UNIVERSAL GRAPH SERIES, CHROMATIC FUNCTIONS, AND THEIR INDEX THEORY

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Abstract. In the present paper, we introduce the concept of universal graph series. We then present four invariants of graphs and discuss some of their properties. In particular, one of these invariants is a generalization of the chromatic symmetric function and a complete invariant for graphs.

1. Introduction

The main focus of the present paper is on undirected simple graphs. Let G and H be graphs. We say that a mapping $f : V_G \to V_H$ between the sets of their vertices is a homomorphism if $\{f(x), f(y)\} \in E_H$ for all $\{x, y\} \in E_G$. Let $\mathrm{Hom}(G, H)$ denote the set of graph homomorphisms from G to H.

Definition 1.1. Let G be a finite simple graph. The chromatic polynomial of G is defined as

$$\chi(G, n) = \sharp \operatorname{Hom}(G, K_n) \ (\forall n \in \mathbb{N}).$$

The chromatic polynomial is not a complete invariant for graphs. Namely, there exist non-isomorphic graph pairs that have the same chromatic polynomial. A typical example is a non-isomorphic pair of trees with m vertices. Let T be a tree with m vertices. Then $\chi(T, n) = n(n-1)^{m-1}$. Hence, it is natural to ask whether there exists a graph invariant stronger than the chromatic polynomial.

In [12], Stanley defined the concept of chromatic symmetric functions:

Definition 1.2 ([12])**.** Let G be a simple graph and $x = \{x_i\}_{i\in\mathbb{N}}$ be countably many indeterminates. The chromatic symmetric function of

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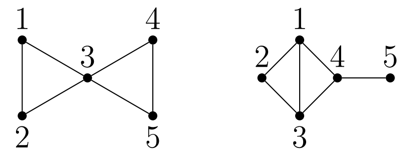


FIGURE 1. Graphs G_1 and G_2

G is defined as follows:

$$X(G) := X(G, x) := \sum_{\varphi \in \mathrm{Hom}(G, K_{\mathbb{N}})} \prod_{v \in V(G)} x_{\varphi(v)},$$

where $K_{\mathbb{N}}$ denotes the infinite complete graph on $\mathbb{N}$.

Remark 1.3. This invariant contains all the information of the chroamtic polynomials since, if $x = \mathbf{1}^n := (\underbrace{1, \ldots, 1}_{n}, 0, \ldots)$, then

$$X(G, \mathbf{1}^n) = \sum_{\varphi \in \mathrm{Hom}(G, K_n)} 1 = \sharp \,\mathrm{Hom}(G, K_n) = \chi(G, n).$$

For example, let K_2 be a complete graph with two vertices. Then

$$X(K_2) = \sum_{i<j} 2x_i x_j.$$

Stanley conjectured that $X(G)$ is a complete invariant for trees:

Conjecture 1.4 ([12])**.** If T_1 and T_2 are non-isomorphic trees then

$$X(T_1) \neq X(T_2).$$

However, $X(G)$ is not a complete invariant for graphs. Stanley gave the following example [12]. Let $V = \{1, 2, 3, 4, 5\}$ and

$$E_1 = \{\{1, 2\}, \{1, 3\}, \{2, 3\}, \{3, 4\}, \{3, 5\}, \{4, 5\}\},$$
$$E_2 = \{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{3, 4\}, \{4, 5\}\}.$$

Then $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ are non-isomorphic (See Figure 1) but

$$X(G_1) = X(G_2).$$

This gives rise to a natural question: is there a generalization of the chromatic symmetric function which is a complete invariant for graphs? This paper aims to provide a candidate generalization that answers this.

In the present paper, we introduce the concept of universal graph series. Then using this concept, we present four invariants of graphs and discuss some of their properties. This concept is motivated by the universal graphs introduced by Rado [11].

Definition 1.5 ([11])**.** Let G be an infinite graph. We say that G is universal if it contains every finite graph as an induced subgraph.

The following is a generalization of the concept of universal graphs:

Definition 1.6. Let $N \subset \mathbb{N}$ and let $\{H_n\}_{n\in N}$ be a family of graphs. We say that $\{H_n\}_{n\in N}$ is a universal graph series if for any simple graph G there exists $n \in N$ such that G is an induced subgraph of H_n. Moreover, the universal graph series $\{H_n\}_{n\in N}$ is said to be induced universal, if H_m is an induced subgraph of H_n for every pair $m, n \in N$ with $m \leq n$.

For example, the family of Kneser graphs $\{K_{\mathbb{N},k}\}_{k=1}^{\infty}$ is induced universal [7] and that of Paley graphs $\{P(q)\}_{q\in\mathcal{P}}$, where $\mathcal{P}$ is the set of all prime powers q with $q \equiv 1 \pmod 4$, are universal [2, 4, 6]. Additionally, the family of Peisert graphs and certain types of generalized Paley graphs are also universal. This is because these graphs include any graph of order n as an induced subgraph when the order of the graph is sufficiently large relative to n [1, 8]. In the present paper, we use two examples as the universal graph series: the family of Kneser graphs and the family of Paley graphs. The definitions of these two graphs will be provided in Sections 4 and 5.

To define our invariants, we first introduce the H-chromatic function, defined as follows:

Definition 1.7. Let x_u $(u \in V(H))$ be indeterminates. We define

$$X_H(G) := X_H(G, x) := \sum_{\varphi\in \mathrm{Hom}(G,H)} \prod_{v\in V(G)} x_{\varphi(v)}.$$

For a given universal graph series $\{H_n\}_{n\in N}$, we define the following invariants for graphs:

Definition 1.8. Let $H = \{H_n\}_{n\in N}$ be a universal graph series and G be a simple graph. Let $\mathcal{G}$ be the set of all the simple graphs.

(1) We define the universal H-chromatic functions of G as follows:

$$\{X_{H_n}(G)\}_{n\in N}.$$

(2) For $\mathcal{A} \subset \mathcal{G}$, we define the H-functional index $I_H(\mathcal{A})$ as

$$I_H(\mathcal{A}) = \min\{t \in N \mid \{X_{H_n}(\bullet)\}_{n=1}^{t} \text{ is a complete invariant for } \mathcal{A}\}.$$

(3) We define the H-induced index $i_H(G)$ as

$$i_H(G) = \min\{n \in N \mid G \text{ is an induced subgraph of } H_n\}.$$

(4) We define the H-index $\widetilde{i}_H(G)$ as
$$\widetilde{i}_H(G) = \min\{n \in N \mid G \text{ is a subgraph of } H_n\}.$$

Remark 1.9. (1) Let $H = \{H_n\}_{n\in N}$ be an induced universal graph series. Then for all $m, n \in N$ with $m \leq n$, $X_{H_n}(\bullet)$ is a stronger invariant than $X_{H_m}(\bullet)$. Namely, for any $G, G' \in \mathcal{G}$, if $X_{H_n}(G) = X_{H_n}(G')$ then $X_{H_m}(G) = X_{H_m}(G')$. For example, as mentioned before, $\{K_{\mathbb{N},k}\}_{k=1}^{\infty}$ is induced universal. Therefore, $X_{K_{\mathbb{N},k+1}}(\bullet)$ is a stronger invariant than $X_{K_{\mathbb{N},k}}(\bullet)$.

(2) Let $K = \{K_{\mathbb{N},k}\}_{k=1}^{\infty}$ be the Kneser graph series. The K-(induced) index was considered in [7]. The definition of the H-(indeced) index is motivated by their work.

Let us explain the main results of the present paper. If we have a universal graph series, then we have a complete invariant for graphs:

Theorem 1.10. *(1) Let H be a universal graph. Then*
$$X_H(\bullet)$$
is a complete invariant for finite graphs.

(2) Let $H = \{H_n\}_{n\in N}$ be a universal graph series. Then
$$\{X_{H_n}(\bullet)\}_{n\in N}$$
is a complete invariant for finite graphs.

