gamma_0(q))^{\mathrm{new}}$ of elliptic cuspidal new forms of weight $l$ and level $q$ with trivial nebentypus. Set
$$\omega_q(f)=\frac{\Gamma(l-1)}{(4\pi)^{l-1}\|f\|^2},$$
where $\|f\|$ denotes the Petersson norm of $f$ as in [44, §2.1.1]. We denote by $\epsilon_{f,r}$ the sign of the functional equation of $L(s,\mathrm{Sym}^r(f))$ for $f\in H_l^*(q)$.

Then, Ricotta and Royer [44] proved the following asymptotic formula in the level aspect without GRH.

**Theorem 1.1.** [44, Theorems A and B][1] *Let $r$ be any positive integer. Let $l \geqslant 2$ be a positive even integer and let $q$ vary in the set of prime numbers. Let $\phi$ be an even Schwartz function on $\mathbb{R}$. Set*

$$\beta_1 = \left(1 - \frac{1}{2(l-2\theta)}\right)\frac{2}{r^2},$$

*where $\theta \in [0, 1/2)$ is the exponent toward the generalized Ramanujan-Petersson conjecture for $\mathrm{GL}_2$ (cf. [44, §3.1]). If $\mathrm{supp}(\hat{\phi}) \subset (-\beta_1, \beta_1)$, then we have*

$$\lim_{q\to\infty}\sum_{f\in H_l^*(q)}\omega_q(f)D(\mathrm{Sym}^r(f),\phi) = \begin{cases} \displaystyle\int_{\mathbb{R}}\phi(x)W(\mathrm{Sp})(x)dx & (r \text{ is even}), \\ \displaystyle\int_{\mathbb{R}}\phi(x)W(\mathrm{O})(x)dx & (r \text{ is odd}). \end{cases}$$

Here the weight factor $\omega_q(f)$ is called the harmonic weight. It can be removable and negligible under GRH (cf. [23, Lemma 2.18]). As an analogous result to Ricotta and Royer as above, we give the density of low-lying zeros of $L(s, \mathrm{Sym}^r(\pi))$ weighted by special values $L(\frac{z+1}{2}, \mathrm{Sym}^2(\pi))$ of symmetric square $L$-functions at each $z \in [0, 1]$. Now we review the ensemble of the special values $\{L(\frac{z+1}{2}, \mathrm{Sym}^2(\pi))\}_{\pi\in\Pi^*_{\mathrm{cus}}(l,\mathfrak{q})}$. The value $L(\frac{z+1}{2}, \mathrm{Sym}^2(\pi))$ is believed to be non-negative due to GRH. Although the non-negativity is still open, it is supported by the asymptotics

$$\sum_{\pi\in\Pi^*_{\mathrm{cus}}(l,\mathfrak{q})}\frac{L(\frac{z+1}{2}, \mathrm{Sym}^2(\pi))}{L(1, \mathrm{Sym}^2(\pi))} \sim C\,\mathrm{N}(\mathfrak{q})(\log \mathrm{N}(\mathfrak{q}))^{\delta_{z,0}}, \qquad \mathrm{N}(\mathfrak{q}) \to \infty$$

for some explicit constant $C > 0$, where $\delta_{z,0}$ is the Kronecker delta. This asymptotic formula follows from the proof of Tsuzuki and the author's result [50, Theorem 1.3] (see also Theorem 2.7). In particular, this average is non-zero for any non-zero prime ideals $\mathfrak{q}$ of $\mathfrak{o}$ such that $\mathrm{N}(\mathfrak{q})$ is sufficiently large. Our main result in the level aspect without GRH is stated as follows.

**Theorem 1.2.** *We assume that the prime $2 \in \mathbb{Q}$ is completely splitting in $F$. Suppose that $l \in 2\mathbb{N}^{\Sigma_\infty}$ satisfies $\underline{l} := \min_{v\in\Sigma_\infty} l_v \geqslant 6$. Let $\mathfrak{q}$ vary in the set of non-zero prime ideals of $\mathfrak{o}$. For $r \in \mathbb{N}$, we assume $\mathrm{Nice}(\pi, r)$ for all $\mathfrak{q}$ and all $\pi \in \Pi^*_{\mathrm{cus}}(l, \mathfrak{q})$. We fix $z \in [0, 1]$ and define $\beta_2 > 0$ by*

$$\beta_2 = \frac{1}{r(r^{\frac{\underline{l}-3-z+2d_F}{2d_F}} + \frac{1}{2})} \times \begin{cases} \frac{1}{2} & (\underline{l} \geqslant d_F + 4), \\ \frac{\underline{l}-3-z}{2d_F} & (6 \leqslant \underline{l} \leqslant d_F + 3). \end{cases}$$

[1]They assumed the extra hypotheses $\mathrm{Nice}(f, r)$ for all prime numbers $q$ and all $f \in H_l^*(q)$ in [44]. But it is no longer needed due to [40].

*Then, for any even Schwartz function $\phi$ on $\mathbb{R}$ with* $\mathrm{supp}(\hat{\phi}) \subset (-\beta_2, \beta_2)$*, we have*

$$\lim_{\mathrm{N}(\mathfrak{q})\to\infty} \frac{1}{\sum_{\pi\in\Pi^*_{\mathrm{cus}}(l,\mathfrak{q})} \frac{L(\frac{z+1}{2},\mathrm{Sym}^2(\pi))}{L(1,\mathrm{Sym}^2(\pi))}} \sum_{\pi\in\Pi^*_{\mathrm{cus}}(l,\mathfrak{q})} \frac{L(\frac{z+1}{2},\mathrm{Sym}^2(\pi))}{L(1,\mathrm{Sym}^2(\pi))} D(\mathrm{Sym}^r(\pi),\phi)$$

$$= \begin{cases} \hat{\phi}(0) - \frac{1}{2}\phi(0) = \int_{\mathbb{R}} \phi(x) W(\mathrm{Sp})(x)dx & (r \textit{ is even and } (r,z) \neq (2,0)), \\ \hat{\phi}(0) + \frac{1}{2}\phi(0) = \int_{\mathbb{R}} \phi(x) W(\mathrm{O})(x)dx & (r \textit{ is odd}), \\ \hat{\phi}(0) - \frac{3}{2}\phi(0) + 2\int_{\mathbb{R}} \hat{\phi}(x)|x|dx = \int_{\mathbb{R}} \phi(x) W'(x)dx & (r = 2 \textit{ and } z = 0), \end{cases}$$

*where $W'(x)$ is defined as*

$$W'(x) = 1 + \frac{\sin(2\pi x)}{2\pi x} - \frac{2\sin^2(\pi x)}{(\pi x)^2}.$$

Here we note that the denominator in the left-hand side of the limit formula above is non-zero if $\mathrm{N}(\mathfrak{q})$ is large as noted before. Compared to Theorem 1.1, Theorem 1.2 can be interpreted as a $z$-interpolation of Ricotta and Royer's result. We remark that our assumption $\underline{l} \geqslant 6$ can be weakened to $\underline{l} \geqslant 4$, in which the condition $z \in [0,1]$ is replaced with $z \in [0, \min(1,\sigma)]$ for any $\sigma \in (0, \underline{l} - 3)$. This condition on $z$ and $l$ is derived from the assumption in [50, Corollary 1.2].

There are some remarks in various directions. Theorem 1.2 for $z = 1$ is a generalization of Theorem 1.1 to the case of Hilbert modular forms without harmonic weight. Although the setting is slightly different, Theorem 1.2 for $z = 1$ is similar to Shin and Templier's result [47, Example 9.13 and Theorem 11.5], where they considered the principal congruence subgroup $\Gamma(\mathfrak{q})$ instead of our level $\Gamma_0(\mathfrak{q})$, the Hecke congruence subgroup.

Theorem 1.2 in the case of the standard $L$-functions ($r = 1$) for Hilbert modular forms without weight factor ($z = 1$) has overlap with Liu and Miller [34], in which they assumed GRH and imposed conditions that the narrow class number of $F$ is one, the weight is the parallel weight $2k \geqslant 4$, and the level is endowed by the congruence subgroup $\Gamma_0(\mathcal{I})$ for a square-free ideal $\mathcal{I}$ of $\mathfrak{o}$. Their contribution is to extend the size of the support of $\hat{\phi}$ beyond $(-1,1)$ under the assumption above.

We remark that, when $r$ is odd, Theorem 1.2 does not distinguish $W(\mathrm{O})$, $W(\mathrm{SO(even)})$ and $W(\mathrm{SO(odd)})$ because $(-\beta_2, \beta_2) \subset [-1,1]$. For determining the symmetry type, we need to calculate the two-level density of low-lying zeros weighted by symmetric square $L$-functions. It can be done by generalizing the trace formula in Theorem 2.1 ([50, Corollary 1.2]) to the case where $S$ and $S(\mathfrak{n})$ used there have common finite places. On the other hand, we do not expect the extension of the support of $\hat{\phi}$ in Theorem 1.2 even if we assume GRH. The extension of the support can be done under GRH if we apply the Petersson trace formula (cf. [25], [34]). However, our method is due to a parameterized Eichler-Selberg trace formula [50] but not the Petersson trace formula.

We summarize the change of densities in Theorem 1.2 when moving $z$ in $[0,1]$. For any $r \neq 2$, the weighted density for the family of $L(s, \mathrm{Sym}^r(\pi))$ for $\pi \in \Pi^*_{\mathrm{cus}}(l,\mathfrak{q})$ is $W(\mathrm{Sp})$ or $W(\mathrm{O})$, which is stationary when $z$ varies in $[0,1]$. Contrary to this, the weighted density for $r = 2$ is stationary as $z \in (0,1]$ but the symmetry type $W(\mathrm{Sp})$ is broken and changed to $W'(x)$ when $z=0$. Hence, we conclude that the change of density of low-lying zeros of $L(s,\mathrm{Sym}^r(\pi))$ occurs only when $r=2$ and the weight factors are essentially central values $L(1/2,\mathrm{Sym}^2(\pi))$.

Our weighted density $W'(x)$ for $(r,z)=(2,0)$ is a new type in the sense that it is not in the list of five types of the densities from random matrix theory. A density function not arising in the list is also seen for families of $L$-functions attached to elliptic curves in Miller [38]. Therefore, Theorem 1.2 gives us a new example of the phenomenon that central $L$-values effect to the change of density of low-lying zeros, as seen in Knightly and Reno [32] for $\mathrm{GL}_2$ and Kowalski, Saha and Tsimerman [33], Dickson [13] for $\mathrm{GSp}_4$. Furthermore, our weighted density $W'(x)$ coincides with the density function $W^1_{\mathrm{USp}}(x)$ in the one-level density of eigenvalues of random matrices in the symplectic group $\mathrm{USp}(2N)$ weighted by central values of the characteristic polynomials of the random matrices in $\mathrm{USp}(2N)$ (cf. [17, Theorem 4]). By these observations, it might be meaningful to suggest the following naive conjecture, which is not in a rigorous form:

Let $\mathcal{F}=\bigcup_{k=1}^{\infty}\mathcal{F}_k$ be a family of indices $\Pi$ of $L$-functions $L(s,\Pi)$ (e.g., a multiset of irreducible automorphic representations of an adelic group such as a harmonic family in the sense of [45]). Assume $\#\mathcal{F}_k<\infty$ for each $k\geqslant 1$ and $\lim_{k\to\infty}\#\mathcal{F}_k=\infty$. Further assume the existence of density $W(\mathcal{F})$ for the one-level density of low-lying zeros of the family of $L$-functions $L(s,\Pi)$, $(\Pi\in\mathcal{F})$, that is,

$$\lim_{k\to\infty}\frac{1}{\#\mathcal{F}_k}\sum_{\Pi\in\mathcal{F}_k}D(\Pi,\phi)=\int_{\mathbb{R}}\phi(x)W(\mathcal{F})(x)dx$$

for Paley-Wiener functions $\phi$, where $D(\Pi,\phi)$ is the one-level density such as (1.1). Let $w:\mathcal{F}\to\mathbb{C}$ be a function such that $\sum_{\Pi\in\mathcal{F}_k}w_\Pi\neq 0$ for any $k\gg 1$ and the weighted one-level density is described as

$$\lim_{k\to\infty}\frac{1}{\sum_{\Pi\in\mathcal{F}_k}w_\Pi}\sum_{\Pi\in\mathcal{F}_k}w_\Pi\,D(\Pi,\phi)=\int_{\mathbb{R}}\phi(x)W_w(\mathcal{F})(x)dx.$$

Then, the density $W_w(\mathcal{F})$ would be changed from $W(\mathcal{F})$ only when $w_\Pi$ essentially has the central value $L(1/2,\Pi)$ as a factor.

As for the naive conjecture above, it might be better to consider the case where the weight factor $w_\Pi$ is a quotient of special values of automorphic $L$-functions or a period integral, as long as $\mathcal{F}$ consists of automorphic representations. To attack the conjecture in the automorphic case, relative trace formulas would be more useful tools rather than the Arthur-Selberg trace formula.

The recent works related to our study are as follows. Fazzari [17] independently suggests a conjecture on the one-level density of low-lying zeros for a family of $L$-functions weighted by powers of central $L$-values, after this article got public on arXiv. Theorem 1.2 for $(r,z)=(2,0)$ supports his conjecture because of the coincidence $W'(x)=W^1_{\mathrm{USp}}(x)$. Fazzari's conjecture and our conjecture as above also have been studied in [49] and [5].

The study in this article is related to the value distribution of symmetric square $L$-functions. As for the value distribution of $L$-functions, the statistics weighted by central values has been studied for the Riemann zeta function ([15], [16]) and for several $L$-functions ([9]). We also note that Bourgade, Nikeghbali and Rouault's articles [7] and [8] are relevant to our study as they compute statistics weighted by central values in random matrix theory.

