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Legendre singularities of sub-Riemannian geodesics associated to Riemannian surfaces

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Abstract. Let M be a surface with a Riemannian metric and UM the unit tangent bundle over M with the canonical contact sub-Riemannian structure $D \subset T(UM)$. In this paper, the complete local classification of singularities, under the Legendre projection $UM \to M$, is given for sub-Riemannian geodesics of (UM, D). Legendre singularities of sub-Riemannian geodesics are classified completely also for another Legendre projection from UM to the space of Riemannian geodesics on M. The duality on Legendre singularities is observed related to the pendulum motion.

1. Introduction

Let M be a C^∞ surface with a Riemannian metric g. Then the unit tangent bundle UM over M has the canonical contact structure $D \subset T(UM)$. Moreover D has a sub-Riemannian structure induced from the Riemannian metric on M. A *sub-Riemannian geodesic* of D (or a *D-geodesic*) is a curve on UM which is tangent to D and is a local minimizer of the sub-Riemannian or Carnot-Carathéodory arc length for the metric on D ([**20**]). Any D-geodesic on UM is known to be an immersion if it is not a constant map. However the projection $\pi : UM \to M$, which is a Legendre projection, restricted to a D-geodesic on UM may have singularities, which are called the *Legendre singularities*.

In this paper we study Legendre singularities of D-geodesics on (UM, D) and give the local classification result which determines the Legendre singularities of D-geodesics completely.

The unit tangent bundle UM has the geodesic flow for the metric g on M and is foliated by the horizontal lifts of Riemannian geodesics on M to UM for the projection $\pi : UM \to M$. We denote by $\mathcal{K}$ the associated foliation on UM. Each leaf of $\mathcal{K}$ is a Legendre curve for the contact structure D and then we have another Legendre projection π', at least locally, from UM to the leaf space $UM/\mathcal{K}$, i.e. *the space of Riemannian geodesics*.

We determine Legendre singularities of D-geodesics on (UM, D) also for the projection π' completely in this paper.

Theorem 1.1. *Let* $\Gamma : (\mathbf{R}, t_0) \to UM$ *be any germ of D-geodesic. Then the composite mapping diagram* $(\Gamma, \pi) : (\mathbf{R}, t_0) \xrightarrow{\Gamma} (UM, \Gamma(t_0)) \xrightarrow{\pi} (M, \pi(\Gamma(t_0)))$ *(resp.*

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$(\Gamma, \pi') : (\mathbf{R}, t_0) \xrightarrow{\Gamma} (UM, \Gamma(t_0)) \xrightarrow{\pi'} (UM/\mathcal{K}, \pi'(\Gamma(t_0))))$ *is Legendre equivalent to one of following normal forms:*

(i) $(c_1, \Pi), c_1 : (\mathbf{R}, 0) \to (\mathbf{R}^3, 0), c_1(t) = (0, 0, 0)$,

(ii) $(c_2, \Pi), c_2 : (\mathbf{R}, 0) \to (\mathbf{R}^3, 0), c_2(t) = (0, 0, t)$,

(iii) $(c_3, \Pi), c_3 : (\mathbf{R}, 0) \to (\mathbf{R}^3, 0), c_3(t) = (t, 0, 0)$,

(iv) $(c_4, \Pi), c_4 : (\mathbf{R}, 0) \to (\mathbf{R}^3, 0), c_4(t) = (\frac{1}{2}t^2, \frac{1}{3}t^3, t)$.

Here $\mathbf{R}^3$ *with coordinates* (x, y, p) *has the canonical contact structure defined by* $dy - pdx = 0$ *and* $\Pi : (\mathbf{R}^3, 0) \to \mathbf{R}^2$ *is the Legendre projection defined by* $\Pi(x, y, p) = (x, y)$.

Moreover the pair of Legendre equivalence classes of (Γ, π) *and* (Γ, π') *is given by* ((i), (i)), ((ii), (iii)), ((iii), (ii)), ((iii), (iii)), ((iii), (iv)) *or* ((iv), (iii)).

In Theorem 1.1, the case (i) means that Γ itself is a constant curve, (ii) (resp. (iii)) means Γ is an embedding to a π-fiber, (resp. π'-fiber), and (iv) means that Γ has the cusp singularities by the Legendre projection π or π'. Note that the projection $\pi \circ \Gamma$ (resp. $\pi' \circ \Gamma$) of any D-geodesic Γ is a front with only cusp singularities, provided it is not a constant map.

It would be worth mentioning the similar problem on singularities of π-projections for *Riemannian* geodesics on the unit tangent bundle UM with the induced Riemannian metric from (M, g). Then, by the uniqueness of geodesics by initial velocity vectors, we see that $\pi \circ \Gamma$ is an immersion or a constant curve for any Riemannian geodesic Γ, and therefore the case (iv) never occurs in the problem for Riemannian geodesics on UM.

The transformation of a Riemannian geodesic of (M, g) to the π'-projection of its Legendre π-lift is a kind of Legendre transformation. For instance, the set of oriented geodesics of the unit sphere S^2 in $\mathbf{R}^3$ is identified to itself by taking the orthogonal cuts of S^2 by the orthogonal planes to unit vectors in S^2. The set of oriented geodesics on the hyperbolic space modelled in the Minkowski 3-space $\mathbf{R}^{2,1}$ is identified to the de-Sitter space $S^{1,1}$. Moreover the space of geodesics on the Euclidean plane $\mathbf{R}^2$ is identified with $S^1 \times \mathbf{R}$ naturally in the framework of projective duality [**9**]. Then Theorem 1.1 provides the complete local classification of projections to both surfaces for any oriented sub-Riemannian geodesics on the unit tangent bundle in each case.

In this paper we investigate locally such Legendre transformations and related "projective duality" on surfaces along the idea in sub-Riemannian contact geometry and geometric control theory, but in classical differential geometric language.

In §2 we recall basic constructions related to Riemannian surfaces, and in §3 we recall some facts in singularities of differentiable mappings. First we show Theorem 1.1 in the flat case in §4 to make clear the basic outline of the proof, as well as to observe a relation to the equation of a simple pendulum. Then, after a preliminary from basic differential geometry of surfaces in §5, we prove Theorem 1.1 in the general case in §6.

For geometric control theory and sub-Riemannian geometry, consult [**2, 1, 20, 21, 13**]. The sub-Riemannian geometry on UM or U^*M, the unit cotangent bundle in the flat case $M = \mathbf{R}^2$ has been investigated in detail, in particular, the problems on conjugate-loci, cut-loci and wavefronts for the sub-Riemannian geodesics were solved in [**19, 22**]. See also the related work [**8**]. Though our aim in this paper to study on Legendre duality of singularities, our method of construction in the present paper essentially follows these preceding works. Based on the results in the present paper, as well as the previous works such as [**8, 19, 22**], it would be expected further studies on global behaviors of Legendre projections of sub-Riemannian geodesics and on possible applications to geometric control systems.

At the end of Introduction, let us mention a native motivation of our problem treated in this paper.

In the snowy season of winter, you will observe many cusp-shaped traces of vehicles on many roads and parking lots usually. Naturally it can be supposed that we control vehicles in a (nearly) optimal way, when we drive and park. Therefore the cuspidal shape of such snow-traces may be regarded as an appearance of generic singularities for solutions to some problem of optimal control theory. For instance:

Problem. Suppose your car is located on a parking place. You are asked to move your car to the very next (right) place. How do you drive and move your car ?

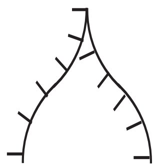

Figure 1.

Maybe you will go forward to a right direction a little and then go back to the proper parking place. Then the trace of your drive wheel will form a cusp-shaped curve (Figure 1). The front direction of the wheel or its left-side normal is determined anytime, so the trace can be regarded as a kind of so-called a "front" or a "frontal" [**10**]. The short lines of Figure 1 indicate the left side directions of the driver, which form a normal field to the trace. The phenomena of the appearance of singularities do not depend on the flatness of the field and you will observe the singularities also on slopes and non-flat parking lots everywhere. The singularities can be understood as Legendre singularities of sub-Riemannian geodesics for general Riemannian surfaces which we have discussed in the present paper.

In this paper, all manifolds and maps are supposed to be of class C^∞ unless otherwise stated.

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2. Basic constructions from Riemannian surfaces

Let M be an oriented 2-dimensional Riemannian manifold with metric g and TM the tangent bundle of M. Let UM be the unit tangent bundle over M,

$$UM := \{(x, v) \in TM \mid x \in M, v \in T_xM, g(v, v) = 1\}.$$

The bundle $\pi : UM \to M$, $\pi(x, v) = x$, is a principal $SO(2) = U(1)$ bundle and is naturally regarded as the orthonormal frame bundle over M. The Levi-Civita (Riemannian) connection on M gives the decomposition

$$T(UM) = H \oplus V$$

into the vertical distribution V of rank 1 and the horizontal distribution H of rank 2. Since π induces an isomorphism $\pi_* : H_{(x,v)} \to T_xM$, the bundle H has the induced Riemannian metric and the orientation. Moreover, for each $x \in M$, the fiber U_xM of π over $x \in M$ is regarded the unit circle of the Euclidean plane T_xM, and therefore the bundle V has the induced metric, which is written as $(d\theta)^2$ using a radian angle parameter θ. Thus UM has the induced Riemannian metric $g + d\theta^2$ from H and V so that $H \perp V$.

