QUASI-GALOIS POINTS, II: ARRANGEMENTS

SATORU FUKASAWA, KEI MIURA AND TAKESHI TAKAHASHI

Abstract. In Part I, the present authors introduced the notion of a quasi-Galois point, for investigating the automorphism groups of plane curves. In this second part, the number of quasi-Galois points for smooth plane curves is described. In particular, sextic or quartic curves with many quasi-Galois points are characterized.

1. Introduction

In Part I [5], the present authors introduced the notion of a *quasi-Galois point* for a plane curve $C \subset \mathbb{P}^2$, for investigating the automorphism group $\mathrm{Aut}(C)$ of C. In this second part, we describe the arrangement of quasi-Galois points. It is inferred that quasi-Galois points are useful to classify algebraic curves.

Let $C \subset \mathbb{P}^2$ be a smooth plane curve of degree $d \geq 4$ over an algebraically closed field K of characteristic zero, and let $P \in \mathbb{P}^2$. We define the set $G[P]$ as the group consisting of all birational transformations of C preserving the fibers of the projection π_P. If $|G[P]| \geq 2$, then we say that P is a *quasi-Galois point*. This is a generalization of the Galois point, which was introduced by Hisao Yoshihara ([4, 11, 17]).

In this second part, the number of quasi-Galois points for smooth plane curves is described. The number $\delta[n]$ (resp., $\delta'[n]$) of quasi-Galois points $P \in C$ (resp., $P \in \mathbb{P}^2 \setminus C$) with $|G[P]| = n$ is described explicitly for any $n \geq 3$, in Theorems 3.4, 3.10 and 3.14. Furthermore, when $d = 4$ or 6, all possibilities of $\delta'[d/2]$ are determined (Theorems 4.1 and 5.12).

2020 *Mathematics Subject Classification.* 14H50, 14H05, 12F10.

Key words and phrases. plane curve, projection, Galois point, quasi-Galois point.

The work of the first author was partially supported by JSPS KAKENHI Grant Numbers 25800002, JP16K05088 and JP19K03438.

The work of the second author was partially supported by JSPS KAKENHI Grant Numbers 26400057 and JP18K03230.

The work of the third author was partially supported by JSPS KAKENHI Grant Numbers 25400059, JP16K05094 and JP19K03441.

Quasi-Galois points are related to reflections or finite unitary reflection groups in group theory. In fact, a generator of the associated group $G[P]$ at a quasi-Galois point is represented by a reflection (see [5, Theorem 2.3] or Fact 2.4). Finite unitary reflection groups are well studied, and were classified in 1950s when such groups are irreducible (Shephard–Todd [14], [8, Theorem 8.29]). Proofs of our results do not depend on the results of them. Some parts of our proofs of Theorems 3.10 and 3.14 are related to the methods of Mitchell in [10, Sections 5 and 6]. Our proofs come from the point of view of algebraic geometry.

2. Preliminaries

We introduce the system $(X : Y : Z)$ of homogeneous coordinates on $\mathbb{P}^2$, and take $x = X/Z, y = Y/Z$. If $P \in C$, then the (projective) tangent line at P is denoted by T_PC. For a projective line $\ell \subset \mathbb{P}^2$ and a point $P \in C \cap \ell$, the intersection multiplicity of C and ℓ at P is denoted by $I_P(C, \ell)$. The line passing through points P and Q is denoted by $\overline{PQ}$, when $P \neq Q$, and the projection from a point $P \in \mathbb{P}^2$ by π_P, which is the rational map from C to $\mathbb{P}^1$ represented by $Q \mapsto \overline{PQ}$. If $Q \in C$, the ramification index of π_P at Q is denoted by e_Q. We note the following elementary fact.

Fact 2.1. *Let $P \in \mathbb{P}^2$, and let $Q \in C$. Then, for π_P we have the following.*

(1) *If $P = Q$, then $e_P = I_P(C, T_PC) - 1$.*
(2) *If $P \neq Q$, then $e_Q = I_Q(C, \overline{PQ})$.*

If $|G[P]| \geq 2$, then the fixed field $K(C)^{G[P]}$ is an intermediate field of $K(C)/\pi_P^*K(\mathbb{P}^1)$, and we have a Galois covering $C \to C/G[P]$.

Remark 2.2. The order $|G[P]|$ divides the degree of π_P.

In general, the following fact holds for a Galois covering $\theta : C \to C'$ with a Galois group G between smooth curves, where $G(P)$ is the stabilizer subgroup of P (see [15, III. 7.2, 8.2]).

Fact 2.3. *Let $\theta : C \to C'$ be a Galois covering of degree d, and let G be the Galois group. Then:*

(1) *The order of $G(P)$ is equal to e_P at P for any point $P \in C$.*
(2) *Let $P, Q \in C$. If $\theta(P) = \theta(Q)$, then $e_P = e_Q$.*

Note that any automorphism is the restriction of a linear transformation (see [1, Appendix A, 17 and 18] or [3]), since C is smooth and of degree $d \geq 4$. According to Part I [5, Remark 2.2 and Theorem 2.3], we have the following fact and two corollaries.

Fact 2.4 ([5], Theorem 2.3)**.** *The group $G[P]$ is a cyclic group. Furthermore, for an integer $n \geq 2$, n divides $|G[P]|$ if and only if there exists a linear transformation ϕ such that*

(1) $\phi(P) = (1:0:0)$,

(2) *there exists an element $\sigma \in G[\phi(P)] \subset \mathrm{Bir}(\phi(C))$ which is represented by the matrix*

$$A_\sigma = \begin{pmatrix} \zeta & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where ζ is a primitive n-th roof of unity, and

(3) *$\phi(C)$ is given by*

$$\sum_i G_{d-ni}(Y,Z)X^{ni} = 0,$$

where G_{d-ni} is a homogeneous polynomial of degree $d-ni$ in variables Y, Z.

Corollary 2.5. *For $\sigma \in G[P] \setminus \{1\}$, we define $F[P] := \{Q \in \mathbb{P}^2 \mid \sigma(Q) = Q\}$. If we use the standard form as in Fact 2.4, $F[P] = \{P\} \cup \{X = 0\}$. In particular, the set $F[P]$ does not depend on σ.*

Corollary 2.6. *Let $P_1, P_2 \in \mathbb{P}^2$. If $P_1 \neq P_2$, then $G[P_1] \cap G[P_2] = \{1\}$.*

We will use the following two well-known facts.

Fact 2.7. *Let $G \subset \mathrm{Aut}(C)$ be a finite subgroup, and let $Q \in C$ be a point. If $\sigma(Q) = Q$ for any $\sigma \in G$, then G is a cyclic group.*

Fact 2.8. *Let G be a finite subgroup of $\mathrm{PGL}(2, K)$. Then G is isomorphic to one of the following:*

(1) *a cyclic group;*
(2) *a dihedral group;*
(3) *the alternating group A_4 of degree four;*
(4) *the symmetric group S_4 of degree four;*
(5) *the alternating group A_5 of degree five.*

To study the number of quasi-Galois points, we introduce some symbols here. The set of all quasi-Galois points $P \in C$ with $|G[P]| = n$ (resp., $|G[P]| \geq n$) is denoted by Δ_n (resp., $\Delta_{\geq n}$). The number of quasi-Galois points $P \in C$ with $|G[P]| = n$ (resp., $|G[P]| \geq n$) is denoted by $\delta[n]$ (resp., $\delta[\geq n]$). Similarly, we define Δ'_n, $\Delta'_{\geq n}$, $\delta'[n]$ and $\delta'[\geq n]$, when we consider the case $P \in \mathbb{P}^2 \setminus C$.

3. The number of quasi-Galois points

Let $P \in \mathbb{P}^2$ be a quasi-Galois point for C with $|G[P]| = n \geq 2$. We consider ramification points for the projection π_P.

Proposition 3.1. *There exist d points $Q_1, \dots, Q_d \in C \cap (F[P] \setminus \{P\})$ such that $P \in T_{Q_i}C$ and $I_{Q_i}(C, T_{Q_i}C) = l_i n$ for some integer $l_i \geq 1$.*

Proof. Let $Q \in C \cap (F[P] \setminus \{P\})$. By Corollary 2.5, $\sigma(Q) = Q$ for each $\sigma \in G[P]$. By Fact 2.3(1), the ramification index at Q for the covering map $C \mapsto C/G[P]$ is equal to n. Since the projection π_P is the composite map of $C \to C/G[P]$ and $C/G[P] \to \mathbb{P}^1$, the ramification index e_Q at Q for π_P is equal to ln for some $l \geq 1$. By Fact 2.1(2), $e_Q = I_Q(C, \overline{PQ}) = ln$ and $\overline{PQ} = T_QC$. Furthermore, the line given by $F[P] \setminus \{P\}$ intersects with C at exactly d points. □

If $P \in C$, we have the following.

Proposition 3.2. *If $P \in C$, then $I_P(C, T_PC) = ln + 1$ for some integer $l \geq 1$.*

Proof. By Corollary 2.5, for any $\sigma \in G[P]$, $\sigma(P) = P$. Then the covering map $C \to C/G[P]$ is ramified at P with index n, by Fact 2.3(1). Since the projection π_P is the composite map of $C \to C/G[P]$ and $C/G[P] \to \mathbb{P}^1$, the ramification index e_P at P is equal to ln for some $l \geq 1$. Note that $e_P = I_P(C, T_PC) - 1$, by Fact 2.1(1). It follows that $I_P(C, T_PC) = ln + 1$. □

Using Fact 2.7, we have the following.

Proposition 3.3. *Let $P_1, P_2 \in \mathbb{P}^2$ be points with $|G[P_1]| = n_1 \geq 2$, $|G[P_2]| = n_2 \geq 2$.*

(1) *If $P_1, P_2 \in C$, then $C \cap F[P_1] \cap F[P_2] \subset \{P_1, P_2\}$. Furthermore, if n_1 and n_2 are not coprime, then $C \cap F[P_1] \cap F[P_2] = \emptyset$.*
(2) *If $P_1, P_2 \in \mathbb{P}^2 \setminus C$, then $C \cap F[P_1] \cap F[P_2] = \emptyset$.*

Proof. Assume that there exists a point $Q \in C \cap F[P_1] \cap F[P_2]$. Note that, by the definition of $F[P_i]$ and Proposition 3.1, points P_1, P_2 and Q are collinear.

First, we assume that n_1 and n_2 are divisible by some integer $n \geq 2$. Since $G[P_1]$ and $G[P_2]$ are cyclic by Fact 2.4, there exist subgroups of $G[P_1]$ and $G[P_2]$ of order n respectively. Let G be the group generated by such subgroups. Then G fixes the point Q. By Fact 2.7, G is a cyclic group. Therefore, by Corollary 2.6, G is a cyclic group of order n^2. However, the cyclic group of order n^2 has a unique subgroup of order n. This is a contradiction. In particular, the latter assertion of (1) follows.

Next, we consider the case where $Q \neq P_1, P_2$. Let $\sigma \in G[P_1] \setminus \{1\}$. Since σ fixes P_1 and Q on the line $\overline{P_1Q} = \overline{P_2Q}$, it follows that $P_3 := \sigma(P_2) \neq P_2$. Then $G[P_3] = \sigma G[P_2]\sigma^{-1}$ and $Q \in C \cap F[P_2] \cap F[P_3]$. By the above discussion, we have a contradiction. Assertions (1) and (2) follow. □

For the number of quasi-Galois points on C, we have the following.

Theorem 3.4. *Let $n \geq 3$. Then*

$$\delta[n] = 0, 1 \text{ or } 4.$$

Furthermore, $\delta[n] = 4$ only if $n = 3$, and $d = 6m + 4$ for some integer $m \geq 0$.

Proof. Let P_1 and $P_2 \in C$ be quasi-Galois points with $|G[P_1]| = |G[P_2]| = n$, and let $\ell = \overline{P_1P_2}$. Note that $\sigma(\ell) = \ell$ for each $\sigma \in G[P_i]$ for $i = 1, 2$. Let

$$G := \{\sigma \in \mathrm{Aut}(C) \mid \sigma(\ell) = \ell\} \subset \mathrm{PGL}(3, K),$$

and let $\varphi : G \to \mathrm{Aut}(\ell) \cong \mathrm{PGL}(2, K)$ be the homomorphism defined by $\sigma \mapsto \sigma|_\ell$. Since $\sigma(P_2) \neq P_2$ for each element $\sigma \in G[P_1] \setminus \{1\}$ by Proposition 3.3, we have $mn+1$ quasi-Galois points $P_1, P_2, \ldots, P_{mn+1}$ on the line ℓ for some integer m. Note that the restriction of φ over $G[P_i]$ is injective for each i. By Fact 2.8, $\varphi(G) = A_4$, S_4 or A_5. Since the stabilizer subgroup $\varphi(G)(P_i)$ of $\varphi(G)$ acts on the projective line ℓ, $\varphi(G)(P_i)$ is a cyclic group such that

$$n \leq |\varphi(G)(P_i)| \leq 5,$$

for each i.

Assume that $n = 5$. Then $|\varphi(G)(P_i)| = 5$ and $\varphi(G) \cong A_5$. Since $\varphi(G)(P_i)$ is a Sylow 5-group, it follows from Proposition 3.3 that $\varphi(G)$ acts on the set $\{P_i\}$ transitively. The orbit-stabilizer theorem (see, for example, [9, p.75]) implies that

$$5(5m + 1) = 60$$

holds. This is a contradiction.

Assume that $n = 4$. Then $|\varphi(G)(P_i)| = 4$ and $\varphi(G) \cong S_4$. Note that S_4 has exactly three cyclic subgroups of order 4. Since $\varphi(G)$ has at least 5 cyclic subgroups of order 4, this is a contradiction.

