Submitted to
Journal of the Mathematical Society of Japan

Dimensions of paramodular forms and compact twist modular forms with involutions

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Abstract. We give an explicit dimension formula for paramodular forms of degree two of prime level with plus or minus sign of the Atkin–Lehner involution of weight $\det^k \mathrm{Sym}(j)$ with $k \geq 3$, as well as a dimension formula for algebraic modular forms of any weight associated with the binary quaternion hermitian maximal lattices in non-principal genus of prime discriminant with fixed sign of the involution. These two formulas are essentially equivalent by a recent result of N. Dummigan, A. Pacetti. G. Rama and G. Tornaría on correspondence between algebraic modular forms and paramodular forms with signs. So we give the formula by calculating the latter. When p is odd, our formula for the latter is based on a class number formula of some quinary lattices by T. Asai and its interpretation to the type number of quaternion hermitian forms given in our previous works. On paramodular forms, we also give a dimensional bias between plus and minus eigenspaces, some list of palindromic Hilbert series, numerical examples for small p and k, and the complete list of primes p such that there is no paramodular cusp form of level p of weight 3 with plus sign. This last result has geometric meaning on moduli of Kummer surface with $(1,p)$ polarization.

1. Introduction

The paramodular forms of degree 2 of level p are Siegel modular forms associated with the moduli space of abelian surfaces with polarization of type $(1,p)$. These forms have recently gathered significant attentions from researchers. For example, denote by B the definite quaternion algebra with discriminant p and $B_A^\times$ its adelization. Then Eichler's classical work established a well-known Hecke equivariant bijection between elliptic new cusp forms of level p and algebraic modular forms on $B_A^\times$ which can be represented as vectors of some harmonic polynomials of 4 variables. Paramodular cusp forms arise in a natural extension of this correspondence to the case of symplectic groups of rank two a la Langlands. In fact, the present author proposed a conjectural correspondence between paramodular forms of prime level p and algebraic modular forms on G_A which is the adelization of the quaternion hermitian group

$$G = \{g \in M_2(B); gg^* = n(g)1_2\}, \qquad g^* = (\overline{g_{ji}}) \text{ for } g = (g_{ij})$$

with respect to the stabilizers of a maximal quaternion hermitian lattice in the non-principal genus, where $\overline{*}$ is the main involution of B. Similarly as in the classical case,

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2020 *Mathematics Subject Classification.* 11F46,11F55,11F72.

Key Words and Phrases. Paramodular forms, algebraic modular forms, dimension formula, Atkin-Lehner involution.

This work was supported by JSPS KAKENHI Grant Number JP19K03424, JP20H00115, JP23K03031, and AIM SQuaRE "Computing paramodular forms". Declaration of interest: none.

such algebraic modular forms in case of weight 0 have relations to some arithmetics of supersingular abelian surfaces as written for example in [**25**]. The conjecture regarding this correspondence was initially proposed in [**17**] for the scalar valued prime level case, later extended to the vector valued case in [**19**], and also to the square-free level case in [**27**]. These conjectures were based on dimensional equalities and numerical examples. In these works, new forms were defined in a slightly different way from the elliptic modular case, and certain concrete lifting behaviours were also conjectured. By the way, later the theory of new forms based on paramodular fixed vectors has been extensively developed in Robert–Schmidt [**42**] and Schmidt [**47**].

Recently, the above conjectural correspondence has been proved by van Hoften [**57**] and Rösner–Weissauer [**43**] independently. Subsequently, the correspondence has been refined to the correspondence between plus and minus eigenspaces of Atkin–Lehner type involutions in Dummigan–Pacetti–Rama–Tornaría [**6**].

The main objective of this paper is to give explicitly dimensions of the above algebraic modular forms and dimensions of paramodular forms of prime level with the Atkin–Lehner plus or minus sign. In these formulas, weights $\det^k Sym(j)$ of paramodular forms are restricted to the case $k \geq 3$ since there exists no algebraic modular forms corresponding to the case $k < 3$. But the formula for $k \geq 3$ is also useful for determining dimensions for weight 2 for concrete cases as explained in Poor–Yuen [**41**]. Their motivation behind [**41**] is a Shimura–Taniyama type conjecture for abelian surfaces, which is a generalization of the fact that the zeta functions of elliptic curves defined over $\mathbb{Q}$ are given by those of elliptic cusp forms of weight 2. Examples of this type of abelian surfaces were first explored in Yoshida [**62**]. It is conjectured more precisely in Brumer–Kramer [**4**] that for any abelian surface A defined over $\mathbb{Q}$ with $End(A) = \mathbb{Z}$ of conductor p, there should exist a paramodular form of weight 2 of level p whose L function gives the L function of A. Explicit calculations of paramodular forms of weight 2 of small concrete levels were carried out in [**41**], which also includes some calculation of dimensions of higher weights of plus and minus eigenvalues. The examples for $k \geq 3$ obtained from their calculations coincide with the data derived from our general formula.

The outline of our paper is as follows. In Section 2, we state our main theorems 2.1 and 2.2 on dimensions without going into details. In Section 3, we prove Theorem 2.2. In Section 4, we prove Theorem 2.1 for $p \neq 2$, 3. In Section 5, using a more direct method, we prove Theorem 2.1 for $p = 2$ and 3. We also give explicit generating functions of dimensions for scalar valued case for $p = 2$, 3 as examples. In Section 6, we quote concrete dimension formulas of algebraic modular forms used in this paper from [**14**] II and explicitly describe the group characters we require. In Section 7, we show that $(-1)^k\big(\dim S_k^+(K(p)) - \dim S_k^-(K(p))\big) \geq 0$ for any $k \geq 3$ in Theorem 7.1. In Section 8, we give several numerical examples of dimensions for small p as well as a certain list of Hilbert series with palindromic property, and also tables of dimensions for $k = 3$, 4, 5, 6, 7, 8, 10 for small p. In particular, we determine all primes p such that $\dim S_3^+(K(p)) = 0$.

2. Main Theorems

In this section, we will state our results on explicit dimension formulas for algebraic modular forms and paramodular forms that have the Atkin–Lehner plus or minus sign without going into details. Precise definitions and most proofs will be given in later sections.

First we start from algebraic modular forms. Let B be the definite quaternion algebra over $\mathbb{Q}$ with prime discriminant p. For a natural number n, define a positive definite quaternion hermitian form $h(x,y)$ of B^n by

$$h(x,y) = \sum_{i=1}^{n} x_i \overline{y_i} \qquad x = (x_i), \quad y = (y_i) \in B^n,$$

where $\overline{*}$ is the main involution of B. This is the unique positive definite quaternion hermitian form on B^n up to base change of B^n over B (See [**52**] Lemma 4.4). For $g = (g_{ij}) \in M_n(B)$, we put $g^* = {}^t\overline{g} = (\overline{g_{ji}})$. We denote by G the group of quaternion hermitian similitudes of degree n defined by

$$G = \{g \in M_n(B); gg^* = n(g)1_n, n(g) \in \mathbb{Q}_+^\times\}.$$

We denote by G_A the adelization of G, and by G_v the local factor of G_A at a place v. In particular, if we define the subgroup G^1 of G by taking elements with $n(g) = 1$, then G^1_∞ is the compact twist $Sp(n)$ of the split symplectic group $Sp(n,\mathbb{R}) \subset SL_{2n}(\mathbb{R})$ of real rank n. For an irreducible representation (ρ, V) of G^1_∞ such that $\rho(\pm 1_n) = 1$, we define a representation of G_A by

$$G_A \to G_\infty \to G_\infty/\{center\} \cong G^1_\infty/\{\pm 1_n\} \xrightarrow{\rho} GL(V). \tag{1}$$

We denote this representation also by ρ by abuse of language. For an open subgroup U of G_A, we define the space $\mathfrak{M}_\rho(U)$ of modular forms on G_A with respect to U of weight ρ by

$$\mathfrak{M}_\rho(U) = \{f : G_A \to V; f(uga) = \rho(u)f(g) \text{ for all } g \in G_A, u \in U, a \in G\}.$$

(The modular forms of this type are nowadays called algebraic modular forms in [**12**]. Classically the Brandt matrices that appeared in the theory of Eichler (and of Jacquet-Langlands) are of this sort. For higher degrees, see also [**32**], [**13**], [**14**].) When $n = 2$, as was shown in [**14**] I Theorem 3 and II Theorem and [**19**], [**22**] Theorem 5.1, the dimension of $\mathfrak{M}_\rho(U)$ is explicitly known for any ρ and for any $U = U_{pg}(p)$ or $U_{npg}(p)$ corresponding to the principal genus, or the non-principal genus of maximal lattices in B^2. For a precise definition, see the first paragraph of Section 4. The irreducible representation ρ of the compact group $G^1_\infty = Sp(2)$ corresponds to a Young diagram parameter (f_1, f_2) with $f_1 \geq f_2 \geq 0$ with $f_1 \equiv f_2 \bmod 2$ (coming from the assumption $\rho(\pm 1_2) = 1$). We write this representation by ρ_{f_1,f_2} and we write

$$\mathfrak{M}_{\rho_{f_1,f_2}}(U) = \mathfrak{M}_{f_1,f_2}(U).$$

Here in this paper we are interested in the case $U = U_{npg}(p)$ since this case has a good correspondence with paramodular forms. Our first problem is to decompose $\mathfrak{M}_{f_1,f_2}(U_{npg}(p))$ under the Atkin–Lehner involution for $U_{npg}(p)$ and give a dimension formula for each eigenspace. So we explain what the involution is in this case. Let O be a maximal order of B. We denote by π a local prime element at p of $O_p = O \otimes_{\mathbb{Z}} \mathbb{Z}_p$. We may assume that $\pi^2 = -p$. Then the Atkin–Lehner involution for $\mathfrak{M}_{f_1,f_2}(U_{npg}(p))$ is defined to be the action of the Hecke double coset $U_{npg}(p)\pi U_{npg}(p) = U_{npg}(p)\pi = \pi U_{npg}(p)$. This action gives an involution (under a natural normalization), since $\pi^2 = -p$. The representation matrix of the action of $U_{npg}(p)\pi$ on $\mathfrak{M}_{f_1,f_2}(U_{npg}(p))$ as a Hecke algebra is denoted by $R_{f_1,f_2}(\pi)$. The $+1$ and -1 eigen subspaces of $\mathfrak{M}_{f_1,f_2}(U_{npg}(p))$ of the matrix $R_{f_1,f_2}(\pi)$ will be denoted by $\mathfrak{M}^{\pm}_{f_1,f_2}(U_{npg}(p))$, respectively. We would like to give dimension formulas for these spaces. By definition, we have

$$Tr(R_{f_1,f_2}(\pi)) = \dim \mathfrak{M}^{+}_{f_1,f_2}(U_{npg}(p)) - \dim \mathfrak{M}^{-}_{f_1,f_2}(U_{npg}(p))$$

and since a formula for

$$\dim \mathfrak{M}_{f_1,f_2}(U_{npg}(p)) = \dim \mathfrak{M}^{+}_{f_1,f_2}(U_{npg}(p)) + \dim \mathfrak{M}^{-}_{f_1,f_2}(U_{npg}(p))$$

has been known in [**14**] II Theorem and [**19**] (and also reproduced in Section 6), all we should do is to give a formula for $Tr(R_{f_1,f_2}(\pi))$. This will be given below as Theorem 2.1.

Before stating the result, we explain some notation that we need in the formula.

For any $g \in G_\infty$, the characteristic polynomial $\phi(x)$ of degree 4 of the image of the embedding $g \in G_\infty \subset M_2(B \otimes_{\mathbb{Q}} \mathbb{R}) \subset M_4(\mathbb{C})$ is said to be a principal polynomial of g. The character value $Tr(\rho_{f_1,f_2}(g))$ of the representation ρ_{f_1,f_2} at $g \in G_\infty$ depends only on the principal polynomial $\phi(x)$ of $g/\sqrt{n(g)} \in G^1_\infty = Sp(2)$. Since we assumed $\rho_{f_1,f_2}(\pm 1_2) = 1$, the character values are the same for characteristic polynomials $\phi(x)$ and $\phi(-x)$. The general formula for the characters is well known and found in [**58**] Theorem 7.8 E. Here, we need the following principal polynomials

$$\begin{aligned}
&\phi_2(x) = (x-1)^2(x+1)^2, \qquad \phi_6(x) = (x^2+1)^2, \quad \phi_9(x) = x^4 + x^2 + 1, \\
&\phi_{11}(x) = x^4 + 1, \quad \phi_{13}(x) = x^4 + \sqrt{5}x^3 + 3x^2 + \sqrt{5}x + 1, \\
&\phi_{14}(x) = (x^2 + \sqrt{2}x + 1)^2, \quad \phi_{15}(x) = x^4 + \sqrt{2}x^3 + x^2 + \sqrt{2}x + 1, \\
&\phi_{16}(x) = (x^2 + \sqrt{2}x + 1)(x^2+1), \quad \phi_{17}(x) = (x^2 + \sqrt{3}x + 1)(x^2+1)
\end{aligned}$$

and denote by χ_i the character values of elements corresponding to $\phi_i(\pm x)$. (We use this strange numbering to maintain consistency with our previous works.) The polynomials ϕ_{14} to ϕ_{17} appear only when $p = 2$ or 3, and $\phi_{13}(x)$ only when $p = 5$. We denote by $h(\sqrt{-d})$ the class number of an imaginary quadratic field $\mathbb{Q}(\sqrt{-d})$ and by $B_{2,\chi}$ the second generalized Bernoulli number for the character χ corresponding to the real quadratic extension $\mathbb{Q}(\sqrt{p})/\mathbb{Q}$. By definition, we have

$$B_{2,\chi} = \frac{1}{f}\sum_{a=1}^{f} \chi(a)a^2 - \sum_{a=1}^{f} \chi(a)a,$$

where f is the conductor of χ, i.e. $f = p$ if $p \equiv 1 \bmod 4$ and $f = 4p$ if $p \equiv 3 \bmod 4$, and the latter sum is always 0 in this case since $\chi(-1) = 1$ ([**1**] p.54). Our first theorem is given below. Here the cases $p = 2$, 3 are slightly exceptional and the formula in these cases will be reproduced as Theorem 5.2 in Section 5 with a different proof. We put $\delta_{ab} = 1$ if $a = b$ and $= 0$ otherwise. We define the quadratic residue symbol $\left(\frac{d}{p}\right)$ for a prime p to be 1, -1 and 0 if p splits unramified, remains prime, and is ramified in $\mathbb{Q}(\sqrt{d})$, respectively.

THEOREM 2.1. *We assume that $n = 2$. Then an explicit formula for $Tr(R_{f_1,f_2}(\pi))$ is given for any $f_1 \geq f_2 \geq 0$ with $f_1 \equiv f_2 \bmod 2$ as follows.*
For $p \equiv 1 \bmod 4$, we have

$$Tr(R_{f_1,f_2}(\pi)) = \frac{\chi_2}{2^5 \cdot 3}\left(9 - 2\left(\frac{2}{p}\right)\right) B_{2,\chi} + \frac{h(\sqrt{-p})}{2^4}\chi_6 + \frac{h(\sqrt{-2p})}{2^3}\chi_{11} + \frac{h(\sqrt{-3p})}{2^2 \cdot 3}\left(3 + \left(\frac{2}{p}\right)\right)\chi_9 + \frac{\delta_{p5}}{5}\chi_{13}.$$

For $p \equiv 3 \bmod 4$ and $p > 3$, we have

$$Tr(R_{f_1,f_2}(\pi)) = \frac{\chi_2}{2^5 \cdot 3} B_{2,\chi} + \frac{h(\sqrt{-p})}{2^4}\left(1 - \left(\frac{2}{p}\right)\right)\chi_6 + \frac{h(\sqrt{-2p})}{2^3}\chi_{11} + \frac{h(\sqrt{-3p})}{2^2 \cdot 3}\chi_9.$$

For $p = 2$, we have

$$Tr(R_{f_1,f_2}(\pi)) = \frac{1}{48}\chi_2 + \frac{1}{16}\chi_6 + \frac{1}{6}\chi_9 + \frac{5}{16}\chi_{11} + \frac{1}{48}\chi_{14} + \frac{1}{6}\chi_{15} + \frac{1}{4}\chi_{16}.$$

For $p = 3$, we have

$$Tr(R_{f_1,f_2}(\pi)) = \frac{1}{24}\chi_2 + \frac{1}{24}\chi_6 + \frac{1}{3}\chi_9 + \frac{1}{4}\chi_{11} + \frac{1}{3}\chi_{17}.$$

Here we have $\chi_i = Tr(\rho_{f_1,f_2}(g_i))$ for any $g_i \in G^1$ whose principal polynomials are given by $\phi_i(\pm x)$ for $i = 2, 6, 9, 11, 13, 14, 15, 16, 17$ defined above.

The proof of Theorem 2.1 will be given in Section 4.

The explicit value of χ_i for each (f_1, f_2) for the case $p \neq 2, 3$ is given as follows as can be easily deduced from the classical result in [**58**] Theorem 7.8 E (See also Section 6). For the formulas for χ_{14} to χ_{17}, see Section 6. We use notation $[a_0, \ldots, a_m; m]_b$ that means the number a_i when $b \equiv i \bmod m$.

$$\chi_2 = (-1)^{f_2}\frac{(f_1+2)(f_2+1)}{2},$$

$$\chi_6 = \frac{(-1)^{(f_1+f_2)/2}}{2} \times \begin{cases} f_1 + 2 & \text{if } f_2 \equiv 0 \bmod 2, \\ -(f_2+1) & \text{if } f_2 \equiv 1 \bmod 2, \end{cases}$$

$$\chi_9 = \begin{cases} [1,0,0,-1,0,0;6]_{f_2} & \text{if } f_1 - f_2 \equiv 0 \bmod 6, \\ [-1,1,0,1,-1,0;6]_{f_2} & \text{if } f_1 - f_2 \equiv 2 \bmod 6, \\ [0,-1,0,0,1,0:6]_{f_2} & \text{if } f_1 - f_2 \equiv 4 \bmod 6, \end{cases}$$

$$\chi_{11} = \begin{cases} (-1)^{(f_1-f_2)/4}[1,-1,0,0;4]_{f_2} & \text{if } f_1-f_2 \equiv 0 \bmod 4, \\ (-1)^{(f_1-f_2-2)/4}[0,1,-1,0;4]_{f_2} & \text{if } f_1-f_2 \equiv 2 \bmod 4, \end{cases}$$

$$\chi_{13} = \begin{cases} [1,2,1,0,0,-1,-2,-1,0,0;10]_{f_2} & \text{if } f_1-f_2 \equiv 0 \bmod 10, \\ [2,0,-1,1,0,-2,0,1,-1,0;10]_{f_2} & \text{if } f_1-f_2 \equiv 2 \bmod 10, \\ [-2,-2,2,2,0,2,2,-2,-2,0;10]_{f_2} & \text{if } f_1-f_2 \equiv 4 \bmod 10, \\ [-1,1,0,-2,0,1,-1,0,2,0;10]_{f_2} & \text{if } f_1-f_2 \equiv 6 \bmod 10, \\ [0,-1,-2,-1,0,0,1,2,1,0;10]_{f_2} & \text{if } f_1-f_2 \equiv 8 \bmod 10. \end{cases}$$

By definition of $\mathfrak{M}^{\pm}_{f_1,f_2}(U_{npg}(p))$, we have

$$\dim \mathfrak{M}^{+}_{f_1,f_2}(U_{npg}(p)) = \frac{1}{2}\Big(\dim \mathfrak{M}_{f_1,f_2}(U_{npg}(p)) + Tr(R_{f_1,f_2}(\pi))\Big), \tag{2}$$

$$\dim \mathfrak{M}^{-}_{f_1,f_2}(U_{npg}(p)) = \frac{1}{2}\Big(\dim \mathfrak{M}_{f_1,f_2}(U_{npg}(p)) - Tr(R_{f_1,f_2}(\pi))\Big). \tag{3}$$

But the dimension for $\dim \mathfrak{M}_{f_1,f_2}(U_{npg}(p))$ is explicitly known for any ρ_{f_1,f_2} as given in [**14**] II (reproduced in Theorem 6.1 in Section 6). So as a corollary of the above theorem, we have explicit dimension formulas for $\dim \mathfrak{M}^{+}_{f_1,f_2}(U_{npg}(p))$ and $\dim \mathfrak{M}^{-}_{f_1,f_2}(U_{npg}(p))$. (By the same sort of calculation given in this paper, we can give an explicit formula also for $\mathfrak{M}^{\pm}_{f_1,f_2}(U_{pg}(p))$, but we omit it here.)

