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The closedness of the product set of two Pukanszky polarizations of exponential Lie groups

By Ali Baklouti and Hidenori Fujiwara

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Abstract. Let $G = \exp \mathfrak{g}$ be an exponential solvable Lie group with Lie algebra $\mathfrak{g}$ and $\mathfrak{g}^*$ the dual vector space of $\mathfrak{g}$. We take two (real) polarizations $\mathfrak{h}_1, \mathfrak{h}_2$ of $\mathfrak{g}$ at $f \in \mathfrak{g}^*$ which satisfy the Pukanszky condition and define a unitary character χ_f of $H_j = \exp(\mathfrak{h}_j)$ $(j = 1, 2)$ by the formula $\chi_f(\exp X) = e^{if(X)}(X \in \mathfrak{h}_j)$. Then, it is well known that the two monomial representations $\pi_j = \mathrm{ind}_{H_j}^G \chi_f$ $(j = 1, 2)$ are mutually equivalent and we even have a natural candidate of the intertwining operator between them. In order to verify that it is a true intertwining operator, the principal obstacle is the convergence of the integral in question. A property which assures this convergence, is the closedness of the simple product set $H_2 H_1$ in G. In this paper, we establish this property, generalizing then a previous proof in the particular case when one of the polarizations is of Vergne type.

1. Introduction

Let $G = \exp \mathfrak{g}$ be an exponential solvable Lie group with Lie algebra $\mathfrak{g}$. We consider $f \in \mathfrak{g}^*$ and two (real) polarizations $\mathfrak{h}_1, \mathfrak{h}_2$ of $\mathfrak{g}$ at f which satisfy the Pukanszky condition. This means that $f + \mathfrak{h}_j^{\perp} = H_j \cdot f$ with $H_j = \exp(\mathfrak{h}_j)(1 \leq j \leq 2)$. Starting from the unitary character χ_f of H_j defined by $\chi_f(\exp X) = e^{if(X)}$ $(X \in \mathfrak{h}_j)$, we induce a monomial representation $\pi_j = \mathrm{ind}_{H_j}^G \chi_f$ of G. Thus, it comes that π_1, π_2 are irreducible and mutually equivalent (cf. [**4**]). For $j = 1, 2$, we denote by $\mathcal{H}_{\pi_j}$ the Hilbert space of π_j. We propose to construct explicitly an intertwining operator between them. Our first step in [**5**] was to establish the relation

$$\mathrm{Tr}\ \mathrm{ad}_{\mathfrak{h}_1/(\mathfrak{h}_1 \cap \mathfrak{h}_2)} X + \mathrm{Tr}\ \mathrm{ad}_{\mathfrak{h}_2/(\mathfrak{h}_1 \cap \mathfrak{h}_2)} X = 0$$

for all $X \in \mathfrak{h}_1 \cap \mathfrak{h}_2$. This leads to

$$\Delta_{H_1,G}(h) = \Delta_{H_2,G}(h) \Delta^2_{H_1 \cap H_2, H_2}(h) \quad (h \in H_1 \cap H_2).$$

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This being set, for $\varphi \in \mathcal{H}_{\pi_1}$ and $g \in G$, the function Φ_g on H_2 defined by

$$\Phi_g(h) = \varphi(gh)\chi_f(h)\Delta_{H_2,G}^{-1/2}(h)$$

verifies the relation

$$\Phi_g(hx) = \Delta_{H_1\cap H_2,H_2}(x)\Phi_g(h) \quad (h \in H_2, x \in H_1 \cap H_2).$$

Hence we are able to describe formally the integral

$$\left(T_{\mathfrak{h}_2\mathfrak{h}_1}\varphi\right)(g) = \oint_{H_2/(H_2\cap H_1)} \varphi(gh)\chi_f(h)\Delta_{H_2,G}^{-1/2}(h)d\nu(h) \quad (g \in G). \tag{1.1}$$

At least on the formal level, it is clair that $T_{\mathfrak{h}_2\mathfrak{h}_1}\varphi$ verifies the covariance condition required to belong to the space $\mathcal{H}_{\pi_2}$ and that $T_{\mathfrak{h}_2\mathfrak{h}_1}$ commutes with the action of G by left translations. In fact, it is not exaggerated [**3**] to say that one of the principal problems is the question of the convergence of the integral (1.1), which will be settled when the following claim holds:

Claim 1.1. *The product H_2H_1 is closed in G.*

Indeed, it turns out from the Pukanszky condition that the simple product set H_2H_1 is locally closed in G and hence that the space H_2H_1/H_1 is homeomorphic to homogeneous space $H_2/(H_1 \cap H_2)$. Therefore, if Claim 1.1 holds, the integral (1.1) is convergent for all continuous function φ with compact support modulo H_1 of $\mathcal{H}_{\pi_1}$.