For example, let $\{K_{\mathbb{N},k}\}_{k\in\mathbb{N}}$ be the Kneser graph series. Then
$$\{X_{K_{\mathbb{N},k}}(\bullet)\}_{k=1}^{\infty}$$
is a complete invariant for graphs. Since $K_{\mathbb{N},1} = K_{\mathbb{N}}$, $X_{K_{\mathbb{N},1}}(\bullet)$ is the chromatic symmetric function $X(\bullet)$ and in [12, Theorem 2.5], the expansion of $X(\bullet)$ in the basis of power sum symmetric functions was provided. The next theorem is a generalization of this formula to $K_{\mathbb{N},k}$ for $k \in \mathbb{N}$ (see Section 4 for the definitions of $\mathcal{A}_S^{(k)}$ and p_λ):

Theorem 1.11. *We have the following:*
$$X_{K_{\mathbb{N},k}}(G) = \sum_{S\subset E(G)} (-1)^{|S|} \sum_{\lambda\in\mathcal{A}_S^{(k)}} p_\lambda.$$

In [7], the upper bound of the Kneser (induced) index was studied. The following theorem provides upper bounds for some Paley (induced) indices. Let $P(q)$ denote the Paley graph with order q, where q is a prime power with $q \equiv 1 \pmod 4$. We denote by K_{k_1,k_2} the complete bipartite graph, with parts of size k_1 and k_2. Let C_k and P_k denote the cycle and path of order k, respectively.

Theorem 1.12. *Let q be an odd prime power. Then $P = \{P((q^2)^{3^n})\}_{n=1}^{\infty}$ is an induced universal graph series.*

(1) *Let $\mathcal{A}_k$ be the set of all simple graphs with at most k vertices. Then we have the following:*

$$I_P(\mathcal{A}_k) \leq \lceil \log_3 \log_q((k-1)2^{k-2}) \rceil.$$

(2) *Let k_1 and k_2 be any integers satisfying $q^{3^{m-1}} - 1 \leq k_1 \leq k_2 \leq q^{3^m} - 1$.*

(a) *Let*

$$\mathcal{H} = \{K_{k_1,k_2}, C_{2k_1}, P_{k_1+k_2-1}\}$$

be a set of graphs. Then, for any graph $G \in \mathcal{H}$, we have the following:

$$i_P(G) \leq \left\lceil \log_3 \log_q \left((q^{3^m} - 3) \binom{q^{3^m} - 1}{\frac{q^{3^m}-1}{2}} + 3 \right) \right\rceil.$$

(b) *We have the following:*

$$i_P(C_{2k_1+1}) \\ \leq \left\lceil \log_3 \log_q \left(2^{q^{3^m}} (q^{3^m} - 2) \binom{q^{3^m} - 1}{\frac{q^{3^m}-1}{2}} + 3 \right) \right\rceil.$$

(3) *For any integer k, we have the following:*

$$\widetilde{i}_P(C_k) = \widetilde{i}_P(P_k) = \left\lceil \log_3 \frac{1}{2}(\log_q k) \right\rceil.$$

This paper is organized as follows. In Section 2, We provide a proof of Theorem 1.10. In Section 3, we present definitions and some basic properties of graph homomorphisms and related topics used in this paper. In Sections 4, we offer a proof of Theorem 1.11. Subsequently, in Sections 5 and 6, we present a proof of Theorem 1.12. Finally, in Section 7, we provide concluding remarks and pose some questions for further research.

2. Proof of Theorem 1.10

In order to prove Theorem 1.10, the following lemma is required.

Lemma 2.1 ([5, p.128 Excercise 11])**.** *Let G_1 and G_2 be finite graphs. If $|\operatorname{Hom}(G_1, F)| = |\operatorname{Hom}(G_2, F)|$ for any finite graph F, then G_1 and G_2 are isomorphic.*

Proof of Theorem 1.10. We only give a proof of (1). The statement (2) can be proved similarly.

Suppose that $X_H(G_1) = X_H(G_2)$. Let F be a finite graph. Since H is a universal graph, F can be regarded as an induced subgraph of H. Then substituting $x_w = 1$ or 0 according as $w \in V(F)$ or not yields $|\operatorname{Hom}(G_1, F)| = |\operatorname{Hom}(G_2, F)|$. Therefore G_1 and G_2 are isomorphic by Lemma 2.1. □

3. Functions regarding homomorphisms and weak homomorphisms

In this section, we define weak homomorphisms and give some basic properties of the functions regarding (weak) homomorphisms, which will be used in Section 4.

3.1. Expansion formulas for $X_H(G)$ and $W_H(G)$. Let G and H be simple graphs. Define

$$\operatorname{Hom}^{\mathrm{w}}(G,H) := \left\{ \varphi\colon V(G) \to V(H) \;\middle|\; \begin{array}{c} \text{If } \{u,v\} \in E(G) \text{ then} \\ \{\varphi(u), \varphi(v)\} \in E(H) \\ \text{or } \varphi(u) = \varphi(v) \end{array} \right\}.$$

A map in $\operatorname{Hom}^{\mathrm{w}}(G,H)$ is called a weak homomorphism.

Define

$$W_H(G) := W_H(G,x) := \sum_{\varphi \in \operatorname{Hom}^{\mathrm{w}}(G,H)} \prod_{v \in V(G)} x_{\varphi(v)}.$$

Both functions $X_H(G)$ and $W_H(G)$ belong to the ring of formal power series $R_H := \mathbb{C}[\![x_w \mid w \in V(H)]\!]$. The automorphism group $\operatorname{Aut}(H)$ naturally acts on R_H and both $X_H(G)$ and $W_H(G)$ are members of the invariant ring $R_H^{\operatorname{Aut}(H)}$.

Proposition 3.1. *$X_H(G) = \sum_{S \subset E(G)} (-1)^{|S|} W_{\overline{H}}(G_S)$, where G_S denotes the spanning subgraph of G with edge set S and $\overline{H}$ denotes the complement of H.*

Proof.

$$\begin{aligned} \sum_{S \subset E(G)} (-1)^{|S|} W_{\overline{H}}(G_S) &= \sum_{S \subset E(G)} (-1)^{|S|} \sum_{\varphi \in \operatorname{Hom}^{\mathrm{w}}(G(S), \overline{H})} \prod_{v \in V(G)} x_{\varphi(v)} \\ &= \sum_{\varphi\colon V(G) \to V(H)} \sum_{S \subset E_\varphi} (-1)^{|S|} \prod_{v \in V(G)} x_{\varphi(v)}, \end{aligned}$$

where

$$\begin{aligned} E_\varphi &:= \left\{ \{u,v\} \in E(G) \;\middle|\; \varphi(u) = \varphi(v) \text{ or } \{\varphi(u), \varphi(v)\} \in E(\overline{H}) \right\} \\ &= \left\{ \{u,v\} \in E(G) \mid \{\varphi(u), \varphi(v)\} \notin E(H) \right\}. \end{aligned}$$

Since

$$\sum_{S \subset E_\varphi} (-1)^{|S|} = \begin{cases} 1 & \text{if } E_\varphi = \emptyset. \\ 0 & \text{otherwise.} \end{cases}$$

Thus

$$\sum_{\varphi\colon V(G) \to V(H)} \sum_{S \subset E_\varphi} (-1)^{|S|} \prod_{v \in V(G)} x_{\varphi(v)} = X_H(G),$$

which proves the assertion. □

4. Power sum expansion of $X_{K_{\mathbb{N},k}}(G, x)$

In this section, we give a proof of Theorem 1.11.

Definition 4.1 (Kneser graph)**.** Kneser graph $K_{\mathbb{N},k}$ is a graph whose vertex set consists of k-subsets of $\mathbb{N}$, and two vertices A and B are adjacent if $A \cap B = \emptyset$.

Note that the automorphism group of Kneser graph $\mathrm{Aut}(K_{\mathbb{N},k})$ is isomorphic to the symmetric group $S_{\mathbb{N}}$ and $X_{K_{\mathbb{N},k}}(G)$ is invariant under the natural action of $S_{\mathbb{N}}$. Define $\mathrm{Sym}^{(k)}$ to be the subring of $R_{K_{\mathbb{N},k}}^{S_{\mathbb{N}}}$ consisting of elements of finite degrees. Since $X_{K_{\mathbb{N},k}}(G)$ is homogeneous of degree $|V(G)|$, $X_{K_{\mathbb{N},k}}(G)$ belongs to $\mathrm{Sym}^{(k)}$. Note that $\mathrm{Sym}^{(1)}$ is the ring of symmetric functions and $X_{K_{\mathbb{N},1}}(G)$ is the chromatic symmetric function of G.

