SYMMETRIZER GROUP OF A PROJECTIVE HYPERSURFACE

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Abstract. To each projective hypersurface which is not a cone, we associate an abelian linear algebraic group called the symmetrizer group of the corresponding symmetric form. This group describes the set of homogeneous polynomials with the same Jacobian ideal and gives a conceptual explanation of results by Ueda–Yoshinaga and Wang. In particular, the diagonalizable part of the symmetrizer group detects Sebastiani-Thom property of the hypersurface and its unipotent part is related to the singularity of the hypersurface.

Keywords: projective hypersurface, Jacobian ideal, Sebastiani-Thom type

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1. Introduction

We work over the complex numbers. Throughout, we fix an integer $d \geq 3$ and a vector space V with $\dim V = n \geq 2$. Let $\mathbb{P}V$ be the projectivization of V, the set of 1-dimensional subspaces of V. For a nonzero vector $v \in V$, we denote by $[v] \in \mathbb{P}V$ the 1-dimensional subspace $\mathbb{C}v \subset V$. We regard the vector space $\mathrm{Sym}^d V^*$ of symmetric d-forms on V as the subspace of the vector space $\otimes^d V^*$ of d-linear forms on V, which consists of all $F \in \otimes^d V^*$ satisfying

$$F(v_1, \ldots, v_d) = F(v_{\sigma(1)}, \ldots, v_{\sigma(d)})$$

for any permutation σ of $\{1, \ldots, d\}$. Denote by $Z(F) \subset \mathbb{P}V$ the hypersurface defined by the homogenous polynomial of degree d corresponding to F.

Definition 1.1. (i) For $F \in \mathrm{Sym}^d V^*$, define the homomorphism $\partial F : V \to \mathrm{Sym}^{d-1} V^*$, called the *Jacobian* of F, by

$$(\partial F(u))(v_1, \ldots, v_{d-1}) := F(u, v_1, \ldots, v_{d-1})$$

for all $u, v_1, \ldots, v_{d-1} \in V$.

(ii) We say that F is *nondegenerate* if $\mathrm{Ker}(\partial F) = 0$, equivalently, if the hypersurface $Z(F)$ is not a cone. Denote by $\mathrm{Sym}^d_o V^*$ the Zariski-open subset of $\mathrm{Sym}^d V^*$ consisting of nondegenerate forms.

(iii) Let $\mathrm{Gr}(n; \mathrm{Sym}^{d-1} V^*)$ be the Grassmannian of n-dimensional subspaces in the vector space $\mathrm{Sym}^{d-1} V^*$. For a nondegenerate form $F \in \mathrm{Sym}^d_o V^*$, the dimension of $\mathrm{Im}(\partial F)$ is n. Let $J(F)$ be the point

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in $\mathrm{Gr}(n; \mathrm{Sym}^{d-1} V^*)$ corresponding to the n-dimensional subspace $\mathrm{Im}(\partial F)$. This defines a morphism

$$J : \mathrm{Sym}^d_o V^* \to \mathrm{Gr}(n; \mathrm{Sym}^{d-1} V^*).$$

Historically, the interest in the morphism J arose from the study of variation of Hodge structures for families of hypersurfaces. In particular, Carlson and Griffiths showed in Section 4.b of [1] that $J^{-1}(J(F)) = \mathbb{C}^\times \cdot F$ for a general $F \in \mathrm{Sym}^d_o V^*$. Obvious examples satisfying $J^{-1}(J(F)) \neq \mathbb{C}^\times \cdot F$ are symmetric forms of Sebastiani-Thom type in the following sense.

Definition 1.2. An element $F \in \mathrm{Sym}^d V^*$ is *of Sebastiani-Thom type*, if there is a direct sum decomposition $V = V_1 \oplus \cdots \oplus V_k$ with $k \geq 2$ such that $F = F_1 + \cdots + F_k$ for some $F_i \in \mathrm{Sym}^d V_i^*, 1 \leq i \leq k$. More precisely, this means that for each $1 \leq i \neq j \leq k$ and $u \in V_i, w \in V_j$,

$$F_i(u, w, v_1, \ldots, v_{d-2}) = 0$$

for all $v_1, \ldots, v_{d-2} \in V$.

For $F = F_1 + \cdots + F_k \in \mathrm{Sym}^d_o V^*$ of Sebastiani-Thom type, we have

$$J(F) = J(c_1 F_1 + \cdots + c_k F_k)$$

for any $c_1, \ldots, c_k \in \mathbb{C}^\times$. Thus $J^{-1}(J(F)) \neq \mathbb{C}^\times \cdot F$. Ueda and Yoshinaga proved the following in [3].