1.2. **Framework.** The usual distribution of low-lying zeros has been mainly analyzed by the usage of the Petersson trace formula and the Kuznetsov trace formula, which are the same in the viewpoint of relative trace formulas given by double integrals along the product of adelic quotients of two unipotent subgroups. Shin and Templier [47] used Arthur's invariant trace formula by estimating orbital integrals to quantify automorphic Plancherel density theorem. Contrary to those trace formulas, the proof of Theorem 1.2 relies on the explicit Jacquet-Zagier type trace formula given by Tsuzuki and the author [50]. This is a generalization of the Eichler-Selberg trace formula to a parameterized version. In our setting, we need to treat not only the main term but also the second main term of the weighted automorphic Plancherel density theorem quantitatively as in Theorem 2.7 in order to study weighted density of low-lying zeros.

This article is organized as follows. In §2, we review the explicit Jacquet-Zagier type trace formula for $\mathrm{GL}_2$ given by Tsuzuki and the author [50], and prove a refinement of the Plancherel density theorem weighted by symmetric square $L$-functions [50, Theorem 1.3] by explicating the error term. In §3, we introduce symmetric $r$th $L$-functions for $\mathrm{GL}_2$ for any $r \in \mathbb{N}$ and prove the main theorem 1.2 on weighted density of low-lying zeros of such $L$-functions.

1.3. **Notation.** Let $\mathbb{N}$ be the set of the positive integers and set $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For a condition P, $\delta(\mathrm{P})$ is the generalized Kronecker delta symbol defined by $\delta(\mathrm{P}) = 1$ if P is true, and $\delta(\mathrm{P}) = 0$ if P is false, respectively. Throughout this article, any fractional ideal of a number field is always supposed to be non-zero. We set $\Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(s/2)$.

## 2. Refinements of equidistributions weighted by symmetric square $L$-functions

In this section, we give a quantitative version of the equidistribution of Hecke eigenvalues weighted by $L(\frac{z+1}{2}, \mathrm{Sym}^2(\pi))$ in [50, Theorem 1.3] as a deduction from the explicit Jacquet-Zagier type trace formula in [50] by estimating error terms more explicitly. For this purpose, let us review the Jacquet-Zagier type trace formula.

Let $F$ be a totally real number field of finite degree $d_F = [F : \mathbb{Q}]$. We suppose that the prime $2 \in \mathbb{Q}$ is completely splitting in $F$. Let $D_F$ be the absolute value of the discriminant of $F/\mathbb{Q}$. The set of archimedean places (resp. non-archimedean places) of $F$ is denoted by $\Sigma_\infty$ (resp. $\Sigma_{\mathrm{fin}}$). For an ideal $\mathfrak{a}$ of $\mathfrak{o}$, we denote by $S(\mathfrak{a})$ the set of all finite places dividing $\mathfrak{a}$ and by $\mathrm{N}(\mathfrak{a})$ the absolute norm of $\mathfrak{a}$, respectively. For any $v \in \Sigma_\infty \cup \Sigma_{\mathrm{fin}}$, let $F_v$ be the completion of $F$ at $v$. The normalized valuation of $F_v$ is denoted by $|\cdot|_v$. For any $v \in \Sigma_{\mathrm{fin}}$, let $q_v$ be the cardinality of the residue field $\mathfrak{o}_v/\mathfrak{p}_v$ of $F_v$, where $\mathfrak{o}_v$ and $\mathfrak{p}_v$ are the integer ring of $F_v$ and its unique maximal ideal, respectively. We fix a uniformizer $\varpi_v$ at every $v \in \Sigma_{\mathrm{fin}}$. Then, $q_v = |\varpi_v|_v^{-1}$ holds.

For an ideal $\mathfrak{a}$ of $\mathfrak{o}$, the symbol $\mathfrak{a} = \square$ means that $\mathfrak{a}$ is a square of a non-zero ideal of $\mathfrak{o}$.

2.1. **Trace formulas.** We review the Jacquet-Zagier type trace formula given in [50]. Let $S$ be a finite subset of $\Sigma_{\mathrm{fin}}$. Let $l = (l_v)_{v \in \Sigma_\infty}$ be a family of positive even integers and let $\mathfrak{n}$ be a square-free ideal of $\mathfrak{o}$ relatively prime to $S$ and $2\mathfrak{o}$. For $l$ and $\mathfrak{n}$, let $\Pi_{\mathrm{cus}}(l, \mathfrak{n})$ denote the set of all irreducible cuspidal automorphic representations $\pi \cong \otimes_v' \pi_v$ of $\mathrm{PGL}_2(\mathbb{A})$ such that $\pi_v$ for each $v \in \Sigma_\infty$ is isomorphic to the discrete series representation of $\mathrm{PGL}_2(\mathbb{R})$ whose minimal $\mathrm{O}(2)$-type is $l_v$ and the conductor $\mathfrak{f}_\pi$ of $\pi$ divides $\mathfrak{n}$. For any $\pi \in \Pi_{\mathrm{cus}}(l, \mathfrak{n})$ and $v \in \Sigma_{\mathrm{fin}} - S(\mathfrak{f}_\pi)$, the Satake parameter of $\pi_v$ is denoted by $(q_v^{-\nu_v(\pi)/2}, q_v^{\nu_v(\pi)/2})$. Notice Blasius' bound $|q_v^{\nu_v(\pi)/2}| = 1$ by [6]. For a complex number $z$, set

$$W_{\mathfrak{n}}^{(z)}(\pi) := \mathrm{N}(\mathfrak{n}\mathfrak{f}_\pi^{-1})^{(1-z)/2} \prod_{v \in S(\mathfrak{n}\mathfrak{f}_\pi^{-1})} \left\{ 1 + \frac{Q(I_v(|\cdot|_v^{z/2})) - Q(\pi_v)^2}{1 - Q(\pi_v)^2} \right\},$$

where $I_v(\chi_v) = \mathrm{Ind}_{B(F_v)}^{\mathrm{GL}_2(F_v)}(\chi_v \boxtimes \chi_v^{-1})$ for a quasi-character $\chi_v : F_v^\times \to \mathbb{C}^\times$ denotes the normalized parabolic induction, and we set

$$Q(\tau) = \frac{a_v + a_v^{-1}}{q_v^{1/2} + q_v^{-1/2}}$$

for any irreducible unitarizable spherical representations $\tau$ of $\mathrm{PGL}_2(F_v)$ with the Satake parameter $(a_v, a_v^{-1})$. Note that $Q(\tau)$ is real and $-1 < Q(\tau) < 1$.

Let $\mathcal{A}_v$ for each $v \in \Sigma_{\mathrm{fin}}$ be the space consisting of all holomorphic functions $\alpha_v$ on $\mathbb{C}/(4\pi i (\log q_v)^{-1}\mathbb{Z})$ such that $\alpha_v(-s_v) = \alpha_v(s_v)$, and set $\mathcal{A}_S = \bigotimes_{v \in S} \mathcal{A}_v$. Then, for any $\alpha \in \mathcal{A}_S$, we define the quantity $\mathbb{I}^0_{\mathrm{cusp}}(\mathfrak{n}|\alpha, z)$ arising in the cuspidal part of the spectral side in the explicit Jacquet-Zagier type trace formula as

$$\mathbb{I}^0_{\mathrm{cusp}}(\mathfrak{n}|\alpha, z) = \frac{1}{2} D_F^{z-1/2} \mathrm{N}(\mathfrak{n})^{(z-1)/2} \sum_{\pi \in \Pi_{\mathrm{cus}}(l, \mathfrak{n})} W_{\mathfrak{n}}^{(z)}(\pi) \frac{L(\frac{z+1}{2}, \mathrm{Sym}^2(\pi))}{L(1, \mathrm{Sym}^2(\pi))} \alpha(\nu_S(\pi))$$

with $\nu_S(\pi) = (\nu_v(\pi))_{v \in S}$. Here $L(s, \mathrm{Sym}^2(\pi))$ is the completed symmetric square $L$-function associated to $\pi$. The following is also needed to describe the spectral side:

$$C(l, \mathfrak{n}) := D_F^{-1} \prod_{v \in \Sigma_\infty} \frac{4\pi}{l_v - 1} \prod_{v \in S(\mathfrak{n})} \frac{1}{1 + q_v}.$$

Next we define the quantities arising in the geometric side of the explicit Jacquet-Zagier type trace formula. For $v \in \Sigma_\infty$, we set

$$\mathcal{O}_v^{+,(z)}(a) = \frac{2\pi}{\Gamma(l_v)} \frac{\Gamma(l_v + \frac{z-1}{2})\Gamma(l_v + \frac{-z+1}{2})}{\Gamma_{\mathbb{R}}(\frac{1+z}{2})\Gamma_{\mathbb{R}}(\frac{1-z}{2})} \delta(|a| > 1)(a^2 - 1)^{1/2} \mathfrak{P}^{1-l_v}_{\frac{z-1}{2}}(|a|)$$

and

$$\mathcal{O}_v^{-,(z)}(a) = \frac{\pi}{\Gamma(l_v)} \Gamma\left(l_v + \tfrac{z-1}{2}\right) \Gamma\left(l_v + \tfrac{-z+1}{2}\right) \mathrm{sgn}(a)(1 + a^2)^{1/2} \{\mathfrak{P}^{1-l_v}_{\frac{z-1}{2}}(ia) - \mathfrak{P}^{1-l_v}_{\frac{z-1}{2}}(-ia)\}$$

for $a \in F_v \cong \mathbb{R}$, where $\mathfrak{P}^\mu_\nu(x)$ is the associated Legendre function of the first kind defined on $\mathbb{C} - (-\infty, 1]$ (cf. [36, §4.1]). Note that the value at $a = 0$ is understood as the limit when $a \to +0$.

For $v \in \Sigma_{\mathrm{fin}}$, let $\varepsilon_\delta$ for $\delta \in F_v^\times$ denote the real-valued character of $F_v^\times$ corresponding to $F_v(\sqrt{\delta})/F_v$ by local class field theory. We define two functions on $F_v^\times$ associated to

$\delta \in F_v^\times$, $\epsilon \in \{0,1\}$ and $z, s \in \mathbb{C}$ satisfying $\mathrm{Re}(s) > (|\mathrm{Re}(z)|-1)/2$ by

$$\mathcal{O}_{v,\epsilon}^{\delta,(z)}(a) = \frac{\zeta_{F_v}(-z)}{L_{F_v}\left(\frac{-z+1}{2},\varepsilon_\delta\right)}\left(\frac{1+q_v^{\frac{z+1}{2}}}{1+q_v}\right)^{\epsilon}|a|_v^{\frac{-z+1}{4}} + \frac{\zeta_{F_v}(z)}{L_{F_v}\left(\frac{z+1}{2},\varepsilon_\delta\right)}\left(\frac{1+q_v^{\frac{-z+1}{2}}}{1+q_v}\right)^{\epsilon}|a|_v^{\frac{z+1}{4}}$$

for all $a \in F_v^\times$, and

$$\mathcal{S}_v^{\delta,(z)}(s;a) = -q_v^{-\frac{s+1}{2}}\frac{\zeta_{F_v}\left(s+\frac{z+1}{2}\right)\zeta_{F_v}\left(s+\frac{-z+1}{2}\right)}{L_{F_v}(s+1,\varepsilon_\delta)}|a|_v^{\frac{s+1}{2}}, \quad (|a|_v \leqslant 1),$$

$$\mathcal{S}_v^{\delta,(z)}(s;a) = -q_v^{-\frac{s+1}{2}}\left\{\frac{\zeta_{F_v}(-z)\zeta_{F_v}\left(s+\frac{z+1}{2}\right)}{L_{F_v}\left(\frac{-z+1}{2},\varepsilon_\delta\right)}|a|_v^{\frac{-z+1}{4}} + \frac{\zeta_{F_v}(z)\zeta_{F_v}\left(s+\frac{-z+1}{2}\right)}{L_{F_v}\left(\frac{z+1}{2},\varepsilon_\delta\right)}|a|_v^{\frac{z+1}{4}}\right\}, \quad (|a|_v > 1).$$

Here $\zeta_{F_v}(s)$ and $L_{F_v}(s,\varepsilon_\delta)$ denote the local $L$-factors attached to the trivial character of $F_v^\times$ and to $\varepsilon_\delta$, respectively. Furthermore, for a function $\alpha_v \in \mathcal{A}_v$ and $a \in F_v^\times$, set

$$\hat{\mathcal{S}}_v^{\delta,(z)}(\alpha_v;a) = \tfrac{1}{2\pi i}\int_{c-2\pi i(\log q_v)^{-1}}^{c+2\pi i(\log q_v)^{-1}} \mathcal{S}_v^{\delta,(z)}(s;a)\,\alpha_v(s)\,\frac{\log q_v}{2}(q_v^{(1+s)/2}-q_v^{(1-s)/2})ds$$

for some $c \in \mathbb{R}$.

Take a function $\alpha \in \mathcal{A}_S$. If $\alpha$ is a pure tensor of the form $\otimes_{v\in S}\alpha_v$, we set

$$\mathbf{B}_{\mathfrak{n}}^{(z)}(\alpha|\Delta;\mathfrak{a}) = \prod_{v\in\Sigma_{\mathrm{fin}}-S\cup S(\mathfrak{n})}\mathcal{O}_{v,0}^{\Delta,(z)}(a_v)\prod_{v\in S(\mathfrak{n})}\mathcal{O}_{v,1}^{\Delta,(z)}(a_v)\prod_{v\in S}\hat{\mathcal{S}}_v^{\Delta,(z)}(\alpha_v,a_v)$$

for any ideal $\mathfrak{a}$ of $\mathfrak{o}$, any $\Delta \in F^\times$ and any $z \in \mathbb{C}$. Here $(a_v)_{v\in\Sigma_{\mathrm{fin}}} \in \prod'_{v\in\Sigma_{\mathrm{fin}}}F_v^\times$ denotes a finite idèle of $F$ such that $\mathfrak{a}\mathfrak{o}_v = a_v\mathfrak{o}_v$ for all $v \in \Sigma_{\mathrm{fin}}$. The quantity $\mathbf{B}_{\mathfrak{n}}^{(z)}(\alpha|\Delta;\mathfrak{a})$ is independent of the choice of $(a_v)_{v\in\Sigma_{\mathrm{fin}}}$. Further we set

$$\Upsilon_v^{(z)}(\alpha_v) = \prod_{v\in S}\tfrac{1}{2\pi i}\int_{c_v-2\pi i(\log q_v)^{-1}}^{c_v+2\pi i(\log q_v)^{-1}}\frac{-q_v^{-(s_v+1)/2}}{1-q_v^{-s_v-(z+1)/2}}\alpha_v(s_v)\frac{\log q_v}{2}(q_v^{(1+s_v)/2}-q_v^{(1-s_v)/2})ds_v$$

with a fixed sufficiently large $c_v \in \mathbb{R}$ for each $v \in S$ and set $\Upsilon^{(z)}(\alpha) = \prod_{v\in S}\Upsilon_v^{(z)}(\alpha_v)$.