Let $\{I_1, \dots, I_n\}$ and $\{J_1, \dots, J_n\}$ be two multisets consisting of elements in $\binom{\mathbb{N}}{k}$. Define an equivalence relation $\sim$ by $\{I_1, \dots, I_n\} \sim \{J_1, \dots, J_n\}$ if there exists $\sigma \in S_{\mathbb{N}}$ such that

$$\{I_1, \dots, I_n\} = \{\sigma(J_1), \dots, \sigma(J_n)\}$$

as multisets.

Definition 4.2. Let $\mathcal{P}_n^{(k)}$ denote the equivalence classes of such multisets discussed above and $\mathcal{P}^{(k)} := \bigsqcup_{n=1}^{\infty} \mathcal{P}_n^{(k)}$.

Let $\lambda \in \mathcal{P}_n^{(k)}$ and $\{I_1, \dots, I_n\}$ a representative of λ. Then λ can be regarded as a k-uniform hyper-multigraph on $V_\lambda = I_1 \cup \cdots \cup I_n$ with edge multiset $E_\lambda = \{I_1, \dots, I_n\}$. Note that the isomorphism type of this hypergraph is independent of the choice of the representative $\{I_1, \dots, I_n\}$. Also, note that $\mathcal{P}_n^{(k)}$ is a finite set for any n, k.

For example, when $k = 1$ the graph λ is a 1-uniform hyper-multigraph. In this case, λ can be identified with the integer partition consisting of multiplicities of hyperedges of λ. Thus $\mathcal{P}^{(1)}$ is the set of integer partitions. When $k = 2$, λ is a 2-uniform hyper-multigraph. Therefore $\mathcal{P}^{(2)}$ is the set of multigraphs without loops or isolated vertices.

Suppose that $\lambda \in \mathcal{P}^{(k)}$. Define the monomial k-fold symmetric function $m_\lambda = m_\lambda^{(k)} \in \mathrm{Sym}^{(k)}$ by

$$m_\lambda \coloneqq \sum_{\{I_1,\dots,I_n\}\in\lambda} x_{I_1}\cdots x_{I_n}.$$

Every homogeneous component of $\mathrm{Sym}^{(k)}$ is finite dimensional and the set $\{m_\lambda\}_\lambda$ forms a linear basis for $\mathrm{Sym}^{(k)}$ over $\mathbb{C}$.

We say that $\lambda \in \mathcal{P}^{(k)}$ is connected if it is connected as a hyper-multigraph. The hyper-multigraph λ can be decomposed into the disjoint union $\lambda = \lambda_1 \sqcup \cdots \sqcup \lambda_\ell$ in the usual manner. Then we define the power sum k-fold symmetric function $p_\lambda = p_\lambda^{(k)} \in \mathrm{Sym}^{(k)}$ by

$$p_\lambda \coloneqq m_{\lambda_1}\cdots m_{\lambda_\ell}.$$

Note that when $k = 1$, p_λ is the usual power sum symmetric function.

Proposition 4.3. *The set $\{p_\lambda\}_{\lambda\in\mathcal{P}^{(k)}}$ forms a linear basis for $\mathrm{Sym}^{(k)}$ over $\mathbb{C}$. In particular, $\mathrm{Sym}^{(k)}$ is freely generated as a $\mathbb{C}$-algebra by $\left\{ p_\lambda \in \mathcal{P}^{(k)} \;\middle|\; \lambda \text{ is connected} \right\}$.*

Proof. Let $\lambda, \mu \in \mathcal{P}_n^{(k)}$. Define a partial order $\leq$ on $\mathcal{P}_n^{(k)}$ by setting $\mu \leq \lambda$ if μ is obtained by identifying some vertices of λ.

Now, let $\lambda \in \mathcal{P}_{n_1}^{(k)}, \mu \in \mathcal{P}_{n_2}^{(k)}$ and consider the product $m_\lambda m_\mu$. There exist $c_\nu \in \mathbb{Z}_{\geq 0}$ for every $\nu \in \mathcal{P}_{n_1+n_2}^{(k)}$ such that

$$m_\lambda m_\mu = \sum_{\nu\in\mathcal{P}_{n_1+n_2}^{(k)}} c_\nu m_\nu.$$

It follows that $c_{\lambda\sqcup\mu} > 0$ and $c_\nu = 0$ unless $\nu \leq \lambda \sqcup \mu$. Hence

$$m_\lambda m_\mu = c_{\lambda\sqcup\mu} m_{\lambda\sqcup\mu} + \sum_{\nu<\lambda\sqcup\mu} c_\nu m_\nu.$$

Therefore for every $\lambda \in \mathcal{P}_n^{(k)}$, there exists a positive integer d_μ for every $\mu \in \mathcal{P}_n^{(k)}$ such that

$$p_\lambda = d_\lambda m_\lambda + \sum_{\mu<\lambda} d_\mu m_\mu.$$

Hence we can order the equivalence classes linearly such that the coefficient matrix of $\{p_\lambda\}_{\lambda\in\mathcal{P}^{(k)}}$ for the basis $\{m_\lambda\}_{\lambda\in\mathcal{P}^{(k)}}$ is upper triangular. Therefore $\{p_\lambda\}_{\lambda\in\mathcal{P}^{(k)}}$ is a basis for $\mathrm{Sym}^{(k)}$. $\square$

Definition 4.4. Let G be a connected graph on n vertices. We say that $\lambda \in \mathcal{P}_n^{(k)}$ is admissible by G if there exists a bijection $\varphi\colon V(G) \to E_\lambda$ such that $\{u, v\} \in E(G)$ implies $\varphi(u) \cap \varphi(v) \neq \emptyset$. Let $\mathcal{A}_G^{(k)}$ denote the elements in $\mathcal{P}_n^{(k)}$ that is admissible by G.

When G is disconnected and $G = G_1 \sqcup \cdots \sqcup G_\ell$ is the decomposition into connected components, define $\mathcal{A}_G^{(k)} \coloneqq \mathcal{A}_{G_1}^{(k)} \times \cdots \times \mathcal{A}_{G_\ell}^{(k)}$.

Lemma 4.5.

$$W_{\overline{K_{\mathbb{N},k}}}(G) = \sum_{\lambda \in \mathcal{A}_G^{(k)}} p_\lambda,$$

where $p_\lambda = p_{\lambda_1} \cdots p_{\lambda_\ell}$ *if* $\lambda = (\lambda_1, \ldots, \lambda_\ell)$.

Proof. First suppose that G is connected. Then

$$W_{\overline{K_{\mathbb{N},k}}}(G) = \sum_{\varphi \in \mathrm{Hom}^{\mathrm{w}}(G, \overline{K_{\mathbb{N},k}})} \prod_{v \in V(G)} x_{\varphi(v)} = \sum_{\lambda \in \mathcal{A}_G^{(k)}} m_\lambda = \sum_{\lambda \in \mathcal{A}_G^{(k)}} p_\lambda.$$

Next, suppose that $G = G_1 \sqcup \cdots \sqcup G_\ell$ is the decomposition into connected components. Then the natural bijection

$$\mathrm{Hom}^{\mathrm{w}}(G, \overline{K_{\mathbb{N},k}}) \simeq \prod_{i=1}^{\ell} \mathrm{Hom}^{\mathrm{w}}(G_i, \overline{K_{\mathbb{N},k}})$$

implies the equality

$$W_{\overline{K_{\mathbb{N},k}}}(G) = \prod_{i=1}^{\ell} W_{\overline{K_{\mathbb{N},k}}}(G_i) = \prod_{i=1}^{\ell} \sum_{\lambda_i \in \mathcal{A}_{G_i}^{(k)}} p_{\lambda_i} = \sum_{\lambda \in \mathcal{A}_G^{(k)}} p_\lambda.$$

□

Theorem 4.6 (Restatement of Theorem 1.11)**.**

$$X_{K_{\mathbb{N},k}}(G) = \sum_{S \subset E(G)} (-1)^{|S|} \sum_{\lambda \in \mathcal{A}_S^{(k)}} p_\lambda,$$

where $\mathcal{A}_S^{(k)}$ *stands for* $\mathcal{A}_{G_S}^{(k)}$ *and* G_S *is the spanning subgraph of* G *with edge set* S.