Now we proceed to our next theme. Since $G^1_\infty = Sp(2)$ is the compact twist of the split symplectic group $Sp(2,\mathbb{R}) \subset SL_4(\mathbb{R})$ of real rank 2, we may expect a nice correspondence between algebraic modular forms on G_A and Siegel cusp forms of degree 2. (A general principle by Langlands, and for this special case asked also by Y. Ihara [**32**].) An explicit correspondence for the non-principal genus for $n = 2$ was conjectured in our previous works [**17**], [**19**], [**27**] with precise comparison of explicit dimension formulas, and this conjecture has been proved by van Hoften in [**57**] Theorem 3 and by Rösner and Weissauer in [**43**] Proposition 12.3, independently by a completely different method. (Other parahoric cases different from the above case have been conjectured in [**22**],[**16**],[**15**], but this is another story.) The corresponding Siegel cusp forms here are so called paramodular forms. Now there also exists the Atkin–Lehner type involution on paramodular forms of level p. Recently, Dummigan, Pacetti, Rama and Tornaría generalized the above correspondence to the case between paramodular forms and algebraic modular forms with given sign of the Atkin–Lehner involution (See [**6**] Theorem 9.6 etc.). (Their theorem includes some general level cases, but here we are concerned only with prime level. See also [**35**].) We will explain more details below. For any positive integer N, we denote by $K(N)$ the paramodular subgroup of $Sp(2,\mathbb{Q})$ of level N defined by

$$K(N) = Sp(2,\mathbb{Q}) \cap \begin{pmatrix} \mathbb{Z} & N\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & N^{-1}\mathbb{Z} \\ \mathbb{Z} & N\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ N\mathbb{Z} & N\mathbb{Z} & N\mathbb{Z} & \mathbb{Z} \end{pmatrix}.$$

Let H_n be the Siegel upper half space of degree n. For any $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2,\mathbb{R})$ and a $V_{k,j}$-valued function F of $Z \in H_2$, we put

$$F|_{k,j}[g] = \rho_{k,j}(CZ+D)^{-1}F(gZ),$$

where $(\rho_{k,j}, V_{k,j})$ is the irreducible representation $\det^k Sym(j)$ of $GL_2(\mathbb{C})$ and $Sym(j)$ is the symmetric tensor representation of degree j. (When compared with algebraic modular forms explained later, we will put $f_1 = j+k-3$ and $f_2 = k-3$, assuming $k \geq 3$.) We denote by $A_{k,j}(K(N))$ the space of paramodular forms belonging to $K(N)$ of weight $\rho_{k,j}$, and by $S_{k,j}(K(N))$ its subspace of cusp forms. By definition, $S_{k,j}(K(N))$ means the vector space of $V_{k,j}$-valued holomorphic functions on H_2 such that $F|_{k,j}[\gamma] = F$ for any $\gamma \in K(N)$ and that vanish at all the cusps. When j is odd, we always have $A_{k,j}(K(N)) = 0$ by $\rho_{k,j}(-1_2) = (-1)^{2k+j} = -1$. When $j = 0$, we simply write $A_{k,0}(K(N)) = A_k(K(N))$ and $S_{k,0}(K(N)) = S_k(K(N))$. The formula for $\dim S_{k,j}(K(N))$ is known for square free N for any $k \geq 3$, $j \geq 0$ (See [**17**], [**20**] for $j = 0$, N =prime, [**19**] for $j > 0$, $k > 4$, N =prime, [**27**] for square free N with $j = 0$, $k \geq 3$ and $j > 0$ and $k > 4$, and by Dan Petersen (colloquial communication) for $k = 3$, 4, $j > 0$. The last case has been also reproved by van Hoften [**57**].

For a prime p, we put

$$\eta = \frac{1}{\sqrt{p}}\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & p & 0 & 0 \\ p & 0 & 0 & 0 \end{pmatrix}.$$

Then $\iota : F \to F|_{k,j}[\eta]$ induces an involution on $S_{k,j}(K(p))$. (This can be regarded also as the action of the Hecke operator associated with $K(p)\eta$.) We denote by $S^{\pm}_{k,j}(K(p)) \subset S_{k,j}(K(p))$ the eigenspaces of ι belonging to eigenvalues $+1$ and -1, respectively. To adjust the lifting part in the correspondence, we need the space $S_k(\Gamma_0(p))$ of elliptic cusp forms of weight k belonging to the group

$$\Gamma_0(p) = SL_2(\mathbb{Z}) \cap \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ p\mathbb{Z} & \mathbb{Z} \end{pmatrix}.$$

We denote by $S^{\pm}_{j+2}(\Gamma_0(p))$ the eigenspaces of $S_{j+2}(\Gamma_0(p))$ belonging to the eigenvalues $+1$ and -1 of the Atkin–Lehner involution W_p defined by

$$f(z)|_{j+2}W_p = p^{-(j+2)/2}z^{-(j+2)}f(-1/pz)$$

on $S_{j+2}(\Gamma_0(p))$. We put $S^{\pm,new}_{j+2}(\Gamma_0(p)) = S^{\pm}_{j+2}(\Gamma_0(p)) \cap S^{new}_{j+2}(\Gamma_0(p))$ where $S^{new}_{j+2}(\Gamma_0(p))$ is the space of new forms.

Now by Theorem 2.1 and the above mentioned result of Dummigan, Pacetti, Rama, Tornaría in [**6**], together with [**57**], [**43**], and other results, we have the following explicitly calculable formula for $\dim S^{\pm}_{k,j}(K(p))$.

THEOREM 2.2. *Let p be any prime. For $k \geq 3$ and even $j \geq 0$, we have an explicit formula for $\dim S^{\pm}_{k,j}(K(p))$. It is given by*

$$\dim S^+_{k,j}(K(p)) = \dim \mathfrak{M}^-_{j+k-3,k-3}(U_{npg}(p)) + \dim S_{k,j}(Sp(2,\mathbb{Z}))$$
$$- \dim S^{+,new}_{j+2}(\Gamma_0(p)) \times \dim S_{2k+j-2}(SL_2(\mathbb{Z})),$$
$$\dim S^-_{k,j}(K(p)) = \dim \mathfrak{M}^+_{j+k-3,k-3}(U_{npg}(p)) + \dim S_{k,j}(Sp(2,\mathbb{Z}))$$
$$- \delta_{j0} \dim S_{2k-2}(SL_2(\mathbb{Z})) - \delta_{j0}\delta_{k3}$$
$$- \dim S^{-,new}_{j+2}(\Gamma_0(p)) \times \dim S_{2k+j-2}(SL_2(\mathbb{Z})),$$

where the main terms $\dim \mathfrak{M}^{\pm}_{j+k-3,k-3}(U_{npg}(p))$ *of the right hand side are explicitly given by Theorem 2.1, Theorem 5.2, Theorem 6.1, and the other parts are explained below.*

In particular, $S_3^+(K(p))$ has a geometric meaning on the moduli of the Kummer surfaces associated to $(1,p)$ polarization (See [**10**] Theorem 1.5), and by using the above formula and estimating bounds of the class numbers and the Bernoulli numbers, we can give the complete list of primes such that $S_3^+(K(p)) = 0$ in Proposition 8.2. This includes a partial result in [**3**] and [**11**] for primes, though they also treated composite levels.

The meaning of the dimensional relations in Theorem 2.2 and its proof will be explained later in Section 3. Here we will explain why this is an explicit formula. The formula for $\dim \mathfrak{M}^{\pm}_{f_1,f_2}(U_{npg}(p))$ is deduced by Theorem 2.1 and (2) and (3) by virtue of [**14**] (reproduced as Theorem 6.1 in Section 6). The formula for $\dim S_{k,j}(Sp(2,\mathbb{Z}))$ is in [**30**], [**31**], [**56**], [**39**] (not known for the case $k = 2$ and big $j > 0$ but this case is also excluded in Theorem 2.2). The dimension $\dim S^{\pm,new}_{j+2}(\Gamma_0(p))$ is essentially in [**61**] 1.6 Theorem, and easily obtained by the formula (4), (5), (6), (7) explained below (See also [**37**]). The formula for $\dim S_k(SL_2(\mathbb{Z}))$ is well known and given by (8) below. So the above Theorem 2.2 gives a really calculable formula for any given prime p, $k \geq 3$ and even $j \geq 0$.

We review the formula for $S^{\pm,new}_{j+2}(\Gamma_0(p))$ in [**61**] for readers' convenience (see also [**37**]). For any prime p and even $k \geq 2$, as well known we have

$$\begin{aligned} \dim S^{new}_k(\Gamma_0(p)) &= \dim S^{+,new}_k(\Gamma_0(p)) + \dim S^{-,new}_k(\Gamma_0(p)) \\ &= \frac{(p-1)(k-1)}{12} + \frac{1}{4}(-1)^{k/2+1}\left(1 - \left(\frac{-1}{p}\right)\right) \\ &\quad + \frac{1}{3}[-1,0,1;3]_k \left(1 - \left(\frac{-3}{p}\right)\right) - \delta_{k2}. \end{aligned} \tag{4}$$

For $p > 3$ and even $k \geq 2$, we have

$$\dim S^{+,new}_k(\Gamma_0(p)) - \dim S^{-,new}_k(\Gamma_0(p)) = (-1)^{k/2}\frac{a_p \cdot h(\sqrt{-p})}{2} + \delta_{k2}, \tag{5}$$

where $h(\sqrt{-p})$ is the class number of $\mathbb{Q}(\sqrt{-p})$ and $a_p = 1$ if $p \equiv 1 \bmod 4$, $a_p = 2$ if $p \equiv 7 \bmod 8$, and $a_p = 4$ if $p \equiv 3 \bmod 8$. We can also write $a_p h(\sqrt{-p}) = h(-p) + h(-4p)$, where $h(-d)$ denotes the class number of quadratic order of discriminant $-d$ (not necessarily maximal), regarding $h(-d) = 0$ if $-d \equiv 2, 3 \bmod 4$.
The case $p = 2$ and 3 for even $k \geq 2$ is given by

$$\dim S_k^{+,new}(\Gamma_0(2)) - \dim S_k^{-,new}(\Gamma_0(2)) = \frac{(-1)^{k/2} - (-1)^{(k-4)(k-2)/8}}{2} + \delta_{k2}, \tag{6}$$

$$\dim S_k^{+,new}(\Gamma_0(3)) - \dim S_k^{-,new}(\Gamma_0(3)) = \delta_{k2} + \begin{cases} -1 & \text{if } k \equiv 2, 6 \bmod 12, \\ 0 & \text{if } k \equiv 4, 10 \bmod 12, \\ 1 & \text{if } k \equiv 0, 8 \bmod 12. \end{cases} \tag{7}$$

It is also well known that

$$\dim S_k(SL_2(\mathbb{Z})) = \frac{k-1}{12} + \frac{1}{4}(-1)^{k/2} + \frac{1}{3}[1, 0, -1; 3]_k - \frac{1}{2} + \delta_{k2}. \tag{8}$$

As a special case of Theorem 2.2 for $(k, j) = (3, 0)$, we have

$$\dim S_3^+(K(p)) = H - T, \quad \dim S_3^-(K(p)) = T - 1,$$

where H is the class number and T is the (G-)type number of the non-principal genus $\mathcal{L}_{npg}$ in B^2 defined in [**23**] p. 370. The formula and numerical tables for H and T were given in [**14**] II and [**24**] p.218. Numerical examples of $\dim S_4^{\pm}(K(p))$ for many primes p have been already given in [**41**] Table 4 and of course our results coincide with these values. More numerical tables will be given in Section 8.

3. Proof of Theorem 2.2

In this section, we prove Theorem 2.2, since this is much shorter than the proof of Theorem 2.1. We also explain the meaning of the dimensional relations in the theorem. Of course these relations can be read as a reflection of the Hecke equivariant bijection, but since this is obvious by [**57**], [**43**] and [**6**], we do not explain such details.

The local completions of groups $K(p)$ and $Sp(2, \mathbb{Z})$ are maximal compact subgroups of $Sp(2, \mathbb{Q}_p)$ and there is no inclusion relation between $K(p)$ and $Sp(2, \mathbb{Z})$ even if we take conjugacy. But still we may consider paramodular forms in $S_{k,j}(K(p))$ coming from $Sp(2, \mathbb{Z})$ as images of $S_{k,j}(Sp(2, \mathbb{Z}))$ and $S_{k,j}(Sp(2, \mathbb{Z}))|_k[\eta] = S_{k,j}(\eta^{-1} Sp(2, \mathbb{Z})\eta) \cong S_{k,j}(Sp(2, \mathbb{Z}))$ by trace operators. These are old forms in the sense of [**42**] and also explained in the much earlier paper [**17**]. In general, old forms of $Sp(2, \mathbb{Z})$ doubly appear in $S_{k,j}(K(p))$, but the Saito-Kurokawa lift from $S_{2k-2}(SL_2(\mathbb{Z}))$ (that exists only when $j = 0$ and k is even) exceptionally appears only once. Indeed, by [**44**] Table 1 and [**50**], [**49**], the possibility of local representations at p which have both $Sp(2, \mathbb{Z}_p)$ fixed vectors and $K(p)$ fixed vectors are (I) and (IIb) in the notation of [**42**]. Here (IIb) corresponds to the Saito–Kurokawa type (P), where $K(p)$ fixed vector is unique up to scalar with plus Atkin–Lehner sign, while (I) corresponds to the general type (G), where there are two $K(p)$ fixed vectors, one is of Atkin–Lehner plus and the other is of Atkin–Lehner minus. This can be seen in [**42**] Table A.15. (See also [**9**], [**47**], [**48**].) So the dimensions of new forms in $S_{k,j}^{\pm}(K(p))$ in the sense of [**42**] is given by

$$\dim S_{k,j}^+(K(p)) - \dim S_{k,j}(Sp(2, \mathbb{Z})),$$
$$\dim S_{k,j}^-(K(p)) - (\dim S_{k,j}(Sp(2, \mathbb{Z})) - \delta_{j0}\delta_{k,even} \dim S_{2k-2}(SL_2(\mathbb{Z}))).$$

Now, [**57**] and [**43**] claim that the general part (i.e. non-lift part) of $\mathfrak{M}_{j+k-3,k-3}(U_{npg}(p))$

and that of new forms of $S_{k,j}(K(p))$ have Hecke equivariant bijection. By [**6**] Theorem 10.1 (i), the general new part of $S^{\pm}_{k,j}(K(p))$ corresponds to the general part of $\mathfrak{M}^{\mp}_{j+k-3,k-3}(U_{npg})$ (where double sign corresponds). So we have to see the Atkin–Lehner signs of the remaining lifting part. When k is odd, there is no Saito-Kurokawa lift from $S_{2k-2}(SL_2(\mathbb{Z}))$ to $S_k(Sp(2,\mathbb{Z}))$, but there exists an Ihara lift from $S_{2k-2}(SL_2(\mathbb{Z}))$ to algebraic modular forms and this lift is injective to $\mathfrak{M}_{k-3,k-3}(U_{npg}(p))$ by [**57**] Theorem 8.2.1 (3) and [**43**] Proposition 12.2 (, though not known if this lift is exactly as constructed in [**32**], [**26**], [**19**]). By virtue of [**6**] Theorem 10.1 (ii), we see that the image of this lift appears in $\mathfrak{M}^{+}_{k-3,k-3}(U_{npg}(p))$. Also, when $k=3$ and $j=0$, $\mathfrak{M}_{0,0}(U_{npg}(p))$ contains a constant function of G_A which belongs to Atkin–Lehner plus, and this does not correspond to a cusp form. So we must subtract

$$\delta_{j0}\delta_{k,odd}\dim S_{2k-2}(SL_2(\mathbb{Z}))+\delta_{j0}\delta_{k3}.$$

from $\dim \mathfrak{M}^{+}_{j+k-3,k-3}(U_{npg}(p))$. Now by [**57**] Theorem 8.2.1 (3) and [**43**] Proposition 12.1, there is an injective Yoshida lifting from $S^{new}_{j+2}(\Gamma_0(p))\times S_{2k+j-2}(SL_2(\mathbb{Z}))$ to $\mathfrak{M}_{j+k-3,k-3}(U_{npg}(p))$. The signature of the image of the Yoshida lift is given in [**6**] Theorem 10.1 (iii). This gives the following injection

$$\begin{aligned}
S^{+,new}_{j+2}(\Gamma_0(p))\times S_{2k+j-2}(SL_2(\mathbb{Z})) &\hookrightarrow \mathfrak{M}^{-}_{j+k-3,k-3}(U_{npg}(p)),\\
S^{-,new}_{j+2}(\Gamma_0(p))\times S_{2k+j-2}(SL_2(\mathbb{Z})) &\hookrightarrow \mathfrak{M}^{+}_{j+k-3,k-3}(U_{npg}(p)).
\end{aligned}$$

Note that there is no Yoshida lift in $S_{k,j}(K(p))$ by [**49**] Lemma 2.2.1. The only remaining part now is a lift from $S^{\pm,new}_{2k-2}(\Gamma_0(p))$. For paramodular forms, this is the Gritsenko lift (paramodular version of Saito-Kurokawa lift) from $S^{\epsilon,new}_{2k-2}(\Gamma_0(p))$ to $S_k(K(p))$. Here we must have $\epsilon=(-1)^k$, since originally the elliptic modular forms here should correspond with Jacobi forms of index p of weight k and the sign of the functional equation should be $-1=(-1)^{k-1}\epsilon$. The Atkin–Lehner sign of Gritsenko lifts is $(-1)^k$ ([**9**] (12), [**48**] Theorem 5.2 (i)). So we have the following injections.

$$\begin{aligned}
S^{+,new}_{2k-2}(\Gamma_0(p)) &\hookrightarrow S^{+}_{k}(K(p)) \quad \text{for even } k,\\
S^{-,new}_{2k-2}(\Gamma_0(p)) &\hookrightarrow S^{-}_{k}(K(p)) \quad \text{for odd } k.
\end{aligned}$$

(By the way, the fact that the sign of the Saito-Kurokawa lift from $S_{2k-2}(SL_2(\mathbb{Z}))$ to $S_k(K(p))$ is plus can be proved also by [**9**] Theorem 3 Corollary.) For the compact twist, we have also a lift from $S^{(-1)^k,new}_{2k-2}(\Gamma_0(p))$ to $\mathfrak{M}_{k-3,k-3}(U_{npg}(p))$ by [**43**] Proposition 12.2. The Atkin–Lehner sign is determined by [**6**] Theorem 10.1 (ii), and we have

$$\begin{aligned}
S^{+,new}_{2k-2}(\Gamma_0(p)) &\hookrightarrow \mathfrak{M}^{-}_{k-3,k-3}(U_{npg}(p)) \quad \text{for even } k,\\
S^{-,new}_{2k-2}(\Gamma_0(p)) &\hookrightarrow \mathfrak{M}^{+}_{k-3,k-3}(U_{npg}(p)) \quad \text{for odd } k.
\end{aligned}$$

Note that the sign is reversed compared with paramodular case. So in the comparison between dimensions of $S^{\pm}_{k,j}(K(p))$ and $\mathfrak{M}^{\mp}_{j+k-3,k-3}(U_{npg}(p))$, this part apparently need not appear. Gathering above considerations, Theorem 2.2 is proved.

4. Proof of Theorem 2.1

First we explain the definition of the non-principal genus. From now on, we consider only the case $n = 2$ for simplicity. Let B be the definite quaternion algebra of prime discriminant p as before. We fix a maximal order O of B. A choice of O is not essential but we fix O that contains a prime element π with $\pi^2 = -p$ for the sake of simplicity (such O always exists). A $\mathbb{Z}$ lattice $L \subset B^2$ (a free $\mathbb{Z}$ submodule of B^2 of rank 8) is said to be a left O lattice if L is a left O module. As in Shimura [**52**], we define a norm $N(L)$ of L as the two sided O ideal spanned by $h(x,y)$ for x, $y \in L$. A left O lattice L is said to be maximal if any O-lattice M with $L \subset M$ and $N(M) = N(L)$ satisfies $M = L$. For any left O lattice L and any prime q, we write $L_q = L \otimes_{\mathbb{Z}} \mathbb{Z}_q$. A genus $\mathcal{L}$ is a set of left O lattices in B^2 such that for any L_1, $L_2 \in \mathcal{L}$, we have $L_{1,q} = L_{2,q} g_q$ for some $g_q \in G_q$ for any prime q. If a lattice in $\mathcal{L}$ is maximal, then so is any other lattice in $\mathcal{L}$. The set of norm $N(L)$ for $L \in \mathcal{L}$ is determined up to a multiplication by $\mathbb{Q}^{\times}$. In our case of discriminant p, there are two genera of maximal lattices, one is called the principal genus $\mathcal{L}_{pg}$ containing O^2, and the other is the one called non-principal genus $\mathcal{L}_{npg}$ containing a maximal lattice L with $N(L) = \pi O = O\pi$ (See [**52**] Proposition 4.6).

Now we fix any genus $\mathcal{L}$. For $L \in \mathcal{L}$, a set of lattices $\{Lg; g \in G\} \subset \mathcal{L}$ is called a class. The number of classes in $\mathcal{L}$ is called the class number of $\mathcal{L}$. For a fixed left O lattice $L \in \mathcal{L}$ and any prime q, we put

$$U(L_q) = \{g_q \in G_q; L_q g_q = L_q\},$$

and define an open subgroup $U(L)$ of G_A by

$$U = U(L) = G_\infty \prod_q U(L_q).$$

For any $g_A = (g_v) \in G_A$, we define a left O lattice $Lg_A \subset B^2$ by

$$Lg_A = \bigcap_{v<\infty} (L_v g_v \cap B^2).$$

Then by $L \to Lg_A$, we see that the class number H is equal to the number of double cosets in $U \backslash G_A / G$, and if we write the representatives of double cosets as

$$G_A = \coprod_{i=1}^{H} U g_i G \qquad \text{(disjoint)}, \tag{9}$$

then the set $\{Lg_i; i = 1, \ldots, H\}$ gives a complete set of representatives of classes in $\mathcal{L}$. If we put $\Gamma_i = g_i^{-1} U g_i \cap G$, then these are finite groups and are (metric preserving) automorphism groups of Lg_i for $1 \leq i \leq H$. Let ρ_{f_1,f_2} be the irreducible representation of $Sp(2)$ corresponding to (f_1, f_2) with $f_1 \equiv f_2 \bmod 2$ with $f_1 \geq f_2 \geq 0$. We define $\mathfrak{M}_{f_1,f_2}(U)$ as in the introduction and call an element of $\mathfrak{M}_{f_1,f_2}(U)$ an algebraic modular form of weight ρ_{f_1,f_2} belonging to U. For later use, we review another non-adelic realization of $\mathfrak{M}_{f_1,f_2}(U)$ (see [**13**] p. 230). Let V be a representation space of ρ_{f_1,f_2}. Then we have

$$\mathfrak{M}_{f_1,f_2}(U) \cong \oplus_{i=1}^{H} V^{\Gamma_i} \qquad \text{(direct sum)},$$

where we put

$$V^{\Gamma_i} = \{v \in V; \rho_{f_1,f_2}(\gamma)v = v \text{ for all } \gamma \in \Gamma_i\}.$$

The above isomorphism is given by the mapping

$$\mathfrak{M}_{f_1,f_2}(U) \ni f \to \sum_{i=1}^{H} \rho(g_i)^{-1} f(g_i) \in \oplus_{i=1}^{H} V_i^{\Gamma_i}.$$

Next we consider an action of the Hecke algebra. For $g \in G_A$, we define an action of $UgU = \coprod_j z_j U$ (disjoint) on $\mathfrak{M}_{f_1,f_2}(U)$ by

$$(R_{f_1,f_2}(UgU)f)(x) = \sum_j \rho_{f_1,f_2}(z_j) f(z_j^{-1}x).$$

(Note that the representation ρ_{f_1,f_2} is defined on G_A by (1).) We interpret the action of UgU on $\mathfrak{M}_{f_1,f_2}(U)$ into an $H \times H$ matrix action on $\oplus_{i=1}^{H} V^{\Gamma_i}$. We put $T_{ij} = G \cap g_i^{-1} U g U g_j$. Then it is clear by definition that we have $\Gamma_i T_{ij} \Gamma_j = T_{ij}$. So we regard T_{ij} as a formal sum

$$T_{ij} = \sum_h \Gamma_i h \Gamma_j = \sum_m h_m \Gamma_j.$$

Then this can be regarded as an operator of V^{Γ_j} to V^{Γ_i} by defining the action on $v_j \in V^{\Gamma_j}$ by

$$T_{ij} v_j = \sum_m \rho_{f_1,f_2}(h_m) v_j \qquad (T_{ij} = \sum_h \Gamma_i h \Gamma_j = \sum_m h_m \Gamma_j). \tag{10}$$

Then by [**13**] Lemma 1, the action of UgU is identified with the action of the matrix $(T_{ij})_{1 \le i,j \le H}$ on $\oplus_{i=1}^{H} V^{\Gamma_i}$ by $(v_1, \ldots, v_H) \to (\sum_{j=1}^{H} T_{ij} v_j)_{1 \le i \le H}$. Here we may also write

$$T_{ij} = \sum_{a \in G \cap g_i^{-1} U g U g_j / \Gamma_j} \rho_{f_1,f_2}(a) | V^{\Gamma_j}.$$

We describe the trace of the action of UgU in this setting. For $x \in G$, we define $\rho_{f_1,f_2}(x)$ by (1) through the diagonal embedding $G \to G_A$. We denote by $Tr(\rho_{f_1,f_2}(x))$ the trace of the representation $\rho_{f_1,f_2}(x)$ on whole V. The following Lemma 4.1 and Corollary 4.2 are almost trivial by definition and has been used many times for the calculation of dimension formulas of algebraic modular forms such as [**14**], [**17**], [**15**], [**19**], [**22**], but to see its connection to $SO(5)$ clearly, we explain some details for safety.