Assume that $n = 3$. Then $|\varphi(G)(P_i)| = 3$. Since $\varphi(G)(P_i)$ is a Sylow 3-group, $\varphi(G)$ acts on the set $\{P_i\}$ transitively. The orbit-stabilizer theorem implies that

$$3(3m+1) = 12, 24 \text{ or } 60.$$

This implies that $m = 1$.

We have to show that ℓ is a unique line containing exactly four quasi-Galois points on C, in the case where $n = 3$. By Lemma 3.5 below, it is inferred that for each four quasi-Galois points on a line ℓ, there exists a quasi-Galois point Q such that $\ell = F[Q] \setminus \{Q\}$. Assume that $\delta[3] \geq 5$. Since quasi-Galois points are not collinear, there exist two lines ℓ and ℓ' containing two quasi-Galois points such that the point $P \in \ell \cap \ell'$ is a quasi-Galois point on C. In this case, there exist exactly four quasi-Galois points for each line ℓ and ℓ'. Then there exist two quasi-Galois points Q and Q' such that $\ell = F[Q] \setminus \{Q\}$ and $\ell' = F[Q'] \setminus \{Q'\}$. This implies that $P \in F[Q] \cap F[Q']$, and hence, this is a contradiction to Proposition 3.3(2). □

Lemma 3.5. *Let ℓ be a line containing four points P_1, P_2, P_3 and $P_4 \in \mathbb{P}^2$ with $|G[P_i]| = 3$ for each i. If the group $\langle G[P_1], G[P_2]\rangle$ acts on the set $\{P_1, P_2, P_3, P_4\}$, then there exists a point $Q \notin \ell$ such that $|G[Q]| = 2m$ for some integer $m \geq 1$, and $F[Q] \setminus \{Q\} = \ell$. Furthermore, $Q \in \mathbb{P}^2 \setminus C$ and d is even.*

Proof. Let $\omega^2 + \omega + 1 = 0$. We can take a system of coordinates so that $P_1 = (1 : 0 : 0)$, $P_2 = (1 : -1 : 0)$, $P_3 = (1 : -\omega^2 : 0)$ and $P_4 = (1 : -\omega : 0)$, and a generator of $G[P_1]$ is represented by

$$\sigma_1 = \begin{pmatrix} \omega & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Let $Q \in F[P_1] \cap F[P_2]$. If $Q \in \ell$, then $\varphi(\langle G[P_1], G[P_2]\rangle)$ admits a cyclic subgroup of order at least $3^2 = 9$, according to Fact 2.7. This is a contradiction. Therefore, $Q \notin \ell$ and we can assume that $Q = (0 : 0 : 1)$. Let $\sigma_2 \in G[P_2]$ be a generator. By the condition that $\sigma_2(Q) = Q$, $\sigma_2(P_2) = P_2$, $\sigma_2(P_1) = P_4$, $\sigma_2(P_3) = P_1$, it follows that σ_2 is represented by

$$\begin{pmatrix} -\omega\alpha & 2\omega^2\alpha & 0 \\ \omega^2\alpha & \alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

for some $\alpha \in K$. Since the projection π_{P_2} from $P_2 = (1 : -1 : 0)$ is represented by $(x : y : 1) \mapsto (x+y : 1)$, the condition $\sigma_2^*(x+y) = x+y$ implies that $\alpha = 1/(2\omega^2+1)$. Note that $\alpha^2 = -1/3$. Then it follows that

$$(\sigma_2\sigma_1^2)^2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and hence, $G[Q]$ contains an element of order 2. Therefore, $|G[Q]|$ is even and $F[Q] \setminus \{Q\} = \ell$. If $Q \in C$, the tangent line at Q contains P_1 and P_2. This is a contradiction. Therefore, $Q \in \mathbb{P}^2 \setminus C$ and d is even. □

Corollary 3.6. *We have*

$$\delta[\geq 3] = 0, 1, 2 \textit{ or } 4.$$

Furthermore, $\delta[\geq 3] = 4$ *only if* $\delta[\geq 3] = \delta[3] = 4$.

Proof. Assume that $\delta[\geq 3] \geq 3$ and $\delta[\geq 4] \geq 1$. Let $P_1, P_2, P_3 \in C$ be different points with $|G[P_1]| \geq 3$, $|G[P_2]| \geq 3$ and $|G[P_3]| \geq 4$. It follows from Theorem 3.4 that $P_3 \in F[P_1] \cap F[P_2]$. By Facts 2.1(2) and 2.3(1), $P_1, P_2 \in T_{P_3}C$. In this case, points P_1, P_2 and P_3 are collinear. By Corollary 2.5, $P_1 \notin F[P_3]$ or $P_2 \notin F[P_3]$. Assume that $P_1 \notin F[P_3]$. In this case, there exists a point $P_1' \in C$ such that $|G[P_1']| = |G[P_1]|$ and $P_3 \in F[P_1']$. By Proposition 3.3, this is a contradiction. □

We consider the number of quasi-Galois points in $\mathbb{P}^2 \setminus C$. To do this, we introduce the notion of "G-pairs". Let $P, P' \in \mathbb{P}^2 \setminus C$ be points such that $P \neq P'$ and $|G[P]|$ and $|G[P']|$ are divisible by $n \geq 2$. We call the pair (P, P') a *G-pair with respect to n* if $\sigma(P') = P'$ and $\sigma'(P) = P$ for generators $\sigma \in G[P]$ and $\sigma' \in G[P']$. By Corollary 2.5, the definition does not depend on the choice of generators.

Lemma 3.7. *Let $n \geq 2$, let $P_1, P_2 \in \mathbb{P}^2 \setminus C$ be different points such that n divides $|G[P_1]|$ and $|G[P_2]|$, and let $\sigma_i \in G[P_i]$ be a generator for $i = 1, 2$. If $\sigma_1(P_2) = P_2$, then $\sigma_2(P_1) = P_1$. In particular, (P_1, P_2) is a G-pair with respect to n.*

Proof. By the assumption, $P_2 \in F[P_1] \setminus \{P_1\}$. It follows from Corollary 2.5 and Proposition 3.1 that the set $F[P_1] \setminus \{P_1\}$ is a line containing d points $Q_1, \dots, Q_d \in C$

with $\overline{P_1Q_i} = T_{Q_i}C$ for each i. Since $F[P_1] \setminus \{P_1\}$ is a line passing through P_2, it follows that $\sigma_2(Q_1) = Q_i$ and $\sigma_2(Q_2) = Q_j$ for some i, j. Since $\overline{P_1Q_1}$ and $\overline{P_1Q_i}$ are tangent lines at Q_1 and Q_i respectively, $\sigma_2(\overline{P_1Q_1}) = \overline{P_1Q_i}$. Then $\sigma_2(\overline{P_1Q_1} \cap \overline{P_1Q_2}) \subset \overline{P_1Q_i} \cap \overline{P_1Q_j} = \{P_1\}$. It follows that $\sigma_2(P_1) = P_1$. □

Proposition 3.8. *There exists a G-pair (P, P') with respect to n, if and only if C is projectively equivalent to the curve defined by*

$$g(x^n, y^n) = 0$$

for some polynomial g. In this case, there exists a point $P'' \in \mathbb{P}^2 \setminus (C \cup \overline{PP'})$ such that pairs (P, P'') and (P', P'') are G-pairs. In particular, $\delta'[\geq n] \geq 3$.

Proof. We consider the if part. According to Fact 2.4, for the defining equation $g(x^n, y^n) = 0$, it follows that $P = (1 : 0 : 0)$ and $P' = (0 : 1 : 0)$ form a G-pair with respect to n.

We prove the only-if part. Assume that (P, P') be a G-pair with respect to n. By the assumption, $P' \in F[P]$ and $P \in F[P']$. By Fact 2.4, for a suitable system of coordinates, we can assume that $P = (1 : 0 : 0)$ and there exists an element $\sigma \in G[P]$ of order n which is represented by the matrix

$$A_\sigma = \begin{pmatrix} \zeta & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where ζ is a primitive n-th root of unity. Then the line given by $F[P] \setminus \{P\}$ is defined by $X = 0$. Since $P' \in F[P] \setminus \{P\}$, $P' = (0 : 0 : 1)$ or $(0 : 1 : a)$ for some $a \in K$. If we take a linear transformation $(X : Y : Z) \mapsto (X : Z : Y)$ or $(X : Y : Z) \mapsto (X : Y : Z - aY)$, we can assume that $P' = (0 : 1 : 0)$. Then there exists an element $\sigma \in G[P']$ of order n which is represented by the matrix

$$A_{\sigma'} = \begin{pmatrix} 1 & 0 & 0 \\ a & \zeta & b \\ 0 & 0 & 1 \end{pmatrix},$$

for some $a, b \in K$. Since the line given by $F[P'] \setminus \{P'\}$ is defined by $aX + (\zeta - 1)Y + bZ = 0$ and $P \in F[P'] \setminus \{P'\}$, it follows that $a = 0$. If we take

$$B = \begin{pmatrix} 1-\zeta & 0 & 0 \\ 0 & 1 & b \\ 0 & 0 & 1-\zeta \end{pmatrix},$$

then

$$B^{-1}A_\sigma B = \begin{pmatrix} \zeta & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ B^{-1}A_{\sigma'}B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \zeta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

By taking the linear transformation represented by B^{-1}, the defining polynomial of C is of the form

$$g(x^n, y^n) = 0.$$

The assertion follows.

In this case, the automorphism

$$\begin{pmatrix} \zeta & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & 0 & 0 \\ 0 & \zeta & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \zeta & 0 & 0 \\ 0 & \zeta & 0 \\ 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \zeta^{-1} \end{pmatrix}$$

acts on C. Then the point $P'' = (0:0:1)$ is a quasi-Galois point with $|G[P'']| \ge n$. We have $\delta'[\ge n] \ge 3$. □

Corollary 3.9. *Let d be even, $n = d/2$ and let (P, P') be a G-pair. Then C is projectively equivalent to the curve defined by*

$$X^{2n} + Y^{2n} + Z^{2n} + aX^nY^n + bY^nZ^n + cZ^nX^n = 0,$$

where $a, b, c \in K$.

Proof. Since $n = d/2$, by Proposition 3.8, the defining polynomial of C is of the form

$$F = X^{2n} + (aY^n + bZ^n)X^n + (\alpha Y^{2n} + \beta Y^nZ^n + \gamma Z^{2n}).$$

We can assume $\alpha = \gamma = 1$. □

We consider the case where there exist two quasi-Galois points $P_1, P_2 \in \mathbb{P}^2 \setminus C$. Let $\ell = \overline{P_1P_2}$, let

$$G := \{\sigma \in \mathrm{Aut}(C) \mid \sigma(\ell) = \ell\} \subset \mathrm{PGL}(3, K),$$

and let $\varphi : G \to \mathrm{Aut}(\overline{P_1P_2}) \cong \mathrm{PGL}(2, K)$ be the homomorphism defined by $\sigma \mapsto \sigma|_\ell$. Note that $\sigma(\ell) = \ell$ for each $\sigma \in G[P_i]$, and the induced homomorphism $G[P_i] \to \varphi(G[P_i])$ is injective, for $i = 1, 2$.

Theorem 3.10. *Assume that $n \geq 4$, and points P_1 and $P_2 \in \mathbb{P}^2 \setminus C$ are different quasi-Galois points with $|G[P_1]| = |G[P_2]| = n$. Then there exists a point $P_1' \in \ell := \overline{P_1P_2}$ such that (P_1, P_1') is a G-pair with respect to n. Furthermore, the following hold.*

(1) *If $n \geq 6$, then $\delta'[\geq n] = 3$, and each two of the three quasi-Galois points form a G-pair. Furthermore, $\delta'[\geq 3] = \delta'[\geq n] = 3$.*
(2) *If $n = 5$, then $\#\Delta_5' \cap \ell = 2$ or 12. Furthermore, if $\#\Delta_5' \cap \ell = 12$, then $\#\Delta_{\geq 5}' \cap (\mathbb{P}^2 \setminus \ell) = \#\Delta_{\geq 10}' \cap (\mathbb{P}^2 \setminus \ell) = 1$. In particular, $\delta'[5] = 2, 3$ or 12.*
(3) *If $n = 4$, then $\#\Delta_4' \cap \ell = 2$ or 6. Furthermore, if $\#\Delta_4' \cap \ell = 6$, then $\#\Delta_{\geq 4}' \cap (\mathbb{P}^2 \setminus \ell) = 1$. In particular, $\delta'[4] = 2, 3, 6$ or 7.*

Proof. Assume that (P_1, P_2) is not a G-pair. Since $\sigma(P_2) \neq P_2$ for each element $\sigma \in G[P_1] \setminus \{1\}$, there exist at least $n+1$ quasi-Galois points $P_1, P_2, \ldots, P_{n+1}$ on the line ℓ. Since $\varphi(G)$ contains at least $(n+1)/2 \geq 2$ subgroups of order $n \geq 3$, by Fact 2.8, $\varphi(G) = A_4$, S_4 or A_5. Then $n \leq 5$.

Assume that $n \geq 6$. Then (P_1, P_2) is a G-pair. By Proposition 3.8, there exists a point P_3 such that (P_1, P_2), (P_2, P_3), (P_3, P_1) are G-pairs. If there exists a point $Q \notin \{P_1, P_2, P_3\}$ with $|G[Q]| \geq 3$ and $Q \in \mathbb{P}^2 \setminus C$, then $P_i \notin F[Q]$ for some i. Then there exists a point P_i' with $|G[P_i]| = |G[P_i']| \geq 6$ such that (P_i, P_i') is not a G-pair on the line $\overline{QP_i}$. This is a contradiction. It follows that $\delta'[\geq 3] = \delta'[\geq n] = 3$. Hereafter, for the case where $n = 3, 4$ or 5, we can assume that $\delta'[mn] \leq 1$ for any $m \geq 2$.

