Spaces of non-resultant systems of real bounded multiplicity determined by a toric variety

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Abstract

For each field $\mathbb{F}$ and positive integers m, n, d with $(m, n) \neq (1, 1)$, Farb and Wolfson defined the certain affine variety $\mathrm{Poly}_n^{d,m}(\mathbb{F})$ as generalizations of spaces first studied by Arnold, Vassiliev, Segal and others. As a natural generalization, for each fan Σ and r-tuple $D = (d_1, \cdots, d_r)$ of positive integers, the authors also defined and considered a more general space $\mathrm{Poly}_n^{D,\Sigma}(\mathbb{F})$, where r is the number of one dimensional cones in Σ. This space can also be regarded as a generalization of the space $\mathrm{Hol}_D^*(S^2, X_\Sigma)$ of based rational curves from the Riemann sphere S^2 to the toric variety X_Σ of degree D, where X_Σ denotes the toric variety (over $\mathbb{C}$) corresponding to the fan Σ.

In this paper, we define a space $\mathrm{Q}_n^{D,\Sigma}(\mathbb{F})$ ($\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$) which is its real analogue and can be viewed as a generalization of spaces considered by Arnold, Vassiliev and others in the context of real singularity theory. We prove that homotopy stability holds for this space and compute the stability dimension explicitly.

1 Introduction

1.1 Historical survey. For a complex manifold X, let $\mathrm{Map}^*(S^2, X) = \Omega^2 X$ (resp. $\mathrm{Hol}^*(S^2, X)$) denote the space of all based continuous maps (resp. based holomorphic maps) from the Riemann sphere S^2 to X. The

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relationship between the topology of the space $\mathrm{Hol}^*(S^2, X)$ and that of the space $\Omega^2 X$ has played a significant role in several different areas of geometry and mathematical physics (e.g. [5], [4]). In particular there arose the question whether the inclusion $\mathrm{Hol}^*(S^2, X) \xrightarrow{\subset} \Omega^2 X$ is a homotopy equivalence (or homology equivalence) up to a certain dimension, which we will refer to as the stability dimension. Since G. Segal [32] studied this problem for the case $X = \mathbb{CP}^m$, a number of mathematicians have investigated various closely related ones (e.g. [1], [16], [18], [23], [24], [28], [29], [30]).

Similar stabilization results appeared in the work of Arnold ([2], [3]), and Vassiliev ([33], [34]) in connection with singularity theory. They considered spaces of polynomials without roots of multiplicity greater than a certain natural number. These spaces are examples of "complement of discriminants" in Vassiliev's terminology [33] (cf. [21]).

Inspired by these results, Farb and Wolfson [14] introduced a new family of spaces $\mathrm{Poly}_n^{d,m}(\mathbb{F})$, which is defined for every field $\mathbb{F}$ and integers $m, n, d \geq 1$ with $(m, n) \neq (1, 1)$. The present authors generalised this further in [27], by considering a fan Σ (or toric variety) and a field $\mathbb{F}$, and defined a space $\mathrm{Poly}_n^{D,\Sigma}(\mathbb{F})$ as follows.

Definition 1.1 ([27])**.** Let $\mathbb{F}$ be a field with its algebraic closure $\overline{\mathbb{F}}$, and let Σ be a fan in $\mathbb{R}^m$ such that

$$\Sigma(1) = \{\rho_1, \cdots, \rho_r\}, \tag{1.1}$$

where $\Sigma(1)$ denotes the set of all one dimensional cones in Σ.[1] Let X_Σ denote the toric variety over $\mathbb{C}$ associated to the fan Σ, and let $\mathbb{N}$ denote the set of all positive integers.

For each r-tuple $D = (d_1, \cdots, d_r) \in \mathbb{N}^r$, let $\mathrm{Poly}_n^{D,\Sigma}(\mathbb{F})$ denote the space of all r-tuples $(f_1(z), \cdots, f_r(z)) \in \mathbb{F}[z]^r$ of monic polynomials satisfying the following two conditions (1.1a) and (1.1b):

(1.1a) $f_i(z) \in \mathbb{F}[z]$ is an $\mathbb{F}$-coefficients monic polynomial of the degree d_i for each $1 \leq i \leq r$.

(1.1b) For each $\sigma = \{i_1, \cdots, i_s\} \in I(\mathcal{K}_\Sigma)$, polynomials $f_{i_1}(z), \cdots, f_{i_s}(z)$ have no common root $\alpha \in \overline{\mathbb{F}}$ of multiplicity $\geq n$.

Here, $\mathcal{K}_\Sigma$ denotes the underlying simplicial complex of the fan Σ on the index set $[r] = \{1, 2, \cdots, r\}$ defined by (2.8), and the set $I(\mathcal{K}_\Sigma)$ is defined by $I(\mathcal{K}_\Sigma) = \{\sigma \subset [r] : \sigma \notin \mathcal{K}_\Sigma\}$ as in (2.2).

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[1] Formal definitions and a description of the notation related to toric varieties and their fans will be given in §2.

Remark 1.2. (i) By using the classical theory of resultants, one can show that $\mathrm{Poly}^{D,\Sigma}_n(\mathbb{F})$ is an affine variety over $\mathbb{F}$ and that it is the complement of the set of solutions of a system of polynomial equations (called a generalized resultant) with integer coefficients. For this reason, we call it *the space of non-resultant systems of bounded multiplicity determined by a toric variety.*

(ii) Let $\mathbf{0}_m = (0, \cdots, 0) \in \mathbb{R}^m$, and let Σ denote the fan in $\mathbb{R}^m$ such that $\Sigma(1) = \{\rho_1, \cdots, \rho_r\}$ as in (1.1). For each $1 \leq k \leq r$, let $\boldsymbol{n}_k \in \mathbb{Z}^m$ denote the primitive generator of ρ_k as in Definition 2.4. Then

$$\mathrm{Poly}^{D,\Sigma}_n(\mathbb{C}) = \mathrm{Hol}^*_D(S^2, X_\Sigma) \quad \text{if } n = 1 \text{ and } \textstyle\sum_{k=1}^r d_k \boldsymbol{n}_k = \mathbf{0}_m, \tag{1.2}$$

where $\mathrm{Hol}^*_D(S^2, X_\Sigma)$ denotes the space of based rational curves $f : S^2 \to X_\Sigma$ of degree D (see [24] for further details). Thus, the space $\mathrm{Poly}^{D,\Sigma}_n(\mathbb{C})$ can be regarded as a generalization of the space $\mathrm{Hol}^*_D(S^2, X_\Sigma)$. □

Now recall the following homotopy stability result.

Theorem 1.3 ([27]). *Let $D = (d_1, \cdots, d_r) \in \mathbb{N}^r$, $n \geq 2$, and let X_Σ be an m dimensional simply connected non-singular toric variety over $\mathbb{C}$ such that the condition (2.19a) holds.*

(i) *If $\sum_{k=1}^r d_k \boldsymbol{n}_k = \mathbf{0}_m$, then the natural map*

$$i_D : \mathrm{Poly}^{D,\Sigma}_n(\mathbb{C}) \to \Omega^2_D X_\Sigma(n) \simeq \Omega^2_0 X_\Sigma(n) \simeq \Omega^2 \mathcal{Z}_{\mathcal{K}_\Sigma}(D^{2n}, S^{2n-1})$$

is a homotopy equivalence through dimension $d_{\mathrm{poly}}(D; \Sigma, n)$, where we denote by $\mathcal{Z}_K(X, A)$ and $X_\Sigma(n)$ the polyhedral product of a pair (X, A) and the orbit space given by (2.12) and Definition 2.3, respectively.

(ii) *If $\sum_{k=1}^r d_k \boldsymbol{n}_k \neq \mathbf{0}_m$, there is a map*

$$j_D : \mathrm{Poly}^{D,\Sigma}_n(\mathbb{C}) \to \Omega^2 \mathcal{Z}_{\mathcal{K}_\Sigma}(D^{2n}, S^{2n-1})$$

which is a homotopy equivalence through dimension $d_{\mathrm{poly}}(D; \Sigma, n)$.

Hereafter we will denote by $\lfloor x \rfloor$ the integer part of a real number x. Moreover, we let $d_{\min} = \min\{d_1, \cdots, d_r\}$ and $r_{\min}(\Sigma)$ denote the positive integers given by (2.38), and $d_{\mathrm{poly}}(D; \Sigma, n)$ denote the positive integer defined by

$$d_{\mathrm{poly}}(D; \Sigma, n) = (2n r_{\min}(\Sigma) - 3) \lfloor d_{\min}/n \rfloor - 2. \tag{1.3}$$ □

1.2 Basic definitions. In this paper, we replace the space $\mathrm{Poly}^{D,\Sigma}_n(\mathbb{F})$ by its *real* analogue $\mathrm{Q}^{D,\Sigma}_n(\mathbb{F})$, where $\mathbb{F} = \mathbb{C}$ or $\mathbb{R}$. Since the space $\mathrm{Q}^{D,\Sigma}_n(\mathbb{F})$ is defined only for the fields $\mathbb{F} = \mathbb{R}$ and $\mathbb{C}$, we will use the letter $\mathbb{K}$ to refer to both of them at the same time.

The formal definition of $\mathrm{Q}^{D,\Sigma}_n(\mathbb{K})$ is given below.

Definition 1.4. Let Σ be a fan in $\mathbb{R}^m$ such that $\Sigma(1) = \{\rho_1, \cdots, \rho_r\}$ as in (1.1).

(i) For each r-tuple $D = (d_1, \cdots, d_r) \in \mathbb{N}^r$ and $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$, let $\mathrm{Q}_n^{D,\Sigma}(\mathbb{K})$ denote the space of all r-tuples $(f_1(z), \cdots, f_r(z)) \in \mathbb{K}[z]^r$ of $\mathbb{K}$-coefficients monic polynomials satisfying the following two conditions (1.3a) and (1.3b):

(1.3a) For each $1 \leq i \leq r$, $f_i(z) \in \mathbb{K}[z]$ is an $\mathbb{K}$-coefficients monic polynomial of the degree d_i.

(1.3b) For each $\sigma = \{i_1, \cdots, i_s\} \in I(\mathcal{K}_\Sigma) = \{\tau \subset [r] : \tau \notin \mathcal{K}_\Sigma\}$, polynomials $f_{i_1}(z), \cdots, f_{i_s}(z)$ have no common *real* root $\alpha \in \mathbb{R}$ of multiplicity $\geq n$ (but may have a common root $\alpha \in \mathbb{C} \setminus \mathbb{R}$ of any multiplicity).

(ii) Let $d(D;\Sigma, n, \mathbb{K})$ denote the positive integer defined by

$$d(D;\Sigma, n, \mathbb{K}) = (nr_{\min}(\Sigma)\dim_{\mathbb{R}}\mathbb{K} - 2)\lfloor d_{\min}/n \rfloor - 2$$

$$(1.4) \qquad = \begin{cases} (2nr_{\min}(\Sigma) - 2)\lfloor d_{\min}/n \rfloor - 2 & \text{if } \mathbb{K} = \mathbb{C}, \\ (nr_{\min}(\Sigma) - 2)\lfloor d_{\min}/n \rfloor - 2 & \text{if } \mathbb{K} = \mathbb{R}. \end{cases}$$

Remark 1.5. (i) It is easy to see that the following inclusion holds:

$$(1.5) \qquad \mathrm{Poly}_n^{D,\Sigma}(\mathbb{K}) \subset \mathrm{Q}_n^{D,\Sigma}(\mathbb{K}) \quad \text{for } \mathbb{K} = \mathbb{R} \text{ or } \mathbb{C}.$$

(ii) Note that the space $\mathrm{Q}_n^{D,\Sigma}(\mathbb{C})$ was already investigated for the case $n = 1$ in [25],[2] and that the space $\mathrm{Q}_n^{D,\Sigma}(\mathbb{K})$ was already extensively studied in [26] for the the case $(X_\Sigma, D) = (\mathbb{CP}^{m-1}, D_m(d))$,[3] where $D_m(d) \in \mathbb{N}^m$ denotes the m-tuple of positive integers defined by

$$(1.6) \qquad D_m(d) = (d, d, \cdots, d) \quad (m\text{-times}). \qquad \square$$

1.3 The main results. In this paper we will study the homotopy type of the space $\mathrm{Q}_n^{D,\Sigma}(\mathbb{K})$ for $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$. In particular, we will show that Atiyah-Jones-Segal type homotopy stability holds for the space $\mathrm{Q}_n^{D,\Sigma}(\mathbb{K})$.

In our result we will need the following two conditions (1.6a) and (1.6b).[4]

(1.6a) $d_{\min} \geq n \geq 1$.

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[2] It is denoted by $\mathrm{Pol}_D^*(S^1, X_\Sigma) = \mathrm{Q}_n^{D,\Sigma}(\mathbb{C})$ for $n = 1$ in [25].

[3] It is denoted by $\mathrm{Q}_n^{d,m}(\mathbb{K}) = \mathrm{Q}_n^{D,\Sigma}(\mathbb{K})$ in [26] for $(X_\Sigma, D) = (\mathbb{CP}^{m-1}, D_m(d))$.

[4] If the condition (1.6a) (resp. (1.6b)) is satisfied, the space $\mathrm{Q}_n^{D,\Sigma}(\mathbb{C})$ (resp. $\mathrm{Q}_n^{D,\Sigma}(\mathbb{R})$) is simply connected (see Corollary 7.7). Moreover, if the condition (1.6a) or (1.6b) is satisfied, the two conditions $\lfloor d_{\min}/n \rfloor \geq 1$ and $d(D;\Sigma, n, \mathbb{K}) \geq 1$ hold. Thus, the main results (Theorem 1.6, Corollary 2.17) are not vacuous. Note that the condition (1.6a) holds if the condition (1.6b) is satisfied.

(1.6b) One of the following three conditions holds:

(i) $d_{\min} \geq n \geq 2$.

(ii) $n = 1$, $d_{\min} \geq 2$, and $r_{\min}(\Sigma) \geq 4$.

(iii) $n = d_{\min} = 1$ and $r_{\min}(\Sigma) \geq 5$.

Then we can state the main result of this article as follows.

Theorem 1.6 (Theorems 2.14 and 2.15)**.** *Let $n \in \mathbb{N}$, let $D = (d_1, \cdots, d_r) \in \mathbb{N}^r$, and let X_Σ be an m dimensional simply connected non-singular toric variety satisfying the condition (2.19a).*

(i) *If the condition (1.6a) is satisfied, the map (given by (2.28) and (10.1))*

$$j_{D,n,\mathbb{C}} : \mathrm{Q}_n^{D,\Sigma}(\mathbb{C}) \to \Omega \mathcal{Z}_{\mathcal{K}_\Sigma}(D^{2n}, S^{2n-1})$$

is a homotopy equivalence through dimension $d(D; \Sigma, n, \mathbb{C})$.

(ii) *If the condition (1.6b) is satisfied, the map (given by (2.33) and (10.3))*

$$j_{D,n,\mathbb{R}} : \mathrm{Q}_n^{D,\Sigma}(\mathbb{R}) \to \Omega \mathcal{Z}_{\mathcal{K}_\Sigma}(D^{n}, S^{n-1})$$

is a homotopy equivalence through dimension $d(D; \Sigma, n, \mathbb{R})$.

Remark 1.7. Let $g : V \to W$ be a based map.

(i) The map $g : V \to W$ is called *a homotopy* (*resp. homology*) *equivalence through dimension N* if the induced homomorphism

$$(1.7) \qquad g_* : \pi_k(V) \to \pi_k(W) \quad (\text{resp. } g_* : H_k(V; \mathbb{Z}) \to H_k(W; \mathbb{Z}))$$

is an isomorphism for any $k \leq N$.

(ii) The map $g : V \to W$ is called *a homotopy* (*resp. homology*) *equivalence up to dimension N* if the induced homomorphism (1.7) is an isomorphism for any $k < N$ and an epimorphism for $k = N$.

(iii) When G is a topological group and a map g is a G-equivariant map between G-spaces V and W, the map g is called a *G-equivariant homotopy* (*resp. G-equivariant homology*) *equivalence through dimension N* if the restriction $g^H = g|V^H : V^H \to W^H$ is a homotopy (resp. homology) equivalence through dimension N for any subgroup $H \subset G$. Here, for each G-space X and a subgroup $H \subset G$, let X^H denote the H-fixed subspace of X defined by

$$(1.8) \qquad X^H = \{x \in X : h \cdot x = x \quad \text{for any } h \in H\}. \qquad \Box$$

1.4 Organization. This paper is organized as follows. In §2 we recall the basic definitions and facts which are needed for the statements of the results of this article. After that, precise statements of the main results (Theorems 2.14, 2.15, and Corollary 2.16) are given. In §3 we recall several basic facts related to polyhedral products and toric varieties. In §4, we define and summarize the main facts about the non-degenerate simplicial resolution. By using this non-degenerate simplicial resolution we construct the Vassiliev spectral sequence. In §5 we define the stabilization maps, and in §6, we construct the truncated spectral sequence induced from the spectral sequence obtained in §4. By using this truncated spectral sequence, we shall prove the homology stability result (Theorems 6.5, 6.8, and Corollary 6.6). In §7 we investigate the connectivity of the space $\mathrm{Q}^{D,\Sigma}_n(\mathbb{K})$. In particular, we prove that the space $\mathrm{Q}^{D,\Sigma}_n(\mathbb{C})$ (resp. $\mathrm{Q}^{D,\Sigma}_n(\mathbb{R})$) is simply connected if the condition (1.6a) (resp. (1.6b)) is satisfied. In §8 we consider the configuration space model for the space $\mathrm{Q}^{D,\Sigma}_n(\mathbb{K})$ and recall the stabilized horizontal scanning map (see Theorem 8.7). In §9 we prove the stability result (Theorem 9.2), and in §10 we use it to prove the main results (Theorems 2.14, 2.15, and Corollary 2.16).

2 Toric varieties and the main results

In this section we recall several basic definitions and facts related to toric varieties (convex rational polyhedral cones, toric varieties, fans of toric varieties, polyhedral products, homogenous coordinate, rational curves on a toric variety etc). Then we use these definitions and notations to give precise statements of the main results of this paper. From now on, we always assume that $\mathbb{K}=\mathbb{C}$ or $\mathbb{R}$. Moreover, if $d_{\min}<n$, $\lfloor d_{\min}/n\rfloor=0$ and $d(D;\Sigma,n,\mathbb{K})=-2<0$. So we also assume that $d_{\min}\geq n\geq 1$.

2.1 Fans, toric varieties and Polyhedral products. *A convex rational polyhedral cone* in $\mathbb{R}^m$ is a subset of $\mathbb{R}^m$ of the form

$$\sigma=\mathrm{Cone}(S)=\mathrm{Cone}(\boldsymbol{m}_1,\cdots,\boldsymbol{m}_s)=\left\{\sum_{k=1}^{s}\lambda_k\boldsymbol{m}_k:\lambda_k\geq 0\right\} \tag{2.1}$$

for a finite set $S=\{\boldsymbol{m}_1,\cdots,\boldsymbol{m}_s\}\subset\mathbb{Z}^m$. The dimension of σ is the dimension of the smallest subspace of $\mathbb{R}^m$ which contains σ. A convex rational polyhedral cone σ is called *strongly convex* if $\sigma\cap(-\sigma)=\{\mathbf{0}_m\}$, where we set $\mathbf{0}_m=\mathbf{0}=(0,0,\cdots,0)\in\mathbb{R}^m$. *A face* τ of a convex rational polyhedral cone σ is a subset $\tau\subset\sigma$ of the form $\tau=\sigma\cap\{\boldsymbol{x}\in\mathbb{R}^m:L(\boldsymbol{x})=0\}$ for some linear form L on $\mathbb{R}^m$, such that $\sigma \subset \{\boldsymbol{x} \in \mathbb{R}^m : L(\boldsymbol{x}) \geq 0\}$. Note that if σ is a strongly convex rational polyhedral cone, so is any of its faces.[5]

Definition 2.1. Let Σ be a finite collection of strongly convex rational polyhedral cones in $\mathbb{R}^m$.

(i) The set Σ is called *a fan* (in $\mathbb{R}^m$) if the following two conditions hold:

(2.1a) Every face τ of $\sigma \in \Sigma$ belongs to Σ.

(2.1b) If $\sigma_1, \sigma_2 \in \Sigma$, $\sigma_1 \cap \sigma_2$ is a common face of each σ_k and $\sigma_1 \cap \sigma_2 \in \Sigma$.

(ii) An m dimensional irreducible normal variety X (over $\mathbb{C}$) is called *a toric variety* if it has a Zariski open subset $\mathbb{T}^m_{\mathbb{C}} = (\mathbb{C}^*)^m$ and the action of $\mathbb{T}^m_{\mathbb{C}}$ on itself extends to an action of $\mathbb{T}^m_{\mathbb{C}}$ on X.

The most significant property of a toric variety is that it is characterized up to isomorphism entirely by its associated fan Σ. We denote by X_Σ the toric variety associated to a fan Σ (see [11] for the details).

(iii) Let K be some set of subsets of $[r]$. Then the set K is called *an abstract simplicial complex* on the index set $[r]$ if the following condition $(\dagger)_K$ holds:

$(\dagger)_K$ $\quad \tau \subset \sigma$ and $\sigma \in K$, then $\tau \in K$.

Remark 2.2. (i) It is well known that there are no holomorphic maps $\mathbb{CP}^1 = S^2 \to \mathbb{T}^m_{\mathbb{C}}$ except the constant maps, and that the fan Σ of $\mathbb{T}^m_{\mathbb{C}}$ is $\Sigma = \{\{\mathbf{0}_m\}\}$. Hence, without loss of generality, we will always assume that $X_\Sigma \neq \mathbb{T}^m_{\mathbb{C}}$, and that any fan Σ in $\mathbb{R}^m$ satisfies the condition $\{\{\mathbf{0}_m\}\} \subsetneqq \Sigma$.

(ii) In this paper by a simplicial complex K we always mean *an abstract simplicial complex* and we always assume that a simplicial complex K contains the empty set $\emptyset$. $\square$

Definition 2.3. Let K be a simplicial complex on the index set $[r] = \{1, 2, \cdots, r\}$, and let (X, A) be a pair of based spaces.

(i) Let $I(K)$ denote the collection of subsets $\sigma \subset [r]$ defined by

$$I(K) = \{\sigma \subset [r] : \sigma \notin K\}. \tag{2.2}$$

(ii) Define *the polyhedral product* $\mathcal{Z}_K(X, A)$ with respect to K by

$$\mathcal{Z}_K(X, A) = \bigcup_{\sigma \in K} (X, A)^\sigma, \qquad \text{where} \tag{2.3}$$
$$(X, A)^\sigma = \{(x_1, \cdots, x_r) \in X^r : x_k \in A \text{ if } k \notin \sigma\}.$$

[5]When S is the emptyset $\emptyset$, we set $\mathrm{Cone}(\emptyset) = \{\mathbf{0}_m\}$ and regard it as a strongly convex rational polyhedral cone in $\mathbb{R}^m$.

(iii) For each subset $\sigma = \{i_1, \cdots, i_s\} \subset [r]$, let $L_\sigma(\mathbb{K}^n)$ denote the subspace of $\mathbb{K}^{nr}$ defined by

$$(2.4) \qquad L_\sigma(\mathbb{K}^n) = \{(\boldsymbol{x}_1, \cdots, \boldsymbol{x}_r) \in (\mathbb{K}^n)^r = \mathbb{K}^{nr} : \boldsymbol{x}_{i_1} = \cdots = \boldsymbol{x}_{i_s} = \mathbf{0}_n\}$$

and let $L_n^K(\mathbb{K})$ denote the subspace of $\mathbb{K}^{nr}$ defined by

$$(2.5) \qquad L_n^K(\mathbb{K}) = \bigcup_{\sigma \in I(K)} L_\sigma(\mathbb{K}^n) = \bigcup_{\sigma \subset [r], \sigma \notin K} L_\sigma(\mathbb{K}^n).$$

It is easy to see that

$$(2.6) \qquad \mathcal{Z}_K(\mathbb{K}^n, (\mathbb{K}^n)^*) = \mathbb{K}^{nr} \setminus L_n^K(\mathbb{K}), \quad \text{where } (\mathbb{K}^n)^* = \mathbb{K}^n \setminus \{\mathbf{0}_n\}.$$

2.2 Homogenous coordinates. Next we recall the basic facts about homogenous coordinates on toric varieties.

Definition 2.4. Let $\Sigma \supsetneqq \{\{\mathbf{0}_m\}\}$ be a fan in $\mathbb{R}^m$ such that $\Sigma(1) = \{\rho_1, \cdots, \rho_r\}$ as in (1.1).

(i) For each $1 \leq k \leq r$, we denote by $\boldsymbol{n}_k \in \mathbb{Z}^m$ *the primitive generator* of ρ_k, such that $\rho_k \cap \mathbb{Z}^m = \mathbb{Z}_{\geq 0} \cdot \boldsymbol{n}_k$. Note that

$$(2.7) \qquad \rho_k = \text{Cone}(\boldsymbol{n}_k) = \{x\boldsymbol{n}_k : x \geq 0\}.$$

(ii) Let $\mathcal{K}_\Sigma$ denote *the underlying simplicial complex of* Σ defined by

$$(2.8) \qquad \mathcal{K}_\Sigma = \Big\{\{i_1, \cdots, i_s\} \subset [r] : \boldsymbol{n}_{i_1}, \boldsymbol{n}_{i_2}, \cdots, \boldsymbol{n}_{i_s} \text{ span a cone in } \Sigma\Big\}.$$

It is easy to see that $\mathcal{K}_\Sigma$ is a simplicial complex on the index set $[r]$.