Note that the parameter θ itself is determined if the base point on the circle is fixed. Therefore if a unit vector field is provided on a connected and simply connected open set $\Omega \subset M$, then the function $\theta : \pi^{-1}(\Omega) \to \mathbf{R}$ is determined, which is periodic along π-fibers with period "2π".

For each $(x, v) \in UM$, T_xM is decomposed as $\langle v\rangle_{\mathbf{R}} \oplus \langle Jv\rangle_{\mathbf{R}}$, where J is the $90°$ rotation with respect to the given orientation of T_xM, and therefore, by setting $K_x = (\pi_*)^{-1}\langle v\rangle_{\mathbf{R}}$ and $L_x = (\pi_*)^{-1}\langle Jv\rangle_{\mathbf{R}}$ for any $x \in M$, we have the decomposition $H = K \oplus L$. Note also that $L = K^{\perp}$ in H and we have the orthogonal decomposition $T(UM) = K \oplus L \oplus V$.

Recall that the *connection form* ω associated to the Levi-Civita connection on UM is characterized as the $SO(2)$-invariant 1-form ω satisfying $\mathrm{Ker}(\omega) = H$ and $\omega(\xi) = 1$ for the unit tangent vector ξ with positive direction along the π-fiber, i.e. the fundamental vector corresponding to $1 \in \mathbf{R} = \mathfrak{so}(2)$ (see [**25**]).

The *canonical bundle* $D \subset T(UM)$ is defined by

$$D := \{(x, v; \xi) \in T(UM) \mid (x, v) \in UM, \xi \in T_{(x,v)}UM, \pi_*(\xi) \in \langle v\rangle_{\mathbf{R}}\}.$$

The distribution D is a contact distribution on UM. Note that $D = K \oplus V$ and $K = D \cap H$.

Note that the geodesic flow on UM induced by the Riemannian metric of M preserves K, L and V respectively and its trajectories, the horizontal lifts of Riemannian geodesics are integral curves of K.

Recall that a contact structure on a manifold W means a subbundle $D \subset TW$ of codimension 1 such that, any local 1-form α on W defining D, satisfies that $d\alpha|_D$ is non-degenerate. Then the dimension of W is odd, say, $2n + 1$ for some n. An immersion $\Gamma : N \to W$ from an n-dimensional manifold N is called a *Legendre immersion* if $d\Gamma(TN) \subset D$. A submersion $\pi : W \to M$ to an $(n+1)$-dimensional manifold M is called a *Legendre projection* if the tangent bundle of any π-fiber is contained in D [**3, 5**]. In this paper we are concerned with the case $n = 1$.

Definition 2.1. A *pseudo-product sub-Riemannian contact structure* D on a 3-dimensional manifold W is a sub-Riemannian contact structure $D \subset TW$ with an orthogonal decomposition $D = K \oplus V$ into subbundles K and V of rank 1 respectively.

Therefore $D \subset T(UM)$ is a pseudo-product sub-Riemannian contact structure on UM.

Let us denote by $N := UM/K$ the local leaf space of K at a point $(x, v) \in UM$ and $\pi' : UM \to UM/K$ the projection. Then we have locally the double Legendre projection

$$M \xleftarrow{\pi} UM \xrightarrow{\pi'} N,$$

for the contact structure D on UM. Note that $\mathrm{Ker}(\pi_*) = V$ and $\mathrm{Ker}(\pi'_*) = K$.

For the general theory of pseudo-product structures or double Legendre projections, see [**12, 23, 24, 27**].

3. Around the recognition of cusps

Recall that two composite mapping diagrams $(\mathbf{R}, t_0) \xrightarrow{\Gamma} (W, \Gamma(t_0)) \xrightarrow{\pi} (M, \pi(\Gamma(t_0))$ and $(\mathbf{R}, t'_0) \xrightarrow{\Gamma'} (W', \Gamma'(t'_0)) \xrightarrow{\pi'} (M', \pi'(\Gamma'(t'_0))$, where W, W' are contact manifolds, are called *Legendre equivalent* if there exist diffeomorphism-germs $\sigma : (\mathbf{R}, t_0) \to (\mathbf{R}, t'_0), \tau : (M, \pi(\Gamma(t_0))) \to (M', \pi'(\Gamma'(t'_0))$, and a contactomorphism-germ $\Phi : (W, \Gamma(t_0)) \to (W', \Gamma'(t'_0))$ such that the diagram

$$\begin{array}{ccccc} (\mathbf{R}, t_0) & \xrightarrow{\Gamma} & (W, \Gamma(t_0)) & \xrightarrow{\pi} & (M, \pi(\Gamma(t_0))) \\ \sigma \downarrow & & \Phi \downarrow & & \tau \downarrow \\ (\mathbf{R}, t'_0) & \xrightarrow{\Gamma'} & (W', \Gamma'(t'_0)) & \xrightarrow{\pi'} & (M', \pi'(\Gamma'(t'_0))) \end{array}$$

is commutative [**3, 5, 4**]. Then the compositions $\pi \circ \Gamma$ and $\pi' \circ \Gamma'$ are right-left equivalent by diffeomorphisms σ and τ.

A map-germ $\gamma : (\mathbf{R}, t_0) \to M$ to a surface is called a *cusp* if γ is right-left equivalent to the standard cusp $(\mathbf{R}, 0) \to (\mathbf{R}^2, 0), t \mapsto (\frac{1}{2}t^2, \frac{1}{3}t^3)$.

We use the following fundamental recognition lemma on cusp singularities.

Lemma 3.1. ([**29**]) *Let $k = (x_1, x_2) : (\mathbf{R}, t_0) \to \mathbf{R}^2$ be a germ of C^∞ curve on the plane. Suppose k is not an immersion at t_0, i.e. $(\dot{x}_1(t_0), \dot{x}_2(t_0)) = (0, 0)$. Then k is a cusp if and only if*

$$\Delta := \begin{vmatrix} \ddot{x}_1 & \dddot{x}_1 \\ \ddot{x}_2 & \dddot{x}_2 \end{vmatrix} (t_0) \neq 0.$$

Proof. Suppose k is not an immersion at t_0. Then we see, by simple direct calculations, that the condition $\Delta \neq 0$ depends only on the right-left equivalence class of k. Then we see if k is cusp then $\Delta \neq 0$ for the normal form of cusp. Now suppose $\Delta \neq 0$. Then we have $(X_1 \circ k)(T) = T^m, (X_2 \circ k)(T) = T^{m+1}a(T)$ for an integer $m \geq 2$ and a C^∞ function-germ $a : (\mathbf{R}, 0) \to \mathbf{R}$, by taking a new coordinate $T = t - t_0$ of $\mathbf{R}$ and a system of coordinates (X_1, X_2) on $(\mathbf{R}^2, k(t_0))$ centered at $k(t_0)$. Since $\Delta \neq 0$, we have $m = 2$ and $a(0) \neq 0$. Then $(X_1 \circ k)(T) = T^2, (X_2 \circ k)(T) = T^3 a(T)$. We see there exist C^∞ function-germs $b(T), c(T)$ such that $a(T) = b(T^2) + Tc(T^2)$ with $b(0) \neq 0$ and $c(0) \neq 0$. Then $(X_2 \circ k)(T) = T^3 a(T^2) + T^4 c(T^2)$, using Malgrange preparation theorem (see [**6, 18**]). Set $Y_1 = X_1, Y_2 = \frac{1}{a(X_1)}\left(X_2 - X_1^2 c(X_1)\right)$. Then the Jacobian $\frac{\partial(Y_1, Y_2)}{\partial(X_1, X_2)} \neq 0$ at $(0, 0)$ and $(Y_1 \circ k)(T) = T^2, (Y_2 \circ k)(T) = T^3$ for the new system of coordinates (Y_1, Y_2). After a linear transformation, we have the result. □

Remark 3.2. It is known that any two Legendre projections $\pi : (W, z_0) \to (M, x_0)$ and $\pi' : (W', z_0') \to (M', x_0')$ are Legendre equivalent, i.e. there exist a diffeomorphism-germ $\tau : (M, x_0) \to (M', x_0')$ and a contactomorphism-germ $\Phi : (W, z_0) \to (W', z_0')$ such that $\tau \circ \pi = \pi' \circ \Phi$ ([**3, 5**]).