Proof. It follows from Proposition 3.1 and Lemma 4.5. □

Example 4.7. Consider the case $G = P_3$ and $k = 2$. The spanning subgraphs of G are isomorphic to one of $3K_1, K_2 \sqcup K_1, P_3$. We have

$$\mathcal{A}^{(2)}_{3K_1} = \mathcal{A}^{(2)}_{K_1} \times \mathcal{A}^{(2)}_{K_1} \times \mathcal{A}^{(2)}_{K_1} = \left\{ \left(\quad \right) \right\},$$
$$\mathcal{A}^{(2)}_{K_2 \sqcup K_1} = \mathcal{A}^{(2)}_{K_2} \times \mathcal{A}^{(2)}_{K_1} = \left\{ \left(\quad \right), \left(\quad \right) \right\},$$
$$\mathcal{A}^{(2)}_{P_3} = \left\{ \quad , \quad , \quad , \quad , \quad \right\}.$$

Therefore

$$X_{K_{\mathbb{N},2}}(P_3) = \sum_{\lambda \in \mathcal{A}^{(2)}_{3K_1}} p_\lambda - 2 \sum_{\lambda \in \mathcal{A}^{(2)}_{K_2 \sqcup K_1}} p_\lambda + \sum_{\lambda \in \mathcal{A}^{(2)}_{P_3}} p_\lambda$$
$$= p - 2p - 2p + p + p + p + p.$$

Example 4.8. Consider the case $G = K_3$ and $k = 2$. The spanning subgraphs of G are isomorphic to one of $3K_1, K_2 \sqcup K_1, P_3, K_3$. Since

$$\mathcal{A}^{(2)}_{K_3} = \left\{ \quad , \quad , \quad , \quad \right\}$$

we have

$$X_{K_{\mathbb{N},2}}(K_3) = \sum_{\lambda \in \mathcal{A}^{(2)}_{3K_1}} p_\lambda - 3 \sum_{\lambda \in \mathcal{A}^{(2)}_{K_2 \sqcup K_1}} p_\lambda + 3 \sum_{\lambda \in \mathcal{A}^{(2)}_{P_3}} p_\lambda - \sum_{\lambda \in \mathcal{A}^{(2)}_{K_3}} p_\lambda$$
$$= p - 3p - 3p + 3p + 2p + 2p + 2p.$$

5. Evaluation of the Paley-induced index

In this section, we first provide a trivial upper bound for the Paley-induced index. Then, we give proofs of Theorem 1.12 (1) and (2a).

The lower and upper bounds of the Kneser-induced index of paths, cycle graphs, and hypercube graphs are shown in [7]. Therefore, we investigate the induced index using another universal graph.

Definition 5.1 (Paley graph)**.** Let q be a prime power such that $q \equiv 1 \pmod 4$. A Paley graph $P(q)$ is a graph whose vertex set consists of the elements of a finite field of order q, where two vertices u and v are adjacent if $u - v \in (\mathbb{F}_q^*)^2$.

It is known that if q is sufficiently large relative to n, $P(q)$ contains any graph with order n. Hence, any infinite sequence of Paley graphs forms a universal graph series. We then investigate the Paley-induced index.

5.1. **A trivial upper bound for the Paley-induced index.** In this subsection, we provide a trivial upper bound for the Paley-induced index.

Let $\mathbb{F}_q$ be a finite field with order q and $P(q)$ be the Paley graph of order q. In the following, we always denote by $P_n(q,k)$ the Paley graph $P(q^{k^n})$.

Proposition 5.2. *Let k be any positive integer. Then $P_n(q,2k+1)$ is an induced subgraph of $P_{n+1}(q,2k+1)$.*

Proof. For the sake of simplicity, we denote $q^{(2k+1)^n}$ as q_0. We consider the induced subgraph of $P_{n+1}(q,2k+1)$ constructed by $\mathbb{F}_{q_0}$. This graph can be denoted as

$$Cay(\mathbb{F}_{q_0}, (\mathbb{F}^*_{q_0^{2k+1}})^2 \cap \mathbb{F}_{q_0}),$$

where $Cay(G,S)$ represents the Cayley graph made by the group G and its subset S. Therefore, if

$$(\mathbb{F}^*_{q_0^{2k+1}})^2 \cap \mathbb{F}_{q_0} = (\mathbb{F}^*_{q_0})^2$$

then this graph is $P_n(q,2k+1)$. Because $\mathbb{F}_{q_0^{2k+1}}$ is an odd-degree extension of $\mathbb{F}_{q_0}$, the above equality holds. Hence, the induced subgraph of $P_{n+1}(q,2k+1)$ constructed by $\mathbb{F}_{q_0}$ is $P_n(q,2k+1)$. □

From Proposition 5.2, $\{P_n(q,2k+1)\}_{n\in\mathbb{N}}$ is induced universal. In the following, we consider the case $P = \{P_n(q^2,3)\}_{n\in\mathbb{N}}$. Then, we can define the Paley-induced index i_P as follows:

$$i_P(G) = \min\{n \in \mathbb{N} \mid G \text{ is an induced subgraph of } P_n(q^2,3)\}.$$

There exists a trivial upper bound for this invariant using the number of vertices in the graph.

Theorem 5.3. *Let G be a graph with order k. Then,*

$$i_P(G) \leq \lceil \log_3 \log_q((k-1)2^{k-2}) \rceil.$$

Proof. From Theorem 7.19 of [9], any graph G with order k is an induced subgraph of $P(q_0)$ if the following inequality holds:

$$q_0 > ((k-1)2^{k-2})^2.$$

Therefore, G is an induced subgraph of $P_n(q^2,3)$ for any n which satisfies

$$(q^2)^{3^n} > ((k-1)2^{k-2})^2 \Leftrightarrow n > \log_3 \log_q((k-1)2^{k-2}).$$

Especially, since $\lceil \log_3 \log_q((k-1)2^{k-2}) \rceil > \log_3 \log_q((k-1)2^{k-2})$, $P_n(q^2,3)$ contains G when $n = \lceil \log_3 \log_q((k-1)2^{k-2}) \rceil$. Hence,

$$i_P(G) \leq \lceil \log_3 \log_q((k-1)2^{k-2}) \rceil.$$

$\square$

Proof of Theorem 1.12 (1). Let $\mathcal{A}_k$ be the set of all simple graphs with at most k vertices. Theorem 5.3 implies the upper bound for $I_P(\mathcal{A}_k)$. Similarly to the discussion in the proof of Theorem 1.10, for any graph $G \in \mathcal{A}_k$, $\{|\operatorname{Hom}(G,F)|\}_{F\in\mathcal{A}_k}$ is also a complete invariant. Therefore, if there exists a graph H such that H contains all graphs in $\mathcal{A}_k$ as induced subgraphs, then $X_H(G)$ is a complete invariant. Hence, from Theorem 5.3,

$$I_P(\mathcal{A}_k) \leq \lceil \log_3 \log_q((k-1)2^{k-2}) \rceil.$$

$\square$

5.2. **Lemmas for a non-trivial upper bound for some Paley-induced indices.** In this subsection, we provide lemmas to prove the statement (2a) in Theorem 1.12. First, we introduce the theorems related to the character sums used in the proofs of the lemmas.

Theorem 5.4 ([15]). *Let f be a polynomial in $\mathbb{F}_q[x]$ of degree d and let σ be a multiplicative character of $\mathbb{F}_q^*$ with order 2. If there does not exist any polynomial g in $\mathbb{F}_q[x]$ such that $f = g^2$, then*

$$\left| \sum_{v\in\mathbb{F}_q} \sigma(f(v)) \right| \leq (d-1)\sqrt{q}.$$

Theorem 5.5 ([1]). *Let f be a polynomial in $\mathbb{F}_q[x]$ of degree d and let σ be a multiplicative character of $\mathbb{F}_q^*$ with order 2. If d is even and there does not exist any polynomial g in $\mathbb{F}_q[x]$ such that $f = g^2$, then*

$$\left| \sum_{v\in\mathbb{F}_q} \sigma(f(v)) \right| \leq 1 + (d-2)\sqrt{q}.$$

Afterwards, we denote K_{k_1,k_2} as the complete bipartite graph, where k_1 and k_2 represent the sizes of the independent sets, respectively, and C_k and P_k denote the cycle and path with order k, respectively. Let p be a prime number and let m and n be any integers with $m < n$. We define $q := p^{3^m}$ and $q_0 := (p^2)^{3^n}$.

Lemma 5.6. *If q_0 satisfies the following inequality:*

$$q_0 > \left((q-3)\binom{q-1}{\frac{q-1}{2}} + 3 \right)^2$$

then $K_{q-1,q-1}$ is an induced subgraph of $P(q_0)$, when $q \geq 5$.