Note that Claim 1.1 holds, in the setting when G is nilpotent or at least one of the polarisations $\mathfrak{h}_j$ is of Vergne type (cf. [**10**]), and also in the case where $\mathfrak{h}_1 + \mathfrak{h}_2$ is a subalgebra of $\mathfrak{g}$ (cf. [**1**]). The aim of the paper is to provide a proof in the general setting of exponential solvable Lie groups.

The outline of the paper is as follows. The next section presents some preliminaries about the representation theory of exponential solvable Lie groups and the orbit method. Section 3 proves first preparations of the proof of the main result proving that Claim 1.1 holds whenever there exists an abelian ideal $\mathfrak{n}$ between the nilpotent radical of $\mathfrak{g}$ and $[\mathfrak{g}, \mathfrak{g}]$ (cf. Proposition 3.2). This is actually crucial for the proof of the general case. Section 4 treats the setting where G is completely solvable, where by means of the techniques of Proposition 3.2, we overcome the technical difficulties raised through the induction procedure, (cf. Theorem 4.1). The last section proves the general case of exponential solvable, and as a consequence the convergence of the intertwining integral (1.1) (cf. Corollary 5.2), and also the composition formula introducing the Maslov index (cf. Corollary 5.3).

2. Backgrounds

2.1. Induced representations

Let dg be a left Haar measure on G and Δ_G the modular function of G so that we have

$$\int_G \varphi(gx^{-1})dg = \Delta_G(x) \int_G \varphi(g)dg \quad (x \in G)$$

for all functions φ belonging to the space $K(G)$ of the continuous functions on G with compact support. We have

$$\Delta_G(x) = |\det(\mathrm{Ad}x)|^{-1} \quad (x \in G).$$

A closed subgroup H with Lie algebra $\mathfrak{h}$ being given, we denote by $\Delta_{H,G}$ the character of H with values in $\mathbb{R}_+$ defined by

$$\Delta_{H,G}(h) = \frac{\Delta_H(h)}{\Delta_G(h)} \quad (h \in H).$$

It follows that, for $X \in \mathfrak{h}$,

$$\Delta_{H,G}(\exp X) = \exp(\mathrm{Tr}\ \mathrm{ad}_{\mathfrak{g}/\mathfrak{h}} X).$$

Let $K(G,H)$ be the space of the numerical continuous functions φ on G with compact support modulo H and which verify $\varphi(gh) = \Delta_{H,G}(h)\varphi(g)$ for all $g \in G$ and $h \in H$. G acting on $K(G,H)$ by left translations, we know that there exists, up to a scalar multiplication, one and only one G-invariant positive linear form. We denote it by $\nu_{G,H}$ or more simply ν and write it in the form of an integral

$$\nu_{G,H}(\varphi) = \oint_{G/H} \varphi(g)d\nu(g).$$

If $\Delta_H = \Delta_G$ on H, $\nu_{G,H}$ is nothing but a G-invariant measure on the homogeneous space G/H.

Let π be a unitary representation of H in the Hilbert space $\mathcal{H}_\pi$, $\mathcal{K}(G,H,\pi)$ the space of continuous functions ψ on G with values in $\mathcal{H}_\pi$, having the compact support modulo H and verifying the covariance condition

$$\psi(gh) = \big(\frac{\Delta_H(h)}{\Delta_G(h)}\big)^{1/2}\pi^{-1}(h)\psi(g) \quad (g \in G, \quad h \in H). \tag{2.1}$$

We define the norm $\|\psi\|$ in $\mathcal{K}(G,H,\pi)$ by

$$\|\psi\| = \left(\oint_{G/H} |\psi(g)|^2 d\nu_{G,H}(g) \right)^{1/2}.$$

Then G acts isometrically in the space $\mathcal{K}(G, H, \pi)$ by left translations and we obtain the induced unitary representation $\tau = \mathrm{Ind}_H^G \pi$ of G in the completion $\mathcal{H}_\tau$ of $\mathcal{K}(G, H, \pi)$.