Lemma 3.3. *Let W be a 3-dimensional contact manifold, $\pi : W \to M$ a Legendre projection and $\Gamma : (\mathbf{R}, t_0) \to W$ a Legendre immersion. Suppose $\pi \circ \Gamma$ is not an immersion at t_0. Then we have that $\pi \circ \Gamma : (\mathbf{R}, t_0) \to M$ is a cusp if and only if the second derivative $(\pi \circ \Gamma)''(t_0) \neq 0$.*

Proof. Assume $\pi \circ \Gamma$ is a cusp. Take a system of local coordinates x_1, x_2, x_3 of W centered at $\Gamma(t_0)$ such that x_1 and x_2 are constant along each π-fibers. Then we have that (x_1, x_2) induces a system of local coordinates of M centered at $\pi \circ \Gamma(t_0)$ and π' is given by $(x_1, x_2, x_3) \mapsto (x_1, x_2)$. Then, by Lemma 3.1, $(x_1''(t_0), x_2''(t_0)) \neq (0, 0)$, for the system of local coordinates (x_1, x_2, x_3). Conversely assume $(\pi \circ \Gamma)''(t_0) \neq 0$. Then the planer curve $(x_1(\Gamma(t)), x_2(\Gamma(t)))$ is singular at t_0 and has a non-vanishing term of second order for the coordinate $T = t - t_0$. Changing the systems of local coordinates (x_1, x_2) and T if necessary, we have $x_1 \circ \Gamma(T) = T^2, x_2 \circ \Gamma(T) = cT^3 + e(T)$, the order of $e(T)$ at 0 being > 3, for some $c \in \mathbf{R}$. Since Legendre lift of the planer curve is unique and must be an immersion which is Legendre equivalent to Γ at t_0, we have $c \neq 0$. Then, by Lemma 3.1 or a direct argument as in the proof of Lemma 3.1, we see that $\pi \circ \Gamma$ is a cusp, i.e., it is right-left equivalent to the normal form of the cusp. □

4. The flat case

First we consider the case $M = \mathbf{R}^2$, the Euclidean plane with coordinates x_1, x_2. Then UM has coordinates x_1, x_2, θ, where θ is the radian angle coordinate for the section $\frac{\partial}{\partial x_1}$. We explain the general basic constructions in sub-Riemannian geometry along this simple situation.

We set

$$V_1 = \cos\theta \frac{\partial}{\partial x_1} + \sin\theta \frac{\partial}{\partial x_2}, \quad V_2 = \frac{\partial}{\partial \theta},$$

which form an orthonormal frame of $D \subset T(UM)$. Let $\Gamma : [a, b] \to UM$ be an absolutely continuous or a piecewise smooth curve such that $\Gamma'(t) \in D$ for almost every $t \in [a, b]$. The sub-Riemannian or Carnot-Caratheodory arc length of Γ is defined by

$$L(\Gamma) = \int_a^b \|\Gamma'(t)\| dt,$$

using the norm of the sub-Riemannian metric on D introduced in §2. It is known the length minimizing problem is equivalent to the energy minimizing problem [**20**].

We represent vectors in $D \subset T(UM)$ using the frame V_1, V_2 as

$$F(x_1, x_2, \theta; u_1, u_2) = u_1 V_1 + u_2 V_2 = u_1 \left(\cos\theta \frac{\partial}{\partial x_1} + \sin\theta \frac{\partial}{\partial x_2} \right) + u_2 \frac{\partial}{\partial \theta}.$$

The parameters u_1, u_2 are regarded as *control parameters*. The *energy function* $E : D \to \mathbf{R}$ is given by using the squared norm of F as

$$E(x_1, x_2, \theta; u_1, u_2) := \frac{1}{2}(u_1^2 + u_2^2).$$

Now we consider the optimal control problem on D-integral curves of minimizing the energy E. Then the Hamiltonian function $H : D \times_{UM} T^*(UM) \to \mathbf{R}$ of the optimal control problem is given by $H(x, v, p) := \langle p, F(v)\rangle + cE(x, v)$ for some constant c (See [**20**] §5.3.1, and [**11**] §6). Here $(x, v) \in D, (x, p) \in T^*(UM)$ and $x \in M$. In coordinates, it is written as

$$H(x_1, x_2, \theta; u_1, u_2; p_1, p_2, \varphi) := u_1(p_1 \cos\theta + p_2 \sin\theta) + u_2\varphi + \frac{1}{2}c(u_1^2 + u_2^2).$$

By the Pontryagin principle, any solution $(x_1(t), x_2(t), \theta(t), u_1(t), u_2(t))$ of the optimal control problem is obtained by the constrained Hamilton equation, as a necessary condition,

$$\dot{x}_1 = \frac{\partial H}{\partial p_1},\ \dot{x}_2 = \frac{\partial H}{\partial p_2},\ \dot{\theta} = \frac{\partial H}{\partial \varphi},\ \ \dot{p}_1 = -\frac{\partial H}{\partial x_1},\ \dot{p}_2 = -\frac{\partial H}{\partial x_2},\ \dot{\varphi} = -\frac{\partial H}{\partial \theta},$$

with constraint $\dfrac{\partial H}{\partial u_1} = 0, \dfrac{\partial H}{\partial u_2} = 0$, for some $(p_1(t), p_2(t), \varphi(t)) \neq 0, c \in \mathbf{R}$.

A curve $\Gamma : (\mathbf{R}, t_0) \to UM$, given by $\Gamma(t) = (x_1(t), x_2(t), \theta(t))$, is called a D-*extremal*, which is called also a D-*geodesic*, if the above constrained Hamilton equation is satisfied for some $p_1(t), p_2(t), \varphi(t), u_1(t), u_2(t)$ and $c \in \mathbf{R}$. A D-extremal is called *abnormal* if $c = 0$ and is called *normal* if $c \neq 0$. See [**11**]. It is known that any normal D-extremal is in fact a local minimizer of arc-length ([**20**] Theorem 1.14).

In our case the condition is explicitly given by

$$\dot{x}_1 = u_1 \cos\theta,\ \dot{x}_2 = u_1 \sin\theta,\ \dot{\theta} = u_2,\ \ \dot{p}_1 = 0,\ \dot{p}_2 = 0,\ \dot{\varphi} = u_1(p_1 \sin\theta - p_2 \cos\theta),$$
$$p_1 \cos\theta + p_2 \sin\theta + cu_1 = 0,\ \ \varphi + cu_2 = 0,\ \ (p_1, p_2, \varphi, c) \neq 0, c \in \mathbf{R}.$$

By the above condition we see that each of p_1 and p_2 is a locally constant on t.

Because our distribution D is a contact structure it is known that there does not exist any non-trivial abnormal D-extremal ([**11**] §5.6). Here a trivial extremal means a locally constant $(x_1(t), x_2(t), \theta(t))$.

To make sure we will check that fact in our simple situation: Suppose there exists an extremal with $c = 0$. Then $p_1 \cos\theta + p_2 \sin\theta = 0$ and $\varphi = 0$. For any t with $u_1(t) \neq 0$, we have $p_1 \sin\theta - p_2 \cos\theta = 0$. Then we have $p_1 = p_2 = \varphi = 0$, which leads a contradiction. Therefore $u_1(t)$ must be 0 almost everywhere. Since $(p_1, p_2) \neq (0, 0)$ and it is a locally constant vector, we have also $(\cos\theta, \sin\theta)$ and so θ must be a locally constant, which implies $u_2(t) = 0$ a.e. also, which means that the extremal is trivial.

Now suppose $c \neq 0$ and seek normal D-extremals. Then, replacing p_1, p_2, φ by $-\frac{1}{c}p_1, -\frac{1}{c}p_2, -\frac{1}{c}\varphi$ respectively, we may set $c = -1$. Then $u_1 = p_1 \cos\theta + p_2 \sin\theta, u_2 = \varphi$. Therefore the D-extremal $(x_1(t), x_2(t), \theta(t), p_1(t), p_2(t), \varphi(t))$ satisfies a system of ordinary differential equations

$$\dot{x}_1 = (p_1 \cos\theta + p_2 \sin\theta)\cos\theta, \quad \dot{x}_2 = (p_1 \cos\theta + p_2 \sin\theta)\sin\theta, \quad \dot{\theta} = \varphi,$$
$$\dot{p}_1 = 0, \quad \dot{p}_2 = 0, \quad \dot{\varphi} = (p_1 \cos\theta + p_2 \sin\theta)(p_1 \sin\theta - p_2 \cos\theta),$$

with C^∞ right hand sides, and any solution is of class C^∞. Suppose that Γ is not an immersion at t_0. Then $(\dot{x}_1(t_0), \dot{x}_2(t_0), \dot{\theta}(t_0)) = (0, 0, 0)$. Then p_1, p_2, φ must be all identically zero, and Γ should be a constant map. Therefore any non-constant D-extremal Γ is an immersion. Moreover we observe that θ satisfies the second order ordinary differential equation

$$\begin{aligned}\ddot{\theta} &= (p_1 \cos\theta + p_2 \sin\theta)(p_1 \sin\theta - p_2 \cos\theta)\\ &= p_1^2 \cos\theta \sin\theta - p_1 p_2 \cos^2\theta + p_1 p_2 \sin^2\theta - p_2^2 \cos\theta \sin\theta.\end{aligned}$$

Suppose the constants $p_1 = p_2 = 0$, then $u_1 = 0$ and $\dot{x_1} = \dot{x_2} = 0$, each of x_1, x_2 being a constant. Moreover $\dot{\varphi} = 0$ and φ is a constant. Therefore $\dot{\theta}$ is a constant and $\theta(t) = at$ for some $a \in \mathbf{R}$. If $a = 0$, then Γ is a constant curve. If $a \neq 0$, then Γ gives a parametrization of a π-fiber over a point on M, and the D-extremal Γ can be regarded as a constant *directed* curve, or a frontal, on the plane endowed with rotating directions of constant angular velocity.