Theorem 1.3. *Let $F \in \mathrm{Sym}^d_o V^*$ be such that the hypersurface $Z(F) \subset \mathbb{P}V$ is nonsingular. Then $J^{-1}(J(F)) \neq \mathbb{C}^\times \cdot F$ if and only if F is of Sebastiani-Thom type.*

By this result, the key issue in understanding symmetric forms satisfying $J^{-1}(J(F)) \neq \mathbb{C}^\times \cdot F$ is to study its relation with the singularity of the hypersurface $Z(F)$. In this direction, Wang proved in [4] the following result.

Theorem 1.4. *Let $F \in \mathrm{Sym}^d_o V^*$ be such that $J^{-1}(J(F)) \neq \mathbb{C}^\times \cdot F$ and F is not of Sebastiani-Thom type. Then the hypersurface $Z(F)$ has a singular point of multiplicity $d-1$, namely, there is a nonzero vector $u \in V$ such that*

$$F(u, u, v_1, \ldots, v_{d-2}) = 0 \text{ for all } v_1, \ldots v_{d-2} \in V.$$

Of course, Theorem 1.4 is reduced to Theorem 1.3 when $d = 3$, but for $d \geq 4$, it gives additional information on the singularity of $Z(F)$.

The proofs of Theorem 1.3 and Theorem 1.4 in [3] and [4] are computational. Our main result is a geometric description of the fibers of the morphism J, which gives a more conceptual explanation of Theorems 1.3 and 1.4. More precisely, we describe the fibers of J as follows.

Theorem 1.5. *Let $x \in \mathrm{Im}(J) \subset \mathrm{Gr}(n; \mathrm{Sym}^{d-1} V^*)$ be a point in the image of the morphism J. Then there is a connected abelian algebraic subgroup $G_x \subset \mathrm{GL}(V)$ canonically associated to x, which contains $\mathbb{C}^\times \cdot \mathrm{Id}_V$, such that the fiber $J^{-1}(x)$ is a principal homogeneous space of the group G_x.*

By the classification of connected abelian groups (Theorems 3.1.1 and 3.4.7 of [2]), we have a decomposition into direct product of algebraic groups

$$G_x/(\mathbb{C}^\times \cdot \mathrm{Id}_V) \;=\; G_x^\times \times G_x^+,$$

where $G_x^\times$ is a diagonalizable group (an algebraic torus) and G_x^+ is a vector group. We have the corresponding decomposition of the Lie algebra

$$\mathfrak{g}_x/(\mathbb{C} \cdot \mathrm{Id}_V) \;=\; \mathfrak{g}_x^\times \oplus \mathfrak{g}_x^+.$$

We prove the following, which is a refinement of Theorems 1.3 and 1.4.

Theorem 1.6. *In the setting of Theorem 1.5,*

(i) *any element $F \in J^{-1}(x)$ is of Sebastiani-Thom type if and only if $\mathfrak{g}_x^\times \neq 0$; and*

(ii) *if $\mathfrak{g}_x^+ \neq 0$, then there is $0 \neq u \in V$ such that $[u]$ is a point of multiplicity $d-1$ on the hypersurface $Z(F)$ for all $F \in J^{-1}(x)$.*

The diagonalizable part $G_x^\times$ of the group G_x is well-explained by Theorem 1.6 (i), but the unipotent part G_x^+ is not fully described by (ii). It would be interesting to find more geometric consequences of the unipotent part G_x^+. We obtain the following result in this direction.

Theorem 1.7. *In the setting of Theorem 1.5, assume that for some $F \in J^{-1}(x)$, the hypersurface $Z(F) \subset \mathbb{P}V$ has only finitely many singular points of multiplicity $d-2$. (This is the case, for example, if $Z(F)$ has only isolated singularities.)*

(i) *The number of points in $\mathbb{P}\mathfrak{g}_x^+$ corresponding to nonzero elements $h \in \mathrm{End}(V)$ satisfying $h^2 = 0$ is less than or equal to the number of singular points of multiplicity $d-2$ on $Z(F)$.*

(ii) *For any $f \in \mathfrak{g}_x^+$, we have $f^3 = 0$.*

It would be interesting to investigate how big $\dim \mathfrak{g}_x^+$ can be, especially in the setting of Theorem 1.7.