By using functions defined above, the quantities in the geometric side are defined as $\mathbb{J}^0_{\mathrm{unip}}(\mathfrak{n}|\alpha,z)$, $\mathbb{J}^0_{\mathrm{hyp}}(\mathfrak{n}|\alpha,z)$ and $\mathbb{J}^0_{\mathrm{ell}}(\mathfrak{n}|\alpha,z)$ in the following way. First set

$$\mathbb{J}^0_{\mathrm{unip}}(\mathfrak{n}|\alpha,z) = D_F^{-\frac{z+2}{4}}\zeta_F(-z)\Upsilon^{(z)}(\alpha)\prod_{v\in S(\mathfrak{n})}\frac{1+q_v^{\frac{z+1}{2}}}{1+q_v}\prod_{v\in\Sigma_\infty}2^{1-z}\pi^{\frac{3-z}{4}}\frac{\Gamma(l_v+\frac{z-1}{2})}{\Gamma(\frac{z+1}{4})\Gamma(l_v)}$$

and

$$\mathbb{J}^0_{\mathrm{hyp}}(\mathfrak{n}|\alpha,z) = \tfrac{1}{2}D_F^{-1/2}\zeta_F(\tfrac{1-z}{2})\sum_{a\in\mathfrak{o}(S)_+^\times-\{1\}}\mathbf{B}_{\mathfrak{n}}^{(z)}(\alpha|1;a(a-1)^{-2}\mathfrak{o})\prod_{v\in\Sigma_\infty}\mathcal{O}_v^{+,(z)}(\tfrac{a+1}{a-1}),$$

where $\zeta_F(s) := \prod_{v\in\Sigma_\infty}\Gamma_{\mathbb{R}}(s)\prod_{v\in\Sigma_{\mathrm{fin}}}\zeta_{F_v}(s)$ is the completed Dedekind zeta function of $F$ and $\mathfrak{o}(S)_+^\times$ is the totally positive unit group of the $S$-integers of $F$.

For any $\Delta \in F^\times$ such that $\sqrt{\Delta} \notin F^\times$, let $\varepsilon_\Delta = \otimes'_{v\in\Sigma_\infty\cup\Sigma_{\mathrm{fin}}}\varepsilon_{\Delta,v}$ be the real-valued character of $F^\times\backslash\mathbb{A}^\times$ corresponding to $F(\sqrt{\Delta})/F$ by class field theory, and $L(s,\varepsilon_\Delta) =$

$\prod_{v\in\Sigma_\infty\cup\Sigma_{\mathrm{fin}}} L_{F_v}(s,\varepsilon_{\Delta,v})$ the completed Hecke $L$-function associated to $\varepsilon_\Delta$. Then, we set

$$\mathbb{J}^0_{\mathrm{ell}}(\mathfrak{n}|\alpha,z)=\frac{1}{2}D_F^{\frac{z-1}{2}}\sum_{(t:n)_F}\mathrm{N}(\mathfrak{d}_\Delta)^{\frac{z+1}{4}}L(\tfrac{z+1}{2},\varepsilon_\Delta)\mathbf{B}^{(z)}_{\mathfrak{n}}(\alpha|\Delta;n\mathfrak{f}_\Delta^{-2})\prod_{v\in\Sigma_\infty}\mathscr{O}_v^{\mathrm{sgn}(\Delta^{(v)}),(z)}(t|\Delta|_v^{-1/2}),$$

where $\mathfrak{d}_\Delta$ is the relative discriminant of $F(\sqrt{\Delta})/F$ and $\mathfrak{f}_\Delta$ is the fractional ideal of $F$ satisfying $\Delta\mathfrak{o}=\mathfrak{d}_\Delta\mathfrak{f}_\Delta^2$. In the summation defining $\mathbb{J}^0_{\mathrm{ell}}(\mathfrak{n}|\alpha,z)$, $(t:n)_F$ runs over the different cosets $\{(ct,c^2n)\in F\times F\mid c\in F^\times\}$ such that $\Delta=t^2-4n\in F^\times-(F^\times)^2$, $(t,n)\in\{(c_vt_v,c_v^2n_v)\mid c_v\in F_v^\times,t_v\in\mathfrak{o}_v,n_v\in\mathfrak{o}_v^\times\}$ for all $v\in\Sigma_{\mathrm{fin}}-S$, and $\mathrm{ord}_v(n\mathfrak{f}_\Delta^{-2})<0$ for all $v\in S(\mathfrak{n})$ with $\varepsilon_{\Delta,v}$ being unramified and non-trivial.

The explicit Jacquet-Zagier type trace formula is given as follows.

**Theorem 2.1.** ([50, Corollary 1.2]) *Let $S$ be a finite subset of $\Sigma_{\mathrm{fin}}$. Let $l=(l_v)_{v\in\Sigma_\infty}$ be a family of positive even integers such that $\min_{v\in\Sigma_\infty}l_v\geqslant 4$, and $\mathfrak{n}$ a square-free ideal of $\mathfrak{o}$ relatively prime to $2\mathfrak{o}$ and $S$. For any $z\in\mathbb{C}$ with $|\mathrm{Re}(z)|<\min_{v\in\Sigma_\infty}l_v-3$, we have*

$$(-1)^{\#S}C(l,\mathfrak{n})\mathbb{I}^0_{\mathrm{cusp}}(\mathfrak{n}|\alpha,z)=D_F^{z/4}\{\mathbb{J}^0_{\mathrm{unip}}(\mathfrak{n}|\alpha,z)+\mathbb{J}^0_{\mathrm{unip}}(\mathfrak{n}|\alpha,-z)\}+\mathbb{J}^0_{\mathrm{hyp}}(\mathfrak{n}|\alpha,z)+\mathbb{J}^0_{\mathrm{ell}}(\mathfrak{n}|\alpha,z).$$

**Remark 2.2.** In [50], It is assumed that $S$ does not include the dyadic places. This assumption is removable by [51, §5].

**Remark 2.3.** *Theorem 2.1 is a generalization of parameterized trace formulas by Zagier* [53]*, by Mizumoto* [39]*, and by Takase* [52]*. In the long version of this article* [48]*, the author proved that the explicit Jacquet-Zagier type trace formula in Theorem 2.1 recovers all of three parameterized trace formulas. Such comparison is not so straightforward and includes non-trivial analytic evaluations. We omit details of the comparison here. If the reader is interested in it, refer to Appendices A, B and C in* [48].

2.2. **Quantitative versions of weighted equidistributions.** In this subsection, we give a quantitative version of weighted equidistribution of the Satake parameters of $\pi\in\Pi_{\mathrm{cus}}(l,\mathfrak{n})$. We give estimates of geometric terms $\mathbb{J}^0_{\mathrm{unip}}(\mathfrak{n}|\alpha,z)$, $\mathbb{J}^0_{\mathrm{hyp}}(\mathfrak{n}|\alpha,z)$ and $\mathbb{J}^0_{\mathrm{ell}}(\mathfrak{n}|\alpha,z)$ by making the dependence on the test function $\alpha$ and $S$ explicit. In what follows, we suppress the dependence on $F$ and $l$ of the implied constants.

Let $S$ be a finite subset of $\Sigma_{\mathrm{fin}}$. For $v\in S$, let $\mathcal{A}^0_v$ be the space of all Laurent polynomials in $q_v^{-s_v/2}$ which are invariant under $q_v^{-s_v/2}\mapsto q_v^{s_v/2}$. Then $\mathcal{A}^0_v$ has a $\mathbb{C}$-basis consisting of $(q_v^{-s_v/2})^n+(q_v^{s_v/2})^n$ for all $n\in\mathbb{N}_0$. Let $\mathcal{A}^0_v[m]$ for $m\in\mathbb{N}_0$ be the subspace of $\mathcal{A}^0_v$ generated by $(q_v^{-s_v/2})^n+(q_v^{s_v/2})^n$ for all $n\in\mathbb{N}_0$ with $0\leqslant n\leqslant m$.

Let $\mathfrak{a}$ be an integral ideal of the form $\mathfrak{a}=\prod_{v\in S}(\mathfrak{p}_v\cap\mathfrak{o})^{n_v}$ with $n_v\in\mathbb{N}$. Set $\mathcal{A}(\mathfrak{a})=\otimes_{v\in S}\mathcal{A}^0_v[n_v]$. We fix a sufficiently large $c>1$ and define $\mathbf{c}=(c_v)_{v\in S}\in\mathbb{R}^S$ by $c_v=c$ for all $v\in S$ as in [50, §8.1]. For $\alpha\in\mathcal{A}_S$, set

$$\|\alpha\|=\left(\frac{1}{2\pi}\right)^{\#S}\int_{\mathbb{L}_S(\mathbf{c})}|\alpha(\mathbf{s})|\,|d\mu_S(\mathbf{s})|,$$

where the right-hand side is a multi-dimensional line integral on the set

$$\mathbb{L}_S(\mathbf{c}):=\prod_{v\in S}\{c_v+it\in\mathbb{C}\mid|t|\leqslant 2\pi(\log q_v)^{-1}\}$$

with respect to the measure

$$|d\mu_S(\mathbf{s})| := \prod_{v \in S} \frac{\log q_v}{2} |(q_v^{(1+s_v)/2} - q_v^{(1-s_v)/2})|\, |ds_v|, \qquad \mathbf{s} = (s_v)_{v \in S}.$$

Let $X_n$ for $n \in \mathbb{N}_0$ be the polynomial defined by $\frac{\sin(n+1)\theta}{\sin\theta} = X_n(2\cos\theta)$.

**Lemma 2.4.** *For any $\alpha(\mathbf{s}) = \prod_{v \in S} X_{m_v}(q_v^{-s/2} + q_v^{s/2}) \in \mathcal{A}(\mathfrak{a})$, we have*

$$\|\alpha\| \ll \prod_{v \in S}(m_v + 1) \ll_\epsilon \mathrm{N}(\mathfrak{a})^\epsilon$$

*for any $\epsilon > 0$, where the implied constant is independent of $\alpha$ and $\mathfrak{a}$.*

*Proof.* The assertion is reduced to the inequality $\max_{x \in [-2,2]} |X_n(x)| \leqslant n + 1$. This inequality is proved by the induction on $n$ with the aid of the estimate $|\sin(n+1)\theta| = |\sin n\theta \cos\theta + \cos n\theta \sin\theta| \leqslant |\sin n\theta| + |\sin\theta|$. □

**Lemma 2.5.** *Fix any $\sigma \in [0, 1]$ and suppose $\underline{l} := \min_{v \in \Sigma_\infty} l_v > \sigma + 3$. For any $\alpha \in \mathcal{A}(\mathfrak{a})$ and $z \in \mathbb{C}$ with $|\mathrm{Re}(z)| \leqslant \sigma$, we have*

$$\mathbb{J}^0_{\mathrm{hyp}}(\mathfrak{n}|\alpha, z) \ll_{\sigma,\epsilon,\epsilon'} \mathrm{N}(\mathfrak{a})^{(\frac{\underline{l}}{2} + d_F - 1 - (\sigma+1)/2 - \epsilon)/d_F + \epsilon'} \|\alpha\|\, \mathrm{N}(\mathfrak{n})^{-\delta+\epsilon}$$

*uniformly in $z$, $\alpha$, $\mathfrak{n}$ and $\mathfrak{a}$ for any sufficiently small $\epsilon, \epsilon' > 0$, where $\delta \in (1/2, 1]$ is defined by*

$$\delta := \begin{cases} 1 & (\sigma \leqslant \underline{l} - 3 - d_F), \\ \frac{1}{2} + \frac{\underline{l} - 3 - \sigma}{2d_F} & (\sigma > \underline{l} - 3 - d_F). \end{cases}$$

*Proof.* This is proved by explicating the implied constant depending on $\alpha$ and $\mathfrak{a}$ in [50, Lemma 8.2]. By the proof of [50, Lemma 8.2], we have the estimate

$$\mathbb{J}^0_{\mathrm{hyp}}(\mathfrak{n}|\alpha, z) \ll_{\sigma,\epsilon} C^{\#S} \|\alpha\| \sum_{\mathfrak{c}|\mathfrak{n}} \Big\{ \prod_{v \in S(\mathfrak{n}\mathfrak{c}^{-1})} \tfrac{4}{q_v+1} \prod_{v \in S(\mathfrak{c})} \tfrac{4(q_v^{1/2}+1)}{q_v+1} \Big\} \sum_{x \in \mathfrak{c}\mathfrak{a}^{-1} - \{0\}} f_\infty(x+1),$$

where $f_\infty((a_v)_{v \in \Sigma_\infty}) = \prod_{v \in \Sigma_\infty} f_v(a_v)$ with $f_v(a_v) = \delta(a_v > 0)(1 + |a_v|_v)^{-\frac{l_v}{2} + \frac{\sigma+1}{2} + \epsilon}$, and $C > 0$ is an absolute constant. By $\sum_{x \in \mathfrak{c}\mathfrak{a}^{-1} - \{0\}} f_\infty(x+1) \ll \mathrm{N}(\mathfrak{a})\mathrm{N}(\mathfrak{c}\mathfrak{a}^{-1})^{\{1 - \underline{l}/2 + (\sigma+1)/2 + \epsilon\}/d_F}$ uniformly for $\mathfrak{c}$ and $\mathfrak{a}$ from [50, Lemma 7.19], we conclude the assertion. □

**Lemma 2.6.** *For any $\alpha \in \mathcal{A}(\mathfrak{a})$, we have*

$$\mathbb{J}^0_{\mathrm{ell}}(\mathfrak{n}|\alpha, z) \ll_\epsilon \mathrm{N}(\mathfrak{a})^\epsilon \|\alpha\|\, \mathrm{N}(\mathfrak{n})^{-\delta'+\epsilon}$$

*uniformly in $z \in \mathbb{C}$ with $|\mathrm{Re}(z)| \leqslant 1$, $\alpha$, $\mathfrak{n}$ and $\mathfrak{a}$ for any sufficiently small $\epsilon > 0$, where $\delta' \in (1/2, 1]$ is defined by*

$$\delta' := \begin{cases} 1 & (\underline{l} > d_F + 2), \\ \frac{1}{2} + \frac{\underline{l} - 2}{2d_F} & (d_F + 2 \geqslant \underline{l}). \end{cases}$$

*Proof.* This is proved by explicating the implied constant depending on $\alpha$ and $\mathfrak{a}$ in [50, Lemma 8.3]. In the proof, the assumption [50, (7.11)] is not imposed although we need it as in [50, §7]. This is because we can take a complete system $\{\mathfrak{a}_j\}_{j=1}^h$ of representatives for the ideal class group of $F$ such that $\{\mathfrak{a}_j\}_{j=1}^h$ are prime ideals relatively prime to $S$ satisfying [50, (7.11)] as long as we fix $\mathfrak{n}$.