Next suppose $(p_1, p_2) \neq (0, 0)$. For example, for $p_1 = 0, p_2 = 1$, then the D-extremal $\Gamma(t) = (\dot{x}_1(t), \dot{x}_2(t), \theta(t))$ satisfies $\dot{x}_1 = \sin\theta\cos\theta, \dot{x}_2 = \sin^2\theta$ and $\ddot{\theta} = -\sin\theta\cos\theta = -\frac{1}{2}\sin 2\theta$. In general we have $\ddot{\theta} = -r\sin(2\theta + \rho)$, where we set $r = \frac{1}{2}(p_1^2 + p_2^2)$, $\cos\rho = -\frac{1}{2r}(p_1^2 - p_2^2)$ and $\sin\rho = \frac{1}{r}p_1 p_2$. Note that r, ρ are constant. If we set $\Theta = 2\theta + \rho$, $\omega = \sqrt{2r}$, then we have

$$\ddot{\Theta} = -\omega^2 \sin\Theta,$$

that is the non-linear equation of a *simple pendulum*. See also [**19, 22**]. Therefore θ can be expressed by elliptic functions. We need only the simple behavior of θ hereafter: When $\dot{\theta} = 0$, then $\ddot{\theta} \neq 0$. When $\ddot{\theta} = 0$, then $\dot{\theta} \neq 0$. We call this behavior the "pendulum duality".

Now we start to show Theorem 1.1.

Proof of Theorem 1.1 in the flat case. Let $\Gamma : (\mathbf{R}, t_0) \to (UM, \Gamma(t_0))$ be a D-geodesic. Set $\Gamma(t) = (x_1(t), x_1(t), \theta(t))$ as above.

Part I. π-Legendre classification of Γ.

We have by setting $c = -1$,

$$\dot{x}_1 = (p_1 \cos\theta + p_2 \sin\theta)\cos\theta, \quad \dot{x}_2 = (p_1 \cos\theta + p_2 \sin\theta)\sin\theta.$$

Let $\widetilde{\Gamma}(t) = (\Gamma(t); u_1(t), u_2(t); p_1(t), p_2(t), \varphi(t))$ be a corresponding extremal. If Γ is a constant curve, then (Γ, π) is Legendre equivalent to the case (i) of Theorem 1.1. If γ is not a constant curve, but $\pi \circ \Gamma = (x_1, x_2)$ is a constant curve. Then (Γ, π) is Legendre equivalent to (ii). Suppose $\pi \circ \Gamma$ is not a constant curve. First suppose $(\dot{x}_1(t_0), \dot{x}_2(t_0)) \neq (0, 0)$, i.e. $\pi \circ \gamma$ is an immersion-germ. Then (Γ, π) is Legendre equivalent to (iii). Now suppose $(\dot{x}_1(t_0), \dot{x}_2(t_0)) = (0, 0)$. Note that, then, we have $p_1 \cos\theta(t_0) + p_2 \sin\theta(t_0) = 0$

and therefore $\ddot{\theta}(t_0) = 0$. Then we have

$$\begin{aligned}
\ddot{x}_1 &= \dot{\theta}\{(-\sin\theta)(p_1\cos\theta + p_2\sin\theta) + \cos\theta(-p_1\sin\theta + p_2\cos\theta)\} \ = \dot{\theta}(-p_1\sin 2\theta + p_2\cos 2\theta),\\
\ddot{x}_2 &= \dot{\theta}\{\cos\theta(p_1\cos\theta + p_2\sin\theta) + \sin\theta(-p_1\sin\theta + p_2\cos\theta)\} \ = \dot{\theta}(p_1\cos 2\theta + p_2\sin 2\theta),\\
\dddot{x}_1 &= \ddot{\theta}(-p_1\sin 2\theta + p_2\cos 2\theta) + \dot{\theta}^2(-2p_1\cos 2\theta - 2p_2\sin 2\theta),\\
\dddot{x}_2 &= \ddot{\theta}(p_1\cos 2\theta + p_2\sin 2\theta) + \dot{\theta}^2(-2p_1\sin 2\theta + 2p_2\cos 2\theta).
\end{aligned}$$

Therefore we have

$$\begin{aligned}
\begin{vmatrix} \ddot{x}_1 & \dddot{x}_1 \\ \ddot{x}_2 & \dddot{x}_2 \end{vmatrix} &= \begin{vmatrix} \dot{\theta}(-p_1\sin 2\theta + p_2\cos 2\theta) & \ddot{\theta}(-p_1\sin 2\theta + p_2\cos 2\theta) + \dot{\theta}^2(-2p_1\cos 2\theta - 2p_2\sin 2\theta) \\ \dot{\theta}(p_1\cos 2\theta + p_2\sin 2\theta) & \ddot{\theta}(p_1\cos 2\theta + p_2\sin 2\theta) + \dot{\theta}^2(-2p_1\sin 2\theta + 2p_2\cos 2\theta) \end{vmatrix}\\
&= 2\dot{\theta}^3 \begin{vmatrix} -p_1\sin 2\theta + p_2\cos 2\theta & -p_1\cos 2\theta - p_2\sin 2\theta \\ p_1\cos 2\theta + p_2\sin 2\theta & -p_1\sin 2\theta + p_2\cos 2\theta \end{vmatrix}\\
&= 2\dot{\theta}^3\{(-p_1\sin 2\theta + p_2\cos 2\theta)^2 + (p_1\cos 2\theta + p_2\sin 2\theta)^2\} = 2\dot{\theta}^3(p_1^2 + p_2^2).
\end{aligned}$$

Thus we have $\Delta = 2\dot{\theta}(t_0)^3(p_1^2 + p_2^2)$. Since $\ddot{\theta}(t_0) = 0$, and since $\theta(t)$ satisfies the above second order ordinary differential equation and is not a constant, we see that $\dot{\theta}(t_0) \neq 0$. Therefore $\Delta \neq 0$, and we see that $\pi \circ \Gamma$ is right-left equivalent to the cusp $t \mapsto (\frac{1}{2}t^2, \frac{1}{3}t^3)$, which has the unique Legendre lift $t \mapsto (\frac{1}{2}t^2, \frac{1}{3}t^3, t)$ to the standard contact manifold $\mathbf{R}^3$ with coordinates (x, y, p) with $dy - pdx = 0$. Thus we have that (Γ, π) is Legendre equivalent to (iv).

Part II. π'-Legendre classification of Γ.

In our flat case, the projection π' is given by $(x_1, x_2, \theta) \mapsto (F, E)$, where

$$F = -x_1\sin\theta + x_2\cos\theta, \quad E = \theta.$$

Note that F and E are independent first integrals of the geodesic flow for the flat metric on $M = \mathbf{R}^2$. We set $f = F \circ \Gamma, e = E \circ \Gamma$. Then we have

$$\begin{aligned}
\dot{f} &= -\dot{\theta}(x_1\cos\theta + x_2\sin\theta),\\
\ddot{f} &= -\dot{\theta}(p_1\cos\theta + p_2\sin\theta) - \dot{\theta}^2(-x_1\sin\theta + x_2\cos\theta) - \ddot{\theta}(-x_1\cos\theta + a_2\sin\theta),\\
\dddot{f} &= \dot{\theta}^2(p_1\cos\theta - p_2\sin\theta) - 2\ddot{\theta}(p_1\cos\theta + p_2\sin\theta) + 3\dot{\theta}\ddot{\theta}(x_1\cos\theta - x_2\sin\theta)\\
&\quad + \dot{\theta}^3(x_1\cos\theta + x_2\sin\theta) - \dddot{\theta}(x_1\cos\theta + x_2\sin\theta).
\end{aligned}$$

If Γ is a constant curve, then (Γ, π') is Legendre equivalent to (i). If θ is a constant function, then (Γ, π') is Legendre equivalent to (ii). If $\dot{\theta}(t_0) \neq 0$, then $\pi' \circ \Gamma$ is an immersion at t_0 and (Γ, π') is Legendre equivalent to (iii). Suppose $\pi' \circ \Gamma$ is not an immersion at t_0. Then $\dot{\theta}(t_0) = 0$. Then we have that

$$\begin{vmatrix} \ddot{f} & \dddot{f} \\ \ddot{e} & \dddot{e} \end{vmatrix}(t_0) = 2\ddot{\theta}(t_0)^2(p_1\cos\theta(t_0) + p_2\sin\theta(t_0)).$$

If $p_1\cos\theta(t_0) + p_2\sin\theta(t_0) = 0$, then $\dot{x}_1(t_0) = \dot{x}_2(t_0) = \dot{\theta}(t_0) = \dot{p}_1(t_0) = \dot{p}_2(t_0) = \dot{\varphi}(t_0) = 0$, which leads that Γ is a constant curve. Thus, if Γ is not a constant curve, then we see $p_1 \cos\theta(t_0) + p_2 \sin\theta(t_0) \neq 0$. Therefore we see that $\Delta = \ddot{\theta}(t_0)^2(p_1 \cos\theta(t_0) + p_2 \sin\theta(t_0)) \neq 0$ whenever $\dot{\theta}(t_0) = 0$. Thus we have that, in this case, (Γ, π') is Legendre equivalent to (iv).