(iii) Let $G_{\Sigma,\mathbb{K}} \subset \mathbb{T}^r_{\mathbb{K}} = (\mathbb{K}^*)^r$ be the subgroup

$$(2.9) \qquad G_{\Sigma,\mathbb{K}} = \{(\mu_1, \cdots, \mu_r) \in \mathbb{T}^r_{\mathbb{K}} : \prod_{k=1}^r (\mu_k)^{\langle \boldsymbol{n}_k, \boldsymbol{m} \rangle} = 1 \text{ for all } \boldsymbol{m} \in \mathbb{Z}^m\},$$

where $\langle \boldsymbol{u}, \boldsymbol{v} \rangle = \sum_{k=1}^m u_k v_k$ for $\boldsymbol{u} = (u_1, \cdots, u_m)$ and $\boldsymbol{v} = (v_1, \cdots, v_m) \in \mathbb{R}^m$.

(iv) There is a natural $G_{\Sigma,\mathbb{K}}$-action on $\mathcal{Z}_{\mathcal{K}_\Sigma}(\mathbb{K}^n, (\mathbb{K}^n)^*)$ given by coordinate-wise multiplication,

$$(2.10) \qquad (\mu_1, \cdots, \mu_r) \cdot (\boldsymbol{x}_1, \cdots, \boldsymbol{x}_r) = (\mu_1 \boldsymbol{x}_1, \cdots, \mu_r \boldsymbol{x}_r)$$

for $((\mu_1, \cdots, \mu_r), (\boldsymbol{x}_1, \cdots, \boldsymbol{x}_r)) \in G_{\Sigma,\mathbb{K}} \times \mathcal{Z}_{\mathcal{K}_\Sigma}(\mathbb{K}^n, (\mathbb{K}^n)^*)$, where we set

$$(2.11) \qquad \mu \boldsymbol{x} = (\mu x_1, \cdots, \mu x_n) \quad \text{if } (\mu, \boldsymbol{x}) = (\mu, (x_1, \cdots, x_n)) \in \mathbb{K}^* \times \mathbb{K}^n.$$

(v) Let $X_{\Sigma,\mathbb{K}}(n)$ denote the corresponding orbit space

$$X_{\Sigma,\mathbb{K}}(n)=\mathcal{Z}_{\mathcal{K}_\Sigma}(\mathbb{K}^n,(\mathbb{K}^n)^*)/G_{\Sigma,\mathbb{K}}, \quad \text{where} \tag{2.12}$$

$$q_{n,\mathbb{K}}:\mathcal{Z}_{\mathcal{K}_\Sigma}(\mathbb{K}^n,(\mathbb{K}^n)^*)\to X_{\Sigma,\mathbb{K}}(n)=\mathcal{Z}_{\mathcal{K}_\Sigma}(\mathbb{K}^n,(\mathbb{K}^n)^*)/G_{\Sigma,\mathbb{K}} \tag{2.13}$$

denotes the corresponding canonical projection. In particular, we also write

$$X_\Sigma(n)=X_{\Sigma,\mathbb{C}}(n) \ \text{ and } \ G_\Sigma=G_{\Sigma,\mathbb{C}} \quad \text{if } \mathbb{K}=\mathbb{C}. \tag{2.14}$$

Theorem 2.5 ([9])**.** *If the set $\{\boldsymbol{n}_k\}_{k=1}^r$ of all primitive generators spans $\mathbb{R}^m$ (i.e. $\sum_{k=1}^r\mathbb{R}\cdot\boldsymbol{n}_k=\mathbb{R}^m$), there is a natural isomorphism*

$$X_\Sigma\cong\mathcal{Z}_{\mathcal{K}_\Sigma}(\mathbb{C},\mathbb{C}^*)/G_{\Sigma,\mathbb{C}}=X_\Sigma(1)=X_{\Sigma,\mathbb{C}}(1). \tag{2.15}$$

Hence, we can identify $X_\Sigma(n)$ with the toric variety X_Σ if $n=1$. □

Remark 2.6. (i) The notion of the quotient in algebraic geometry has several variants (see [9, Theorem 2.1], [11, Theorem 5.1.11] for further details). However, in this paper we only consider *smooth* varieties (as in the assumptions in section 2.3).

(ii) Let Σ be a fan in $\mathbb{R}^m$ as in Definition 2.4. Note that the fan Σ is completely determined by the pair $(\mathcal{K}_\Sigma,\{\boldsymbol{n}_k\}_{k=1}^r)$ (see [24, Remark 2.3] for the details). □

For each $1\leq i\leq r$, let $F_i=(f_{1;i},\cdots,f_{n;i})\in\mathbb{K}[z_0,\cdots,z_s]^n$ be an n-tuple of homogenous polynomials of the same degree d_i satisfying the following condition:

(†) For each $\sigma\in I(\mathcal{K}_\Sigma)$, the homogenous polynomials $\{f_{k;i}\}_{k\in\sigma}$ have no common *real* root except $\mathbf{0}_{s+1}\in\mathbb{R}^{s+1}$.

In this situation, consider the map

$$F=(F_1,\cdots,F_r):\mathbb{R}^{s+1}\setminus\{\mathbf{0}_{s+1}\}\to(\mathbb{K}^n)^r=\mathbb{K}^{rn} \quad \text{given by}$$

$$\begin{cases} F(\boldsymbol{x}) & =(F_1(\boldsymbol{x}),\cdots F_r(\boldsymbol{x})) \quad \text{for } \boldsymbol{x}\in\mathbb{R}^{m+1}\setminus\{\mathbf{0}_{m+1}\}, \\ F_i(\boldsymbol{x}) & =(f_{1;i}(\boldsymbol{x}),f_{2;i}(\boldsymbol{x}),\cdots,f_{n;i}(\boldsymbol{x})) \quad \text{for } 1\leq i\leq r. \end{cases} \tag{2.16}$$

By the assumption (†), the homogenous polynomials $\{f_{k;i}\}_{k\in\sigma}$ have no common real root except $\mathbf{0}_{s+1}\in\mathbb{R}^{s+1}$ for each $1\leq i\leq r$ and each $\sigma\in I(\mathcal{K}_\Sigma)$. Thus, we see that the image of the map F is contained in $\mathcal{Z}_{\mathcal{K}_\Sigma}(\mathbb{K}^n,(\mathbb{K}^n)^*)$, and we may regard the map F as the map

$$F=(F_1,\cdots,F_r):\mathbb{R}^{s+1}\setminus\{\mathbf{0}_{s+1}\}\to\mathcal{Z}_{\mathcal{K}_\Sigma}(\mathbb{K}^n,(\mathbb{K}^n)^*). \tag{2.17}$$

The following lemma, whose proof we postpone until the end of §3, plays a key role in the proof of the main result of this paper.

Lemma 2.7 (cf. [10], Theorem 3.1; [20], Lemma 2.6)**.** *Suppose that the set* $\{\boldsymbol{n}_k\}_{k=1}^r$ *of all primitive generators spans* $\mathbb{R}^m$*. For each* $1 \leq i \leq r$*, let* $F_i = (f_{1;i}, \cdots, f_{n;i}) \in \mathbb{K}[z_0, \cdots, z_s]^n$ *be an* n*-tuple of homogenous polynomials of the same degree* d_i *satisfying the above condition* (†)*.*

Then there is a unique map $f : \mathbb{R}\mathrm{P}^s \to X_{\Sigma,\mathbb{K}}(n)$ *such that the diagram*

$$\begin{array}{ccc} \mathbb{R}^{s+1} \setminus \{\mathbf{0}_{s+1}\} & \xrightarrow{(F_1,\cdots,F_r)} & \mathcal{Z}_{\mathcal{K}_\Sigma}(\mathbb{K}^n, (\mathbb{K}^n)^*) \\ {\scriptstyle \gamma_{s,\mathbb{R}}} \downarrow & & {\scriptstyle q_{n,\mathbb{K}}} \downarrow \\ \mathbb{R}\mathrm{P}^s & \xrightarrow{\quad f \quad} & X_{\Sigma,\mathbb{K}}(n) \end{array} \tag{2.18}$$

is commutative if and only if the following condition holds:

$$\sum_{k=1}^r d_k \boldsymbol{n}_k = \mathbf{0}_m. \tag{2.19}$$

Here, we identify $X_{\Sigma,\mathbb{K}}(n) = \mathcal{Z}_{\mathcal{K}_\Sigma}(\mathbb{K}^n, (\mathbb{K}^n)^*)/G_{\Sigma,\mathbb{K}}$*,* $\gamma_{s,\mathbb{R}} : \mathbb{R}^{s+1} \setminus \{\mathbf{0}_{s+1}\} \to \mathbb{R}\mathrm{P}^s$ *denotes the canonical Hopf fibering, and the map* $F = (F_1, \cdots, F_r)$ *is given by (2.17).*

2.3 Assumptions. (i) From now on, let $\Sigma \supsetneqq \{\{\mathbf{0}_m\}\}$ be a fan in $\mathbb{R}^m$ with the set of one dimensional cones $\Sigma(1) = \{\rho_1, \cdots, \rho_r\}$ as in Definition 2.4. We also assume that X_Σ is a simply connected and smooth (not necessarily compact) toric variety satisfying the following condition:

(2.19a) There is an r-tuple $D_* = (d_1^*, \cdots, d_r^*) \in \mathbb{N}^r$ such that $\sum_{k=1}^r d_k^* \boldsymbol{n}_k = \mathbf{0}_m$, where $\boldsymbol{n}_k \in \mathbb{Z}^m$ denotes the primitive generator of ρ_k for each $1 \leq k \leq r$.

(ii) In this paper, we always identify

$$X_{\Sigma,\mathbb{K}}(n) = \mathcal{Z}_{\mathcal{K}_\Sigma}(\mathbb{K}^n, (\mathbb{K}^n)^*)/G_{\Sigma,\mathbb{K}} \qquad (\mathbb{K} = \mathbb{R} \text{ or } \mathbb{C}), \tag{2.20}$$

and for each $(\boldsymbol{a}_1, \cdots, \boldsymbol{a}_r) \in \mathcal{Z}_{\mathcal{K}_\Sigma}(\mathbb{K}^n, (\mathbb{K}^n)^*) \subset (\mathbb{K}^n)^r$, we let $[\boldsymbol{a}_1, \cdots, \boldsymbol{a}_r] \in X_{\Sigma,\mathbb{K}}(n)$ denote the point given by this equivalence class.

(iii) We also make the identification $\mathbb{R}\mathrm{P}^1 = S^1 = \mathbb{R} \cup \infty$, and let $\boldsymbol{e} = (1, 1, \cdots, 1) \in \mathbb{K}^n$. Under this identification, we choose the points ∞ and $* = [\boldsymbol{e}, \cdots, \boldsymbol{e}]$ as the base-points of $\mathbb{R}\mathrm{P}^1$ and $X_{\Sigma,\mathbb{K}}(n)$, respectively. □

Remark 2.8. (i) Consider the toric variety $X_\Sigma = \mathbb{C}^2$. Its fan Σ is given by

$$\Sigma = \Big\{\{\mathbf{0}_2\}, \rho_1 = \mathrm{Cone}(\boldsymbol{e}_1), \rho_2 = \mathrm{Cone}(\boldsymbol{e}_2), \mathrm{Cone}(\boldsymbol{e}_1, \boldsymbol{e}_2)\Big\},$$

where $\mathbf{0}_2 = (0,0)$, $\boldsymbol{e}_1 = (1,0)$, and $\boldsymbol{e}_2 = (0,1)$. The set of one dimensional cones is $\Sigma(1) = \{\rho_1, \rho_2\}$, and the primitive generator of ρ_k $(k = 1,2)$ is $\boldsymbol{n}_k = \boldsymbol{e}_k$. Thus $r = 2$ and

$$\sum_{k=1}^{r} d_k \boldsymbol{n}_k = \sum_{k=1}^{2} d_k \boldsymbol{e}_k = \mathbf{0}_2 \Leftrightarrow (d_1, d_2) = (0,0).$$

Hence, when $X_\Sigma = \mathbb{C}^2$, there are no non-constant maps $f : \mathbb{R}\mathrm{P}^s \to X_\Sigma = \mathbb{C}^2$ satisfying the condition (2.19), which is why we need the condition (2.19a).

(ii) It follows from [11, Theorem 12.1.10] that X_Σ is simply connected if and only if the following condition (††) holds:

(††) The set $\{\boldsymbol{n}_k\}_{k=1}^r$ of all primitive generators spans $\mathbb{Z}^m$ over $\mathbb{Z}$, i.e. $\sum_{k=1}^r \mathbb{Z} \cdot \boldsymbol{n}_k = \mathbb{Z}^m$.

Thus we see that if X_Σ is simply connected then the set $\{\boldsymbol{n}_k\}_{k=1}^r$ of all primitive generators spans $\mathbb{R}^m$. In particular, if X_Σ is a compact smooth toric variety then X_Σ is simply connected (see Lemma 3.8).

(iii) Note that the space $X_{\Sigma,\mathbb{R}}(n)$ is simply connected if $n \geq 2$ (see Lemma 7.1). However, the space $X_{\Sigma,\mathbb{R}} = X_{\Sigma,\mathbb{R}}(1)$ is not simply connected in general. For example, consider $X_\Sigma = \mathbb{CP}^k$ for $k \geq 2$. Then $X_{\Sigma,\mathbb{R}} = X_{\Sigma,\mathbb{R}}(1) = \mathbb{R}\mathrm{P}^k$, which not simply connected. However, the loop space $\Omega X_{\Sigma,\mathbb{R}} = \Omega\mathbb{R}\mathrm{P}^k$ has two path-components $\Omega_\epsilon \mathbb{R}\mathrm{P}^k$ $(\epsilon = 0,1)$ and there is a homotopy equivalence $\Omega_\epsilon \mathbb{R}\mathrm{P}^k \simeq \Omega S^k$ for each $\epsilon = 0,1$. So each path-component $\Omega_\epsilon \mathbb{R}\mathrm{P}^k$ is simply connected if $k \geq 3$.

(iv) Let $s = 1$ and suppose that all assumptions of Lemma 2.7 and the condition (2.19) hold and that the coefficient of $(z_0)^{d_i}$ of the homogenous polynomial $f_{j;i}(z_0, z_1) \in \mathbb{K}[z_0, z_1]$ is 1 for $1 \leq j \leq n$ and $1 \leq i \leq r$. By Lemma 2.7, we obtain the map $f : \mathbb{R}\mathrm{P}^1 \to X_{\Sigma,\mathbb{K}}(n)$ such that the diagram (2.18) is commutative. Note that $f_{j;i}(z_0, z_1) = (z_1)^{d_i} f_{j;i}(z_0/z_1, 1)$ for each j, i. By setting $z = z_0/z_1$, we can identify each homogenous polynomial $f_{j;i}(z_0, z_1) \in \mathbb{K}[z_0, z_1]$ with the monic polynomial $g_{j;i}(z) = f_{j;i}(z, 1) \in \mathbb{K}[z]$ of degree d_i in one variable z. Since

$$\lim_{|\alpha| \to \infty} [F_1(\alpha), \cdots, F_r(\alpha)] = \lim_{|\alpha| \to \infty} [\alpha^{-d_1} F_1(\alpha), \cdots, \alpha^{-d_r} F_r(\alpha)] = [\boldsymbol{e}, \cdots, \boldsymbol{e}]$$

(by (3.16)), this map f is the based map $(\mathbb{R}\mathrm{P}^1, \infty) \to (X_{\Sigma,\mathbb{K}}(n), *)$ given by

$$f(\alpha) = \begin{cases} [F_1(\alpha), \cdots, F_r(\alpha)] & \text{if } \alpha \in \mathbb{R} \\ [\boldsymbol{e}, \boldsymbol{e}, \cdots, \boldsymbol{e}] & \text{if } \alpha = \infty \end{cases} \tag{2.21}$$

for $\alpha \in \mathbb{R} \cup \infty = \mathbb{R}\mathrm{P}^1 = S^1$, where $\boldsymbol{e} = (1, \cdots, 1) \in \mathbb{K}^n$ and we set $F_i(\alpha) = (g_{1;i}(\alpha), \cdots, g_{n;i}(\alpha))$ for each $1 \leq i \leq r$. □

2.4 Spaces of algebraic maps of real bounded multiplicity. Now we can define the space of algebraic maps. Recall that we use $\mathbb{K}$ to denote $\mathbb{C}$ or $\mathbb{R}$.

Definition 2.9. For a monic polynomial $f(z) \in \mathbb{K}[z]$ of degree d, let $F_n(f)(z)$ denote the n-tuple of monic polynomials of the same degree d defined by

$$F_n(f)(z) = (f(z), f(z) + f'(z), f(z) + f''(z), \cdots, f(z) + f^{(n-1)}(z)). \tag{2.22}$$

Note that a monic polynomial $f(z) \in \mathbb{K}[z]$ has a root $\alpha \in \mathbb{C}$ of multiplicity $\geq n$ iff $F_n(f)(\alpha) = \mathbf{0}_n \in \mathbb{C}^n$.

Remark 2.10. (i) Note that $\mathrm{Q}_n^{D,\Sigma}(\mathbb{C})$ is path-connected, and $\mathrm{Q}_n^{D,\Sigma}(\mathbb{R})$ is path-connected if $(n, r_{\min}(\Sigma)) \neq (1,2)$ (this will be explained in the proof of Lemma 7.1 and Remark 8.4).

(ii) Let $\mathbb{Z}_2 = \{\pm 1\}$ denote the multiplicative cyclic group of order 2, and let $X^{\mathbb{Z}_2}$ denote the $\mathbb{Z}_2$-fixed point set of a $\mathbb{Z}_2$-space X as in (1.8). Complex conjugation on $\mathbb{C}$ extends to a $\mathbb{Z}_2$-actions on the spaces $\mathcal{Z}_{\mathcal{K}_\Sigma}(\mathbb{C}^n, (\mathbb{C}^n)^*)$ and $\mathrm{Q}_n^{D,\Sigma}(\mathbb{C})$ so that

$$\mathcal{Z}_{\mathcal{K}_\Sigma}(\mathbb{R}^n, (\mathbb{R}^n)^*) = \mathcal{Z}_{\mathcal{K}_\Sigma}(\mathbb{C}^n, (\mathbb{C}^n)^*)^{\mathbb{Z}_2}, \quad \mathrm{Q}_n^{D,\Sigma}(\mathbb{R}) = \mathrm{Q}_n^{D,\Sigma}(\mathbb{C})^{\mathbb{Z}_2}. \tag{2.23}$$

It is easy to see that complex conjugation on $\mathbb{C}$ also naturally extends to a $\mathbb{Z}_2$-action on $X_\Sigma(n) = \mathcal{Z}_{\mathcal{K}_\Sigma}(\mathbb{C}^n, (\mathbb{C}^n)^*)/G_\Sigma$ so that

$$X_{\Sigma,\mathbb{R}}(n) = X_\Sigma(n)^{\mathbb{Z}_2}. \tag{2.24}$$

It follows from the definitions of the above actions that the following diagram is commutative:

$$\begin{array}{ccc}
\mathrm{Q}_n^{D,\Sigma}(\mathbb{R}) & \xrightarrow{q_{n,\mathbb{R}}} & X_{\Sigma,\mathbb{R}}(n) = \mathcal{Z}_{\mathcal{K}_\Sigma}(\mathbb{R}^n, (\mathbb{R}^n)^*)/G_{\Sigma,\mathbb{R}} \\
{\scriptstyle i_n^D}\downarrow\cap & & {\scriptstyle i_n^X}\downarrow\cap \\
\mathrm{Q}_n^{D,\Sigma}(\mathbb{C}) & \xrightarrow{q_{n,\mathbb{C}}} & X_\Sigma(n) = \mathcal{Z}_{\mathcal{K}_\Sigma}(\mathbb{C}^n, (\mathbb{C}^n)^*)/G_{\Sigma,\mathbb{C}}
\end{array} \tag{2.25}$$

where let i_n^D and i_n^X denote the corresponding inclusion maps.

Note that $\mathrm{Q}_n^{D,\Sigma}(\mathbb{C})$ (resp. $\mathrm{Q}_n^{D,\Sigma}(\mathbb{R})$) is simply connected if the condition (1.6a) (resp. (1.6b)) is satisfied (this will be proved in Corollary 7.7).

Definition 2.11. Suppose that the condition (2.19a) holds, and let $D = (d_1, \cdots, d_r) \in \mathbb{N}^r$ be an r-tuple of positive integers satisfying the condition

$$\sum_{k=1}^{r} d_k \boldsymbol{n}_k = \mathbf{0}_m. \tag{2.26}$$

(i) First, consider the case $\mathbb{K} = \mathbb{C}$. By Lemma 2.7 and (2.21), one can define a map

$$i_{D,n,\mathbb{C}} : \mathrm{Q}_n^{D,\Sigma}(\mathbb{C}) \to \Omega X_\Sigma(n) \qquad \text{by} \tag{2.27}$$

$$i_{D,n,\mathbb{C}}(f)(\alpha) = \begin{cases} [F_n(f_1)(\alpha), F_n(f_2)(\alpha), \cdots, F_n(f_r)(\alpha)] & \text{if } \alpha \in \mathbb{R} \\ [\boldsymbol{e}, \boldsymbol{e}, \cdots, \boldsymbol{e}] & \text{if } \alpha = \infty \end{cases}$$

for $f = (f_1(z), \cdots, f_r(z)) \in \mathrm{Q}_n^{D,\Sigma}(\mathbb{C})$ and $\alpha \in \mathbb{R} \cup \infty = S^1$, where we set $\boldsymbol{e} = (1, 1, \cdots, 1) \in \mathbb{C}^n$.

Since the space $\mathrm{Q}_n^{D,\Sigma}(\mathbb{C})$ is simply connected and $\Omega q_{n,\mathbb{C}}$ is a universal covering (by (ii) of Remark 2.10 and (ii) of Corollary 3.10), the map $i_{D,n,\mathbb{C}}$ lifts to the space $\Omega \mathcal{Z}_{\mathcal{K}_\Sigma}(D^{2n}, S^{2n-1})$, and there is a based map

$$j_{D,n,\mathbb{C}} : \mathrm{Q}_n^{D,\Sigma}(\mathbb{C}) \to \Omega \mathcal{Z}_{\mathcal{K}_\Sigma}(D^{2n}, S^{2n-1}) \simeq \Omega \mathcal{Z}_{\mathcal{K}_\Sigma}(\mathbb{C}^n, (\mathbb{C}^n)^*) \tag{2.28}$$

such that the following equality holds:

$$\Omega q_{n,\mathbb{C}} \circ j_{D,n,\mathbb{C}} = i_{D,n,\mathbb{C}}. \tag{2.29}$$

(ii) Next, consider the case $\mathbb{K} = \mathbb{R}$.

Recall the $\mathbb{Z}_2$-action on the spaces $\mathrm{Q}_n^{D,\Sigma}(\mathbb{C})$ and X_Σ induced from complex conjugation on $\mathbb{C}$, and note that the map $i_{D,n,\mathbb{C}}$ is a $\mathbb{Z}_2$-equivariant map. Then, by (2.23) and (2.24), we see that

$$i_{D,n,\mathbb{C}}(\mathrm{Q}_n^{D,\Sigma}(\mathbb{R})) \subset \Omega X_\Sigma(n)^{\mathbb{Z}_2} = \Omega X_{\Sigma,\mathbb{R}}(n). \tag{2.30}$$

Thus, the restriction $i_{D,n,\mathbb{C}}|\mathrm{Q}_n^{D,\Sigma}(\mathbb{R})$ defines a map

$$i_{D,n,\mathbb{R}} = i_{D,n,\mathbb{C}}|\mathrm{Q}_n^{D,\Sigma}(\mathbb{R}) : \mathrm{Q}_n^{D,\Sigma}(\mathbb{R}) \to \Omega X_{\Sigma,\mathbb{R}}(n) \tag{2.31}$$

such that the following diagram is commutative:

$$\begin{array}{ccccc} \mathrm{Q}_n^{D,\Sigma}(\mathbb{R}) & \xrightarrow{i_{D,n,\mathbb{R}}} & \Omega X_{\Sigma,\mathbb{R}}(n) & \xleftarrow[\simeq]{\Omega q_{n,\mathbb{R}}} & \Omega \mathcal{Z}_{\mathcal{K}_\Sigma}(\mathbb{R}^n, (\mathbb{R}^n)^*) \\ {\scriptstyle i_n^D}\downarrow & & {\scriptstyle \Omega i_n^X}\downarrow & & {\scriptstyle \Omega j_n}\downarrow \\ \mathrm{Q}_n^{D,\Sigma}(\mathbb{C}) & \xrightarrow{i_{D,n,\mathbb{C}}} & \Omega X_\Sigma(n) & \xleftarrow{\Omega q_{n,\mathbb{C}}} & \Omega \mathcal{Z}_{\mathcal{K}_\Sigma}(\mathbb{C}^n, (\mathbb{C}^n)^*) \end{array} \tag{2.32}$$

where the $j_n : \mathcal{Z}_{\mathcal{K}_\Sigma}(\mathbb{R}^n, (\mathbb{R}^n)^*) \xrightarrow{\subset} \mathcal{Z}_{\mathcal{K}_\Sigma}(\mathbb{C}^n, (\mathbb{C}^n)^*)$ denotes the inclusion map. Note that $\Omega q_{n,\mathbb{R}}$ is a homotopy equivalence (this will be proved in Corollary 3.10). Thus, there is a based map

$$j_{D,n,\mathbb{R}} : \mathrm{Q}_n^{D,\Sigma}(\mathbb{R}) \to \Omega \mathcal{Z}_{\mathcal{K}_\Sigma}(\mathbb{R}^n, (\mathbb{R}^n)^*) \simeq \Omega \mathcal{Z}_{\mathcal{K}_\Sigma}(D^n, S^{n-1}) \tag{2.33}$$

which satisfies the following equality:

$$\Omega q_{n,\mathbb{R}} \circ j_{D,n,\mathbb{R}} = i_{D,n,\mathbb{R}} \quad \text{(up to homotopy)}. \tag{2.34}$$

Remark 2.12. When $\sum_{k=1}^{r} d_k \boldsymbol{n}_k = \mathbf{0}_m$, by (2.28) and (2.34), we obtain the map

$$j_{D,n,\mathbb{K}} : \mathrm{Q}_n^{D,\Sigma}(\mathbb{K}) \to \Omega\mathcal{Z}_{\mathcal{K}_\Sigma}(\mathbb{K}^n, (\mathbb{K}^n)^*) \simeq \Omega\mathcal{Z}_{\mathcal{K}_\Sigma}(D^{d(\mathbb{K})n}, S^{d(\mathbb{K})n-1}), \tag{2.35}$$

where the number $d(\mathbb{K})$ is defined by

$$d(\mathbb{K}) = \dim_{\mathbb{R}} \mathbb{K} = \begin{cases} 2 & \text{if } \mathbb{K} = \mathbb{C}, \\ 1 & \text{if } \mathbb{K} = \mathbb{R}. \end{cases} \tag{2.36}$$

2.5 The numbers $r_{\min}(\Sigma)$ and $d(D;\Sigma,n,\mathbb{K})$. Before stating the main results of this paper, we need to define the positive integers $r_{\min}(\Sigma)$ and $d(D;\Sigma,n,\mathbb{K})$ (which already appeared in the statements of our results).