By the proof of [50, Lemma 8.3] with the aid of [50, Lemma 8.1], we have

$$\mathbb{J}^0_{\rm ell}(\mathfrak{n}|\alpha, z) \ll C^{\#S}\|\alpha\| \sum_{(\mathfrak{c},\mathfrak{n}_1,i,\varepsilon,\nu)} \Xi^{(z,\mathfrak{c})}(\mathfrak{c},\mathfrak{n}_1,i,\varepsilon n_{i,\nu})$$

with an absolute constant $C>0$, where $\Xi^{(z,\mathfrak{c})}(\mathfrak{c},\mathfrak{n}_1,i,\varepsilon n_{i,\nu})$ is the series defined in [50, pp.3026–3027] and $(\mathfrak{c},\mathfrak{n}_1,i,\varepsilon,\nu)$ varies so that $\mathfrak{n}_1$ and $\mathfrak{c}$ are integral ideals such that $\mathfrak{n}_1|\mathfrak{n}$ and $\mathfrak{c}|\mathfrak{n}_1$, $\varepsilon \in \mathfrak{o}^\times/(\mathfrak{o}^\times)^2$, $\nu \in \mathbb{N}_0^S$ with $\nu_v \leqslant {\rm ord}_v(\mathfrak{a})$ for all $v\in S$, and $1\leqslant i\leqslant h$ with $n_{i,\nu}\mathfrak{o} = \mathfrak{a}_i\prod_{v\in S}\mathfrak{p}^{\nu_v}$. When we estimate $\sum_{(\mathfrak{c},\mathfrak{n}_1,i,\varepsilon,\nu)}\Xi^{(z,\mathfrak{c})}(\mathfrak{c},\mathfrak{n}_1,i,\varepsilon n_{i,\nu})$ in the same way as in [50, Lemma 7.20], the dependence on $\mathfrak{a}$ and $S=S(\mathfrak{a})$ occurs in the following three cases; the factor $\prod_{v\in S}(\cdots)$ in [50, p.3028][2] derived from [50, Lemma 7.14 (3)], the sum over $\nu\in\mathbb{N}_0^S$ in [50, p.3032] and the second term of the majorant in [50, Lemma 7.20]. The implied constant depending on $\mathfrak{a}, S$ in the first case is $16^{\#S}\ll_\epsilon {\rm N}(\mathfrak{a})^\epsilon$ by [50, Lemma 7.14 (3)]. The constant in the second case is

$$\sum_{\substack{\nu=(\nu_v)_{v\in S}\\ 0\leqslant\nu_v\leqslant{\rm ord}_v(\mathfrak{a})(\forall v\in S)}} \{\prod_{v\in S} q_v^{\nu_v(\frac{-c_v+\varrho(z)}{4}+\epsilon)}\}|{\rm N}(n_{i,\nu})|^{\frac12+\frac{\underline{L}(z)-1}{2d_F}} \ll (\sum_{\mathfrak{c}|\mathfrak{a}}1){\rm N}(\mathfrak{a})^{\frac{1-c}{4}+\epsilon+1/2+\frac{\underline{l}-2-2\epsilon}{2d_F}} \ll_\epsilon 1,$$

where $\varrho(z) = \max(|{\rm Re}(z)|,1)$ and $\underline{L}(z) = \underline{l}-\frac{1+\varrho(z)}{2}-2\epsilon$ (cf. [50, Lemma 7.20]). Here we note $\varrho(z)=1$ and $\underline{L}(z)=\underline{l}-1-2\epsilon$ by $|{\rm Re}(z)|\leqslant 1$, and $c$ can be taken so that $\frac{1-c}{4}+\epsilon+1/2+\frac{\underline{l}-2-2\epsilon}{2d_F}<0$. The constant in the third case is

$$\sum_{\nu\in\{0,1\}^S}\prod_{v\in S}\max(1, q_v^{\frac{-{\rm Re}(c_v)+|{\rm Re}(z)|}{4}+\epsilon}) = \sum_{\nu\in\{0,1\}^S} 1 = 2^{\#S} \ll {\rm N}(\mathfrak{a})^\epsilon$$

by virtue of $|{\rm Re}(z)|\leqslant 1$ and $c>1$. Combining these, we obtain

$$\mathbb{J}^0_{\rm ell}(\mathfrak{n}|\alpha, z) \ll_\epsilon {\rm N}(\mathfrak{a})^\epsilon\|\alpha\| \times \left\{{\rm N}(\mathfrak{a})^\epsilon\sum_{j=1}^h \Sigma(\mathfrak{n}, N_j) + {\rm N}(\mathfrak{a})^\epsilon {\rm N}(\mathfrak{n})^{-1+\epsilon}\right\}$$

uniformly for $|{\rm Re}(z)|\leqslant 1$, where $\Sigma(\mathfrak{n},N_j)$ is the series defined in [50, p.3038] and estimated in the same way as [50, p.3038–3039]. Thus we are done. $\square$

Let $\mathfrak{n}$ be a square-free ideal of $\mathfrak{o}$ such that $\mathfrak{n}\neq\mathfrak{o}$. For any $f\in C([-2,2]^S)$, we set

$$\Lambda_{\mathfrak{n}}^{(z)}(f) = \frac{1}{{\rm M}(\mathfrak{n})^{\delta(z=0)}}\prod_{v\in S(\mathfrak{n})}\frac{q_v^{(z-1)/2}}{1+q_v^{(z+1)/2}}\sum_{\pi\in\Pi_{\rm cus}(l,\mathfrak{n})} W_{\mathfrak{n}}^{(z)}(\pi)\frac{L(\frac{z+1}{2},{\rm Sym}^2(\pi))}{L(1,{\rm Sym}^2(\pi))}f(\mathbf{x}_S(\pi)),$$

where

$${\rm M}(\mathfrak{n}) = \sum_{v\in S(\mathfrak{n})}\frac{\log q_v}{1+q_v^{-1/2}}, \qquad \mathbf{x}_S(\pi) = (q_v^{-\nu_v(\pi)/2}+q_v^{\nu_v(\pi)/2})_{v\in S}.$$

We remark that $\mathbf{x}_S(\pi)$ is an element of $[-2,2]^S$ by Blasius' bound $|q_v^{\nu_v(\pi)/2}|=1$ ([6]).

[2]In [50, p.3029], the factor $\prod_{v\in S}|\Delta_v^0|_v\times\prod_{v\in\Sigma_{\rm fin}-S}|4^{-1}|_v$ depending on $S$ occurs. This is negligible since it is estimated as $\prod_{v\in S}|\Delta_v^0|_v\times\prod_{v\in\Sigma_\infty\cup S}|4|_v\leqslant 4^{d_F}$.

Let $\{\mathfrak{a}_j\}_{j=1}^h$ be a complete system of representatives for the ideal class group of $F$ consisting of prime ideals relatively prime to $S$, due to the Chebotarev density theorem. We denote by $\zeta_{F,\mathrm{fin}}(z) := \prod_{v\in\Sigma_{\mathrm{fin}}}\zeta_{F_v}(s)$ the non-completed Dedekind zeta function of $F$ and $\mathrm{CT}_{z=1}\zeta_{F,\mathrm{fin}}(z)$ denotes the constant term of the Laurent expansion of $\zeta_{F,\mathrm{fin}}(z)$ at $z=1$.

For any $\alpha\in\mathcal{A}$, we denote by $f_\alpha$ the function $f_\alpha : [-2,2]^S \to \mathbb{C}$ determined by $f_\alpha((q_v^{-s_v/2}+q_v^{s_v/2})_{v\in S}) = \alpha(\mathbf{s})$. We quantitatively refine the weighted equidistribution theorem [50, Theorem 1.3 (1)] restricted to the function space $\mathcal{A}(\mathfrak{a})$ by making the dependence on $\mathfrak{a}$ explicit as follows.

**Theorem 2.7.** *Let $l=(l_v)_{v\in\Sigma_\infty}$ be a family of positive even integers such that $\underline{l} := \min_{v\in\Sigma_\infty} l_v \geqslant 4$. Let $\mathfrak{a}$ be an ideal of the form $\prod_{v\in S}(\mathfrak{p}_v\cap\mathfrak{o})^{n_v}$ with $n_v\in\mathbb{N}$. Take any $\alpha\in\mathcal{A}(\mathfrak{a})$. Let $\mathfrak{n}\neq\mathfrak{o}$ be a square-free ideal of $\mathfrak{o}$ relatively prime to $2\prod_{j=1}^h\mathfrak{a}_j$ and $S$. For a fixed $\sigma\in\mathbb{R}$ with $0<\sigma<\underline{l}-3$, take $z\in[0,\min(1,\sigma)]$. Set*

$$C_l^{(z)} := 2D_F^{3/2}\{2^{\frac{1-z}{2}}\pi^{-\frac{3z+1}{4}}\Gamma(\tfrac{z+3}{4})\}^{d_F}\prod_{v\in\Sigma_\infty}\frac{\Gamma(l_v+\frac{z-1}{2})}{4\pi\Gamma(l_v-1)}$$

*as in* [50, p.2985]*. Then, we have*

$$\begin{aligned}&\Lambda_{\mathfrak{n}}^{(z)}(f_\alpha)\\ &=\zeta_{F,\mathrm{fin}}(1+z)C_l^{(z)}(-1)^{\#S}\Upsilon^{(z)}(\alpha)\left(1+D_F^{-3z/2}\prod_{v\in S(\mathfrak{n})}\frac{1+q_v^{\frac{-z+1}{2}}}{1+q_v^{\frac{z+1}{2}}}\right) &(2.1)\\ &\quad+\|\alpha\|\mathcal{O}_{\epsilon,\epsilon'}\left(\mathrm{N}(\mathfrak{a})^{(\underline{l}/2+d_F-1-(\sigma+1)/2-\epsilon)/d_F+\epsilon'}\mathrm{N}(\mathfrak{n})^{-\delta_1+\epsilon}\right), &(2.2)\end{aligned}$$

*for any sufficiently small $\epsilon,\epsilon'>0$ if $z>0$. Furthermore, we have*

$$\begin{aligned}&\Lambda_{\mathfrak{n}}^{(0)}(f_\alpha)\\ &=\frac{1}{\mathrm{M}(\mathfrak{n})}2\,\mathrm{Res}_{z=1}\,\zeta_{F,\mathrm{fin}}(z)\,C_l^{(0)}\Bigg[(-1)^{\#S}\Upsilon^{(0)}(\alpha)\Bigg\{d_F\left(-\frac{1}{2}\log 2-\frac{3}{4}\log\pi+\frac{1}{4}\psi\left(\frac{3}{4}\right)\right) &(2.3)\\ &\quad+\frac{1}{2}\sum_{v\in\Sigma_\infty}\psi(l_v-\tfrac{1}{2})\Bigg\}+(-1)^{\#S}\sum_{v\in S}\frac{d}{dz}\bigg|_{z=0}\Upsilon_v^{(z)}(\alpha)\prod_{\substack{w\in S\\ w\neq v}}\Upsilon_w^{(0)}(\alpha)\Bigg] &(2.4)\\ &\quad+\frac{1}{\mathrm{M}(\mathfrak{n})}2\,\mathrm{CT}_{z=1}\zeta_{F,\mathrm{fin}}(z)\,C_l^{(0)}(-1)^{\#S}\Upsilon^{(0)}(\alpha) &(2.5)\\ &\quad+\mathrm{Res}_{z=1}\,\zeta_{F,\mathrm{fin}}(z)\left(1+\frac{3\log D_F}{2\mathrm{M}(\mathfrak{n})}\right)C_l^{(0)}(-1)^{\#S}\Upsilon^{(0)}(\alpha) &(2.6)\\ &\quad+\|\alpha\|\,\mathcal{O}_{\epsilon,\epsilon'}\left(\mathrm{N}(\mathfrak{a})^{(\underline{l}/2+d_F-1-(\sigma+1)/2-\epsilon)/d_F+\epsilon'}\frac{\mathrm{N}(\mathfrak{n})^{-\delta_1+\epsilon}}{\mathrm{M}(\mathfrak{n})}\right) &(2.7)\end{aligned}$$

*for any sufficiently small $\epsilon,\epsilon'>0$ if $z=0$. Here the implied constants above are independent of $\mathfrak{n}$ and $\mathfrak{a}$, and we set $\delta_1=\min(\delta,\delta')-1/2\in(0,1/2]$, where $\delta$ and $\delta'$ are as in Lemmas 2.5 and 2.6, respectively. The function $\psi$ denotes the digamma function.*