The last claim on the combination of Legendre singularities for π and π' is obtained just by observing the pendulum duality on points $t = t_0$ where $\dot{\theta}(t_0) = 0, \ddot{\theta}(t_0) \neq 0$ and points $t = t_1$ where $\dot{\theta}(t_1) \neq 0, \ddot{\theta}(t_1) = 0$, which appears as the Legendre duality in our case. □

Remark 4.1. It is the geometry of the curve $\pi \circ \Gamma(t) = (x_1(t), x_2(t))$ on $\mathbf{R}^2$, which is the projection to $\mathbf{R}^2$ of a D-geodesic $\Gamma(t) = (x_1(t), x_2(t), \theta(t))$. We see $\pi \circ \Gamma$ is singular at $t = t_0$ when $p_1 \cos\theta(t_0) + p_2 \sin\theta(t_0) = 0$. For instance we see the curvature of the plane curve $\pi \circ \Gamma(t) = (x_1(t), x_2(t))$ is given by

$$\kappa(t) = \frac{\dot{\theta}(t)}{|p_1 \cos\theta(t) + p_2 \sin\theta(t)|},$$

by simple calculations, if $\pi \circ \Gamma$ is an immersion at t. Therefore we see $\pi \circ \Gamma(t)$ has an inflection point at $t = t_0$ if $\dot{\theta}(t_0) = 0$. Moreover we have that, if $\pi \circ \Gamma$ has a cusp at $t = t_0$, then the *cuspidal curvature* κ_c of $\pi \circ \Gamma$ at $t = t_0$ is given by

$$\kappa_c = 2\,(\operatorname{sign}\dot{\theta})\,\frac{|\dot{\theta}|^{\frac{1}{2}}}{(p_1^2 + p_2^2)^{\frac{1}{4}}}.$$

For the cuspidal curvature see [**29**].

Further we observe that, for any non-constant solution of the equation of pendulum, the points t where $\dot{\theta}(t) = 0$ and $\ddot{\theta}(t) = 0$ appear alternately. Note that, for any D-geodesic Γ, we have an inflection point $t = t_0$ where $\theta(t_0) = 0$ and a cusp point $t = t_1$ on $\pi \circ \Gamma$ where $p_1 \cos\theta(t_0) + p_2 \sin\theta(t_0) = 0$ and so $\ddot{\theta}(t_0) = 0$, and $\theta(t_0) \neq 0$. For a sub-Riemannian geodesic $\Gamma : \mathbf{R} \to U\mathbf{R}^2$, if the variation of the angle $\theta(t)$ is small, then the inflection points, where $\dot{\theta} = 0$, and the cusp points, where $\ddot{\theta} = 0$, appear alternately along the non-constant projection $\pi \circ \Gamma$, which may be called a "zigzag" curve [**29**]. Moreover we observe the directions of cusps are all parallel. This will provide a severe restriction on the front curve $\pi \circ \Gamma$. An example of the projection of D-geodesic is illustrated roughly like the following figure (Figure 2).

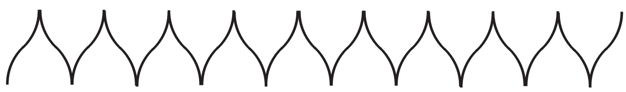

Figure 2

See illustrations also in [**1, 19, 22**].

5. Geodesic parallel coordinates

To analyze the equation of sub-Riemannian geodesics on UM in the case of general Riemannian surface M, it is useful to take a system of special local coordinates on M, which are called a system of *geodesic parallel coordinates*, also called *geodesic coordinates* or *Fermi coordinates* [**7, 14, 15**]. Here we recall it to make sure, omitting its proof.

Let (M, g) be a 2-dimensional Riemannian manifold, $p \in M$ and $v \in T_pM$ a unit tangent vector. For a system of local coordinates (x_1, x_2), we set $g(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}) = g_{ij}$.

LEMMA 5.1. *There exists a system of local coordinates (x_1, x_2) centered at p satisfying that*
(1) $g_{11} = 1, g_{12} = g_{21} = 0$ *so that g is of form $(dx_1)^2 + g_{22}(dx_2)^2$ and moreover g_{22} satisfies the conditions $g_{22}(0, x_2) = 1$ and $\dfrac{\partial g_{22}}{\partial x_1}(0, x_2) = 0$.*
(2) $\dfrac{\partial g_{ij}}{\partial x_k}(p) = 0$ *for any $i, j, k = 1, 2$, and all connection coefficients (Christoffel symbols) Γ^k_{ij} vanish at p.*
(3) $v = \frac{\partial}{\partial x_1}|_p$ *and the curve $x_1(t) = t, x_2(t) = a$, a being a constant, gives a Riemannian geodesic on M. If $c = 0$, it is the Riemannian geodesic starting from p with the initial velocity vector v.*
(4) *Let θ be the angle function with the base section $\frac{\partial}{\partial x_1}$. Then the generating vector field V of the geodesic flow on UM satisfies $\langle d\theta, V\rangle(x_1, x_2, 0) = 0$.*

Note that Lemma 5.1(4) is important for the proof in the next section.

On appropriate local frames of TM and on D for our purpose, we have:

LEMMA 5.2. *There exists a local orthonormal frame v_1, v_2 of TM on a neighborhood of p such that, for some geodesic parallel coordinates x_1, x_2, they are written as*

$$v_1 = k(x_1, x_2)\frac{\partial}{\partial x_1} + \ell(x_1, x_2)\frac{\partial}{\partial x_2}, \quad v_2 = m(x_1, x_2)\frac{\partial}{\partial x_1} + n(x_1, x_2)\frac{\partial}{\partial x_2},$$

with all of first order partial derivatives of k, ℓ, m, n vanished at p. Moreover we have the associated local orthonormal frame V_1, V_2 of $D \subset T(UM)$ written as

$$V_1 = v_1 \cos\theta + v_2 \sin\theta, \quad V_2 = \frac{\partial}{\partial \theta}.$$

PROOF. In general, if we set $k = \frac{1}{\sqrt{g_{11}}}, \ell = 0, m = -\frac{g_{12}}{\sqrt{g_{11}}\sqrt{g_{11}g_{12} - g_{12}^2}}$ and $n = \frac{\sqrt{g_{11}}}{\sqrt{g_{11}}\sqrt{g_{11}g_{12} - g_{12}^2}}$, then $v_1 = k\frac{\partial}{\partial x_1} + \ell\frac{\partial}{\partial x_1}, v_2 = m\frac{\partial}{\partial x_1} + n\frac{\partial}{\partial x_2}$ form a local orthonormal frame. If (x_1, x_2) is a system of geodesic parallel coordinates, then we see $k = 1, \ell = 0, m = 0$ and $n = \frac{1}{\sqrt{g_{22}}}$. Then, for the exterior derivatives, we have $dk = d\ell = dm = 0$ and $dn = -\frac{n}{2g_{22}}dg_{22}$. Thus, by Lemma 5.1 (2), we have the first half. The second half is clear. □

6. The case of general Riemannian surfaces

Let us study the case with a general Riemannian surface (M, g). Let $\Gamma : (\mathbf{R}, t_0) \to UM$ be any curve-germ. Let (x_1, x_2) be a system of geodesic parallel coordinates of M

centered at $\pi(\Gamma(t_0))$. Let v_1, v_2 be a local orthonormal frame for g on M, and V_1, V_2 be the associated local orthonormal frame of $D \subset T(UM)$ as in Lemma 5.2.

Then Hamiltonian for D-geodesics is given by

$$H = u_1\{(kp_1 + \ell p_2)\cos\theta + (mp_1 + np_2)\sin\theta\} + u_2\varphi + \frac{1}{2}c(u_1^2 + u_2^2),$$

where $c \in \mathbf{R}$, and the equation for the extremal $(x_1, x_2, \theta, p_1, p_2, \varphi)$ is written as

$$\begin{cases} \dot{x}_1 = u_1(k\cos\theta + m\sin\theta), \quad \dot{x}_2 = u_1(\ell\cos\theta + n\sin\theta), \quad \dot{\theta} = u_2, \\ \dot{p}_1 = -u_1\{(\dfrac{\partial k}{\partial x_1}p_1 + \dfrac{\partial \ell}{\partial x_1}p_2)\cos\theta + (\dfrac{\partial m}{\partial x_1}p_1 + \dfrac{\partial n}{\partial x_1}p_2)\sin\theta\}, \\ \dot{p}_2 = -u_1\{(\dfrac{\partial k}{\partial x_2}p_1 + \ell_{x_2}p_2)\cos\theta + (\dfrac{\partial m}{\partial x_2}p_1 + \dfrac{\partial n}{\partial x_2}p_2)\sin\theta\}, \\ \dot{\varphi} = -u_1\{-(kp_1 + \ell p_2)\sin\theta + (mp_1 + np_2)\cos\theta\}, \end{cases}$$

with the constraint

$$(kp_1 + \ell p_2)\cos\theta + (mp_1 + np_2)\sin\theta + cu_1 = 0, \quad \varphi + cu_2 = 0.$$

Suppose $c = 0$. Then, by the constraint, $(kp_1 + \ell p_2)\cos\theta + (mp_1 + np_2)\sin\theta = 0$ and $\varphi = 0$. For any t with $u_1(t) \neq 0$, we have $-(kp_1 + \ell p_2)\sin\theta + (mp_1 + np_2)\cos\theta = 0$. Then we have

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} k & \ell \\ m & n \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Then we have $p_1 = p_2 = \varphi = 0$, which leads a contradiction. Therefore $u_1(t)$ must be 0 almost everywhere. Then $x_1(t), x_2(t), p_1(t), p_2(t)$ are locally constants and therefore $(\cos\theta(t), \sin\theta(t))$ and $\theta(t)$ must be a constant. Thus we have checked directly that any non-trivial D-geodesic is normal in our case.