Let $n = 5$. Assume that there does not exist a G-pair on the line ℓ. Then there exist $5m+1$ subgroups of $\varphi(G) \cong A_5$ of order five for some integer m. Since such groups are Sylow 5-groups, $\varphi(G)$ acts transitively on the set $\{P_1, \ldots, P_{5m+1}\}$ of all quasi-Galois points on the line ℓ. The orbit-stabilizer theorem implies that $5(5m+1) = 60$. This is a contradiction. Therefore, there exists a G-pair (P, P') on the line ℓ. By Proposition 3.8, there exists a quasi-Galois point P'' with $F[P''] \setminus \{P''\} = \ell$. Since $P_1 \in \ell = F[P''] \setminus \{P''\}$, it follows from Lemma 3.7 that (P_1, P'') is a G-pair. By Proposition 3.8, there exists a quasi-Galois point $P_1' \in \ell \cap F[P_1]$. Then (P_1, P_1') is a G-pair. In particular, $\#\Delta_5' \cap \ell$ is even. If $\#\Delta_5 \cap \ell \geq 3$, then $\varphi(G) \cong A_5$. Since there exist exactly six subgroups of A_5 of order 5, we have exactly 12 quasi-Galois points on the line ℓ.

We consider the case where $\#\Delta_5' \cap \ell = 12$. Then $\varphi(G) \cong A_5$. Note that (P_1, P_1'), (P_1', P'') and (P'', P_1) are G-pairs, and $F[P''] \setminus \{P''\} = \ell$. By Lemma 3.11 below,

$|G[P'']| = 10m$ for some $m \geq 1$. In our situation, a point R with $|G[R]| = 10m$ is unique, by assertion (1). This implies that $\mathrm{Aut}(C)$ fixes P''. Let $R \neq P''$ be a point with $|G[R]| = 5$. Then $F[R] \ni P''$. By Lemma 3.7, (R, P'') is a G-pair. Since $F[P''] \setminus \{P''\} = \ell$, it follows that $R \in \ell$. The proof of assertion (2) is completed.

Let $n = 4$. Assume that there does not exist a G-pair on the line ℓ. Then there exist at least 5 cyclic subgroups of $\varphi(G) \cong S_4$ of order four. This is a contradiction, since S_4 has exactly three cyclic subgroups of order four. Therefore, there exists a G-pair (P, P') on the line ℓ. Similarly to the previous paragraph, there exists a point $P_1' \in \ell$ such that (P_1, P_1') is a G-pair. If $\#\Delta_4' \cap \ell \geq 3$, then $\varphi(G) \cong S_4$. Then, by the action of $G[P_1]$, we have 6 such points on ℓ. Since S_4 has exactly three cyclic subgroups of order four, we have exactly 6 quasi-Galois points on this line. Note that, by Proposition 3.8, there exists a point $P_1'' \notin \ell$ such that (P_1, P_1'), (P_1', P_1'') and (P_1'', P_1) are G-pairs.

Assume that $P_2 \in \ell$ is a quasi-Galois point with $|G[P_2]| = 4$ and $P_2 \neq P_1, P_1'$, and that $R \notin \ell$ is a quasi-Galois point with $|G[R]| = 4$. If $R \notin \overline{P_1' P_1''}$, then there exists a quasi-Galois point $R' \in \overline{P_1 R}$ such that (P_1, R') is a G-pair, and hence, R' must be in $\overline{P_1' P_1''}$. Therefore, we can assume that $R \in \overline{P_1' P_1''}$ with $R \neq P_1', P_1''$. Let $\eta \in G[P_2]$ be the involution. By Lemma 3.12 below, $\eta(P_1) = P_1'$ and $\eta(P_1') = P_1$. Since $\eta(P_1'') = P_1''$, it follows that $\eta(R) \in \overline{P_1'' P_1}$. By Lemma 3.12 again, $(R, \eta(R))$ is a G-pair. Since $P_1, \eta(R) \in F[R] \setminus \{R\}$, it follows that $F[R] \setminus \{R\} = \overline{P_1 \eta(R)} = \overline{P_1'' P_1} = F[P_1'] \setminus \{P_1'\}$. By Proposition 3.3(2), this is a contradiction. Therefore, $\Delta_4' \subset (\Delta_4' \cap \ell) \cup \{P_1''\}$. □

Lemma 3.11. *Let ℓ be a line containing 12 quasi-Galois points $P \in \mathbb{P}^2$ with $|G[P]| = 5$, let $P, P' \in \ell$ form a G-pair, and let $P'' \in F[P] \cap F[P']$. If $\varphi(G) \cong A_5$, then $|G[P'']| = 10m$ for some integer $m \geq 1$.*

Proof. We can assume that points $P = (1 : 0 : 0)$, $P' = (0 : 1 : 0)$ form a G-pair with $P, P' \in \ell$ and $|G[P]| = |G[P']| = 5$, and $\sigma \in G[P]$, $\sigma' \in G[P']$ are generators represented by

$$A_\sigma = \begin{pmatrix} \zeta & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ A_{\sigma'} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \zeta & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where ζ is a primitive 5-th roof of unity. Further, we can assume that $P'' = (0 : 0 : 1) \in F[P] \cap F[P']$. Let $\overline{\sigma} = \varphi(\sigma)$. Since $\varphi(G) \cong A_5$, it follows that there exists an involution $\overline{\tau} \in \varphi(G)$ such that $\overline{\sigma}(\overline{\tau}\overline{\sigma}\overline{\tau})\overline{\sigma} = \overline{\tau}$. We consider $\overline{\tau}$ as an element of $\mathrm{PGL}(2, K)$. Since $\overline{\tau}$ is an involution and $\overline{\tau}$ does not fix $(1 : 0)$ or $(0 : 1)$, it follows that $\overline{\tau}$ is represented by the matrix

$$A_{\overline{\tau}} = \begin{pmatrix} 1 & b \\ c & -1 \end{pmatrix}$$

for some $b, c \in K$. Let

$$B = \begin{pmatrix} \sqrt{\frac{b}{c}} & 0 \\ 0 & 1 \end{pmatrix}.$$

Then

$$B^{-1}A_{\overline{\tau}}B = \begin{pmatrix} \sqrt{\frac{b}{c}} & b \\ b & -\sqrt{\frac{b}{c}} \end{pmatrix}.$$

Therefore, we can assume that

$$A_{\overline{\tau}} = \begin{pmatrix} 1 & \alpha \\ \alpha & -1 \end{pmatrix}$$

for some $\alpha \in K$. It follows that

$$A_{\overline{\tau}}A_{\overline{\sigma}}A_{\overline{\tau}} = \begin{pmatrix} \zeta+\alpha^2 & (\zeta-1)\alpha \\ (\zeta-1)\alpha & \zeta\alpha^2+1 \end{pmatrix},\ A_{\overline{\sigma}}(A_{\overline{\tau}}A_{\overline{\sigma}}A_{\overline{\tau}})A_{\overline{\sigma}} = \begin{pmatrix} \zeta^2(\zeta+\alpha^2) & \zeta(\zeta-1)\alpha \\ \zeta(\zeta-1)\alpha & \zeta\alpha^2+1 \end{pmatrix}.$$

Since $\overline{\sigma}(\overline{\tau\sigma\tau})\overline{\sigma} = \overline{\tau}$, it follows that

$$\alpha^2 = 1-\zeta-\zeta^4.$$

The fixed locus of the linear transformation $\overline{\tau\sigma\tau}$ consists of two points

$$(1:\alpha),\ (\alpha:-1).$$

Since $\overline{\tau\sigma\tau}$ is of order five and is contained in $\varphi(G)$, it follows that $P_2 := (1:\alpha:0)$ is a quasi-Galois point with $|G[P_2]| = 5$ and there exists a generator $\sigma_2 \in G[P_2]$ such that $\varphi(\sigma_2) = \overline{\tau\sigma\tau}$. Let $P_2' := (\alpha:-1:0)$. Then $F[P_2] = \{P_2\} \cup \{X+\alpha Y = 0\}$. With the condition that $\sigma(-\alpha t : t : 1) = (-\alpha t : t : 1)$ for any $t \in K$ being considered, it is inferred that σ_2 is represented by the matrix

$$A_{\sigma_2} = \begin{pmatrix} \zeta+\alpha^2 & (\zeta-1)\alpha & 0 \\ (\zeta-1)\alpha & \zeta\alpha^2+1 & 0 \\ 0 & 0 & \alpha^2+1 \end{pmatrix}.$$

Then

$$A_\sigma A_{\sigma_2} A_\sigma = \begin{pmatrix} \zeta^3+\zeta^2\alpha^2 & (\zeta^2-\zeta)\alpha & 0 \\ (\zeta^2-\zeta)\alpha & \zeta\alpha^2+1 & 0 \\ 0 & 0 & \alpha^2+1 \end{pmatrix}.$$

Note that $\zeta^3+\zeta^2\alpha^2=\zeta(\zeta-1)$ and $\zeta\alpha^2+1=\zeta(1-\zeta)$. It follows that

$$(A_\sigma A_{\sigma_2} A_\sigma)^2 \sim \begin{pmatrix} \zeta^2(\zeta-1)^2 & 0 & 0 \\ 0 & \zeta^2(\zeta-1)^2 & 0 \\ 0 & 0 & \alpha^2+1 \end{pmatrix} \in G[P''].$$

It can be confirmed that

$$\frac{\alpha^2+1}{\zeta^2(\zeta-1)^2}=-\zeta^2.$$

Since $(\sigma\sigma_2\sigma)^2$ is of order 10, it follows that $|G[P'']|=10m$ for some $m \ge 1$. □

Lemma 3.12. *Let $P, R \in \mathbb{P}^2 \setminus C$ be points with $|G[P]|=|G[R]|=4$, and let $\eta \in G[P]$ be an involution. If $\eta(R) \ne R$, then $(R, \eta(R))$ is a G-pair.*

Proof. Let $\ell'=\overline{PR}$. Similar to the definition of G and φ, we define

$$G_{\ell'} := \{\sigma \in \mathrm{Aut}(C) \mid \sigma(\ell')=\ell'\} \subset \mathrm{PGL}(3,K),$$

and $\varphi_{\ell'}: G_{\ell'} \to \mathrm{Aut}(\ell'); \sigma \mapsto \sigma|_{\ell'}$. Since $R \not\in F[P]$, $\varphi_{\ell'}(\langle G[P], G[R]\rangle) \cong S_4$. Note that S_4 has exactly three cyclic subgroup of order four, and $\varphi_{\ell'}(G[P])$ acts on the set of subgroups of order four different from $\varphi_{\ell'}(G[P])$. This implies that $\varphi_{\ell'}(\eta)$ fixes each cyclic subgroup of order four. Then $\varphi_{\ell'}(G[\eta(R)])=\varphi_{\ell'}(\eta G[R]\eta^{-1})=\varphi_{\ell'}(G[R])$. This implies that $F[R] \ni \eta(R)$. By Lemma 3.7, $(R, \eta(R))$ is a G-pair. □

Corollary 3.13. *Assume that $d \ge 8$, d is even, and $n=d/2$. Then, $\delta'[\ge n] \ge 2$ if and only if C is projectively equivalent to the curve defined by*

$$X^{2n}+Y^{2n}+Z^{2n}+aX^nY^n+bY^nZ^n+cZ^nX^n=0,$$

where $a,b,c \in K$. In this case, if $d \ge 10$ (resp., $d=8$), then $\delta'[\ge n]=3$ (resp., $\delta'[\ge 4]=3$ or 7).

Proof. The former assertion is derived from Corollary 3.9 and Theorem 3.10. The latter assertion for the case where $d \ge 12$ or $d=8$ is obvious, by Theorem 3.10. Assume that $d=10$. Let $P \in \mathbb{P}^2 \setminus C$ be a point with $|G[P]| \ge 5$. By Proposition 3.1, there exist $d=10$ points $Q \in C \cap (F[P] \setminus \{P\})$ such that $P \in T_QC$ and $I_Q(C, T_QC) \ge 5$. Therefore, for each quasi-Galois point $P \in \mathbb{P}^2 \setminus C$ with $|G[P]| \ge 5$, we need at least $10 \times (5-2)=30$ flexes with multiplicities. It follows from Proposition 3.3 that there exists no point $Q \in C$ such that $Q \in F[P_1] \cap F[P_2]$ for different quasi-Galois points P_1 and P_2 with $|G[P_1]| \ge 5$ and $|G[P_2]| \ge 5$. By the flex formula [12, Theorem 1.5.10], we have $\delta'[\ge 5] \times 30 \le 3d(d-2)=240$. Therefore, $\delta'[\ge 5] \le 8$. By Theorem 3.10, it follows that $\delta'[\ge 5]=3$. □

Theorem 3.14. *Assume that $n = 3$, and points P_1 and $P_2 \in \mathbb{P}^2 \setminus C$ are different quasi-Galois points with $|G[P_1]| = |G[P_2]| = 3$. Let $\ell = \overline{P_1P_2}$. Then the following hold.*

(1) $\#\Delta'_3 \cap \ell = 2, 4, 8$ *or* 20. *Furthermore, if* $\#\Delta'_3 \cap \ell = 8$ *or* 20, *then there exists* $P'_1 \in \ell$ *such that* (P_1, P'_1) *is a G-pair.*
(2) *If* $\#\Delta'_3 \cap \ell = 8$, *then* $\delta'[3] = 8$ *and there exists a unique integer* $m \geq 1$ *such that* $\delta'[6m] = 1$.
(3) *If* $\#\Delta'_3 \cap \ell = 20$, *then* $\delta'[3] = 20$ *and there exists a unique integer* $m \geq 1$ *such that* $\delta'[6m] = 1$.
(4) *If* $\#\Delta'_3 \cap \ell = 4$, *then* $\delta'[3] = 4$ *or* 12.

In particular, $\delta'[3] = 2, 3, 4, 8, 12$ *or* 20.