*Proof.* By the explicit Jacquet-Zagier type trace formula in Theorem 2.1, we have

$$\begin{aligned}\Lambda_{\mathfrak{n}}^{(z)}(f_\alpha) =&2D_F^{1/2-z}\frac{1}{\mathrm{M}(\mathfrak{n})^{\delta(z=0)}}\left(\prod_{v\in S(\mathfrak{n})}\frac{1}{1+q_v^{\frac{z+1}{2}}}\right)\mathbb{I}^0_{\mathrm{cusp}}(\mathfrak{n}|\alpha,z)\\ =&\frac{2D_F^{1/2-z}}{\mathrm{M}(\mathfrak{n})^{\delta(z=0)}}\left(\prod_{v\in S(\mathfrak{n})}\frac{1}{1+q_v^{\frac{z+1}{2}}}\right)(-1)^{\#S}C(l,\mathfrak{n})^{-1}\\ &\times\left[D_F^{z/4}\{\mathbb{J}^0_{\mathrm{unip}}(\mathfrak{n}|\alpha,z)+\mathbb{J}^0_{\mathrm{unip}}(\mathfrak{n}|\alpha,-z)\}+\mathbb{J}^0_{\mathrm{hyp}}(\mathfrak{n}|\alpha,z)+\mathbb{J}^0_{\mathrm{ell}}(\mathfrak{n}|\alpha,z)\right].\end{aligned}$$

Let us evaluate the unipotent terms $\mathbb{J}^0_{\mathrm{unip}}(\mathfrak{n}|\alpha,\pm z)$. When $0<z<1$, we obtain

$$\begin{aligned}&\frac{2D_F^{1/2-z}}{\mathrm{M}(\mathfrak{n})^{\delta(z=0)}}\left(\prod_{v\in S(\mathfrak{n})}\frac{1}{1+q_v^{\frac{z+1}{2}}}\right)(-1)^{\#S}C(l,\mathfrak{n})^{-1}D_F^{z/4}\mathbb{J}^0_{\mathrm{unip}}(\mathfrak{n}|\alpha,z)\\ =&\zeta_{F,\mathrm{fin}}(1+z)C_l^{(z)}\times(-1)^{\#S}\Upsilon^{(z)}(\alpha).\end{aligned}\tag{2.8}$$

Indeed, the functional equation $\zeta_F(s)=D_F^{1/2-s}\zeta_F(1-s)$ leads us to

$$\zeta_F(-z)=D_F^{1/2+z}\Gamma_{\mathbb{R}}(1+z)^{d_F}\zeta_{F,\mathrm{fin}}(1+z)=D_F^{1/2+z}\{\pi^{(-1-z)/2}\Gamma(\tfrac{1+z}{2})\}^{d_F}\zeta_{F,\mathrm{fin}}(1+z).$$

Hence the left-hand side of (2.8) is transformed into

$$2D_F^{3/2}\left\{\prod_{v\in\Sigma_\infty}\frac{l_v-1}{4\pi}2^{1-z}\pi^{(3-z)/4}\frac{\Gamma(l_v+\frac{z-1}{2})}{\Gamma(\frac{z+1}{4})\Gamma(l_v)}\pi^{(-1-z)/2}\Gamma(\tfrac{1+z}{2})\right\}\zeta_{F,\mathrm{fin}}(1+z)(-1)^{\#S}\Upsilon^{(z)}(\alpha).$$

This coincides with the right-hand side of (2.8) with the aid of $\Gamma(l_v)=(l_v-1)\Gamma(l_v-1)$ and the duplication formula $\Gamma(\frac{1+z}{2})=2^{(-1+z)/2}\pi^{-1/2}\Gamma(\frac{1+z}{4})\Gamma(\frac{3+z}{4})$.

In the same way, we have

$$\begin{aligned}&\frac{2D_F^{1/2-z}}{\mathrm{M}(\mathfrak{n})^{\delta(z=0)}}\left(\prod_{v\in S(\mathfrak{n})}\frac{1}{1+q_v^{\frac{z+1}{2}}}\right)(-1)^{\#S}C(l,\mathfrak{n})^{-1}D_F^{z/4}\mathbb{J}^0_{\mathrm{unip}}(\mathfrak{n}|\alpha,-z)\\ =&\left(\prod_{v\in S(\mathfrak{n})}\frac{1+q_v^{\frac{-z+1}{2}}}{1+q_v^{\frac{z+1}{2}}}\right)D_F^{-3z/2}\times\zeta_{F,\mathrm{fin}}(1-z)C_l^{(-z)}\times(-1)^{\#S}\Upsilon^{(-z)}(\alpha).\end{aligned}$$

Next let us consider the case $z=0$. We start from the expression

$$D_F^{-3z/2}\prod_{v\in S(\mathfrak{n})}\frac{1+q_v^{\frac{-z+1}{2}}}{1+q_v^{\frac{z+1}{2}}}=1+k_1z+\mathcal{O}(z^2),$$

$$\zeta_{F,\mathrm{fin}}(1+z)=\frac{c_{-1}}{z}+c_0+\mathcal{O}(z),$$

$$C_l^{(z)}\,(-1)^{\#S}\Upsilon^{(z)}(\alpha)=a_0+a_1z+\mathcal{O}(z^2),$$

where we set

$$k_1=-\frac{3}{2}\log D_F-\mathrm{M}(\mathfrak{n}),\quad c_{-1}=\mathrm{Res}_{s=1}\,\zeta_{F,\mathrm{fin}}(s),\quad c_0=\mathrm{CT}_{s=1}\zeta_{F,\mathrm{fin}}(s),$$

$$a_0=C_l^{(0)}\,(-1)^{\#S}\Upsilon^{(0)}(\alpha),$$

$$a_1 = C_l^{(0)}(-1)^{\#S}\Upsilon^{(0)}(\alpha) \times \Bigg[d_F\left\{-\frac{1}{2}\log 2 - \frac{3}{4}\log\pi + \frac{1}{4}\psi\left(\frac{3}{4}\right)\right\} + \frac{1}{2}\sum_{v\in\Sigma_\infty}\psi(l_v - \tfrac{1}{2}) + \sum_{v\in S}\frac{\frac{d}{dz}|_{z=0}\Upsilon_v^{(z)}(\alpha)}{\Upsilon_v^{(z)}(\alpha)}\Bigg].$$

Note that $k_1$ and $a_1$ are given by the use of the logarithmic derivative. Hence we evaluate the unipotent terms for $z=0$ as

$$\Bigg[2D_F^{1/2-z}\frac{1}{\mathrm{M}(\mathfrak{n})^{\delta(z=0)}}\Bigg(\prod_{v\in S(\mathfrak{n})}\frac{1}{1+q_v^{\frac{z+1}{2}}}\Bigg)(-1)^{\#S}C(l,\mathfrak{n})^{-1} \times D_F^{z/4}\{\mathbb{J}^0_{\mathrm{unip}}(\mathfrak{n}|\alpha,z)+\mathbb{J}^0_{\mathrm{unip}}(\mathfrak{n}|\alpha,-z)\}\Bigg]\Bigg|_{z=0} = \frac{1}{\mathrm{M}(\mathfrak{n})}\{2(c_{-1}a_1+c_0a_0)-k_1c_{-1}a_0\}.$$

By combining these with Lemmas 2.5 and 2.6, we are done. □

**Remark 2.8.** We need to correct [50, Theorem 1.3] as follows. In the assertion, the supremum of the form $\sup_{z\in[0,\min(1,\sigma)]}$ is considered. However, this should be replaced with $\sup_{z\in\{0\}\cup[\epsilon,\min(1,\sigma)]}$ with any fixed $\epsilon\in(0,\min(1,\sigma))$, since $\mathbb{J}^0_{\mathrm{unip}}(\mathfrak{n}|\alpha,-z)$ for $z\neq 0$ yields the error term
$\zeta_{F,\mathrm{fin}}(1-z)\mathrm{N}(\mathfrak{n})^{-z+\epsilon}\Upsilon^{(-z)}(\alpha)$, which is not bounded in $z\in(0,1]$.

We note the identity $(-1)^{\#S}\Upsilon^{(z)}(\alpha) = \langle\otimes_{v\in S}\lambda_v^{(z)}, f_\alpha\rangle$, where $\otimes_{v\in S}\lambda_v^{(z)}$ is the measure on $[-2,2]^S$ with

$$\langle\lambda_v^{(z)}, f\rangle := \int_{-2}^{2} f(x)\frac{1+q_v^{(z+1)/2}}{\pi}\frac{\sqrt{1-x^2/4}}{(q_v^{(1+z)/4}+q_v^{-(1+z)/4})^2-x^2}dx, \quad f\in C([-2,2])$$

as in [50, §1.2.1]. Indeed, this identity is deduced from the following.

**Lemma 2.9.** *Let $z$ be a real number such that $z\in[-1,1]$. For $m_v\in\mathbb{N}_0$ and $\alpha_v(s) = X_{m_v}(q_v^{-s/2}+q_v^{s/2})$, we have*

$$-\Upsilon_v^{(z)}(\alpha_v) = \langle\lambda_v^{(z)}, f_{\alpha_v}\rangle = \delta(m_v\in 2\mathbb{N}_0)\, q_v^{-m_v(z+1)/4}.$$

*Proof.* It follows from a direct computation. For the reader, we show the detail as follows. By taking $c_v = c = 0$ and noting $\alpha_v(-s) = \alpha_v(s)$, we have

$$\begin{aligned}\Upsilon_v^{(z)}(\alpha_v) =& \frac{1}{2\pi i}\left(\int_{0i}^{\frac{2\pi}{\log q_v}i} + \int_{-\frac{2\pi}{\log q_v}i}^{0i}\right)\frac{-q_v^{-s/2}q_v^{-1/2}}{1-q_v^{-s}q_v^{-(z+1)/2}}\alpha_v(s)\times\frac{\log q_v}{2}(q_v^{(1+s)/2}-q_v^{(1-s)/2})ds \\ =& \frac{1}{2\pi i}\int_{0i}^{\frac{2\pi}{\log q_v}i}\left(\frac{-q_v^{-s/2}q_v^{-1/2}}{1-q_v^{-s}q_v^{-(z+1)/2}} + \frac{q_v^{-s/2}q_v^{-1/2}}{1-q_v^{s}q_v^{-(z+1)/2}}\right)\alpha_v(s) \\ &\times\frac{\log q_v}{2}(q_v^{(1+s)/2}-q_v^{(1-s)/2})ds \\ =& -\frac{1}{2\pi i}\int_{-2}^{2}\frac{(1+q_v^{(z+1)/2})i\sqrt{4-x^2}}{(q_v^{(z+1)/4}+q_v^{-(z+1)/4})^2-x^2}X_{m_v}(x)dx.\end{aligned}$$

The last equality is deduced by the change of variables $x = q_v^{-s/2} + q_v^{s/2}$. The integral in the last line of the equalities above is equal to $-\langle \lambda_v^{(z)}, f_{\alpha_v}\rangle$. By applying [46, §2.3] for $q = q_v^{\frac{z+1}{4}}$, we obtain the assertion. □

Define $f_{\mathfrak{a}} \in C([-2,2]^S)$ by $f_{\mathfrak{a}}((x_v)_{v\in S}) = \prod_{v\in S} X_{n_v}(x_v)$ for an integral ideal $\mathfrak{a} = \prod_{v\in S}(\mathfrak{p}_v \cap \mathfrak{o})^{n_v}$. By Theorem 2.7, we have the following.

**Corollary 2.10.** *Fix any $\sigma \in (0,1]$ and suppose $\underline{l} > \sigma + 3$. For $0 < z \leqslant 1$, we have*

$$\frac{\Lambda_{\mathfrak{n}}^{(z)}(f_{\mathfrak{a}})}{\Lambda_{\mathfrak{n}}^{(z)}(1)} = \delta(\mathfrak{a} = \square)\mathrm{N}(\mathfrak{a})^{(-z-1)/4} + \mathcal{O}_{\epsilon,\epsilon'}(\mathrm{N}(\mathfrak{a})^{(\underline{l}/2+d_F-1-(\sigma+1)/2)/d_F-\epsilon/d_F+\epsilon'}\mathrm{N}(\mathfrak{n})^{-\delta_1+\epsilon})$$

*for sufficiently small $\epsilon, \epsilon' > 0$, where the implied constant is independent of $\mathfrak{n}$, $\mathfrak{a}$ and $z \in (0,1]$.*

*When $z = 0$, we have*

$$\frac{\Lambda_{\mathfrak{n}}^{(0)}(f_{\mathfrak{a}})}{\Lambda_{\mathfrak{n}}^{(0)}(1)} = \delta(\mathfrak{a} = \square)\mathrm{N}(\mathfrak{a})^{-1/4} - \frac{1}{2}(\log \mathrm{N}(\mathfrak{a}))\delta(\mathfrak{a} = \square)\mathrm{N}(\mathfrak{a})^{-1/4}C_l^{(0)}\frac{\mathrm{Res}_{z=1}\,\zeta_{F,\mathrm{fin}}(z)}{\mathrm{M}(\mathfrak{n})D(\mathfrak{n})}$$
$$+ \mathcal{O}_{\epsilon,\epsilon'}\left(\mathrm{N}(\mathfrak{a})^{(\underline{l}/2+d_F-1-(\sigma+1)/2)/d_F-\epsilon/d_F+\epsilon'}\frac{\mathrm{N}(\mathfrak{n})^{-\delta_1+\epsilon}}{\mathrm{M}(\mathfrak{n})}\right),$$

*where*

$$D(\mathfrak{n}) := \frac{1}{\mathrm{M}(\mathfrak{n})}2\ \mathrm{Res}_{z=1}\,\zeta_{F,\mathrm{fin}}(z)\,C_l^{(0)}\left\{d_F\left(-\frac{1}{2}\log 2 - \frac{3}{4}\log\pi + \frac{1}{4}\psi\left(\frac{3}{4}\right)\right)\right.$$
$$\left. + \frac{1}{2}\sum_{v\in\Sigma_\infty}\psi(l_v - \tfrac{1}{2})\right\}$$
$$+ \frac{1}{\mathrm{M}(\mathfrak{n})}2\,\mathrm{CT}_{z=1}\zeta_{F,\mathrm{fin}}(z)\,C_l^{(0)} + \mathrm{Res}_{z=1}\,\zeta_{F,\mathrm{fin}}(z)\left(1 + \frac{3\log D_F}{2\mathrm{M}(\mathfrak{n})}\right)C_l^{(0)},$$