Definition 2.13. (i) We say that a set $S = \{\boldsymbol{n}_{i_1}, \cdots, \boldsymbol{n}_{i_s}\}$ is *a primitive collection* if $\mathrm{Cone}(S) \notin \Sigma$ and $\mathrm{Cone}(T) \in \Sigma$ for any proper subset $T \subsetneqq S$.

(ii) For each r-tuple $D = (d_1, \cdots, d_r) \in \mathbb{N}^r$, define the positive integer $d(D,\Sigma,n,\mathbb{K})$ by

$$d(D;\Sigma,n,\mathbb{K}) = \begin{cases} (2nr_{\min}(\Sigma) - 2)\lfloor \frac{d_{\min}}{n} \rfloor - 2 & \text{if } \mathbb{K} = \mathbb{C} \\ (nr_{\min}(\Sigma) - 2)\lfloor \frac{d_{\min}}{n} \rfloor - 2 & \text{if } \mathbb{K} = \mathbb{R} \end{cases} \tag{2.37}$$

as in (1.4), where $r_{\min}(\Sigma)$ and $d_{\min}$ are the positive integers given by

$$\begin{cases} r_{\min}(\Sigma) = \min\{s \in \mathbb{N} : \{\boldsymbol{n}_{i_1}, \cdots, \boldsymbol{n}_{i_s}\} \text{ is a primitive collection}\}, \\ d_{\min} = \min\{d_1, d_2, \cdots, d_r\}. \end{cases} \tag{2.38}$$

Note that

$$r_{\min}(\Sigma) \geq 2. \tag{2.39}$$

2.6 The main results. The space $\mathrm{Q}_n^{D,\Sigma}(\mathbb{C})$ has already been extensively studied in the case $n = 1$ in [25]. The main purpose of this paper is to generalize the results of [25] to the space $\mathrm{Q}_n^{D,\Sigma}(\mathbb{K})$ for $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$ and for any $n \geq 1$. These generalizations are stated below as the next two theorems and their corollaries.

Theorem 2.14 (The case $\mathbb{K} = \mathbb{C}$)**.** *Let $n \in \mathbb{N}$, let $D = (d_1, \cdots, d_r) \in \mathbb{N}^r$, and let X_Σ be an m dimensional simply connected non-singular toric variety such that the two conditions (2.19a) and (1.6a) hold.*

(i) *If $\sum_{k=1}^{r} d_k \boldsymbol{n}_k = \mathbf{0}_m$, then the map (given by (2.28))*

$$j_{D,n,\mathbb{C}} : \mathrm{Q}_n^{D,\Sigma}(\mathbb{C}) \to \Omega\mathcal{Z}_{\mathcal{K}_\Sigma}(D^{2n}, S^{2n-1})$$

is a homotopy equivalence through dimension $d(D;\Sigma,n,\mathbb{C})$.

(ii) *If* $\sum_{k=1}^r d_k\boldsymbol{n}_k\neq \mathbf{0}_m$, *there is a map*

$$j_{D,n,\mathbb{C}}:\mathrm{Q}_n^{D,\Sigma}(\mathbb{C})\to \Omega\mathcal{Z}_{\mathcal{K}_\Sigma}(D^{2n},S^{2n-1})$$

which is a homotopy equivalence through dimension $d(D;\Sigma,n,\mathbb{C})$.[6]

Theorem 2.15 (The case $\mathbb{K}=\mathbb{R}$)**.** *Let* $n\in\mathbb{N}$, *let* $D=(d_1,\cdots,d_r)\in\mathbb{N}^r$, *and let* X_Σ *be an* m *dimensional simply connected non-singular toric variety such that the two conditions* (*2.19a*) *and* (*1.6b*) *hold.*

(i) *If* $\sum_{k=1}^r d_k\boldsymbol{n}_k= \mathbf{0}_m$, *then the map* (*given by* (*2.33*))

$$j_{D,n,\mathbb{R}}:\mathrm{Q}_n^{D,\Sigma}(\mathbb{R})\to \Omega\mathcal{Z}_{\mathcal{K}_\Sigma}(D^{n},S^{n-1})$$

is a homotopy equivalence through dimension $d(D;\Sigma,n,\mathbb{R})$.

(ii) *If* $\sum_{k=1}^r d_k\boldsymbol{n}_k\neq \mathbf{0}_m$, *there is a map*

$$j_{D,n,\mathbb{R}}:\mathrm{Q}_n^{D,\Sigma}(\mathbb{R})\to \Omega\mathcal{Z}_{\mathcal{K}_\Sigma}(D^{n},S^{n-1})$$

which is a homotopy equivalence through dimension $d(D;\Sigma,n,\mathbb{R})$.

Corollary 2.16. *Let* $n\in\mathbb{N}$, *let* $D=(d_1,\cdots,d_r)\in\mathbb{N}^r$, *and let* X_Σ *be an* m *dimensional simply connected non-singular toric variety such that the two conditions* (*2.19a*) *and* (*1.6a*) *hold.*

(i) *If* $\sum_{k=1}^r d_k\boldsymbol{n}_k= \mathbf{0}_m$, *then the map* $i_{D,n,\mathbb{C}}:\mathrm{Q}_n^{D,\Sigma}(\mathbb{C})\to\Omega X_\Sigma(n)$ *induces an isomorphism*

$$(i_{D,n,\mathbb{C}})_*:\pi_s(\mathrm{Q}_n^{D,\Sigma}(\mathbb{C}))\xrightarrow{\cong}\pi_s(\Omega X_\Sigma)\cong\pi_{s+1}(X_\Sigma(n))$$

for any $2\leq s\leq d(D;\Sigma,n,\mathbb{C})$.

(ii) *If* $\sum_{k=1}^r d_k\boldsymbol{n}_k\neq \mathbf{0}_m$, *the map* $i_{D,n,\mathbb{C}}:\mathrm{Q}_n^{D,\Sigma}(\mathbb{C})\to\Omega X_\Sigma(n)$ *defined by*

$$i_{D,n,\mathbb{C}}:=\Omega q_{n,\mathbb{C}}\circ j_{D,n,\mathbb{C}} \tag{2.40}$$

induces an isomorphism

$$(i_{D,n,\mathbb{C}})_*:\pi_s(\mathrm{Q}_n^{D,\Sigma}(\mathbb{C}))\xrightarrow{\cong}\pi_s(\Omega X_\Sigma(n))\cong\pi_{s+1}(X_\Sigma(n))$$

for any $2\leq s\leq d(D;\Sigma,n,\mathbb{C})$.

[6] This map has to be constructed in a slightly different way from the one in (i) but we shall use the same notation for both.

Consider the $\mathbb{Z}_2$-action on the spaces $\mathrm{Q}_n^{D,\Sigma}(\mathbb{C})$ and $\mathcal{Z}_{\mathcal{K}_\Sigma}(D^{2n}, S^{2n-1})$ induced from the complex conjugation on $\mathbb{C}$, where we identify

$$D^{2n} = \{(x_1, \cdots, x_n) \in \mathbb{C}^n : \sum_{k=1}^{n} |x_k|^2 \leq 1\}. \tag{2.41}$$

Note that we can regard the space D^{2n} as a $\mathbb{Z}_2$-space whose $\mathbb{Z}_2$ action is given by the complex conjugation.

$$(-1) \cdot (x_1, \cdots, x_n) = (\overline{x_1}, \cdots, \overline{x_n}) \quad \text{for } (x_1, \cdots, x_n) \in D^{2n}. \tag{2.42}$$

Since $\mathrm{Q}_n^{D,\Sigma}(\mathbb{R}) = \mathrm{Q}_n^{D,\Sigma}(\mathbb{C})^{\mathbb{Z}_2}$, $\mathcal{Z}_{\mathcal{K}_\Sigma}(D^n, S^{n-1}) = \mathcal{Z}_{\mathcal{K}_\Sigma}(D^{2n}, S^{2n-1})^{\mathbb{Z}_2}$, and $j_{D,n,\mathbb{R}} = (j_{D,n,\mathbb{C}})^{\mathbb{Z}_2}$, we also obtain the following result.

Corollary 2.17. *Let $n \in \mathbb{N}$, let $D = (d_1, \cdots, d_r) \in \mathbb{N}^r$, and let X_Σ be an m dimensional simply connected non-singular toric variety satisfying the two conditions (2.19a) and (1.6b). Then the map*

$$j_{D,n,\mathbb{C}} : \mathrm{Q}_n^{D,\Sigma}(\mathbb{C}) \to \Omega \mathcal{Z}_{\mathcal{K}_\Sigma}(D^{2n}, S^{2n-1})$$

is a $\mathbb{Z}_2$-equivariant homotopy equivalence through dimension $d(D; \Sigma, n, \mathbb{R})$. □

Finally, we easily obtain the following result.

Corollary 2.18. *Let $n \in \mathbb{N}$, let $D = (d_1, \cdots, d_r) \in \mathbb{N}^r$, and let X_Σ be a simply connected compact non-singular toric variety such that the the condition (2.19a) holds. Let $\Sigma(1)$ denote the set of all one dimensional cones in Σ, and let Σ_1 be any fan in $\mathbb{R}^m$ satisfying the condition*

$$\Sigma(1) \subset \Sigma_1 \subsetneqq \Sigma. \tag{2.43}$$

(i) *If the condition (1.6a) holds and $\sum_{k=1}^{r} d_k \boldsymbol{n}_k = \mathbf{0}_m$, then the map*

$$j_{D,n,\mathbb{C}} : \mathrm{Q}_n^{D,\Sigma_1}(\mathbb{C}) \to \Omega \mathcal{Z}_{\Sigma_1}(D^{2n}, S^{2n-1})$$

is a homotopy equivalence through the dimension $d(D; \Sigma_1, n, \mathbb{C})$.

Moreover, the map $i_{D,n,\mathbb{C}} : \mathrm{Q}_n^{D,\Sigma_1}(\mathbb{C}) \to \Omega X_{\Sigma_1}$ induces an isomorphism

$$(i_{D,n,\mathbb{C}})_* : \pi_s(\mathrm{Q}_n^{D,\Sigma_1}(\mathbb{C})) \xrightarrow{\cong} \pi_s(\Omega X_{\Sigma_1}(n)) \cong \pi_{s+1}(X_{\Sigma_1}(n))$$

for any $2 \leq s \leq d(D; \Sigma_1, n, \mathbb{C})$.

(ii) *If the condition (1.6a) holds and $\sum_{k=1}^{r} d_k \boldsymbol{n}_k \neq \mathbf{0}_m$, then there is a map*

$$j_{D,n,\mathbb{C}} : \mathrm{Q}_n^{D,\Sigma_1}(\mathbb{C}) \to \Omega \mathcal{Z}_{\mathcal{K}_{\Sigma_1}}(D^{2n}, S^{2n-1})$$

which is a homotopy equivalence through dimension $d(D;\Sigma_1,n,\mathbb{C})$*. Moreover, the map* $i_{D,n,\mathbb{C}}:\mathrm{Q}_n^{D,\Sigma_1}(\mathbb{C})\to\Omega X_{\Sigma_1}$ *defined by*

$$i_{D,n,\mathbb{C}}:=\Omega q_{n,\mathbb{C}}\circ j_{D,n,\mathbb{C}} \tag{2.44}$$

induces an isomorphism

$$(i_{D,n,\mathbb{C}})_*:\pi_s(\mathrm{Q}_n^{D,\Sigma_1}(\mathbb{C}))\xrightarrow{\cong}\pi_s(\Omega X_{\Sigma_1}(n))\cong\pi_{s+1}(X_{\Sigma_1}(n))$$

for any $2\leq s\leq d(D;\Sigma_1,n,\mathbb{C})$*.*

(iii) *If the condition (1.6b) holds and* $\sum_{k=1}^r d_k\boldsymbol{n}_k=\mathbf{0}_m$*, then the map*

$$j_{D,n,\mathbb{R}}:\mathrm{Q}_n^{D,\Sigma_1}(\mathbb{R})\to\Omega\mathcal{Z}_{\Sigma_1}(D^n,S^{n-1})$$

is a homotopy equivalence through the dimension $d(D;\Sigma_1,n,\mathbb{K})$*.*

(iv) *If the condition (1.6b) holds and* $\sum_{k=1}^r d_k\boldsymbol{n}_k\neq\mathbf{0}_m$*, there is a map*

$$j_{D,n,\mathbb{R}}:\mathrm{Q}_n^{D,\Sigma_1}(\mathbb{R})\to\Omega\mathcal{Z}_{\mathcal{K}_{\Sigma_1}}(D^n,S^{n-1})$$

which is a homotopy equivalence through dimension $d(D;\Sigma_1,n,\mathbb{R})$*.* □

3 Basic facts about toric varieties

In this section, we recall some basic definitions and known results.

Definition 3.1 ([7], Definition 6.27, Example 6.39)**.** Let K be a simplicial complex on the index set $[r]$, and let $I(K)=\{\sigma\subset[r]:\sigma\notin K\}$ as in (2.2).

(i) An element $\sigma\in I(K)$ is called *a minimal non-face of* K if $\tau\in K$ for any proper subset $\tau\subsetneqq\sigma$.

(ii) Then we denote by $I_{\min}(K)$ the set of all minimal non-faces of K. It is easy to see that the following equality holds.

$$K=\{\sigma\subset[r]:\tau\not\subset\sigma\text{ for any }\tau\in I_{\min}(K)\}. \tag{3.1}$$

(iii) We denote by $\mathcal{Z}_K$ and $DJ(K)$ *the moment-angle complex* of K and *the Davis-Januszkiewicz space* of K ([12]) which are defined by

$$\mathcal{Z}_K=\mathcal{Z}_K(D^2,S^1),\quad DJ(K)=\mathcal{Z}_K(\mathbb{CP}^\infty,*). \tag{3.2}$$ □

Remark 3.2. Let Σ be a fan in $\mathbb{R}^m$ and let X_Σ be a smooth toric variety such that the condition (2.19a) holds. Then it is easy to see that $\{\boldsymbol{n}_{i_1},\boldsymbol{n}_{i_2},\cdots,\boldsymbol{n}_{i_s}\}$ is primitive if and only if $\sigma=\{i_1,i_2,\cdots,i_s\}\in I_{\min}(\mathcal{K}_\Sigma)$. Thus, we also obtain the following equality:

$$r_{\min}(\Sigma)=\min\{\mathrm{card}(\sigma):\sigma\in I(\mathcal{K}_\Sigma)\}. \tag{3.3}$$ □

Lemma 3.3 ([7]; Corollary 6.30, Theorems 6.33, 8.9). *Let K be a simplicial complex on the index set $[r]$.*

(i) *The space $\mathcal{Z}_K$ is 2-connected, and there is a fibration sequence*

$$\mathcal{Z}_K \longrightarrow DJ(K) \xrightarrow{\subset} (\mathbb{CP}^\infty)^r. \tag{3.4}$$

(ii) *There are $\mathbb{T}^r$-equivariant deformation retraction*

$$ret : \mathcal{Z}_K(\mathbb{K}^n, (\mathbb{K}^n)^*) \xrightarrow{\simeq} \mathcal{Z}_K(D^{d(\mathbb{K})n}, S^{d(\mathbb{K})n-1}). \tag{3.5}$$

where we set $\mathbb{T}^r = (S^1)^r$. □

Lemma 3.4 ([31]). *Let Σ be a fan in $\mathbb{R}^m$ defining smooth toric variety X_Σ such that the condition (2.19a) holds.*

(i) *There is an isomorphism*

$$G_{\Sigma,\mathbb{K}} \cong \mathbb{T}_{\mathbb{K}}^{r-m} = (\mathbb{K}^*)^{r-m}. \tag{3.6}$$

(ii) *The group $G_{\Sigma,\mathbb{K}}$ acts on the space $\mathcal{Z}_{\mathcal{K}_\Sigma}(\mathbb{K}^n, (\mathbb{K}^n)^*)$ freely as in (2.10) and there is a principal $G_{\Sigma,\mathbb{K}}$-bundle sequence*

$$G_{\Sigma,\mathbb{K}} \longrightarrow \mathcal{Z}_{\mathcal{K}_\Sigma}(\mathbb{K}^n, (\mathbb{K}^n)^*) \xrightarrow{q_{n,\mathbb{K}}} X_{\Sigma,\mathbb{K}}. \tag{3.7}$$

(iii) *If $\mathbb{K} = \mathbb{R}$, there is a homotopy equivalence $\mathbb{T}_{\mathbb{R}}^r \simeq (\mathbb{Z}_2)^{r-m}$ and the map $q_{n,\mathbb{R}}$ is a covering projection with fiber $(\mathbb{Z}_2)^{r-m}$ (up to homotopy).*

Proof. First, consider the case $\mathbb{K} = \mathbb{C}$. Then the assertions (i) and (ii) follow from [31, (6.2) page 527; Proposition 6.7].

Next, let $\mathbb{K} = \mathbb{R}$. Since $G_\Sigma = G_{\Sigma,\mathbb{C}} \cong (\mathbb{C}^*)^{r-m}$ and $G_{\Sigma,\mathbb{R}} = G_\Sigma \cap (\mathbb{R}^*)^r$, we have an isomorphism $G_{\Sigma,\mathbb{R}} \cong (\mathbb{R}^*)^{r-m} = \mathbb{T}_{\mathbb{R}}^{r-m}$. Since $G_{\Sigma,\mathbb{C}}$ acts on the space $\mathcal{Z}_{\mathcal{K}_\Sigma}(\mathbb{C}^n, (\mathbb{C}^n)^*)$ freely, the subgroup $\mathbb{G}_{\Sigma,\mathbb{R}}$ also acts on the space $\mathcal{Z}_{\mathcal{K}_\Sigma}(\mathbb{R}^n, (\mathbb{R}^n)^*)$ freely and we obtain the $G_{\Sigma,\mathbb{R}}$-principal fibration sequence (3.7) for the case $\mathbb{K} = \mathbb{R}$. This proves (i) and (ii) for the case $\mathbb{K} = \mathbb{R}$. Since $G_{\Sigma,\mathbb{R}} \simeq (\mathbb{Z}_2)^{r-m}$, $q_{n,\mathbb{R}}$ is a covering projection with fiber $(\mathbb{Z}_2)^{r-m}$ and we obtain (iii). □

Definition 3.5 (c.f. [27], (5.26)). Let $n \geq 2$, and let Σ be a fan in $\mathbb{R}^m$ defining smooth toric variety X_Σ such that the condition (2.19a) holds.

(i) Let $\mathcal{K}_\Sigma(n)$ denote the simplicial complex on the index set $[r] \times [n]$ defined by

$$\mathcal{K}_\Sigma(n) = \{\tau \subset [r] \times [n] : \sigma \times [n] \not\subset \tau \text{ for any } \sigma \in I(\mathcal{K}_\Sigma)\}. \tag{3.8}$$

(ii) For each $(i,j)\in[r]\times[n]$, let $\boldsymbol{n}_{i,j}\in\mathbb{Z}^{mn}$ denote the lattice vector defined by

$$\boldsymbol{n}_{i,j}=(\boldsymbol{a}_1,\cdots,\boldsymbol{a}_n),\ \text{where we set } \boldsymbol{a}_k=\begin{cases}\boldsymbol{n}_i & (k=j)\\ \boldsymbol{0}_m & (k\neq j)\end{cases} \tag{3.9}$$

and define a fan $F_n(\Sigma)$ in $\mathbb{R}^{mn}$ by

$$F_n(\Sigma)=\{c_\tau:\tau\in\mathcal{K}_\Sigma(n)\}, \tag{3.10}$$

where c_τ denotes the strongly convex rational polyhedral cone given by

$$c_\tau=\mathrm{Cone}(\{\boldsymbol{n}_{i,j}:(i,j)\in\tau\})=\Big\{\sum_{(i,j)\in\tau}x_{i,j}\boldsymbol{n}_{i,j}:x_{i,j}\geq 0\Big\}. \tag{3.11}$$

Lemma 3.6. *Let $n\geq 2$.*

(i) *If $\mathbb{T}^r=(S^1)^r$, there is a $\mathbb{T}^r$-equivariant homeomorphism*

$$\mathcal{Z}_{\mathcal{K}_\Sigma}(D^{2n},S^{2n-1})\cong\mathcal{Z}_{\mathcal{K}_\Sigma(n)}(D^2,S^1). \tag{3.12}$$

(ii) *If $\mathbb{T}^r_{\mathbb{C}}=(\mathbb{C}^*)^r$, there is a $\mathbb{T}^r_{\mathbb{C}}$-equivariant homeomorphsim*

$$\mathcal{Z}_{\mathcal{K}_\Sigma}(\mathbb{C}^n,(\mathbb{C}^n)^*)\simeq\mathcal{Z}_{\mathcal{K}_\Sigma(n)}(\mathbb{C},\mathbb{C}^*). \tag{3.13}$$

(iii) *The space $\mathcal{Z}_{\mathcal{K}_\Sigma}(D^{2n},S^{2n-1})$ is 2-connected.*

Proof. (i) Let $J=(n,n,\cdots,n)\in\mathbb{N}^r$ and let $\mathcal{K}_\Sigma(J)$ denote the simplical complex on the index set $[r]\times[n]$ defined by [6, Definition 2.1].[7] Then it follows from [6, Definition 2.1] that the following equality holds:

$$I_{\min}(\mathcal{K}_\Sigma(J))=\{\tau\times[n]:\tau\in I_{\min}(\mathcal{K}_\Sigma)\}. \tag{3.14}$$

Hence, by (3.1) and (3.8), we obtain the following equality:

$$\mathcal{K}_\Sigma(J)=\{\sigma\subset[r]\times[n]:\tau\times[n]\not\subset\sigma \text{ for any } \tau\in I_{\min}(\mathcal{K}_\Sigma)\}.$$

Thus, we have $\mathcal{K}_\Sigma(J)=\mathcal{K}_\Sigma(n)$ by (3.14). Hence, by [6, Theorem 7.5], there is a $\mathbb{T}^r$-equivariant homeomorphism $\mathcal{Z}_{\mathcal{K}_\Sigma}(D^{2n},S^{2n-1})\cong\mathcal{Z}_{\mathcal{K}_\Sigma(n)}(D^2,S^1)$, and the assertion (i) follows.

[7] More precisely, if we set $(K,m)=(\mathcal{K}_\Sigma,r)$ and $J=(j_1,j_2,\cdots,j_r)=(n,n,\cdots,n)$ (r-times) in the notation of [6, Definition 2.1], we obtain a simplicial complex $K(J)$ on the index set $[r]\times[n]$.

(ii) By [27, (5.32)] that there is a homeomorphism

$$\mathcal{Z}_{\mathcal{K}_\Sigma}(\mathbb{C}^n, (\mathbb{C}^n)^*) \cong \mathcal{Z}_{\mathcal{K}_{\mathcal{K}_\Sigma(n)}}(\mathbb{C}, \mathbb{C}^*).$$

One can easily check that this homeomorphism is $\mathbb{T}^r_{\mathbb{C}}$-equivariant, and we obtain assertion (ii).

(iii) It follows from (i) and (ii) that there are the following homotopy equivalences

$$\mathcal{Z}_{\mathcal{K}_\Sigma}(D^{2n}, S^{2n-1}) \simeq \mathcal{Z}_{\mathcal{K}_\Sigma}(\mathbb{C}^n, (\mathbb{C}^n)^*) \cong \mathcal{Z}_{\mathcal{K}_\Sigma(n)}(\mathbb{C}, \mathbb{C}^*) \simeq \mathcal{Z}_{\mathcal{K}_\Sigma(n)}(D^2, S^1).$$

Since the moment-angle complex $\mathcal{Z}_{\mathcal{K}_\Sigma(n)}(D^2, S^1)$ is 2-connected by [7, Theorem 6.33], the space $\mathcal{Z}_{\mathcal{K}_\Sigma}(D^{2n}, S^{2n-1})$ is also 2-connected. □

Definition 3.7 ([11]). Let Σ be a fan in $\mathbb{R}^m$. Then a cone $\sigma \in \Sigma$ is called *smooth* if it is generated by a subset of a basis of $\mathbb{Z}^m$.