LEMMA 4.1. *For $i = 1, \ldots, H$, define $T_{ii} = G \cap g_i^{-1} U g U g_i$ as before. Then we have*

$$Tr(R_{f_1,f_2}(UgU)) = \sum_{i=1}^{H} \frac{\sum_{a\in T_{ii}} Tr(\rho_{f_1,f_2}(a))}{\#(\Gamma_i)},$$

where $\#(\Gamma_i)$ *is the cardinality of* Γ_i.

Proof. Notation being as in (10), the action of T_{ii} on $v_i \in V^{\Gamma_i}$ is given by

$$T_{ii}v_i = \sum_m \rho_{f_1,f_2}(h_m)v_i = \sum_m \sum_{\gamma\in\Gamma_i} \frac{\rho_{f_1,f_2}(h_m\gamma)v_i}{\#(\Gamma_i)} = \frac{\sum_{a\in T_{ii}} \rho_{f_1,f_2}(a)v_i}{\#(\Gamma_i)}.$$

Now we must consider a relation between the trace of the above action of T_{ii} on V^{Γ_i} and $Tr(\rho_{f_1,f_2}(a))$ on V. We define an action of T_{ii} on $v \in V$ by

$$T_{ii}v = \sum_{a\in T_{ii}} \rho_{f_1,f_2}(a)v.$$

Since $\Gamma_i T_{ii} = T_{ii}$, we see that $T_{ii}v \in V^{\Gamma_i}$ for any $v \in V$. So considering the representation matrix of T_{ii} with respect to a basis of V obtained by prolonging a basis of V^{Γ_i}, it is clear that

$$\sum_{a\in T_{ii}} Tr(\rho_{f_1,f_2}(a)|V) = \sum_{a\in T_{ii}} Tr(\rho_{f_1,f_2}(a)|V^{\Gamma_i}). \qquad \square$$

By definition (1) of the representation ρ, if we put $\hat{g} = g/\sqrt{n(g)}$ for $g \in G$, then we have $Tr(\rho_{f_1,f_2}(g)) = Tr(\rho_{f_1,f_2}(\widehat{g}))$. Here if $\Phi(x)$ is the principal polynomial of $g \in G$ (defined as the characteristic polynomial of $g \in G \subset M_2(B) \subset M_4(\mathbb{C})$), the principal polynomial of $\widehat{g}$ is given by $\phi(x) = n(g)^{-2}\Phi(\sqrt{n(g)}x)$. It is well known that the character value $Tr(\rho_{f_1,f_2}(\widehat{g}))$ depends only on the principal polynomial $\phi(x)$ of $\widehat{g}$. For any such polynomial $\phi(x)$, we write $\chi_{f_1,f_2}(\phi) = Tr(\rho_{f_1,f_2}(\widehat{g})) = Tr(\rho_{f_1,f_2}(g))$ where $g \in G$ is as above. Since we assumed $\rho_{f_1,f_2}(\pm 1_2) = 1$, we have $Tr(\rho_{f_1,f_2}(\widehat{g})) = Tr(\rho_{f_1,f_2}(-\widehat{g}))$, so we have $\chi_{f_1,f_2}(\phi(x)) = \chi_{f_1,f_2}(\phi(-x))$. Let $G(\phi)$ be the set of all elements g of G such that $\phi(x)$ or $\phi(-x)$ is the principal polynomial of $\widehat{g}$. We also put

$$Tr(R_{f_1,f_2}(UgU), \phi) = \sum_{i=1}^{H} \frac{\sum_{a\in T_{ii}\cap G(\phi)} Tr(\rho_{f_1,f_2}(a))}{\#(\Gamma_i)}.$$

Then by Lemma 4.1, we have the following corollary.

Corollary 4.2. *We have*

$$Tr(R_{f_1,f_2}(UgU), \phi) = \chi_{f_1,f_2}(\phi) \times \sum_{i=1}^{H} \frac{\#(T_{ii} \cap G(\phi))}{\#(\Gamma_i)}. \qquad (11)$$

This corollary is very useful since the right hand side is a product of the term depending on the representation and the term independent of the representation. We have $\chi_{0,0}(\phi) = 1$ for any ϕ, so the general formula is reduced to the case of the trivial representation as far as the formula for $Tr(R_{0,0}(UgU))$ is given as a sum of explicit terms for each ϕ. (This is usually true for any trace formula).

Now we specialize our consideration to the case when $\mathcal{L} = \mathcal{L}_{npg}$. We consider the double coset $R(\pi) = U_{npg}(p)\pi U_{npg}(p) = U_{npg}(p)\pi = \pi U_{npg}(p)$, where π is identified with an element of G_A by the diagonal embedding. We write $R_{f_1,f_2}(R(\pi)) = R_{f_1,f_2}(\pi)$. Our aim is to give a formula for $Tr(R_{f_1,f_2}(\pi))$ on $\mathfrak{M}_{f_1,f_2}(U_{npg}(p))$. In this case, for any element $a \in T_{ii}$, we have $n(a) = p$. By Corollary 4.2, the main part of the calculation of $Tr(R_{f_1,f_2}(\pi))$ for ρ_{f_1,f_2} is to give the value of the right hand side of (11) for $(f_1, f_2) = (0, 0)$ for each f. This calculation is based on two things. One is the relation $2T = H + Tr(R_{0,0}(\pi))$ proved in [**23**] Theorem 3.6 between the class number H of $\mathcal{L}_{npg}$, the type number T of $\mathcal{L}_{npg}$(the definition will be reviewed soon), and $Tr(R_{0,0}(\pi))$. The other is an equality between the type number T of $\mathcal{L}_{npg}$ and the class number of quinary lattices of some genus of det $= 2p$ proved in [**24**] Theorem 4.5 for $p \neq 2$. (The case $p = 2$ will be treated separately in Section 5.) In [**24**], our calculation of T depends on the class number formula of Asai in [**2**]. But this time, we should be more careful since we must calculate the right hand side of Corollary 4.2 for each principal polynomial. So we will compare the class number formula in [**2**] and a formula of $2T$ given by $H + Tr(R_{0,0}(\pi))$ for the Hecke operator of G for each principal polynomial. For that purpose, we will review some details on the type number below.

We fix a set of representatives of $U_{npg}(p)\backslash G_A/G$ as in (9) and for a fixed $L \in \mathcal{L}_{npg}$, put $L_i = Lg_i$ $(i = 1, \ldots, H)$. We define the right order R_i of L_i by

$$R_i = \{z \in M_2(B); L_i z \subset L_i\}.$$

The classical meaning of the type number is the number of isomorphism classes of maximal orders in an algebra. But in our case, all maximal orders in $M_2(B)$ are conjugate to $M_2(O)$ since the class number of $M_2(B)$ is 1 by the strong approximation theorem of $SL_2(B)$ (See [**33**], [**34**]). But here, instead of $GL_2(B)$ conjugacy, we consider the G conjugacy of R_i. The (G-)type number T of $\mathcal{L}_{npg}$ is defined to be a number of G-conjugacy classes in $\{R_i\}_{1\leq i\leq H}$. In [**23**] we proved that $2T = H + Tr(R_{0,0}(\pi))$. So we have $T \leq H \leq 2T$. For each principal polynomial Φ of an element appearing in $g_i^{-1}(U_{npg}(p) \cup U_{npg}(p)\pi)g_i \cap G$ for some i, put $\phi(x) = n(g)^{-2}\Phi(\sqrt{n(g)}x)$ as before. Then the contribution to T of the "$\phi(x)$ and $\phi(-x)$-part" in $T = (H+Tr(R_{0,0}(\pi)))/2$ is defined to be

$$T(\phi) = \frac{1}{2}\sum_{i=1}^{H} \frac{\#(\Gamma_i \cap G(\phi)) + \#(g_i^{-1}U_{npg}(p)\pi g_i \cap G(\phi))}{\#(\Gamma_i)}, \tag{12}$$

where we write $\Gamma_i = g_i^{-1}U_{ngp}(p)g_i \cap G$. Now we will compare this to the class number formula of quinary lattices. Let $\mathcal{M} = \mathcal{M}(1, p)$ be the genus of lattices with determinant $2p$ in the 5 dimensional positive definite quadratic space W defined in [**24**] p. 215. We have shown in [**24**] that the class number of $\mathcal{M}$ is the type number T of $\mathcal{L}_{npg}$ if $p \neq 2$. The class number formula for $\mathcal{M}$ can be explained as follows. We denote by M_i $(i = 1, \ldots, T)$ a complete set of representatives of classes in $\mathcal{M}$. Although Asai in [**2**] used the orthogonal group $O(W)$ to define a class, we have $O(W) = SO(W)\cup(-1_5)SO(W)$ for the special orthogonal group $SO(W)$, and the class number for both $O(W)$ and $SO(W)$

are the same and the formulas are identical, so we explain the $SO(W)$ formulation. For each $1 \le i \le T$, we denote by $Aut(M_i)$ the group of automorphisms of M_i in $SO(W)$. Then we have the trivial identity

$$T = \sum_{i=1}^{T} \frac{\#(Aut(M_i))}{\#(Aut(M_i))}.$$

Let $\widetilde{h}(x)$ be a principal polynomial of an element of $SO(W)$ of degree 5. It is of the shape

$$\widetilde{h}(x) = (x-1)h(x)$$

for some degree 4 monic reciprocal polynomial $h(x)$. We denote by $SO(W, \widetilde{h})$ the set of elements of $SO(W)$ whose principal polynomial is $\widetilde{h}(x)$. We put

$$T(\widetilde{h}) = \sum_{i=1}^{T} \frac{\#(Aut(M_i) \cap SO(W, \widetilde{h}))}{\#(Aut(M_i))}.$$

Naturally we have $T = \sum_{\widetilde{h}} T(\widetilde{h})$. In Asai [**2**], he gave a formula for $T(\widetilde{h})$ for each $\widetilde{h}$ as a contribution of the case $C_{\pm i}$ where $C_{\pm i}$ means a principal polynomial of $\pm\widetilde{g}$ of some element $\widetilde{g} \in O(W)$. Here one of $\{\widetilde{g}, -\widetilde{g}\}$ is an element of $SO(W)$ and Asai's formula is the same as the contribution of $SO(W)$ belonging to one of C_i or C_{-i}. Now we will interpret each $T(\widetilde{h})$ into the "ϕ-part" $T(\phi)$ of T defined by (12). To explain this, we review some parts of [**24**]. The even Clifford algebra $C_2(W)$ of W can be identified with $M_2(B)$, W with a linear subspace of $M_2(B)$ over $\mathbb{Q}$, and the even Clifford group Γ_2 with G. The inner automorphism $W \ni w \mapsto gwg^{-1} \in W$ for $g \in G$ induces an isomorphism $G/\{\mathbb{Q}^\times 1_2\} \cong SO(W)$ ([**24**] p. 210). To compare with the formulation by G, we must describe $Aut(M_i)$ in terms of G. Fix L to be a representative of $\mathcal{L}_{npg}$ and R the right order of L. Then we have

Lemma 4.3 ([**24**] Lemma 4.1 and Corollary 4.4). *There exists a lattice $M \in \mathcal{M}$ such that for any $g_A = (g_v) \in G_A$, we have $g_v M_v g_v^{-1} = M_v$ if and only if $g_v R_v g_v^{-1} = R_v$ for any finite place v of $\mathbb{Q}$, where we put $M_v = M \otimes_{\mathbb{Z}} \mathbb{Z}_v$ and $R_v = R \otimes_{\mathbb{Z}} \mathbb{Z}_v$.*

For a representative $g_i = (g_{i,v})$ of the double coset in (9), we put

$$R_i = g_i^{-1} R g_i = \cap_{v<\infty}(g_{i,v}^{-1} R_v g_{i,v} \cap M_2(B)).$$

This is the right order of $L_i = Lg_i$. Changing numbers i if necessary, we assume that $R_1, \ldots, R_T$ are representatives of types (i.e. G conjugacy classes). Then $M_i = g_i^{-1} M g_i$ ($i = 1, \ldots, T$) are also the representatives of classes in $\mathcal{M}$. By Lemma 4.3, we see that any element of $Aut(M_i)$ comes from $g \in G$ with $gg_i^{-1}Rg_i g^{-1} = g_i^{-1} R g_i$. This means that $Rg_i g g_i^{-1}$ is a two sided ideal of R, and it is well known that any two sided ideal of R_v is spanned by $\mathbb{Q}_v^\times$ and besides π if $v = p$ up to $R_v^\times$. By definition we have $U_{npg}(p) = G_\infty \prod_{v<\infty}(R_v^\times \cap G_v)$, and since $\mathbb{Q}_A^\times = \mathbb{Q}^\times \mathbb{R}_+^\times \prod_{v<\infty} \mathbb{Z}_v^\times$ and $(\mathbb{R}^\times 1_2) \prod_{v<\infty}(\mathbb{Z}_v^\times 1_2) \subset U_{npg}(p)$, this means that $g_i(mg)g_i^{-1} \in U_{npg}(p) \cup U_{npg}(p)\pi$ for some $m \in \mathbb{Q}^\times$. Writing the natural projection of G to $G/\mathbb{Q}^{\times}1_2 \cong SO(W)$ by ι, we have $\iota(m1_2) = 1_5$, so we have

$$Aut(M_i) = \iota(g_i^{-1}(U_{npg}(p) \cup U_{npg}(p)\pi)g_i \cap G).$$

Here by definition we have

$$g_i^{-1}U_{npg}(p)g_i \cap G = \Gamma_i,$$

so $\iota(\Gamma_i)$ is always contained in $Aut(M_i)$. The problem is the part $g_i^{-1}U_{npg}(p)\pi g_i \cap G$. To explain this part more clearly, we review the relation of the type number and the class number of $\mathcal{L}_{npg}$ written in [**24**]. We write the right orders of $L_i = Lg_i$ by R_i ($i = 1, \ldots, H$). Assume that $gR_ig^{-1} = R_j$ for some $g \in G$. By the same reason we explained above, this means that

$$g_i(mg)g_j^{-1} \in U_{npg}(p) \cup U_{npg}(p)\pi$$

for some $m \in \mathbb{Q}^{\times}$. So R_i is G conjugate to R_j if and only if

$$g_i^{-1}(U_{npg}(p) \cup U_{npg}(p)\pi)g_j \cap G \neq \emptyset. \tag{13}$$

Now we fix i and see which $j \neq i$ satisfies (13). If $i \neq j$, then $g_i^{-1}U_{npg}(p)g_j \cap G = \emptyset$ since $\{g_i\}_{1 \leq i \leq h}$ is a set of representatives of $U\backslash G_A/G$. The condition $g_i^{-1}U_{npg}(p)\pi g_j \cap G \neq \emptyset$ is equivalent to

$$\pi U_{npg}(p)g_jG = U_{npg}(p)\pi g_jG = U_{npg}(p)g_iG. \tag{14}$$

Since

$$U_{npg}(p)g_iG \subset G_A = \pi G_A = \coprod_{l=1}^{H} \pi U_{npg}(p)g_lG = \coprod_{l=1}^{H} U_{npg}(p)\pi g_lG \quad \text{(disjoint)},$$

there exists unique j that satisfies (14). This means that we have two cases. The first one is the case that $j = i$ in (14) and we have

$$U_{npg}(p)\pi g_iG = U_{npg}(p)g_iG.$$

This means that $g_i^{-1}U_{npg}(p)\pi g_i \cap G \neq \emptyset$. Besides, if $g,\ g' \in g_i^{-1}U_{npg}(p)\pi g_i \cap G$, then writing $g = g_i^{-1}u\pi g_i$ and $g' = g_i^{-1}u'\pi g_i$ for $u,\ u' \in U_{npg}(p)$, we have $g^{-1}g' = g_i^{-1}\pi^{-1}u^{-1}u'\pi g_i \in G$, but we have $\pi^{-1}u^{-1}u'\pi \in \pi^{-1}U_{npg}(p)\pi = U_{npg}(p)$, so $g^{-1}g' \in \Gamma_i$. This means that

$$g_i^{-1}U_{npg}(p)\pi g_i \cap G = g_0\Gamma_i$$

for some $g_0 \in G$. We may assume that the set of such i is $\{1, \ldots, t\}$. Then for these i, we have

$$Aut(M_i) = \iota(\Gamma_i) \cup \iota(g_0\Gamma_i).$$

This is a disjoint union in $SO(W)$. Indeed, for any element $g_0 \in G$ with $n(g_0) = p$, we have $\iota(g_0) \notin \iota(G^1)$, because for any element $g_0' \in \mathbb{Q}^\times G^1$, we have $n(g_0') \in (\mathbb{Q}^\times)^2$. So we have $\#(Aut(M_i)) = 2\#(\iota(\Gamma_i))$. Next we consider the second case that $j \neq i$ in (14). This case, we have a pair of right orders R_i and R_j which are G conjugate one another, and we may take $(i, j) = (i, i + (H - t)/2)$ for $i = t + 1, \ldots, (H - t)/2$. Here we have

$$Aut(M_i) = \iota(\Gamma_i).$$

We also have $T = t + (H - t)/2$ and $t = Tr(R_{0,0}(\pi))$. Now for a principal polynomial $\widetilde{h}$, we have

$$2T(\widetilde{h}) = 2\sum_{i=1}^{T} \frac{\#(\iota(\Gamma_i) \cap SO(W, \widetilde{h})) + \#(\iota(g_i^{-1}U_{npg}(p)\pi g_i \cap G) \cap SO(W, \widetilde{h}))}{\#(\iota(\Gamma_i)) + \#(\iota(g_i^{-1}U_{npg}(p)\pi g_i \cap G))}.$$

For $i = 1, \ldots, t$, the denominator is $2\#(\iota(\Gamma_i))$. For $i = t+1, \ldots, T-t$, the denominator is $\#(\iota(\Gamma_i))$. Besides, when $gg_iRg_i^{-1}g^{-1} = g_jRg_j^{-1}$ for some (i, j) with $1 \leq i \neq j \leq H$ and $g \in G$, we have $\Gamma_i \cong \Gamma_j$, so $\#(\iota(\Gamma_i)) = \#(\iota(\Gamma_j))$. Now we compare principal polynomials $f(x)$ of $\widehat{g} = g/\sqrt{n(g)}$ for $g \in G$ and $\widetilde{h}(x)$ of $\iota(g)$. If the eigenvalues of an element of $Sp(2)$ is ϵ_1, ϵ_2, ϵ_1^{-1}, ϵ_2^{-1}, then eigenvalues of the image in $SO(5)$ is 1, $\epsilon_1\epsilon_2$, $\epsilon_1\epsilon_2^{-1}$, $\epsilon_1^{-1}\epsilon_2$, $\epsilon_1^{-1}\epsilon_2^{-1}$, so for the principal polynomial

$$f(x) = x^4 + c_1x^3 + c_2x^2 + c_1x + 1$$

of an element $\widehat{g} \in G^1_\infty$, the principal polynomial of $\iota(g) \in SO(W)$ is given by

$$\widetilde{\phi}(x) = (x - 1)h(x), \quad h(x) = x^4 - (c_2 - 2)x^3 + (c_1^2 - 2c_2 + 2)x^2 - (c_2 - 2)x + 1.$$

Here for $\phi(x)$ and $\phi(-x)$, we have the same $\widetilde{\phi}(x)$. This is clear from the above calculation and also by $Ker(\iota|Sp(2)) = \{\pm 1_2\}$. We have $\#(\Gamma_i) = 2\#(\iota(\Gamma_i))$ and

$$\#(\Gamma_i \cap G(\phi)) = 2(\#(\iota(\Gamma_i) \cap SO(W, \widetilde{\phi})),$$
$$\#(g_i^{-1}U_{npg}(p)\pi g_i \cap G(\phi)) = 2\#(\iota(g_i^{-1}U_{npg}(p)\pi g_i \cap G) \cap SO(W, \widetilde{\phi})),$$

so we can rewrite the above formula for $2T(\widetilde{\phi})$ as

Lemma 4.4. *We have*

$$2T(\widetilde{\phi}) = \sum_{i=1}^{H} \left(\frac{\#(\Gamma_i \cap G(f))}{\#(\Gamma_i)} + \frac{\#(g_i^{-1}U_{npg}(p)\pi g_i \cap G(f))}{\#(\Gamma_i)} \right) = 2T(\phi).$$

This Lemma means that $T(\phi)$ can be obtained from Asai's formula for $T(\widetilde{\phi})$. The principal polynomials of elements of Γ_i and of elements of $g_i^{-1}U_{npg}(p)\pi g_i \cap G$ are different, because the constant terms $n(g)^2$ are different. But sometimes the principal polynomials for $\widehat{g}$ are the same. So both contribute to the same $T(\widetilde{\phi})$. But this does not matter, since the character of the elements depend only on $\widetilde{\phi}(x)$, or $\phi(\pm x)$.