Proof. Since $\mathbb{F}_{q_0}$ is an even-degree extension of $\mathbb{F}_q$, $\mathbb{F}_q \subset (\mathbb{F}_{q_0}^*)^2$.

Let x be any element of quadratic non-residues of $\mathbb{F}_{q_0}$, and define A as the vertex set of $P(q_0)$ as follows:

$$A \coloneqq \{ax \mid a \in \mathbb{F}_q\}.$$

From the definition of A, for any two elements in A, the difference between them is ax, where a is some element in $\mathbb{F}_q$. Therefore, the induced subgraph of $P(q_0)$ constructed by A is an independent set with order q. Next, we assume that there exists a vertex y such that $y \in V \setminus A$ and y is adjacent to all elements of $A \setminus \{0\}$. Similarly, we define B as the vertex set of $P(q_0)$ as follows:

$$B \coloneqq \{by \mid b \in \mathbb{F}_q\}.$$

Because y is not adjacent to 0, y is a quadratic non-residue, and an induced subgraph of $P(q_0)$ constructed by B is also an independent set with order q. Furthermore, $by - ax = b(y - b^{-1}ax)$ and $y - b^{-1}ax$ is a quadratic residue because of the definition of y. This implies that for any element in B, it is adjacent to all elements of $A \setminus \{0\}$. Therefore, the induced subgraph constructed by $(A \cup B) \setminus \{0\}$ is $K_{q-1,q-1}$. Additionally, y exists if and only if a vertex, denoted as $y' = y + x$, also exists which is adjacent to all elements of $A \setminus \{x\}$. Hence, we consider the condition of q_0 under which a vertex y' exists.

Let $D = \{x\}, C = A \setminus D$ and σ be a multiplicative character of $\mathbb{F}_{q_0}^*$ with order 2, where we define $\sigma(0) = 0$. We consider the function τ as follows:

$$\tau(v) = \prod_{c \in C}(1 + \sigma(v - c)) \prod_{d \in D}(1 - \sigma(v - d)).$$

When $\tau(v) = 2^q$, then v is adjacent to all vertices of C and not adjacent to any vertices of D. Here, if $v \in \mathbb{F}_{q_0} \setminus A$, then

$$\tau(v) \in \{0, 2^q\}.$$

Therefore, the condition for the existence of the desired vertex y' is the following inequality:

$$\sum_{v \in \mathbb{F}_{q_0} \setminus A} \tau(v) > 0.$$

Additionally, in the case $v \in A$, $\tau(v)$ is 0. Therefore, we consider the bound for q_0 which satisfies the following inequality:

$$\sum_{v \in \mathbb{F}_{q_0}} \tau(v) > 0.$$

Expanding the product that defines $\tau(v)$, we obtain

$$\sum_{v\in\mathbb{F}_{q_0}} \tau(v) = q_0 + \sum_{\emptyset\neq S\subset A} (-1)^{|S\cap D|} \sum_{v\in\mathbb{F}_{q_0}} \sigma(f_S(v)),$$

where S represents any subset of A, and $f_S(v) = \prod_{s\in S}(v-s)$. Let $\mathcal{S}$ be the power set of A. Then, we can define a bijection φ_a using some element $a \in \mathbb{F}_q$ as follows:

$$\varphi_a : \mathcal{S} \longrightarrow \mathcal{S}; S \mapsto aS.$$

Additionally, because $\sigma(a) = 1$,

$$\begin{aligned}
\sum_{v\in\mathbb{F}_{q_0}} \sigma(f_{aS}(v)) &= \sum_{v\in\mathbb{F}_{q_0}} \sigma\left(\prod_{x\in aS}(v-x)\right) \\
&= \sum_{v\in\mathbb{F}_{q_0}} \sigma(a)^{|S|}\sigma\left(\prod_{x\in S}(a^{-1}v-x)\right) \\
&= \sum_{v\in\mathbb{F}_{q_0}} \sigma\left(\prod_{x\in S}(v-x)\right) \\
&= \sum_{v\in\mathbb{F}_{q_0}} \sigma(f_S(v)).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\sum_{\emptyset\neq S\subset A} (-1)^{|S\cap D|} \sum_{v\in\mathbb{F}_{q_0}} \sigma(f_S(v)) &= \sum_{\emptyset\neq S\subset A} (-1)^{|(aS)\cap(aD)|} \sum_{v\in\mathbb{F}_{q_0}} \sigma(f_S(v)) \\
&= \sum_{\emptyset\neq S\subset A} (-1)^{|S\cap\{ax\}|} \sum_{v\in\mathbb{F}_{q_0}} \sigma(f_{a^{-1}S}(v)) \\
&= \sum_{\emptyset\neq S\subset A} (-1)^{|S\cap\{ax\}|} \sum_{v\in\mathbb{F}_{q_0}} \sigma(f_S(v)).
\end{aligned}$$

From the above,

$$\begin{aligned}
&\sum_{\emptyset\neq S\subset A} (-1)^{|S\cap D|} \sum_{v\in\mathbb{F}_{q_0}} \sigma(f_S(v)) \\
&\quad= \sum_{\emptyset\neq S\subset A} \sum_{a\in\mathbb{F}_q^*} \frac{(-1)^{|S\cap\{ax\}|}}{q-1} \sum_{v\in\mathbb{F}_{q_0}} \sigma(f_S(v)) \\
&\quad= \sum_{\emptyset\neq S\subset A\setminus\{0\}} \sum_{a\in\mathbb{F}_q^*} \frac{(-1)^{|S\cap\{ax\}|}}{q-1} \sum_{v\in\mathbb{F}_{q_0}} (\sigma(f_S(v)) + \sigma(f_{S\cup\{0\}}(v))).
\end{aligned}$$

Furthermore, the following equality holds when $S \subset A \setminus \{0\}$:

$$\begin{aligned}\sum_{a\in\mathbb{F}_q^*}(-1)^{|S\cap\{ax\}|} &= |S|(-1)^1 + q - 1 - |S|(-1)^0\\ &= q-1-2|S|.\end{aligned}$$

Since either $|S|$ or $|S\cup\{0\}|$ is always even, from Theorem 5.4 and Theorem 5.5,

$$\begin{aligned}\sum_{v\in\mathbb{F}_{q_0}}(\sigma(f_S(v))+\sigma(f_{S\cup\{0\}}(v))) &\geq -\left|\sum_{v\in\mathbb{F}_{q_0}}(\sigma(f_S(v))+\sigma(f_{S\cup\{0\}}(v)))\right|\\ &\geq -(1+2\sqrt{q_0}(|S|-1))\end{aligned}$$

Therefore,

$$\sum_{v\in\mathbb{F}_{q_0}}\tau(v) \geq q_0 - \sum_{\emptyset\neq S\subset A\setminus\{0\}}\frac{|q-1-2|S||}{q-1}(1+2(|S|-1)\sqrt{q_0}). \tag{1}$$

On the other hand,

$$\begin{aligned}&\sum_{\emptyset\neq S\subset A\setminus\{0\}}\frac{|q-1-2|S||}{q-1}(1+2(|S|-1)\sqrt{q_0})\\ &\quad=\sum_{i=1}^{q-1}\frac{|q-1-2i|}{q-1}(1+2(i-1)\sqrt{q_0})\binom{q-1}{i}\\ &\quad=\left(\sum_{i=1}^{\frac{q-1}{2}}-\sum_{i=\frac{q+1}{2}}^{q-1}\right)\left(\frac{q-1-2i}{q-1}(1+2(i-1)\sqrt{q_0})\binom{q-1}{i}\right),\end{aligned}$$

where $(\sum_{i=1}^{x}-\sum_{i=x+1}^{y})f(i) = (\sum_{i=1}^{x}f(i)) - (\sum_{i=x+1}^{y}f(i))$. From the properties of binomial coefficients,

$$\begin{aligned}\frac{2(q-1-2i)(i-1)}{q-1}\binom{q-1}{i} &= \left(\binom{q-1}{i}-2\binom{q-2}{i-1}\right)2(i-1)\\ &= 2\left(-\binom{q-1}{i}+(q-1)\binom{q-2}{i-1}-2(q-2)\binom{q-3}{i-2}\right).\end{aligned}$$