Proof. Assume that there does not exist a G-pair on the line ℓ. Then there exist $3m + 1$ quasi-Galois points for some integer m by the actions associated with one quasi-Galois point. Then, by Fact 2.8,

$$3(3m + 1) = 12, 24 \text{ or } 60.$$

This implies that $m = 1$ and $\#\Delta'_3 \cap \ell = 4$.

We assume that $\#\Delta'_3 \cap \ell \geq 5$. Then there exists a G-pair (P, P') on the line ℓ. By Proposition 3.8, there exists a quasi-Galois point P'' with $F[P''] \setminus \{P''\} = \ell$. Since $P_1 \in \ell = F[P''] \setminus \{P''\}$, it follows from Lemma 3.7 that (P_1, P'') is a G-pair. By Proposition 3.8, there exists a quasi-Galois point $P'_1 \in \ell \cap F[P_1]$. Then (P_1, P'_1) is a G-pair.

By the discussion in the previous paragraph, $\#\Delta'_3 \cap \ell$ is even, and hence, $\#\Delta'_3 \cap \ell \geq 8$. By Fact 2.8, $\varphi(G) = A_4, S_4$ or A_5. Since $\varphi(G[P])$ is a Sylow 3-group of $\varphi(G)$, each subgroup of order three is realized as $\varphi(G[P])$ ($= \varphi(G[P'])$) for some exactly two quasi-Galois points $P, P' \in \Delta'_3 \cap \ell$. If $\varphi(G) \cong A_4$ or S_4 (resp., $\varphi(G) \cong A_5$), then the number of subgroup of order three is 4 (resp., 10). Therefore, the number of quasi-Galois points on ℓ is 8 or 20. Assertion (1) follows.

Assume that $\#\Delta'_3 \cap \ell = 8$. Then $\varphi(G) \cong A_4$ or S_4. In this case, for points P_1 and P_2 such that (P_1, P_2) is not a G-pair, $\varphi(\langle G[P_1], G[P_2]\rangle) \cong A_4$. Then the orbit A_4P_1 has length four. Note that (P_1, P'_1), (P'_1, P''_1) and (P''_1, P_1) are G-pairs. By Lemma 3.5, there exists a point $Q \notin \ell$ such that $|G[Q]|$ is even and $F[Q] \setminus \{Q\} = \ell$. Then $Q = P''_1$, because the intersection point of tangent lines at d points of C on the line ℓ is unique. Since $G[P''_1]$ contains elements of order three and two, the order

$|G[P_1'']|$ is equal to $6m$ for some m. In our situation, a point R with $|G[R]| = 6m$ is unique, by Theorem 3.10(1). This implies that $\mathrm{Aut}(C)$ fixes P_1''. Let $R \neq P_1''$ be a point with $|G[R]| = 3$. Then $F[R] \ni P_1''$. By Lemma 3.7, (R, P_1'') is a G-pair. Since $F[P_1''] \setminus \{P_1''\} = \ell$, it follows that $R \in \ell$. Assertion (2) follows.

Assume that $\#\Delta_3' \cap \ell = 20$. Then $\varphi(G) \cong A_5$. Note that all subgroups of A_5 of order three are realized as the image $\varphi(G[P])$ of associated groups $G[P]$ of quasi-Galois points $P \in \Delta_3' \cap \ell$ under the restriction φ. This implies that there exists a pair of points $P_1, P_2 \in \Delta_3' \cap \ell$ such that (P_1, P_2) is not a G-pair and $\varphi(\langle G[P_1], G[P_2]\rangle) \cong A_4$. The same argument as assertion (2) can be applied to assertion (3).

We consider assertion (4). Assume that $\#\Delta_3' \cap \ell = 4$. According to assertions (1), (2) and (3), we can assume that for all lines $\ell' \subset \mathbb{P}^2$, $\#\Delta_3' \cap \ell' = 0$, 1, 2 or 4. By Lemma 3.5, there exists a quasi-Galois point $Q \notin \ell$ such that $|G[Q]|$ is even and $F[Q] \setminus \{Q\} = \ell$. Let $\tau \in G[Q]$ be an involution. We prove that there does not exist a line $\ell' \ni Q$ with $\#\Delta_3' \cap \ell' = 4$. Assume by contradiction that $\#\Delta_3' \cap \ell' = 4$ and $\Delta_3' \cap \ell' = \{P_1', P_2', P_3', P_4'\}$. Note that $\ell' \cap \ell \cap \Delta_3' = \emptyset$, by considering the action of τ. Let $Q' \notin \ell'$ be a quasi-Galois point such that $|G[Q']|$ is even and $F[Q'] \setminus \{Q'\} = \ell'$. Since the point Q' is contained in the tangent line for any point in $C \cap \ell'$ and $G[Q]$ acts on ℓ', it follows that $Q' \in F[Q]$, namely, $Q' \in \ell$. Note that $G[Q']$ acts on the set $\Delta_3' \cap \ell$. The group $G[Q']$ does not fix any point in $\Delta_3' \cap \ell$, since $Q' \notin \Delta_3'$ and $\ell' \cap \ell \cap \Delta_3' = \emptyset$. Let $\tau' \in G[Q']$ be an involution. Then there exist a quasi-Galois point $P \in \Delta_3' \cap \ell$ and an automorphism $\sigma \in G[P]$ such that three points $Q', \sigma(Q'), \sigma^2(Q')$ are different. Let $Q_2' = \sigma(Q'), Q_3' = \sigma^2(Q')$. Then $\tau_2' := \sigma\tau'\sigma^{-1} \in G[Q_2']$ and $\tau_3' := \sigma^2\tau'\sigma^{-2} \in G[Q_3']$. Since τ' and σ act on $\Delta_3' \cap \ell$, it follows that $\tau'|_\ell$, $(\sigma\tau'\sigma^{-1})|_\ell$, $(\sigma^2\tau'\sigma^{-2})|_\ell$ are different involutions on ℓ. Since the number of involutions acting on four points given by $\Delta_3' \cap \ell$ not fixing any point of them is at most three, it follows that $(\tau_2'\tau_3')|_\ell = \tau'|_\ell$. Note that (Q_2', Q_3') is not a G-pair, since if (Q_2', Q_3') is a G-pair, then $\tau_2'|_\ell = \tau_3'|_\ell$. It follows from Lemma 3.7 that $\tau_3'(Q_2') \neq Q_2'$. If $\tau_2'\tau_3'$ is an involution as an automorphism of $\mathbb{P}^2$, then $\tau_3'(Q_2')$ is a quasi-Galois point with $\tau_3'\tau_2'\tau_3' = \tau_2' \in G[\tau_3'(Q_2')] \cap G[Q_2']$. By Corollary 2.6, this is a contradiction. Therefore, the order of $\tau_2'\tau_3'$ is at least 3. Since $(\tau_2'\tau_3')(\ell \cap \ell') = \tau'(\ell \cap \ell') = \ell \cap \ell'$ and $\tau_2'\tau_3'(Q) = Q$, it follows that $(\tau_2'\tau_3')(\ell') = \ell'$. It follows that $\tau_2'\tau_3'$ acts on $\Delta_3' \cap \ell'$ faithfully, namely, $\tau_2'\tau_3'|_{\ell'}$ is of order four. Let $G' \subset \mathrm{Aut}(\ell')$ be the group arising from the restrictions of all automorphisms in $\langle G[P_1'], G[P_2'], \tau_2'\tau_3'\rangle$ on the line ℓ'. Then $G' \cong S_4$. As the group S_4, the stabilizer subgroup of P_1' is S_3. As a finite subgroup of $\mathrm{Aut}(\ell')$, the stabilizer subgroup of P_1' is a cyclic group. This is a contradiction.

Assume that there exists a point $P \in \Delta_3'$ with $P \not\in \ell = \overline{P_1P_2}$. Then there exists a point $P' \in \Delta_3' \cap \overline{QP}$ with $P' \not\in \ell \cup \{P\}$, according to the action of an involution in $G[Q]$. Since the group $\langle G[P_1], G[P_2]\rangle$ acts on the set of all lines passing through Q, it follows that $\delta'[3] \geq 12$. If (P, P') is not a G-pair, then $\#\Delta_3' \cap \overline{QP} = 4$. According to the above discussion, this is a contradiction. Therefore, (P, P') is a G-pair. There exists a point $P'' \not\in \overline{QP}$ such that (P, P'') and (P', P'') are G-pairs. Since $P'' \in F[P] \cap F[P'] \subset F[Q] \setminus \{Q\} = \ell$, it follows that $P'' \in \Delta_3' \cap \ell$ or $|G[P'']| \geq 6$. For the latter case, by the action of $G[P'']$, there exist at least six points $P''' \in \Delta_3' \cap \ell$ with $|G[P''']| = 3$. This is a contradiction. Therefore, $P'' \in \Delta_3' \cap \ell$ holds. This implies that $\delta'[3] = 12$. □

4. Curves of degree six

We consider the case where $d = 6$ and $n = 3$. We determine the number $\delta'[3]$.

Theorem 4.1. *Let $C \subset \mathbb{P}^2$ be a smooth plane curve of degree $d = 6$. Then*

$$\delta'[3] = 0, 1, 2, 3, 4, 8 \text{ or } 12.$$

Furthermore, the following hold.

(1) $\delta'[3] = 12$ *if and only if C is projectively equivalent to the curve defined by*

$$X^6 + Y^6 + Z^6 - 10(X^3Y^3 + Y^3Z^3 + Z^3X^3) = 0.$$

(2) $\delta'[3] = 8$ *if and only if C is projectively equivalent to the curve defined by*

$$X^6 + 20X^3Y^3 - 8Y^6 + Z^6 = 0.$$

(3) $\delta'[3] = 4$ *if and only if C is projectively equivalent to the curve defined by*

$$Z^6 + a(X^3Y + Y^4)Z^2 + (X^6 + 20X^3Y^3 - 8Y^6) = 0$$

for some $a \in K \setminus \{0\}$, and C is not in the case (1).

Proof. Let $P \in \mathbb{P}^2 \setminus C$ be a point with $|G[P]| = 3$. By Proposition 3.1, there exist $d = 6$ points $Q \in C \cap (F[P] \setminus \{P\})$ such that $P \in T_QC$ and $I_Q(C, T_QC) \geq 3$. Therefore, for each quasi-Galois point $P \in C$ with $|G[P]| = 3$, we need at least 6 flexes. It follows from Proposition 3.3 that there exists no point $Q \in C$ such that $Q \in F[P_1] \cap F[P_2]$ for different quasi-Galois points P_1 and P_2 with $|G[P_1]| =$

$|G[P_2]| = 3$. By the flex formula [12, Theorem 1.5.10], we have $\delta'[3] \times 6 \le 72$. Therefore, $\delta'[3] \le 12$.

Assume that $\delta'[3] \ge 5$. First, we prove that there exists a G-pair. Assume by contradiction that $\sigma(P_2) \ne P_2$ for any quasi-Galois points P_1, P_2 and any generator $\sigma \in G[P_1]$. By Theorem 3.14, if a line contains two quasi-Galois points, then we obtain other two quasi-Galois points on the line. We consider a quasi-Galois point P and a line $\ell \not\ni P$ containing four quasi-Galois points P_1, P_2, P_3, P_4. In this case, the lines $\overline{PP_i}$ contains four quasi-Galois points for $i = 1, 2, 3, 4$. Then it is inferred that there exist $3 \times 4 + 1 = 13$ quasi-Galois points. This is a contradiction. Therefore, there exists a G-pair (P, P'). According to Proposition 3.8, for a suitable system of coordinates, we can assume that $P = (1 : 0 : 0)$, $P' = (0 : 1 : 0)$, generators σ of $G[P]$ and σ' of $G[P']$ are given by the matrices

$$A_\sigma = \begin{pmatrix} \omega & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ A_{\sigma'} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

respectively, where $\omega^2 + \omega + 1 = 0$, and C is given by

$$X^6 + aY^6 + bZ^6 + cX^3Y^3 + dY^3Z^3 + eZ^3X^3 = 0,$$

where $a, b, c, d, e \in K$. Then $P'' = (0 : 0 : 1)$ is also quasi-Galois.

Next, we consider the case where there exists a quasi-Galois point not contained in the set $S := \overline{PP'} \cup \overline{P'P''} \cup \overline{P''P} = \{XYZ = 0\}$. Let $(\alpha : \beta : 1)$ be a quasi-Galois point on $\mathbb{P}^2 \setminus S$. By using the linear transformation given by $(X : Y : Z) \mapsto ((1/\alpha)X : (1/\beta)Y : Z)$, we can assume that $(\alpha : \beta : 1) = (1 : 1 : 1)$. Then points

$$P_{ij} := (\omega^i : \omega^j : 1)$$

are quasi-Galois for $i, j = 0, 1, 2$, and the set $\{P, P', P''\} \cup \{P_{ij} \mid i, j = 0, 1, 2\}$ consists of all quasi-Galois points for C.

We compute a generator $\tau \in G[P_{00}]$, where $P_{00} = (1 : 1 : 1)$. It follows that $\tau(P) = P_{i0}$, $\tau(P') = P_{0j}$ and $\tau(P'') = P_{kk}$ for some $i, j, k \ne 0$. We can assume that $k = 2$ and $\tau(P'') = (1 : 1 : \omega)$. Then τ is represented by the matrix

$$A_\tau = \begin{pmatrix} \lambda\omega^i & \mu & 1 \\ \lambda & \mu\omega^j & 1 \\ \lambda & \mu & \omega \end{pmatrix},$$

for some $\lambda, \mu \in K \setminus \{0\}$. By using the condition $\tau((1:1:1)) = (1:1:1)$, it follows that

$$\lambda = \frac{\omega - 1}{\omega^i - 1},\ \mu = \frac{\omega - 1}{\omega^j - 1}.$$

If $i = 2$, then $\lambda = 1/(\omega+1) = -\omega$ and $\tau((1:0:1)) = (0:1:0)$. Since $(1:0:1)$ is not quasi-Galois, this is a contradiction. It follows that $i = 1$ and $\lambda = 1$. Similarly, it follows that $j = 1$ and $\mu = 1$.