In the case where $\pi = \chi$ is a unitary character of H, τ is the induced monomial representation $\mathrm{Ind}_H^G \chi$.

2.2. The orbit method theory

The Kirillov-Bernat-Vergne orbit method makes it possible to parameterize the unitary dual $\widehat{G}$ of G by the space of coadjoint orbits of G in $\mathfrak{g}^*$. For $\ell \in \mathfrak{g}^*$, let B_ℓ be the bilinear form on $\mathfrak{g}$ defined by $B_\ell(X, Y) = \ell([X, Y])$ and $\mathfrak{g}(\ell)$ the radical of B_ℓ. We take a polarization $\mathfrak{b} \subset \mathfrak{g}$ at ℓ (a totally isotropic subalgebra of $\mathfrak{g}$ (or Lagrangian) with respect of B_ℓ of maximal dimension equals to $\frac{1}{2}(\dim \mathfrak{g} + \dim \mathfrak{g}(\ell))$), satisfying Pukanszky's condition $B \cdot \ell = \ell + \mathfrak{b}^\perp$, where $B := \exp \mathfrak{b}$. We define the unitary character $\chi_\ell : B \to U(1)$ associated to ℓ by

$$\chi_\ell(\exp X) := e^{i\ell(X)} \quad (X \in \mathfrak{b}), \tag{2.2}$$

and denote by $\pi_{\ell,\mathfrak{b}}$ the induced unitary representation $\mathrm{Ind}_B^G \chi_\ell$ of G. Then $\pi_{\ell,\mathfrak{b}}$ is an irreducible representation of G and its equivalence class $[\pi_{\ell,\mathfrak{b}}]$ only depends upon the coadjoint orbit Ω_ℓ through ℓ. Every unitary and irreducible representation π of G is equivalent to the representation $\pi_{\ell,\mathfrak{b}}$ induced from a character χ_ℓ and a Pukanszky polarization $\mathfrak{b}$ at ℓ. Moreover, the Kirillov-Bernat-Vergne mapping

$$\begin{aligned} \theta_G : \mathfrak{g}^* &\longrightarrow \widehat{G} \\ \Omega_\ell &\longmapsto \pi_{\ell,\mathfrak{b}} \end{aligned} \tag{2.3}$$

which factors through the quotient to:

$$\begin{aligned} \bar{\theta}_G : \mathfrak{g}^*/G &\longrightarrow \widehat{G} \\ \Omega_\ell &\longmapsto [\pi_{\ell,\mathfrak{b}}] =: \pi_{\Omega_\ell} \end{aligned} \tag{2.4}$$

is a homeomorphism. For more details, see [**6**].

3. Orientation of our study

We begin with groping our way to find an orientation.

Example 3.1. Let $\mathfrak{g} = \langle x, y, z, a \rangle_{\mathbb{R}} : [x, y] = z, [x, a] = a$. Then, the derived algebra $[\mathfrak{g}, \mathfrak{g}]$ is commutative. At the point $f = z^*$ in $\mathfrak{g}^*$, the subalgebra $\mathfrak{h} = \langle x, a, z \rangle_{\mathbb{R}}$ is a polarization satisfying the Pukanszky condition and $\mathfrak{h}$ is not contained in the nilpotent radical of $\mathfrak{g}$.

3.1. Indices and notations

Let

$$\mathcal{S} : \{0\} = \mathfrak{g}_0 \subset \mathfrak{g}_1 \subset \cdots \subset \mathfrak{g}_{n-1} \subset \mathfrak{g}_n = \mathfrak{g}, \dim \mathfrak{g}_j = j \ (0 \leq j \leq n)$$

be a good sequence of subalgebras of $\mathfrak{g}$ which passes a nilpotent ideal $\mathfrak{n} = \mathfrak{g}_{j_0}$ containing $[\mathfrak{g}, \mathfrak{g}]$. What is to say, if $\mathfrak{g}_j$ is not an ideal of $\mathfrak{g}$, then $\mathfrak{g}_{j\pm1}$ are ideals of $\mathfrak{g}$ and the adjoint representation of $\mathfrak{g}$ on $\mathfrak{g}_{j+1}/\mathfrak{g}_{j-1}$ is irreducible. If $\mathfrak{g}_j$ and $\mathfrak{g}_{j-1}$ are ideals of $\mathfrak{g}$, we obtain a root $\alpha_j : \mathfrak{g} \to \mathbb{R}$ of $\mathfrak{g}$:

$$\mathrm{ad}(X)(X_j) - \alpha_j(X)X_j \in \mathfrak{g}_{j-1}, \ X \in \mathfrak{g}, \ X_j \in \mathfrak{g}_j \setminus \mathfrak{g}_{j-1}.$$