The proof of Theorem 1.5 is rather simple, once one realizes what the group G_x should be. We describe the group G_x and prove Theorem 1.5 in Section 2. Theorems 1.6 and 1.7 are proved in Section 3.

2. Symmetrizer group of a symmetric form

Definition 2.1. For $F \in \mathrm{Sym}^d V^*$ and $g \in \mathrm{End}(V)$, define $F^g \in \otimes^d V^*$ by

$$F^g(v_1, \ldots, v_d) := F(g \cdot v_1, v_2, \ldots, v_d) \text{ for } v_1, \ldots, v_d \in V.$$

We say that g is a *symmetrizer* of F if $F^g \in \mathrm{Sym}^d V^*$, namely,

$$\begin{aligned} F(g \cdot v_1, v_2, v_3, \ldots, v_d) &= F(v_1, g \cdot v_2, v_3, \ldots, v_d) \\ &= F(v_1, v_2, g \cdot v_3, \ldots, v_d) \\ &= \cdots \\ &= F(v_1, v_2, v_3, \ldots, g \cdot v_d). \end{aligned}$$

The subspace $\mathfrak{g}_F \subset \mathrm{End}(V)$ of all symmetrizers of F is called the *symmetrizer algebra* of F and the intersection $G_F := \mathfrak{g}_F \cap \mathrm{GL}(V)$ is called the *symmetrizer group* of F.

The two names, the symmetrizer algebra and the symmetrizer group, in Definition 2.1 are justified by the next two propositions.

Proposition 2.2. *In Definition 2.1, the following holds.*

(i) *If $g, h \in \mathfrak{g}_F$, then $g \circ h \in \mathfrak{g}_F$. In particular, the vector space $\mathfrak{g}_F$ is a subalgebra under the composition in* $\mathrm{End}(V)$*, hence a Lie subalgebra of* $\mathrm{End}(V)$.

(ii) *If F is nondegenerate, then $g \circ h = h \circ g$ for any $g, h \in \mathfrak{g}_F$, namely, the Lie algebra $\mathfrak{g}_F$ is abelian.*

(iii) *If $h \in \mathfrak{g}_F$, then $F(u, w, v_1, \ldots, v_{d-2}) = 0$ for any $u \in \mathrm{Im}(h), w \in$* $\mathrm{Ker}(h)$ *and $v_1, \ldots, v_{d-2} \in V$.*

Proof. Write $gh = g \circ h$ for simplicity. Then for $g, h \in \mathfrak{g}_F$,

$$\begin{aligned} F(gh \cdot v_1, v_2, v_3, \ldots, v_d) &= F(h \cdot v_1, v_2, g \cdot v_3, \ldots, v_d) \\ &= F(v_1, h \cdot v_2, g \cdot v_3, \ldots, v_d) \\ &= F(v_1, gh \cdot v_2, v_3, \ldots, v_d). \end{aligned}$$

Thus $gh \in \mathfrak{g}_F$, proving (i).

Now assume that F is nondegenerate. Then

$$\begin{aligned} F(gh \cdot v_1, v_2, v_3, \ldots, v_d) &= F(h \cdot v_1, g \cdot v_2, v_3, \ldots, v_d) \\ &= F(v_1, hg \cdot v_2, v_3, \ldots, v_d) \\ &= F(hg \cdot v_1, v_2, v_3, \ldots, v_d), \end{aligned}$$

where the last equality uses (i). Thus $F((hg - gh) \cdot v_1, v_2, \ldots, v_d) = 0$ for all $v_1, \ldots, v_d \in V$. By the nondegeneracy of F, this implies $hg = gh$, proving (ii).

To prove (iii), write $u = h \cdot u'$ for some $u' \in V$. Then for $w \in \mathrm{Ker}(h)$,

$$\begin{aligned} F(u, w, v_1, \ldots, v_{d-2}) &= F(h \cdot u', w, v_1, \ldots, v_{d-2}) \\ &= F(u', h \cdot w, v_1, \ldots, v_{d-2}) = 0. \end{aligned}$$

□

Proposition 2.3. *For $F \in \mathrm{Sym}^d V^*$, the intersection $G_F = \mathfrak{g}_F \cap \mathrm{GL}(V)$ is a connected subgroup of* $\mathrm{GL}(V)$*, corresponding to the Lie subalgebra $\mathfrak{g}_F \subset$* $\mathrm{End}(V) = \mathfrak{gl}(V)$*. It is an abelian group if F is nondegenerate.*