Furthermore,

$$\sum_{i=1}^{\frac{q-1}{2}} \binom{q-1}{i} - \sum_{i=\frac{q+1}{2}}^{q-1} \binom{q-1}{i} = \binom{q-1}{\frac{q-1}{2}} - 1,$$

$$\sum_{i=1}^{\frac{q-1}{2}} \binom{q-2}{i-1} - \sum_{i=\frac{q+1}{2}}^{q-1} \binom{q-2}{i-1} = 0,$$

$$\sum_{i=2}^{\frac{q-1}{2}} \binom{q-3}{i-2} - \sum_{i=\frac{q+1}{2}}^{q-1} \binom{q-3}{i-2} = -\binom{q-3}{\frac{q-3}{2}}.$$

Hence,

$$\begin{aligned}
&\left(\sum_{i=1}^{\frac{q-1}{2}} - \sum_{i=\frac{q+1}{2}}^{q}\right) \frac{q-1-2i}{q-1}(1+2(i-1)\sqrt{q_0}) \binom{q-1}{i} \\
&= \left(\binom{q-1}{\frac{q-1}{2}} - 1\right) + 2\sqrt{q_0}\left(-\binom{q-1}{\frac{q-1}{2}} + 1 + 2(q-2)\binom{q-3}{\frac{q-3}{2}}\right) \\
&= \sqrt{q_0}\left((q-3)\binom{q-1}{\frac{q-1}{2}} + 2\right) + \binom{q-1}{\frac{q-1}{2}} - 1.
\end{aligned}$$

Based on the above,

$$\sum_{v\in\mathbb{F}_{q_0}} \tau(v) \geq q_0 - \sqrt{q_0}\left((q-3)\binom{q-1}{\frac{q-1}{2}} + 2\right) - \binom{q-1}{\frac{q-1}{2}} + 1.$$

The desired vertex exists if the right-hand side of this inequality is greater than 0. Especially, when $q \geq 5$, if q_0 satisfies the following inequality:

$$q_0 > \left((q-3)\binom{q-1}{\frac{q-1}{2}} + 3\right)^2$$

then the desired vertex exists and $K_{q-1,q-1}$ is an induced subgraph of $P(q_0)$. □

Corollary 5.7. *For any integers k_1 and k_2 such that $0 < k_1 \leq k_2 < q-1$, if q_0 satisfies the following inequality:*

$$q_0 > \left((q-3)\binom{q-1}{\frac{q-1}{2}} + 3\right)^2$$

then K_{k_1,k_2} is an induced subgraph of $P(q_0)$, where $q \geq 5$.

Proof. K_{k_1,k_2} is an induced subgraph of $K_{q-1,q-1}$, and from Lemma 5.6, $K_{q-1,q-1}$ is an induced subgraph of $P(q_0)$ under this condition. $\square$

Lemma 5.8. *If q_0 satisfies the following inequality:*

$$q_0 > \left((q-3)\binom{q-1}{\frac{q-1}{2}} + 3\right)^2$$

then $C_{2(q-1)}$ is an induced subgraph of $P(q_0)$, where $q \geq 5$.

Proof. In the same manner of Lemma 5.6, let x be some quadratic non-residue in $\mathbb{F}_{q_0}$, and define $A := \{ax \mid a \in \mathbb{F}_q\}$ as the vertex subset of $P(q_0)$. The induced subgraph constructed by A forms an independent set with order q.

Let α be a primitive root of $\mathbb{F}_q$, and we assume the existence of a vertex y which is adjacent to x and αx, and not adjacent to other vertices of A. From the property of A, if such a vertex exists then $y \in \mathbb{F}_{q_0} \setminus A$.

We also define $B := \{ay \mid a \in \mathbb{F}_q\}$ as the vertex subset of $P(q_0)$. Similarly, B is also an independent set with order q.

Because y is adjacent to only two vertices in A, $\alpha^i y$ is also adjacent to only two vertices in A, $\alpha^i x$ and $\alpha^{i+1}x$. Therefore, the induced subgraph constructed by $(A \cup B) \setminus \{0\}$ forms $C_{2(q-1)}$. Using the same argument as in Lemma 5.6, the requirement for the existence of y is exactly the same as the requirement for the existence of $y' = y - x$, which is only adjacent to 0 and $(\alpha - 1)x$ in A. We consider the condition of y'.

Let $C = \{0, (\alpha - 1)x\}$ and $D = A \setminus C$. Similar to Lemma 5.6, we consider

$$\tau(v) = \prod_{c \in C}(1 + \sigma(v - c)) \prod_{d \in D}(1 - \sigma(v - d)).$$

Then, the desired vertex y' exists if

$$\sum_{v \in \mathbb{F}_{q_0} \setminus A} \tau(v) > 0.$$

Since for any vertex $v \in A, \tau(v) = 0$, we can rephrase the above inequality as follows:

$$\sum_{v \in \mathbb{F}_{q_0}} \tau(v) = q_0 + \sum_{\emptyset \neq S \subset A} (-1)^{|S \cap D|} \sum_{v \in \mathbb{F}_{q_0}} \sigma(f_S(v)) > 0.$$

Similar to the discussion of Lemma 5.6,

$$\sum_{\emptyset \neq S \subset A} (-1)^{|S \cap D|} \sum_{v \in \mathbb{F}_{q_0}} \sigma(f_S(v))$$
$$= \sum_{\emptyset \neq S \subset A} \sum_{a \in \mathbb{F}_q^*} \frac{(-1)^{|S \cap aD|}}{q-1} \sum_{v \in \mathbb{F}_{q_0}} \sigma(f_S(v))$$
$$= \sum_{\emptyset \neq S \subset A \setminus \{0\}} \frac{(q-1-2|S|)}{q-1} \sum_{v \in \mathbb{F}_{q_0}} (\sigma(f_S(v)) + \sigma(f_{S \cup \{0\}}(v))).$$

Therefore, from Theorem 5.4 and Theorem 5.5,

$$\sum_{v \in \mathbb{F}_{q_0}} \tau(v) \geq q_0 - \sqrt{q_0} \sum_{\emptyset \neq S \subset A \setminus \{0\}} \frac{|q-1-2|S||}{q-1}(1 + 2(|S|-1)\sqrt{q_0}).$$

This inequality is exactly the same as inequality (1) in Lemma 5.6. Hence, we obtain an equivalent inequality to Lemma 5.6, and the proof is complete. ☐

Corollary 5.9. *Let k be an integer such that $0 < k < q-1$. If q_0 satisfies the following inequality:*

$$q_0 > \left((q-3) \binom{q-1}{\frac{q-1}{2}} + 3 \right)^2$$

then C_{2k+2} is an induced subgraph of $P(q_0)$, where $q \geq 5$.

Proof. Let α be a primitive root of $\mathbb{F}_q$. From Lemma 5.8, there exist vertex sets $A = \{ax \mid a \in \mathbb{F}_q\}$ and $B = \{ay \mid a \in \mathbb{F}_q\}$ in $P(q_0)$ under this condition, where y is only adjacent to x and αx in A. Additionally, we also know there is a vertex $y' = y - x$ which is only adjacent to 0 and $(\alpha - 1)x$ in A. We consider the adjacency between y' and B. When $a \neq 1$, because

$$y' - ay = (1-a)(y - (1-a)^{-1}x)$$

and $1-a$ is a quadratic residue, y' is adjacent to ay if and only if $a = 1-\alpha^{-1}$ or $a = 0$. This implies that y' is only adjacent to $0, (\alpha-1)x$, and $\alpha^{-1}(\alpha-1)y$ in $A \cup B$. Especially, $z = (\alpha-1)^{-1}y'$ is only adjacent to $0, x$ and $\alpha^{-1}y$, and $\alpha^k z$ is only adjacent to $0, \alpha^k x$, and $\alpha^{k-1}y$. Furthermore, because z is a quadratic residue, z is adjacent to $\alpha^k z$. Hence, the induced subgraph obtained by

$$\{x, y, \alpha x, \alpha y, \alpha^2 x, \alpha^2 y, \ldots, \alpha^{k-2}x, \alpha^{k-2}y, \alpha^{k-1}x, \alpha^{k-1}y, \alpha^k z, z\}$$

is C_{2k+2}. Therefore, if q_0 satisfies the inequality of Lemma 5.8 then C_{2k+2} is an induced subgraph of $P(q_0)$. ☐

Corollary 5.10. *Let k be an integer such that $k < 2(q-1)$. If q_0 satisfies the following inequality:*

$$q_0 > \left((q-3)\binom{q-1}{\frac{q-1}{2}}+3\right)^2$$

then P_k is an induced subgraph of $P(q_0)$, where $q \geq 5$.