Lemma 3.8 ([11]). *Let X_Σ be a toric variety determined by a fan Σ in $\mathbb{R}^m$.*

(i) *X_Σ is compact if and only if $\mathbb{R}^m = \bigcup_{\sigma\in\Sigma} \sigma$.*

(ii) *X_Σ is smooth if and only if every cone $\sigma \in \Sigma$ is smooth.* □

Lemma 3.9. *Let $n \geq 2$. Then the space $X_\Sigma(n)$ is homeomorphic to the smooth toric variety $X_{F_n(\Sigma)}$ associated to the fan $F_n(\Sigma)$, and $\mathcal{K}_\Sigma(n)$ is the underlying simplicial complex of the fan $F_n(\Sigma)$.*

Proof. To see this, consider the toric variety $X_{F_n(\Sigma)}$ determined by the fan $F_n(\Sigma)$. By considering the homogenous coordinate representation of $X_{F_n(\Sigma)}$, we easily see that there is a homeomorphism $X_{F_n(\Sigma)} \cong X_\Sigma(n)$. Moreover, one can easily show that $X_{F_n(\Sigma)}$ is non-singular (by using Lemma 3.8). Thus, $X_{F_n(\Sigma)}$ is a smooth toric variety. Moreover, by (3.10) we easily see that $\mathcal{K}_\Sigma(n)$ is the underlying simplicial complex of the fan $F_n(\Sigma)$. □

Corollary 3.10. *Let Σ be a fan in $\mathbb{R}^m$ defining smooth toric variety X_Σ such that the condition (2.19a) holds.*

(i) *The map $\Omega q_{n,\mathbb{C}} : \Omega\mathcal{Z}_{\mathcal{K}_\Sigma}(\mathbb{C}^n, (\mathbb{C}^n)^*) \longrightarrow \Omega X_\Sigma(n)$ is a universal covering (up to homotopy) with fiber $\mathbb{Z}^{r-m}$.*

(ii) *The map $\Omega q_{n,\mathbb{R}} : \Omega\mathcal{Z}_{\mathcal{K}_\Sigma}(\mathbb{R}^n, (\mathbb{R}^n)^*) \xrightarrow{\simeq} \Omega X_{\Sigma,\mathbb{R}}(n)$ is a homotopy equivalence.*

(iii) *There is the following fibration sequence (up to homotopy)*

$$\mathbb{T}^{mn}_{\mathbb{C}} \longrightarrow X_\Sigma(n) \longrightarrow DJ(\mathcal{K}_\Sigma(n)). \tag{3.15}$$

Proof. (i) It follows easily from Lemma 3.4, that the map $\Omega q_{n,\mathbb{C}}$ is a covering projection with fiber $\mathbb{Z}^{r-m}$. Since $\Omega \mathrm{Q}_n^{D,\Sigma}(\mathbb{C})$ is simply connected (by (i)), $\Omega q_{n,\mathbb{C}}$ is a universal covering with fiber $\mathbb{Z}^{r-m}$.

(ii) The assertion (ii) easily follows from (iii) of Lemma 3.4.

(iii) The assertion (iii) follows from Lemmas 3.6, 3.9 and [24, Proposition 4.4]. $\square$

Lemma 3.11 ([24]; Lemma 3.4)**.** *If the condition* (*2.19a*) *is satisfied, the space* X_Σ *is simply connected and* $\pi_2(X_\Sigma) = \mathbb{Z}^{r-m}$*.* $\square$

We end this section with a proof of Lemma 2.7.

Proof of Lemma 2.7. Consider the map $F = (F_1, \cdots, F_r)$ is given by (2.17). We let $\mathbb{K} = \mathbb{C}$, as the proof for $\mathbb{K} = \mathbb{R}$ is completely analogous. It suffices to show that $F(\lambda \boldsymbol{x}) = F(\boldsymbol{x})$ up to $G_{\Sigma,\mathbb{C}}$-action for any $(\lambda, \boldsymbol{x}) \in \mathbb{R}^* \times (\mathbb{R}^{s+1} \setminus \{\mathbf{0}_{s+1}\})$ iff $\sum_{k=1}^r d_k \boldsymbol{n}_k = \mathbf{0}_m$.

Since all homogenous polynomials $\{f_{k;i}\}_{k=1}^n$ have the same degree d_i, for each $(\lambda, \boldsymbol{x}) \in \mathbb{R}^* \times \mathbb{R}^{s+1}$,

$$\begin{aligned} F_i(\lambda \boldsymbol{x}) &= (f_{1;i}(\lambda \boldsymbol{x}), \cdots, f_{n;i}(\lambda \boldsymbol{x})) = (\lambda^{d_i} f_{1;i}(\boldsymbol{x}), \cdots, \lambda^{d_i} f_{n;i}(\boldsymbol{x})) \\ &= \lambda^{d_i}(f_{1;i}(\boldsymbol{x}), \cdots, f_{n;i}(\boldsymbol{x})) = \lambda^{d_i} F_i(\boldsymbol{x}). \end{aligned}$$

Thus, we have

$$\begin{aligned} F(\lambda \boldsymbol{x}) &= (F_1(\lambda \boldsymbol{x}), \cdots, F_r(\lambda \boldsymbol{x})) = (\lambda^{d_1} F_1(\boldsymbol{x}), \cdots, \lambda^{d_r} F_r(\boldsymbol{x})) \\ &= (\lambda^{d_1}, \cdots, \lambda^{d_r}) \cdot (F_1(\boldsymbol{x}), \cdots, F_r(\boldsymbol{x})) = (\lambda^{d_1}, \cdots, \lambda^{d_r}) \cdot F(\boldsymbol{x}). \end{aligned}$$

Hence, it remains to show that $(\lambda^{d_1}, \cdots, \lambda^{d_r}) \in G_{\Sigma,\mathbb{C}}$ for any $\lambda \in \mathbb{R}^*$ iff $\sum_{k=1}^r d_k \boldsymbol{n}_k = \mathbf{0}_m$. However, $(\lambda^{d_1}, \cdots, \lambda^{d_r}) \in G_{\Sigma,\mathbb{C}}$ for any $\lambda \in \mathbb{R}^*$ iff

$$\prod_{k=1}^r (\lambda^{d_k})^{\langle \boldsymbol{n}_k, \boldsymbol{m} \rangle} = \lambda^{\langle \sum_{k=1}^r d_k \boldsymbol{n}_k, \boldsymbol{m} \rangle} = 1 \ \text{ for any } \boldsymbol{m} \in \mathbb{Z}^m \ \Leftrightarrow \ \sum_{k=1}^r d_k \boldsymbol{n}_k = \mathbf{0}_m$$

and this completes the proof. $\square$

The following result easily follows from the proof of Lemma 2.7.

Corollary 3.12. *If* $D = (d_1, \cdots, d_r) \in \mathbb{N}^r$ *and* $\sum_{k=1}^r d_k \boldsymbol{n}_k = \mathbf{0}_m$*,*

$$(\lambda^{d_1}, \lambda^{d_2}, \cdots, \lambda^{d_r}) \in G_{\Sigma,\mathbb{K}} \qquad \textit{for any } \lambda \in \mathbb{K}^*. \tag{3.16}$$ $\square$

4 The Vassiliev spectral sequence

4.1 Simplicial resolutions. First, recall the definitions of the non-degenerate simplicial resolution and the associated truncated simplicial resolution ([22], [28], [29], [33], [34]).

Definition 4.1. (i) For a finite set $\boldsymbol{v} = \{v_1, \cdots, v_l\} \subset \mathbb{R}^N$, let $\sigma(\boldsymbol{v})$ denote the convex hull spanned by $\boldsymbol{v}$. Let $h : X \to Y$ be a surjective map such that $h^{-1}(y)$ is a finite set for any $y \in Y$, and let $i : X \to \mathbb{R}^N$ be an embedding. Let $\mathcal{X}^\Delta$ and $h^\Delta : \mathcal{X}^\Delta \to Y$ denote the space and the map defined by

$$(4.1)\quad \mathcal{X}^\Delta = \big\{(y,u) \in Y \times \mathbb{R}^N : u \in \sigma(i(h^{-1}(y)))\big\} \subset Y \times \mathbb{R}^N,\ h^\Delta(y,u) = y.$$

The pair $(\mathcal{X}^\Delta, h^\Delta)$ is called *the simplicial resolution of* (h, i). In particular, it is called *a non-degenerate simplicial resolution* if for each $y \in Y$ any k points of $i(h^{-1}(y))$ span $(k-1)$-dimensional simplex of $\mathbb{R}^N$.

(ii) For each $k \geq 0$, let $\mathcal{X}^\Delta_k \subset \mathcal{X}^\Delta$ be the subspace of the union of the $(k-1)$-skeletons of the simplices over all the points y in Y given by

$$(4.2)\quad \mathcal{X}^\Delta_k = \big\{(y,u) \in \mathcal{X}^\Delta : u \in \sigma(\boldsymbol{v}), \boldsymbol{v} = \{v_1, \cdots, v_l\} \subset i(h^{-1}(y)), l \leq k\big\}.$$

We make the identification $X = \mathcal{X}^\Delta_1$ by identifying $x \in X$ with the pair $(h(x), i(x)) \in \mathcal{X}^\Delta_1$, and we note that there is an increasing filtration

$$(4.3)\qquad \emptyset = \mathcal{X}^\Delta_0 \subset X = \mathcal{X}^\Delta_1 \subset \mathcal{X}^\Delta_2 \subset \cdots \subset \mathcal{X}^\Delta_k \subset \cdots \subset \bigcup_{k=0}^{\infty} \mathcal{X}^\Delta_k = \mathcal{X}^\Delta.$$

Since the map $h^\Delta : \mathcal{X}^\Delta \to Y$ is a proper map, it extends to the map $h^\Delta_+ : \mathcal{X}^\Delta_+ \to Y_+$ between the one-point compactifications, where X_+ denotes the one-point compactification of a locally compact space X. □

Definition 4.2. Let $h : X \to Y$ be a surjective semi-algebraic map between semi-algebraic spaces, $j : X \to \mathbb{R}^N$ be a semi-algebraic embedding, and let $(\mathcal{X}^\Delta, h^\Delta : \mathcal{X}^\Delta \to Y)$ denote the associated non-degenerate simplicial resolution of (h, j). Then for each positive integer $k \geq 1$, we denote by $h^\Delta_k : X^\Delta(k) \to Y$ *the truncated* (*after the k-th term*) *simplicial resolution* of Y as in [29]. Note that that there is a natural filtration

$$X^\Delta_0 \subset X^\Delta_1 \subset \cdots \subset X^\Delta_l \subset X^\Delta_{l+1} \subset \cdots \subset X^\Delta_k \subset X^\Delta_{k+1} = X^\Delta_{k+2} = \cdots = X^\Delta(k),$$

where $X^\Delta_0 = \emptyset$, $X^\Delta_l = \mathcal{X}^\Delta_l$ if $l \leq k$ and $X^\Delta_l = X^\Delta(k)$ if $l > k$. □

4.2 Vassiliev spectral sequences. Next, we shall construct the Vassiliev spectral sequence for computing the homology of the space $\mathrm{Q}_n^{D,\Sigma}(\mathbb{K})$.

From now on, we always assume that Σ is a fan in $\mathbb{R}^m$ such that X_Σ is simply connected toric variety satisfying the condition (2.19a). Moreover, let $D=(d_1,\cdots,d_r)\in\mathbb{N}^r$ will always be a fixed r-tuple of positive integers.

Definition 4.3. (i) For each $d\in\mathbb{N}$, let $\mathrm{P}_d^{\mathbb{K}}\subset\mathbb{K}[z]$ denote the space of all monic polynomials $f(z)=z^d+a_1z^{d-1}+\cdots+a_d\in\mathbb{K}[z]$ of degree d. Then for each $D=(d_1,\cdots,d_r)\in\mathbb{N}^r$, let $\mathrm{P}_D^{\mathbb{K}}$ denote the space of r-tuples of monic polynomials defined by

$$\mathrm{P}_D^{\mathbb{K}}=\mathrm{P}_{d_1}^{\mathbb{K}}\times\mathrm{P}_{d_2}^{\mathbb{K}}\times\cdots\times\mathrm{P}_{d_r}^{\mathbb{K}}. \tag{4.4}$$

(ii) For each $\mathrm{f}=(f_1(z),\cdots,f_r(z))\in\mathrm{P}_D^{\mathbb{K}}$, let $F_{(n)}(\mathrm{f})(z)$ denote the rn-tuple of monic polynomials defined by

$$F_{(n)}(\mathrm{f})(z)=(F_n(f_1)(z),\cdots,F_n(f_r)(z))\in\mathbb{K}[z]^{rn}, \tag{4.5}$$

where we denote by $F_n(f_i)(z)$ the n-tuple of monic polynomials of degree d_i given by

$$F_n(f_i)(z)=(f_i(z),f_i(z)+f_i'(z),f_i(z)+f_i''(z),\cdots,f_i(z)+f_i^{(n-1)}(z)) \tag{4.6}$$

for each $1\leq i\leq r$ (as in (2.22)).

(iii) Let Σ_D denote *the discriminant* of $\mathrm{Q}_n^{D,\Sigma}(\mathbb{K})$ in $\mathrm{P}_D^{\mathbb{K}}$ given by the complement

$$\begin{aligned}\Sigma_D&=\mathrm{P}_D^{\mathbb{K}}\setminus\mathrm{Q}_n^{D,\Sigma}(\mathbb{K})\\&=\{\mathrm{f}=(f_1(z),\cdots,f_r(z))\in\mathrm{P}_D^{\mathbb{K}}:F_{(n)}(\mathrm{f})(x)\in L_n^{\mathcal{K}_\Sigma}(\mathbb{K})\text{ for some }x\in\mathbb{R}\},\end{aligned}$$

where $L_n^{\mathcal{K}_\Sigma}(\mathbb{K})$ denotes the set given by $K=\mathcal{K}_\Sigma$ in (2.5).

(iv) Let $Z_D\subset\Sigma_D\times\mathbb{R}$ denote *the tautological normalization* of Σ_D consisting of all pairs $(\mathrm{f},x)=((f_1(z),\dots,f_r(z)),x)\in\Sigma_D\times\mathbb{R}$ satisfying the condition $F_{(n)}(\mathrm{f})(x)=(F_n(f_1)(x),\cdots,F_n(f_r)(x))\in L_n^{\mathcal{K}_\Sigma}(\mathbb{K})$. Projection on the first factor gives a surjective map $\pi_D:Z_D\to\Sigma_D$. □

Remark 4.4. Let $\sigma_k\in[r]$ for $k=1,2$. It is easy to see that $L_{\sigma_1}(\mathbb{K}^n)\subset L_{\sigma_2}(\mathbb{K}^n)$ if $\sigma_1\supset\sigma_2$. Letting

$$Pr(\Sigma)=\{\sigma=\{i_1,\cdots,i_s\}\subset[r]:\{\boldsymbol{n}_{i_1},\cdots,\boldsymbol{n}_{i_s}\}\text{ is a primitive collection}\},$$

we see that $L_n^{\mathcal{K}_\Sigma}(\mathbb{K})=\bigcup_{\sigma\in Pr(\Sigma)}L_\sigma(\mathbb{K}^n)$, and by using (2.38) we obtain the equality

$$\dim L_n^{\mathcal{K}_\Sigma}(\mathbb{K})=nd(\mathbb{K})(r-r_{\min}(\Sigma))=\begin{cases}2n(r-r_{\min}(\Sigma)) & \text{if }\mathbb{K}=\mathbb{C},\\ n(r-r_{\min}(\Sigma)) & \text{if }\mathbb{K}=\mathbb{R}.\end{cases} \tag{4.7}$$

Our goal in this section is to construct, by means of the *non-degenerate* simplicial resolution of the discriminant, a spectral sequence converging to the homology of $\mathrm{Q}_n^{D,\Sigma}(\mathbb{K})$.

Definition 4.5. (i) For an r-tuple $D=(d_1,\cdots,d_r)\in\mathbb{N}^r$ of positive integers, let $N(D)$ denote the positive integer given by

$$N(D)=\sum_{k=1}^{r}d_k. \tag{4.8}$$

(ii) For each based space X, let $F(X,d)$ denote *the ordered configuration space of distinct d points in X* defined by

$$F(X,d)=\{(x_1,\cdots,x_d)\in X^d : x_i\neq x_j \text{ if } i\neq j\}. \tag{4.9}$$

Note that the symmetric group S_d of d-letters acts on $F(X,d)$ freely by permuting coordinates. Let $C_d(X)$ denote *the unordered configuration space of d-distinct points in X* given by the orbit space

$$C_d(X)=F(X,d)/S_d. \tag{4.10}$$

(iii) Let $L_{k;\Sigma,\mathbb{K}}\subset(\mathbb{R}\times L_n^{\mathcal{K}_\Sigma}(\mathbb{K}))^k$ denote the subspaces defined by

$$L_{k;\Sigma,\mathbb{K}}=\{((x_1,s_1),\cdots,(x_k,s_k))\in(\mathbb{R}\times L_n^{\mathcal{K}_\Sigma}(\mathbb{K}))^k : x_i\neq x_j \text{ if } i\neq j\}.$$

The symmetric group S_k on k letters acts on the space $L_{k;\Sigma,\mathbb{K}}$ by permuting k-elements., and let $C_{k;\Sigma,\mathbb{K}}$ denote the orbit space defined by

$$C_{k;\Sigma,\mathbb{K}}=L_{k;\Sigma,\mathbb{K}}/S_k. \tag{4.11}$$

Note that the space $C_{k;\Sigma,\mathbb{K}}$ is a cell-complex of dimension (by (4.7))

$$\dim C_{k;\Sigma,\mathbb{K}}=\begin{cases}k+2kn(r-r_{\min}(\Sigma)) & \text{if } \mathbb{K}=\mathbb{C},\\ k+kn(r-r_{\min}(\Sigma)) & \text{if } \mathbb{K}=\mathbb{R}.\end{cases} \tag{4.12}$$

(iv) Let $(\mathcal{X}^D,\pi_D^\Delta:\mathcal{X}^D\to\Sigma_D)$ be the non-degenerate simplicial resolution associated to the surjective map $\pi_D:Z_D\to\Sigma_D$ with the natural increasing filtration as in Definition 4.1,

$$\emptyset=\mathcal{X}_0^D\subset\mathcal{X}_1^D\subset\mathcal{X}_2^D\subset\cdots\subset\mathcal{X}^D=\bigcup_{k=0}^{\infty}\mathcal{X}_k^D. \qquad \square$$

By [33, Lemma 1 (page 90)], the map π^{Δ}_{D} extends to a homology equivalence $\pi^{\Delta}_{D+}:\mathcal{X}^{D}_{+}\xrightarrow{\simeq}\Sigma_{D+}$. Since $\mathcal{X}^{D}_{k\ +}/\mathcal{X}^{D}_{k-1+}\cong(\mathcal{X}^{D}_{k}\setminus\mathcal{X}^{D}_{k-1})_{+}$, we have a spectral sequence

$$\left\{E^{k,s}_{t;D}, d_t : E^{k,s}_{t;D}\to E^{k+t,s+1-t}_{t;D}\right\}\Rightarrow H^{k+s}_{c}(\Sigma_D;\mathbb{Z}), \tag{4.13}$$

where $E^{k,s}_{1;D}=H^{k+s}_c(\mathcal{X}^D_k\setminus\mathcal{X}^D_{k-1};\mathbb{Z})$ and $H^k_c(X;\mathbb{Z})$ denotes the cohomology group with compact supports given by $H^k_c(X;\mathbb{Z})=\tilde{H}^k(X_+;\mathbb{Z})$.

Since there is a homeomorphism $\mathrm{P}^{\mathbb{K}}_D\cong\mathbb{K}^{N(D)}\cong\mathbb{R}^{d(\mathbb{K})N(D)}$, by Alexander duality there is a natural isomorphism

$$\tilde{H}_k(\mathrm{Q}^{D,\Sigma}_n(\mathbb{K});\mathbb{Z})\cong H^{d(\mathbb{K})N(D)-k-1}_c(\Sigma_D;\mathbb{Z})\quad\text{for any } k. \tag{4.14}$$

By reindexing we obtain a spectral sequence

$$\left\{E^{t;D}_{k,s},\tilde{d}^t : E^{t;D}_{k,s}\to E^{t;D}_{k+t,s+t-1}\right\}\Rightarrow H_{s-k}(\mathrm{Q}^{D,\Sigma}_n(\mathbb{K});\mathbb{Z}), \tag{4.15}$$

where $E^{1;D}_{k,s}=H^{d(\mathbb{K})N(D)+k-s-1}_c(\mathcal{X}^D_k\setminus\mathcal{X}^D_{k-1};\mathbb{Z})$.

Lemma 4.6. *If $d_{\min}\geq n$ and $1\leq k\leq\lfloor\frac{d_{\min}}{n}\rfloor$, the space $\mathcal{X}^D_k\setminus\mathcal{X}^D_{k-1}$ is homeomorphic to the total space of a real affine bundle $\xi_{D,k,n}$ over $C_{k;\Sigma,\mathbb{K}}$ with rank $l_{D,k,n}=d(\mathbb{K})(N(D)-nrk)+k-1$.*

Proof. Since the proof is completely analogous to that of [27, Lemma 4.9], we omit detail of the proof. □

Lemma 4.7. *If $d_{\min}\geq n$ and $1\leq k\leq\lfloor\frac{d_{\min}}{n}\rfloor$, there is a natural isomorphism*

$$E^{1;D}_{k,s}\cong H^{d(\mathbb{K})nrk-s}_c(C_{k;\Sigma,\mathbb{K}};\pm\mathbb{Z}),$$

where the twisted coefficients system $\pm\mathbb{Z}$ comes from the Thom isomorphism.

Proof. Suppose that $1\leq k\leq\lfloor\frac{d_{\min}}{n}\rfloor$. By Lemma 4.6, there is a homeomorphism $(\mathcal{X}^D_k\setminus\mathcal{X}^D_{k-1})_+\cong T(\xi_{D,k})$, where $T(\xi_{D,k,n})$ denotes the Thom space of $\xi_{D,k,n}$. Since $(d(\mathbb{K})N(D)+k-s-1)-l_{D,k,n}=d(\mathbb{K})nrk-s$, the Thom isomorphism gives a natural isomorphism

$$E^{1;d}_{k,s}\cong\tilde{H}^{d(\mathbb{K})N(D)+k-s-1}(T(\xi_{d,k,n});\mathbb{Z})\cong H^{d(\mathbb{K})nrk-s}_c(C_{k;\Sigma,\mathbb{K}};\pm\mathbb{Z}),$$

and the assertion follows. □

5 Stabilization maps

We will now define two stabilization maps

$$
(5.1)\qquad \begin{cases} s_{D,D+\boldsymbol{a}}:\mathrm{Q}_n^{D,\Sigma}(\mathbb{C})\to \mathrm{Q}_n^{D+\boldsymbol{a},\Sigma}(\mathbb{C}) \\ s^{\mathbb{R}}_{D,D+\boldsymbol{a}}:\mathrm{Q}_n^{D,\Sigma}(\mathbb{R})\to \mathrm{Q}_n^{D+\boldsymbol{a},\Sigma}(\mathbb{R}) \end{cases} \quad \text{for each } \boldsymbol{a}\neq \mathbf{0}_r\in (\mathbb{Z}_{\geq 0})^r.
$$

Definition 5.1. (i) For an r-tuple $D=(d_1,\cdots,d_r)\in\mathbb{N}^r$, let $U_D\subset\mathbb{C}$ denote the subspace defined by

$$
(5.2)\qquad U_D=\{w\in\mathbb{C}:\mathrm{Re}(w)<N(D)\},
$$

and let $\varphi_D:\mathbb{C}\xrightarrow{\cong}U_D$ be any homeomorphism (which we now fix) satisfying the following two conditions:

$$
(5.3)\qquad \varphi_D(\mathbb{R})=(-\infty,N(D)) \ \text{ and } \ \varphi_D(\overline{\alpha})=\overline{\varphi_D(\alpha)} \quad \text{for any } \alpha\in\mathrm{H}_+,
$$

where $N(D)$ is the positive integer given by (4.8), and $\mathrm{H}_+\subset\mathbb{C}$ denotes the upper half plane in $\mathbb{C}$ given by

$$
(5.4)\qquad \mathrm{H}_+=\{\alpha\in\mathbb{C}:\ \mathrm{Im}\ \alpha>0\}.
$$

(ii) Now let us choose and fix any r points $(x_1,\cdots,x_r)\in(\mathbb{C}\setminus U_D)^r$ satisfying the condition $x_i\neq x_j$ if $i\neq j$.

For each monic polynomial $f(z)=\prod_{k=1}^d(z-\alpha_k)\in\mathbb{C}[z]$ of degree d, let $\varphi_D(f)$ denote the monic polynomial of the same degree d given by

$$
(5.5)\qquad \varphi_D(f)=\prod_{k=1}^d(z-\varphi_D(\alpha_k)).
$$

(iii) For each r-tuple $\boldsymbol{a}=(a_1,\cdots,a_r)\neq\mathbf{0}_r\in(\mathbb{Z}_{\geq 0})^r$, define the stabilization map

$$
(5.6)\qquad \begin{aligned} s_{D,D+\boldsymbol{a}}:&\mathrm{Q}_n^{D,\Sigma}(\mathbb{C})\to\mathrm{Q}_n^{D+\boldsymbol{a},\Sigma}(\mathbb{C}) \quad \text{by} \\ s_{D,D+\boldsymbol{a}}(f)&=(\varphi_D(f_1)(z-x_1)^{a_1},\cdots,\varphi_D(f_r)(z-x_r)^{a_r}) \end{aligned}
$$

for $f=(f_1(z),\cdots,f_r(z))\in\mathrm{Q}_n^{D,\Sigma}(\mathbb{C})$.

Remark 5.2. (i) Note that the definition of the map $s_{D,D+\boldsymbol{a}}$ depends on the choice of the homeomorphism φ_D and the r-tuple $(x_1,\cdots,x_r)\in(\mathbb{C}\setminus U_D)^r$ of points, but one can show that the homotopy type of it does not depend on these choices.