Proof. Since G_F is a Zariski open subset in the vector space $\mathfrak{g}_F$, it is connected. To check that it is a subgroup, it suffices to show, by Proposition 2.2, that $g \in G_F$ implies $g^{-1} \in G_F$. For $v_i \in V$, write $u_i := g^{-1} \cdot v_i$. Then

$$\begin{aligned} F(g^{-1} \cdot v_1, v_2, v_3, \ldots, v_d) &= F(u_1, g \cdot u_2, v_3, \ldots, v_d) \\ &= F(g \cdot u_1, u_2, v_3, \ldots, v_d) \\ &= F(v_1, g^{-1} \cdot v_2, v_3, \ldots, v_d). \end{aligned}$$

This shows that $g^{-1} \in G_F$. $\square$

The proof of the following proposition is straightforward.

Proposition 2.4. *When $F = F_1 + \cdots + F_k \in \mathrm{Sym}^d V^*$ is of Sebastiani-Thom type in the notation of Definition 1.2,*

$$\mathfrak{g}_F = \mathfrak{g}_{F_1} \oplus \cdots \oplus \mathfrak{g}_{F_k} \text{ and } G_F = G_{F_1} \times \cdots \times G_{F_k},$$

where the products mean those coming from the decomposition $V = V_1 \oplus \cdots \oplus V_k$.

Proposition 2.5. *For $F \in \mathrm{Sym}^d V^*$ and $g \in G_F$,*

(i) $G_{F^g} = G_F$;
(ii) $\mathrm{Ker}(\partial F^g) = g^{-1} \cdot \mathrm{Ker}(\partial F)$; *and*
(iii) *F^g is nondegenerate if and only if F is nondegenerate.*

Assume that F is nondegenerate, then

(iv) *$F = F^g$ if and only if $g = \mathrm{Id}_V$; and*
(v) *$J(F) = J(F^g)$.*

Proof. To prove (i), pick $h \in G_F$. Then

$$F^g(h \cdot v_1, \ldots, v_d) = F(gh \cdot v_1, \ldots, v_d)$$

is symmetric in $v_1, \ldots, v_d$ because $gh \in G_F$. Thus $h \in G_{F^g}$, proving $G_F \subset G_{F^g}$. In particular, if $f \in G_{F^g}$, then $g^{-1}f \in G_{F^g}$. Consequently,

$$F(f \cdot v_1, \ldots, v_d) = F^g(g^{-1}f \cdot v_1, \ldots, v_d)$$

is symmetric in $v_1, \ldots, v_d$. This shows $f \in G_F$, proving $G_{F^g} \subset G_F$.

Note that $u \in \mathrm{Ker}(\partial F^g)$ if and only if

$$F^g(u, v_1, \ldots, v_{d-1}) = F(g \cdot u, v_1, \ldots, v_{d-1}) = 0 \text{ for all } v_1, \ldots, v_{d-1} \in V.$$

This is equivalent to saying $g \cdot u \in \mathrm{Ker}(\partial F)$. This proves (ii). (iii) is immediate from (ii).

Now assume that F is nondegenerate and $F = F^g$ for some $g \in G_F$. Then

$$0 = F(v_1, v_2, \ldots, v_d) - F(g \cdot v_1, v_2, \ldots, v_d) = F((\mathrm{Id}_V - g) \cdot v_1, v_2, \ldots, v_d)$$

for all $v_1, v_2, \ldots, v_{d-1} \in V$. By the nondegeneracy of F, this implies $g = \mathrm{Id}_V$, proving (iv).

To check (v), for each $u \in V$ and $v_1, \ldots, v_{d-1} \in V$,

$$(\partial F(u))(v_1, \ldots, v_{d-1}) = F(u, v_1, \ldots, v_{d-1}) = F^g(g^{-1} \cdot u, v_1, \ldots, v_{d-1}).$$

This means $\partial F(u) = \partial F^g(g^{-1} \cdot u)$. Thus $\mathrm{Im}(\partial F) = \mathrm{Im}(\partial F^g)$, implying $J(F) = J(F^g)$. $\square$

The following is the converse of Proposition 2.5 (iv).