*and the implied constant is independent of $\mathfrak{n}$ and $\mathfrak{a}$.*

*Proof.* We follow the proof of [32, Proposition 3.1]. Denote by $F_{\mathfrak{a}}$ the main term (2.1) (resp. the sum of (2.3), (2.4), (2.5) and (2.6)) for $z \neq 0$ (resp. $z = 0$) in Theorem 2.7, and put $E_{\mathfrak{a}} = \Lambda_{\mathfrak{n}}^{(z)}(f_{\mathfrak{a}}) - F_{\mathfrak{a}}$. Then, we see

$$\frac{\Lambda_{\mathfrak{n}}^{(z)}(f_{\mathfrak{a}})}{\Lambda_{\mathfrak{n}}^{(z)}(1)} = \frac{F_{\mathfrak{a}} + E_{\mathfrak{a}}}{F_{\mathfrak{o}} + E_{\mathfrak{o}}} = \frac{F_{\mathfrak{a}}}{F_{\mathfrak{o}}} + \frac{E_{\mathfrak{a}} - \frac{F_{\mathfrak{a}}}{F_{\mathfrak{o}}}E_{\mathfrak{o}}}{F_{\mathfrak{o}} + E_{\mathfrak{o}}}.$$

The first term above yields the main term of the assertion. Indeed, for $z \neq 0$, the explicit form of the main term in the assertion is given by Lemma 2.9. We estimate the second term of the assertion as

$$\frac{E_{\mathfrak{a}} - \frac{F_{\mathfrak{a}}}{F_{\mathfrak{o}}}E_{\mathfrak{o}}}{F_{\mathfrak{o}} + E_{\mathfrak{o}}}$$
$$\ll \frac{E_{\mathfrak{a}} + \delta(\mathfrak{a} = \square)\mathrm{N}(\mathfrak{a})^{(-z-1)/4}E_{\mathfrak{o}}}{F_{\mathfrak{o}} + E_{\mathfrak{o}}}$$
$$\ll_{\epsilon,\epsilon'} \frac{1}{F_{\mathfrak{o}} + E_{\mathfrak{o}}}(\mathrm{N}(\mathfrak{a})^{(\underline{l}/2+d_F-1-(\sigma+1)/2)/d_F-\epsilon/d_F+\epsilon'}\mathrm{N}(\mathfrak{n})^{-\delta_1+\epsilon} + \delta(\mathfrak{a} = \square)\mathrm{N}(\mathfrak{a})^{(-z-1)/4}\mathrm{N}(\mathfrak{n})^{-\delta_1+\epsilon})$$

$$\ll \mathrm{N}(\mathfrak{a})^{(\underline{l}/2+d_F-1-(\sigma+1)/2)/d_F-\epsilon/d_F+\epsilon'}\mathrm{N}(\mathfrak{n})^{-\delta_1+\epsilon},$$

where $\|\alpha\|$ in the error term (2.2) is negligible by Lemma 2.4. This completes the proof for $z \neq 0$. Next consider the case $z = 0$. The first term $\dfrac{F_{\mathfrak{a}}}{F_{\mathfrak{o}}}$ is similarly evaluated as

$$\frac{D(\mathfrak{n})(-1)^{\#S}\Upsilon^{(0)}(\alpha) + \frac{2}{\mathrm{M}(\mathfrak{n})}\operatorname{Res}_{z=1}\zeta_{F,\mathrm{fin}}(z)C_l^{(0)}(-1)^{\#S}\sum_{v\in S}\frac{d}{dz}\big|_{z=0}\Upsilon_v^{(z)}(\alpha)\prod_{\substack{w\in S\\ w\neq v}}\Upsilon_w^{(0)}(\alpha)}{D(\mathfrak{n})}$$

$$= \delta(\mathfrak{a} = \square)\mathrm{N}(\mathfrak{a})^{-1/4} - \frac{1}{4}(\log \mathrm{N}(\mathfrak{a}))\delta(\mathfrak{a} = \square)\mathrm{N}(\mathfrak{a})^{-1/4}C_l^{(0)}\frac{2\operatorname{Res}_{z=1}\zeta_{F,\mathrm{fin}}(z)}{\mathrm{M}(\mathfrak{n})D(\mathfrak{n})}.$$

Here we take $\alpha \in \mathcal{A}(\mathfrak{a})$ such that $f_{\mathfrak{a}} = f_\alpha$. By $\lim_{\mathrm{N}(\mathfrak{n})\to\infty}(F_{\mathfrak{o}} + E_{\mathfrak{o}}) = \operatorname{Res}_{z=1}\zeta_{F,\mathrm{fin}}(z)C_l^{(0)}$, the error term is estimated as

$$\frac{E_{\mathfrak{a}} - \frac{F_{\mathfrak{a}}}{F_{\mathfrak{o}}}E_{\mathfrak{o}}}{F_{\mathfrak{o}} + E_{\mathfrak{o}}} \ll_\epsilon \frac{E_{\mathfrak{a}} + \delta(\mathfrak{a} = \square)\mathrm{N}(\mathfrak{a})^{-1/4+\epsilon}E_{\mathfrak{o}}}{F_{\mathfrak{o}} + E_{\mathfrak{o}}} \ll_\epsilon \mathrm{N}(\mathfrak{a})^{c+\epsilon}\frac{\mathrm{N}(\mathfrak{n})^{-\delta_1+\epsilon}}{\mathrm{M}(\mathfrak{n})}.$$

Here $\|\alpha\|$ in (2.7) is negligible by Lemma 2.4. This completes the proof for $z = 0$. $\square$

We recall that $\{\mathfrak{a}_j\}_{j=1}^h$ is the complete system of representatives for the ideal class group of $F$ consisting of prime ideals relatively prime to $S$, as is fixed in §2.2. From now, $\mathfrak{n}$ is assumed to be a prime ideal $\mathfrak{q}$ of $\mathfrak{o}$ relatively prime to $2\prod_{j=1}^h \mathfrak{a}_j$ and $S$. We also recall the definition of $f_{\mathfrak{a}} \in C([-2,2]^S)$:

$$f_{\mathfrak{a}}((x_v)_{v\in S}) = \prod_{v\in S} X_{n_v}(x_v)$$

for an integral ideal $\mathfrak{a} = \prod_{v\in S}(\mathfrak{p}_v \cap \mathfrak{o})^{n_v}$ with $n_v \in \mathbb{N}_0$.

**Lemma 2.11.** *For $z \in [0,1]$, we have*

$$I_{\mathfrak{q}}^{(z)}(f_{\mathfrak{a}}) := \frac{1}{\mathrm{M}(\mathfrak{q})^{\delta(z=0)}}\frac{\mathrm{N}(\mathfrak{q})^{(z-1)/2}}{1 + \mathrm{N}(\mathfrak{q})^{(z+1)/2}}\sum_{\pi\in\Pi_{\mathrm{cus}}(l,\mathfrak{o})} W_{\mathfrak{q}}^{(z)}(\pi)\frac{L(\frac{z+1}{2}, \mathrm{Sym}^2(\pi))}{L(1, \mathrm{Sym}^2(\pi))}f_{\mathfrak{a}}(\mathbf{x}_S(\pi))$$
$$\ll_\epsilon \mathrm{N}(\mathfrak{q})^{(-1-z)/2}\mathrm{N}(\mathfrak{a})^\epsilon,$$

*for any $\epsilon > 0$, where the implied constant is independent of $\mathfrak{a}$ and $\mathfrak{q}$.*

*Proof.* Recall the inequality $-1 < Q(\tau) < 1$ for any irreducible unitarizable spherical representations $\tau$ of $\mathrm{PGL}_2(F_v)$ at any $v \in \Sigma_{\mathrm{fin}}$. The assertion follows from the inequality

$$0 < W_{\mathfrak{q}}^{(z)}(\pi) = \mathrm{N}(\mathfrak{q})^{(1-z)/2}\left\{2 - \frac{1 - Q(I(|\cdot|_{\mathfrak{q}}^{z/2}))}{1 - Q(\pi_{\mathfrak{q}})^2}\right\} \leqslant 2\,\mathrm{N}(\mathfrak{q})^{(1-z)/2}$$

for any $\pi \in \Pi_{\mathrm{cus}}(l, \mathfrak{o})$ and the estimate $\max_{x\in[-2,2]}|X_m(x)| \leqslant m + 1$. $\square$

Let us define $\Lambda_{\mathfrak{q}}^{(z),*}(f)$ for any $f \in C([-2,2]^S)$ similarly to $\Lambda_{\mathfrak{q}}^{(z)}(f)$ but restricting the range of the summation from $\Pi_{\mathrm{cus}}(l,\mathfrak{q})$ to $\Pi_{\mathrm{cus}}^*(l,\mathfrak{q})$. Here $\Pi_{\mathrm{cus}}^*(l,\mathfrak{q})$ is the set of all $\pi \in \Pi_{\mathrm{cus}}(l,\mathfrak{q})$ such that $\mathfrak{f}_\pi = \mathfrak{q}$ as in §1.1. Thus we have the identity $\Lambda_{\mathfrak{q}}^{(z),*}(f) = \Lambda_{\mathfrak{q}}^{(z)}(f) - I_{\mathfrak{q}}^{(z)}(f)$ for any $f \in C([-2,2]^S)$, where $I_{\mathfrak{q}}^{(z)}(f)$ is defined in the same way as in Lemma 2.11.

**Corollary 2.12.** *Let $\mathfrak{a}=\prod_{v\in S}(\mathfrak{p}_v\cap\mathfrak{o})^{m_v}$ be an ideal of $\mathfrak{o}$. Let $\mathfrak{q}$ vary in the set of prime ideals of $\mathfrak{o}$ relatively prime to $2\mathfrak{a}\prod_{j=1}^{h}\mathfrak{a}_j$. For $0<z\leqslant 1$, we have*

$$\frac{\Lambda_{\mathfrak{q}}^{(z),*}(f_{\mathfrak{a}})}{\Lambda_{\mathfrak{q}}^{(z),*}(1)}=\delta(\mathfrak{a}=\square)\mathrm{N}(\mathfrak{a})^{(-z-1)/4}+\mathcal{O}_{\epsilon,\epsilon'}(\mathrm{N}(\mathfrak{a})^{(\underline{l}/2+d_F-1-(\sigma+1)/2)/d_F-\epsilon/d_F+\epsilon'}\mathrm{N}(\mathfrak{q})^{-\delta_1+\epsilon})$$

*for any sufficiently small $\epsilon,\epsilon'>0$, where the implied constant is independent of $\mathfrak{n}$, $\mathfrak{a}$ and $z$. When $z=0$, we have*

$$\begin{aligned}\frac{\Lambda_{\mathfrak{q}}^{(0),*}(f_{\mathfrak{a}})}{\Lambda_{\mathfrak{q}}^{(0),*}(1)}=&\,\delta(\mathfrak{a}=\square)\mathrm{N}(\mathfrak{a})^{-1/4}-\delta(\mathfrak{a}=\square)\frac{1}{2}\mathrm{N}(\mathfrak{a})^{-1/4}\frac{\log\mathrm{N}(\mathfrak{a})}{\log\mathrm{N}(\mathfrak{q})}\left\{1+\mathcal{O}\left(\frac{1}{\log\mathrm{N}(\mathfrak{q})}\right)\right\}\\&+\mathcal{O}_{\epsilon,\epsilon'}\left(\mathrm{N}(\mathfrak{a})^{(\underline{l}/2+d_F-1-(\sigma+1)/2)/d_F-\epsilon/d_F+\epsilon'}\frac{\mathrm{N}(\mathfrak{q})^{-\delta_1+\epsilon}}{\log\mathrm{N}(\mathfrak{q})}\right),\end{aligned}$$

*where the implied constant is independent of $\mathfrak{n}$ and $\mathfrak{a}$.*