(ii) Let $\boldsymbol{a},\boldsymbol{b}\in(\mathbb{Z}_{\geq 0})^r$ be any two r-tuples such that $\boldsymbol{a},\boldsymbol{b}\neq \mathbf{0}_r$. Then it is easy to see that the equality

$$(s_{D+\boldsymbol{a},D+\boldsymbol{a}+\boldsymbol{b}})\circ(s_{D,D+\boldsymbol{a}})=s_{D,D+\boldsymbol{a}+\boldsymbol{b}}\quad \text{(up to homotopy)} \tag{5.7}$$

holds. Thus we mostly only consider the stabilization map $s_{D,D+\boldsymbol{e}_i}$ for each $1\leq i\leq r$, where $\boldsymbol{e}_1=(1,0,\cdots,0),\boldsymbol{e}_2=(0,1,0,\cdots,0),\cdots,\boldsymbol{e}_r=(0,0,\cdots,0,1)\in\mathbb{R}^r$ denote the standard orthogonal basis of $\mathbb{R}^r$.

(iii) From (5.3) it easily follows that

$$\varphi_D(f)\in\mathbb{R}[z]\quad \text{if } f=f(z)\in\mathbb{R}[z]. \tag{5.8}$$

Thus, for each r-tuple $\boldsymbol{a}=(a_1,\cdots,a_r)\neq\mathbf{0}_r\in(\mathbb{Z}_{\geq 0})^r$, one can easily show that the following holds:

$$s_{D,D+\boldsymbol{a}}(\mathrm{Q}_n^{D,\Sigma}(\mathbb{R}))\subset \mathrm{Q}_n^{D+\boldsymbol{a},\Sigma}(\mathbb{R}). \tag{5.9}$$

Definition 5.3. By (5.9), one can define the stabilization map

$$s^{\mathbb{R}}_{D,D+\boldsymbol{a}}:\mathrm{Q}_n^{D,\Sigma}(\mathbb{R})\to \mathrm{Q}_n^{D+\boldsymbol{a},\Sigma}(\mathbb{R})\quad \text{by the restriction} \tag{5.10}$$
$$s^{\mathbb{R}}_{D,D+\boldsymbol{a}}=s_{D,D+\boldsymbol{a}}|\mathrm{Q}_n^{D,\Sigma}(\mathbb{R}).$$

Remark 5.4. From the definition (5.6) and (5.8) we see that the following equality holds:

$$s^{\mathbb{R}}_{D,D+\boldsymbol{a}}=(s_{D,D+\boldsymbol{a}})^{\mathbb{Z}_2}\quad \text{for each } \boldsymbol{a}\neq\mathbf{0}_r\in(\mathbb{Z}_{\geq 0})^r. \quad\Box \tag{5.11}$$

6 Homology stability

We will now consider homology stability of the space $\mathrm{Q}_n^{D,\Sigma}(\mathbb{K})$.

6.1 The case $\mathbb{K}=\mathbb{C}$. First, consider the case $\mathbb{K}=\mathbb{C}$. Let $1\leq i\leq r$ and consider the stabilization map

$$s_{D,D+\boldsymbol{e}_i}:\mathrm{Q}_n^{D,\Sigma}(\mathbb{C})\to \mathrm{Q}_n^{D+\boldsymbol{e}_i,\Sigma}(\mathbb{C}). \tag{6.1}$$

It is easy to see that it extends to an open embedding

$$s_{D,i}:\mathbb{C}\times \mathrm{Q}_n^{D,\Sigma}(\mathbb{C})\to \mathrm{Q}_n^{D+\boldsymbol{e}_i,\Sigma}(\mathbb{C}) \tag{6.2}$$

by adding the points from the infinity as in Definition 5.1. It also naturally extends to an open embedding $\tilde{s}_{D,i}:\mathbb{C}\times \mathrm{P}^{\mathbb{C}}_D\to \mathrm{P}^{\mathbb{C}}_{D+\boldsymbol{e}_i}$ and by restriction we obtain an open embedding

$$\tilde{s}_{D,i}:\mathbb{C}\times\Sigma_D\to\Sigma_{D+\boldsymbol{e}_i}. \tag{6.3}$$

Since one-point compactification is contravariant for open embeddings, this map induces a map in the opposite direction

$$\tilde{s}_{D,i+} : (\Sigma_{D+\boldsymbol{e}_i})_+ \to (\mathbb{C} \times \Sigma_D)_+ = S^2 \wedge \Sigma_{D+}. \tag{6.4}$$

We obtain the following commutative diagram

$$\begin{array}{ccc} \tilde{H}_k(\mathrm{Q}_n^{D,\Sigma}(\mathbb{C});\mathbb{Z}) & \xrightarrow{(s_{D,D+\boldsymbol{e}_i})_*} & \tilde{H}_k(\mathrm{Q}_n^{D+\boldsymbol{e}_i,\Sigma}(\mathbb{C});\mathbb{Z}) \\ {\scriptstyle AD_1}\downarrow \cong & & {\scriptstyle AD_2}\downarrow \cong \\ H_c^{2N(D)-k-1}(\Sigma_D;\mathbb{Z}) & \xrightarrow{(\tilde{s}_{D,i+})^*} & H_c^{2N(D)-k+1}(\Sigma_{D+\boldsymbol{e}_i};\mathbb{Z}). \end{array} \tag{6.5}$$

Here, AD_k ($k = 1, 2$) denote the corresponding Alexander duality isomorphisms and $\tilde{s}^*_{D,i+}$ denotes the composite of the suspension isomorphism with the homomorphism $(\tilde{s}_{D+})^*$ given by

$$H_c^M(\Sigma_D;\mathbb{Z}) \xrightarrow{\cong} H_c^{M+2}(\mathbb{C}\times\Sigma_D;\mathbb{Z}) \xrightarrow{(\tilde{s}_{D,i+})^*} H_c^{M+2}(\Sigma_{D+\boldsymbol{e}_i};\mathbb{Z}), \tag{6.6}$$

where $M = 2N(D) - k - 1$.

By the universality of the non-degenerate simplicial resolution [28], the map $\tilde{s}_{D,i}$ also naturally extends to a filtration preserving open embedding

$$\tilde{s}_{D,i} : \mathbb{C} \times \mathcal{X}^D \to \mathcal{X}^{D+\boldsymbol{e}_i} \tag{6.7}$$

between non-degenerate simplicial resolutions. This induces a filtration preserving map

$$(\tilde{s}_{D,i})_+ : \mathcal{X}_+^{D+\boldsymbol{e}_i} \to (\mathbb{C}\times\mathcal{X}^D)_+ = S^2 \wedge \mathcal{X}^D_+, \tag{6.8}$$

and we finally obtain the homomorphism of spectral sequences

$$\{\tilde{\theta}^t_{k,s} : E^{t;D}_{k,s} \to E^{t;D+\boldsymbol{a}}_{k,s}\}, \quad \text{where} \tag{6.9}$$

$$\begin{cases} \{E^{t;D}_{k,s}, \tilde{d}^t : E^{t;D}_{k,s} \to E^{t;D}_{k+t,s+t-1}\} & \Rightarrow H_{s-k}(\mathrm{Q}_n^{D,\Sigma}(\mathbb{C});\mathbb{Z}), \\ \{E^{t;D+\boldsymbol{e}_i}_{k,s}, \tilde{d}^t : E^{t;D+\boldsymbol{e}_i}_{k,s} \to E^{t;D+\boldsymbol{e}_i}_{k+t,s+t-1}\} & \Rightarrow H_{s-k}(\mathrm{Q}_n^{D+\boldsymbol{e}_i,\Sigma}(\mathbb{C});\mathbb{Z}), \end{cases}$$

$$\begin{cases} E^{1;D}_{k,s} & = H_c^{2N(D)+k-1-s}(\mathcal{X}^D_k \setminus \mathcal{X}^D_{k-1};\mathbb{Z}), \\ E^{1;D+\boldsymbol{e}_i}_{k,s} & = H_c^{2N(D)+k+1-s}(\mathcal{X}^{D+\boldsymbol{e}_i}_k \setminus \mathcal{X}^{D+\boldsymbol{e}_i}_{k-1};\mathbb{Z}). \end{cases}$$

Lemma 6.1. *If $1 \le i \le r$ and $0 \le k \le \lfloor \frac{d_{\min}}{n} \rfloor$, $\tilde{\theta}^1_{k,s} : E^{1;D}_{k,s} \to E^{1;D+\boldsymbol{e}_i}_{k,s}$ is an isomorphism for any s.*

Proof. Since the proof is completely analogous to that of [27, Lemma 4.13], we omit its details. □

Now we consider the spectral sequences induced by truncated simplicial resolutions.

Definition 6.2. Let X^{Δ} denote the truncated (after the $\lfloor \frac{d_{\min}}{n} \rfloor$-th term) simplicial resolution of Σ_D with the natural filtration

$$\emptyset = X^{\Delta}_0 \subset X^{\Delta}_1 \subset \cdots \subset X^{\Delta}_{\lfloor d_{\min}/n \rfloor} \subset X^{\Delta}_{\lfloor d_{\min}/n \rfloor+1} = X^{\Delta}_{\lfloor d_{\min}/n \rfloor+2} = \cdots = X^{\Delta},$$

where $X^{\Delta}_k = \mathcal{X}^D_k$ if $k \leq \lfloor \frac{d_{\min}}{n} \rfloor$ and $X^{\Delta}_k = X^{\Delta}$ if $k \geq \lfloor \frac{d_{\min}}{n} \rfloor + 1$.

Similarly, let Y^{Δ} denote the truncated (after the $\lfloor \frac{d_{\min}}{n} \rfloor$-th term) simplicial resolution of $\Sigma_{D+\boldsymbol{e}_i}$ with the natural filtration

$$\emptyset = Y^{\Delta}_0 \subset Y^{\Delta}_1 \subset \cdots \subset Y^{\Delta}_{\lfloor d_{\min}/n \rfloor} \subset Y^{\Delta}_{\lfloor d_{\min}/n \rfloor+1} = Y^{\Delta}_{\lfloor d_{\min}/n \rfloor+2} = \cdots = Y^{\Delta},$$

where $Y^{\Delta}_k = \mathcal{X}^{D+\boldsymbol{e}_i}_k$ if $k \leq \lfloor \frac{d_{\min}}{n} \rfloor$ and $Y^{\Delta}_k = Y^{\Delta}$ if $k \geq \lfloor \frac{d_{\min}}{n} \rfloor + 1$.

By [29, §2 and §3], we obtain the following *truncated spectral sequences*

$$(6.10) \qquad \begin{cases} \{E^{t;\mathbb{C}}_{k,s}, d^t : E^{t;\mathbb{C}}_{k,s} \to E^{t;\mathbb{C}}_{k+t,s+t-1}\} & \Rightarrow H_{s-k}(\mathrm{Q}^{D,\Sigma}_n(\mathbb{C});\mathbb{Z}), \\ \{\,'E^{t;\mathbb{C}}_{k,s},\ d^t :\ 'E^{t;\mathbb{C}}_{k,s} \to\ 'E^{t}_{k+t,s+t-1}\} & \Rightarrow H_{s-k}(\mathrm{Q}^{D+\boldsymbol{e}_i,\Sigma}_n(\mathbb{C});\mathbb{Z}), \end{cases}$$

where

$$(6.11) \qquad \begin{cases} E^{1;\mathbb{C}}_{k,s} &= H^{2N(D)+k-1-s}_c(X^{\Delta}_k \setminus X^{\Delta}_{k-1};\mathbb{Z}), \\ 'E^{1;\mathbb{C}}_{k,s} &= H^{2N(D)+k+1-s}_c(Y^{\Delta}_k \setminus Y^{\Delta}_{k-1};\mathbb{Z}). \end{cases}$$

By the naturality of truncated simplicial resolutions, the filtration preserving map $\tilde{s}_{D,i} : \mathbb{C} \times \mathcal{X}^D \to \mathcal{X}^{D+\boldsymbol{e}_i}$ gives rise to a natural filtration preserving map $\tilde{s}'_{D,i} : \mathbb{C} \times X^{\Delta} \to Y^{\Delta}$ which, in a way analogous to (6.9), induces a homomorphism of spectral sequences

$$(6.12) \qquad \{\theta^t_{k,s} : E^{t;\mathbb{C}}_{k,s} \to\ 'E^{t;\mathbb{C}}_{k,s}\}.$$

Lemma 6.3. (i) *If $k < 0$ or $k \geq \lfloor \frac{d_{\min}}{n} \rfloor + 2$, $E^{1;\mathbb{C}}_{k,s} =\ 'E^{1;\mathbb{C}}_{k,s} = 0$ for any s.*

(ii) *$E^{1;\mathbb{C}}_{0,0} =\ 'E^{1;\mathbb{C}}_{0,0} = \mathbb{Z}$ and $E^{1;\mathbb{C}}_{0,s} =\ 'E^{1;\mathbb{C}}_{0,s} = 0$ if $s \neq 0$.*

(iii) *If $1 \leq k \leq \lfloor \frac{d_{\min}}{n} \rfloor$, there are isomorphisms*

$$E^{1;\mathbb{C}}_{k,s} \cong\ 'E^{1;\mathbb{C}}_{k,s} \cong H^{2nrk-s}_c(C_{k;\Sigma};\pm\mathbb{Z}).$$

(iv) *If $1 \leq k \leq \lfloor \frac{d_{\min}}{n} \rfloor$, $E^{1;\mathbb{C}}_{k,s} =\ 'E^{1;\mathbb{C}}_{k,s} = 0$ for any $s \leq (2nr_{\min}(\Sigma) - 1)k - 1$.*

(v) *If $k = \lfloor \frac{d_{\min}}{n} \rfloor + 1$, $E^{1;\mathbb{C}}_{k,s} =\ 'E^{1;\mathbb{C}}_{k,s} = 0$ for any $s \leq (2nr_{\min}(\Sigma)-1)\lfloor \frac{d_{\min}}{n} \rfloor - 1$.*

Proof. Let us write $r_{\min} = r_{\min}(\Sigma)$ and $d'_{\min} = \lfloor \frac{d_{\min}}{n} \rfloor$. Since the proofs of both cases are identical, it suffices to prove the assertions for $E^{1;\mathbb{C}}_{k,s}$.

(i), (ii), (iii): Since $X^{\Delta}_k = X^{\Delta}$ for any $k \geq d'_{\min}+2$, the assertions (i) and (ii) are clearly true. Since $X^{\Delta}_k = \mathcal{X}^D_k$ for any $k \leq d'_{\min}$, the assertion (iii) easily follows from Lemma 4.7.

(iv) Suppose that $1 \leq k \leq d'_{\min}$. By using the equality (4.12),

$$2nrk - s > \dim C_{k;\Sigma} \;\Leftrightarrow\; s \leq (2nr_{\min}-1)k-1.$$

Thus, the assertion (iv) follows from the isomorphism given by (iii).

(v) By Lemma [29, Lemma 2.1], we see that

$$\begin{aligned}\dim(X^{\Delta}_{d'_{\min}+1} \setminus X^{\Delta}_{d'_{\min}}) &= \dim(\mathcal{X}^D_{d'_{\min}} \setminus \mathcal{X}^D_{d'_{\min}-1}) + 1 = l_{D,d'_{\min},n} + \dim C_{d'_{\min};\Sigma} + 1 \\ &= 2N(D) + 2d'_{\min} - 2nr_{\min}d'_{\min}.\end{aligned}$$

Since $E^{1;\mathbb{C}}_{d'_{\min}+1,s} = H_c^{2N(D)+d'_{\min}-s}(X^{\Delta}_{d'_{\min}+1} \setminus X^{\Delta}_{d'_{\min}};\mathbb{Z})$ (by (6.11)) and

$$\begin{aligned}2N(D)+d'_{\min}-s > \dim(X^{\Delta}_{d'_{\min}+1} \setminus X^{\Delta}_{d'_{\min}}) &= 2N(D)+2d'_{\min}-2nr_{\min}d'_{\min} \\ \Leftrightarrow s < (2nr_{\min}-1)d'_{\min} &\Leftrightarrow s \leqq (2nr_{\min}-1)d'_{\min}-1,\end{aligned}$$

we see that $E^{1;\mathbb{C}}_{d'_{\min}+1,s} = 0$ for any $s \leq (2nr_{\min}-1)d'_{\min}-1$. □

Lemma 6.4. *If* $0 \leq k \leq \lfloor \frac{d_{\min}}{n} \rfloor$, $\theta^1_{k,s} : E^{1;\mathbb{C}}_{k,s} \xrightarrow{\cong} {}'E^{1;\mathbb{C}}_{k,s}$ *is an isomorphism for any* s.

Proof. Since $(X^{\Delta}_k, Y^{\Delta}_k) = (\mathcal{X}^D_k, \mathcal{X}^{D+\boldsymbol{e}_i}_k)$ for $k \leq \lfloor \frac{d_{\min}}{n} \rfloor$, the assertion follows from Lemma 6.1. □

Theorem 6.5. *For each* $1 \leq i \leq r$, *the stabilization map*

$$s_{D,D+\boldsymbol{e}_i} : \mathrm{Q}^{D,\Sigma}_n(\mathbb{C}) \to \mathrm{Q}^{D+\boldsymbol{e}_i,\Sigma}_n(\mathbb{C})$$

is a homology equivalence through dimension $d(D;\Sigma,n,\mathbb{C})$.

Proof. We write $r_{\min} = r_{\min}(\Sigma)$ and $d'_{\min} = \lfloor \frac{d_{\min}}{n} \rfloor$ as in the proof of Lemma 6.3. Without loss of generality, we may assume that $d_{\min} \geq n \geq 1$.

Let us consider the homomorphism $\theta^t_{k,s} : E^{t;\mathbb{C}}_{k,s} \to {}'E^{t;\mathbb{C}}_{k,s}$ of truncated spectral sequences given in (6.12). By using the commutative diagram (6.5) and the comparison theorem for spectral sequences, we see that it suffices to prove that the positive integer $d(D;\Sigma,n,\mathbb{C})$ has the following property:

(†) $\theta^{\infty}_{k,s}$ is an isomorphism for all (k,s) such that $s-k \leq d(D;\Sigma,n,\mathbb{C})$.

By Lemma 6.3, we can easily see that:

$(\dagger)_1$ if $k<0$ or $k\geq d'_{\min}+1$, $\theta^{\infty}_{k,s}$ is an isomorphism for all (k,s) such that $s-k\leq d(D;\Sigma,n,\mathbb{C})$.

Next, assume that $0\leq k\leq d'_{\min}$, and investigate the conditions that ensure that $\theta^{\infty}_{k,s}$ is an isomorphism. Note that the groups $E^{1;\mathbb{C}}_{k_1,s_1}$ and $'E^{1;\mathbb{C}}_{k_1,s_1}$ are not known for $(u,v)\in\mathcal{S}_1=\{(d'_{\min}+1,s)\in\mathbb{Z}^2:s\geq(2nr_{\min}-1)d'_{\min}\}$. By considering the differentials d^1's of $E^{1;\mathbb{C}}_{k,s}$ and $'E^{1;\mathbb{C}}_{k,s}$, and applying Lemma 6.4, we see that $\theta^2_{k,s}$ is an isomorphism if $(k,s)\notin\mathcal{S}_1\cup\mathcal{S}_2$, where

$$\mathcal{S}_2=\{(u,v)\in\mathbb{Z}^2:(u+1,v)\in\mathcal{S}_1\}=\{(d'_{\min},v)\in\mathbb{Z}^2:v\geq(2nr_{\min}-1)d'_{\min}\}.$$

A similar argument shows that $\theta^3_{k,s}$ is an isomorphism if $(k,s)\notin\bigcup_{t=1}^3\mathcal{S}_t$, where $\mathcal{S}_3=\{(u,v)\in\mathbb{Z}^2:(u+2,v+1)\in\mathcal{S}_1\cup\mathcal{S}_2\}$. Continuing in the same fashion: considering the differentials d^t's on $E^{t;\mathbb{C}}_{k,s}$ and $'E^{t;\mathbb{C}}_{k,s}$ and applying the inductive hypothesis, we see that $\theta^{\infty}_{k,s}$ is an isomorphism if $(k,s)\notin\mathcal{S}:=\bigcup_{t\geq 1}\mathcal{S}_t=\bigcup_{t\geq 1}A_t$, where A_t denotes the set

$$A_t=\left\{\ (u,v)\in\mathbb{Z}^2\ \middle|\ \begin{array}{l}\text{There are positive integers } l_1,\cdots,l_t \text{ such that,}\\ 1\leq l_1<l_2<\cdots<l_t,\ u+\sum_{j=1}^t l_j=d'_{\min}+1,\\ v+\sum_{j=1}^t(l_j-1)\geq(2nr_{\min}-1)d'_{\min}\end{array}\right\}.$$

Note that if this set was empty for every t, then, of course, the conclusion of Theorem 6.5 would hold in all dimensions (this is known to be false in general). If $A_t\neq\emptyset$, it is easy to see that

$$\begin{aligned}a(t)&=\min\{s-k:(k,s)\in A_t\}=(2nr_{\min}-1)d'_{\min}-(d'_{\min}+1)+t\\&=(2nr_{\min}-2)d'_{\min}+t-1=d(D;\Sigma,n,\mathbb{C})+t+1.\end{aligned}$$

Hence, we obtain that $\min\{a(t):t\geq 1,A_t\neq\emptyset\}=d(D;\Sigma,n,\mathbb{C})+2$. Since $\theta^{\infty}_{k,s}$ is an isomorphism for any $(k,s)\notin\bigcup_{t\geq 1}A_t$ for each $0\leq k\leq d'_{\min}$, we have the following:

$(\dagger)_2$ If $0\leq k\leq d'_{\min}$, $\theta^{\infty}_{k,s}$ is an isomorphism for any (k,s) such that $s-k\leq d(D;\Sigma,n,\mathbb{C})+1$.

Then, by $(\dagger)_1$ and $(\dagger)_2$, we know that $\theta^{\infty;\mathbb{C}}_{k,s}:E^{\infty;\mathbb{C}}_{k,s}\xrightarrow{\cong}{}'E^{\infty;\mathbb{C}}_{k,s}$ is an isomorphism for any (k,s) if $s-k\leq d(D;\Sigma,n,\mathbb{C})$. Hence, by $(\dagger)$ we have the desired assertion and this completes the proof of Theorem 6.5. □

Corollary 6.6. *For each $\boldsymbol{a}\neq\mathbf{0}_r\in(\mathbb{Z}_{\geq 0})^r$, the stabilization map*

$$s_{D,D+\boldsymbol{a}}:\mathrm{Q}^{D,\Sigma}_n(\mathbb{C})\to\mathrm{Q}^{D+\boldsymbol{a},\Sigma}_n(\mathbb{C})$$

is a homology equivalence through dimension $d(D;\Sigma,n,\mathbb{C})$.

Proof. The assertion easily follows from (5.7) and Theorem 6.5. □

6.2 The case $\mathbb{K}=\mathbb{R}$. Next, we shall consider the case $\mathbb{K}=\mathbb{R}$. By using exactly the same approach as in Lemmas 4.6, 4.7, 6.1, 6.3, 6.4, Theorem 6.5, and Corollary 6.6, we can obtain the following result.

Lemma 6.7. *There is the following truncated spectral sequence*

(6.13) $$\left\{E^{t;\mathbb{R}}_{k,s}, d^t : E^{t;\mathbb{R}}_{k,s} \to E^{t;\mathbb{R}}_{k+t,s+t-1}\right\} \Rightarrow H_{s-k}(\mathrm{Q}^{D,\Sigma}_n(\mathbb{R});\mathbb{Z})$$

satisfying the following conditions:

(i) *If $k<0$ or $k \geq \lfloor \frac{d_{\min}}{n}\rfloor +2$, $E^{1;\mathbb{R}}_{k,s}=0$ for any s.*

(ii) *$E^{1;\mathbb{R}}_{0,0}=\mathbb{Z}$ and $E^{1;\mathbb{R}}_{0,s}=0$ if $s\neq 0$.*

(iii) *If $1\leq k\leq \lfloor \frac{d_{\min}}{n}\rfloor$, there is a natural isomorphism*

$$E^{1;\mathbb{R}}_{k,s} \cong H^{nrk-s}_c(C_{k;\Sigma,\mathbb{R}};\pm\mathbb{Z}).$$

(iv) *If $1\leq k\leq \lfloor \frac{d_{\min}}{n}\rfloor$, $E^{1;\mathbb{R}}_{k,s}=0$ for any $s\leq (nr_{\min}(\Sigma)-1)k-1$.*

(v) *If $k=\lfloor \frac{d_{\min}}{n}\rfloor+1$, $E^{1;\mathbb{R}}_{k,s}=0$ for any $s\leq (nr_{\min}(\Sigma)-1)\lfloor \frac{d_{\min}}{n}\rfloor-1$.* □

Theorem 6.8. *For each $\boldsymbol{a}\neq \mathbf{0}_r \in (\mathbb{Z}_{\geq 0})^r$, the stabilization map*

$$s^{\mathbb{R}}_{D,D+\boldsymbol{a}} : \mathrm{Q}^{D,\Sigma}_n(\mathbb{R}) \to \mathrm{Q}^{D+\boldsymbol{a},\Sigma}_n(\mathbb{R})$$

is a homology equivalence through dimension $d(D;\Sigma,n,\mathbb{R})$, where $d(D;\Sigma,n,\mathbb{R})$ denotes the integer given by (2.37).

Proof. This assertion can be proved by using the spectral sequence (6.13) in exactly the same way as in the case of $\mathrm{Q}^{D,\Sigma}_n(\mathbb{C})$, so we omit the details. □

7 Connectivity

Lemma 7.1. (i) *The space $\mathrm{Q}^{D,\Sigma}_n(\mathbb{C})$ is simply connected.*

(ii) *If $n\geq 2$, the space $\mathrm{Q}^{D,\Sigma}_n(\mathbb{R})$ is simply connected. If $n=1$ and $r_{\min}(\Sigma)\geq 3$, the fundamental group $\pi_1(\mathrm{Q}^{D,\Sigma}_n(\mathbb{R}))$ is abelian.*

Proof. Note that an element of $\pi_1(\mathrm{Q}^{D,\Sigma}_n(\mathbb{K}))$ can be represented by an r-tuple $(\eta_1,\cdots,\eta_r)$ of strings of r-different colors where each η_k $(1\leq k\leq r)$ has total multiplicity d_k, as in the case of strings representing elements of the classical braid group $\mathrm{Br}_d=\pi_1(C_d(\mathbb{C}))$ [19]. However, in our case an r-tuple $(\eta_1,\cdots,\eta_r)$ of strings of r-different colors can move continuously representing the same element of the fundamental group,[8] as long as the following situation $(*)_\sigma$ does not occur for each $\sigma=\{i_1,\cdots,i_s\}\in I(\mathcal{K}_\Sigma)$:

$(*)_\sigma$ The strings $\{\eta_i\}_{i\in\sigma}$ of s-different colors with multiplicity $\geq n$ pass through a single point of the real line $\mathbb{R}$.