Now suppose $c \neq 0$. By replacing $-\frac{1}{c}p_1, -\frac{1}{c}p_2, -\frac{1}{c}\varphi$ by p_1, p_2, φ respectively, we may set $c = -1$. Then we have

$$u_1 = (kp_1 + \ell p_2)\cos\theta + (mp_1 + np_2)\sin\theta, \quad u_2 = \varphi.$$

Thus we have a first order ordinary differential equation (*)

$$\begin{cases} \dot{x}_1 = \{(kp_1 + \ell p_2)\cos\theta + (mp_1 + np_2)\sin\theta\}(k\cos\theta + m\sin\theta), \\ \dot{x}_2 = \{(kp_1 + \ell p_2)\cos\theta + (mp_1 + np_2)\sin\theta\}(\ell\cos\theta + n\sin\theta), \\ \dot{\theta} = \varphi, \\ \dot{p}_1 = -\{(kp_1 + \ell p_2)\cos\theta + (mp_1 + np_2)\sin\theta\} \\ \qquad \cdot\{(\frac{\partial k}{\partial x_1}p_1 + \frac{\partial \ell}{\partial x_1}p_2)\cos\theta + (\frac{\partial m}{\partial x_1}p_1 + \frac{\partial n}{\partial x_1}p_2)\sin\theta\}, \\ \dot{p}_2 = -\{(kp_1 + \ell p_2)\cos\theta + (mp_1 + np_2)\sin\theta\} \\ \qquad \cdot\{(\frac{\partial k}{\partial x_2}p_1 + \frac{\partial \ell}{\partial x_2}p_2)\cos\theta + (\frac{\partial m}{\partial x_2}p_1 + \frac{\partial n}{\partial x_2}p_2)\sin\theta\}, \\ \dot{\varphi} = -\{(kp_1 + \ell p_2)\cos\theta + (mp_1 + np_2)\sin\theta\}\{-(kp_1 + \ell p_2)\sin\theta + (mp_1 + np_2)\cos\theta\}. \end{cases}$$

In our general case, we have $\ddot{\theta} = -r\sin(2\theta+\rho)$, where $r = r(t) = \frac{1}{2}\{(kp_1+\ell p_2)^2 + (mp_1+np_2)^2\}$ and $\rho = \rho(t)$ satisfies $\sin\rho = -(kp_1+\ell p_2)(mp_1+np_2)/r$ and $\cos\rho = \frac{1}{2}\{(kp_1+\ell p_2)^2 - (mp_1+np_2)^2\}/r$.

Note that, contrary to the flat case treated in §4, r and ρ may depend on t in general. Therefore we could say that θ follows, not the equation of simple pendulum, but a generalized equation of pendulum.

Now we return to show Theorem 1.1.

Remark 6.1. It is known that any normal sub-Riemannian geodesic $(x(t), \theta(t))$ is obtained as the projection of a solution $x(t), \theta(t), p(t), \varphi(t)$ of the Hamilton equation

$$\dot{x} = \frac{\partial \widetilde{H}}{\partial p},\ \dot{\theta} = \frac{\partial \widetilde{H}}{\partial \varphi}, \quad \dot{p} = -\frac{\partial \widetilde{H}}{\partial x},\ \dot{\varphi} = -\frac{\partial \widetilde{H}}{\partial \theta},$$

on $T^*(UM)$ for the another Hamiltonian

$$\widetilde{H}(x,\theta,p,\varphi) = \frac{1}{2}\left(\langle p, V_1\rangle^2 + \langle \varphi, V_2\rangle^2\right).$$

See [**20**] Theorem 1.14. One can check that the same result is obtained also by analyzing the above Hamilton equation.

Proof of Theorem 1.1 in the general case.

Let $\Gamma : (\mathbf{R}, t_0) \to (UM, \Gamma(t_0))$ be a D-geodesic and $\widetilde{\Gamma} : (\mathbf{R}, t_0) \to (T^*(UM), \widetilde{\Gamma}(t_0))$ sh be a corresponding extremal for some $c \neq 0$. Set $\Gamma(t) = (x_1(t), x_2(t), \theta(t))$ and $\widetilde{\Gamma}(t) = (\Gamma(t); p_1(t), p_2(t), \varphi(t))$. We set

$$\begin{aligned} A(x_1,x_2,\theta,p_1,p_2) &:= (kp_1+\ell p_2)\cos\theta + (mp_1+np_2)\sin\theta, \\ B(x_1,x_2,\theta,p_1,p_2) &:= -(kp_1+\ell p_2)\sin\theta + (mp_1+np_2)\cos\theta. \end{aligned}$$

Part I. π-Legendre classification of Γ. If $(\dot{x}_1(t_0), \dot{x}_2(t_0)) \neq (0,0)$, then we have the case (iii). Suppose $(\dot{x}_1(t_0), \dot{x}_2(t_0)) = (0,0)$. Then $A(x_1(t_0), x_2(t_0), \theta(t_0), p_1(t_0), p_2(t_0)) = 0$ at t_0. Assume $\dot{\theta}(t_0) = 0$. Then we have $\dot{x}_1(t_0) = \dot{x}_2(t_0) = \dot{\theta}(t_0) = \dot{p}_1(t_0) = \dot{p}_2(t_0) = \dot{\varphi}(t_0) = 0$. Therefore, by the uniqueness of solution, we see $\widetilde{\Gamma}$ itself is a constant curve, and Γ is also constant, then we have (i). Suppose $\dot{\theta}(t_0) \neq 0$. Set $a = \dot{\theta}(t_0)$. If $B(x_1(t_0), x_2(t_0), \theta(t_0), p_1(t_0), p_2(t_0)) = 0$ at t_0, then we have $(p_1(t_0), p_2(t_0)) = (0,0)$. Then, by the uniqueness of solution to the initial value problem of the ordinary differential equation (*), we have that identically

$$x_1(t) = x_1(t_0),\ x_2(t) = x_2(t_0),\ \theta(t) = at+b,\ p_1(t) = 0,\ p_2(t) = 0,\ \varphi(t) = a,$$

for a constant b. Then we see $\pi \circ \Gamma$ is a constant map. Thus we have the case (ii).

Suppose $B(x_1(t_0), x_2(t_0), \theta(t_0), p_1(t_0), p_2(t_0)) \neq 0$ at t_0. Now we calculate Δ as in Lemma 3.1.

$$\dot{x}_1 = \left(\frac{\partial A}{\partial p_1}\right) A, \quad \dot{x}_2 = \left(\frac{\partial A}{\partial p_2}\right) A,$$

$$\ddot{x}_1 = \left(\frac{\partial A}{\partial p_1}\right)^{\cdot} A + \left(\frac{\partial A}{\partial p_1}\right) \dot{A}, \quad \ddot{x}_2 = \left(\frac{\partial A}{\partial p_2}\right)^{\cdot} A + \left(\frac{\partial A}{\partial p_2}\right) \dot{A},$$

$$\dddot{x}_1 = \left(\frac{\partial A}{\partial p_1}\right)^{\cdot\cdot} A + 2\left(\frac{\partial A}{\partial p_1}\right)^{\cdot} \dot{A} + \left(\frac{\partial A}{\partial p_1}\right) \ddot{A}, \quad \dddot{x}_2 = \left(\frac{\partial A}{\partial p_2}\right)^{\cdot\cdot} A + 2\left(\frac{\partial A}{\partial p_2}\right)^{\cdot} \dot{A} + \left(\frac{\partial A}{\partial p_2}\right) \ddot{A}.$$

At $t = t_0$, we have $A = 0, \dot{k} = \dot{\ell} = \dot{m} = \dot{n} = 0, \dot{p}_1 - \dot{p}_2 = 0$. Therefore we have

$$\Delta := \begin{vmatrix} \ddot{x}_1 & \dddot{x}_1 \\ \ddot{x}_2 & \dddot{x}_2 \end{vmatrix} (t_0) = \dot{A}(t_0) \begin{vmatrix} \frac{\partial A}{\partial p_1} & 2\left(\frac{\partial A}{\partial p_1}\right)^{\cdot} \dot{A} + \left(\frac{\partial A}{\partial p_1}\right) \ddot{A} \\ \frac{\partial A}{\partial p_2} & 2\left(\frac{\partial A}{\partial p_2}\right)^{\cdot} \dot{A} + \left(\frac{\partial A}{\partial p_2}\right) \ddot{A} \end{vmatrix} (t_0) = 2\dot{A}(t_0)^2 \begin{vmatrix} \frac{\partial A}{\partial p_1} & \left(\frac{\partial A}{\partial p_1}\right)^{\cdot} \\ \frac{\partial A}{\partial p_2} & \left(\frac{\partial A}{\partial p_2}\right)^{\cdot} \end{vmatrix} (t_0).$$

We have, at $t = t_0$,

$$\dot{A} = \{-(kp_1 + \ell p_2)\sin\theta + (mp_1 + np_2)\cos\theta\}\dot{\theta} = B\dot{\theta},$$

and

$$\begin{vmatrix} \frac{\partial A}{\partial p_1} & \left(\frac{\partial A}{\partial p_1}\right)^{\cdot} \\ \frac{\partial A}{\partial p_2} & \left(\frac{\partial A}{\partial p_2}\right)^{\cdot} \end{vmatrix} (t_0) = \dot{\theta}(t_0)(kn - \ell m)(t_0).$$

Therefore we have

$$\Delta = 2B(t_0)^2(kn - \ell m)(t_0)\dot{\theta}(t_0)^3 \neq 0.$$

Therefore $\pi \circ \Gamma$ is right-left equivalent to the cusp and this is the case (v).