*Proof.* Invoking $\Lambda_{\mathfrak{q}}^{(z),*}(f_{\mathfrak{a}})=\Lambda_{\mathfrak{q}}^{(z)}(f_{\mathfrak{a}})-I_{\mathfrak{q}}^{(z)}(f_{\mathfrak{a}})$, the same proof of Corollary 2.10 goes through with the aid of Lemma 2.11. We remark

$$\frac{C_l^{(0)}\operatorname{Res}_{z=1}\zeta_{F,\mathrm{fin}}(z)}{D(\mathfrak{q})}=1+\mathcal{O}\left(\frac{1}{\mathrm{M}(\mathfrak{q})}\right)=1+\mathcal{O}\left(\frac{1}{\log\mathrm{N}(\mathfrak{q})}\right)$$

as $\mathrm{N}(\mathfrak{q})\to\infty$. $\square$

## 3. Weighted distributions of low-lying zeros

3.1. **Symmetric power $L$-functions.** Let $F$ be a totally real number field such that $2\in\mathbb{Q}$ is completely splitting in $F$ as in §2. Let $l=(l_v)_{v\in\Sigma_\infty}$ be a family of positive even integers and $\mathfrak{q}$ a prime ideal of $\mathfrak{o}$, and fix $r\in\mathbb{N}$. For $\pi\in\Pi_{\mathrm{cus}}^*(l,\mathfrak{q})$, we define the completed symmetric power $L$-function $L(s,\mathrm{Sym}^r(\pi))=\prod_{v\in\Sigma_\infty\cup\Sigma_{\mathrm{fin}}}L(s,\mathrm{Sym}^r(\pi_v))$ as in [12, §3] (see also [44, §2.1.4]). First we define the local $L$-factors of $\mathrm{Sym}^r(\pi_v)$ as follows. For $v\in\Sigma_\infty$, set

$$L(s,\mathrm{Sym}^r(\pi_v))=\prod_{j=0}^{\frac{r-1}{2}}\Gamma_{\mathbb{R}}\left(s+(2j+1)\frac{l_v-1}{2}\right)\Gamma_{\mathbb{R}}\left(s+1+(2j+1)\frac{l_v-1}{2}\right)$$

if $r$ is odd, and

$$L(s,\mathrm{Sym}^r(\pi_v))=\Gamma_{\mathbb{R}}(s+\mu_r)\prod_{j=1}^{\frac{r}{2}}\Gamma_{\mathbb{R}}\left(s+j(l_v-1)\right)\Gamma_{\mathbb{R}}\left(s+1+j(l_v-1)\right)$$

with $\mu_r=\delta(r/2\in 2\mathbb{N}_0+1)\in\{0,1\}$ if $r$ is even. For $v\in\Sigma_{\mathrm{fin}}-S(\mathfrak{q})$, set

$$L(s,\mathrm{Sym}^r(\pi_v))=\det\left(1_{r+1}-q_v^{-s}\,\mathrm{Sym}^r\left(\begin{smallmatrix}q_v^{-\nu_v(\pi)/2}&0\\0&q_v^{\nu_v(\pi)/2}\end{smallmatrix}\right)\right)^{-1}=\prod_{j=0}^{r}(1-(q_v^{\nu_v(\pi)/2})^{2j-r}q_v^{-s})^{-1},$$

where $1_{r+1}$ is the $(r+1)\times(r+1)$ unit matrix and $\mathrm{Sym}^r:\mathrm{GL}_2(\mathbb{C})\to\mathrm{GL}_{r+1}(\mathbb{C})$ is the $r$th symmetric tensor representation. At $v=\mathfrak{q}$, the conductor of $\pi_v$ equals $\mathfrak{p}_v=\mathfrak{q}\mathfrak{o}_v$ and $\pi_v$

is isomorphic to $\chi_v \otimes \mathrm{St}_2$, where $\mathrm{St}_2$ is the Steinberg representation of $\mathrm{GL}_2(F_v)$ and $\chi_v$ is an unramified character of $F_v^\times$ such that $\chi_v^2$ is trivial. Set

$$L(s, \mathrm{Sym}^r(\pi_v)) = (1 - \chi_v(\varpi_v)^r q_v^{-s-r/2})^{-1}.$$

Then, $L(s, \mathrm{Sym}^r(\pi))$ is expected to have an analytic continuation to $\mathbb{C}$ and the functional equation $L(s, \mathrm{Sym}^r(\pi)) = \epsilon_{\pi,r}(D_F^{r+1}\mathrm{N}(\mathfrak{q})^r)^{1/2-s}L(1-s, \mathrm{Sym}^r(\pi))$, where $\epsilon_{\pi,r} \in \{\pm 1\}$ is defined as

$$\epsilon_{\pi,r} = \begin{cases} \{\prod_{v\in\Sigma_\infty}\prod_{j=0}^{(r-1)/2} i^{(2j+1)(l_v-1)+1}\}(-\chi_{\mathfrak{q}}(\varpi_{\mathfrak{q}})^r)^r & (r \in 2\mathbb{N}_0+1), \\ 1 & (r \in 2\mathbb{N}) \end{cases}$$

by [12, §3]. The local $L$-factors defined above are compatible with the local Langlands correspondence for $\mathrm{GL}_n$.

We recall that $\{\mathfrak{a}_j\}_{j=1}^h$ is the complete system of representatives for the ideal class group of $F$ consisting of prime ideals relatively prime to $S$ which is fixed in §2.2. Throughout this article, we assume $\mathrm{Nice}(\pi, r)$ in the introduction for all $\pi \in \Pi_{\mathrm{cus}}^*(l, \mathfrak{q})$ and for all prime ideals $\mathfrak{q}$ relatively prime to $2\prod_{j=1}^h \mathfrak{a}_j$ for a fixed $l$.

Here we review known results related to the hypothesis $\mathrm{Nice}(\pi, r)$. In the case $r = 1$, this hypothesis is well-known to be true. In our setting, $\pi$ is non-CM since $\mathfrak{f}_\pi$ is square-free and the central character of $\pi$ is trivial. Thus, $\mathrm{Nice}(\pi, r)$ is known for $r = 2$ by Gelbart and Jacquet [21, (9.3) Theorem], $r = 3$ by Kim and Shahidi [29, Corollary 6.4] and $r = 4$ by Kim [28, Theorem B and §7.2], respectively. For any $r \in \mathbb{N}$, the meromorphy of $L(s, \mathrm{Sym}^r(\pi))$ and its functional equation can be proved by the potential automorphy of the Galois representation $\mathrm{Sym}^r \circ \rho_\pi$, where $\rho_\pi$ is the Galois representation attached to $\pi$. See Harris, Shepherd-Barron and Taylor [24] for non-CM elliptic curves over a totally real number field with multiplicative reduction at a finite place, Gee [20] for non-CM elliptic modular forms of weight 3 with a twisted Steinberg representation at a prime, Barnet-Lamb, Geraghty, Harris and Taylor [4] for non-CM elliptic modular forms of general weight, level and nebentypus, and Barnet-Lamb, Gee and Geraghty [3] for Hilbert modular forms.

Automorphy of $\mathrm{Sym}^r(\pi)$ with higher $r$ has been studied and is known in several cases. When $\pi$ is attached to a holomorphic elliptic cusp form of level 1, $\mathrm{Sym}^5(\pi)$ is automorphic by Dieulefait [14]. For some class of totally real number fields $F$ and for any non-CM regular $C$-algebraic irreducible cuspidal automorphic representation $\pi$ of $\mathrm{GL}_2(\mathbb{A}_F)$, the lift $\mathrm{Sym}^r(\pi)$ is automorphic when $r = 5, 7$ by Clozel and Thorne [10, Corollary 1.3] and when $r = 6, 8$ by Clozel and Thorne [11, Corollary 1.2]. As a recent remarkable work, if we restrict our case to elliptic modular forms ($F = \mathbb{Q}$), the hypothesis $\mathrm{Nice}(\pi, r)$ holds true for all $r \in \mathbb{N}$, all $\pi \in \Pi_{\mathrm{cus}}^*(l, \mathfrak{q})$ and all prime ideals $\mathfrak{q}$ since $\mathrm{Sym}^r(\pi)$ is automorphic and cuspidal by Newton and Thorne [40]. They generalized it to [41], which covers general levels of elliptic modular forms. Very recently, they announced in [42] the automorphy of symmetric power liftings of Hilbert modular forms. Therefore we strongly believe that $\mathrm{Nice}(\pi, r)$ is no longer needed.

Let $\phi$ be an even Schwartz function on $\mathbb{R}$ whose Fourier transform $\hat{\phi}$ is compactly supported. In the same manner as [23, Lemma 2.6] and [44, Proposition 3.8], the explicit formula for $L(s,\mathrm{Sym}^r(\pi))$ à la Weil is stated as

$$\begin{aligned}
&D(\mathrm{Sym}^r(\pi),\phi)\\
&=\hat{\phi}(0)+\frac{(-1)^{r+1}}{2}\phi(0)-\frac{2}{\log Q(\mathrm{Sym}^r(\pi))}\sum_{v\in\Sigma_{\mathrm{fin}}-S(\mathfrak{q})}\lambda_\pi(\mathfrak{p}_v^r)\frac{\log q_v}{q_v^{1/2}}\hat{\phi}\left(\frac{\log q_v}{\log Q(\mathrm{Sym}^r(\pi))}\right)\\
&\quad-\sum_{m=0}^{r-1}(-1)^m\frac{2}{\log Q(\mathrm{Sym}^r(\pi))}\sum_{v\in\Sigma_{\mathrm{fin}}-S(\mathfrak{q})}\lambda_\pi(\mathfrak{p}_v^{2(r-m)})\frac{\log q_v}{q_v}\hat{\phi}\left(\frac{2\log q_v}{\log Q(\mathrm{Sym}^r(\pi))}\right)\\
&\quad+\mathcal{O}\left(\frac{1}{\log Q(\mathrm{Sym}^r(\pi))}\right),\qquad \mathrm{N}(\mathfrak{q})\to\infty.
\end{aligned}$$

Here $Q(\mathrm{Sym}^r(\pi)):=(\prod_{v\in\Sigma_\infty}l_v^{2\lfloor\frac{r+1}{2}\rfloor})\mathrm{N}(\mathfrak{q})^r$ is the analytic conductor of $\mathrm{Sym}^r(\pi)$ and set

$$\lambda_\pi(\mathfrak{p}_v^m)=\sum_{j=0}^{r}(q_v^{-m\nu_v(\pi)/2})^j(q_v^{m\nu_v(\pi)/2})^{r-j},\qquad m\in\mathbb{N}.$$

Then $\lambda_\pi(\mathfrak{p}_v^m)=X_m(\lambda_\pi(\mathfrak{p}_v))$ holds, where $X_m$ is the polynomial defined by $\frac{\sin(m+1)\theta}{\sin\theta}=X_m(2\cos\theta)$ as in §2.2. We remark that the explicit formula à la Weil above is still valid even if $Q(\mathrm{Sym}^r(\pi))$ is replaced with $Q_r:=\mathrm{N}(\mathfrak{q})^r$.

We consider the averaged one-level density of low-lying zeros of $L(s,\mathrm{Sym}^r(\pi))$ weighted by special values $L(\frac{z+1}{2},\mathrm{Sym}^2(\pi))$. For $l=(l_v)_{v\in\Sigma_\infty}$ with $\underline{l}:=\min_{v\in\Sigma_\infty}l_v\geqslant 4$, a prime ideal $\mathfrak{q}$, $z\in[0,\min(1,\sigma)]$ with a fixed $\sigma\in(0,\underline{l}-3)$, and a map $A_\bullet:\Pi_{\mathrm{cus}}^*(l,\mathfrak{q})\to\mathbb{C}\ ;\ \pi\mapsto A_\pi$, set

$$\mathcal{E}_z(A_\bullet)=\frac{1}{\sum_{\pi\in\Pi_{\mathrm{cus}}^*(l,\mathfrak{q})}\frac{L(\frac{z+1}{2},\mathrm{Sym}^2(\pi))}{L(1,\mathrm{Sym}^2(\pi))}}\sum_{\pi\in\Pi_{\mathrm{cus}}^*(l,\mathfrak{q})}\frac{L(\frac{z+1}{2},\mathrm{Sym}^2(\pi))}{L(1,\mathrm{Sym}^2(\pi))}A_\pi.$$

**Proposition 3.1.** *For any $r\in\mathbb{N}$ and $z\in(0,\min(1,\sigma)]$, set*

$$\beta_0=\frac{\delta_1}{r\{r(\frac{\underline{l}}{2}+d_F-1-\frac{z+1}{2})\frac{1}{d_F}+\frac{1}{2}\}}>0,$$

*where $\delta_1>0$ is the number defined in Theorem 2.7. Then, for any even Schwartz function $\phi$ on $\mathbb{R}$ with $\mathrm{supp}(\hat{\phi})\subset(-\beta_0,\beta_0)$, we have*

$$\mathcal{E}_z(D(\mathrm{Sym}^r(\bullet),\phi))=\hat{\phi}(0)+\frac{(-1)^{r+1}}{2}\phi(0)+\mathcal{O}\left(\frac{1}{\log Q_r}\right),\qquad \mathrm{N}(\mathfrak{q})\to\infty.$$

*Proof.* We assume $\beta>0$ and $\mathrm{supp}(\hat{\phi})\subset[-\beta,\beta]$, where $\beta>0$ is suitably chosen later. We start evaluation from the expression

$$\begin{aligned}
&\mathcal{E}_z(D(\mathrm{Sym}^r(\bullet),\phi)) &&(3.1)\\
&=\hat{\phi}(0)+\frac{(-1)^{r+1}}{2}\phi(0)+\mathcal{O}\left(\frac{1}{\log Q_r}\right)\\
&\quad-\frac{2}{\log Q_r}\sum_{v\in\Sigma_{\mathrm{fin}}-S(\mathfrak{q})}\mathcal{E}_z(\lambda_\bullet(\mathfrak{p}_v^r))\frac{\log q_v}{q_v^{1/2}}\hat{\phi}\left(\frac{\log q_v}{\log Q_r}\right) &&(3.2)
\end{aligned}$$

(3.3) $$-\sum_{m=0}^{r-1}(-1)^m\frac{2}{\log Q_r}\sum_{v\in\Sigma_{\mathrm{fin}}-S(\mathfrak{q})}\mathcal{E}_z(\lambda_\bullet(\mathfrak{p}_v^{2(r-m)}))\frac{\log q_v}{q_v}\hat{\phi}\left(\frac{2\log q_v}{\log Q_r}\right)$$

as $\mathrm{N}(\mathfrak{q})\to\infty$. We denote the terms (3.2) and (3.3) by $M^{(1)}$ and $M^{(2)}$, respectively. Since $\mathrm{N}(\mathfrak{q})$ tends to infinity, we may assume that $\mathfrak{q}$ is relatively prime to $2\mathfrak{a}\prod_{j=1}^{h}\mathfrak{a}_j$. Set

$$A=(\underline{l}/2+d_F-1-(\sigma+1)/2)/d_F-\epsilon/d_F+\epsilon'$$

for any fixed sufficiently small $\epsilon,\epsilon'>0$. For any fixed $v\in\Sigma_{\mathrm{fin}}-S(\mathfrak{q})$, we have $\lambda_\pi(\mathfrak{p}_v^n)=X_n(\lambda_\pi(\mathfrak{p}_v))=f_{\mathfrak{a}_v}(\mathbf{x}_{\{v\}}(\pi))$, where $\mathfrak{a}_v:=(\mathfrak{p}_v\cap\mathfrak{o})^n$ for any fixed $n\in\mathbb{N}$. As for the weight factor in $\Lambda_{\mathfrak{q}}^{(z),*}(f_{\mathfrak{a}_v})$, the identity $W_{\mathfrak{q}}^{(z)}(\pi)=1$ holds when $\pi\in\Pi_{\mathrm{cus}}(l,\mathfrak{q})$ with $\mathfrak{f}_\pi=\mathfrak{q}$ (i.e., $\pi\in\Pi^*_{\mathrm{cus}}(l,\mathfrak{q})$). Therefore Corollary 2.12 for $S=\{v\}$ and $\mathfrak{a}=\mathfrak{a}_v$ yields

$$\mathcal{E}_z(\lambda_\bullet(\mathfrak{p}_v^r))=\frac{\Lambda_{\mathfrak{q}}^{(z),*}(f_{\mathfrak{a}_v})}{\Lambda_{\mathfrak{q}}^{(z),*}(1)}=\delta(r\in 2\mathbb{N})(q_v^r)^{-(z+1)/4}+\mathcal{O}_{\epsilon,\epsilon'}((q_v^r)^A\mathrm{N}(\mathfrak{q})^{-\delta_1+\epsilon}).$$