Note that

$$A_\sigma A_\tau A_\sigma A_\tau = \begin{pmatrix} 3\omega & 0 & 0 \\ 0 & 0 & 3\omega \\ 0 & 3\omega & 0 \end{pmatrix}.$$

The linear transformation given by $(X:Y:Z) \mapsto (X:Z:Y)$ acts on C. Similarly, $\sigma'\tau\sigma'\tau$ acts on C by $(X:Y:Z) \mapsto (Z:Y:X)$. Therefore, the defining equation of C is of the form

$$F = X^6 + Y^6 + Z^6 + a(X^3Z^3 + Y^3Z^3 + Z^3X^3) = 0$$

for some $a \in K$. We consider the action by τ. Polynomials $(\tau^{-1})^*F$ and F are the same up to a constant. We consider the coefficient of X^4YZ. It follows that the coefficient of X^4YZ is 30ω for $(\omega X+Y+Z)^6, (X+\omega Y+Z)^6$ and $(X+Y+\omega Z)^6$. The coefficient is 3ω for $(\omega X+Y+Z)^3(X+\omega Y+Z)^3$, $(X+\omega Y+Z)^3(X+Y+\omega Z)^3$ and $(X+Y+\omega Z)^3(\omega X+Y+Z)^3$. It follows that $a = -10$.

We consider the case where all quasi-Galois points are contained in the set S. If there exist two quasi-Galois points $P_2, P_3 \not\in \{P, P', P''\}$ which are contained in $X = 0$ and $Y = 0$ respectively, then (P_2, P_3) is not a G-pair, since (P_2, P) is a G-pair, (P_2, P') is not a G-pair, and $P_3 \in \overline{PP''}$. We can find quasi-Galois points in $\mathbb{P}^2 \setminus S$ by the actions associated with P_2. Therefore, we can assume that quasi-Galois points different from $\neq P, P', P''$ are contained in one line $\subset S$. We can assume that such a line is $\overline{PP'}$. Then, by Theorem 3.14, $\#\Delta'_3 \cap \overline{PP'} = 8$. Furthermore, $|G[P'']| = 6$, that is, P'' is a Galois point. Let $P_1 = P$. We use the same symbols in the proof of Lemma 3.5. It follows that $\sigma_2\sigma_1^2$ is represented by the matrix

$$A_{\sigma_2\sigma_1^2} = \begin{pmatrix} -\alpha & 2\omega^2\alpha & 0 \\ \omega\alpha & \alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Note that the restriction of $\sigma_2\sigma_1^2$ on the line $\overline{PP'}$ is of order two, and fixed points of $\sigma_2\sigma_1^2$ on the line $\overline{PP'}$ are $((-1+\sqrt{3})\omega^2 : 1 : 0)$ and $((-1-\sqrt{3})\omega^2 : 1 : 0)$. Since

A_4 acts on the six points given by $C \cap \overline{PP'}$ and contains exactly three elements of order two, it follows that

$$C \cap \overline{PP'} = \{(-1+\sqrt{3})\omega^i : 1 : 0) \mid i = 0,1,2\} \cup \{((-1-\sqrt{3})\omega^i : 1 : 0) \mid i = 0,1,2\}.$$

Then the defining equation of C is of the form

$$F(X,Y,Z) = X^6 + 20X^3Y^3 - 8Y^6 + Z^6 = 0.$$

Assume that $\delta'[3] = 4$. Then the four quasi-Galois points P_1, P_2, P_3, P_4 are contained in a unique line ℓ, and $\varphi(G) \cong A_4$. We can assume that $P_1 = (1:0:0)$, $P_2 = (1:-1:0)$, $\sigma_1 \in G[P_1]$ and $\sigma_2 \in G[P_2]$ are generators represented by

$$A_{\sigma_1} = \begin{pmatrix} \omega & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ A_{\sigma_2} = \begin{pmatrix} -\omega\alpha & 2\omega^2\alpha & 0 \\ \omega^2\alpha & \alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

respectively. By Lemma 3.5, it follows that $|G[Q]| \ge 2$ and $\ell = F[Q] \setminus \{Q\}$ for the point $Q = (0:0:1) \in F[P_1] \cap F[P_2]$. Then the defining equation of C is of the form

$$F(X,Y,Z) = Z^6 + aY^2Z^4 + (bX^3Y + cY^4)Z^2 + G(X,Y) = 0,$$

where $a, b, c \in K$ and G is a homogeneous polynomial of degree 6. Similarly to the previous paragraph, we can assume that

$$G(X,Y) = X^6 + 20X^3Y^3 - 8Y^6.$$

By comparing the coefficient of Y^2Z^4 of $F(-\omega\alpha X + 2\omega^2\alpha Y, \omega^2\alpha X + \alpha Y, Z)$ and $F(X,Y,Z)$, it follows that $a = 0$. By comparing the coefficient of X^4Z^2 of $F(-\omega\alpha X + 2\omega^2\alpha Y, \omega^2\alpha X + \alpha Y, Z)$ and $F(X,Y,Z)$, it follows that $b = c$. Then the defining equation of C is of the form

$$Z^6 + c(X^3Y + Y^4)Z^2 + (X^6 + 20X^3Y^3 - 8Y^6) = 0.$$

On the contrary, we consider the curve C given by

$$X^6 + Y^6 + Z^6 - 10(X^3Y^3 + Y^3Z^3 + Z^3X^3) = 0.$$

By Fact 2.4, points $P = (1:0:0)$, $P' = (0:1:0)$ are quasi-Galois, and groups $G[P], G[P']$ are generated by the linear transformations σ, σ' given by

$$A_{\sigma} = \begin{pmatrix} \omega & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ A_{\sigma'} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

respectively. Let τ be the linear transformation given by the matrix

$$A_\tau = \begin{pmatrix} \omega & 1 & 1 \\ 1 & \omega & 1 \\ 1 & 1 & \omega \end{pmatrix}.$$

Then $\tau(C) = C$ and

$$\tau^* \left(\frac{x-y}{y-1} \right) = \frac{x-y}{y-1}.$$

Therefore, the point $(1:1:1)$ is quasi-Galois on $\mathbb{P}^2 \setminus \{XYZ = 0\}$. By considering the actions of σ and σ', it follows that $\delta'[3] \geq 12$. Since we confirmed $\delta'[3] \leq 12$ in the first paragraph, it follows that $\delta'[3] = 12$.

We consider the curve C defined by

$$F(X, Y, Z) = X^6 + 20X^3Y^3 - 8Y^6 + Z^6 = 0.$$

To prove $\delta'[3] = 8$, we have to prove that the linear transformation σ_2 represented by

$$A_{\sigma_2} = \begin{pmatrix} -\omega\alpha & 2\omega^2\alpha & 0 \\ \omega^2\alpha & \alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

acts on C. To do this, we prove that the linear transformation $\sigma_2\sigma_1^2$ represented by

$$A_{\sigma_2\sigma_1^2} = \begin{pmatrix} -\alpha & 2\omega^2\alpha & 0 \\ \omega\alpha & \alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

acts on C. Let $G(X, Y) = X^6 + 20X^3Y^3 - 8Y^6$. It is easily verified that the coefficient of X^6 for $G(-\alpha X + 2\omega^2\alpha Y, \omega\alpha X + \alpha Y)$ is 1. Further, it is inferred that the set

$$\{(-1+\sqrt{3})\omega^i : 1 : 0) \mid i = 0, 1, 2\} \cup \{((-1-\sqrt{3})\omega^i : 1 : 0) \mid i = 0, 1, 2\}$$

is invariant under the action of $A_{\sigma_2\sigma_1^2}$. The claim follows.

We consider the curve defined by

$$F(X, Y, Z) = Z^6 + a(X^3Y + Y^4)Z^2 + (X^6 + 20X^3Y^3 - 8Y^6) = 0,$$

where $a \in K \setminus \{0\}$. Assume that C is not in the case (1). It is obvious that $F(\omega X, Y, Z) = F$. According to the previous paragraph, for the polynomial $G(X, Y) =$

$X^6 + 20X^3Y^3 - 8Y^6$,

$$G(-\omega\alpha X + 2\omega^2\alpha Y, \omega^2\alpha X + \alpha Y) = G(X, Y).$$

It is not difficult to confirm that for the polynomial $H(X,Y) = X^3Y + Y^4$,

$$H(-\omega\alpha X + 2\omega^2\alpha Y, \omega^2\alpha X + \alpha Y) = H(X, Y)$$

(see also [6, Lemma 1]). Therefore, the linear transformation represented by

$$A_{\sigma_2} = \begin{pmatrix} -\omega\alpha & 2\omega^2\alpha & 0 \\ \omega^2\alpha & \alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

acts on C, and hence, $\delta'[3] = 4$ or 8. If $\delta'[3] = 8$, then the point $(0:0:1)$ must be a Galois point. This forces $a = 0$. The claim $\delta'[3] = 4$ follows. □

Remark 4.2. If $\delta'[3] = 2$, then $\delta'[6] = 1$. If $\delta'[3] = 4$, then $\delta'[2] = 1$.

Remark 4.3. We can prove that if the curve defined by

$$Z^6 + a(X^3Y + Y^4)Z^2 + (X^6 + 20X^3Y^3 - 8Y^6) = 0$$

is projectively equivalent to the curve defined by

$$X^6 + Y^6 + Z^6 - 10(X^3Y^3 + Y^3Z^3 + Z^3X^3) = 0,$$

then $a^3 = 18000$.

As an application, on the automorphism group $\mathrm{Aut}(C)$, we have the following (see Part I [5] for the definition of $G_3(C)$).

Theorem 4.4. *Let C be the plane curve defined by $X^6 + Y^6 + Z^6 - 10(X^3Y^3 + Y^3Z^3 + Z^3X^3) = 0$. Then*

$$G_3(C) = \mathrm{Aut}(C).$$

Proof. Let $P = (1:0:0)$, $P' = (0:1:0)$, $P'' = (0:0:1)$, and let $P_{ij} = (\omega^i : \omega^j : 1)$ for $i, j = 0, 1, 2$, where $\omega^2 + \omega + 1 = 0$. Then the set $\Delta' := \{P, P', P''\} \cup \{P_{ij} \mid i, j = 0, 1, 2\}$ consists of all quasi-Galois points P with $|G[P]| = 3$.

Let $\sigma \in \mathrm{Aut}(C) \subset \mathrm{Aut}(\mathbb{P}^2)$. Then σ acts on Δ'. If $\sigma(P) = P'$ or P'', then there exists $\phi_1 \in G_3(C)$ such that $\phi_1\sigma(P) = P$, since the automorphisms $(X:Y:Z) \mapsto (Y:X:Z)$ and $(X:Y:Z) \mapsto (Z:Y:X)$ are contained in $G_3(C)$ as in the proof of Theorem 4.1. If $\sigma(P) = P_{ij}$ for some i, j, then there exists $\phi_2 \in G[P_{kj}]$ for $k \neq i$ such that $\phi_2\sigma(P) = P$. Therefore, there exists $\phi \in G_3(C)$ such that

$\phi\sigma(P) = P$. The line $F[P] \setminus \{P\}$ is a unique line ℓ such that $I_Q(C, \overline{PQ}) = 3$ for any $Q \in C \cap \ell$. By this fact and $\phi\sigma(P) = P$, $\phi\sigma(F[P] \setminus \{P\}) = F[P] \setminus \{P\}$. Since $\Delta' \cap F[P] \setminus \{P\} = \{P', P''\}$ and the automorphism $(X : Y : Z) \mapsto (X : Z : Y)$ is contained in $G_3(C)$, there exists $\phi_3 \in G_3(C)$ such that $\phi_3\sigma(P) = P$, $\phi_3\sigma(P') = P'$ and $\phi_3\sigma(P'') = P''$. Then $\phi_3\sigma$ is represented by the matrix of the form

$$\begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

for some $\alpha, \beta \in K$. By considering the action on the defining equation, it follows that $\alpha^3 = 1$ and $\beta^3 = 1$. If we take

$$\phi_4 = \begin{pmatrix} \alpha^{-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in G[P],\ \phi_5 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \beta^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix} \in G[P'],$$

then $\phi_5\phi_4\phi_3\sigma = 1$ on $\mathbb{P}^2$. Therefore, $\sigma = \phi_3^{-1}\phi_4^{-1}\phi_5^{-1} \in G_3(C)$. □

Remark 4.5. For the curve defined by $X^6+Y^6+Z^6-10(X^3Y^3+Y^3Z^3+Z^3X^3) = 0$, it is known that the group $\mathrm{Aut}(C)$ is isomorphic to the Hessian group of order 216 ([2]).

Theorem 4.6. *Let C be the plane curve defined by $X^6 + 20X^3Y^3 - 8Y^6 + Z^6 = 0$. Then there exist two exact sequences*

$$0 \to \mathbb{Z}/6\mathbb{Z} \to G_3(C) \to A_4 \to 1,$$

$$0 \to \mathbb{Z}/6\mathbb{Z} \to \mathrm{Aut}(C) \to S_4 \to 1.$$

In particular, $|\mathrm{Aut}(C)| = 144$ *and* $|G_3(C)| = 72$.