Proof. P_k is an induced subgraph of $C_{2(q-1)}$, and from Lemma 5.8, $C_{2(q-1)}$ is an induced subgraph of $P(q_0)$ under this condition. □

Lemma 5.11. *Let k be an integer such that $k \leq q-1$. If q_0 satisfies the following inequality:*

$$q_0 > \left(2^q(q-2)\binom{q-1}{\frac{q-1}{2}}+3\right)^2$$

then C_{2k+1} is an induced subgraph of $P(q_0)$.

Proof. From the following condition

$$q_0 > \left(2^q(q-2)\binom{q-1}{\frac{q-1}{2}}+3\right)^2 > \left((q-3)\binom{q-1}{\frac{q-1}{2}}+3\right)^2,$$

q_0 satisfies the condition of Lemma 5.8. Therefore, there exist vertex subsets A and B of $P(q_0)$ as considered in Lemma 5.8, that is,

$$A = \{ax \mid a \in \mathbb{F}_q\}, B = \{ay \mid a \in \mathbb{F}_q\},$$

and the induced subgraph of $A \cup B \setminus \{0\}$ is $C_{2(q-1)}$.

We assume the existence of a vertex z which is adjacent to 0 and x, and not adjacent to other vertices in $A \cup B$. Let α be a primitive root of $\mathbb{F}_q$. Then, $\alpha^{k-1}z$ is also adjacent to only 0 and $\alpha^{k-1}x$ in $A \cup B$, and because z is a quadratic residue, z is adjacent to $\alpha^{k-1}z$. Therefore, the induced subgraph constructed by the vertex set

$$\{x, y, \alpha x, \alpha y, \alpha^2 x, \alpha^2 y, \ldots, \alpha^{k-2}x, \alpha^{k-2}y, \alpha^{k-1}x, \alpha^{k-1}z, z\}$$

is C_{2k+1}. Hence, we consider the condition of q_0 under which such a vertex z exists.

In the same manner as other lemmas, let $C = \{0, x\}$ and $D = (A \cup B) \setminus C$, and define $\tau(v)$ as the same in other Lemmas. Since all vertices in $A \cup B$ are not adjacent to 0, the desired vertex exists when

$$\sum_{v \in \mathbb{F}_{q_0}} \tau(v) = q_0 + \sum_{\emptyset \neq S \subset A \cup B} (-1)^{|S \cap D|} \sum_{v \in \mathbb{F}_{q_0}} \sigma(f_S(v)) > 0.$$

We evaluate the second term on the left-hand side.

Let $\mathcal{S}$ be the power set of $A \cup B$. Additionally, we consider a bijection φ_a using an element $a \in \mathbb{F}_q$, as described in Lemma 5.6. Following the same discussion as in Lemma 5.6, we obtain

$$\sum_{\emptyset \neq S \subset A\cup B} (-1)^{|S\cap D|} \sum_{v \in \mathbb{F}_{q_0}} \sigma(f_S(v)) = \sum_{\emptyset \neq S \subset A\cup B} (-1)^{|S\cap aD|} \sum_{v \in \mathbb{F}_{q_0}} \sigma(f_S(v)).$$

From the above,

$$\sum_{\emptyset \neq S \subset A\cup B} (-1)^{|S\cap D|} \sum_{v \in \mathbb{F}_{q_0}} \sigma(f_S(v)) = \sum_{\emptyset \neq S \subset A\cup B} \sum_{a \in \mathbb{F}_q^*} \frac{(-1)^{|S\cap aD|}}{q-1} \sum_{v \in \mathbb{F}_{q_0}} \sigma(f_S(v)).$$

Furthermore,

$$\begin{aligned}
\sum_{\emptyset \neq S \subset (A\cup B)\setminus\{0\}} \sum_{a \in \mathbb{F}_q^*} (-1)^{|S\cap aD|} &= \sum_{\emptyset \neq S \subset (A\cup B)\setminus\{0\}} \left(s_a(-1)^{s-1} + (q-1-s_a)(-1)^s\right) \\
&= \sum_{\emptyset \neq S \subset (A\cup B)\setminus\{0\}} (q-1-2s_a)(-1)^s,
\end{aligned}$$

where $|S| = s$ and $|S \cap A| = s_a$. From Theorem 5.4 and Theorem 5.5,

$$\begin{aligned}
(-1)^s \sum_{v \in \mathbb{F}_{q_0}} \sigma(f_S(v) + \sigma(f_{S\cup\{0\}}(v))) &\geq - \left| \sum_{v \in \mathbb{F}_{q_0}} (\sigma(f_S(v)) + \sigma(f_{S\cup\{0\}}(v))) \right| \\
&\geq -(1 + 2(|S|-1)\sqrt{q_0}).
\end{aligned}$$

Therefore,

$$\sum_{\emptyset \neq S \subset A\cup B} (-1)^{|S\cap D|} \sum_{v \in \mathbb{F}_{q_0}} \sigma(f_S(v)) \geq - \sum_{\emptyset \neq S \subset (A\cup B)\setminus\{0\}} \frac{(q-1-2s_a)(1+2(|S|-1)\sqrt{q_0})}{q-1}$$

and we obtain

$$\sum_{v \in \mathbb{F}_{q_0}} \tau(v) \geq q_0 - \sum_{\emptyset \neq S \subset (A\cup B)\setminus\{0\}} \frac{|q-1-2s_a|(1+2(|S|-1)\sqrt{q_0})}{q-1}.$$

We consider the value of $V = \sum_{\emptyset \neq S \subset (A\cup B)\setminus\{0\}} \frac{|q-1-2s_a|(1+2(|S|-1)\sqrt{q_0})}{q-1}$. Removing the absolute value of V, we obtain

$$V = -(-2\sqrt{q_0}+1) + \left(\sum_{i=0}^{\frac{q-1}{2}} - \sum_{i=\frac{q+1}{2}}^{q-1} \right) \sum_{j=0}^{q-1} \frac{q-1-2i}{q-1} (1 + 2(i+j-1)\sqrt{q_0}) \binom{q-1}{i} \binom{q-1}{j}.$$

From Lemma 5.6, we obtain

$$\left(\sum_{i=0}^{\frac{q-1}{2}} - \sum_{i=\frac{q+1}{2}}^{q}\right) \frac{q-1-2i}{q-1}(1+2(i-1)\sqrt{q_0}) \binom{q-1}{i} = \sqrt{q_0}(q-3)\binom{q-1}{\frac{q-1}{2}} + \binom{q-1}{\frac{q-1}{2}}$$

and

$$\left(\sum_{i=0}^{\frac{q-1}{2}} - \sum_{i=\frac{q+1}{2}}^{q}\right) \frac{q-1-2i}{q-1} \binom{q-1}{i} = \binom{q-1}{\frac{q-1}{2}}.$$

Note that this summation starts with $i = 0$. Therefore,

$$\begin{aligned} V &= 2\sqrt{q_0} - 1 + \sum_{j=0}^{q-1} \left(\sqrt{q_0}(q-3)\binom{q-1}{\frac{q-1}{2}} - \binom{q-1}{\frac{q-1}{2}}\right)\binom{q-1}{j} + 2j\sqrt{q_0}\binom{q-1}{\frac{q-1}{2}}\binom{q-1}{j} \\ &= 2\sqrt{q_0}\left(2^{q-1}(q-2)\binom{q-1}{\frac{q-1}{2}} + 1\right) + 2^{q-1}\binom{q-1}{\frac{q-1}{2}} - 1. \end{aligned}$$

Based on the above,

$$\begin{aligned} \sum_{v \in \mathbb{F}_{q_0}} \tau(v) &\geq q_0 - V \\ &= q_0 - 2\sqrt{q_0}\left(2^{q-1}(q-2)\binom{q-1}{\frac{q-1}{2}} + 1\right) - 2^{q-1}\binom{q-1}{\frac{q-1}{2}} + 1. \end{aligned}$$

Henceforth, when $q \geq 3$, if q_0 satisfies the following inequality:

$$q_0 > \left(2^q(q-2)\binom{q-1}{\frac{q-1}{2}} + 3\right)^2,$$

then $\sum_{v \in \mathbb{F}_{q_0}} \tau(v) > 0$ and the desired vertex exists. □

5.3. **Non-trivial upper bound of the Paley-induced index for some graphs.** In this subsection, we provide a proof of statements (2a) and (2b) in Theorem 1.12 using the lemmas in subsection 5.2.