Hence $M^{(1)}$ is evaluated as

$$\begin{aligned}M^{(1)}&=-\frac{2}{\log Q_r}\sum_{v\in\Sigma_{\mathrm{fin}}-S(\mathfrak{q})}\{\delta(r\in 2\mathbb{N})(q_v^r)^{-(z+1)/4}+\mathcal{O}_{\epsilon,\epsilon'}((q_v^r)^A\mathrm{N}(\mathfrak{q})^{-\delta_1+\epsilon})\}\frac{\log q_v}{q_v^{1/2}}\hat{\phi}\left(\frac{\log q_v}{\log Q_r}\right)\\ &=-\frac{2\delta(r\in 2\mathbb{N})}{\log Q_r}\sum_{v\in\Sigma_{\mathrm{fin}}}\frac{\log q_v}{q_v^{1/2+r/4+rz/4}}+\mathcal{O}\left(\frac{1}{\log Q_r}\right)+\mathcal{O}\Bigg(\sum_{\substack{v\in\Sigma_{\mathrm{fin}}-S(\mathfrak{q})\\ q_v\leqslant Q_r^\beta}}q_v^{rA}\frac{\log q_v}{q_v^{1/2}}\frac{\mathrm{N}(\mathfrak{q})^{-\delta_1+\epsilon}}{\log Q_r}\Bigg)\\ &=\mathcal{O}\left(\frac{\delta(r\in 2\mathbb{N})}{\log Q_r}\sum_{v\in\Sigma_{\mathrm{fin}}}\frac{\log q_v}{q_v^{1+rz/4}}\right)+\mathcal{O}\left(\frac{\mathrm{N}(\mathfrak{q})^{r\beta(rA+1/2)}\mathrm{N}(\mathfrak{q})^{-\delta_1+\epsilon}}{\log Q_r}\right),\qquad \mathrm{N}(\mathfrak{q})\to\infty.\end{aligned}$$

Here we use the inequality $1/2+r/4+rz/4\geqslant 1+rz/4>1$ for $r\geqslant 2$ and the asymptotics

(3.4) $$\sum_{\substack{v\in\Sigma_{\mathrm{fin}}\\ q_v\leqslant x}}q_v^a\log q_v=\frac{1}{a+1}x^{a+1}+\mathcal{O}\left(\frac{x^{a+1}}{\log x}\right)=\mathcal{O}(x^{a+1}),\qquad x\to\infty$$

for $a>-1$ deduced from the prime ideal theorem and partial summation. Consequently, the estimate $M^{(1)}\ll\frac{1}{\log Q_r}$ holds as long as $\beta\leqslant\frac{\delta_1-\epsilon}{r(rA+1/2)}$.

Furthermore, we obtain $M^{(2)}\ll\frac{1}{\log Q_r}$ because of the estimate

$$\begin{aligned}&\sum_{v\in\Sigma_{\mathrm{fin}}-S(\mathfrak{q})}\{(q_v^{2(r-m)})^{-(z+1)/4}+\mathcal{O}((q_v^{r-m})^A\mathrm{N}(\mathfrak{q})^{-\delta_1+\epsilon})\}\frac{\log q_v}{q_v}\hat{\phi}\left(\frac{\log q_v}{\log Q_r}\right)\\ &\ll\sum_{v\in\Sigma_{\mathrm{fin}}}\frac{\log q_v}{q_v^{1+\frac{(r-m)(z+1)}{2}}}+(\mathrm{N}(\mathfrak{q})^{r\beta})^{rA}\mathrm{N}(\mathfrak{q})^{-\delta_1+\epsilon}\ll 1\end{aligned}$$

for any $0\leqslant m\leqslant r-1$ by virtue of Corollary 2.12 and (3.4), as long as $\beta\leqslant\frac{\delta_1-\epsilon}{r^2A}$. By removing $\epsilon$ and $\epsilon'$ from two inequalities on $\beta$ as above, we are done. □

Next let us consider the central value case $z=0$.

**Proposition 3.2.** *For any $r \in \mathbb{N}$ and $z = 0$, let $\beta_0 > 0$ be the same as in Proposition 3.1 for $z = 0$. Then, for any even Schwartz function $\phi$ on $\mathbb{R}$ with $\mathrm{supp}(\hat{\phi}) \subset (-\beta_0, \beta_0)$, we have*

$$\mathcal{E}_0(D(\mathrm{Sym}^r(\bullet), \phi)) = \hat{\phi}(0) + \frac{(-1)^{r+1}}{2}\phi(0) + \delta_{r,2}\{-\phi(0) + 2\int_{-\infty}^{\infty} \hat{\phi}(x)|x|dx\} + \mathcal{O}\left(\frac{1}{\log \mathrm{N}(\mathfrak{q})}\right)$$

*as $\mathrm{N}(\mathfrak{q}) \to \infty$ with the implied constant independent of $\mathfrak{q}$, where $\delta_{r,2} := \delta(r = 2)$.*

*Proof.* We assume $\beta > 0$ and $\mathrm{supp}(\hat{\phi}) \subset [-\beta, \beta]$, where $\beta > 0$ is suitably chosen later. The formula (3.1) is valid for $z = 0$, and we define $M^{(1)}$ and $M^{(2)}$ in the same way as the proof of Proposition 3.1. With the aid of Corollary 2.12 for $z = 0$, the term $M^{(1)}$ is evaluated as

$$\begin{aligned} M^{(1)} =& -\frac{2}{\log Q_r}\sum_{v \in \Sigma_{\mathrm{fin}} - S(\mathfrak{q})} \left\{\delta(r \in 2\mathbb{N})q_v^{-r/4} - \frac{\delta(r \in 2\mathbb{N})}{2} q_v^{-r/4}\frac{\log(q_v^r)}{\log \mathrm{N}(\mathfrak{q})}\right. \\ &\left. \times\left\{1 + \mathcal{O}\left(\frac{1}{\log \mathrm{N}(\mathfrak{q})}\right)\right\} + \mathcal{O}_{\epsilon,\epsilon'}\left(q_v^{rA}\frac{\mathrm{N}(\mathfrak{q})^{-\delta_1+\epsilon}}{\log \mathrm{N}(\mathfrak{q})}\right)\right\}\frac{\log q_v}{q_v^{1/2}}\hat{\phi}\left(\frac{\log q_v}{\log Q_r}\right) \\ =& -2\delta(r \in 2\mathbb{N})\sum_{v \in \Sigma_{\mathrm{fin}}} \hat{\phi}\left(\frac{\log q_v}{\log Q_r}\right)\frac{\log q_v}{q_v^{1/2+r/4}\log Q_r} \\ &+ \delta(r \in 2\mathbb{N})\left\{1 + \mathcal{O}\left(\frac{1}{\log \mathrm{N}(\mathfrak{q})}\right)\right\}\frac{r}{\log \mathrm{N}(\mathfrak{q})}\sum_{v \in \Sigma_{\mathrm{fin}}}\hat{\phi}\left(\frac{\log q_v}{\log Q_r}\right)\frac{(\log q_v)^2}{q_v^{1/2+r/4}\log Q_r} \\ &+ \mathcal{O}\left(\frac{1}{\log Q_r}\right) + \mathcal{O}_{\epsilon,\epsilon'}\left(\frac{1}{\log Q_r}\sum_{\substack{v \in \Sigma_{\mathrm{fin}} - S(\mathfrak{q}) \\ q_v \leqslant Q_r^{\beta}}} q_v^{rA}\frac{\log q_v}{q_v^{1/2}}\frac{\mathrm{N}(\mathfrak{q})^{-\delta_1+\epsilon}}{\log \mathrm{N}(\mathfrak{q})}\right) \end{aligned}$$

as $\mathrm{N}(\mathfrak{q}) \to \infty$ for any fixed $\epsilon, \epsilon' > 0$. When $r \geqslant 3$, then $1/2 + r/4 > 1$ is satisfied and hence $M^{(1)}$ is estimated as $\mathcal{O}(\frac{1}{\log \mathrm{N}(\mathfrak{q})})$ with the aid of the estimate

$$\sum_{\substack{v \in \Sigma_{\mathrm{fin}} - S(\mathfrak{q}) \\ q_v \leqslant Q_r^{\beta}}} q_v^{rA}\frac{\log q_v}{q_v^{1/2}}\mathrm{N}(\mathfrak{q})^{-\delta_1+\epsilon} \ll (Q_r^{\beta})^{rA+1/2}\mathrm{N}(\mathfrak{q})^{-\delta_1+\epsilon} \ll 1$$

from the asymptotics (3.4) under $\beta \leqslant \frac{\delta_1 - \epsilon}{r(rA+1/2)}$. The case $r = 1$ is similarly estimated. Consequently we have $M^{(1)} \ll \frac{1}{\log \mathrm{N}(\mathfrak{q})}$ when $r \neq 2$. Next let us consider the case $r = 2$. By $1/2 + r/4 = 1$, we need two asymptotics for evaluating $M^{(1)}$:

$$\sum_{v \in \Sigma_{\mathrm{fin}}} \hat{\phi}\left(\frac{\log q_v}{\log Q_2}\right)\frac{\log q_v}{q_v \log Q_2} = \frac{1}{2}\phi(0) + \mathcal{O}\left(\frac{1}{\log Q_2}\right), \qquad \mathrm{N}(\mathfrak{q}) \to \infty,$$

$$\sum_{v \in \Sigma_{\mathrm{fin}}} \hat{\phi}\left(\frac{\log q_v}{\log Q_2}\right)\frac{(\log q_v)^2}{q_v(\log Q_2)^2} = \frac{1}{2}\int_{-\infty}^{\infty}\hat{\phi}(x)|x|dx + \mathcal{O}\left(\frac{1}{\log Q_2}\right), \qquad \mathrm{N}(\mathfrak{q}) \to \infty.$$

These are proved by the prime ideal theorem and partial summation (cf. [38, Lemma 4.4 i), iii)]). Hence, a direct computation yields

$$\begin{aligned} M^{(1)} =& -\phi(0) + \frac{\log Q_2}{\log \mathrm{N}(\mathfrak{q})}\int_{-\infty}^{\infty}\hat{\phi}(x)|x|dx + \mathcal{O}\left(\frac{1}{\log \mathrm{N}(\mathfrak{q})}\right) + \mathcal{O}_{\epsilon,\epsilon'}\left(\frac{1+\mathrm{N}(\mathfrak{q})^{2\beta(2A+1/2)-\delta_1+\epsilon}}{\log Q_2}\right) \\ =& -\phi(0) + 2(1+\mathrm{N}(\mathfrak{q})^{-1/2})\int_{-\infty}^{\infty}\hat{\phi}(x)|x|dx + \mathcal{O}_{\epsilon,\epsilon'}\left(\frac{1}{\log \mathrm{N}(\mathfrak{q})}\right) \end{aligned}$$

as $\mathrm{N}(\mathfrak{q}) \to \infty$ under $\beta \leqslant \frac{\delta_1-\epsilon}{2(2A+1/2)}$. This gives the evaluation of $M^{(1)}$ for $r=2$.

The term $M^{(2)}$ for any $r \in \mathbb{N}$ is estimated by $\mathcal{O}(\frac{1}{\log \mathrm{N}(\mathfrak{q})})$ similarly to Proposition 3.1 under $\beta \leqslant \frac{\delta_1-\epsilon}{r(rA+1/2)}$. Thus we are done by removing $\epsilon$ and $\epsilon'$ from the inequalities on $\beta$ above. $\square$

Theorem 1.2 is proved by Propositions 3.1 and 3.2. The explicit form of $\beta_2$ in Theorem 1.2 is given by $\beta_0$ in Proposition 3.1 and $\delta_1 = \min(\delta,\delta') - 1/2$ in Theorem 2.7 with the aid of Lemmas 2.5 and 2.6. We note that the expression of $W'(x)$ in Theorem 1.2 is deduced from Proposition 3.2 for $r=2$ and the formula

$$\int_{\mathbb{R}}\hat{\phi}(x)|x|dx = \int_{\mathbb{R}}\phi(x)\times\widehat{|\cdot|\mathrm{ch}_{[-1,1]}}(x)dx = \int_{\mathbb{R}}\phi(x)\left(2\frac{\sin(2\pi x)}{2\pi x} - \frac{\sin^2(\pi x)}{(\pi x)^2}\right)dx$$

by using $\mathrm{supp}(\hat{\phi}) \subset (-\beta_2,\beta_2) \subset [-1,1]$, where $\mathrm{ch}_{[-1,1]}$ is the characteristic function of $[-1,1]$.

As a remark, if we specialize the parameter $z$ to $z=1$, Theorem 1.2 becomes a formula similar to [47, Theorem 11.5] for $G = \mathrm{PGL}_2$ and $\mathrm{Sym}^r : {}^L\mathrm{PGL}_2 \to \mathrm{GL}_{r+1}(\mathbb{C})$, although the principal congruence subgroup $\Gamma(\mathfrak{q})$ is considered there. Now Hypotheses 11.2 and 11.4 in [47] are satisfied and Hypothesis 10.1 in [47] is replaced with $\mathrm{Nice}(\pi, r)$ in our setting. Furthermore, the Frobenius-Schur indicator $s(\mathrm{Sym}^r)$ of $\mathrm{Sym}^r : \mathrm{SL}_2(\mathbb{C}) \to \mathrm{GL}_{r+1}(\mathbb{C})$ is equal to $(-1)^r$ (cf. [18, Exercise 11.33]).

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