(i) In the case $\mathbb{K}=\mathbb{C}$, we can continuously deform the strings $(\eta_1,\cdots,\eta_r)$ and, if necessary, make them pass through one another in $\mathbb{C}\setminus\mathbb{R}$, so that any collection of strings can be continuously deformed to a trivial one. Thus, the space $\mathrm{Q}^{D,\Sigma}_n(\mathbb{C})$ is path-connected and simply connected.

(ii) Let $\mathbb{K}=\mathbb{R}$. If $n\geq 2$, a similar argument as above shows that the fundamental group must be trivial, since any string of multiplicity $\geq n$ can be split into stings of multiplicity less than n (by the continuous deformation). Thus, the space $\mathrm{Q}^{D,\Sigma}_n(\mathbb{R})$ is path-connected and simply connected if $n\geq 2$.

Next, consider the case $n=1$ with $r_{\min}(\Sigma)\geq 3$. Then the space $\mathrm{Q}^{D,\Sigma}_1(\mathbb{R})$ is path-connected and $\pi_1(\mathrm{Q}^{D,\Sigma}_1(\mathbb{R}))$ is commutative. To see this, let $a,b\in\pi_1(\mathrm{Q}^{D,\Sigma}_1(\mathbb{R}))$ be any two elements, and suppose that r-tuple $(\eta_1,\cdots,\eta_r)$ of strings of r-different colors which represents the product $a\cdot b\in\pi_1(\mathrm{Q}^{D,\Sigma}_n(\mathbb{R}))$. Let $\sigma\in I(\mathcal{K}_\Sigma)$ and $\{i,j\}\subset\sigma$. Since $\mathrm{card}(\sigma)\geq r_{\min}(\Sigma)\geq 3$ (by (3.3)), there is some number $k\in\sigma$ such that $k\notin\{i,j\}$. But this means that the i-th string and the j-th string can pass through one another on the real line, as long as they both don't pass through the k-th string at the same time. By using this fact, we see that the i-th string and the j-th string can pass though one another and change the order on the real line (by the continuous deformation). Thus, the r-tuple $(\eta_1,\cdots,\eta_r)$ of strings can be deformed continuously to an r-tuple of strings representing the product $b\cdot a$. Thus, we proved that the space $\mathrm{Q}^{D,\Sigma}_n(\mathbb{R})$ is path-connected and $\pi_1(\mathrm{Q}^{D,\Sigma}_n(\mathbb{R}))$ is commutative if $n=1$ and $r_{\min}(\Sigma)\geq 3$. □

Remark 7.2. The space $\mathrm{Q}^{D,\Sigma}_n(\mathbb{R})$ is not path-connected if $(n,r_{\min}(\Sigma))=(1,2)$. But each of its path-components is simply connected.

To see this, suppose that $(n,r_{\min}(\Sigma))=(1,2)$. Since $r_{\min}(\Sigma)=2$, there has to exist $\sigma\in I(\mathcal{K}_\Sigma)$ such that $\sigma=\{i,j\}$ (by (3.3)). Since $n=1$, this means that particles on the real line corresponding to the i-th and the j-th polynomial cannot cross one another on the real line (i.e. the i-th and the j-th polynomials cannot have common real roots). Thus, $\mathrm{Q}^{D,\Sigma}_1(\mathbb{R})$ is not path-connected. However, since there are no restrictions on the movement of roots (particles) within a connected component, each path-component is simply connected. □

[8]Let $f(z)\in\mathbb{R}[z]$ be a real coefficient polynomial and $\alpha\in\mathbb{C}\setminus\mathbb{R}$ be a complex root of $f(z)$ of multiplicity n_α. Then $\overline{\alpha}$ is a root of $f(z)$ of the same multiplicity n_α. Thus, in the case $\mathbb{K}=\mathbb{R}$, each string η_k moves symmetrically along the real axis $\mathbb{R}$.

Lemma 7.3. (i) *If* $k < 0$, *or* $k \geq \lfloor \frac{d_{\min}}{n} \rfloor + 2$, *or* $k = 0$ *and* $s \neq 0$, $E^{1;\mathbb{K}}_{k,s} = 0$.

(ii) *If* $1 \leq k \leq \lfloor \frac{d_{\min}}{n} \rfloor$ *and* $s - k \leq (d(\mathbb{K})nr_{\min}(\Sigma) - 2)k - 1$, $E^{1;\mathbb{K}}_{k,s} = 0$.

(iii) *If* $k = \lfloor \frac{d_{\min}}{n} \rfloor + 1$ *and* $s - k \leq (d(\mathbb{K})nr_{\min}(\Sigma) - 2)\lfloor d_{\min}/n \rfloor - 2$, $E^{1;\mathbb{K}}_{k,s} = 0$.

Proof. The assertions follow from Lemmas 6.3 and 6.13. □

Lemma 7.4. (i) *If* $\lfloor \frac{d_{\min}}{n} \rfloor \geq 2$,

$$\tilde{H}_i(\mathrm{Q}^{D,\Sigma}_n(\mathbb{K}); \mathbb{Z}) = 0 \quad \textit{for any } i \leq d(\mathbb{K})nr_{\min}(\Sigma) - 3.$$

(ii) *If* $\lfloor \frac{d_{\min}}{n} \rfloor = 1$,

$$\tilde{H}_i(\mathrm{Q}^{D,\Sigma}_n(\mathbb{K}); \mathbb{Z}) = 0 \quad \textit{for any } i \leq d(\mathbb{K})nr_{\min}(\Sigma) - 4.$$

Proof. Let us write $d'_{\min} = \lfloor \frac{d_{\min}}{n} \rfloor$. Consider the spectral sequences (6.10) and (6.13). Define the integer $a(k)$ by

$$a(k) = (d(\mathbb{K})nr_{\min}(\Sigma) - 2)n_0(k) - \epsilon(k) \quad \text{for each } 1 \leq k \leq d'_{\min} + 1,$$

where $n_0(k)$ and $\epsilon(k)$ denote the integers given by

$$(n_0(k), \epsilon(k)) = \begin{cases} (k, 1) & \text{if } 1 \leq k \leq d'_{\min}, \\ (d'_{\min}, 2) & \text{if } k = d'_{\min} + 1. \end{cases}$$

Then, by Lemma 7.3, we see that $E^{1;\mathbb{K}}_{k,s} = 0$ for any $(k, s) \neq (0, 0)$ if $s - k \leq m_0 = \min\{a(k) : 1 \leq k \leq d'_{\min} + 1\}$ is satisfied. Hence, $\tilde{H}_k(\mathrm{Q}^{D,\Sigma}_n(\mathbb{K}); \mathbb{Z}) = 0$ for any $k \leq m_0$. We also see that

$$\begin{aligned} m_0 &= \min\{a(k) : 1 \leq k \leq d'_{\min} + 1\} = \min\{a(1), a(d'_{\min} + 1)\} \\ &= \begin{cases} d(\mathbb{K})nr_{\min}(\Sigma) - 3 & \text{if } d'_{\min} \geq 2, \\ d(\mathbb{K})nr_{\min}(\Sigma) - 4 & \text{if } d'_{\min} = 1. \end{cases} \end{aligned}$$

Hence, we obtain the assertions (i) and (ii). □

Corollary 7.5. (i) *If* $n \geq 2$ *and* $\lfloor \frac{d_{\min}}{n} \rfloor \geq 2$, $\mathrm{Q}^{D,\Sigma}_n(\mathbb{C})$ *is* $(2nr_{\min}(\Sigma) - 3)$-*connected.*

(ii) *If* $n \geq 2$ *and* $\lfloor \frac{d_{\min}}{n} \rfloor = 1$, $\mathrm{Q}^{D,\Sigma}_n(\mathbb{C})$ *is* $(2nr_{\min}(\Sigma) - 4)$-*connected.*

(iii) *If* $n = 1$ *and* $d_{\min} \geq 2$, $\mathrm{Q}^{D,\Sigma}_n(\mathbb{C})$ *is* $(2r_{\min}(\Sigma) - 3)$-*connected.*

(iv) *Let* $n = d_{\min} = 1$. *Then* $\mathrm{Q}^{D,\Sigma}_n(\mathbb{C})$ *is* $(2r_{\min}(\Sigma) - 4)$-*connected if* $r_{\min}(\Sigma) \geq 3$, *and it is simply connected if* $r_{\min}(\Sigma) = 2$.

Proof. Since $\mathrm{Q}_n^{D,\Sigma}(\mathbb{C})$ is simply connected (by Lemma 7.1), the assertions follow from the Hurewicz Theorem and Lemma 7.4. □

Corollary 7.6. (i) *Let* $n \geq 2$ *and* $\lfloor \frac{d_{\min}}{n} \rfloor \geq 2$. *Then* $\mathrm{Q}_n^{D,\Sigma}(\mathbb{R})$ *is* $(nr_{\min}(\Sigma)-3)$*-connected.*

(ii) *Let* $n \geq 2$ *and* $\lfloor \frac{d_{\min}}{n} \rfloor = 1$. *Then* $\mathrm{Q}_n^{D,\Sigma}(\mathbb{R})$ *is* $(nr_{\min}(\Sigma) - 4)$*-connected if* $nr_{\min}(\Sigma) \geq 5$. *and it is simply connected if* $n = r_{\min}(\Sigma) = 2$.

(iii) *Let* $n = 1$, $d_{\min} \geq 2$. *Then* $\mathrm{Q}_n^{D,\Sigma}(\mathbb{R})$ *is* $(r_{\min}(\Sigma) - 3)$*-connected if* $r_{\min}(\Sigma) \geq 4$.

(iv) *Let* $n = d_{\min} = 1$. *Then* $\mathrm{Q}_n^{D,\Sigma}(\mathbb{R})$ *is* $(r_{\min}(\Sigma) - 4)$*-connected if* $r_{\min}(\Sigma) \geq 5$.

Proof. If $n = 1$ and $r_{\min}(\Sigma) \geq 3$, the group $\pi_1(\mathrm{Q}_n^{D,\Sigma}(\mathbb{R}))$ is commutative (by Lemma 7.1), and there is an isomorphism $\pi_1(\mathrm{Q}_n^{D,\Sigma}(\mathbb{R})) \cong H_1(\mathrm{Q}_n^{D,\Sigma}(\mathbb{R}); \mathbb{Z})$. Thus the assertions follow from Lemmas 7.1 and 7.4. □

Corollary 7.7. (i) *If the condition* (*1.6a*) *holds, the space* $\mathrm{Q}_n^{D,\Sigma}(\mathbb{C})$ *is simply connected.*

(ii) *If the condition* (*1.6b*) *holds, the space* $\mathrm{Q}_n^{D,\Sigma}(\mathbb{R})$ *is simply connected.*

Proof. The assertions follow from Lemma 7.1 and Corollary 7.6. □

Corollary 7.8. *Let* $\boldsymbol{a} \neq \mathbf{0}_r \in (\mathbb{Z}_{\geq 0})^r$.

(i) *If the condition* (*1.6a*) *holds, the stabilization map*

$$s_{D,D+\boldsymbol{a}} : \mathrm{Q}_n^{D,\Sigma}(\mathbb{C}) \to \mathrm{Q}_n^{D+\boldsymbol{a},\Sigma}(\mathbb{C})$$

is a homotopy equivalence through dimension $d(D; \Sigma, n, \mathbb{C})$.

(ii) *If the condition* (*1.6b*) *holds, the stabilization map*

$$s^{\mathbb{R}}_{D,D+\boldsymbol{a}} : \mathrm{Q}_n^{D,\Sigma}(\mathbb{R}) \to \mathrm{Q}_n^{D+\boldsymbol{a},\Sigma}(\mathbb{R})$$

is a homotopy equivalence through dimension $d(D; \Sigma, n, \mathbb{R})$.

(iii) *If the condition* (*1.6b*) *holds, the stabilization map*

$$s_{D,D+\boldsymbol{a}} : \mathrm{Q}_n^{D,\Sigma}(\mathbb{C}) \to \mathrm{Q}_n^{D+\boldsymbol{a},\Sigma}(\mathbb{C})$$

is a $\mathbb{Z}_2$*-equivariant homotopy equivalence through dimension* $d(D; \Sigma, n, \mathbb{R})$.

Proof. The assertions (i) and (ii) follow from Theorem 6.8, Corollaries 6.6 and 7.7. Since $d(D; \Sigma, n, \mathbb{R}) < d(D; \Sigma, n, \mathbb{C})$ and $(s_{D,D+\boldsymbol{a}})^{\mathbb{Z}_2} = s^{\mathbb{R}}_{D,D+\boldsymbol{a}}$, the assertion (iii) follows from (i) and (ii). □

8 Scanning maps

In this section we study a configuration space model of $\mathrm{Q}_n^{D,\Sigma}(\mathbb{K})$ and the corresponding scanning map.

8.1 Configuration space models. First, consider the following configuration space model of $\mathrm{Q}_n^{D,\Sigma}(\mathbb{K})$.

Definition 8.1. For a positive integer $d \geq 1$ and a based space X, let $\mathrm{SP}^d(X)$ denote *the d-th symmetric product* of X defined as the orbit space

$$\mathrm{SP}^d(X) = X^d/S_d, \tag{8.1}$$

where the symmetric group S_d of d letters acts on the d-fold product X^d in the natural manner.

Remark 8.2. (i) Note that an element $\eta \in \mathrm{SP}^d(X)$ may be identified with a formal linear combination

$$\eta = \sum_{k=1}^{s} n_k x_k, \tag{8.2}$$

where $\{x_k\}_{k=1}^s \in C_s(X)$ and $\{n_k\}_{k=1}^s \subset \mathbb{N}$ with $\sum_{k=1}^s n_k = d$. In this situation we shall refer to η as a configuration (or 0-cycle) of points, the point $x_k \in X$ having a multiplicity n_k.

(ii) For example, when $X = \mathbb{C}$, we have the natural homeomorphism

$$\psi_d : \mathrm{P}_d^{\mathbb{C}} \xrightarrow{\cong} \mathrm{SP}^d(\mathbb{C}) \tag{8.3}$$

given by using the above identification

$$\psi_d(f(z)) = \sum_{k=1}^{s} d_k \alpha_k \qquad \text{for } f(z) = \textstyle\prod_{k=1}^s (z-\alpha_k)^{d_k} \in \mathrm{P}_d^{\mathbb{C}}. \qquad \Box \tag{8.4}$$

Definition 8.3. (i) For a closed subspace $A \subset X$, let $\mathrm{SP}^d(X,A)$ denote the quotient space

$$\mathrm{SP}^d(X,A) = \mathrm{SP}^d(X)/\sim \tag{8.5}$$

where the equivalence relation $\sim$ is defined by

$$\xi \sim \eta \Leftrightarrow \xi \cap (X \setminus A) = \eta \cap (X \setminus A) \quad \text{for } \xi, \eta \in \mathrm{SP}^d(X). \tag{8.6}$$

Thus, the points of A are ignored. When $A \neq \emptyset$, by adding a point in A we have the natural inclusion $\mathrm{SP}^d(X,A) \subset \mathrm{SP}^{d+1}(X,A)$. Thus, when $A \neq \emptyset$, one can define the space $\mathrm{SP}^\infty(X,A)$ by the union

$$\mathrm{SP}^\infty(X,A) = \bigcup_{d\geq 0} \mathrm{SP}^d(X,A), \tag{8.7}$$

where we set $\mathrm{SP}^0(X,A) = \{\emptyset\}$ and $\emptyset$ denotes the empty configuration.

(ii) From now on, we always assume that $X \subset \mathbb{C}$. For each r-tuple $D = (d_1, \cdots, d_r) \in \mathbb{N}^r$, we let $\mathrm{SP}^D(X) = \prod_{i=1}^r \mathrm{SP}^{d_i}(X)$ and define a space $\mathcal{Q}^\Sigma_{D,n}(X)$ by

$$\mathcal{Q}^\Sigma_{D,n}(X) = \{(\xi_1, \cdots, \xi_r) \in \mathrm{SP}^D(X) : \text{the condition } (*)^\Sigma_n \text{ holds}\}, \tag{8.8}$$

where the condition $(*)^\Sigma_n$ is given by

$(*)^\Sigma_n$: The configuration $(\bigcap_{k\in\sigma} \xi_k) \cap \mathbb{R}$ contains no point $x \in X$ of multiplicity $\geq n$ for any $\sigma \in I(\mathcal{K}_\Sigma)$.

(iii) When $A \subset X$ is a closed subspace, define an equivalence relation "$\sim$" on the space $\mathcal{Q}^\Sigma_{D,n}(X)$ by

$$(\xi_1, \cdots, \xi_r) \sim (\eta_1, \cdots, \eta_r) \quad \text{if} \quad \xi_i \cap (X \setminus A) = \eta_i \cap (X \setminus A)$$

for each $1 \leq j \leq r$. Let $\mathcal{Q}^\Sigma_{D,n}(X,A)$ be the quotient space defined by

$$\mathcal{Q}^\Sigma_{D,n}(X,A) = \mathcal{Q}^\Sigma_{D,n}(X)/\sim . \tag{8.9}$$

When $A \neq \emptyset$, by adding points in A we get a natural inclusion

$$\mathcal{Q}^\Sigma_{D,n}(X,A) \subset \mathcal{Q}^\Sigma_{D+\boldsymbol{e}_i,n}(X,A) \quad \text{for each } 1 \leq i \leq r, \tag{8.10}$$

where $D + \boldsymbol{e}_i = (d_1, \cdots, d_{i-1}, d_i + 1, d_{i+1}, \cdots, d_r)$.

Thus, when $A \neq \emptyset$, one can define a space $\mathcal{Q}^\Sigma_n(X,A)$ as the union

$$\mathcal{Q}^\Sigma_n(X,A) = \bigcup_{D\in\mathbb{N}^r} \mathcal{Q}^\Sigma_{D,n}(X,A), \tag{8.11}$$

where the empty configuration $(\emptyset, \cdots, \emptyset)$ is the base-point of $\mathcal{Q}^\Sigma_n(X,A)$.

Remark 8.4. (i) Let $D = (d_1, \cdots, d_r) \in \mathbb{N}^r$. Then by using the identification (8.3) we easily obtain a homeomorphism

$$\begin{array}{ccc} \mathrm{Q}^{D,\Sigma}_n(\mathbb{C}) & \xrightarrow[\cong]{\Psi_D} & \mathcal{Q}^\Sigma_{D,n}(\mathbb{C}) \\ (f_1(z), \cdots, f_r(z)) & \longrightarrow & (\psi_{d_1}(f_1(z)), \cdots, \psi_{d_r}(f_r(z))) \end{array} \tag{8.12}$$

(ii) Now let $\varphi_D : \mathbb{C} \xrightarrow{\cong} U_D$ and $\boldsymbol{x}_D = (x_{D,1}, \cdots, x_{D,r}) \in F(\mathbb{C} \setminus \overline{U_D}, r)$ be the homeomorphism and the point used in defining the stabilization map s_D given in Definition 5.1. We define a map

$$(8.13) \qquad s_D^{\Sigma} : \mathcal{Q}_{D,n}^{\Sigma}(\mathbb{C}) \to \mathcal{Q}_{D+\mathbf{e},n}^{\Sigma}(\mathbb{C}) \qquad \text{by}$$
$$s_D^{\Sigma}(\xi_1, \cdots, \xi_r) = (\varphi_D(\xi_1) + x_{D,1}, \cdots, \varphi_D(\xi_r) + x_{D,r})$$

for $(\xi_1, \cdots, \xi_r) \in \mathcal{Q}_{D,n}^{\Sigma}$, where we write

$$(8.14) \qquad \mathbf{e} = (1, 1, \cdots, 1) \in \mathbb{N}^r,$$

and $\varphi_D(\xi) = \sum_{k=1}^{s} n_k \varphi_D(x_k)$ if $\xi = \sum_{k=1}^{s} n_k x_k \in \mathrm{SP}^d(\mathbb{C})$ and $(n_k, x_k) \in \mathbb{N} \times \mathbb{C}$ with $\sum_{k=1}^{s} n_k = d$.

By using the above homeomorphism (8.12), we now obtain the following commutative diagram

$$(8.15) \qquad \begin{array}{ccc} \mathrm{Q}_n^{D,\Sigma}(\mathbb{C}) & \xrightarrow{s_{D,D+\mathbf{e}}} & \mathrm{Q}_n^{D+\mathbf{e},\Sigma}(\mathbb{C}) \\ \Psi_D \downarrow \cong & & \Psi_{D+\mathbf{e}} \downarrow \cong \\ \mathcal{Q}_{D,n}^{\Sigma}(\mathbb{C}) & \xrightarrow{s_D^{\Sigma}} & \mathcal{Q}_{D+\mathbf{e},n}^{\Sigma}(\mathbb{C}) \end{array}$$

(iv) Note that $\mathcal{Q}_{D,n}^{\Sigma}(\mathbb{C})$ is path-connected. Indeed, for any two points $\xi_0, \xi_1 \in \mathcal{Q}_{D,n}^{\Sigma}(\mathbb{C})$, one can construct a path $\omega : [0,1] \to \mathcal{Q}_{D,n}^{\Sigma}(\mathbb{C})$ such that $\omega(i) = \xi_i$ for $i \in \{0,1\}$ by means of the string representation used in [17, §Appendix]. Thus the space $\mathrm{Q}_n^{D,\Sigma}(\mathbb{C})$ is also path-connected. By choosing the path ω in a $\mathbb{Z}_2$-equivariant way, one can show that $\mathrm{Q}_n^{D,\Sigma}(\mathbb{R})$ is also path-connected if $n \geq 2$ or if $n = 1$ and $r_{\min}(\Sigma) \geq 3 \Leftrightarrow (n, r_{\min}(\Sigma)) \neq (1,2)$ (see also the proof of Lemma 7.1 and Remark 7.2). □

Definition 8.5. Define the stabilized space $\mathrm{Q}_n^{D+\infty,\Sigma}(\mathbb{C})$ as the colimit

$$(8.16) \qquad \mathrm{Q}_n^{D+\infty,\Sigma}(\mathbb{C}) = \lim_{k\to\infty} \mathrm{Q}_n^{D+k\mathbf{e},\Sigma}(\mathbb{C}),$$

where the colimit is taken over the family of stabilization maps

$$(8.17) \qquad \{s_{D+k\mathbf{e},D+(k+1)\mathbf{e}} : \mathrm{Q}_n^{D+k\mathbf{e},\Sigma}(\mathbb{C}) \to \mathrm{Q}_n^{\Sigma,D+(k+1)\mathbf{e}}(\mathbb{C})\}_{k\geq 0} \qquad \square$$

8.2 Scanning maps. Now we are ready to define the scanning map. From now on, we identify $\mathbb{C} = \mathbb{R}^2$ in a usual way.

Definition 8.6. For a rectangle X in $\mathbb{C} = \mathbb{R}^2$, let σX denote the union of the sides of X which are parallel to the y-axis, and for a subspace $Z \subset \mathbb{C} = \mathbb{R}^2$, let $\overline{Z}$ be the closure of Z. From now on, let I denote the interval $I = [-1, 1]$ and let $0 < \epsilon < \frac{1}{1000000}$ be any fixed real number.

For each $x \in \mathbb{R}$, let $V(x)$ be the set defined by

$$(8.18) \qquad \begin{aligned} V(x) &= \{w \in \mathbb{C} : |\mathrm{Re}(w) - x| < \epsilon, |\mathrm{Im}(w)| < 1\} \\ &= (x-\epsilon, x+\epsilon) \times (-1, 1), \end{aligned}$$

and let us identify $I \times I = I^2$ with the closed unit rectangle $\{t + s\sqrt{-1} \in \mathbb{C} : -1 \leq t, s \leq 1\}$ in $\mathbb{C}$.

For each $D = (d_1, \cdots, d_r) \in \mathbb{N}^r$, we define *the horizontal scanning map*

$$(8.19) \qquad sc_D : \mathcal{Q}^{\Sigma}_{D,n}(\mathbb{C}) \to \Omega\mathcal{Q}^{\Sigma}_n(I^2, \partial I \times I) = \Omega\mathcal{Q}^{\Sigma}_n(I^2, \sigma I^2)$$

as follows. For each r-tuple $\alpha = (\xi_1, \cdots, \xi_r) \in \mathcal{Q}^{\Sigma}_{D,n}(\mathbb{C})$ of configurations, let $sc_D(\alpha) : \mathbb{R} \to \mathcal{Q}^{\Sigma}_n(I^2, \partial I \times I) = \mathcal{Q}^{\Sigma}_n(I^2, \sigma I^2)$ denote the map given by

$$\mathbb{R} \ni x \mapsto (\xi_1 \cap \overline{V}(x), \cdots, \xi_r \cap \overline{V}(x)) \in \mathcal{Q}^{\Sigma}_n(\overline{V}(x), \sigma\overline{V}(x)) \cong \mathcal{Q}^{\Sigma}_n(I^2, \sigma I^2),$$

where we use the canonical identification $(\overline{V}(x), \sigma\overline{V}(x)) \cong (I^2, \sigma I^2)$.