If $\mathfrak{g}_j$ is not an ideal of $\mathfrak{g}$, then we take a subspace $\mathfrak{v}_j$ of dimension 2 of $\mathfrak{g}_{j+1}$ such that $\mathfrak{g}_{j+1} = \mathfrak{v}_j + \mathfrak{g}_{j-1}$ and there exists a homomorphism α_j of $\mathfrak{g}$ to the algebra of the endomorphisms of $\mathfrak{v}_j$ such that

$$\mathrm{ad}(X)v - \alpha_j(X)v \in \mathfrak{g}_{j-1}, \ X \in \mathfrak{g}, \ v \in \mathfrak{v}_j.$$

We define for a subspace $\mathfrak{v}$ of $\mathfrak{g}$ the index set $I^{\mathfrak{v}} \subset \{1, 2, \ldots, n\}$ by

$$\begin{aligned} I^{\mathfrak{v}} &= \{1 \leq j \leq n; \mathfrak{v} + \mathfrak{g}_j = \mathfrak{v} + \mathfrak{g}_{j-1}\} \\ &= \{1 \leq j \leq n; \mathfrak{v} \cap \mathfrak{g}_j \neq \mathfrak{v} \cap \mathfrak{g}_{j-1}\}. \end{aligned}$$

Choose for $j \in I^{\mathfrak{h}_2} \backslash I^{\mathfrak{h}_1 \cap \mathfrak{h}_2}$ an element $U_j \in \mathfrak{h}_2 \cap \mathfrak{g}_j$ outside of $\mathfrak{g}_{j-1}$ and determine the integer $k = k(j) \leq j$ as the smallest index such that

$$(U_j + (\mathfrak{h}_2 \cap \mathfrak{g}_{j-1}) + \mathfrak{h}_1) \cap \mathfrak{g}_k \neq \emptyset.$$

Then, $k \geq 1$. This gives us, retaking U_j again if necessary, an element $S_j \in \mathfrak{h}_1$ and an element $V_j \in \mathfrak{g}_{k(j)} \setminus \mathfrak{g}_{k(j)-1}$ such that

$$U_j = V_j + S_j.$$

We see that $j \in I^{\mathfrak{h}_2} \backslash I^{\mathfrak{h}_1}$ if and only if $k(j) = j$. In this case, we can take $S_j = 0$ and $V_j = U_j$. Because of the minimality of $k(j)$, we know that $k(j) \notin I^{\mathfrak{h}_1}$ and that the mapping

$$I^{\mathfrak{h}_2} \cap I^{\mathfrak{h}_1} \backslash I^{\mathfrak{h}_1 \cap \mathfrak{h}_2} \ni j \mapsto k(j) \in I^{\mathfrak{h}_2 + \mathfrak{h}_1} \backslash (I^{\mathfrak{h}_1} \cup I^{\mathfrak{h}_2})$$

is injective.

We set

$$\widehat{I}^{\mathfrak{h}_2} = \{j \in I^{\mathfrak{h}_2} \backslash I^{\mathfrak{h}_1 \cap \mathfrak{h}_2}; k(j) \leq j_0\}.$$

For $0 \leq j \leq j_0$, we put $\mathfrak{m}_j = \mathfrak{g}_j + \mathfrak{h}_1$ and

$$I^{\mathfrak{h}_2}_{\mathfrak{m}} = \{1 \leq j \leq j_0; \mathfrak{h}_2 \cap \mathfrak{m}_j \neq \mathfrak{h}_2 \cap \mathfrak{m}_{j-1}\}.$$

Let $j \in I_{\mathfrak{m}}^{\mathfrak{h}_2}$. Let us show that there exists X_j in $\mathfrak{h}_2 \cap \mathfrak{m}_j$ outside of $\mathfrak{h}_2 \cap \mathfrak{m}_{j-1}$ such that we have $[X_j, \mathfrak{g}_j] \subset \mathfrak{g}_{j-1}$. When we can take X_j in $\mathfrak{n}$, our assertion is trivial.