Part II. π'-Legendre classification of Γ.

Next we analyse (Γ, π'). If $\pi' \circ \Gamma$ is an immersion at t_0, then we have (iii). If Γ is a constant curve, then we have (i). If $\pi' \circ \Gamma$ is a constant curve, then since Γ is an immersion, we have (ii).

Now suppose $\pi' \circ \Gamma$ is not an immersion at t_0. We take geodesic parallel coordinates around $\pi \circ \Gamma(t_0)$ and local frame v_1, v_2 as in Lemma 5.1.

Set

$$R(x_1, x_2, \theta) := k(x_1, x_2)\cos\theta + m(x_1, x_2)\sin\theta,$$

and

$$S(x_1, x_2, \theta) := \ell(x_1, x_2)\cos\theta + n(x_1, x_2)\sin\theta.$$

Then the vector field which generates the geodesic flow in a neighborhood of $\Gamma(t_0)$ is written as

$$V = R\frac{\partial}{\partial x_1} + S\frac{\partial}{\partial x_2} + Q\frac{\partial}{\partial \theta},$$

for some function $Q = Q(x_1, x_2, \theta)$. We use the geodesic parallel coordinates around $\pi\circ\Gamma(t_0)$. Then, by Lemma 5.1(4), we see $Q(x_1, x_2, 0) = 0$.

The projection π' is locally expressed by taking a pair of independent first integrals (F, E) of V in a neighborhood of $\Gamma(t_0)$ by flow-box theorem. We set $\pi'\circ\Gamma(t) = (F(\Gamma(t)), E(\Gamma(t))) =: (f(t), e(t))$. Then we have

$$\begin{cases} \dot f(t) = \left(\dfrac{\partial F}{\partial x_1}\circ\Gamma\right)(t)\ \dot x_1(t) + \left(\dfrac{\partial F}{\partial x_2}\circ\Gamma\right)(t)\ \dot x_2(t) + \left(\dfrac{\partial F}{\partial \theta}\circ\Gamma\right)(t)\ \dot\theta(t), \\ \dot e(t) = \left(\dfrac{\partial E}{\partial x_1}\circ\Gamma\right)(t)\ \dot x_1(t) + \left(\dfrac{\partial E}{\partial x_2}\circ\Gamma\right)(t)\ \dot x_2(t) + \left(\dfrac{\partial E}{\partial \theta}\circ\Gamma\right)(t)\ \dot\theta(t), \end{cases}$$

$$\begin{cases} \ddot f(t) = \left(\dfrac{\partial^2 F}{\partial x_1^2}\circ\Gamma\right)(t)\ \dot x_1(t)^2 + \left(\dfrac{\partial^2 F}{\partial x_2^2}\circ\Gamma\right)(t)\ \dot x_2(t)^2 + \left(\dfrac{\partial^2 F}{\partial \theta^2}\circ\Gamma\right)(t)\ \dot\theta(t)^2 \\ \qquad +2\left(\dfrac{\partial^2 F}{\partial x_1\partial x_2}\circ\Gamma\right)(t)\ \dot x_1(t)\dot x_2(t) + 2\left(d\partial^2 F\partial x_1\partial\theta\circ\Gamma\right)(t)\ \dot x_1(t)\dot\theta(t) + 2\left(\dfrac{\partial^2 F}{\partial x_2\partial\theta}\circ\Gamma\right)(t)\ \dot x_2(t)\dot\theta(t) \\ \qquad +\left(\dfrac{\partial F}{\partial x_1}\circ\Gamma\right)(t)\ \ddot x_1(t) + \left(\dfrac{\partial F}{\partial x_2}\circ\Gamma\right)(t)\ \ddot x_2(t) + \left(\dfrac{\partial F}{\partial \theta}\circ\Gamma\right)(t)\ \dot\theta(t), \\ \ddot e(t) = \left(\dfrac{\partial^2 E}{\partial x_1^2}\circ\Gamma\right)(t)\ \dot x_1(t)^2 + \left(\dfrac{\partial^2 E}{\partial x_2^2}\circ\Gamma\right)(t)\ \dot x_2(t)^2 + \left(\dfrac{\partial^2 E}{\partial \theta^2}\circ\Gamma\right)(t)\ \ddot\theta(t)^2 \\ \qquad +2\left(\dfrac{\partial^2 E}{\partial x_1\partial x_2}\circ\Gamma\right)(t)\ \dot x_1(t)\dot x_2(t) + 2\left(\dfrac{\partial^2 E}{\partial x_1\partial\theta}\circ\Gamma\right)(t)\ \dot x_1(t)\dot\theta(t) + 2\left(\dfrac{\partial^2 K}{\partial x_2\partial\theta}\circ\Gamma\right)(t)\ \dot x_2(t)\dot\theta(t) \\ \qquad +\left(\dfrac{\partial E}{\partial x_1}\circ\Gamma\right)(t)\ \ddot x_1(t) + \left(\dfrac{\partial E}{\partial x_2}\circ\Gamma\right)(t)\ \ddot x_2(t) + \left(\dfrac{\partial E}{\partial \theta}\circ\Gamma\right)(t)\ \ddot\theta(t). \end{cases}$$

Since $\pi'\circ\Gamma$ is not an immersion at t_0, we have $(\dot f(t_0), \dot e(t_0)) = (0,0)$. Moreover we have

$$\dot x_1(t_0) = k(kp_1+\ell p_2)|_{t=t_0}, \dot x_2(t_0) = \ell(kp_1+\ell p_2)|_{t=t_0}, \dot\theta(t_0) = 0.$$

Moreover, since $\dot p_1(t_0) = \dot p_2(t_0) = 0$ and all partial derivatives of first order of k, ℓ, m, n vanish at $\pi\circ\Gamma(t_0)$, we see $\ddot x_1(t_0) = \ddot x_2(t_0) = 0$. On the other hand we have $\ddot\theta(t_0) \neq 0$. Thus we have

$$\begin{cases} \ddot f(t_0) = \dfrac{\partial^2 F}{\partial x_1^2}k^2(kp_1+\ell p_2)^2 + 2\dfrac{\partial^2 F}{\partial x_1\partial x_2}k\ell(kp_1+\ell p_2)^2 + \dfrac{\partial^2 F}{\partial x_2^2}\ell^2(kp_1+\ell p_2)^2 + \dfrac{\partial F}{\partial\theta}\ddot\theta\,\Big|_{t=t_0} \\ \qquad = k(kp_1+\ell p_2)^2\left(k\dfrac{\partial^2 F}{\partial x_1^2} + \ell\dfrac{\partial^2 F}{\partial x_1\partial x_2}\right) + \ell(kp_1+\ell p_2)^2\left(k\dfrac{\partial^2 F}{\partial x_1\partial x_2} + \ell\dfrac{\partial^2 F}{\partial x_1^2}\right) + \dfrac{\partial F}{\partial\theta}\ddot\theta\,\Big|_{t=t_0}, \\ \ddot e(t_0) = \dfrac{\partial^2 E}{\partial x_1^2}k^2(kp_1+\ell p_2)^2 + 2\dfrac{\partial^2 E}{\partial x_1\partial x_2}k\ell(kp_1+\ell p_2)^2 + \dfrac{\partial^2 E}{\partial x_2^2}\ell^2(kp_1+\ell p_2)^2 + \dfrac{\partial E}{\partial\theta}\ddot\theta\,\Big|_{t=t_0} \\ \qquad = k(kp_1+\ell p_2)^2\left(k\dfrac{\partial^2 E}{\partial x_1^2} + \ell\dfrac{\partial^2 E}{\partial x_1\partial x_2}\right) + \ell(kp_1+\ell p_2)^2\left(k\dfrac{\partial^2 E}{\partial x_1\partial x_2} + \ell\dfrac{\partial^2 E}{\partial x_1^2}\right) + \dfrac{\partial E}{\partial\theta}\ddot\theta\,\Big|_{t=t_0}. \end{cases}$$

Because F and E are first integrals of V, as functions on x_1, x_2, θ,

$$R\frac{\partial F}{\partial x_1} + S\frac{\partial F}{\partial x_2} + Q\frac{\partial F}{\partial\theta} = 0, \quad R\frac{\partial E}{\partial x_1} + S\frac{\partial E}{\partial x_2} + Q\frac{\partial E}{\partial\theta} = 0.$$