Since $\lim_{x \to \pm\infty} sc_D(\alpha)(x) = (\emptyset, \cdots, \emptyset)$, by setting $sc_D(\alpha)(\infty) = (\emptyset, \cdots, \emptyset)$ we obtain a based map $sc_D(\alpha) \in \Omega\mathcal{Q}^{\Sigma}_n(I^2, \sigma I^2)$, where we identify $S^1 = \mathbb{R} \cup \infty$ and we choose the empty configuration $(\emptyset, \cdots, \emptyset)$ as the base-point of $\mathcal{Q}^{\Sigma}_n(I^2, \sigma I^2)$. One can show that the following diagram is homotopy commutative:

$$(8.20) \qquad \begin{array}{ccc} \mathcal{Q}^{\Sigma}_{D+k\mathbf{e},n}(\mathbb{C}) & \xrightarrow{sc_{D+k\mathbf{e}}} & \Omega\mathcal{Q}^{\Sigma}_n(I^2, \sigma I^2) \\ {\scriptstyle s^{\Sigma}_{D+k\mathbf{e}}}\downarrow & & \| \\ \mathcal{Q}^{\Sigma}_{D+(k+1)\mathbf{e},n}(\mathbb{C}) & \xrightarrow{sc_{D+(k+1)\mathbf{e}}} & \Omega\mathcal{Q}^{\Sigma}_n(I^2, \sigma I^2) \end{array}$$

By using the above diagram and by identifying $\mathrm{Q}^{D+k\mathbf{e},\Sigma}_n(\mathbb{C})$ with $\mathcal{Q}^{\Sigma}_{D+k\mathbf{e},n}(\mathbb{C})$, we finally obtain *the stable horizontal scanning map*

$$(8.21) \qquad S^H = \lim_{k \to \infty} sc_{D+k\mathbf{e}} : \mathrm{Q}^{D+\infty,\Sigma}_n(\mathbb{C}) \to \Omega\mathcal{Q}^{\Sigma}_n(I^2, \sigma I^2),$$

where $\mathrm{Q}^{D+\infty,\Sigma}_n(\mathbb{C})$ is defined in (8.16).

Theorem 8.7 ([32], (cf. [15], [26])). *The stable horizontal scanning map*

$$S^H : \mathrm{Q}^{D+\infty,\Sigma}_n(\mathbb{C}) \xrightarrow{\simeq} \Omega\mathcal{Q}^{\Sigma}_n(I^2, \sigma I^2)$$

is a homotopy equivalence.

Proof. The proof is analogous to the one given in [32, Prop. 3.2, Lemma 3.4] and [15, Prop. 2]. However, as it appears to be probably most difficult and least familiar part of the article [32], we gave a rigorous proof in [26, Theorem 5.6] (see also [26, Remark 5.8]). □

Definition 8.8. (i) We define the stabilized space $\mathrm{Q}_n^{D+\infty,\Sigma}(\mathbb{R})$ as the colimit

$$\mathrm{Q}_n^{D+\infty,\Sigma}(\mathbb{R}) = \lim_{k\to\infty} \mathrm{Q}_n^{D+k\mathbf{e},\Sigma}(\mathbb{R}), \tag{8.22}$$

where the colimit is taken over the family of stabilization maps

$$\{s^{\mathbb{R}}_{D+k\mathbf{e},D+(k+1)\mathbf{e}} : \mathrm{Q}_n^{D+k\mathbf{e},\Sigma}(\mathbb{R}) \to \mathrm{Q}_n^{\Sigma,D+(k+1)\mathbf{e}}(\mathbb{R})\}_{k\geq 0}. \tag{8.23}$$

(ii) Recall that there is a $\mathbb{Z}_2$-action on the space $\mathrm{Q}_n^{D,\Sigma}(\mathbb{C})$ induced from the complex conjugation on $\mathbb{C}$. Then by using (5.11), one can easily see the following:

$$\mathrm{Q}_n^{D+\infty,\Sigma}(\mathbb{R}) = (\mathrm{Q}_n^{D+\infty,\Sigma}(\mathbb{C}))^{\mathbb{Z}_2}. \tag{8.24}$$

Moreover, since $s^{\mathbb{R}}_{D,k} = (s_{D,k})^{\mathbb{Z}_2}$ as in Remark 5.4, one can define the horizontal scanning map

$$S^{\mathbb{Z}_2} = \lim_{k\to\infty} (sc_{D+k\mathbf{e}})^{\mathbb{Z}_2} : \mathrm{Q}_n^{D+\infty,\Sigma}(\mathbb{R}) \to \Omega\mathcal{Q}_n^{\Sigma}(I^2,\sigma I^2)^{\mathbb{Z}_2} \tag{8.25}$$

in the same way as in (8.21).

Since $\mathrm{Q}_n^{D,\Sigma}(\mathbb{R}) = \mathrm{Q}_n^{D,\Sigma}(\mathbb{C})^{\mathbb{Z}_2} \subset \mathrm{Q}_n^{D,\Sigma}(\mathbb{C})$, one can identify the space $\mathrm{Q}_n^{D+\infty,\Sigma}(\mathbb{R})$ with a subspace of $\mathrm{Q}_n^{D+\infty,\Sigma}(\mathbb{C})$. By means of this identification, we can also identify

$$\mathrm{Q}_n^{D+\infty,\Sigma}(\mathbb{R}) = \mathrm{Q}_n^{D+\infty,\Sigma}(\mathbb{C})^{\mathbb{Z}_2} \quad\text{and}\quad S^{\mathbb{Z}_2} = S^H|\mathrm{Q}_n^{D+\infty,\Sigma}(\mathbb{R}) = (S^H)^{\mathbb{Z}_2}. \tag{8.26}$$

Theorem 8.9 ([32], (cf. [15], [26]))**.** *The stable horizontal scanning map*

$$(S^H)^{\mathbb{Z}_2} : \mathrm{Q}_n^{D+\infty,\Sigma}(\mathbb{R}) \xrightarrow{\simeq} \Omega\mathcal{Q}_n^{\Sigma}(I^2,\sigma I^2)^{\mathbb{Z}_2}$$

is a homotopy equivalence if $(n, r_{\min}(\Sigma)) \neq (1,2)$.

Proof. If $(n, r_{\min}(\Sigma)) \neq (1,2)$, the group $\pi_1(\mathrm{Q}_n^{D,\Sigma}(\mathbb{R}))$ is an abelian group. The proof is completely analogous to that of Theorem 8.7. □

Corollary 8.10. *The stable horizontal scanning map*

$$S^H : \mathrm{Q}_n^{D+\infty,\Sigma}(\mathbb{C}) \xrightarrow{\simeq} \Omega\mathcal{Q}_n^{\Sigma}(I^2,\sigma I^2)$$

is a $\mathbb{Z}_2$-equivariant homotopy equivalence if $(n, r_{\min}(\Sigma)) \neq (1,2)$.

Proof. The assertion follows from Theorems 8.7 and 8.9. □

9 The stable result

In this section we prove the stable theorem (Theorem 9.2).

Definition 9.1. From now on, let $\boldsymbol{a} = (a_1, a_2, \cdots, a_r) \in \mathbb{N}^r$ such that $\sum_{k=1}^r a_k \boldsymbol{n}_k = \mathbf{0}_m$, and let $D = (d_1, \cdots, d_r) \in \mathbb{N}^r$ be an r-tuple of positive integers such that $\sum_{k=1}^r d_k \boldsymbol{n}_k = \mathbf{0}_m$. Then it is easy to see that the following two diagram is homotopy commutative for $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$:

$$\begin{array}{ccc} \mathrm{Q}_n^{D,\Sigma}(\mathbb{K}) & \xrightarrow{j_{D,n,\mathbb{K}}} & \Omega\mathcal{Z}_{\mathcal{K}_\Sigma}(\mathbb{K}^n, (\mathbb{K}^n)^*) \simeq \Omega\mathcal{Z}_{\mathcal{K}_\Sigma}(D^{d(\mathbb{K})n}, S^{d(\mathbb{K})n-1}) \\ {\scriptstyle s^{\mathbb{K}}_{D,D+\boldsymbol{a}}}\downarrow & & \| \\ \mathrm{Q}_n^{D+\boldsymbol{a},\Sigma}(\mathbb{K}) & \xrightarrow{j_{D+\boldsymbol{a},n,\mathbb{K}}} & \Omega\mathcal{Z}_{\mathcal{K}_\Sigma}(\mathbb{K}^n, (\mathbb{K}^n)^*) \simeq \Omega\mathcal{Z}_{\mathcal{K}_\Sigma}(D^{d(\mathbb{K})n}, S^{d(\mathbb{K})n-1}) \end{array}$$

where we set

$$s^{\mathbb{K}}_{D,D+\boldsymbol{a}} = s_{D,D+\boldsymbol{a}} \quad \text{if } \mathbb{K} = \mathbb{C}. \tag{9.1}$$

Hence, for $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, we obtain the stabilized map

$$j_{D+\infty,n,\mathbb{K}} : \mathrm{Q}_n^{D+\infty,\Sigma}(\mathbb{K}) \to \Omega\mathcal{Z}_{\mathcal{K}_\Sigma}(D^{d(\mathbb{K})n}, S^{d(\mathbb{K})n-1}), \tag{9.2}$$

where we set

$$j_{D+\infty,n,\mathbb{K}} = \lim_{t\to\infty} j_{D+t\boldsymbol{a},D+(t+1)\boldsymbol{a},\mathbb{K}}. \tag{9.3}$$

The main purpose of this section is to prove the following result.

Theorem 9.2. *Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, and let $D = (d_1, \cdots, d_r) \in \mathbb{N}^r$ be an r-tuple of positive integers such that $\sum_{k=1}^r d_k \boldsymbol{n}_k = \mathbf{0}_m$. Then the stabilized map*

$$j_{D+\infty,n,\mathbb{K}} : \mathrm{Q}_n^{D+\infty,\Sigma}(\mathbb{K}) \xrightarrow{\simeq} \Omega\mathcal{Z}_{\mathcal{K}_\Sigma}(D^{d(\mathbb{K})n}, S^{d(\mathbb{K})n-1})$$

is a homotopy equivalence.

Before proving Theorem 9.2 we need the following definition and lemma.

Definition 9.3. Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ as before. Now we identify $\mathbb{C} = \mathbb{R}^2$ in a usual way and let us write $U = \{w \in \mathbb{C} : |\mathrm{Re}(w)| < 1, |\mathrm{Im}(w)| < 1\} = (-1,1) \times (-1,1)$ and $I = [-1,1]$.

(i) For an open set $X \subset \mathbb{C}$, let $F_n^{\mathbb{K}}(X)$ denote the space of r-tuples $(f_1(z), \cdots, f_r(z)) \in \mathbb{K}[z]^r$ of (not necessarily monic) polynomials satisfying the following condition $(*)_{n,\mathbb{R}}$:

$(*)_{n,\mathbb{R}}$ For any $\sigma = \{i_1, \cdots, i_s\} \in I(\mathcal{K}_\Sigma)$, the polynomials $f_{i_1}(z), \cdots, f_{i_s}(z)$ have no common roots of multiplicity $\geq n$ in $X \cap \mathbb{R}$.

(ii) Let $ev_{0,\mathbb{K}} : F_n^{\mathbb{K}}(U) \to \mathcal{Z}_{\mathcal{K}_\Sigma}(\mathbb{K}^n, (\mathbb{K}^n)^*)$ denote the map given by evaluation at 0, i.e.

$$ev_{0,\mathbb{K}}(f_1(z), \cdots, f_r(z)) = (F_n(f_1)(0), \cdots, F_n(f_r)(0)) \tag{9.4}$$

for $(f_1(z), \cdots, f_r(z)) \in F_n^{\mathbb{K}}(U)$, where $F_n(f_i)(z)$ denotes the n-tuple of monic polynomials of the same degree d_i given by (4.6).

(iii) Let $\tilde{F}_n^{\mathbb{K}}(U) \subset F_n^{\mathbb{K}}(U)$ denote the subspace of all $(f_1(z), \cdots, f_r(z)) \in F_n^{\Sigma,\mathbb{K}}(U)$ such that no $f_i(z)$ is identically zero.

Let $ev_{\mathbb{K}} : \tilde{F}_n^{\mathbb{K}}(U) \to \mathcal{Z}_{\mathcal{K}_\Sigma}(\mathbb{K}^n, (\mathbb{K}^n)^*)$ denote the map given by the restriction

$$ev_{\mathbb{K}} = ev_{0,\mathbb{K}}|\tilde{F}_n^{\mathbb{K}}(U). \tag{9.5}$$

It is easy to see that the following two equality holds:

$$ev_{\mathbb{R}} = (ev_{\mathbb{C}})^{\mathbb{Z}_2}. \tag{9.6}$$

(iv) Note that the group $\mathbb{T}_{\mathbb{K}}^r = (\mathbb{K}^*)^r$ acts freely on the space $\tilde{F}_n^{\mathbb{K}}(U)$ in the natural way, and let

$$p_{\mathbb{K}} : \tilde{F}_n^{\mathbb{K}}(U) \to \tilde{F}_n^{\mathbb{K}}(U)/\mathbb{T}_{\mathbb{K}}^r \tag{9.7}$$

denote the natural projection, where $\tilde{F}_n^{\mathbb{K}}(U)/\mathbb{T}_{\mathbb{K}}^r$ denotes the corresponding orbit space.

Lemma 9.4. *Let X_Σ be a simply connected non-singular toric variety such that the condition (2.19a) is satisfied.*

(i) *If the condition (1.6a) is satisfied, the space $\mathrm{Q}_n^{D+\infty,\Sigma}(\mathbb{C})$ is simply connected. Similarly, if the condition (1.6b) is satisfied, the space $\mathrm{Q}_n^{D+\infty,\Sigma}(\mathbb{R})$ is simply connected.*

(ii) *The map $ev_{\mathbb{K}} : \tilde{F}_n^{\mathbb{K}}(U) \xrightarrow{\simeq} \mathcal{Z}_{\mathcal{K}_\Sigma}(\mathbb{K}^n, (\mathbb{K}^n)^*)$ is a homotopy equivalence.*

Proof. (i) The assertion easily follows from Corollary 7.7.

(ii) For each $\boldsymbol{b} = (b_0, b_1, \cdots, b_{n-1}) \in \mathbb{K}^n$, let $f_{\boldsymbol{b}}(z) \in \mathbb{K}[z]$ denote the polynomial of degree $\leq n$ defined by

$$f_{\boldsymbol{b}}(z) = b_0 + \sum_{k=1}^{n-1} \frac{b_k - b_0}{k!} z^k. \tag{9.8}$$

Let $i_0 : \mathcal{Z}_{\mathcal{K}_\Sigma}(\mathbb{K}^n, (\mathbb{K}^n)^*) \to F_n^{\mathbb{K}}(U)$ be the inclusion map given by

$$i_0(\boldsymbol{b}_1, \cdots, \boldsymbol{b}_r) = \big(f_{\boldsymbol{b}_1}(z), \cdots, f_{\boldsymbol{b}_r}(z)\big) \tag{9.9}$$

for $(\boldsymbol{b}_1, \cdots, \boldsymbol{b}_r) \in \mathcal{Z}_{\mathcal{K}_\Sigma}(\mathbb{K}^n, (\mathbb{K}^n)^*)$. Since the degree of each polynomial $f_{\boldsymbol{b}_1}(z)$ has at most $n-1$, it has no root of multiplicity $\geq n$. Thus, the map i_0 is well-defined, and clearly the equality $ev_0 \circ i_0 = \mathrm{id}$ holds.

Let $f : F_n^{\mathbb{K}}(U) \times [0,1] \to F_n^{\mathbb{K}}(U)$ be the homotopy given by

$$f((f_1, \cdots, f_r), t) = (f_{1,t}(z), \cdots, f_{r,t}(z)),$$

where $f_{i,t}(z) = f_i(tz)$. This gives a homotopy between the map $i_0 \circ ev_{0,\mathbb{K}}$ and the identity map, and this proves that the map

$$ev_{0,\mathbb{K}} : F_n^{\mathbb{K}}(U) \xrightarrow{\simeq} \mathcal{Z}_{\mathcal{K}_\Sigma}(\mathbb{K}^n, (\mathbb{K}^n)^*)$$

is a deformation retraction. Since $F_n^{\mathbb{K}}(U)$ is an infinite dimensional manifold and the complement of $\tilde{F}_n^{\mathbb{K}}(U)$ is a closed submanifold of $F_n^{\mathbb{K}}(U)$ of infinite codimension, it follows from [13, Theorem 2] that the inclusion

$$i_n^{\Sigma,\mathbb{K}} : \tilde{F}_n^{\mathbb{K}}(U) \xrightarrow{\simeq} F_n^{\mathbb{K}}(U) \tag{9.10}$$

is a homotopy equivalence. Hence the restriction $ev_{\mathbb{K}} = ev_{0,\mathbb{K}} \circ i_n^{\Sigma,\mathbb{K}}$ is also a homotopy equivalence. □

Definition 9.5. Note that $(\overline{U}, \sigma\overline{U}) = (I^2, \sigma I^2) = (I \times I, \partial I \times I)$. Let

$$\begin{cases} w_n^{\mathbb{C}} : \tilde{F}_n^{\mathbb{C}}(U) \to \mathcal{Q}_n^{\Sigma}(\overline{U}, \sigma\overline{U}) = \mathcal{Q}_n^{\Sigma}(I^2, \sigma I^2) \\ w_n^{\mathbb{R}} : \tilde{F}_n^{\mathbb{R}}(U) \to \mathcal{Q}_n^{\Sigma}(\overline{U}, \sigma\overline{U})^{\mathbb{Z}_2} = \mathcal{Q}_n^{\Sigma}(I^2, \sigma I^2)^{\mathbb{Z}_2} \end{cases} \tag{9.11}$$

denote the natural maps which assigns to an r-tuple $(f_1(z), \cdots, f_r(z)) \in \tilde{F}_n^{\mathbb{K}}(U)$ ($\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$) the r-tuple of their configurations represented by their real roots which lie in $\overline{U} = I^2$. These maps clearly induce the maps

$$\begin{cases} v_n^{\mathbb{C}} : \tilde{F}_n^{\mathbb{C}}(U)/\mathbb{T}_{\mathbb{C}}^r \to \mathcal{Q}_n^{\Sigma}(\overline{U}, \sigma\overline{U}) = \mathcal{Q}_n^{\Sigma}(I^2, \sigma I^2) \\ v_n^{\mathbb{R}} : \tilde{F}_n^{\mathbb{R}}(U)/\mathbb{T}_{\mathbb{R}}^r \to \mathcal{Q}_n^{\Sigma}(\overline{U}, \sigma\overline{U})^{\mathbb{Z}_2} = \mathcal{Q}_n^{\Sigma}(I^2, \sigma I^2)^{\mathbb{Z}_2} \end{cases} \tag{9.12}$$

such that the following diagram is commutative:

$$\begin{array}{ccccccc} \tilde{F}_n^{\mathbb{C}}(U) & \xrightarrow{w_n^{\mathbb{C}}} & \mathcal{Q}_n^{\Sigma}(\overline{U}, \sigma\overline{U}) & \xleftarrow[\supset]{} & \mathcal{Q}_n^{\Sigma}(\overline{U}, \sigma\overline{U})^{\mathbb{Z}_2} & \xleftarrow{w_n^{\mathbb{R}}} & \tilde{F}_n^{\mathbb{R}}(U) \\ {\scriptstyle p_{\mathbb{C}}}\downarrow & & \| & & \| & & {\scriptstyle p_{\mathbb{R}}}\downarrow \\ \tilde{F}_n^{\mathbb{C}}(U)/\mathbb{T}_{\mathbb{C}}^r & \xrightarrow{v_n^{\mathbb{C}}} & \mathcal{Q}_n^{\Sigma}(\overline{U}, \sigma\overline{U}) & \xleftarrow[\supset]{} & \mathcal{Q}_n^{\Sigma}(\overline{U}, \sigma\overline{U})^{\mathbb{Z}_2} & \xleftarrow{v_n^{\mathbb{R}}} & \tilde{F}_n^{\mathbb{R}}(U)/\mathbb{T}_{\mathbb{R}}^r \end{array}$$

Lemma 9.6. *Any fiber of the map $w_n^{\mathbb{K}}$ is homotopy equivalent to the space $\mathbb{T}_{\mathbb{K}}^r$.*

Proof. Any fiber of the map $w_n^{\mathbb{K}}$ is homeomorphic to the space $fib(r)$ consisting of all r-tuples $(f_1(z), \cdots, f_r(z)) \in \mathbb{K}[z]^r$ of $\mathbb{K}$-coefficients polynomials such that each polynomial $f_i(z)$ has no root in U. It suffices to show that there is a homotopy equivalence

$$fib(r) \simeq \mathbb{T}_{\mathbb{K}}^r. \tag{9.13}$$

First define the inclusion map $j_0 : \mathbb{T}_{\mathbb{K}}^r \to fib(r)$ by $j_0(\boldsymbol{x}) = (x_1, \cdots, x_r)$ for $\boldsymbol{x} = (x_1, \cdots, x_r) \in \mathbb{T}_{\mathbb{K}}^r$. Next, let $f = (f_1(z), \cdots, f_r(z)) \in fib(r)$ be any element. Since $0 \in U$, $(f_1(0), \cdots, f_r(0)) \in \mathbb{T}_{\mathbb{K}}^r$. Hence, one can define the evaluation map $\epsilon_0 : fib(r) \to \mathbb{T}_{\mathbb{K}}^r$ by $\epsilon_0(\mathrm{f}) = (f_1(0), \cdots, f_r(0))$ for $\mathrm{f} = (f_1(z), \cdots, f_r(z)) \in fib(r)$. It is easy to see that $\epsilon_0 \circ j_0 = \mathrm{id}_{\mathbb{T}_{\mathbb{K}}^r}$.

Now consider the map $j_0 \circ \epsilon_0$. Note that if a polynomial $g(z) \in \mathbb{K}[z]$ has a root $\alpha \in \mathbb{C} \setminus U$ and $0 < t \leq 1$, the polynomial $g(tz)$ has a root $\alpha/t \in \mathbb{C} \setminus U$. Thus, one can define the homotopy $F : fib(r) \times [0,1] \to fib(r)$ by $F(\mathrm{f}, t) = (f_1(tz), \cdots, f_r(tz))$ for $(\mathrm{f}, t) = ((f_1(z), \cdots, f_r(z)), t) \in fib(r) \times [0,1]$. It is easy to see that the map F gives a homotopy between the maps $j_0 \circ \epsilon_0$ and $\mathrm{id}_{fib(r)}$. Hence, we see that the map $\epsilon_0 : fib(r) \xrightarrow{\simeq} \mathbb{T}_{\mathbb{K}}^r$ is a desired homotopy equivalence. $\square$

Lemma 9.7. *The map $w_n^{\mathbb{C}} : \tilde{F}_n^{\mathbb{C}}(U) \to \mathcal{Q}_n^{\Sigma}(\overline{U}, \sigma\overline{U})$ is a quasifibration with fiber $\mathbb{T}_{\mathbb{C}}^r$. Similarly, the map $w_n^{\mathbb{R}} : \tilde{F}_n^{\mathbb{R}}(U) \to \mathcal{Q}_n^{\Sigma}(\overline{U}, \sigma\overline{U})^{\mathbb{Z}_2}$ is a quasifibration with fiber $\mathbb{T}_{\mathbb{R}}^r$.*

Proof. Since the proofs are completely analogous, we give one only for the map $w_n^{\mathbb{C}}$. The assertion may be proved by using the well-known Dold-Thom criterion. Recall that the base space $B = \mathcal{Q}_n^{\Sigma}(\overline{U}, \sigma\overline{U})$ consists of r-tuple of configurations $(\xi_1, \cdots, \xi_r)$ satisfying the condition

- $(\dagger)_\Sigma$ The configuration $(\cap_{k \in \sigma} \xi_k) \cap \mathbb{R} \cap (\overline{U} \setminus \sigma\overline{U})$ contains no points of multiplicity $\geq n$ for any $\sigma \in I(\mathcal{K}_\Sigma)$.

For each r-tuple $(d_1, \cdots, d_r) \in (\mathbb{Z}_{\geq 0})^r$ of non-negative integers, we denote by $B_{\leq d_1, \cdots, \leq d_r}$ the subspace of B consisting of all r-tuples $(\xi_1, \cdots, \xi_r) \in B$ satisfying the condition

$$\deg(\xi_k \cap \mathbb{R} \cap (\overline{U} \setminus \sigma\overline{U})) \leq d_k \quad \text{for each } 1 \leq k \leq r. \tag{9.14}$$

We filter the base space B by an increasing family of subspaces $\{B_{\leq d_1, \cdots, \leq d_r}\}$. It suffices to prove that each restriction

$$w_n | w_n^{-1}(B_{\leq d_1, \cdots, \leq d_r}) : w_n^{-1}(B_{\leq d_1, \cdots, \leq d_r}) \to B_{\leq d_1, \cdots, \leq d_r} \tag{9.15}$$

is a quasifibration. Its proof is essentially analogous to that of [26, Lemma 5.13] (cf. [32, Lemmas 3.3, 3.4]). The difference is only in the condition on the configurations considered. In the case of [26, Lemma 5.13], we considered the m-tuple $(\xi_1, \cdots, \xi_m)$ of configurations which satisfies the condition $(\dagger)_1$, where

$(\dagger)_1$ The configuration $(\cap_{k=1}^m \xi_k) \cap \mathbb{R} \cap (\overline{U} \setminus \sigma\overline{U})$ contains no points of multiplicity $\geq n$.