Proof. Let ℓ be the line defined by $Z = 0$, which contains 8 points P with $|G[P]| = 3$. Since ℓ is a unique line containing 8 points P with $|G[P]| = 3$, there exists a homomorphism $\varphi : \mathrm{Aut}(C) \to \mathrm{Aut}(\ell) \cong \mathbb{P}^1$. Since $\varphi(G_3(C)) = \varphi(\langle G[P_1], G[P_2]\rangle)$ for each points P_1 and P_2 such that (P_1, P_2) is not a G-pair, it follows that $\varphi(G_3(C)) \cong A_4$. Since $Q = (0 : 0 : 1)$ is a unique Galois point, the group $\mathrm{Aut}(C)$ fixes Q. This implies that $\mathrm{Ker}\ \varphi = G[Q] \cong \mathbb{Z}/6\mathbb{Z}$. The former exact sequence is obtained. On the other hand, the linear transformation

$$\begin{pmatrix} 0 & \sqrt{2}i & 0 \\ \frac{1}{\sqrt{2}i} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

acts on C, where $i^2 = -1$. This implies that $\varphi(\mathrm{Aut}(C)) \cong S_4$. The latter exact sequence is obtained. □

5. Curves of degree four

In this section, we assume that C is smooth and of degree $d = 4$. The set $\Delta'_{\ge 2}$ of all quasi-Galois points in $\mathbb{P}^2 \setminus C$ for C is denoted by Δ'. If $P \in \Delta'$, then there exists a unique involution in $G[P]$, since $G[P]$ is a cyclic group of order 2 or 4. First, we note the following.

Lemma 5.1. *If $P \in \Delta'$, then we have the following.*

(1) *There exist exactly four lines $\ell \ni P$ such that $C \cap \ell$ consists of one or two points, and the tangent line at each point of $C \cap \ell$ is equal to ℓ.*
(2) *There does not exist a line $\ell \ni P$ such that $I_Q(C, \ell) = 3$ for some $Q \in C \cap \ell$.*

Proof. Let $\sigma \in G[P]$ be the involution. The projection π_P is the composite map of $g_P : C \to C/\sigma$ and $f_P : C/\sigma \to \mathbb{P}^1$. Since g_P is ramified at exactly four points by Corollary 2.5 and Fact 2.3(1), by Hurwitz formula, the genus of the smooth model of C/σ is equal to 1. Then $f_P : C/\sigma \to \mathbb{P}^1$ has exactly four ramification points. Therefore, we have (1). Assertion (2) is obvious, since π_P is the composite map of double coverings g_P and f_P. □

We recall the notion of G-pairs and the following proposition (see Proposition 3.8 and Corollary 3.9 in Section 3).

Proposition 5.2. *Let (P, P') be a G-pair. Then there exists a linear transformation ϕ such that $\phi(P) = (1 : 0 : 0)$, $\phi(P') = (0 : 1 : 0)$, and $\phi(C)$ is given by*

$$X^4 + Y^4 + Z^4 + aX^2Y^2 + bY^2Z^2 + cZ^2X^2 = 0,$$

where $a, b, c \in K$. In this case, there exists a quasi-Galois point P'' with $\phi(P'') = (0 : 0 : 1)$ such that (P', P'') and (P'', P) are G-pairs. In particular, $C \cap \overline{PP'}$ consists of exactly four points.

Furthermore, if P is a Galois point, then we can take $a = c = 0$.

Proof. Assertions except for the last one are derived from Proposition 3.8 and Corollary 3.9. We consider the last assertion. Assume that $|G[P]| = 4$. In the proof of Proposition 3.8, we can take

$$B^{-1}A_{\sigma_1}B = \begin{pmatrix} i & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ B^{-1}A_{\sigma_2}B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \zeta & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where $i^2 = -1$ and $\zeta^2 = \pm 1$. Then the defining equation of the form

$$X^4 + Y^4 + Z^4 + bY^2Z^2 = 0$$

for some $b \in K$. □

Let $\ell \subset \mathbb{P}^2$ be a projective line. We would like to calculate the number of quasi-Galois points on the line ℓ. We treat the cases $\# C \cap \ell = 4, 3, 2$ and 1 separately.

Proposition 5.3. *Let ℓ be a line with $\# C \cap \ell = 4$. Then $\# \Delta' \cap \ell = 0, 1, 2, 4$ or 6. Furthermore, if $\# \Delta' \cap \ell = 2$ (resp., 4, 6), then we have exactly one (resp., two, three) G-pair.*

Proof. Let $C \cap \ell = \{Q_1, Q_2, Q_3, Q_4\}$. We consider the possibilities of involutions acting on $C \cap \ell$. There are at most three types:

(1) $Q_1 \leftrightarrow Q_2$, $Q_3 \leftrightarrow Q_4$,
(2) $Q_1 \leftrightarrow Q_3$, $Q_2 \leftrightarrow Q_4$,
(3) $Q_1 \leftrightarrow Q_4$, $Q_2 \leftrightarrow Q_3$.

If $P_1, P_2 \in \Delta' \cap \ell$, and involutions $\sigma_1 \in G[P_1]$ and $\sigma_2 \in G[P_2]$ are of type (1), then we have $\sigma_1|_\ell = \sigma_2|_\ell$. Then $\sigma_1(P_2) = \sigma_2(P_2) = P_2$ and $\sigma_2(P_1) = \sigma_1(P_1) = P_1$, i.e. (P_1, P_2) is a G-pair. For each types (1)-(3) we have at most two quasi-Galois points, and hence, $\# \Delta' \cap \ell \le 6$.

Let $\sigma_1 \in G[P_1]$ and $\sigma_2 \in G[P_2]$ give involutions of types (1) and (2) respectively. Then $\sigma_1\sigma_2\sigma_1(Q_1) = Q_3$, and hence, $\sigma_1\sigma_2\sigma_1$ is of type (2). Since $\sigma_1(P_2) \ne P_2$ and $\sigma_1(P_2)$ is quasi-Galois, $(P_2, \sigma_1(P_2))$ is a G-pair. Similarly, $(P_1, \sigma_2(P_1))$ is a G-pair. We have two G-pairs.

Assume that $\# \Delta' \cap \ell \ge 5$. There are at least two G-pairs. We can assume that (P_1, P_2) and (P_3, P_4) are G-pairs, and give involutions on ℓ of type (1) and (2) respectively. Let P_5 be another quasi-Galois point, and let $\sigma_i \in G[P_i]$ be the involution. Then $\sigma_1(P_5) \ne P_5$ and the involution $\sigma_1\sigma_5\sigma_1 \in G[\sigma_1(P_5)]$ gives an involution on ℓ of type (3). Therefore, $\sigma_1(P_5) \ne P_1, \dots, P_5$. We have $\# \Delta' \cap \ell = 6$. □

Hereafter, we consider the curve C defined by

$$F = X^4 + Y^4 + Z^4 + aX^2Y^2 + bY^2Z^2 + cZ^2X^2 = 0.$$

Then $P = (1 : 0 : 0)$, $P' = (0 : 1 : 0)$, $P'' = (0 : 0 : 1) \in \Delta'$. The lines $F[P] \setminus \{P\}$, $F[P'] \setminus \{P'\}$ and $F[P''] \setminus \{P''\}$ are defined by $X = 0$, $Y = 0$ and $Z = 0$ respectively. Since C is smooth, we have $a \neq \pm 2$, $b \neq \pm 2$ and $c \neq \pm 2$.

Proposition 5.4. *We have the following.*

1. *If there exist two G-pairs on the line defined by $Z = 0$, then $b = \pm c$. Furthermore, when $c = -b$, we take the linear transformation given by $X \mapsto iX$, where $i^2 = -1$, so that we have the defining equation with $c = b$.*
2. *If there exist three G-pairs on the line defined by $Z = 0$, then $b = c = 0$.*

Proof. Assume that (P_1, P_1') and (P_2, P_2') are two G-pairs on the line $Z = 0$. Then the point $P_1'' = (0 : 0 : 1)$ is contained in $F[P_1] \cap F[P_2]$. Let $\sigma_1 \in G[P_1]$ and $\sigma_2 \in G[P_2]$ be involutions. Then $\sigma_1\sigma_2$ satisfies

$$P_1 \leftrightarrow P_1', \ P_2 \leftrightarrow P_2'$$

(see the second paragraph of the proof of Proposition 5.3). Since $\sigma_1\sigma_2(F[P_1]) = F[P_1']$, we have $\sigma_1\sigma_2(P_1'') = P_1''$. Then $\sigma_1\sigma_2$ is represented by the matrix

$$\begin{pmatrix} 0 & \lambda & 0 \\ \mu & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

for some $\lambda, \mu \in K$. Then $((\sigma_1\sigma_2)^{-1})^*F$ and F are the same up to a constant. Therefore, we have

$$\lambda^4Y^4 + \mu^4X^4 + Z^4 + a\lambda^2\mu^2X^2Y^2 + b\mu^2X^2Z^2 + c\lambda^2Y^2Z^2 = F.$$

Considering the coefficients of Y^4 and Y^2Z^2, we have $\lambda^2 = \pm 1$ and $b = \pm c$.

Assume that (P_1, P_1'), (P_2, P_2') and (P_3, P_3') are three G-pairs on the line $Z = 0$. Let $\sigma_2 \in G[P_2]$ and $\sigma_3 \in G[P_3]$ be involutions. Then $\sigma_2\sigma_3$ satisfies

$$P_1 \to P_1, \ P_1' \to P_1', \ P_2 \leftrightarrow P_2', \ P_3 \leftrightarrow P_3'$$

(see the second paragraph of the proof of Proposition 5.3). Since $\sigma_2\sigma_3(P_1) = P_1$ and $\sigma_2\sigma_3(P_1') = P_1'$, we have $\sigma_2\sigma_3(P_1'') = P_1''$. Note that the order of $\sigma_2\sigma_3$ is at least 3 and the order of the restriction $(\sigma_2\sigma_3)|_{\{Z=0\}}$ on the line $Z=0$ is two. Then $\sigma_2\sigma_3$ is represented by the matrix

$$\begin{pmatrix} -\eta & 0 & 0 \\ 0 & \eta & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where $\eta^2 \neq 1$. Then $((\sigma_2\sigma_3)^{-1})^*F$ and F are the same up to a constant. Therefore, we have

$$\eta^4X^4 + \eta^4Y^4 + Z^4 + a\eta^4X^2Y^2 + b\eta^2Y^2Z^2 + c\eta^2Z^2X^2 = F.$$

Considering the coefficients of Y^2Z^2 and Z^2X^2, we have $b=c=0$. □

On the contrary, we have the following.

Proposition 5.5. *Let $a, b \in K$, and let C be the smooth plane curve given by*

$$X^4 + Y^4 + Z^4 + aX^2Y^2 + bY^2Z^2 + bZ^2X^2 = 0.$$

Then we have the following.

(1) *Points $(1:0:0)$, $(0:1:0)$, $(1:1:0)$ and $(1:-1:0)$ are quasi-Galois points. Furthermore, if $b \neq 0$, they are not Galois.*
(2) *If $b=0$, then points $(\pm i:1:0)$ are quasi-Galois, where $i^2=-1$. Furthermore, we have the following.*
   - *If $a \neq 0$, then points $(1:0:0)$ and $(0:1:0)$ are not Galois.*
   - *If $a \neq 6$, then points $(\pm 1:1:0)$ are not Galois.*
   - *If $a \neq -6$, then points $(\pm i:1:0)$ are not Galois.*
   - *If $a=0$ or ± 6, then there exists a linear transformation ϕ such that $\phi(\{Z=0\}) = \{Z=0\}$ and $\phi(C)$ is the Fermat curve $X^4+Y^4+Z^4=0$.*
(3) *If $b=0$, then $\delta'[2]=6$ or 12. Furthermore, $\delta'[2]=12$ if and only if C is projectively equivalent to the Fermat curve $X^4+Y^4+Z^4=0$.*

Proof. We consider points $(\pm 1:1:0)$. We set

$$\tilde{X} = \frac{1}{2}(X+Y),\ \tilde{Y} = \frac{1}{2}(X-Y),\ \tilde{Z} = Z$$

and take the linear transformation $\phi : (X:Y:Z) \mapsto (\tilde{X}:\tilde{Y}:\tilde{Z})$. Then $\phi^{-1}((1:1:0)) = (1:0:0)$, $\phi^{-1}((-1:1:0)) = (0:1:0)$, and $\phi^{-1}(C)$ is given by

$$G = (2+a)X^4 + (2+a)Y^4 + Z^4 + (12-2a)X^2Y^2 + 2bY^2Z^2 + 2bX^2Z^2 = 0.$$

By Theorem 2.4, $\phi^{-1}((\pm 1 : 1 : 0))$ are quasi-Galois. Therefore, $(\pm 1 : 0 : 0)$ are quasi-Galois. Furthermore, if $\phi^{-1}((1 : 1 : 0))$ is Galois, then the matrix

$$\begin{pmatrix} i & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

acts on G. This implies $12 - 2a = 0$ and $2b = 0$.

Let $b = 0$. We consider points $(\pm i : 1 : 0)$. We set

$$\tilde{X} = \frac{1}{2}(X + iY),\ \tilde{Y} = \frac{1}{2}(X - iY),\ \tilde{Z} = Z$$

and take the linear transformation $\phi : (X : Y : Z) \mapsto (\tilde{X} : \tilde{Y} : \tilde{Z})$. Then $\phi^{-1}((i : 1 : 0)) = (1 : 0 : 0)$, $\phi^{-1}((-i : 1 : 0)) = (0 : 1 : 0)$, and $\phi^{-1}(C)$ is given by

$$H = (2 - a)X^4 + (2 - a)Y^4 + Z^4 + (12 + 2a)X^2Y^2 = 0.$$

By Theorem 2.4, $\phi^{-1}((\pm i : 1 : 0))$ are quasi-Galois. Therefore, $(\pm i : 0 : 0)$ are quasi-Galois. Furthermore, if $a \neq -6$, then $(\pm i : 0 : 0)$ are not Galois.