For more precise calculations, it would be safer to give a list of tables of principal polynomials $\Phi(x)$, $\phi(x)$ and $\widetilde{\phi}(x)$ of $g \in G$, $\widehat{g}$, and $\iota(g)$. We consider possible principal polynomials of elements in $U_{npg}(p) \cup U_{npg}(p)\pi$. The principal polynomials of elements in Γ_i has been listed in [**14**] I p. 590 and also in Section 6 after Theorem 6.1. So we see the case $U_{npg}(p)\pi$. For any prime $q \neq p$, we have $U(L_q) = G_q \cap GL_2(O_q)$. At p, the prime element π of O we defined before is also a prime of $O_p = O \otimes_{\mathbb{Z}} \mathbb{Z}_p$. We put

$$G_p^* = \left\{ g \in M_2(B_p); g \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} g^* = n(g) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}.$$

Then we have $G_p \cong G_p^*$ by $GL_2(O_p)$ conjugation, and

$$U_{npg}(L_p) \cong G_p^* \cap \begin{pmatrix} O_p & \pi^{-1}O_p \\ \pi O_p & O_p \end{pmatrix}^{\times}.$$

Hence by the condition that g is conjugate to $U_{npg}\pi$ and $g^* = pg^{-1}$, the principal polynomial $F(x)$ of an element g appearing in $g_i^{-1}U\pi g_i \cap G$ should be of the form

$$\Phi(x) = x^4 + pax^3 + pbx^2 + p^2ax + p^2$$

for some rational integers a, b. The possible principal polynomials Φ of $g \in U_{npg}(p)\pi$ and corresponding $\phi(x)$ are as in Table 1.

Table 1. Polynomials for $U_{npg}\pi$.

p	$\Phi(x)$	$\phi(x)$
general	$(x^2 - p)^2$	$(x-1)^2(x+1)^2$
general	$(x^2 + p)^2$	$(x^2+1)^2$
general	$x^4 + p^2$	$x^4 + 1$
general	$x^4 - px^2 + p^2$	$x^4 - x^2 + 1$
general	$x^4 + px^2 + p^2$	$x^4 + x^2 + 1$
$p = 5$	$x^4 \pm 5x^3 + 15x^2 \pm 25x + 25$	$x^4 \pm \sqrt{5}x^3 + 3x^2 \pm \sqrt{5}x + 1$
$p = 3$	$(x^2 \pm 3x + 3)^2$	$(x^2 \pm \sqrt{3}x + 1)^2$
$p = 3$	$(x^2 \pm 3x + 3)(x^2 + 3)$	$(x^2 \pm \sqrt{3}x + 1)(x^2 + 1)$
$p = 2$	$(x^2 \pm 2x + 2)^2$	$(x^2 \pm \sqrt{2}x + 1)^2$
$p = 2$	$(x^2 \pm 2x + 2)(x^2 + 2)$	$(x^2 \pm \sqrt{2}x + 1)(x^2 + 1)$
$p = 2$	$x^4 \pm 2x^3 + 2x^2 \pm 4x + 4$	$x^4 \pm \sqrt{2}x^3 + x^2 \pm \sqrt{2}x + 1$

This list can be obtained by using the inequalities in the following lemma.

LEMMA 4.5. *We fix p. Then a polynomial*

$$\Phi(x) = x^4 + pax^3 + pbx^2 + p^2ax + p^2$$

is a principal polynomial of an element g of G only if the following conditions are satisfied.

$$pa^2 \leq 16,$$
$$4b \leq pa^2 + 8,$$

$$pa^2 - 4 \leq 2b,$$
$$4pa^2 \leq (b+2)^2.$$

In particular, for a fixed p, there are only finitely many integers a, $b \in \mathbb{Z}$ such that $\Phi(x)$ can be a principal polynomial of an element of G.

PROOF. Since $g \in G$ and $g/\sqrt{p} \in Sp(2)$, if we put

$$h(X) = p^{-2}\Phi(\sqrt{p}X) = X^4 + a\sqrt{p}X^3 + bX^2 + a\sqrt{p}X + 1,$$

then the roots of $h(x) = 0$ should be of absolute value 1. For any root x of $h(X) = 0$, writing $y = x + x^{-1}$, we have

$$y^2 + a\sqrt{p}y + b - 2 = 0. \tag{15}$$

Since $|x| = 1$, we see y is a real number, so we have $pa^2 - 4(b-2) \geq 0$, which gives the second inequality. If we denote by α and β the roots of (15), then these are real and the roots of the equation

$$X^2 - \alpha X + 1 = 0 \text{ or } X^2 - \beta X + 1 = 0$$

are imaginary or ± 1. In both cases, we have $\alpha^2, \beta^2 \leq 4$. So we have

$$8 \geq \alpha^2 + \beta^2 = pa^2 - 2(b-2), \qquad 0 \leq (4-\alpha^2)(4-\beta^2) = (b+2)^2 - 4pa^2,$$

which leads to the third and the fourth inequalities. □

Now assume that a, $b \in \mathbb{Z}$ satisfy the condition in the above lemma. For any prime, we have $|a| \leq 4/\sqrt{p} \leq 4/\sqrt{2}$ so $|a| = 0$, 1, 2. If $a = 0$, then we have $-2 \leq b \leq 2$. If $(a,b) = (0, \pm 2)$, then we have $\Phi(x) = x^4 \pm 2px^2 + p^2 = (x^2 \pm p)^2$. If $(a,b) = (0, \pm 1)$, then $F(x) = x^4 \pm px^2 + p^2$. If $(a,b) = (0,0)$, then $\Phi(x) = x^4 + p^2$. If $|a| = 1$, then by the second and the third condition, we have $p/2 - 2 \leq b \leq p/4 + 2$, so we have $p \leq 16$. If $p = 7$, then $1.5 \leq b \leq 3.75$, so $b = 2$ or 3. But the last condition $28 \leq (b+2)^2$ is not satisfied for $b = 2$ and 3. In the same way, for $p = 11$, we have $b = 4$ and for $p = 13$, we have $b = 5$, violating the last condition. So we have $p = 2$, 3, or 5. For $p = 5$, we have $b = 1$, 2, 3. By $20 \leq (b+2)^2$, we have $b = 3$. So we have $\Phi(x) = x^4 \pm 5x^3 + 15x^2 \pm 5x + 25$. For $p = 3$, we have $b = 0$, 1, 2. Since $12 \leq (b+2)^2$, we have $b = 2$. This gives $\Phi(x) = x^4 \pm 3x^2 + 6x^2 \pm 9x + 9 = (x^2 \pm 3x + 3)(x^2 + 3)$. For $p = 2$, we have $b = -1$, 0, 1, 2. By $8 \leq (b+2)^2$, we have $b = 1$ or $b = 2$. Then $\Phi(x) = x^4 \pm 2^3 + 2x^2 \pm 4x + 4$ for $b = 1$ and $\Phi(x) = (x^2 \pm 2x + x)(x^2 + 1)$ for $b = 2$. Finally we see the case $|a| = 2$. This can happen only when $p = 2$ or 3 since $2 \leq 4/\sqrt{p}$. When $p = 3$, then $b = 4$ or 5. Since $48 \leq (b+2)^2$, we should have $b = 5$. So $\Phi(x) = x^4 \pm 6x^3 + 15x^2 \pm 18x + 9 = (x^2 \pm 3x + 3)^2$. When $p = 2$, we have $b = 2$, 3, 4, and by $32 \leq (b+2)^2$, we have $b = 4$. This means that $\Phi(x) = x^4 \pm 4x^3 + 8x^2 \pm 8x + 4 = (x^2 \pm 2x + 2)^2$. So we have only polynomials $\Phi(x)$ that we have listed.

Of course Table 1 is a possible list and we are not claiming that these really occur always.

Now we list all the possible principal polynomials appearing in the calculation of H and T in the following table. Here, in the first column of Table 2, we write the principal polynomials $\phi(x)$ of $\hat{g} \in G^1_\infty$ coming from $g \in U_{npg}(p)$ and $U_{npg}(p)\pi$. The second column is the principal polynomial $\widetilde{\phi}(x)$ of elements of $SO(5)$ corresponding to the image of $\hat{g}$ by the projection $Sp(2) \to SO(5)$. The notation $C_{\pm i}$ is Asai's notation in [**2**] Lemma 4.16 to indicate principal polynomials. As we mentioned, he used $O(5)$ formulation instead of $SO(5)$, which causes the suffix $\pm$ of $C_{\pm i}$, and for $SO(5)$, we need one of C_i or C_{-i}.

Table 2. Table of principal polynomials of $SO(5)$.

$\phi(x)$, $Sp(2)$	$\widetilde{\phi}(x)$, $SO(5)$	Asai
$(x \pm 1)^4$	$(x-1)^5$	C_1
$(x-1)^2(x+1)^2$	$(x-1)(x+1)^4$	C_2
$(x^2+1)^2$	$(x-1)^3(x+1)^2$	C_{-3}
$(x \pm 1)^2(x^2+1)$	$(x-1)(x^2+1)^2$	C_4
$x^4 \pm 2\sqrt{2}x^3 + 4x^2 \pm 2\sqrt{2}x + 1$	$(x-1)^3(x^2+1)$	C_5
$(x \pm 1)^2(x^2 \mp x + 1)$	$(x-1)(x^2+x+1)^2$	C_6
$(x^2 \pm x + 1)^2$	$(x-1)^3(x^2+x+1)$	C_7
$(x \pm 1)^2(x^2 \pm x + 1)$	$(x-1)(x^2-x+1)^2$	C_8
$x^4 \pm 2\sqrt{3}x^3 + 5x^2 \pm 2\sqrt{3}x + 1$	$(x-1)^3(x^2-x+1)$	C_9
x^4+1	$(x-1)(x+1)^2(x^2+1)$	C_{-10}
x^4-x^2+1	$(x-1)(x+1)^2(x^2+x+1)$	C_{-11}
$(x^2+x+1)(x^2-x+1)$	$(x-1)(x+1)^2(x^2-x+1)$	C_{-12}
$x^4 \pm \sqrt{2}x^3 + 2x^2 \pm \sqrt{2}x + 1$	$(x-1)(x^4+1)$	C_{13}
$x^4 \pm x^3 + x^2 \pm x + 1$	$(x-1)(x^4+x^3+x^2+x+1)$	C_{14}
$x^4 \pm \sqrt{5}x^3 + 3x^2 \pm \sqrt{5}x + 1$	$(x-1)(x^4-x^3+x^2-x+1)$	C_{15}
$(x^2+1)(x^2 \pm x + 1)$	$(x-1)(x^4-x^2+1)$	C_{16}
$x^4 \pm \sqrt{6}x^3 + 3x^2 \pm \sqrt{6}x + 1$	$(x-1)(x^2+1)(x^2-x+1)$	C_{17}
$x^4 \pm \sqrt{2}x^3 + x^2 \pm \sqrt{2}x + 1$	$(x-1)(x^2+1)(x^2+x+1)$	C_{18}
$x^4 \pm \sqrt{3}x^3 + 2x^2 \pm \sqrt{3}x + 1$	$(x-1)(x^2+x+1)(x^2-x+1)$	C_{-19}

The polynomials in the table below are those that do not appear in the usual dimension formula of the class number H of the non-principal genus coming from elements of some Γ_i and newly needed for the type number.

As can be seen, these appear only when $p = 2$, 3 or 5. Actually, the contributions to T from C_9, C_{13}, C_{17}, C_{18} are known to be all zero in Asai [**2**] Theorem 4.17 for $d = p$ with $p \neq 2$ in the notation there, so we do not need these. For $p = 2$, we do not use Asai's result anyway. The case $p = 2$, 3 will be treated separately in the next section and the case $p = 5$ is included in the proof below.

PROOF OF THEOREM 2.1. Here we prove Theorem 2.1 under the assumption that $p \neq 2$, 3. By Corollary 4.2, we have

$$Tr(R_{f_1,f_2}(\pi), \phi) = \chi_{f_1,f_2}(\phi) Tr(R_{0,0}(\pi), \phi),$$

Table 3. Principal polynomials of $Sp(2)$.

$\hat{g} \in Sp(2)$	$g \in G$	Asai
$x^4 \pm 2\sqrt{2}x^3 + 4x^2 \pm 2\sqrt{2}x + 1$	$(x^2 \pm 2x + 2)^2$	C_5
$x^4 \pm 2\sqrt{3}x^3 + 5x^2 \pm 2\sqrt{3}x + 1$	$(x^2 \pm 3x + 3)^2$	C_9
$x^4 \pm \sqrt{2}x^3 + 2x^2 \pm \sqrt{2}x + 1$	$(x^2 \pm 2x + 2)(x^2 + 2)$	C_{13}
$x^4 \pm \sqrt{5}x^3 + 3x^2 \pm \sqrt{5}x + 1$	$x^4 \pm 5x^3 + 15x^2 \pm 25x + 25$	C_{15}
$x^4 \pm \sqrt{6}x^3 + 3x^2 \pm \sqrt{6}x + 1$	$x^4 + 6x^3 + 18x^2 + 36x + 36$	C_{17}
$x^4 \pm \sqrt{2}x^3 + x^2 \pm \sqrt{2}x + 1$	$x^4 \pm 2x^3 + 2x^2 \pm 4x + 4$	C_{18}
$x^4 \pm \sqrt{3}x^3 + 2x^2 \pm \sqrt{3}x + 1$	$(x^2 \pm 3x + 3)(x^2 + 3)$	C_{-19}

so if we calculate $Tr(R_{0,0}(\pi), \phi)$ for each principal polynomial ϕ, then we obtain the case of general (f_1, f_2) by the formula for $\chi_{f_1,f_2}(\phi)$ given later. So we assume that $f_1 = f_2 = 0$ and ρ_{f_1,f_2} is the trivial representation. All we should do is to calculate a contribution to $Tr(R_{0,0}(\pi)) = 2T - H$ for each principal polynomial $\phi(\pm x)$ or $\widetilde{\phi}(x)$. When $p \neq 2$, the formula for $T(\phi)$ of $\mathcal{L}_{npg}$ is equal $T(\widetilde{\phi})$ by Lemma 4.4. To calculate $T(\widetilde{\phi})$, we use the class number formula given in Asai [**2**] Theorem 4.17. On the other hand, contribution of each conjugacy classes to the class number H has been given in Theorem in [**14**] II (which is reproduced in the appendix for prime discriminant case). The list of principal polynomials are given in Table 2, so we calculate $T(\tilde{\phi})$ for each C_i. For $p \neq 2$, 3, the conjugacy classes C_5, C_9, C_{13}, C_{17}, C_{18}, C_{-19} in Tables 2 and 3 does not appear. So the remaining cases are C_1, C_2, C_{-3}, C_4, C_6, C_7, C_8, C_{-10}, C_{-11}, C_{12} C_{14}, C_{15}, and C_{16}. The corresponding character values are χ_1, χ_2, χ_6, χ_3, χ_4, χ_7, χ_5, χ_{11}, χ_{12}, χ_9, χ_{10}, χ_{13} and χ_8, respectively. We must subtract contribution of each conjugacy class to H from $2T$. This contribution to H is given by Theorem 6.1 in Section 6, and under the assumption that $p \neq 2$, 3, only conjugacy classes C_1, C_{-3}, C_7, C_{-10}, C_{-11} and C_{14} in Table 2 contribute to H. The contribution of each C_i to T is given in [**2**] Theorem 4.17, so we will see these. In the notation of Theorem 4.17, we have $d = p$ and $\det(L) = 2p$ and ϵ_v is the Hasse invariant of L for each place v in the sense of Serre [**51**]. For our lattice, we have $\epsilon_p = \epsilon_2 = -1$ and $\epsilon_q = 1$ for any other prime q, assuming that $p \neq 2$. Here we have $\epsilon_\infty = 1$ since L is positive definite and if q is prime to the discriminant $2p$ then of course we have $\epsilon_q = 1$. The fact that $\epsilon_2 = -1$ can be checked by the fact that the lattice in question at 2 is isometric to

$$2p \perp \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \perp \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and this is diagonalized to $\mathrm{diag}(2p, 2, -2, 2, -2)$. We have $\epsilon_p = -1$ by the product formula of the Hasse invariant. In [**2**], the notations $\epsilon(n)$ and $\omega(n)$ mean, as in [**51**] 3.2, elements in $\{0, 1\}$ such that

$$\epsilon(n) \equiv (n-1)/2 \bmod 2,$$
$$\omega(n) \equiv (n^2-1)/8 \bmod 2.$$

So for a prime p we have

$$\left(\frac{-1}{p}\right) = (-1)^{\epsilon(p)}, \qquad \left(\frac{2}{p}\right) = (-1)^{\omega(p)}.$$

We also define

$$\mathfrak{M}(m) = \frac{h(-m)}{2^t \cdot u}$$

where $h(-m)$ is the class number of $\mathbb{Q}(\sqrt{-m})$, u is the order of the unit group of the ring of integers of $Q(\sqrt{-m})$ and t is the number of the divisors of the discriminant of $\mathbb{Q}(\sqrt{-m})$. Below, we use freely the notation used in [**2**] from p. 284 to p. 290 for each conjugacy class C_i. For the general formulas for $h(L, C_i)$ we use here, please refer the corresponding pages of [**2**] since we do not copy them here to avoid complication. For C_1, we have $d = p$, $\epsilon_p = -1$, and $e = 1$, so the contribution to T is $2/(2^8 \cdot 3^2 \cdot 5)(p^2-1) = (p^2-1)/5760$ and the contribution to H is $(p^2-1)/2880$ as in Theorem 6.1 from the coefficient of χ_1, so for $2T - H$, we have

$$2\frac{p^2-1}{5760} - \frac{p^2-1}{2880} = 0.$$

For C_2, in the notation of [**2**] p. 284 (2), we have $e = \delta = 1$ and the contribution to T is

$$\begin{array}{ll} \dfrac{1}{2^6 \cdot 3}\left(9 - 2\left(\frac{2}{p}\right)\right) & \text{if } p \equiv 1 \bmod 4, \\ \dfrac{1}{2^6 \cdot 3} B_{2,\chi} & \text{if } p \equiv 3 \bmod 4, \end{array}$$

and the contribution to H of finite order elements with principal polynomial $(x-1)^2(x+1)^2$ is 0 if $p \neq 2$, so the contribution to $2T - H$ is the twice of the above values. For C_3, we have $\delta = 1$ or p and $S_1 = S_2 = \emptyset$ in p.285 of [**2**]. We have $a = 1$ and $-\left(\frac{2}{p}\right)$ for $\delta = 1$, p, respectively. (By definition, the product on the empty set is 1 in the definition of a in [**2**].) We have

$$\mathfrak{M}(1) = \frac{1}{8}, \quad \mathfrak{M}(p) = h(-p) \times \begin{cases} 1/8 & p \equiv 1 \bmod 4, \\ 1/4 & \text{if } p \equiv 3 \bmod 4. \end{cases}$$

For $\delta = 1$, we have $(-1)^{\epsilon(p)+\omega(1)} = \left(\frac{-1}{p}\right)$. So in this case we have

$$A = \frac{1}{4}\left(4 + \left(\frac{-1}{p}\right)\right))$$

where A is as in [**2**] p. 285 line 6, and

$$\begin{aligned} \mathfrak{M}(1) \times \frac{1}{6} \times A \times \frac{1}{2}\left(p + \epsilon_p\left(\frac{-1}{p}\right)\right) &= \frac{1}{8} \times \frac{1}{12}\left(p - \left(\frac{-1}{p}\right)\right) \times \left(1 + \frac{1}{2^2}\left(\frac{-1}{p}\right)\right) \\ &= \frac{1}{2^5 \cdot 3}\left(p - \left(\frac{-1}{p}\right)\right) + \frac{1}{2^7 \cdot 3}\left(p\left(\frac{-1}{p}\right) - 1\right). \end{aligned}$$

The contribution of the part $\delta = 1$ to $2T$ is the four times of the above and is equal to

$$\frac{1}{2^3\cdot 3}\left(p-\left(\frac{-1}{p}\right)\right)+\frac{1}{2^5\cdot 3}\left(p\left(\frac{-1}{p}\right)-1\right),$$

which is exactly the coefficients of χ_6 in H in Theorem 6.1. So these parts cancel with each other. The remaining part is the case $\delta=p$. When $\delta=p\equiv 1 \bmod 4$, We have

$$a(-1)^{\epsilon(p)+\omega(p)}=-\left(\frac{2}{p}\right)\left(\frac{-2}{p}\right)=-1,$$

so

$$A=2^{-2}(4-1)=\frac{3}{2^2}$$

in this case. If $p\equiv 3 \bmod 4$, then we see $1+a(-1)^{\omega(p)}=0$, so we have

$$A=\frac{1}{2^3\cdot 3}\left(1-\left(\frac{2}{p}\right)\right).$$

Gathering these data, we have

$$h(L,C_3,p)=h(-p)\times\begin{cases}2^{-6} & \text{if } p\equiv 1 \bmod 4,\\ 2^{-6}\left(1-\left(\frac{2}{p}\right)\right) & \text{if } p\equiv 3 \bmod 4.\end{cases}$$

So the contribution $4h(L,C_3,p)$ to $2T$ is exactly the coefficient of χ_6 in Theorem 2.1. In the case of C_4, both conditions (i) and (ii) of (4) in [**2**] p. 285 do not hold, so the contribution to T is 0, The same for H. So this term gives 0. In the case C_6 either, the conditions (i) and (ii) do not hold and the contributions to T and H are 0. For C_7, noting that $\left(\frac{p}{3}\right)=\left(\frac{-3}{p}\right)$, we have $A=3+\left(\frac{-3}{p}\right)$ in (7) of [**2**] p. 286 and

$$h(L,C_7)=\frac{1}{2^5\cdot 3^2}\left(p-\left(\frac{-3}{p}\right)\right)\left(3+\left(\frac{-3}{p}\right)\right).$$