On the other hand, in our case, we need to consider r-tuple $(\xi_1, \cdots, \xi_r)$ of configurations satisfying the condition $(\dagger)_\Sigma$. If we replace the condition in the proof of [26, Lemma 5.13], by $(\dagger)_\Sigma$. we can prove that each restriction (9.15) is a quasifibration by using the same argument. $\square$

Corollary 9.8. *The map* $v_n^{\mathbb{C}} : \tilde{F}_n^{\mathbb{C}}(U)/\mathbb{T}^r_{\mathbb{C}} \xrightarrow{\simeq} \mathcal{Q}_n^{\Sigma}(\overline{U}, \sigma\overline{U})$ *is a homotopy equivalence. Similarly, the map* $v_n^{\mathbb{R}} : \tilde{F}_n^{\mathbb{R}}(U)/\mathbb{T}^r_{\mathbb{R}} \xrightarrow{\simeq} \mathcal{Q}_n^{\Sigma}(\overline{U}, \sigma\overline{U})^{\mathbb{Z}_2}$ *is also a homotopy equivalence.*

Proof. Since the proofs are analogous, we give the one only for the map $v_n^{\mathbb{C}}$. Let F_n denote the homotopy fiber of the map $w_n^{\mathbb{C}}$. It follows from [8, Lemma 2.1] that there is the following homotopy commutative diagram

$$
(9.16)\qquad
\begin{array}{ccccc}
\mathbb{T}^r_{\mathbb{C}} & \xrightarrow[=]{} & \mathbb{T}^r_{\mathbb{C}} & \longrightarrow & * \\
\| & & \downarrow & & \downarrow \\
\mathbb{T}^r_{\mathbb{C}} & \longrightarrow & \tilde{F}_n^{\mathbb{C}}(U) & \xrightarrow{w_n^{\mathbb{C}}} & \mathcal{Q}_n^{\Sigma}(\overline{U}, \sigma\overline{U}) \\
\downarrow & & p_{\mathbb{C}} \downarrow & & \| \\
F_n & \longrightarrow & \tilde{F}_n^{\mathbb{C}}(U)/\mathbb{T}^r_{\mathbb{C}} & \xrightarrow{v_n^{\mathbb{C}}} & \mathcal{Q}_n^{\Sigma}(\overline{U}, \sigma\overline{U})
\end{array}
$$

where all above vertical and horizontal sequences are fibration sequences. From this diagram, we easily see that F_n is contractible. Thus, $v_n^{\mathbb{C}}$ is a homotopy equivalence. $\square$

Now we can give the proof of Theorem 9.2.

Proof of Theorem 9.2. Let $D = (d_1, \cdots, d_r) \in \mathbb{N}^r$ and $\boldsymbol{a} = (a_1, \cdots, a_r) \in \mathbb{N}^r$ be two r-tuples of positive integers such that $\sum_{k=1}^r d_k \boldsymbol{n}_k = \sum_{k=1}^r a_k \boldsymbol{n}_k = \mathbf{0}_m$.

First, we shall prove the assertion for case $\mathbb{K} = \mathbb{C}$.
It follows from Lemma 9.4 and Lemma 3.6 that two spaces $\mathrm{Q}_n^{D+\infty,\Sigma}(\mathbb{C})$ and $\Omega\mathcal{Z}_{\mathcal{K}_\Sigma}(D^{2n}, S^{2n-1})$ are simply connected. Thus, it suffices to prove that the map $j_{D+\infty,n,\mathbb{C}}$ induces an isomorphism

$$
(j_{D+\infty,n,\mathbb{C}})_* : \pi_k(\mathrm{Q}_n^{D+\infty,\Sigma}(\mathbb{C})) \xrightarrow{\cong} \pi_k(\Omega\mathcal{Z}_{\mathcal{K}_\Sigma}(D^{2n}, S^{2n-1})) \quad \text{for any } k \geq 2.
$$

Let us identify $\mathbb{C}=\mathbb{R}^2$ and let $U=(-1,1)\times(-1,1)$ as before. We define the scanning map $scan:\tilde{F}^{\mathbb{C}}_n(\mathbb{C})\to \mathrm{Map}(\mathbb{R},\tilde{F}^{\mathbb{C}}_n(U))$ by

$$scan(f_1(z),\cdots,f_r(z))(w)=(f_1(z+w),\cdots,f_r(z+w)) \tag{9.17}$$

for $(f_1(z),\cdots,f_r(z)),w)\in \tilde{F}^{\mathbb{C}}_n(\mathbb{C})\times\mathbb{R}$, and consider the diagram

$$\begin{array}{ccc}
\tilde{F}^{\mathbb{C}}_n(U) & \xrightarrow[\simeq]{ev_{\mathbb{C}}} & \mathcal{Z}_{\mathcal{K}_\Sigma}(D^{2n},S^{2n-1}) \\
{\scriptstyle p_{\mathbb{C}}}\downarrow & & \\
\tilde{F}^{\mathbb{C}}_n(U)/\mathbb{T}^r_{\mathbb{C}} & \xrightarrow[\simeq]{v^{\mathbb{C}}_n} & \mathcal{Q}^{\Sigma}_n(\overline{U},\sigma\overline{U})
\end{array}$$

This induces the commutative diagram below

$$\begin{array}{ccccc}
\tilde{F}^{\mathbb{C}}_n(\mathbb{C}) & \xrightarrow{scan} & \mathrm{Map}(\mathbb{R},\tilde{F}^{\mathbb{C}}_n(U)) & \xrightarrow[\simeq]{(ev_{\mathbb{C}})_\#} & \mathrm{Map}(\mathbb{R},\mathcal{Z}_{\mathcal{K}_\Sigma}(D^{2n},S^{2n-1})) \\
{\scriptstyle p_{\mathbb{C}}}\downarrow & & {\scriptstyle (p_{\mathbb{C}})_\#}\downarrow & & \\
\tilde{F}^{\mathbb{C}}_n(\mathbb{C})/\mathbb{T}^r_{\mathbb{C}} & \xrightarrow{scan} & \mathrm{Map}(\mathbb{R},\tilde{F}^{\mathbb{C}}_n(U)/\mathbb{T}^r_{\mathbb{C}}) & \xrightarrow[\simeq]{(v^{\mathbb{C}}_n)_\#} & \mathrm{Map}(\mathbb{R},\mathcal{Q}^{\Sigma}_n(\overline{U},\sigma\overline{U}))
\end{array}$$

Observe that $\mathrm{Map}(\mathbb{R},\cdot)$ can be replaced by $\mathrm{Map}^*(S^1,\cdot)$ by extending from $\mathbb{R}$ to $S^1=\mathbb{R}\cup\infty$ (as base-point preserving maps). Thus by setting

$$\begin{cases}
\widehat{j_{D,n,\mathbb{C}}}:\mathrm{Q}^{D,\Sigma}_n(\mathbb{C})\xrightarrow{\subset}\tilde{F}^{\mathbb{C}}_n(\mathbb{C})\xrightarrow{scan}\mathrm{Map}^*(S^1,\tilde{F}^{\mathbb{C}}_n(U))=\Omega\tilde{F}^{\mathbb{C}}_n(U) \\
\widehat{j'_{D,n,\mathbb{C}}}:E^{\Sigma,\mathbb{R}}_{D,n}(\mathbb{C})\xrightarrow{\subset}\tilde{F}^{\mathbb{C}}_n(\mathbb{C})\xrightarrow{scan}\mathrm{Map}^*(S^1,\tilde{F}^{\mathbb{C}}_n(U)/\mathbb{T}^r_{\mathbb{C}})=\Omega(\tilde{F}^{\mathbb{C}}_n(U)/\mathbb{T}^r_{\mathbb{C}})
\end{cases}$$

we obtain the following commutative diagram

$$\begin{array}{ccccc}
\mathrm{Q}^{D,\Sigma}_n(\mathbb{C}) & \xrightarrow{\widehat{j_{D,n,\mathbb{C}}}} & \Omega\tilde{F}^{\mathbb{C}}_n(U) & \xrightarrow[\simeq]{\Omega ev_{\mathbb{C}}} & \Omega\mathcal{Z}_{\mathcal{K}_\Sigma}(D^{2n},S^{2n-1}) \\
\cong\downarrow & & {\scriptstyle \Omega p_{\mathbb{C}}}\downarrow & & \\
\mathcal{Q}^{\Sigma}_{D,n}(\mathbb{C}) & \xrightarrow{\widehat{j'_{D,n,\mathbb{C}}}} & \Omega(\tilde{F}^{\mathbb{C}}_n(U)/\mathbb{T}^r_{\mathbb{C}}) & \xrightarrow[\simeq]{\Omega v^{\mathbb{C}}_n} & \Omega\mathcal{Q}^{\Sigma}_n(\overline{U},\sigma\overline{U})
\end{array} \tag{9.18}$$

If we identify $\mathrm{Q}^{D+\infty,\Sigma}_n(\mathbb{C})$ with the colimit $\lim\limits_{t\to\infty}\mathcal{Q}^{\Sigma}_{D+t\boldsymbol{a},n}(\mathbb{C})$, by replacing D by $D+t\boldsymbol{a}$ ($t\in\mathbb{N}$) and letting $t\to\infty$, we obtain the following homotopy commutative diagram:

$$\begin{array}{ccccc}
\mathrm{Q}^{D+\infty,\Sigma}_n(\mathbb{C}) & \xrightarrow{\widehat{j_{D+\infty,n,\mathbb{C}}}} & \Omega\tilde{F}^{\mathbb{C}}_n(U) & \xrightarrow[\simeq]{\Omega ev_{\mathbb{C}}} & \Omega\mathcal{Z}_{\mathcal{K}_\Sigma}(D^{2n},S^{2n-1}) \\
\| & & {\scriptstyle \Omega p_{\mathbb{C}}}\downarrow & & \\
\mathrm{Q}^{D+\infty,\Sigma}_n(\mathbb{C}) & \xrightarrow{\widehat{j'_{D+\infty,n,\mathbb{C}}}} & \Omega(\tilde{F}^{\mathbb{C}}_n(U)/\mathbb{T}^r_{\mathbb{C}}) & \xrightarrow[\simeq]{\Omega v^{\mathbb{C}}_n} & \Omega\mathcal{Q}^{\Sigma}_n(\overline{U},\sigma\overline{U})
\end{array} \tag{9.19}$$

where we set $\widehat{j_{D+\infty,n,\mathbb{C}}}=\lim_{t\to\infty}\widehat{j_{D+t\boldsymbol{a},n,\mathbb{C}}}$ and $\widehat{j'_{D+\infty,n,\mathbb{C}}}=\lim_{t\to\infty}\widehat{j'_{D+t\boldsymbol{a},n,\mathbb{C}}}$.
Since $(\Omega ev_{\mathbb{C}})\circ\widehat{j_{D+t\boldsymbol{a},n,\mathbb{C}}}=j_{D+t\boldsymbol{a},n,\mathbb{C}}$ and $(\Omega v_n^{\mathbb{C}})\circ\widehat{j'_{D+t\boldsymbol{a},n,\mathbb{C}}}=sc_{D+t\boldsymbol{a}}$ (by identifying $\mathrm{Q}_n^{D+t\boldsymbol{a},\Sigma}(\mathbb{C})$ with the space $E^{\Sigma}_{D+t\boldsymbol{a},n}(\mathbb{C})$), we also obtain the following two equalities:

$$j_{D+\infty,n,\mathbb{C}}=(\Omega ev_{\mathbb{C}})\circ\widehat{j_{D+\infty,n,\mathbb{C}}},\quad S^H=(\Omega v_n^{\mathbb{C}})\circ\widehat{j'_{D+\infty,n,\mathbb{C}}}. \tag{9.20}$$

Since the map $ev_{\mathbb{C}}$ is a homotopy equivalence, it suffices to prove that the map

$$(\dagger\dagger)_{\mathbb{C}} \qquad \widehat{j_{D+\infty,n,\mathbb{C}}}:\mathrm{Q}_n^{D+\infty,\Sigma}(\mathbb{C})\longrightarrow\Omega\tilde{F}_n^{\mathbb{C}}(U)$$

induces an isomorphism on the homotopy group $\pi_k(\ )$ for any $k\geq 2$.

Since $S^H=(\Omega v_n^{\mathbb{C}})\circ\widehat{j'_{D+\infty,n,\mathbb{C}}}$ and $\Omega v_n^{\mathbb{C}}$ are homotopy equivalences (by Theorem 8.7 and Corollary 9.8), the map $\widehat{j'_{D+\infty,n,\mathbb{C}}}$ is a homotopy equivalence. Since $p_{\mathbb{C}}$ is a fibration with fiber $\mathbb{T}^r_{\mathbb{C}}$, the map $\Omega p_{\mathbb{C}}$ induces an isomorphism on the homotopy group $\pi_k(\ )$ for any $k\geq 2$. Hence, by using the equality $(\Omega p_{\mathbb{C}})\circ\widehat{j_{D+\infty,n,\mathbb{C}}}=\widehat{j'_{D+\infty,n,\mathbb{C}}}$ (up to homotopy), we see that the map $\widehat{j_{D+\infty,n,\mathbb{C}}}$ induces an isomorphism on the homotopy group $\pi_k(\ )$ for any $k\geq 2$. This completes the proof for the case $\mathbb{K}=\mathbb{C}$.

Next, consider the case $\mathbb{K}=\mathbb{R}$. This proof is almost identical to the case $\mathbb{K}=\mathbb{C}$ but since $\Omega p_{\mathbb{R}}$ is a homotopy equivalence, it is actually easier.

We define the scanning map $sca:\tilde{F}_n^{\mathbb{R}}(\mathbb{C})\to\mathrm{Map}(\mathbb{R},\tilde{F}_n^{\mathbb{R}}(U))$ by

$$sca(f_1(z),\cdots,f_r(z))(w)=(f_1(z+w),\cdots,f_r(z+w)) \tag{9.21}$$

for $(f_1(z),\cdots,f_r(z)),w)\in\tilde{F}_n^{\mathbb{R}}(\mathbb{C})\times\mathbb{R}$. Now we consider the diagram

$$\begin{array}{ccc}
\tilde{F}_n^{\mathbb{R}}(U) & \xrightarrow[\simeq]{ev_{\mathbb{R}}} & \mathcal{Z}_{\mathcal{K}_\Sigma}(D^n,S^{n-1}) \\
{\scriptstyle p_{\mathbb{R}}}\downarrow & & \\
\tilde{F}_n^{\mathbb{R}}(U)/\mathbb{T}^r_{\mathbb{R}} & \xrightarrow[\simeq]{v_n^{\mathbb{R}}} & \mathcal{Q}_n^{\Sigma}(\overline{U},\sigma\overline{U})^{\mathbb{Z}_2}
\end{array}$$

This induces the commutative diagram below

$$\begin{array}{ccccc}
\tilde{F}_n^{\mathbb{R}}(\mathbb{C}) & \xrightarrow{sca} & \mathrm{Map}(\mathbb{R},\tilde{F}_n^{\mathbb{R}}(U)) & \xrightarrow[\simeq]{(ev_{\mathbb{R}})_\#} & \mathrm{Map}(\mathbb{R},\mathcal{Z}_{\mathcal{K}_\Sigma}(D^n,S^{n-1})) \\
{\scriptstyle p_{\mathbb{R}}}\downarrow & & {\scriptstyle (p_{\mathbb{R}})_\#}\downarrow & & \\
\tilde{F}_n^{\mathbb{R}}(\mathbb{C})/\mathbb{T}^r_{\mathbb{R}} & \xrightarrow{sca} & \mathrm{Map}(\mathbb{R},\tilde{F}_n^{\mathbb{R}}(U)/\mathbb{T}^r_{\mathbb{R}}) & \xrightarrow[\simeq]{(v_n^{\mathbb{R}})_\#} & \mathrm{Map}(\mathbb{R},\mathcal{Q}_n^{\Sigma}(\overline{U},\sigma\overline{U})^{\mathbb{Z}_2})
\end{array}$$

Observe that $\mathrm{Map}(\mathbb{R},\cdot)$ can be replaced by $\mathrm{Map}^*(S^1,\cdot)$ by extending from $\mathbb{R}$ to $S^1=\mathbb{R}\cup\infty$ (as base-point preserving maps). Thus by setting

$$\begin{cases} \widehat{j_{D,n,\mathbb{R}}}:\mathrm{Q}_n^{D,\Sigma}(\mathbb{R}) \xrightarrow{\subset} \tilde{F}_n^{\mathbb{R}}(\mathbb{C}) \xrightarrow{sca} \Omega\tilde{F}_n^{\mathbb{R}}(U) \\ \widehat{j'_{D,n,\mathbb{R}}}:\mathcal{Q}_{D,n}^{\Sigma}(\mathbb{C})^{\mathbb{Z}_2} \xrightarrow{\subset} \tilde{F}_n^{\mathbb{R}}(\mathbb{C}) \xrightarrow{sca} \Omega(\tilde{F}_n^{\mathbb{R}}(U)/\mathbb{T}_{\mathbb{R}}^r) \end{cases}$$

we obtain the following commutative diagram

$$\text{(9.22)}\qquad \begin{array}{ccccc} \mathrm{Q}_n^{D,\Sigma}(\mathbb{R}) & \xrightarrow{\widehat{j_{D,n,\mathbb{R}}}} & \Omega\tilde{F}_n^{\mathbb{R}}(U) & \xrightarrow[\simeq]{\Omega ev_{\mathbb{R}}} & \Omega\mathcal{Z}_{\mathcal{K}_\Sigma}(D^n,S^{n-1}) \\ \cong\big\downarrow & & \Omega p_{\mathbb{R}}\big\downarrow\simeq & & \\ \mathcal{Q}_{D,n}^{\Sigma}(\mathbb{C})^{\mathbb{Z}_2} & \xrightarrow{\widehat{j'_{D,n,\mathbb{R}}}} & \Omega(\tilde{F}_n^{\mathbb{R}}(U)/\mathbb{T}_{\mathbb{R}}^r) & \xrightarrow[\simeq]{\Omega v_n^{\mathbb{R}}} & \Omega\mathcal{Q}_n^{\Sigma}(\overline{U},\sigma\overline{U})^{\mathbb{Z}_2} \end{array}$$

If we identify $\mathrm{Q}_n^{D+\infty,\Sigma}(\mathbb{R})$ with the colimit $\lim\limits_{t\to\infty}\mathcal{Q}_{D+t\boldsymbol{a},n}^{\Sigma}(\mathbb{C})^{\mathbb{Z}_2}$, by replacing D by $D+t\boldsymbol{a}$ $(t\in\mathbb{N})$ and letting $t\to\infty$, we obtain the following homotopy commutative diagram:

$$\text{(9.23)}\qquad \begin{array}{ccccc} \mathrm{Q}_n^{D+\infty,\Sigma}(\mathbb{R}) & \xrightarrow{\widehat{j_{D+\infty,n,\mathbb{R}}}} & \Omega\tilde{F}_n^{\mathbb{R}}(U) & \xrightarrow[\simeq]{\Omega ev_{\mathbb{R}}} & \Omega\mathcal{Z}_{\mathcal{K}_\Sigma}(D^n,S^{n-1}) \\ \| & & \Omega p_{\mathbb{R}}\big\downarrow\simeq & & \\ \mathrm{Q}_n^{D+\infty,\Sigma}(\mathbb{R}) & \xrightarrow{\widehat{j'_{D+\infty,n,\mathbb{R}}}} & \Omega(\tilde{F}_n^{\mathbb{R}}(U)/\mathbb{T}_{\mathbb{R}}^r) & \xrightarrow[\simeq]{\Omega v_n^{\mathbb{R}}} & \Omega\mathcal{Q}_n^{\Sigma}(\overline{U},\sigma\overline{U})^{\mathbb{Z}_2} \end{array}$$

where we set $\widehat{j_{D+\infty,n,\mathbb{R}}}=\lim\limits_{t\to\infty}\widehat{j_{D+t\boldsymbol{a},n,\mathbb{R}}}$ and $\widehat{j'_{D+\infty,n,\mathbb{R}}}=\lim\limits_{t\to\infty}\widehat{j'_{D+t\boldsymbol{a},n,\mathbb{R}}}$.

Since $(\Omega ev_{\mathbb{R}})\circ\widehat{j_{D+t\boldsymbol{a},n,\mathbb{R}}}=j_{D+t\boldsymbol{a},n,\mathbb{R}}$ and $(\Omega v_n^{\mathbb{R}})\circ\widehat{j'_{D+t\boldsymbol{a},n,\mathbb{R}}}=(sc_{D+t\boldsymbol{a}})^{\mathbb{Z}_2}$, we also obtain the following two equalities:

$$\text{(9.24)}\qquad j_{D+\infty,n,\mathbb{R}}=(\Omega ev_{\mathbb{R}})\circ\widehat{j_{D+\infty,n,\mathbb{R}}},\quad (S^H)^{\mathbb{Z}_2}=(\Omega v_n^{\mathbb{R}})\circ\widehat{j'_{D+\infty,n,\mathbb{R}}}.$$

Since the map $ev_{\mathbb{R}}$ is a homotopy equivalence, it suffices to prove that the map

$$(\dagger\dagger)_{\mathbb{R}}\qquad \widehat{j_{D+\infty,n,\mathbb{R}}}:\mathrm{Q}_n^{D+\infty,\Sigma}(\mathbb{C})\longrightarrow\Omega\tilde{F}_n^{\mathbb{R}}(U)$$

is a homotopy equivalence.

Since $(S^H)^{\mathbb{Z}_2}=(\Omega v_n^{\mathbb{R}})\circ\widehat{j'_{D+\infty,n,\mathbb{R}}}$ and $\Omega v_n^{\mathbb{R}}$ are homotopy equivalences (by Theorem 8.9 and Corollary 9.8), the map $\widehat{j'_{D+\infty,n,\mathbb{C}}}$ is a homotopy equivalence. On the other hand, since $p_{\mathbb{R}}$ is a covering projection with fiber $(\mathbb{Z}_2)^r$ (up to homotopy), the map $\Omega p_{\mathbb{R}}$ is a homotopy equivalence. Hence, by using the diagram (9.23), we see that the map $\widehat{j_{D+\infty,n,\mathbb{R}}}$ is a homotopy equivalence. This completes the proof of Theorem 9.2. $\square$

10 Proofs of the main results

Now we give the proofs of the main results (Theorems 2.14, 2.15, and Corollary 2.16).

Proofs of Theorem 2.14. (i) Suppose that $\sum_{k=1}^{r} d_k\boldsymbol{n}_k = \mathbf{0}_n$. Then the assertion (i) easily follows from Corollary 7.8 and Theorem 9.2.

(ii) Next assume that $\sum_{k=1}^{r} d_k\boldsymbol{n}_k \neq \mathbf{0}_n$. Recall from (2.19a) that there is an r-tuple $D_* = (d_1^*, \cdots, d_r^*) \in \mathbb{N}^r$ such that $\sum_{k=1}^{r} d_k^*\boldsymbol{n}_k = \mathbf{0}_n$. If we choose a sufficiently large integer $m_0 \in \mathbb{N}$, then the condition $d_k < m_0 d_k^*$ holds for each $1 \leq k \leq r$. Then consider the map $j_{D,n,\mathbb{C}} : \mathrm{Q}_n^{D,\Sigma}(\mathbb{C}) \to \Omega\mathcal{Z}_{\mathcal{K}_\Sigma}(D^{2n}, S^{2n-1})$ defined by

$$j_{D,n,\mathbb{C}} = j_{D_0,n,\mathbb{C}} \circ s_{D,D_0}, \tag{10.1}$$

where $D_0 = m_0 D_* = (m_0 d_1^*, m_0 d_2^*, \cdots, m_0 d_r^*)$ and $j_{D,n,\mathbb{C}}$ is given by the composite of the following maps

$$j_{D,n,\mathbb{C}} : \mathrm{Q}_n^{D,\Sigma}(\mathbb{C}) \xrightarrow{s_{D,D_0}} \mathrm{Q}_n^{D_0,\Sigma}(\mathbb{C}) \xrightarrow{j_{D_0,n,\mathbb{C}}} \Omega\mathcal{Z}_{\mathcal{K}_\Sigma}(D^{2n}, S^{2n-1}). \tag{10.2}$$

Since the maps s_{D,D_0} and $j_{D_0,\mathbb{C}}$ are homotopy equivalences through dimensions $d(D;\Sigma,n,\mathbb{C})$ and $d(D_0;\Sigma,n,\mathbb{C})$, respectively (by Corollary 7.8 and Theorem 2.14), by using $d(D;\Sigma,n,\mathbb{C}) \leq d(D_0;\Sigma,n,\mathbb{C})$ the map $j_{D,n,\mathbb{C}}$ is a homotopy equivalence through dimension $d(D;\Sigma,n,\mathbb{C})$. □

Proof of Theorem 2.15. (i) Suppose that $\sum_{k=1}^{r} d_k\boldsymbol{n}_k = \mathbf{0}_n$. Then the assertion (i) easily follows from Corollary 7.8 and Theorem 9.2.

(ii) This is proved in an analogous way to the proof of (ii) of Theorem 2.14. Indeed, under the same assumption as in (ii) of Theorem 2.14, we define a map $j_{D,n,\mathbb{R}} : \mathrm{Q}_n^{D,\Sigma}(\mathbb{R}) \to \Omega\mathcal{Z}_{\mathcal{K}_\Sigma}(D^n, S^{n-1})$ by

$$j_{D,n,\mathbb{R}} = j_{D_0,n,\mathbb{R}} \circ s_{D,D_0}^{\mathbb{R}}. \tag{10.3}$$

Since $d(D;\Sigma,n,\mathbb{R}) \leq d(D_0;\Sigma,n,\mathbb{R})$, it is easy to see that this map is a homotopy equivalence through dimension $d(D;\Sigma,n,\mathbb{R})$. □

Proof of Corollary 2.16. Consider the map of composite

$$\Omega\mathcal{Z}_{\mathcal{K}_\Sigma}(D^{2n}, S^{2n-1}) \underset{\simeq}{\longrightarrow} \Omega\mathcal{Z}_{\mathcal{K}_\Sigma}(\mathbb{C}^n, (\mathbb{C}^n)^*) \xrightarrow{\Omega q_{n,\mathbb{C}}} \Omega X_\Sigma(n).$$

Since $\Omega q_{n,\mathbb{C}}$ is a universal covering up to homotopy (by Corollary 3.10), the assertions easily follow from Theorem 2.14. □

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