We prove (3). Now, we have six quasi-Galois points on the line $Z = 0$. By the defining equation, we infer that $Q = (0 : 0 : 1)$ is an outer Galois point and the set $F[Q] \setminus \{Q\}$ is given by $Z = 0$. Assume that $\delta'[2] > 6$. Then there exists a quasi-Galois point $R \in \mathbb{P}^2 \setminus (\{Z = 0\} \cup \{Q\})$. Let $\tau \in G[R]$ be the involution. If $\tau(Q) = Q$, then by Lemma 3.7, (R, Q) is a G-pair. Then R must lie on the line $Z = 0$. This is a contradiction. If $\tau(Q) \neq Q$ is not a G-pair, then we have two Galois points. It follows from a theorem of Yoshihara [17] that C is projectively equivalent to the Fermat curve. In this case, it is known that $\delta'[2] = 12$ ([11, 5]). □

Corollary 5.6. *If $\delta'[\geq 2] \geq 2$ and $\delta'[4] \geq 1$, then there exists a line ℓ such that $\#\Delta' \cap \ell = 6$.*

Proof. If $\delta'[4] \geq 1$, then it follows from a theorem of Yoshihara [17, Theorem 4' and Proposition 5'] that $\delta'[4] = 1$ or 3, and $\delta'[4] = 3$ implies that C is the Fermat curve. For the Fermat curve, the required line exists, by Proposition 5.5. We can assume that $\delta'[4] = 1$. Let P be a Galois point and let R be a quasi-Galois point. Since $\delta'[4] = 1$, then $F[R] \ni P$, that is, (R, P) is a G-pair. By Proposition 5.2, the defining equation is of the form

$$X^4 + Y^4 + Z^4 + bY^2Z^2 = 0$$

for some $b \in K$. By Proposition 5.5, the line defined by $X = 0$ is the required line. □

We consider the case where $C \cap \ell$ consists of three points.

Proposition 5.7. *If $\#C \cap \ell = 3$, then $\#\Delta' \cap \ell = 0$ or 1.*

Proof. Let $C \cap \ell = \{Q_1, Q_2, Q_3\}$, let $T_{Q_1}C = \ell$, and let $P_1, P_2 \in \ell$ be different quasi-Galois points. Then $Q_1 \in C \cap F[P_1] \cap F[P_2]$. This is a contradiction to Proposition 3.3. □

We consider the case where $C \cap \ell$ consists of two points.

Proposition 5.8. *Let $C \cap \ell = \{Q_1, Q_2\}$, where $Q_1 \neq Q_2$.*

1. *If $I_{Q_1}(C, \ell) = 3$, then $\#\Delta' \cap \ell = 0$.*
2. *If $T_{Q_1}C = T_{Q_2}C = \ell$, then $\#\Delta' \cap \ell = 0, 1$ or 3.*
3. *If $\#\Delta' \cap \ell = 3$, then there exists an automorphism $\sigma \in \mathrm{Aut}(C)$ of order three such that the fixed locus of σ coincides with the set $\{Q_1, Q_2, R\}$, where R is the point given by $R \in F[P]$ for any $P \in \Delta' \cap \ell$.*

Proof. Assertion (1) is derived from Lemma 5.1(2). We consider assertion (2). Let $P_1, P_2 \in \ell$ be quasi-Galois points, and let $\sigma_i \in G[P_i]$ be the involution. By Proposition 5.2, (P_1, P_2) is not a G-pair. By Lemma 3.7, $\sigma_1(P_2) \neq P_2$, and hence, $\#\Delta' \cap \ell \geq 3$. We consider $\sigma_1\sigma_2$. Then $\sigma_1\sigma_2(Q_1) = Q_1$ and $\sigma_1\sigma_2(Q_2) = Q_2$. Let R be the intersection point of the lines $F[P_1] \setminus \{P_1\}$ and $F[P_2] \setminus \{P_2\}$. Then $\sigma_1\sigma_2(R) = \sigma_1(R) = R$. If $R \in C$, then $T_RC \ni P_1, P_2$. Therefore, $R \notin C$. For a suitable system of coordinates, we can assume that $Q_1 = (1:0:0), Q_2 = (0:1:0)$ and $R = (0:0:1)$. Consider the action of $\sigma_1\sigma_2$ on the lines $\overline{Q_1R}$ and $\overline{Q_2R}$. Since $\sigma_1\sigma_2$ fixes Q_1, Q_2 and R, $\sigma_1\sigma_2|_{\overline{Q_iR}}$ is identity if $\sigma_1\sigma_2$ fixes some point of $C \cap \overline{Q_iR}$ other than Q_i. Therefore, the restriction $\sigma_1\sigma_2|_{\overline{Q_iR}}$ is identity or of order three for $i = 1, 2$. Then $\sigma_1\sigma_2$ is represented by the matrix

$$A_{\sigma_1\sigma_2} = \begin{pmatrix} \zeta & 0 & 0 \\ 0 & \eta & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where ζ and η are cubic roots of 1. Since Q_1 and Q_2 are not inner Galois (by Facts 2.1 and 2.3(2)) and $R \notin C$, we have $\eta \neq 1$, $\zeta \neq 1$ and $\zeta \neq \eta$. This implies that $\eta = \zeta^2$.

Let $P_3 := \sigma_1\sigma_2(P_2)$ $(= \sigma_1(P_2))$. Since $\sigma_1\sigma_2(F[P_2]) = F[P_3]$, $R \in F[P_3]$. Assume by contradiction that $\#\Delta' \cap \ell \ge 4$. Let $P_4 \ne P_1, P_2, P_3$ be quasi-Galois, and let $\sigma_4 \in G[P_4]$ be the involution. Since $\sigma_1\sigma_4(Q_i) = Q_i$ for $i = 1, 2$ and the order of $\sigma_1\sigma_4$ is three, we have $\sigma_1\sigma_4|_\ell = \sigma_1\sigma_2|_\ell$ or $(\sigma_1\sigma_2)^2|_\ell$. Then we find that $\sigma_1\sigma_4(R) = R$, and hence, $\sigma_1\sigma_4 = \sigma_1\sigma_2$ or $(\sigma_1\sigma_2)^2$ on $\mathbb{P}^2$. We have $\sigma_4 = \sigma_2 \in G[P_2]$ or $\sigma_4 = \sigma_2\sigma_1\sigma_2 \in G[P_3]$. This is a contradiction.

The condition as in assertion (3) is satisfied for the automorphism $\sigma_1\sigma_2$. □

We consider the case where $C \cap \ell$ consists of a unique point.

Proposition 5.9. *If $\#C \cap \ell = 1$, then $\#\Delta' \cap \ell = 0$ or 1.*

Proof. Let $C \cap \ell = \{Q\}$, and let P_1, P_2 be different quasi-Galois points. Then $Q \in F[P_1] \cap F[P_2]$. This is a contradiction to Proposition 3.3. □

Here, we assume that there does not exist a line ℓ such that $\#\Delta' \cap \ell = 6$. This condition is equivalent to the one that there does not exist a Galois point, under the assumption that $\delta'[\ge 2] \ge 2$, by Proposition 5.5 and Corollary 5.6. We introduce the notions of "G-triple" and "G-triangle" here. We call a triple (P, P', P'') a G-triple, if each two of points P, P', P'' form a G-pair. We call the set $\overline{PP'} \cup \overline{P'P''} \cup \overline{P''P} \subset \mathbb{P}^2$ a G-triangle via the triple (P, P', P'').

Lemma 5.10. *Let (P, P', P'') be a G-triple. Assume that R is a quasi-Galois point not in the G-triangle $\overline{PP'} \cup \overline{P'P''} \cup \overline{P''P}$. Then one of three lines $\overline{RP}$, $\overline{RP'}$ and $\overline{RP''}$ is not a multiple tangent line, that is, one of them contains at least three points of C.*

Proof. For a suitable system, we can assume that $P = (1:0:0)$, $P' = (0:1:0)$, $P'' = (0:0:1)$ and $R = (1:1:1)$. Then the defining equation of C is of the form

$$F = X^4 + Y^4 + Z^4 + aX^2Y^2 + bY^2Z^2 + cZ^2X^2 = 0,$$

where $a, b, c \in K$ and $F(1,1,1) = 3 + a + b + c \ne 0$. Assume that the three lines are multiple tangent lines. By the condition that the line $\overline{RP}$ is a multiple tangent line, it follows that

$$D_1(1,1) = (a+c)^2 - 4(b+2) = 0,$$

where $D_1(Y, Z)$ is the discriminant

$$(aY^2 + cZ^2)^2 - 4(Y^4 + bY^2Z^2 + Z^4).$$

By the symmetry, we have the equations

$$D_2 = (a+b)^2 - 4(c+2) = 0, \text{ and } D_3 = (b+c)^2 - 4(a+2) = 0.$$

By the relation $D_1 - D_2 = 0$,

$$(b-c)(2a+b+c+4) = 0.$$

Similarly,

$$(a-b)(a+b+2c+4) = 0, \text{ and } (c-a)(a+2b+c+4) = 0.$$

Assume that $a = b = c$. Then $3a + 3 \neq 0$ and $(2a)^2 - 4(a+2) = 0$. This implies that $a = 2$. This is a contradiction to the smoothness.

We can assume that $a \neq b$. Then $a + b + 2c + 4 = 0$. If $b = c$, then $a = -3c - 4$ and $(-2c-4)^2 - 4(c+2) = 0$. Then $c = -2$ or $a = b = c = -1$. The former is a contradiction to the smoothness, and the latter is a contradiction to $a \neq b$. Therefore, $b \neq c$. Then

$$2a + b + c + 4 = a + b + 2c + 4 = 0,$$

and $a = c$. Therefore, $b = -3c - 4$ and $(-2c-4)^2 - 4(c+2) = 0$. Then $c = -2$ or $c = -1$. The former is a contradiction to the smoothness, and the latter is a contradiction to $b \neq c$. □

Proposition 5.11. *Assume that there exists a G-triple (P, P', P''), and $\delta'[4] = 0$. Then*

$$\delta'[2] = 3, 5, 9 \text{ or } 21.$$

Furthermore, the following hold.

(1) $\delta'[2] = 21$ *if and only if C is projectively equivalent to the curve defined by*

$$X^4 + Y^4 + Z^4 + a(X^2Y^2 + Y^2Z^2 + Z^2X^2) = 0,$$

where $a \in K$ satisfies $a^2 + 3a + 18 = 0$.

(2) $\delta'[2] = 9$ *if and only if C is projectively equivalent to the curve defined by*

$$X^4 + Y^4 + Z^4 + a(X^2Y^2 + Y^2Z^2 + Z^2X^2) = 0,$$

where $a \in K \setminus \{0, -1\}$ and $a^2 + 3a + 18 \neq 0$.

(3) $\delta'[2] = 5$ *if and only if C is projectively equivalent to the curve defined by*

$$X^4 + Y^4 + Z^4 + aX^2Y^2 + b(Y^2Z^2 + Z^2X^2) = 0,$$

where $a, b \in K$, $b \neq 0$ and $b \neq \pm a$.

Proof. Assume that there exists a quasi-Galois point $R \notin \overline{PP'} \cup \overline{P'P''} \cup \overline{P''P}$. By Proposition 5.10, we can assume that $\overline{RP''}$ is not a multiple tangent line. By Propositions 5.3, 5.7 and 5.8, it follows that there exist four quasi-Galois points on the line $\overline{RP''}$, and that the point P_2 given $\overline{PP'} \cap \overline{RP''}$ is a quasi-Galois point. For the involution $\sigma \in G[P]$, $\sigma(P_2)$ is a quasi-Galois point and the line $\overline{\sigma(P_2)P''}$ contains four quasi-Galois points. It follows that the triple $(P_2, \sigma(P_2), P'')$ is a G-triple such that two edges of the G-triangle contain four quasi-Galois points. In this case, it follows form Proposition 5.4 that C is projectively equivalent to the curve defined by

$$X^4 + Y^4 + Z^4 + a(X^2Y^2 + Y^2Z^2 + Z^2X^2) = 0,$$

and the triangle $\overline{P_2\sigma(P_2)} \cup \overline{\sigma(P_2)P''} \cup \overline{P''P_2}$ contains 9 quasi-Galois points. By taking a suitable system of coordinates, we can assume that $P = (1 : 0 : 0)$, $P' = (0 : 1 : 0)$, $P'' = (0 : 0 : 1)$, C is defined by

$$X^4 + Y^4 + Z^4 + a(X^2Y^2 + Y^2Z^2 + Z^2X^2) = 0$$

for some $a \in K \setminus \{0\}$, and $\#\Delta' \cap (\overline{PP'} \cup \overline{P'P''} \cup \overline{P''P}) = 9$.

Assume that $\delta'[2] \geq 10$. Let R be a quasi-Galois point with $R \notin \overline{PP'} \cup \overline{P'P''} \cup \overline{P''P}$. By Proposition 5.10, we can assume that $\overline{RP}$ is not a multiple tangent line. It follows from Proposition 5.5 that $P_2 := (0 : 1 : 1)$ and $P_2' := (0 : -1 : 1) \in F[P] \setminus \{P\} = \{X = 0\}$ are quasi-Galois. Note that (P_2, P_2') is a G-pair, since (P', P'') is a G-pair. We can assume that $R \in \overline{PP_2'}$ with $R \neq P, P_2'$. Let $\tau \in G[R]$ be the involution. Note that $\tau(P_2') = P$, since (P, P_2') is a G-pair. Since (P, P_2) and (P_2, P_2') are G-pairs, $F[P_2] \setminus \{P_2\} = \overline{PP_2'} \ni R$, and hence, (P_2, R) is a G-pair. Therefore $\tau(P_2) = P_2$. Since $\tau((0 : 1 : 1)) = (0 : 1 : 1)$, $\tau((0 : -1 : 1)) = (1 : 0 : 0)$, and $\tau((1 : 0 : 0)) = (0 : -1 : 1)$, τ is represented by the matrix

$$\begin{pmatrix} 0 & \frac{2}{\lambda} & -\frac{2}{\lambda} \\ \lambda & 1 & 1 \\ -\lambda & 1 & 1 \end{pmatrix},$$

where $\lambda \in K$. Then $(\tau^{-1})^*F$ and F are the same up to a constant. Here

$$\begin{aligned}(\tau^{-1})^*F &= \left(\frac{2}{\lambda}\right)^4 (Y-Z)^4 + (\lambda X+Y+Z)^4 + (-\lambda X+Y+Z)^4 \\ &\quad +a\left(\frac{2}{\lambda}\right)^2 (Y-Z)^2(\lambda X+Y+Z)^2 + a(\lambda X+Y+Z)^2(-\lambda X+Y+Z)^2 \\ &\quad +a\left(\frac{2}{\lambda}\right)^2 (-\lambda X+Y+Z)^2(Y-Z)^2.\end{aligned}$$

The coefficient of X^2YZ is

$$12\lambda^2 + 12\lambda^2 - 2a\left(\frac{2}{\lambda}\right)^2\lambda^2 - 4a\lambda^2 - 2a\left(\frac{2}{\lambda}\right)^2\lambda^2 = 0.$$

We have $\lambda^2 = 4a/(6-a)$. The coefficient of Y^3Z is

$$-4\left(\frac{2}{\lambda}\right)^4 + 4 + 4 + 4a = 0.$$

We have $a^3 + a^2 + 12a - 36 = 0$. Since $a \neq 2$, we have $a^2 + 3a + 18 = 0$.