We see that $4h(L,C_7)$ (the contribution to $4T$) is exactly the coefficient of χ_7 of H in Theorem 6.1, so the contribution to $2T-H$ is 0. In the case of C_8, the condition (ii) in (8) of [**2**] p.286 is not satisfied and the contribution to T is 0. The contribution of $\phi_5(x)$ to H is also 0, so no contribution to $2T-H$. In the case of C_{-10}, we have $\delta=1$ or p in [**2**] p, 287 (10) (i) and if $\delta=1$, then by the condition (ii), the contribution is 0 unless $\left(\frac{2}{p}\right)=-1$, i,e, unless $p\equiv 3,5 \bmod 8$. In this case, we have $a=0$ and $\mathfrak{M}(2)=h(\sqrt{-2})/2\cdot 2=1/4$. So the contribution of the case $\delta=1$ to $2T$ is $4h(L,C_{10},2)=4\times(1/2^2)\times(1/4)=1/4$. for $p\equiv\pm 3 \bmod 8$. But the coefficient of χ_{11} in H is $(1/8)(1-(2/p))$ so this is 0 and 1/4 for $p\equiv\pm 1 \bmod 8$ and $p\equiv\pm 3 \bmod 8$, respectively. So the contribution of these parts to $2T-H$ is 0. The case $\delta=p$ remains, and in this case, (ii) in [**2**] p. 287 (10) means no condition. We have $\mathfrak{M}(2p)=h(\sqrt{-2p})/(2^2\cdot 2)$ and $h(L,C_{10},2p)=h(\sqrt{-2p})/2^5$, so the contribution to $2T$ is $4h(L,C_{10},2p)=h(\sqrt{-2p})/2^3$. This is nothing but the coefficient of χ_{11} in Theorem 2.1. In the case of C_{11}, in [**2**] p. 287 (11), we have $a=b=0$, $c=-1$. We have $\delta=1$ or p. If $\delta=p$, then $A=1-1=0$, so $h(L,C_{11},p)=0$. So we assume

$\delta = 1$ hereafter. Then $A = 1 - \left(\frac{p}{3}\right) = 1 - \left(\frac{-3}{p}\right)$, and by the condition (ii), we have $-1 = \left(\frac{3}{p}\right)$. We have $\mathfrak{M}(1) = 1/8$, so the contribution to $2T$ is

$$4h(L, C_{11}, 1) = \frac{1}{12}\left(1 - \left(\frac{-3}{p}\right)\right)$$

if $(3/p) = -1$ and 0 otherwise. In other words, this is equal to $1/6$ if $p \equiv 5 \bmod 12$ and 0 otherwise. On the other hand, the coefficient of χ_{12} in H is

$$\frac{1}{24}\left(1 - \left(\frac{3}{p}\right) + \left(\frac{-1}{p}\right) - \left(\frac{-3}{p}\right)\right).$$

This is also $1/6$ if $p \equiv 5 \bmod 12$ and 0 otherwise. So the contribution to $2T - H$ is 0. In the case of C_{12}, the coefficient of χ_9 in H is 0. In [**2**] p. 288 (12), we always have $a = b = 0$ and $c = -1$ in our setting. By the condition (i), we have $\delta = 1$, 3, p or $3p$. If $3 \nmid \delta$, we have $A = 0$ by definition, so we should have $\delta = 3$ or $3p$. If $\delta = 3$, this contradicts to the condition (ii) since $\epsilon_p = -1$ but $(3^2/p) = 1$. So the only possible case is $\delta = 3p$. In this case we have $A = 2$. We have

$$\mathfrak{M}(3p) = h(\sqrt{-3p}) \times \begin{cases} 2^{-3} & \text{if } p \equiv 1 \bmod 4, \\ 2^{-4} & \text{if } p \equiv 3 \bmod 4. \end{cases}$$

We first consider the case $p \equiv 1 \bmod 4$. Then since we have $(-1)^{\omega(3p)} = -\left(\frac{2}{p}\right)$, we have

$$h(L, C_{12}, 3p) = \frac{h(\sqrt{-3p})}{2^4 \cdot 3}\left(3 + \left(\frac{2}{p}\right)\right).$$

If $p \equiv 3 \bmod 4$, then

$$h(L, C_{12}, 3p) = \frac{h(\sqrt{-3p})}{2^4 \cdot 3}.$$

So the contribution to $2T - H$ given by $4h(L, C_{12}, 3p)$ is as in Theorem 2.1. In the case of C_{14}, then in [**2**] p. 289 (14), we have $a = 0$ and if $p \neq 5$, then we should have $(5/p) = -1$, that is, $p \equiv 2$, $3 \bmod 5$ and in this case $h(L, C_{14}) = 1/10$. If $p = 5$, then $h(L, C_{14}) = 1/20$ So the contribution to $2T$ is $4h(L, C_{14}) = 1/5$, $2/5$ and 0 for $p = 5$, $p \equiv 2, 3 \bmod 5$, $p \equiv \pm 1 \bmod 5$, respectively. This cancels with the corresponding coefficients of χ_{10} in H. In the case of C_{-15}, by [**2**] p, 289 (15), the contribution to T vanishes if $p \neq 5$ and is $1/10$ if $p = 5$. So the contribution to $2T$ for $p = 5$ is $1/5$ and 0 otherwise. On the other hand, the contribution to H is 0 since the polynomial $x^4 \pm \sqrt{5}x^3 + 3x^2 \pm \sqrt{5}x + 1$ does not appear for H. So the contribution to $2T - H$ is as stated in Theorem 2.1 for χ_{13}. In the case of C_{16}, by [**2**] p. 289 (16), there is no contribution to T, By Theorem 6.1, there is no contribution from χ_8 to H either. So the contribution to $2T - H$ is 0. □

5. The case $p = 2$ and 3

In this section, we give a formula for $Tr(R_{f_1,f_2}(\pi))$ for general (f_1, f_2) for $p = 2$ and $p = 3$ by a different method. Applying [**6**] and Theorem 2.2, this also gives a dimension formula for $S^{\pm}_{k,j}(K(p))$ for $p = 2$, 3 and $k \geq 3$ as before. For the scalar valued cases $\dim S^{\pm}_k(K(2))$ and $\dim S^{\pm}_k(K(3))$, this means that we reprove the formula that has already been known in [**28**], [**5**], [**29**].

For the case $p = 3$, we can calculate $Tr(R_{f_1,f_2}(\pi))$ by the same method as in the proof of Theorem 2.1, but if $p = 2$, we have a problem since the correspondence between T and the class number of quinary lattices explained before is not known in [**23**]. So we need a different method. Here we use more direct method for both $p = 2$ and 3. We know that the class number $H = 1$ for the non-principal genus $\mathcal{L}_{npg}$ for both $p = 2$ and 3 (See [**14**]), so $G_A = U_{npg}G$ and $\Gamma_1 = G \cap U_{npg}$. We have $\#(\Gamma_1) = 1920$ for $p = 2$ and 720 for $p = 3$. Besides, we have $R(\pi) = U_{npg}\pi$ for $\pi \in O$, and since $\pi 1_2 \in G$, we see that

$$G \cap U_{npg}\pi = \pi(G \cap U_{npg}) = \pi\Gamma_1.$$

So in order to obtain $Tr(R(\pi))$, all we should do is to count the number of elements $\gamma \in \pi\Gamma_1$ for each fixed principal polynomial. We can concretely describe these elements by a direct calculation. First we give a table of principal polynomials and number of corresponding elements in $\pi\Gamma$, and then state our theorem

Lemma 5.1. *For a fixed polynomial $\Phi(x)$, the number of elements $\gamma \in \pi\Gamma_1$ such that $\Phi(x)$ is their principal polynomial is given in the second column in Tables 4 and 5 for $p = 2$, 3. We put $\phi(x) = p^{-2}\Phi(x/\sqrt{p})$ as before. The last column is the character values $Tr(\rho_{f_1,f_2}(g))$ of elements of the corresponding row, where the notation χ_i is explained in Section 6.*

Theorem 5.2. *For $p = 2$, we have*

$$Tr(R_{f_1,f_2}(\pi)) = \frac{1}{48}\chi_2 + \frac{1}{16}\chi_6 + \frac{1}{6}\chi_9 + \frac{5}{16}\chi_{11} + \frac{1}{48}\chi_{14} + \frac{1}{6}\chi_{15} + \frac{1}{4}\chi_{16}.$$

For $p = 3$, we have

$$Tr(R_{f_1,f_2}(\pi)) = \frac{1}{24}\chi_2 + \frac{1}{24}\chi_6 + \frac{1}{3}\chi_9 + \frac{1}{4}\chi_{11} + \frac{1}{3}\chi_{17}.$$

Theorem 5.2 is an easy corollary of Lemma 5.1, so we prove the lemma.

Before proving Lemma 5.1, we write a formula for principal polynomial by using matrix coefficients.

Lemma 5.3. *For $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$ with $gg^* = n(g)1_2$, the principal polynomial $\Phi(x)$ of g is given by*

$$\Phi(x) = x^4 - (Tr(a) + Tr(d))x^3 + (Tr(a)Tr(d) - N(b + \overline{c}) + 2n(g))x^2 - (Tr(a) + Tr(d))n(g)x + n(g)^2.$$

Table 4. Number of elements for $p = 2$.

$\Phi(x)$	$\pi\Gamma_1$	$\phi(x)$	character values
$(x^2-2)^2$	40	$(x-1)^2(x+1)^2$	χ_2
$(x^2+2)^2$	120	$(x^2+1)^2$	χ_6
x^4+2x^2+4	320	x^4+x^2+1	χ_9
x^4+4	600	x^4+1	χ_{11}
$(x^2\pm 2x+2)^2$	40	$(x^2\pm\sqrt{2}x+1)^2$	χ_{14}
$x^4\pm 2x^3+2x^2\pm 4x+4$	320	$x^4\pm\sqrt{2}x^3+x^2+\sqrt{2}x+1$	χ_{15}
$(x^2\pm 2x+2)(x^2+2)$	480	$(x^2\pm\sqrt{2}x+1)(x^2+1)$	χ_{16}

Table 5. Number of elements for $p = 3$.

$\Phi(x)$	$\pi\Gamma_1$	$\phi(x)$	character values
$(x^2-3)^2$	30	$(x-1)^2(x+1)^2$	χ_2
$(x^2+3)^2$	30	(x^2+1)	χ_6
$(x^2+3x+3)(x^2+3)$	120	$(x^2+\sqrt{3}x+1)(x^2+1)$	χ_{17}
$(x^2-3x+3)(x^2+3)$	120	$(x^2-\sqrt{3}x+1)(x^2+1)$	χ_{17}
x^4+9	180	x^4+1	χ_{11}
x^4+3x^2+9	240	x^4+x^2+1	χ_9

We also have

$$\Phi(x) = x^4 + Tr(g)x^3 + (1/2)(Tr(g)^2 - Tr(g^2))x^2 - Tr(g)n(g)x + n(g)^2.$$

PROOF. Put $\Phi(x) = x^4 + a_1x^3 + a_2x^2 + a_3x + a_4$. Here g^* has the same principal polynomial $\Phi(X)$, and since $g^* = n(g)g^{-1}$, we have $x^4\Phi(n(g)x^{-1}) = \Phi(x)$. So we have $a_4 = n(g)^2$ and $a_3 = a_1n(g)$. Now put $y = x + n(g)x^{-1}$. Then we have

$$x^{-2}\Phi(x) = y^2 + a_1y + a_2 - 2n(g).$$

Since

$$g + n(g)g^{-1} = g + g^* = \begin{pmatrix} Tr(a) & b+\bar{c} \\ c+\bar{b} & Tr(d) \end{pmatrix},$$

and the components of this matrix are commutative, the characteristic polynomial of $g + n(g)g^{-1}$ is given by

$$y^2 - (Tr(a) + Tr(d))y + Tr(a)Tr(d) - N(b+\bar{c}),$$

Comparing this with $x^{-2}\Phi(X)$, we prove the first formula. Noting that

$$Tr(g)^2 = Tr(a)^2 + Tr(d)^2 + 2Tr(a)Tr(d), \quad Tr(g^2) = Tr(a^2) + Tr(d^2) + 2Tr(bc),$$
$$N(b+\bar{c}) = N(b) + N(c) + Tr(bc),$$
$$N(a) + N(b) = N(c) + N(d) = n(g),$$

$$Tr(a)^2 - Tr(a^2) = 2N(a), \quad Tr(d)^2 - Tr(d^2) = 2N(d),$$

we have the second formula. □

PROOF OF LEMMA 5.1. When $p = 2$, a maximal order O is taken to be

$$O = \mathbb{Z} + \mathbb{Z}i + \mathbb{Z}j + \mathbb{Z}\frac{1+i+j+k}{2},$$

where $i^2 = j^2 = -1$, $ij = -ji = k$. We have

$$O^\times = \{\pm 1, \pm i, \pm j, \pm k, (\pm 1 \pm i \pm j \pm k)/2\}.$$

The quaternion hermitian matrix corresponding to a lattice in the non-principal genus $\mathcal{L}_{npg}$ is given by

$$gg^* = \begin{pmatrix} 2 & r \\ -r & 2 \end{pmatrix} \text{ where } r = i - k, \quad g = \begin{pmatrix} 1 & -1 \\ 0 & r \end{pmatrix}.$$

The unit group Γ_1 is a set of matrices $g^{-1}\epsilon g$ such that $\epsilon \in M_2(O)$ and $\epsilon g g^* \epsilon^* = gg^*$. This is given by the following set of elements (1) to (5) (see [**16**] p.592. Here we slightly modify the notation, but essentially the same).

$$(1) \quad \begin{pmatrix} r^{-1}a & -r^{-1}aa_0 \\ r^{-1}a & r^{-1}aa_0 \end{pmatrix}, \qquad (2) \quad \begin{pmatrix} r^{-1}a & r^{-1}aa_0 \\ -r^{-1}a & r^{-1}aa_0 \end{pmatrix},$$
$$(3) \quad \begin{pmatrix} a & 0 \\ 0 & aa_0 \end{pmatrix}, \qquad (4) \quad \begin{pmatrix} 0 & aa_0 \\ a & 0 \end{pmatrix},$$
$$(5) \quad \begin{pmatrix} (1+r^{-1}x)a & r^{-1}xaa_0 \\ r^{-1}xa & (1+r^{-1}x)aa_0 \end{pmatrix},$$

where $a \in O^\times$, $a_0 \in \{\pm 1, \pm i, \pm j, \pm k\}$ and $x \in \{-i, k, (\pm 1 - i \pm j + k)/2\}$. The number of elements are 192,192,192,192 and 1152 for each (1), (2), (3), (4) and (5), respectively. We have $|\Gamma_1| = 1920$.

For $\pi = r = i - k$, by calculating the principal polynomials of elements in $\pi\Gamma_1$, we have the result in Lemma 5.1. Actual calculation is quite long, so we sketch the point. Using Lemma 5.3, we can count the elements having each principal polynomial. For example, if $\gamma \in \Gamma_1$ is as in (4) above, then for $g = \pi\gamma = r\gamma$, we have

$$g = \begin{pmatrix} 0 & raa_0 \\ ra & 0 \end{pmatrix}.$$

So we have $Tr(g) = 0$ and $Tr(g^2) = Tr(raa_0ra) + Tr(raraa_0) = 2Tr((ra)^2a_0)$. Here $N(ra) = 2$ and the norm 2 elements in O are the following 24 elements.

$$\pm(1 \pm i), \pm(1 \pm j), \pm(1 \pm k), \pm(i \pm j), \pm(i \pm k), \pm(j \pm k).$$

Here we have $(ra)^2 = \pm 2i, \pm 2j, \pm 2k$ for the first 12 elements, (each for two a), and -2 for the last 12 elements. First we assume $(ra)^2 = -2$. Then $Tr((ra)^2a_0) = 0$ for $a_0 = \pm i$,

$\pm j$, $\pm k$. In this case we have $\Phi(X) = x^4 + 4$. This is for $12 \times 6 = 72$ elements. For $a_0 = \pm 1$ we have $Tr((ra)^2 a_0) = \mp 2$. This case, we have $\Phi(X) = (x^2 \mp 2)^2$ for 12 elements for each. Next assume that $(ra)^2 = \pm 2i$. Then we have $Tr(g^2) \neq 0$ only when $a_0 = \pm i$. For $a_0 = \pm i$, we have $(ra)^2 a_0 = \pm 2$. So we have $\Phi(x) = x^4 + 4$ for $4 \times 6 = 24$ elements and $\Phi(x) = x^4 + 4x^2 + 4$ for 4 elements, $\Phi(x) = x^4 - 4x^2 + 4$ for 4 elements. This is the same for $(ra)^2 = \pm 2j$ and $\pm 2k$. So as a total, the principal polynomials of $\pi\gamma$ for $\gamma \in \Gamma_1$ in (4) is given by the following table.

$x^4 + 4$	144
$x^4 + 4x^2 + 4 = (x^2 + 2)^2$	24
$x^4 - 4x^2 + 4 = (x^2 - 2)^2$	24

The case (4) is the simplest case, but continuing the same sort of (more complicated) calculation for (1), (2), (3), (5), we obtain the number of elements for each principal polynomial as in the following table. Here the numbers for (1) and (2) coincide since elements in (1) and (2) are conjugate by diag$(1, -1)$.

principal polynomial	(3)	(4)	(1)	(2)	(5)	total	total/1920
$x^4 + 4$	24	144	54	54	324	600	5/16
$x^4 + 4x^3 + 8x^2 + 8x + 4$	12	0	1	1	6	20	1/96
$x^4 - 4x^3 + 8x^2 - 8x + 4$	12	0	1	1	6	20	1/96
$x^4 - 2x^3 + 4x^2 - 4x + 4$	48	0	24	24	144	240	1/8
$x^4 + 2x^2 + 4x^2 + 4x + 4$	48	0	24	24	144	240	1/8
$x^4 + 4x^2 + 4$	48	24	6	6	36	120	1/16
$x^4 - 4x^2 + 4$	0	24	2	2	12	40	1/48
$x^4 - 2x^3 + 2x^2 - 4x + 4$	0	0	20	20	120	160	1/12
$x^4 + 2x^3 + 2x^2 + 4x + 4$	0	0	20	20	120	160	1/12
$x^4 + 2x^2 + 4$	0	0	40	40	240	320	1/6
total	192	192	192	192	1152	1920	1

From this table, we obtain Table 4. We omit the details.

When $p = 3$, we can take

$$O = \mathbb{Z} + \mathbb{Z}\frac{1+\alpha}{2} + \mathbb{Z}\beta + \mathbb{Z}\frac{(1+\alpha)\beta}{2}, \qquad \alpha^2 = -3, \beta^2 = -1, \alpha\beta = -\beta\alpha.$$

We have

$$O^\times = \{\pm 1, \pm\beta, (\pm 1 \pm \alpha)/2, (\pm 1 \pm \alpha)\beta/2\}.$$

The quaternion hermitian matrix corresponding to a lattice in $\mathcal{L}_{npg}$ is given by

$$gg^* = \begin{pmatrix} 3 & -(1+\beta)\alpha \\ (1+\beta)\alpha & 3 \end{pmatrix}, \quad g = \begin{pmatrix} 1 & 1+\beta \\ 0 & \alpha \end{pmatrix}.$$

Then $U_{npg}(p)\pi \cap G$ is given by the following elements.

$$\begin{aligned}
&(1) \quad \begin{pmatrix} \beta\alpha a & 0 \\ 0 & \alpha a a_0 \end{pmatrix} \quad a \in O^\times, a_0 \in \{1, (-1 \pm \alpha)/2\}, \\
&(2) \quad \begin{pmatrix} 0 & \beta\alpha a a_0 \\ \alpha a & 0 \end{pmatrix} \quad a \in O^\times, a_0 \in \{1, (-1 \pm \alpha)/2\}, \\
&(3) \quad \begin{pmatrix} \epsilon_1 & -\epsilon_1 \overline{c_2} \epsilon_2 \\ c_2 & \epsilon_2 \end{pmatrix} \quad \begin{array}{l} c_2 \in O, N(c_2) = 2, \epsilon_1, \epsilon_2 \in O^\times, \\ \epsilon_1 \equiv -(1-\beta)c_2 \bmod \alpha, \epsilon_2 \equiv c_2(1+\beta) \bmod \alpha, \end{array} \\
&(4) \quad \begin{pmatrix} c_2 & \epsilon_2 \\ \epsilon_1 & -\epsilon_2 \overline{c_2} \epsilon_2 \end{pmatrix} \quad \begin{array}{l} c_2 \in O, N(c_2) = 2, \epsilon_1, \epsilon_2 \in O^\times, \\ \epsilon_1 \equiv (1+\beta)c_2 \bmod \alpha, \epsilon_2 \equiv c_2(1+\beta) \bmod \alpha. \end{array}
\end{aligned}$$

The number of elements in (1), (2), (3), (4) are 36, 36, 324, 324, respectively. By similar method as in the case $p = 2$, we have the following number of principal polynomials in each case.

principal polynomial	(1)	(2)	(3)	(4)	total	total/720
$(x^2+3)^2$	12	0	18	0	30	1/24
$(x^2+3)(x^2+3x+3)$	12	0	36	72	120	1/6
$(x^2+3)(x^2-3x+3)$	12	0	36	72	120	1/6
$(x^2-3)^2$	0	12	18	0	30	1/24
x^4+9	0	0	144	36	180	1/4
x^4+3x^2+9	0	24	72	144	240	1/3
total	36	36	324	324	720	1

□

For $p = 2$, 3, we can compare dimensions of scalar valued paramodular cusp forms of weight k and algebraic modular forms of weight $\rho_{k-3,k-3}$ ($k \geq 3$) with involutions directly by [**28**], [**5**], [**29**] without using [**6**]. We note that $S_{j+2}(\Gamma_0(p)) = 0$ for $j = 0$ and $p = 2$, 3, so the Yoshida lift part does not appear in the comparison and the relation becomes much more simple. For algebraic modular forms for $p = 2$, 3, we have the following result from the calculation of $Tr(R(\pi))$ given as above and the result in [**14**] (see Section 6).