Proposition 2.6. *Let $F, \widetilde{F} \in \mathrm{Sym}^d V^*$ be nondegenerate symmetric forms satisfying $J(F) = J(\widetilde{F})$. Then there exists $g \in G_F$ such that $\widetilde{F} = F^g$.*

Proof. Let $g \in \mathrm{GL}(V)$ be the composite

$$V \stackrel{\partial \widetilde{F}}{\longrightarrow} \mathrm{Im}(\partial \widetilde{F}) = \mathrm{Im}(\partial F) \stackrel{(\partial F)^{-1}}{\longrightarrow} V.$$

Then $g \in G_F$ and $F^g = \widetilde{F}$ because

$$\begin{aligned} F^g(v_1, v_2, \ldots, v_d) &= F(g \cdot v_1, v_2, \ldots, v_d) \\ &= (\partial F(g \cdot v_1))(v_2, \ldots, v_d) \\ &= (\partial F \circ g(v_1))(v_2, \ldots, v_d) \\ &= (\partial \widetilde{F}(v_1))(v_2, \ldots, v_d) \\ &= \widetilde{F}(v_1, v_2, \ldots, v_d) \end{aligned}$$

for all $v_1, \ldots, v_d \in V$. □

The following direct corollary of Propositions 2.5 and 2.6 implies Theorem 1.5.

Corollary 2.7. *For each point $x \in \mathrm{Im}(J) \subset \mathrm{Gr}(n; \mathrm{Sym}^{d-1} V^*)$, define $G_x := G_F \subset \mathrm{GL}(V)$ for any $F \in J^{-1}(x)$. Then G_x does not depend on the choice of $F \in J^{-1}(x)$ and the fiber $J^{-1}(x)$ is a principal homogeneous space of G_x.*

3. Diagonalizable and unipotent components of the symmetrizer group

In this section, we fix a nondegenerate form $F \in \mathrm{Sym}^d V^*$. The connected abelian group $G_F/(\mathbb{C}^\times \cdot \mathrm{Id}_V)$ has a canonical decomposition

$$G_F/(\mathbb{C}^\times \cdot \mathrm{Id}_V) = G_F^\times \times G_F^+,$$

where $G_F^\times$ is an algebraic torus and G_F^+ is a vector group. Let

$$\mathfrak{g}_F/(\mathbb{C} \cdot \mathrm{Id}_V) = \mathfrak{g}_F^\times \oplus \mathfrak{g}_F^+$$

be the corresponding decomposition of the Lie algebra, where $\mathfrak{g}_F^\times$ (resp. $\mathfrak{g}_F^+$) consists of semi-simple (resp. nilpotent) elements. The next proposition implies Theorem 1.6 (i).

Proposition 3.1. *If $\mathfrak{g}_F^\times \neq 0$, then F is of Sebastiani-Thom type. More precisely, let*

$$V = V_1 \oplus \cdots \oplus V_k, \; k \geq 2$$

be the decomposition into distinct weight spaces of the diagonalizable subgroup $\widetilde{G}_F^\times \subset G_F \subset \mathrm{GL}(V)$, which is the inverse image of the diagonalizable subgroup $G_F^\times$. Then $F = F_1 + \cdots + F_k$ for some $F_i \in \mathrm{Sym}^d V_i^*$.

Proof. We claim that if $v_i \in V_i$ and $v_j \in V_j$ for $i \neq j$, then

$$F(v_i, v_j, u_1, \ldots, u_{d-2}) = 0 \text{ for any } u_1, \ldots, u_{d-2} \in V.$$

By the claim, if we set $F_i = F|_{V_i}$, then we obtain $F = F_1 + \cdots + F_k$.

To prove the claim, let λ_i be the weight of V_i for $1 \leq i \leq k$. Then for any $g \in \widetilde{G}_F^{\times}$,

$$\begin{aligned}\lambda_i(g) F(v_i, v_j, u_1, \ldots, u_d) &= F(g \cdot v_i, v_j, u_1, \ldots, u_{d-2}) \\ &= F(v_i, g \cdot v_j, u_1, \ldots, u_{d-2}) \\ &= \lambda_j(g) F(v_i, v_j, u_1, \ldots, u_{d-2}).\end{aligned}$$

Since $\lambda_i \neq \lambda_j$, the claim follows. □

Propositions 2.4 and 3.1 show that the genuinely interesting part of the symmetrizer group is G_F^+. The next proposition implies Theorem 1.6 (ii).

Proposition 3.2. *Assume that* $\mathfrak{g}_F^+ \neq 0$.