Proof of Theorem 1.12 (2a). From Corollary 5.7, Corollary 5.9, and Corollary 5.10, if

$$(q^{3^n})^2 > \left((q^{3^m} - 3)\binom{q^{3^m} - 1}{\frac{q^{3^m}-1}{2}} + 3\right)^2,$$

that is,

$$n > \log_3 \log_q \left((q^{3^m} - 3)\binom{q^{3^m} - 1}{\frac{q^{3^m}-1}{2}} + 3\right),$$

then K_{k_1,k_2}, C_{2k_1} and $P_{k_1+k_2-1}$ are induced subgraphs of $P_n(q^2,3)$. Since the right-hand side is not an integer,

$$i_P(G) \leq \left\lceil \log_3 \log_q \left((q^{3^m} - 3) \binom{q^{3^m} - 1}{\frac{q^{3^m}-1}{2}} + 3 \right) \right\rceil .$$

□

Remark 5.12. As the order of the graph increases, the evaluation of (2a) in Theorem 1.12 becomes stronger compared to the upper bound provided in Theorem 5.3. Specifically, when the order is $2(q^{3^m} - 1)$, the evaluation becomes approximately half of the upper bound obtained from Theorem 5.3. The upper bounds obtained from these theorems and corollary are stronger than that of Theorem 5.3 when the order k satisfies the following inequality:

$$(k-1)2^{k-2} \geq (q^{3^m} - 3) \binom{q^{3^m} - 1}{\frac{q^{3^m}-1}{2}} + 3.$$

Proof of Theorem 1.12 (2b). From Lemma 5.11, if

$$n > \log_3 \log_q \left(2^{q^{3^m}} (q^{3^m} - 2) \binom{q^{3^m} - 1}{\frac{q^{3^m}-1}{2}} + 3 \right),$$

then C_{2k_1+1} is an induced subgraph of $P_n(q^2,3)$. Since the right-hand side is not an integer, we obtain the desired inequality. □

Remark 5.13. When

$$k_1 2^{2k_1} \geq 2^{q^{3^m}} (q^{3^m} - 2) \binom{q^{3^m} - 1}{\frac{q^{3^m}-1}{2}} + 3,$$

the bound obtained from (2b) in Theorem 1.12 is stronger than that from Theorem 5.3. Because

$$\binom{q^{3^m} - 1}{\frac{q^{3^m}-1}{2}} \leq \frac{2^{q^{3^m}-1}}{\sqrt{q^{3^m}}},$$

when q^{3^m} is sufficiently large, there exist many k_1 that satisfy such a condition.

6. The value of the Paley index for cycles and paths

In this section, we provide trivial upper bounds for the Paley index. Also, we give a proof of Theorem 1.12 (3).

Similarly to the Paley-induced index, we can define the Paley index $\widetilde{i}_P$ as follows:

$$\widetilde{i}_P(G) = \min\{n \in \mathbb{N} \mid G \text{ is a subgraph of } P_n(q^2,3)\}.$$

Note that the Kneser index of any graph is 1. This is because $K_{\mathbb{N},1}$ is a complete graph with infinitely many vertices. Therefore, we consider only the Paley index.

Theorem 6.1. *Let G be a graph with order k. Then,*

$$\widetilde{i}_P(G) \leq \lceil \log_3 \log_q k \rceil.$$

Proof. From Proposition 6.13 of [9] (see also [4]), because the independence number of $P_n(q^2, 3)$ is q^{3^n} and the Paley graph is self-complementary, $P_n(q^2, 3)$ contains a complete graph with order q^{3^n}. Obviously, a complete graph with such order contains any graph with order k, where k is any integer smaller than q^{3^n}. In other words, any graph with order k is a subgraph of $P_n(q^2, 3)$ when n satisfies the following inequality:

$$n \geq \log_3 \log_q k.$$

From the definition of $\widetilde{i}_P$, we obtain the desired upper bound. □

Next, we calculate the Paley index for cycles by utilizing the property of pancyclicity in the Paley graph.

Definition 6.2 ([3])**.** An undirected graph G with order $n \geq 3$ is pancyclic if it contains a k-cycle as a subgraph for every $k \in \{3, 4, \dots, n\}$.

In [14], they proved a sufficient condition for pancyclicity. For any vertex x in a graph, $N(x)$ represents the neighborhood set of x and $d(x)$ represents the degree of x.

Theorem 6.3 ([14])**.** *Let G be a 2-connected graph of order $n \geq 6$. Suppose that $|N(x) \cup N(y)| + d(z) \geq n$ for every triple independent vertices x, y, z of G. Then G is pancyclic or isomorphic to the complete bipartite graph $K_{\frac{n}{2},\frac{n}{2}}$.*

From the properties of the Paley graph, we can easily prove that for any triple independent vertices x, y, z in the Paley graph with order $q \geq 6$,

$$|N_G(x) \cup N_G(y)| + d_G(z) = \frac{5q - 5}{4} \geq q.$$

Therefore, from Theorem 6.3, we easily demonstrate the pancyclicity of the Paley graph. From this, we can determine the Paley index of cycle graphs and paths.

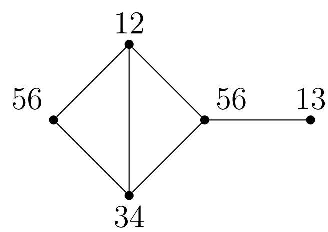
Proof of Theorem 1.12 (3). Because $P_n(q^2, 3)$ is pancyclic, if $(q^2)^{3^n} \geq k$, in other words

$$n \geq \left\lceil \log_3 \frac{1}{2}(\log_q k) \right\rceil,$$

then $P_n(q^2,3)$ contains C_k as a subgraph. Hence, we obtain $\widetilde{i}_P(C_k) \leq \lceil \log_3 \frac{1}{2}(\log_q k) \rceil$. Additionally, if $(q^2)^{3^n} < k$ then obviously $P_n(q,3)$ cannot contain C_k. This implies $\widetilde{i}_P(C_k) \geq \lceil \log_3 \frac{1}{2}(\log_q k) \rceil$, and we have $\widetilde{i}_P(C_k) = \log_3 \frac{1}{2}(\log_q k)$. Because P_k is a subgraph of C_k, we have also derived $\widetilde{i}_P(P_k) \leq \lceil \log_3 \frac{1}{2}(\log_q k) \rceil$ and completing the proof. $\square$

7. Remarks and questions

Remark 7.1. $X_{K_{\mathbb{N},2}}(\bullet)$ distinguish G_1 from G_2 in Figure 1. Indeed there is a homomorphism from G_2 to $K_{\mathbb{N},2}$ given as follows,



while there is no homomorphism from G_1 to $K_{\mathbb{N},2}$ whose image consists of $\{12, 13, 34, 56, 56\}$.

Remark 7.2. The integer sequence of the number of connected graphs in $\mathcal{P}_n^{(2)}$ begins from

$$1, 2, 5, 12, 33, 103, 333, 1183, 4442, 17576, \ldots$$

It is recorded as A076864 in OEIS [10].

The integer sequence consisting of cardinalities of $\mathcal{P}_n^{(2)}$ (or equivalently, the dimensions homogeneous parts of $\mathrm{Sym}^{(2)}$) starts from

$$1, 3, 8, 23, 66, 212, 686, 2389, 8682, 33160, \ldots$$

and it is A050535 in OEIS [10].

Question: Does $\mathcal{A}_\bullet^{(2)}$ distinguish trees? If so, we can conclude that $X_{K_{\mathbb{N},2}}(\bullet)$ distinguish trees by Theorem 1.11.

Observation: Let T be a tree. Suppose that $\lambda \in \mathcal{A}_T^{(2)}$ is also a tree. Then every leaf of T corresponds with a leaf edge of λ. In particular, assume that the number of leaves of T and λ are the same. Then T may be reconstructed by removing a leaf from λ.

Question: Is there a pair G_1, G_2 of non-isomorphic graphs such that $X_{K_{\mathbb{N},2}}(G_1) = X_{K_{\mathbb{N},2}}(G_2)$?

Remark 7.3. Let $\mathcal{T}$ be the set of all trees and i be the Kneser-functional index of $\mathcal{T}$. Conjecture 1.4 means that $i = 1$.

Question: Can we give an upper bound for i?

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