By taking the differentials by x_1 and x_2 of the right hand sides of the above equations respectively, we have

$$\begin{cases} \dfrac{\partial R}{\partial x_1}\dfrac{\partial F}{\partial x_1} + \dfrac{\partial S}{\partial x_1}\dfrac{\partial F}{\partial x_2} + \dfrac{\partial Q}{\partial x_1}\dfrac{\partial F}{\partial \theta} + R\dfrac{\partial^2 F}{\partial x_1^2} + S\dfrac{\partial^2 F}{\partial x_1 \partial x_2} + Q\dfrac{\partial^2 F}{\partial x_1 \partial \theta} = 0, \\ \dfrac{\partial R}{\partial x_2}\dfrac{\partial F}{\partial x_1} + \dfrac{\partial S}{\partial x_2}\dfrac{\partial F}{\partial x_2} + \dfrac{\partial Q}{\partial x_2}\dfrac{\partial F}{\partial \theta} + R\dfrac{\partial^2 F}{\partial x_1 \partial x_2} + S\dfrac{\partial^2 F}{\partial x_2^2} + Q\dfrac{\partial^2 F}{\partial x_2 \partial \theta} = 0, \\ \dfrac{\partial R}{\partial x_1}\dfrac{\partial E}{\partial x_1} + \dfrac{\partial S}{\partial x_1}\dfrac{\partial E}{\partial x_2} + \dfrac{\partial Q}{\partial x_1}\dfrac{\partial E}{\partial \theta} + R\dfrac{\partial^2 E}{\partial x_1^2} + S\dfrac{\partial^2 E}{\partial x_1 \partial x_2} + Q\dfrac{\partial^2 E}{\partial x_1 \partial \theta} = 0, \\ \dfrac{\partial R}{\partial x_2}\dfrac{\partial E}{\partial x_1} + \dfrac{\partial S}{\partial x_2}\dfrac{\partial E}{\partial x_2} + \dfrac{\partial Q}{\partial x_2}\dfrac{\partial E}{\partial \theta} + R\dfrac{\partial^2 E}{\partial x_1 \partial x_2} + S\dfrac{\partial^2 E}{\partial x_2^2} + Q\dfrac{\partial^2 E}{\partial x_2 \partial \theta} = 0. \end{cases}$$

At the point $\Gamma(t_0)$, we have that all of $\dfrac{\partial R}{\partial x_1}, \dfrac{\partial R}{\partial x_2}, \dfrac{\partial S}{\partial x_1}, \dfrac{\partial S}{\partial x_2}, Q, \dfrac{\partial Q}{\partial x_1}, \dfrac{\partial Q}{\partial x_2}$ vanish. Moreover we have $R(\Gamma(t_0)) = k(\pi \circ \Gamma(t_0)), S(\Gamma(t_0) = \ell(\pi \circ \Gamma(t_0))$. Therefore we obtain

$$\begin{aligned} k\frac{\partial^2 F}{\partial x_1^2} + \ell\frac{\partial^2 F}{\partial x_1 \partial x_2}\bigg|_{t=t_0} &= 0, \quad k\frac{\partial^2 F}{\partial x_1 \partial x_2} + \ell\frac{\partial^2 F}{\partial x_1^2}\bigg|_{t=t_0} = 0, \\ k\frac{\partial^2 E}{\partial x_1^2} + \ell\frac{\partial^2 E}{\partial x_1 \partial x_2}\bigg|_{t=t_0} &= 0, \quad k\frac{\partial^2 E}{\partial x_1 \partial x_2} + \ell\frac{\partial^2 E}{\partial x_1^2}\bigg|_{t=t_0} = 0. \end{aligned}$$

Therefore we have

$$\ddot{f}(t_0) = \frac{\partial F}{\partial \theta}(t_0)\ddot{\theta}(t_0), \quad \ddot{e}(t_0) = \frac{\partial E}{\partial \theta}(t_0)\ddot{\theta}(t_0).$$

Since $\dfrac{\partial}{\partial \theta}$ does not belongs to the kernel of the differential of $(F, E) : (UM, \Gamma(t_0)) \to \mathbf{R}^2$ at $\Gamma(t_0)$, we have $(\dfrac{\partial F}{\partial \theta}(t_0), \dfrac{\partial E}{\partial \theta}(t_0)) \neq (0, 0)$. Since $\ddot{\theta}(t_0) \neq 0$, we have $(\ddot{f}(t_0), \ddot{e}(t_0)) \neq (0, 0)$. Therefore, by Lemma 3.3, we have $\pi' \circ \Gamma$ is a cusp.

The last statement on the combination of singularities of $\pi \circ \Gamma$ and $\pi' \circ \Gamma$ follows by remarking that the combinations ((iii), (iii)), ((iv), (iv)) never occur because $\mathrm{Ker}(d\pi) \cap \mathrm{Ker}(d\pi') = V \cap K = \{0\}$. This completes the proof of Theorem 1.1.

References

[1] Agrachev A., Barilari D., Bascain U., *A Comprehensive Introduction to Sub-Riemannian Geometry,* Cambridge studies in adv. math., **181**, Cambridge Univ. Press (2020).

[2] Agrachev A., Sachkov Y., *Control Theory from the Geometric Viewpoint,* Encyclopaedia of Mathematical Sciences **87**, Springer-Verlag, Berlin, Heidelberg, New York (2010).

[3] Arnold V.I., *Singularities of Caustics and Wave Fronts,* Kluwer Academic Publishers, (1990).

[4] Arnold V.I., Goryunov V.V., Lyashko O.V., Vasiliev V.A., *Singularity Theory II, Classification and Applications,* Dynamical Systems VIII, Encyclopaedia of Math. Sci. **39**, Springer-Verlag, (1993).

[5] Arnold V.I., Gusein-Zade S.M., Varchenko A., *Singularities of Differentiable Maps, I,* Birkhäuser, (1985).

[6] Bröcker T., Lander L., *Differentiable Germs and Catastrophes,* London Math. Soc. Lecture Note Series, Cambridge Univ. Press, **17** (1975).

[7] do Carmo M.P., *Differential Geometry of Curves and Surfaces,* Rev. & updated 2nd ed., Dover Publ. (2016).

[8] Duits R., Ghosh A., Haije T.M., Sachkov Y., *Cuspless sub-Riemannian geodesics within the Euclidean motion group $SE(d)$,* in Neuromathematics of Vision, ed. by G. Citti and A. Sarti,

Springer-Verlag (2014), Chapter 5, pp. 173–215.

[9] Ishikawa G., *Generic bifurcations of framed curves in a space form and their envelopes,* Topology and its Applications, **159** (2012), 492–500.

[10] Ishikawa G., *Singularities of frontals,* in "Singularities in Generic Geometry", Advanced Studies in Pure Mathematics **78-2**, Mathematical Society of Japan (2018), pp. 55–106. `http://arxiv.org/abs/1609.00488/`

[11] Ishikawa G., Kitagawa Y., Yukuno W., *Duality on geodesics of Cartan distributions and sub-Riemannian pseudo-product structures,* Demonstratio Math. **48-2** (2015), 193–216.

[12] Ishikawa G., Machida Y., *Monge-Ampère systems with Lagrangian pair,* Symmetry, Integrability and Geometry: Methods and Applications (SIGMA), **11** (2015), 081, 32 pages.

[13] Kitagawa Y., *The Infinitesimal automorphisms of a homogeneous subriemannian contact manifold,* Nara Women's University Annual reports of Graduate School of Humanities and Science **20** (2005), 147–163.

[14] Klingenberg W., *A Course on Differential Geometry,* Graduate Texts in Math., 51, Springer Science+Business Media (1978).

[15] Kobayashi S., *Differential Geometry of Curves and Surfaces,* Springer Undergraduate Math. Series, Springer (2019)

[16] Kreyszig E., *Differential Geometry,* Revised edition, Dover Publ. (1991).

[17] Kühnel, *Differential Geometry: Curves – Surfaces – Manifolds,* 3rd ed., Student Math. Library 77, Amer. Math. Soc. (2015).

[18] Martinet J., *Singularities of Smooth Functions and Maps,* transl. by C.P. Simon, London Math. Soc. Lecture Note Series **58**, Cambridge Univ. Press (1982).

[19] Moiseev I., Sachkov Y., *Maxwell strata in sub-Riemannian problem on the group of motions of a plane,* ESAIM: Control, Optimisation and Calculus of Variations, **16** (2010), 380–399.

[20] Montgomery R., *A Tour of Subriemannian Geometries, Their Geodesics and Applications,* Mathematical Surveys and Monographs **91**, Amer. Math. Soc. (2006).

[21] Morimoto T., *Cartan connection associated with a subriemannian structure,* Differential Geom. its Appl., **26**(2008), 75–78.

[22] Sachkov Y.L., *Conjugate and cut time in the sub-Riemannian problem on the group of motions of a plane,* ESAIM: Control, Optimisation and Calculus of Variations, **16** (2010), 1018–1039.

[23] Sato H., Yamaguchi K., *Lie contact manifolds,* in Geometry of manifolds (Matsumoto, 1988), Perspect. Math., **8**, Academic Press, Boston, MA, 1989. 191–238,

[24] Sato H., Yamaguchi K., *Lie contact manifolds II,* Math. Ann., **297-1** (1993), 33–57.

[25] Singer I.M., Thorpe J.A., *Lecture Notes on Elementary Topology and Geometry,* Springer (1976).

[26] Tabachnikov S., *Geometry of Lagrangian and Legendrian 2-web,* Differential Geom. Appl. **3-3** (1993), 265–284.

[27] Takeuchi M., *Lagrangean contact structures on projective cotangent bundles,* Osaka J. Math., **31** (1994), 837–860.

[28] Tapp K., *Differential Geometry of Curves and Surfaces,* Undergraduate Texts in Math., Springer International Publ. (2016).

[29] Umehara M., Saji K., Yamada K., *Differential Geometry of Curves and Surfaces with Singularities,* transl. by W. Rossman, Series in Algebraic and Differential Geometry **1**, World Scientific Publ., (2021).

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