$$\sum_{f=0}^{\infty} \dim \mathfrak{M}_{f,f}(U_{npg}(2)) t^f = \frac{(1+t^5)(1+t^{20})}{(1-t^4)(1-t^6)(1-t^8)(1-t^{10})},$$

$$\sum_{f=0}^{\infty} \dim \mathfrak{M}^+_{f,f}(U_{npg}(2)) t^f = \frac{1+t^{25}}{(1-t^4)(1-t^6)(1-t^8)(1-t^{10})},$$

$$\sum_{f=0}^{\infty} \dim \mathfrak{M}^-_{f,f}(U_{npg}(2)) t^f = \frac{t^5(1+t^{15})}{(1-t^4)(1-t^6)(1-t^8)(1-t^{10})},$$

$$\sum_{f=0}^{\infty} \dim \mathfrak{M}_{f,f}(U_{npg}(3)) t^f = \frac{(1+t^5)(1+t^{15})}{(1-t^3)(1-t^4)(1-t^6)(1-t^{10})},$$

$$\sum_{f=0}^{\infty} \dim \mathfrak{M}^+_{f,f}(U_{npr}(3)) t^f = \frac{(1+t^8)(1+t^{15})}{(1-t^4)(1-t^6)^2(1-t^{10})},$$

$$\sum_{f=0}^{\infty} \dim \mathfrak{M}^{-}_{f,f}(U_{npr}(3))t^f = \frac{(t^3+t^5)(1+t^{15})}{(1-t^4)(1-t^6)^2(1-t^{10})}.$$

On the other hand, for paramodular forms of level 2 and 3, the dimensions for plus and minus space (including non-cusp forms) has been given in [**28**], [**5**], [**29**]. The structure of cusps of paramodular varieties is well known (see for example [**18**] Proposition 5.1 to 5.3). For level p, there are two one-dimensional cusps isomorphic to the compactification of $SL_2(\mathbb{Z})\backslash H_1$ that cross at one point. For a fixed k, the modular forms of one variable on this boundary consist of a pair of elliptic cusp forms of $SL_2(\mathbb{Z})$, which gives one to plus part and other to minus part of $A_k(K(p))$, and the Eisenstein series besides, which belong to the plus part. This fact and the surjectivity of generalized Siegel Φ-operator by Satake [**45**] show that $\dim S_k^{\pm}(K(p))$ is given by

$$\dim S_k^{+}(K(p)) = \dim A_k^{+}(K(p)) - \dim A_k(SL_2(\mathbb{Z})),$$
$$\dim S_k^{-}(K(p)) = \dim A_k^{-}(K(p)) - \dim S_k(SL_2(\mathbb{Z})).$$

Here we give concrete results only for cusp forms. For the dimensions of the whole space including non cusp forms, see [**29**] p. 113.

$$\sum_{k=0} \dim S_k^{+}(K(2))t^k = \frac{t^8+t^{10}+t^{12}-t^{20}+t^{23}+t^{33}}{(1-t^4)(1-t^6)(1-t^8)(1-t^{12})},$$
$$\sum_{k=0} \dim S_k^{-}(K(2))t^k = \frac{t^{11}+t^{20}+t^{21}+t^{22}+t^{24}-t^{32}}{(1-t^4)(1-t^6)(1-t^8)(1-t^{12})},$$
$$\sum_{k=0}^{\infty} \dim S_k^{+}(K(3))t^k = \frac{t^6+t^8+t^{10}+t^{12}-t^{18}+t^{21}+t^{23}+t^{31}}{(1-t^4)(1-t^6)^2(1-t^{12})},$$
$$\sum_{k=0}^{\infty} \dim S_k^{-}(K(3))t^k = \frac{t^9+t^{11}+t^{18}+t^{19}+t^{20}+t^{22}+t^{24}-t^{30}}{(1-t^4)(1-t^6)^2(1-t^{12})}.$$

Observing these, we have the following dimensional relations without assuming the results of [**6**]. (Actually these relations had been obtained before independently of [**6**].)

PROPOSITION 5.4. *For $p=2$ and $p=3$, and any integer $k \geq 3$, we have*

$$\dim S_k^{-}(K(p)) - \dim S_k(Sp(2,\mathbb{Z})) + \delta_{k,even}\cdot \dim S_{2k-2}(SL_2(\mathbb{Z}))$$
$$= \dim \mathfrak{M}^{+}_{k-3,k-3}(U_{npr}(p)) - \delta_{k,3} - \delta_{k,odd}\cdot \dim S_{2k-2}(SL_2(\mathbb{Z})).$$
$$\dim S_k^{+}(K(p)) - \dim S_k(Sp(2,\mathbb{Z})) = \dim \mathfrak{M}^{-}_{k-3,k-3}(U_{npr}(p)),$$

where $(\delta_{k,even},\delta_{k,odd}) = (1,0)$ if k is even and $=(0,1)$ if k is odd. Here the left hand side are the dimensions of new forms in the sense of [**42**]. *The term $\dim S_{2k-2}(SL_2(\mathbb{Z}))$ is an adjustment for Saito–Kurokawa lift or Ihara lift (which is the compact version of Saito–Kurokawa lift though it had been introduced much earlier).*

6. Appendix on dimensions and characters

We review formulas for $\dim \mathfrak{M}_{f_1,f_2}(U_{npg}(p))$ from [**14**] II p. 696 Theorem for readers convenience. We also give explicit formulas of character values of irreducible representations of $Sp(2)$ at elements of $Sp(2)$ that we need in the theorems.

THEOREM 6.1. *We assume that p is any prime including* 2, 3, 5. *Then we have*

$$\begin{aligned}
&\dim \mathfrak{M}_{j+k-3,k-3}(U_{npg}(p)) \\
&= \frac{p^2-1}{2880}\chi_1 + \frac{\delta_{p2}}{192}\chi_2 + \frac{\delta_{p2}}{16}\chi_3 + \frac{\delta_{p3}}{9}\chi_4 \\
&\quad + \left(\frac{1}{2^3\cdot 3}\left(p - \left(\frac{-1}{p}\right)\right) + \frac{1}{2^5\cdot 3}\left(p\left(\frac{-1}{p}\right) - 1\right)\right)\chi_6 \\
&\quad + \left(\frac{1}{2^3\cdot 3}\left(p - \left(\frac{-3}{p}\right)\right) + \frac{1}{2^3\cdot 3^2}\left(p\left(\frac{-3}{p}\right) - 1\right)\right)\chi_7 \\
&\quad + \frac{\delta_{p2}}{6}\chi_9 + \frac{\chi_{10}}{5}\left(1 - \left(\frac{p}{5}\right)\right) + \frac{\chi_{11}}{8}\left(1 - \left(\frac{2}{p}\right)\right) \\
&\quad + \frac{\chi_{12}}{24}\left(1 - \left(\frac{3}{p}\right) + \left(\frac{-1}{p}\right) - \left(\frac{-3}{p}\right)\right).
\end{aligned}$$

Here we understand $(n/p) = 0$ if p ramifies in $\mathbb{Q}(\sqrt{n})$, $= -1$ if p remains prime, and $= 1$ otherwise. So for example we have

$$\left(\frac{-1}{2}\right) = \left(\frac{3}{2}\right) = \left(\frac{-3}{3}\right) = \left(\frac{5}{5}\right) = \left(\frac{2}{2}\right) = 0, \qquad \left(\frac{-3}{2}\right) = -1.$$

Here χ_i are character values of the elements of $Sp(2)$ whose principal polynomials are $\phi_i(\pm x)$, where $\phi_i(x)$ are given as follows.

$$\begin{aligned}
&\phi_1(x) = (x-1)^4, && \phi_2(x) = (x-1)^2(x+1)^2, \\
&\phi_3(x) = (x-1)^2(x^2+1), && \phi_4(x) = (x-1)^2(x^2+x+1), \\
&\phi_5(x) = (x-1)^2(x^2-x+1), && \phi_6(x) = (x^2+1)^2, \\
&\phi_7(x) = (x^2+x+1)^2, && \phi_8(x) = (x^2+1)(x^2+x+1), \\
&\phi_9(x) = (x^2+x+1)(x^2-x+1), && \phi_{10}(x) = x^4+x^3+x^2+x+1, \\
&\phi_{11}(x) = x^4+1, && \phi_{12}(x) = x^4-x^2+1, \\
&\phi_{13}(x) = x^4+\sqrt{5}x^3+3x^2+\sqrt{5}x+1, && \phi_{14}(x) = (x^2+\sqrt{2}x+1)^2, \\
&\phi_{15}(x) = x^4+\sqrt{2}x^3+x^2+\sqrt{2}x+1, && \phi_{16}(x) = (x^2+\sqrt{2}x+1)(x^2+1), \\
&\phi_{17}(x) = (x^2+\sqrt{3}x+1)(x^2+1).
\end{aligned}$$

These polynomials are possible principal polynomials of all the elements of finite order of G for $n = 2$ and of $g/\sqrt{p}$ with $n(g) = p$ appearing in $U_{npg}\pi$. Of course the character formula is classical and written in [**58**], but explicit calculation is sometimes complicated and we give them below for readers' convenience. We write the Young diagram parameter (f_1, f_2) for an irreducible representation of $Sp(2)$. By the general formula of [**58**] p. 219 Theorem 7.8E, for a principal polynomial $\phi(x)$ of any element in $Sp(2)$, the character value of ρ_{f_1,f_2} is given by

$$p_{f_1}(p_{f_2}+p_{f_2-2})-p_{f_2-1}(p_{f_1+1}+p_{f_1-1}), \tag{16}$$

where p_f is defined by

$$\frac{1}{\phi(x)}=\sum_{f=0}^{\infty}p_f x^f$$

with $p_f=0$ for any $f<0$.

Since it would be convenient to use the notation suitable for the calculation of paramodular forms $S_{k,j}(K(p))$ of weight $\det^k Sym(j)$, we put $(f_1,f_2)=(j+k-3,k-3)$ ($k\geq 3$, $j\geq 0$, j even) and write the character values below for these parameters k, j. In the following table, χ_i are up to constant the same as $C_i(k,j)$ in [**27**] p. 604, and also χ_i for $i=2,6,9,11$ are reproductions of those in the first section, noting that $f_1=j+k-3$, $f_2=k-3$. The concrete results for χ_{13} to χ_{17} are newly calculated by using the above formula (16), writing p_f explicitly for each ϕ. The details of the calculation are omitted here but we will add hints for the calculation later.

$$\chi_1=\frac{(j+1)(k-2)(j+k-1)(j+2k-3)}{6},$$

$$\chi_2=(-1)^{k-3}\frac{(k-2)(k+j-1)}{2},$$

$$\chi_3=\frac{1}{2}[(-1)^{j/2}(k-2),-(j+k-1),-(-1)^{j/2}(k-2),j+k-1;4]_k,$$

$$\chi_4=\frac{1}{3}\big((j+k-1)[1,-1,0;3]_k+(k-2)[1,0,-1;3]_{j+k}\big),$$

$$\chi_5=(j+k-1)[-1,-1,0,1,1,0;6]_k+(k-2)[1,0,-1,-1,0,1;6]_{j+k},$$

$$\chi_6=\frac{1}{2}(-1)^{(2k+j-6)/2}\times[-k+2,j+k-1;2]_k,$$

$$\chi_7=\begin{cases}[(2k+j-3),(2k+2j-2),(2k-4);3]_k/3 & \text{if } j\equiv 0 \bmod 3,\\ [-(2k+2j-2),-(2k+j-3),-(2k-4);3]_k/3 & \text{if } j\equiv 1 \bmod 3,\\ [j+1,-(j+1),0;3]_k/3 & \text{if } j\equiv 2 \bmod 3,\end{cases}$$

$$\chi_8=\begin{cases}[-1,0,0,1,1,1,1,0,0,-1,-1,-1;12]_k & \text{if } j\equiv 0 \bmod 12,\\ [1,-1,0,-1,-1,0,-1,1,0,1,1,0;12]_k & \text{if } j\equiv 2 \bmod 12,\\ [-1,1,0,0,1,-1,1,-1,0,1,-1,1;12]_k & \text{if } j\equiv 4 \bmod 12,\\ [1,0,0,1,-1,1,-1,0,0,-1,1,-1;12]_k & \text{if } j\equiv 6 \bmod 12,\\ [-1,-1,0,-1,1,0,1,1,0,1,-1,0;12]_k & \text{if } j\equiv 8 \bmod 12,\\ [1,1,0,0,-1,-1,-1,-1,0,0,1,1;12]_k & \text{if } j\equiv 10 \bmod 12,\end{cases}$$

$$\chi_9=\begin{cases}[-1,0,0,1,0,0;6]_k & \text{if } j\equiv 0 \bmod 6,\\ [1,-1,0,-1,1,0;6]_k & \text{if } j\equiv 2 \bmod 6,\\ [0,1,0,0,-1,0;6]_k & \text{if } j\equiv 4 \bmod 6,\end{cases}$$

$$\chi_{10}=\begin{cases}[-1,0,0,1,0;5]_k & \text{if } j\equiv 0 \bmod 10,\\ [1,-1,0,0,0;5]_k & \text{if } j\equiv 2 \bmod 10,\\ 0 & \text{if } j\equiv 4 \bmod 10,\\ [0,0,0,-1,1;5]_k & \text{if } j\equiv 6 \bmod 10,\\ [0,1,0,0,-1;5]_k & \text{if } j\equiv 8 \bmod 10,\end{cases}$$

$$\chi_{11} = \begin{cases} [-1,0,0,1;4]_k & \text{if } j \equiv 0 \bmod 8, \\ [1,-1,0,0;4]_k & \text{if } j \equiv 2 \bmod 8, \\ [1,0,0,-1;4]_k & \text{if } j \equiv 4 \bmod 8, \\ [-1,1,0,0;4]_k & \text{if } j \equiv 6 \bmod 8, \end{cases}$$

$$\chi_{12} = (-1)^{j/2} \times \begin{cases} [-1,0,0,1,-2,2;6]_k & \text{if } j \equiv 0 \bmod 6, \\ [-1,1,0;3]_k & \text{if } j \equiv 2 \bmod 6, \\ [2,-1,0,0,1,-2;6]_k & \text{if } j \equiv 4 \bmod 6, \end{cases}$$

$$\chi_{13} = \begin{cases} [-1,0,0,1,2,1,0,0,-1,-2;10]_k & \text{if } j \equiv 0 \bmod 10, \\ [1,-1,0,2,0,-1,1,0,-2,0;10]_k & \text{if } j \equiv 2 \bmod 10, \\ [-2,-2,0,-2,-2,2,2,0,2,2;10]_k & \text{if } j \equiv 4 \bmod 10, \\ [0,2,0,-1,1,0,-2,0,1,-1;10]_k & \text{if } j \equiv 6 \bmod 10, \\ [2,1,0,0,-1,-2,-1,0,0,1;10]_k & \text{if } j \equiv 8 \bmod 10, \end{cases}$$

$$\chi_{14} = \begin{cases} (-1)^{j/4}[j+k-1,j+k-1,k-2,k-2;4]_k & \text{if } j \equiv 0 \bmod 4, \\ (-1)^{(j-2)/4}[j+k-1,k-2,k-2,j+k-1;4]_k & \text{if } j \equiv 2 \bmod 4, \end{cases}$$

$$\chi_{15} = (-1)^{[j/12]} \times$$

$$\begin{cases} [-1,0,0,1,0,-2,1,2,-2,-1,2,0;12]_k & \text{if } j \equiv 0 \bmod 12, \\ [1,-1,0;3]_k & \text{if } j \equiv 2 \bmod 12, \\ [0,-1,0,2,-1,-2,2,1,-2,0,1,0;12]_k & \text{if } j \equiv 4 \bmod 12, \\ [1,-2,0,1,0,0,-1,0,2,-1,-2,2;12]_k & \text{if } j \equiv 6 \bmod 12, \\ [1,-1,0;3]_k & \text{if } j \equiv 8 \bmod 12, \\ [0,-1,0,0,1,0,-2,1,2,-2,-1,2;12]_k & \text{if } j \equiv 10 \bmod 12, \end{cases}$$

$$\chi_{16} = \begin{cases} [-1,0,0,1,1,0,0,-1;8]_k & \text{if } j \equiv 0 \bmod 8, \\ [1,-1,0,0,-1,1,0,0;8]_k & \text{if } j \equiv 2 \bmod 8, \\ [-1,0,0,-1,1,0,0,1;8]_k & \text{if } j \equiv 4 \bmod 8, \\ [1,1,0,0,-1,-1,0,0;8]_k & \text{if } j \equiv 6 \bmod 8, \end{cases}$$

$$\chi_{17} = \begin{cases} [-1,0,0,1,1,-1;6]_k & \text{if } j \equiv 0 \bmod 12, \\ [1,-1,0;3]_k & \text{if } j \equiv 2 \bmod 12, \\ [-1,-1,0,0,1,1;6]_k & \text{if } j \equiv 4 \bmod 12, \\ [1,0,0,-1,-1,1;6]_k & \text{if } j \equiv 6 \bmod 12, \\ [-1,1,0;3]_k & \text{if } j \equiv 8 \bmod 12, \\ [1,1,0,0,-1,-1;6]_k & \text{if } j \equiv 10 \bmod 12. \end{cases}$$

Calculation for χ_{13} to χ_{17} are based on the following observation. Multiplying suitable polynomials to the numerators and denominators of $1/\phi_i(x)$, we have

$$\frac{1}{\phi_{13}(x)} = \frac{1+2x^2-2x^4-x^6+\sqrt{5}(-x+x^5)}{1-x^{10}},$$

$$\frac{1}{\phi_{14}(x)} = \frac{(1+4x^2+x^4)-2\sqrt{2}x(1+x^2)}{(1+x^4)^2},$$

$$\frac{1}{\phi_{15}(x)} = \frac{1+x^2+2x^4+x^6+x^8-\sqrt{2}(x+x^3+x^5+x^7)}{1+x^{12}},$$

$$\frac{1}{\phi_{16}(x)} = \frac{1 - x^4 + \sqrt{2}(-x + x^3)}{1 - x^8},$$
$$\frac{1}{\phi_{17}(x)} = \frac{1 + x^2 - x^6 - x^8 + \sqrt{3}(-x + x^7)}{1 - x^{12}}.$$

Noting that

$$\frac{1}{(1 + x^4)^2} = \sum_{n=0}^{\infty} (-1)^n (n + 1) x^{4n},$$

the formula for p_f defined by $\phi_{14}(x) = \sum_{f=0}^{\infty} p_f x^f$ is given as follows.

$$p_f = \begin{cases} (-1)^{f/4} & f \equiv 0 \bmod 4, \\ -2^{-1}\sqrt{2}(-1)^{(f-1)/4}(f + 3) & f \equiv 1 \bmod 4, \\ (-1)^{(f-2)/4}(f + 2) & f \equiv 2 \bmod 4, \\ -2^{-1}\sqrt{2}(-1)^{(f-3)/4}(f + 1) & f \equiv 3 \bmod 4. \end{cases}$$

Then calculating (16) for each f mod 4 separately, we can show that χ_{14} is given as above. In the other cases, the formula for p_f are simpler and depend on f mod 10, f mod 24, f mod 8 and f mod 12 for ϕ_{13}, ϕ_{15}, ϕ_{16} and ϕ_{17}, respectively. The character values χ_i can be calculated similarly as before, divided by cases.

7. Bias of the dimensions of plus and minus

In [**37**], for square free N, the dimensions of $S_k^{+,new}(\Gamma_0(N))$ and $S_k^{-,new}(\Gamma_0(N))$ of elliptic cusp forms have been compared and it has been shown that we always have

$$(-1)^{k/2}\big(\dim S_k^{+,new}(\Gamma_0(N)) - \dim S_k^{-,new}(\Gamma_0(N))\big) \geq 0.$$

(See [**37**] Corollary 2.3. Note that the definition of the sign there is the sign of the functional equation, which is the Atkin–Lehner sign times $(-1)^{k/2}$. Since we used the Atkin–Lehner sign for the definition, we put $(-1)^{k/2}$ in the above.)

In this section, we similarly compare $\dim S_k^{+}(K(p))$ and $\dim S_k^{-}(K(p))$, and give the same sort of results. By Theorem 2.2 and 2.1, it is almost trivial that $\dim S_k^{+}(K(p))$ and $\dim S_k^{-}(K(p))$ grow proportionally to p^2k^3 as k and p go to infinity. But if we see the difference $\dim S_k^{+}(K(p)) - \dim S_k^{-}(K(p))$, then by Theorem 2.2, this is close to

$$\dim \mathfrak{M}_{k-3,k-3}^{-}(U_{npg}(p)) - \dim \mathfrak{M}_{k-3,k-3}^{+}(U_{npg}(p)).$$

This is $-TrR_{k-3,k-3}(\pi)$ by definition, and its formula is given in Theorem 2.1. As we will see later, $B_{2,\chi}$ and class numbers are approximately $p^{3/2}$ and $\sqrt{p}\log(p)$ up to constant, respectively, so we can see that the main term of the above difference is a positive constant times $-\chi_2 B_{2,\chi}$ for big p or k, which is approximately $(-1)^k p^{3/2}(k-2)(k-1)$ up to constant. So it is almost clear that $(-1)^k\big(\dim S_k^{+}(K(p)) - \dim S_k^{-}(K(p))\big)$ goes to infinity for $k \to \infty$ or $p \to \infty$. By seeing details of the difference, we have more exact result given as follows.

Theorem 7.1. *For any integer $k \geq 3$ and any prime p, we have*

$$(-1)^k\big(\dim S_k^+(K(p)) - \dim S_k^-(K(p))\big) \geq 0,$$

The list of $k \geq 3$ and p such that $\dim S_k^+(K(p)) = \dim S_k^-(K(p))$ is given as follows.

$$\begin{aligned}(p,k) =&(2,3),(2,4),(2,5),(2,6),(2,7),(2,9),(2,13),\\ &(3,3),(3,4),(3,5),(3,7),(5,3),(5,4),(7,3),(11,3).\end{aligned}$$

These are exactly the cases such that $S_k(K(p)) = 0$.

Now the rest of this section is devoted to the proof of this theorem.