(i) *There exists at least one nonzero element* $h \in \mathfrak{g}_F^+$ *satisfying* $h^2 = 0$.
(ii) *For* $0 \neq h \in \mathfrak{g}_F^+$ *satisfying* $h^2 = 0$, *every point of* $\mathbb{P}\mathrm{Im}(h) \subset \mathbb{P}V$ *is a singular point of* $Z(F)$ *with multiplicity* $d-2$.

Proof. Pick $0 \neq f \in \mathfrak{g}_F^+$. All powers $f^k, k \geq 2$, belong to $\mathfrak{g}_F^+$ by Proposition 2.2. Since f is nilpotent, there is an integer $\ell > 1$ satisfying $f^\ell = 0$. Then $h := f^{\ell-1}$ satisfies $h^2 = 0$. This proves (i).

In (ii), from Proposition 2.2 (iii) and $\mathrm{Im}(h) \subset \mathrm{Ker}(h)$, we have

$$F(u, u, v_1, \ldots, v_{d-2}) = 0$$

for all $u \in \mathrm{Im}(h)$ and $v_1, \ldots, v_{d-2} \in V$. Thus $[u] \in \mathbb{P}V$ is a singular point of $Z(F)$ with multiplicity $d-2$. □

We reformulate Theorem 1.7 as follows.

Theorem 3.3. *Assume that* $Z(F)$ *has only finitely many singular points of multiplicity* $d-2$.

(i) *If* $0 \neq h \in \mathfrak{g}_F^+$ *satisfies* $h^2 = 0$, *then* $\dim \mathrm{Im}(h) = 1$.
(ii) *If* $h, \widetilde{h} \in \mathfrak{g}_F^+$ *are nonzero elements satisfying* $h^2 = \widetilde{h}^2 = 0$ *and* $\mathrm{Im}(h) = \mathrm{Im}(\widetilde{h})$, *then* $[h] = [\widetilde{h}] \in \mathbb{P}\mathfrak{g}_F^+$.
(iii) *The number of points in* $\mathbb{P}\mathfrak{g}_F^+$ *corresponding to nonzero elements* $h \in \mathfrak{g}_F^+$ *satisfying* $h^2 = 0$ *is less than or equal to the number of singular points of multiplicity* $d-2$ *on* $Z(F)$.
(iv) *For any* $f \in \mathfrak{g}_F^+$, *we have* $f^3 = 0$.

Proof. By Proposition 3.2, if $h^2 = 0$, then every point on $\mathbb{P}\mathrm{Im}(h)$ is a singular point of $Z(F)$ with multiplicity $d-2$. By the assumption that there are only finitely many singular points of multiplicity $d-2$, we see $\dim \mathrm{Im}(h) = 1$, proving (i).

To prove (ii), pick a nonzero element $u \in \mathrm{Im}(h) = \mathrm{Im}(\widetilde{h})$. By Proposition 2.2 (iii), for all $v_1, \ldots, v_{d-2} \in V$,

$$F(u, \mathrm{Ker}(h), v_1, \ldots, v_{d-2}) = 0 = F(u, \mathrm{Ker}(\widetilde{h}), v_1, \ldots, v_{d-2}).$$

If $V = \mathrm{Ker}(h) + \mathrm{Ker}(\widetilde{h})$, we have $F(u, V, v_1, \ldots, v_{d-2}) = 0$, a contradiction to the nondegeneracy of F. Thus $V \neq \mathrm{Ker}(h) + \mathrm{Ker}(\widetilde{h})$, which implies $\mathrm{Ker}(h) = \mathrm{Ker}(\widetilde{h})$. It follows that $[h] = [\widetilde{h}]$.

(iii) follows from (ii), because $\mathbb{P}\mathrm{Im}(h) \in \mathbb{P}V$ is a singular point of $Z(F)$ with multiplicity $d-2$ by Proposition 3.2 (ii).

To prove (iv), assume the contrary that for some $f \in \mathfrak{g}_F^+$ and an integer $\ell \geq 4$, the elements $f, f^2, \ldots, f^{\ell-1}$ are nonzero and $f^\ell = 0$. Then $h := f^{\ell-1}$ and $\widetilde{h} := f^{\ell-2}$ satisfy $h^2 = \widetilde{h}^2 = 0$. Since $\mathrm{Im}(h) \subset \mathrm{Im}(\widetilde{h})$, we see by (i) and (ii) that $\widetilde{h} = ch$ for some $c \in \mathbb{C}^\times$. In other words, we have $f^{\ell-2} = cf^{\ell-1}$. Hence $f^{\ell-1} = cf^\ell = 0$, a contradiction. $\square$

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