On the contrary, let $a^2 + 3a + 18 = 0$, and let C be the plane curve given by

$$F = X^4 + Y^4 + Z^4 + a(X^2Y^2 + Y^2Z^2 + Z^2X^2) = 0.$$

By Proposition 5.5, we have 9 quasi-Galois points on the union of the lines $X = 0$, $Y = 0$ and $Z = 0$. Let λ be a solution of $\lambda^2 = 4a/(6-a)$, and let τ be the involution given by

$$\begin{pmatrix} 0 & \frac{2}{\lambda} & -\frac{2}{\lambda} \\ \lambda & 1 & 1 \\ -\lambda & 1 & 1 \end{pmatrix}.$$

Then τ acts on C. It is inferred that τ is an involution, $\tau(\{Y+Z=0\}) = \{Y+Z=0\}$, and τ is not identity on this line. By the proof of [5, Proposition 2.6], τ is the involution of some quasi-Galois point R on the line $Y+Z=0$ other than $(1:0:0)$ or $(0:-1:1)$. By considering the actions associated with points $(1:0:0)$, $(0:1:0)$ and $(0:0:1)$, we have four quasi-Galois points not in $\{XYZ=0\}$. Note that R is different from $(\pm 1 : \pm 1 : 1)$. Using the linear transformation given by $(X:Y:Z) \mapsto (Z:X:Y)$, we have 4×3 additional quasi-Galois points. Therefore, we have $\delta'[2] \geq 9 + 12 = 21$.

We prove that $\delta'[2] \leq 21$. Let $P \in \Delta'$ and let $\ell \ni P$ be a line. If $\#\Delta' \cap \ell \geq 2$, then $\#\Delta' \cap \ell = 2, 3$ or 4, by Propositions 5.3, 5.7, 5.8 and 5.9. If $\#\Delta' \cap \ell = 3$, then ℓ is a multiple tangent line, and hence, it follows from Lemma 5.1 that such lines

ℓ are at most 4. If $\#\Delta' \cap \ell = 2$ or 4, then there exists a point $P' \in \ell$ such that $P' \in F[P] \setminus \{P\}$, by Proposition 5.3. Since $\#\Delta' \cap (F[P] \setminus \{P\}) \le 4$, such lines ℓ are at most 4. Therefore, $\delta'[2] \le 1 + 4 \times 2 + 4 \times 3 = 21$.

Assume that $\Delta' \subset \overline{PP'} \cup \overline{P'P''} \cup \overline{P''P}$ and $\delta'[2] \ge 4$. Let $R \in \Delta' \setminus \{P, P', P''\}$. We can assume that $R \in \overline{PP'}$. By Proposition 5.3, there exist four quasi-Galois points on the line $\overline{PP'}$. For a suitable system of coordinates, we can assume that the line $\overline{PP'}$ is defined by $Z = 0$ and C is defined by

$$X^4 + Y^4 + Z^4 + aX^2Y^2 + b(Y^2Z^2 + Z^2X^2) = 0,$$

where $a, b \in K$. By Proposition 5.5, it follows that $b \ne 0$. In this case, $\delta'[2] \ge 5$. If $\delta'[2] > 5$, then the line $\overline{P'P''}$ or $\overline{P''P}$ contains four quasi-Galois points. It follows from Proposition 5.4 that $a = \pm b$. In this case, C is projectively equivalent to the curve defined by

$$X^4 + Y^4 + Z^4 + a(X^2Y^2 + Y^2Z^2 + Z^2X^2) = 0$$

where $a \ne 0$ and $a^2 + 3a + 18 \ne 0$, and $\delta'[2] = 9$. For the smoothness, we need the condition $a \ne -1$. If $\delta'[2] = 5$, then $b \ne \pm a$, by Proposition 5.4. □

Theorem 5.12. *Let $C \subset \mathbb{P}^2$ be a smooth curve of degree four. Then*

$$\delta'[2] = 0, 1, 3, 5, 6, 9, 12 \text{ or } 21.$$

Furthermore, the following hold.

(1) $\delta'[2] = 21$ *if and only if C is projectively equivalent to the curve defined by*

$$X^4 + Y^4 + Z^4 + a(X^2Y^2 + Y^2Z^2 + Z^2X^2) = 0,$$

where $a \in K$ satisfies $a^2 + 3a + 18 = 0$.

(2) $\delta'[2] = 12$ *if and only if C is projectively equivalent to the Fermat curve.*

(3) $\delta'[2] = 9$ *if and only if C is projectively equivalent to the curve defined by*

$$X^4 + Y^4 + Z^4 + a(X^2Y^2 + Y^2Z^2 + Z^2X^2) = 0,$$

where $a \in K \setminus \{0, -1\}$ and $a^2 + 3a + 18 \ne 0$.

(4) $\delta'[2] = 6$ *if and only if C is projectively equivalent to the curve defined by*

$$X^4 + Y^4 + Z^4 + aX^2Y^2 = 0,$$

where $a \in K \setminus \{0\}$ and $a \ne \pm 6$.

(5) $\delta'[2] = 5$ *if and only if* C *is projectively equivalent to the curve defined by*

$$X^4 + Y^4 + Z^4 + aX^2Y^2 + b(Y^2Z^2 + Z^2X^2) = 0,$$

where $a, b \in K$, $b \neq 0$ *and* $b \neq \pm a$.

Proof. Assume that $\delta'[\geq 2] \geq 2$ and $\delta'[4] \geq 1$. By Proposition 5.5 and Corollary 5.6, C is projectively equivalent to the curve defined by

$$X^4 + Y^4 + Z^4 + aX^2Y^2 = 0.$$

Furthermore, if $a = 0$ or ± 6, then $\delta'[2] = 12$ and C is the Fermat curve, otherwise, $\delta'[2] = 6$. The assertion follows.

Hereafter, we assume that $\delta'[4] = 0$ and $\delta'[2] \geq 2$. By Proposition 5.5, there does not exist a line ℓ such that $\#\ell \cap \Delta' = 6$. If two Galois points form a G-pair, then, by Proposition 5.2, there exists a G-triple and $\delta'[2] \geq 3$. Therefore, $\delta'[2] \geq 3$ in any case. If there exists a G-triple, then the assertion follows by Proposition 5.11. We can assume that there does not exist a G-pair.

Assume that $\delta'[2] \geq 4$. By Propositions 5.3, 5.7, 5.8 and 5.9, there exists a line ℓ containing exactly three quasi-Galois points P_1, P_2 and P_3. Let P_4 be a quasi-Galois point with $P_4 \notin \ell$. Then the line $\overline{P_iP_4}$ contains exactly three quasi-Galois points for each i. Therefore, $\delta'[2] \geq 7$. It follows from Proposition 5.8 that there exists an automorphism σ of order three such that $\sigma(\ell) = \ell$ and the fixed point of σ not in ℓ coincides with the point given by $F[P_1] \cap F[P_2]$. If $\delta'[2] = 7$ or 8, then σ acts on $7 - 3 = 4$ or $8 - 3 = 5$ points. Therefore, σ fixes some quasi-Galois point P. Then $P \in F[P_1]$ and hence, (P, P_1) is a G-pair. This is a contradiction. It follows that $\delta'[2] \geq 9$.

Assume that $\delta'[2] = 9$. Since there does not exist a G-pair, the line $\overline{P_1P_2}$ contains exactly three quasi-Galois points for each pair of different quasi-Galois points P_1 and P_2. We consider the set

$$I := \{(P, \ell) \in \Delta' \times \check{\mathbb{P}}^2 \mid P \in \ell,\ \#\Delta' \cap \ell = 3\}$$

with projections $p_1 : I \to \Delta'$ and $p_2 : I \to \check{\mathbb{P}}^2$, where $\check{\mathbb{P}}^2$ is the dual projective plane. Since $\#I = 9 \times 4 = 36$ and each fiber of p_2 contains exactly 3 points, it follows that $\#p_2(I) = 12$. Let $\ell \in p_2(I)$ and let σ be an automorphism of order three on the line ℓ as in Proposition 5.8. Since the automorphism σ acts on the set $p_2(I) \setminus \{\ell\}$ and $\#p_2(I) \setminus \{\ell\} = 11$, there exists a line $\ell' \in p_2(I) \setminus \{\ell\}$ such that $\sigma(\ell') = \ell'$. Then $\sigma(\ell \cap \ell') = \ell \cap \ell'$. By Proposition 5.8, the point given by $\ell \cap \ell'$ is contained in C.

Since the tangent line at the point given by $\ell \cap \ell'$ coincides with lines ℓ and ℓ'. This is a contradiction to the smoothness. It follows that $\delta'[2] \geq 10$.

Assume that $\delta'[2] \geq 10$. Let $P \in \Delta'$. Since lines containing P and another two quasi-Galois points are at most 4, there exists a line containing four quasi-Galois points. In this case, there exists a G-pair, by Proposition 5.3. This is a contradiction. $\square$

Remark 5.13. It is known that the curve defined by

$$X^4 + Y^4 + Z^4 + a(X^2Y^2 + Y^2Z^2 + Z^2X^2) = 0,$$

where $a \in K$ satisfies $a^2+3a+18 = 0$, is projectively equivalent to the Klein quartic $X^3Y + Y^3Z + Z^3X = 0$ ([7], [13]).

Remark 5.14. For $d = 5$ and $n = 2$, the third author determined the number $\delta[2]$ ([16]).

References

[1] E. Arbarello, M. Cornalba, P. A. Griffiths and J. Harris, *Geometry of algebraic curves*, Vol. I. Grundlehren der Mathematischen Wissenschaften, **267**, Springer-Verlag, New York, 1985.

[2] M. Artebani and I. Dolgachev, The Hesse pencil of plane cubic curves, Enseign. Math. (2) **55** (2009), 235–273.

[3] H. C. Chang, On plane algebraic curves, Chinese J. Math. **6** (1978), 185–189.

[4] S. Fukasawa, Galois points for a plane curve in arbitrary characteristic, Proceedings of the IV Iberoamerican conference on complex geometry, Geom. Dedicata **139** (2009), 211–218.

[5] S. Fukasawa, K. Miura and T. Takahashi, Quasi-Galois points, I: Automorphism groups of plane curves, Tohoku Math. J. **71** (2019), 487–494.

[6] M. Kanazawa, T. Takahashi and H. Yoshihara, The group generated by automorphisms belonging to Galois points of the quartic surface, Nihonkai Math. J. **12** (2001), 89–99.

[7] F. Klein, Ueber die Transformation siebenter Ordnung der elliptischen Functionen, Math. Ann. **14** (1879), 428–471.

[8] G. I. Lehrer and D. E. Taylor, *Unitary reflection groups*, Cambridge University Press, New York, 2009.

[9] R. Miranda *Algebraic curves and Riemann surfaces*, Graduate Studies in Mathematics **5**, Amer. Math. Soc., Providence, RI, 1995.

[10] H. H. Mitchell, Determination of the ordinary and modular ternary linear groups, Trans. Amer. Math. Soc. **12** (1911), 207–242.

[11] K. Miura and H. Yoshihara, Field theory for function fields of plane quartic curves, J. Algebra **226** (2000), 283–294.

[12] M. Namba, *Geometry of projective algebraic curves*, Marcel Dekker, Inc., New York, 1984.

[13] R. Rodriguez and V. González-Aguilera, Fermat's quartic curve, Klein's curve and the tetrahedron, *Extremal Riemann Surfaces* (San Francisco, CA, 1995), 43–62, Contemp. Math., **201**, Amer. Math. Soc., Providence, RI, 1997.

[14] G. C. Shephard and J. A. Todd, Finite unitary reflection groups, Canad. J. Math. **6** (1954), 274–304.

[15] H. Stichtenoth, *Algebraic function fields and codes*, Universitext, Springer-Verlag, Berlin, 1993.

[16] T. Takahashi, Projection of a non-singular plane quintic curve and the dihedral group of order eight, Rend. Sem. Mat. Univ. Padova **135** (2016), 39–61.

[17] H. Yoshihara, Function field theory of plane curves by dual curves, J. Algebra **239** (2001), 340–355.

Faculty of Science, Yamagata University, Kojirakawa-machi 1-4-12, Yamagata 990-8560, Japan

Email address: s.fukasawa@sci.kj.yamagata-u.ac.jp

Department of Mathematics, National Institute of Technology, Ube College, Ube, Yamaguchi 755-8555, Japan

Email address: kmiura@ube-k.ac.jp

Education Center for Engineering and Technology, Faculty of Engineering, Niigata University, Niigata 950-2181, Japan

Email address: takeshi@eng.niigata-u.ac.jp