Proof. We have

$$\dim S_2^{-,new}(\Gamma_0(p)) - \dim S_2^{+,new}(\Gamma_0(p)) = \frac{a_p h(\sqrt{-p})}{2} - 1$$

for a_p defined in (5). We put

$$f(p,k) = (-1)^k\big(\dim S_k^+(K(p)) - \dim S_k^-(K(p))\big).$$

Then by Theorem 2.2, we have

$$f(p,k) = (-1)^{k-1}\left(Tr(R_{k-3,k-3}(\pi)) - \frac{a_p h(\sqrt{-p})}{2} \times \dim S_{2k-2}(SL_2(\mathbb{Z}))\right) - \delta_{k3}.$$

By Theorem 2.1, noting that χ_2 is a second order polynomial of k and the other character values χ_i in Theorem 2.1 are at most linear with respect to k, we see that $f(p,k)$ is a second order polynomial of k depending on k mod 60 for a fixed p. So for any concrete p, the evaluation of $f(p,k)$ is just an elementary calculus on quadratic polynomials. Indeed for small p like $p = 2$, 3, 5, the result in Theorem 7.1 is easily obtained. For example, for $p = 5$, we have $B_{2,\chi} = 4/5$ and $h(\sqrt{-5}) = h(\sqrt{-10}) = h(\sqrt{-15}) = 2$, $\dim S_2(\Gamma_0(5)) = 0$, the formula for $f(5,k)$ is explicitly written down as

$$\begin{aligned}f(5,k) =& \frac{11(k-1)(k-2)}{2^4 \cdot 3 \cdot 5} + \frac{1}{2^4}\begin{cases} -(k-2) & (k\text{: even}) \\ (k-1) & (k\text{: odd})\end{cases}\\ &+ \frac{(-1)^{k-1}}{2^2}[-1,0,,0,1;4]_k + \frac{(-1)^{k-1}}{3}[-1,0,0,1,0,0;6]_k\\ &+ \frac{(-1)^{k-1}}{5}[-1,0,0,1,2,1,0,0,-1,-2;10]_k\\ &+ (-1)^k \dim S_{2k-2}(SL_2(\mathbb{Z})) - \delta_{k3}.\end{aligned}$$

The third and the fourth terms are always non-negative and the fifth term is not less than $-2/5$. It is easy to see that

$$\frac{k-8}{6} \leq \dim S_{2k-2}(SL_2(\mathbb{Z})) \leq \frac{k-1}{6}$$

for any k, and by using this evaluation, we see that $f(5,k)>0$ for $k\geq 7$. For $k=3$, 4, 5, 6, by calculating the formula directly, we see $f(5,3)=f(5,4)=0$ and $f(5,5)=f(5,6)=1$. So we proved the result for $p=5$. The case $p=2$ and 3 are similarly done and we omit the details. So in the rest of the proof, we assume that $7\leq p$. Our strategy is to show first that $f(p,k)>0$ for any prime $p\geq 7$ if k is big enough, and that $f(p,k)>0$ for any $k\geq 3$ if p is big enough. If we can say this, then there remain only finitely many (p,k) that are not covered by the above range. Then we calculate $f(p,k)$ directly for these remaining (p,k) and show $f(p,k)\geq 0$ directly. First, for $p\geq 7$, we have

$$\begin{aligned} f(p,k) &= c_2(p)B_{2,\chi}(k-1)(k-2)+c_6(p)h(\sqrt{-p})\begin{cases}-(k-2) & (k\text{: even})\\ (k-1) & (k\text{: odd})\end{cases}\\ &+c_9(p)h(\sqrt{-2p})(-1)^{k-1}\chi_9+c_{11}(p)h(\sqrt{-3p})(-1)^{k-1}\chi_{11}\\ &+(-1)^k\frac{a_p h(\sqrt{-p})}{2}\dim S_{2k-2}(SL_2(\mathbb{Z}))-\delta_{k3}, \end{aligned}$$

where $c_i(p)$ are non-negative constant independent of k but depending on p mod 12. Here by the formula of χ_9 and χ_{11}, we see that $(-1)^{k-1}\chi_9\geq 0$ and $(-1)^{k-1}\chi_{11}\geq 0$, so we may ignore these terms for rough evaluation. We evaluate $c_2(p)B_{2,\chi}$ first. We denote by D_0 the discriminant of $\mathbb{Q}(\sqrt{p})$ and by χ the real character corresponding to $\mathbb{Q}(\sqrt{p})$. Then we have

$$B_{2,\chi}=\frac{L(2,\chi)D_0\sqrt{D_0}}{\pi^2}$$

by [**1**] Theorem 9.6, where $L(s,\chi)$ is the Dirichlet L function with respect to the character χ. Since

$$L(2,\chi)=\prod_p(1-\chi(p)p^{-2})^{-1}\geq\prod_p(1+p^{-2})^{-1}=\frac{\zeta(4)}{\zeta(2)}=\frac{\pi^2}{15},$$

we have

$$B_{2,\chi}=\frac{L(2,\chi)D_0\sqrt{D_0}}{\pi^2}\geq\frac{D_0\sqrt{D_0}}{15}.$$

For $p\equiv 1 \bmod 4$, we have

$$c_2(p)=\frac{1}{2^6\cdot 3}\left(9-2\left(\frac{2}{p}\right)\right)\geq\frac{7}{2^6\cdot 3}$$

and $D_0=p$, so we have

$$c_2(p)B_{2,\chi}\geq\frac{7p^{3/2}}{2^6\cdot 3^2\cdot 5}.$$

On the other hand, if $p\equiv 3 \bmod 4$, then $c_2(p)=1/(2^6\cdot 3)$ and $D_0=4p$, so

$$c_2(p)B_{2,\chi}\geq\frac{8p^{3/2}}{2^6\cdot 3^2\cdot 5}.$$

So we have $c_2(p)B_{2,\chi} \geq 7p^{3/2}/(2^6 \cdot 3^2 \cdot 5)$ in any case.

We explain an evaluation of the class numbers needed later. Let $D < 0$ be the fundamental discriminant of an imaginary imaginary quadratic field $\mathbb{Q}(\sqrt{D})$, and χ_D is the character associated to $\mathbb{Q}(\sqrt{D})$. Then it is well known that for $|D| > 3$, we have

$$L(1, \chi_D) = \frac{\pi h(\sqrt{D})}{\sqrt{|D|}}$$

([**1**] p. 164). On the other hand, by Siegel [**53**] Section 15, we have

$$|L(1, \chi_D)| < 2\log(|D|) + \frac{1}{2}$$

so

$$h(\sqrt{D}) < \frac{1}{\pi}\left(2\sqrt{|D|}\log|D| + \frac{1}{2}\sqrt{|D|}\right). \tag{17}$$

For further evaluation, it is convenient to treat the case k even and odd separately. First we assume that k is even. Then we have $a_p \geq 1$, $c_6(p) \leq 1/2^4$ and $\dim S_{2k-2}(SL_2(\mathbb{Z})) \geq (k-8)/6$, so

$$\begin{aligned} f(p,k) &\geq \frac{7p^{3/2}(k-1)(k-2)}{2^6 \cdot 3^2 \cdot 5} - \frac{h(\sqrt{-p})(k-2)}{16} + \frac{h(\sqrt{-p})(k-8)}{12} \\ &= \frac{7p^{3/2}(k-1)(k-2)}{2^6 \cdot 3^2 \cdot 5} + \frac{h(\sqrt{-p})(k-26)}{48}. \end{aligned}$$

The right hand side is positive if $k \geq 26$. So we assume that $4 \leq k \leq 25$. The last term is bigger than $(-11/24)h(\sqrt{-p})$, which is the value at $k = 4$. The first term is not smaller than the value at $k = 4$. By (17), for both $p \equiv \pm 1 \bmod 4$, we have

$$h(\sqrt{-p}) \leq \frac{1}{\pi}(4\sqrt{p}\log(4p) + \sqrt{p}).$$

So if we put,

$$g(x) = \frac{7x^{3/2}(4-1)(4-2)}{2^6 \cdot 3^2 \cdot 5} - \frac{11}{24\pi}(4\sqrt{x}\log(4x) + \sqrt{x}),$$

then $f(p,k) \geq g(p)$ for $7 \leq p$. We have

$$\frac{d(g(x)/\sqrt{x})}{dx} = -\frac{11}{6\pi x} + \frac{7}{480}.$$

This is a monotonously decreasing function for $x > 0$, and equal to 0 at $x = \frac{880}{7\pi} = 40.01...$, so $g(x)/\sqrt{x}$ is increasing for $x \geq 41$. Since we have $g(293) = 0.05...$, we have $g(x) > 0$ for $x \geq 293$. As a whole, for any $k \geq 4$ and any $p \geq 293$, and for $k \geq 26$ and any $p \geq 7$, we have $f(p,k) > 0$. For $4 \leq k \leq 25$ and $7 \leq p < 293$, we see $f(p,k) > 0$ by checking directly by the explicit formula.

Next we consider the case when k is odd. We assume $p \geq 7$ as before. We have

$c_6(p) \geq 0$, so in this case, we have

$$f(p,k) \geq \frac{7p^{3/2}}{2^6 \cdot 3^2 \cdot 5} - \frac{a_p h(\sqrt{-p})}{2} \dim S_{2k-2}(SL_2(\mathbb{Z})).$$

For $k \leq 6$, we have $S_{2k-2}(SL_2(\mathbb{Z})) = 0$ and $f(p,k) > 0$, so we may assume $k \geq 7$. Since $(k-1)/6 \geq \dim S_{2k-2}(SL_2(\mathbb{Z}))$ and $a_p \leq 4$, we have

$$f(p,k) \geq \frac{7p^{3/2}}{2^6 \cdot 3^2 \cdot 5}(k-1)(k-2) - \frac{k-1}{3} h(\sqrt{-p}).$$

So if we put

$$\begin{aligned} g(x,k) &= \frac{7x^{3/2}(k-2)}{2^6 \cdot 3^2 \cdot 5} - \frac{1}{3\pi}(4\sqrt{x}\log(4x) + \sqrt{x}) \\ &= \frac{\sqrt{x}(-960 + 7(k-2)\pi x - 3840\log(4x))}{2880\pi}, \end{aligned}$$

then $f(p,k) \geq (k-1)g(x,k)$. We put

$$h(x,k) = -960 + 7(k-2)\pi x - 3840\log(4x).$$

Then

$$h_x(x,k) = \frac{dh(x,k)}{dx} = \frac{-3840}{x} + 7\pi(k-2).$$

So the function $h_x(x,k)$ increases monotonously with respect to x for a fixed k, and bigger than 0 for $x \geq 35$ if $k \geq 7$. So for a fixed $k \geq 7$, the function $g(x,k)/\sqrt{x}$ increases monotonously for $x \geq 35$. We have

$$g(250,7) = 0.0055... > 0$$

so we have

$$0 < g(x,7) \leq g(x,k)$$

for any $x > 250$ and $k \geq 7$. On the other hand, we have

$$h_x(7,27) = 1.2... > 0$$

so if $k \geq 27$, then $h_x(x,k) > 0$ for any $x \geq 7$. So for any $k \geq 27$, $h(x,k)$ is increasing monotonously for $x \geq 7$. We have

$$h(7,92) = 98.75... > 0$$

so $g(x,92) > 0$ for any $x \geq 7$, but for a fixed $x > 0$ and $k_1 \leq k_2$, we see easily by definition that $g(x,k_1) \leq g(x,k_2)$, so $g(x,k) > 0$ for any $x \geq 7$ and $k \geq 92$. So the remaining cases are (x,k) with $x < 250$ and $k < 92$. For these finitely many cases, by calculating $f(p,k)$ directly, we see that $f(p,k) \geq 0$ and $= 0$ only when (k,p) are listed in Theorem 7.1. $\square$

8. Numerical Examples

We give several numerical examples of dimensions. The calculation in this section are mostly done by using computer software Mathematica [**60**] and PARI-GP [**38**].

8.1. The case $p = 5$ and $p = 7$

First we assume $p = 5$ for a while. By Theorem 2.1 and 6.1, we have

$$\sum_{f=0}^{\infty} Tr(R_{f,f}(\pi))t^f = \frac{1+t^{11}}{(1-t^2)(1-t^4)(1+t^3)(1+t^5)},$$
$$\sum_{f=0}^{\infty} \dim \mathfrak{M}_{f,f}(U_{npg}(5))t^f = \frac{1+t^{11}}{(1-t^2)(1-t^3)(1-t^4)(1-t^5)},$$

so we have

$$\sum_{f=0}^{\infty} \dim \mathfrak{M}^+_{f,f}(U_{npg}(5))t^f = \frac{(1+t^8)(1+t^{11})}{(1-t^2)(1-t^4)(1-t^6)(1-t^{10})},$$
$$\sum_{f=0}^{\infty} \dim \mathfrak{M}^-_{f,f}(U_{npg}(5))t^f = \frac{(t^3+t^5)(1+t^{11})}{(1-t^2)(1-t^4)(1-t^6)(1-t^{10})}.$$

Next, from these result, we will give dimension formulas for paramodular forms of plus and minus signs. We have

$$\sum_{k=0}^{\infty} \dim S_k(Sp(2,\mathbb{Z}))t^k = \frac{t^{10}+t^{12}-t^{22}+t^{35}}{(1-t^4)(1-t^6)(1-t^{10})(1-t^{12})},$$

and

$$\sum_{k=0}^{\infty} \dim S_{2k-2}(SL_2(\mathbb{Z})))t^k = \frac{t^7}{(1-t^2)(1-t^3)} = \frac{t^7+t^{10}}{(1-t^2)(1-t^6)},$$

and $S_2(\Gamma_0(5)) = 0$. Using these and Theorem 2.2, we have

$$\sum_{k=0}^{\infty} \dim S^+_k(K(5))t^k = \frac{Q^{(5)}_+(t)}{(1-t^4)(1-t^6)(1-t^{10})(1-t^{12})},$$
$$\sum_{k=0}^{\infty} \dim S^-_k(K(5))t^k = \frac{Q^{(5)}_-(t)}{(1-t^4)(1-t^6)(1-t^{10})(1-t^{12})},$$

where

$$\begin{aligned} Q^{(5)}_+(t) =& t^6+2t^8+3t^{10}+3t^{12}+2t^{14}+2t^{16}+t^{17}+t^{18} \\ &+2t^{19}+2t^{21}-t^{22}+2t^{23}+2t^{25}+2t^{27}+t^{29}+t^{35}, \\ Q^{(5)}_-(t) =& t^5+t^7+t^9+2t^{11}+2t^{13}+t^{14}+2t^{15}+t^{16}+t^{17}+t^{18} \\ &+t^{19}+2t^{20}+t^{21}+3t^{22}+t^{23}+3t^{24}+t^{26}+t^{28}+t^{30}-t^{34}. \end{aligned}$$

By the way, adjusting non cusp forms by [**45**], we have

$$\sum_{k=0}^{\infty}\dim A_k^{+}(K(5))t^k = \frac{P_{+}^{(5)}(t)}{(1-t^4)(1-t^6)(1-t^{10})(1-t^{12})},$$
$$\sum_{k=0}^{\infty}\dim A_k^{-}(K(5))t^k = \frac{P_{-}^{(5)}(t)}{(1-t^4)(1-t^6)(1-t^{10})(1-t^{12})},$$
$$\sum_{k=0}^{\infty}\dim A_k(K(5))t^k = \frac{P^{(5)}(t)}{(1-t^4)(1-t^5)(1-t^6)(1-t^{12})},$$

where

$$\begin{aligned}P_{+}^{(5)}(t) =& 1+t^6+2t^8+2t^{10}+2t^{12}+2t^{14}+2t^{16}+t^{17}+t^{18}+\\ & 2t^{19}+2t^{21}+2t^{23}+2t^{25}+2t^{27}+t^{29}+t^{35},\\ P_{-}^{(5)}(t) =& t^5+t^7+t^9+2t^{11}+t^{12}+2t^{13}+t^{14}+2t^{15}+t^{16}+t^{17}+t^{18}\\ & +t^{19}+2t^{20}+t^{21}+2t^{22}+t^{23}+2t^{24}+t^{26}+t^{28}+t^{30},\\ P^{(5)}(t) =& 1+t^6+t^7+2t^8+t^9+2t^{10}+t^{11}+2t^{12}+2t^{14}+2t^{16}+2t^{18}\\ & +t^{19}+2t^{20}+t^{21}+2t^{22}+t^{23}+t^{24}+t^{30}.\end{aligned}$$

The last formula is easily deduced also from the dimension formula in [**17**] and has been explicitly written in [**36**]. Note that the denominator for $A_k(K(5))$ above is different from those for $A_k^{\pm}(K(5))$.

Next we consider the case $p=7$. By using the formula in Theorem 2.1 and Theorem 6.1 for $Tr_{k-3,\,k-3}(R(\pi))$ and $\dim \mathfrak{M}_{k-3,\,k-3}(U_{npg}(7))$, we have

$$\sum_{f=0}^{\infty}\dim \mathfrak{M}_{f,f}^{+}(U_{npg}(7))t^f = \frac{1+t^4+t^6+t^8+t^{11}+t^{13}+t^{15}+t^{19}}{(1-t^2)(1-t^4)(1-t^6)(1-t^{10})},$$
$$\sum_{f=0}^{\infty}\dim \mathfrak{M}_{f,f}^{-}(U_{npg}(7))t^f = \frac{t+t^3+t^5+t^9+t^{10}+t^{14}+t^{16}+t^{18}}{(1-t^2)(1-t^4)(1-t^6)(1-t^{10})}.$$

In the same way as before, we can calculate $\dim S_k^{\pm}(K(7))$ and $\dim A_k^{\pm}(K(7))$ for $k\geq 3$ using the above formulas. We know that $A_1(K(p))=0$ and $A_2(K(p))=S_2(K(p))$ for general prime p. Since $\dim S_8(K(7))=2$ and $\dim A_6(K(7))=3$, we have $A_2(K(7))=S_2(K(7))=0$. By $S_2(\Gamma_0(7))=0$, using Theorem 2.2, we have

$$\sum_{k=0}^{\infty}\dim S_k^{+}(K(7))t^k = \frac{Q_{+}^{(7)}(t)}{(1-t^4)^2(1-t^6)(1-t^{12})},$$
$$\sum_{k=0}^{\infty}\dim S_k^{-}(K(7))t^k = \frac{Q_{-}^{(7)}(t)}{(1-t^4)^2(1-t^6)(1-t^{12})},$$

where

$$\begin{aligned}Q_+^{(7)}(t) =&t^4+2t^6+2t^8+2t^{10}+2t^{12}+t^{13}+t^{14}\\&+t^{15}+t^{17}+2t^{19}+2t^{21}+2t^{23}+t^{29},\\Q_-^{(7)}(t) =&t^5+2t^7+2t^9+2t^{11}+t^{13}+t^{14}+t^{15}\\&+2t^{16}+t^{17}+2t^{18}+2t^{20}+2t^{22}+2t^{24}-t^{28}.\end{aligned}$$

Counting the modular forms on the boundary, we have

$$\begin{aligned}\sum_{k=0}^{\infty}\dim A_k^+(K(7))t^k &= \frac{P_+^{(7)}(t)}{(1-t^4)^2(1-t^6)(1-t^{12})},\\\sum_{k=0}^{\infty}\dim A_k^-(K(7))t^k &= \frac{P_-^{(7)}(t)}{(1-t^4)^2(1-t^6)(1-t^{12})},\\\sum_{k=0}^{\infty}\dim A_k(K(7))t^k &= \frac{P^{(7)}(t)}{(1-t^4)^2(1-t^6)(1-t^{12})}.\end{aligned}$$

$$\begin{aligned}P_+^{(7)}(t) =&1+2t^6+2t^8+2t^{10}+t^{12}+t^{13}+t^{14}+t^{15}+t^{16}\\&+t^{17}+2t^{19}+2t^{21}+2t^{23}+t^{29},\\P_-^{(7)}(t) =&t^5+2t^7+2t^9+2t^{11}+t^{12}+t^{13}+t^{14}+t^{15}+t^{16}\\&+t^{17}+2t^{18}+2t^{20}+2t^{22}+t^{24},\\P^{(7)}(t) =&1+t^5+2t^6+2t^7+2t^8+2t^9+2t^{10}+2t^{11}+2t^{12}+2t^{13}\\&+2t^{14}+2t^{15}+2t^{16}+2t^{17}+2t^{18}+2t^{19}+2t^{20}+2t^{21}\\&+2t^{22}+2t^{23}+t^{24}+t^{29}.\end{aligned}$$

The last formula is easily deduced also from [**17**] and has been explicitly written in [**59**].

Explicit generators of the rings $\oplus_{k=0}^{\infty}A_k^+(K(p))$ for $p=5$ and 7 and their relations have been given in [**59**], though the generating functions of dimensions have not been given in the paper [**59**]. Replying to the author's question, Professor Williams kindly informed the author that the above generating functions for $\dim A_k^+(K(p))$ for $p=5$, 7 can be obtained also by his method and the results are completely the same.

8.2. Small primes p and palindromic Hilbert series

We consider the graded rings $A(K(p)) = \oplus_{k=0}^{\infty}A_k(K(p))$ and $A^+(K(p)) = \oplus_{k=0}^{\infty}A_k^+(K(p))$. For any Noetherian graded integral domain $B=\oplus_{k=0}^{\infty}B_k$ over $\mathbb{C}$ with $B_0=\mathbb{C}$, we call $F(B,t)=\sum_{k=0}^{\infty}\dim B_k t^k$ the Hilbert series of B. We say that $F(B,t)$ is palindromic if $F(B,1/t)=(-1)^m t^{\ell}F(B,t)$ for some $\ell>0$, where m is the Krull dimension of B. Under the assumption that B is a Cohen-Macaulay ring, it is known that B is a Gorenstein ring if and only if $F(B,t)$ is palindromic ([**54**] Theorem 4.4, [**55**] p. 503. The author learned these references from R. Schmidt). For $p=2$, 3, we see that $A(K(p))$ and $A^+(K(p))$ are Cohen-Macaulay since the rings are free over rings generated by 4 algebraically independent forms by [**28**] Theorem 1, Corollary 1 and [**5**] Lemma 5.4, 5.5. They are also Gorenstein since their Hilbert series are palindromic. For $p = 5, 7$, the ring $A(K(p))$ are Gorenstein by B. Williams [**59**] Corollary 16 and 24. He also checked that $A^+(K(5))$ and $A^+(K(7))$ are Gorenstein (private communication).

Cris Poor asked the author for which primes p, the Hilbert series of $A(K(p))$ and $A^+(K(p))$ are palindromic. The proposition below answer this for small primes. We denote by $J_{2,p}$ the space of Jacobi forms of $SL_2(\mathbb{Z})$ of weight 2 of index p.

PROPOSITION 8.1. *Let p be a prime less than 100. Among this range, we have the following results.*
(i) $F(A(K(p)), t)$ is palindromic if and only if $p = 2, 3, 5, 7, 13$.
(ii) $F(A^+(K(p)), t)$ is palindromic if and only if $p = 2, 3, 5, 11, 13, 17, 19, 23, 29, 31, 41, 47, 59, 71$.
(iii) The primes in (i) are exactly those such that $S_2(\Gamma_0(p)) = 0$. The primes in (ii) are exactly those such that $J_{2,p} = 0$.

By [**54**], [**55**] and the palindromic property, it is natural to ask if $A(K(p))$ and $A^+(K(p))$ for primes in the above Proposition are Gorenstein. Seeing the Hilbert series, our guess is that the rings $A(K(p))$ and $A^+(K(p))$ are not Cohen-Macaulay for $p < 100$ except for the above listed primes. The theoretical meaning of (iii) is not clear.

The proof of Proposition 8.1 merely consists of explicit calculations of all the Hilbert series for this range by Theorem 2.1 and 2.2 except for the point that we need $\dim A_1(K(p))$ and $\dim A_2(K(p))$. We know that we always have $A_1(K(p)) = 0$ (e.g. [**20**]). We see easily that $A_2(K(p)) = S_2(K(p))$ by the boundary structure of the Satake compactification of $K(p)\backslash H_2$ and by the fact $A_2(SL_2(\mathbb{Z})) = 0$. For the range of $p < 277$, the space $S_2(K(p))$ are all spanned by the Gritsenko lift from $J_{2,p}$ by [**41**], so we have $\dim S_2(K(p)) = \dim J_{2,p}$ for such p. These are all Atkin–Lehner plus. The latter dimension is given by the table

p	≤ 31	37	41	43	47	53	59	61	67	71	73	79	83	89	97
$\dim J_{2,p}$	0	1	0	1	0	1	0	1	2	0	2	1	1	1	3

([**7**] p. 132). We omit the details of the proof.

8.3. Examples of $j = 2$

We consider the case $j = 2$. We have $\dim S_4(\Gamma_0(2)) = \dim S_4(\Gamma_0(3)) = 0$, so there is no Yoshida lifting to $\mathfrak{M}^+_{f+2,f}(U_{npg}(2))$. Since $j > 0$, there is no Saito–Kurokawa lift. So for $p = 2$ and 3 and $k \geq 3$. we have

$$\dim S^+_{k,2}(K(p)) = \dim S_{k,2}(Sp(2,\mathbb{Z})) + \dim \mathfrak{M}^-_{k-1,k-3}(U_{npg}(p)),$$
$$\dim S^-_{k,2}(K(p)) = \dim S_{k,2}(Sp(2,\mathbb{Z})) + \dim \mathfrak{M}^+_{k-1,k-3}(U_{npg}(p)).$$

Here we can calculate $\dim \mathfrak{M}_{k-1,k-3}$ by Theorem 5.2 and 6.1, and we have

$$\sum_{k=0}^{\infty} \dim S_{k,2}(Sp(2,\mathbb{Z}))t^k = \frac{t^{14} + 2t^{16} + t^{18} + t^{22} - t^{26} - t^{28} + t^{21} + t^{23} + t^{27} + t^{29} - t^{33}}{(1-t^4)(1-t^6)(1-t^{10})(1-t^{12})}.$$

(See [**56**], [**46**], [**21**]). So we can give the following generating functions.

$$\sum_{k=0}^{\infty} \dim S_{k,2}^{+}(K(2))t^k = \frac{P_{+,2}^{(2)}(t)}{(1-t^2)(1-t^6)(1-t^8)(1-t^{12})},$$
$$\sum_{k=0}^{\infty} \dim S_{k,2}^{-}(K(2))t^k = \frac{P_{-,2}^{(2)}(t)}{(1-t^2)(1-t^6)(1-t^8)(1-t^{12})},$$
$$\sum_{k=0}^{\infty} \dim S_{k,2}^{+}(K(3))t^k = \frac{P_{+,2}^{(3)}(t)}{(1-t^2)(1-t^4)(1-t^6)(1-t^{12})},$$
$$\sum_{k=0}^{\infty} \dim S_{k,2}^{-}(K(3))t^k = \frac{P_{-,2}^{(3)}(t)}{(1-t^2)(1-t^4)(1-t^6)(1-t^{12})},$$

where

$$P_{+,2}^{(2)}(t) = t^{10} + 2t^{14} + t^{15} + t^{18} + t^{19} + t^{23} - t^{24} + t^{27} - t^{29},$$
$$P_{-,2}^{(2)}(t) = t^{11} + t^{14} + 2t^{15} + t^{16} - t^{17} + t^{18} + t^{19} + t^{22} - t^{24} + t^{26} - t^{28},$$
$$P_{+,2}^{(3)}(t) = t^{8} + t^{10} + t^{13} + t^{14} + t^{15} + t^{16} - t^{20} + t^{21} + t^{23} - t^{25},$$
$$P_{-,2}^{(3)}(t) = t^{9} + t^{11} + t^{12} + t^{14} + t^{15} + t^{16} + t^{22} - t^{24}.$$

The result for $k = 0, 1, 2$ are obtained by the dimension formula for higher k. For example, we have $\dim A_4(K(2)) = 1$, so if $\dim S_{2,2}(K(2)) \neq 0$, then this contradicts to the formula $\dim S_{6,2}(K(2)) = 0$. Arguments for the other cases are similar.

8.4. The case of small *k*

For several small $k \geq 3$ with $j = 0$ and some primes p, we give tables for $\dim \mathfrak{M}_{k-3,\,k-3}^{\pm}(U_{npg}(p))$ and $\dim S_k^{\pm}(K(p))$. In all the tables below, we put

$$H = \dim \mathfrak{M}_{k-3,\,k-3}(U_{npg}(p)), \quad R = Tr(R_{k-3,\,k-3}(\pi)),$$
$$M^{+} = \dim \mathfrak{M}_{k-3,\,k-3}^{+}(U_{npg}(p)), \quad M^{-} = \dim \mathfrak{M}_{k-3,\,k-3}^{-}(U_{npg}(p)),$$
$$S_k^{\pm} = \dim S_k^{\pm}(K(p)), \quad s_2^{\pm} = \dim S_2^{\pm,new}(\Gamma_0(p)),$$

where k is fixed for each table.

Numerical examples for the case $k = 3$ has been given in in [**24**] p. 218. We denote the class number and the type number of discriminant p of the non-principal genus by $H(p)$ and $T(p)$. Since we have $S_{2k-2}(SL_2(\mathbb{Z})) = 0$ for $k = 3$, we have $\dim S_3^{+}(K(p)) = H(p) - T(p)$ and $\dim S_3^{-}(K(p)) = T(p) - 1$ by Theorem 2.2. Now we determine all primes p such that $\dim S_3^{+}(K(p)) = 0$. This has some geometric meaning. Let $K(N)^*$ be the maximal extension of $K(N)$ in $Sp(2,\mathbb{R})$ of order $2^{\nu(N)}$ where $\nu(N)$ is the number of prime divisors of N. In [**10**], it was proved that $K(N)^* \backslash H_2$ is the moduli space of the Kummer surfaces associated to $(1, N)$ polarizations. In particular, elements of $S_3(K(N)^*)$ give canonical differential forms of the Satake compactification of $K(N)^* \backslash H_2$ (See [**8**]). It has been that $\dim S_3(K(N)^*) = 0$ for $N \leq 40$ in [**3**] and also that $\dim S_3(K(N)^*) \geq 1$

for $N = 167$, 173, 197, 213, 285 in [**11**]. When $N = p$ is a prime, then we have $S_3(K(p)^*) = S_3^+(K(p))$. Now our new result for $k = 3$ is given as follows.

PROPOSITION 8.2. *Let p be a prime. Then we have*

$$\dim S_3^+(K(p)) = 0$$

if and only if p is any prime such that $p \le 163$ or $p = 179$, 181, 191, 193, 199, 211, 229, 241.

By the way, we have $\dim S_3^+(K(p)) = 1$ if $p = 167$, 173, 197, 223, 233, 239, 251, 271, 277, 281, 313, 331, 337 and $\dim S_3^+(K(p)) = 2$ if $p = 227$, 257, 263, 269, 283, 349, 379, 409, 421.

PROOF OF PROPOSITION 8.2. For $p = 2$, 3, 5, we know $S_3^+(K(p)) = 0$ for direct calculation of the formula in Theorem 2.2. Now assuming $p > 5$, by rough estimation first we show that $\dim S_3^+(K(p)) = H(p) - T(p) = (H(p) - Tr(R_{0,0}(\pi)))/2 > 0$ for any prime $p \ge 3702$. On the other hand, for each prime $p \le 3701$, we calculate $\dim S_3^+(K(p)) = (H(p) - Tr(R_{0,0}(\pi)))/2$ directly from Theorem 2.1 and Theorem 6.1 and can show concretely that it is non-negative. This completes the proof.

Now we explain how to obtain the estimation for big p given above. First of all, for $k = 3$, any χ_i in the formula in $H(p)$ and $Tr(R_{0,0}(\pi))$ is 1.

Denote by D_0 the discriminant of the real quadratic field $\mathbb{Q}(\sqrt{p})$ and by χ the character associated to $\mathbb{Q}(\sqrt{p})$. Then by [**1**] Theorem 9.6, we have

$$L(2,\chi) = \frac{\pi^2}{D_0\sqrt{D_0}} B_{2,\chi}.$$

By evaluating the Euler product, we have $0 < L(2,\chi) < \zeta(2) = \pi^2/6$, so we have

$$B_{2,\chi} < \frac{p^{3/2}}{6} \quad \text{if } p \equiv 1 \bmod 4,$$
$$B_{2,\chi} < \frac{4p^{3/2}}{3} \quad \text{if } p \equiv 3 \bmod 4.$$

So since

$$\frac{4}{3} \le \frac{11}{6} \quad \text{and} \quad \frac{1}{6}\left(9 - 2\left(\frac{2}{p}\right)\right) \le \frac{11}{6},$$

the term in $Tr(R_{0,0}(\pi))$ containing $B_{2,\chi}$ is bounded by $11p^{3/2}/(2^6 \cdot 3^2)$ from the above. Considering the evaluation (17) for cases $p \equiv 1 \bmod 4$ and $p \equiv 3 \bmod 4$, we can give a common upper bound for $h(\sqrt{-p})$, $h(\sqrt{-2p})$ and $h(\sqrt{-3p})$. By these inequalities we can estimate $Tr(R_{0,0}(\pi))$ in Theorem 2.1 from the above as follows.

$$Tr(R_{0,0}(\pi)) < \frac{11p^{3/2}}{2^6 \cdot 3^2} + \frac{1}{2^4\pi}(4\sqrt{p}\log(p) + \sqrt{p})$$

$$+\frac{1}{2^3\pi}(4\sqrt{2p}\log(8p)+\sqrt{2p})+\frac{1}{3\pi}(4\sqrt{3p}\log(12p)+\sqrt{3p}).$$

Here $Tr(R_{0,0}(\pi)) = 2T(p) - H(p)$. On the other hand, by Theorem 6.1, comparing the part for $(-1/p) = 1$ and -1, and also for $(-3/p) = +1$ and -1, for $p > 5$ we see that

$$H(p) > \frac{p^2-1}{2880}+\frac{1}{36}(p+1)+\frac{1}{32}(p+1)=\frac{p^2-1}{2880}+\frac{17(p+1)}{288}.$$

So if we put

$$\begin{aligned} f(x) = &\frac{x^2-1}{2880}+\frac{17(x+1)}{288}-\frac{11x^{3/2}}{2^6\cdot 3^2}-\frac{1}{2^4\pi}(4\sqrt{x}\log(x)+\sqrt{x}) \\ &-\frac{1}{2^3\pi}(4\sqrt{2x}\log(8x)+\sqrt{2x})-\frac{1}{3\pi}(4\sqrt{3x}\log(12x)+\sqrt{3x}), \end{aligned}$$

then we have $2\dim S_3^+(K(p)) > f(p)$ for $p \geq 7$. For $x > 0$, we have

$$\frac{df(x)}{dx}=\frac{g(x)}{5760\pi\sqrt{x}},$$

where

$$\begin{aligned} g(x) = &-1620-3240\sqrt{2}-8640\sqrt{3}+340\pi\sqrt{x}-165\pi x+4\pi x^{3/2} \\ &-720\log(x)-3840\sqrt{3}\log(12x)-1440\sqrt{2}\log(8x). \end{aligned}$$

We also have

$$g'(x) := \frac{dg(x)}{dx}=\frac{-720-1440\sqrt{2}-3840\sqrt{3}+\pi(170\sqrt{x}-165x+6x^{3/2})}{x}.$$

It is easy to see that the numerator of $g'(x)$ increases monotonously for $x > 317$ by elementary calculus, and since we have $g'(800) = 21.90... > 0$, we have $g'(x) > 0$ for $x > 800$. This means that $g(x)$ increases monotonously for $x > 800$. We also have $g(1940) = 2047.64... > 0$, so for $x > 1940$, the function $f(x)$ increases monotonously. Since we have $f(3702) = 0.39... > 0$, we have $f(x) > 0$ for $x \geq 3702$. So we are done. □

Next we study the cases $k = 4, 5, 6, 8$ for small p. We have $S_k(Sp(2,\mathbb{Z})) = 0$ and $S_{2k-2}(SL_2(\mathbb{Z})) = 0$ in these cases, so we have the following simple relations.

$$\dim S_k^+(K(p)) = \dim \mathfrak{M}^-_{k-3,\,k-3}(U_{npg}(p)),$$
$$\dim S_k^-(K(p)) = \dim \mathfrak{M}^+_{k-3,\,k-3}(U_{npg}(p)).$$

A table of $\dim S_4^{\pm}(K(p))$ has been given in [**41**] Table 4 in p. 28 by elaborate calculation of constructing paramodular forms, but Table 6 below was obtained by our theoretical result independently. It is nice to see that the results coincide. (We added $p = 601$ and 607 as a small tip to their table.)

The numerical tables in the cases $k = 5, 6, 8$ are given in Table 7, 8, 10, respectively.

By the way, when $k = 8$, using our formula, we have $\dim S_8^+(K(277)) = 1761$ and $\dim S_8^-(K(277)) = 768$. These do not coincide with the numbers given in [**41**] page 31. The authors are aware of the error and will publish a correction [**40**].

When $k = 7$, we have $\dim S_{2k-2}(SL_2(\mathbb{Z})) = 1$ and $S_7(Sp(2,\mathbb{Z})) = 0$, so we have

$$\dim S_7^+(K(p)) = \dim \mathfrak{M}_{4,4}^-(U_{npg}(p)) - \dim S_2^{+,new}(\Gamma_0(p)),$$
$$\dim S_7^-(K(p)) = \dim \mathfrak{M}_{4,4}^+(U_{npg}(p)) - 1 - \dim S_2^{-,new}(\Gamma_0(p)).$$

The numerical table for $k = 7$ is given in Table 9.

When $k = 10$, we have $\dim S_{10}(Sp(2,\mathbb{Z})) = \dim S_{18}(SL_2(\mathbb{Z})) = 1$, so the old form in $S_{10}(K(p))$ comes from the Saito–Kurokawa lift and we have also the Yoshida lift part in $\mathfrak{M}_{7,7}$. So we have

$$\dim S_{10}^+(K(p)) = 1 + \dim \mathfrak{M}_{7,7}^-(U_{npg}(p)) - \dim S_2^{+,new}(\Gamma_0(p)),$$
$$\dim S_{10}^-(K(p)) = \dim \mathfrak{M}_{7,7}^+(U_{npg}(p)) - \dim S_2^{-,new}(\Gamma_0(p)).$$

The numerical table for $k = 10$ is given in Table 11.

Acknowledgement: The author would like to thank Neil Dummigan for informing the interesting result in his joint work [**6**] and for answering author's question in detail, as well as his big interest around the problem, and also to Cris Poor and David S. Yuen for constantly asking the author the dimension formula with involution and sometimes assuring numerical correctness of our calculation by their examples. The author would also like to thank an anonymous referee for their very careful reading and corrections.

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Table 6. The case $k = 4$.

p	7	11	13	17	19	23	29	31	37	41	43	47
H	1	1	2	2	3	3	4	6	8	7	9	8
R	−1	−1	−2	−2	−3	−3	−4	−6	−8	−7	−9	−8
S_4^+	1	1	2	2	3	3	4	6	8	7	9	8
S_4^-	0	0	0	0	0	0	0	0	0	0	0	0

p	53	59	61	67	71	73	79	83	89	97	101	103
H	10	11	16	17	15	21	22	19	23	32	28	33
R	−10	−11	−16	−17	−15	−21	−22	−17	−23	−32	−26	−31
S_4^+	10	11	16	17	15	21	22	18	23	32	27	32
S_4^-	0	0	0	0	0	0	0	1	0	0	1	1

p	107	109	113	127	131	137	139	149	151	157	163	167
H	29	38	34	46	41	47	53	54	61	68	69	63
R	−25	−38	−32	−42	−35	−43	−49	−48	−57	−62	−61	−47
S_4^+	27	38	33	44	38	45	51	51	59	65	65	55
S_4^-	2	0	1	2	3	2	2	3	2	3	4	8

p	173	179	181	191	193	197	199	211	223	227	229	233
H	70	71	86	80	96	88	97	107	118	109	128	119
R	−54	−59	−80	−66	−88	−68	−85	−93	−94	−73	−114	−93
S_4^+	62	65	83	73	92	78	91	100	106	91	121	106
S_4^-	8	6	3	7	4	10	6	7	12	18	7	13

p	239	241	251	257	263	269	271	277	281	283	293	307
H	120	140	131	142	143	154	166	178	167	179	180	207
R	−90	−126	−95	−106	−97	−114	−132	−144	−131	−131	−118	−147
S_4^+	105	133	113	124	120	134	149	161	149	155	149	177
S_4^-	15	7	18	18	23	20	17	17	18	24	31	30

p	311	313	317	331	337	347	349	353	359	367	373	379
H	195	221	208	237	252	239	268	254	255	286	302	303
R	−131	−179	−140	−185	−202	−145	−210	−170	−165	−196	−224	−225
S_4^+	163	200	174	211	227	192	239	212	210	241	263	264
S_4^-	32	21	34	26	25	47	29	42	45	45	39	39

p	383	389	397	401	409	419	421	431	433	439	443	449
H	288	304	338	322	357	341	376	360	396	397	379	398
R	−164	−208	−240	−226	−279	−203	−290	−214	−290	−269	−215	−268
S_4^+	226	256	289	274	318	272	333	287	343	333	297	333
S_4^-	62	48	49	48	39	69	43	73	53	64	82	65

p	457	461	463	467	469	487	491	499	503	509	521	523
H	437	418	438	419	440	481	461	503	483	504	527	549
R	−319	−252	−286	−223	−242	−305	−265	−341	−243	−296	−325	−325
S_4^+	378	335	362	321	341	393	363	422	363	400	426	437
S_4^-	59	83	76	98	99	88	98	81	120	104	101	112

p	541	547	557	563	569	571	577	587	593	599	601	607
H	596	597	598	599	623	647	672	649	674	675	725	726
R	−416	−359	−322	−287	−381	−413	−444	−311	−362	−363	−497	−404
S_4^+	506	478	460	443	502	530	558	480	518	519	611	565
S_4^-	90	119	138	156	121	117	114	169	156	156	114	161

Table 7. The case $k = 5$.

p	7	11	13	17	19	23	29	31	37	41	43	47	53	59	61	67	71
H	1	2	3	4	5	5	9	10	14	15	16	16	22	24	31	33	33
R	1	2	3	4	5	5	9	10	14	15	16	14	20	22	31	31	29
S_5^+	0	0	0	0	0	0	0	0	0	0	0	1	1	1	0	1	2
S_5^-	1	2	3	4	5	5	9	10	14	15	16	15	21	23	31	32	31

Table 8. The case $k = 6$.

p	7	11	13	17	19	23	29	31	37	41	43	47
H	2	3	5	6	8	9	14	17	24	25	29	30
R	-2	-3	-5	-6	-8	-9	-14	-17	-24	-23	-27	-24
S_6^+	2	3	5	6	8	9	14	17	24	24	28	27
S_6^-	0	0	0	0	0	0	0	0	0	1	1	3

Table 9. The case $k = 7$.

p	7	11	13	17	19	23	29	31	37	41	43	47
H	3	5	8	10	13	15	24	28	40	43	49	52
R	3	5	8	10	13	13	22	26	36	37	41	36
M^+	3	5	8	10	13	14	23	27	38	40	45	44
M^-	0	0	0	0	0	1	1	1	2	3	4	8
s_2^+	0	0	0	0	0	0	0	0	1	0	1	0
s_2^-	0	1	0	1	1	2	2	2	1	3	2	4
S_7^+	0	0	0	0	0	1	1	1	1	3	3	8
S_7^-	2	3	7	8	11	11	20	24	36	36	42	39

Table 10. The case $k = 8$.

p	7	11	13	17	19	23	29	31	37	41	43	47
H	4	6	10	14	17	22	34	40	57	64	72	80
R	-4	-6	-10	-12	-17	-18	-28	-36	-49	-48	-56	-50
S_8^+	4	6	10	13	17	20	31	38	53	56	64	65
S_8^-	0	0	0	1	0	2	3	2	4	8	8	15

Table 11. The case $k = 10$.

p	7	11	13	17	19	23	29	31	37	41	43	47
H	6	12	18	26	34	44	70	82	116	134	150	170
R	-6	-10	-16	-20	-28	-30	-48	-60	-82	-82	-94	-84
M^+	0	1	1	3	3	7	11	11	17	26	28	43
M^-	6	11	17	23	31	37	59	71	99	108	122	127
s_2^+	0	0	0	0	0	0	0	0	1	0	1	0
s_2^-	0	1	0	1	1	2	2	2	1	3	2	4
S_{10}^+	7	12	18	24	32	38	60	72	99	109	122	128
S_{10}^-	0	0	1	2	2	5	9	9	16	23	26	39

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