We first prove the following:

Proposition 3.2. *Let $G = \exp \mathfrak{g}$ be an exponential solvable Lie group with Lie algebra $\mathfrak{g}$ such that there exists an abelian ideal $\mathfrak{n}$ between the nilpotent radical of $\mathfrak{g}$ and $[\mathfrak{g}, \mathfrak{g}]$. Let $f \in \mathfrak{g}^*$, $\mathfrak{h}_1, \mathfrak{h}_2$ two polarizations at f satisfying the Pukanszky condition and $H_j = \exp \mathfrak{h}_j$ for $j = 1, 2$. Then, Claim 1.1 holds.*

Proof. Let us proceed by induction on $\dim G$. If there exists a non-trivial ideal on which f vanishes, then this ideal is contained in $\mathfrak{h}_1 \cap \mathfrak{h}_2$ and we can descend to the quotient by this ideal and apply there the induction hypothesis, because the image of $\mathfrak{n}$ in the quotient is found between the nilpotent radical and the derived algebra. Suppose hereafter that there is no such an ideal, what brings that the dimension of the center $\mathfrak{z}$ of $\mathfrak{g}$ is inferior or equal to 1. If there exists a minimal ideal $\mathfrak{a} \neq \{0\}$, then, since f does not vanish on $\mathfrak{a}$, the subalgebra

$$\widetilde{\mathfrak{g}} = \{X \in \mathfrak{g}; f([X, \mathfrak{a}]) = \{0\}\}$$

turns out to be proper in $\mathfrak{g}$. Since the Pukanszky condition requires that $\mathfrak{h}_1, \mathfrak{h}_2$ are contained in $\widetilde{\mathfrak{g}}$, the induction hypothesis suffices for us. These observations apply to the general setting without any assumption and we can assume in what follows $\mathfrak{z} = \mathbb{R}Z$ with $f(Z) = 1$.

In this way, we are led to the following situation: $\mathfrak{g}_1 = \mathfrak{z}$, f does not vanish on $\mathfrak{g}_1$ and $\mathfrak{a} = \mathfrak{g}_2$ or $\mathfrak{a} = \mathfrak{g}_3$ is a minimal non-central ideal of $\mathfrak{g}$. In order to simplify the notation, designate $\widetilde{\mathfrak{g}}$ by $\mathfrak{k}$, which is a proper subalgebra of $\mathfrak{g}$. Remark that $\mathfrak{g} = \mathfrak{k} + [\mathfrak{g}, \mathfrak{g}]$ except the case where $\mathfrak{a} = \mathfrak{g}_2$ and where $[\mathfrak{g}, \mathfrak{a}] = \mathfrak{z}$, namely that $\alpha_2 = 0$. In this last case, $\mathfrak{k}$ is an ideal of $\mathfrak{g}$ and our hypothesis on the existence of $\mathfrak{n}$ regulate this eventuality. All taken into account, we obtain a good sequence of subalgebras of $\mathfrak{k}$ by taking the intersection $\mathcal{S}'$ of $\mathcal{S}$ with $\mathfrak{k}$ and $\mathfrak{k} \cap \mathfrak{n}$ is found between the nilpotent radical of $\mathfrak{k}$ and $[\mathfrak{k}, \mathfrak{k}]$.

This taken in mind, if $\mathfrak{a} \subset \mathfrak{h}_1 \cap \mathfrak{h}_2$, it suffices for us to apply the induction hypothesis to the subgroup $K = \exp \mathfrak{k}$. For $i = 1, 2$, put $\mathfrak{h}_j^0 = \mathfrak{h}_j \cap \mathfrak{k}$ and $\mathfrak{h}_j' = \mathfrak{h}_j^0 + \mathfrak{a}$. When $\mathfrak{h}_2$ contains $\mathfrak{a}$ and $\mathfrak{h}_1$ does not contain $\mathfrak{a}$, it suffices for us to apply the induction hypothesis to the data $(\mathfrak{k}, \mathfrak{h}_1', \mathfrak{h}_2, \mathcal{S}')$.

Suppose that $\mathfrak{h}_1$ contains $\mathfrak{a}$ and so does not $\mathfrak{h}_2$. Since $\mathfrak{g}_{j_0}$ is commutative and $\mathfrak{g}_{j_0}$ contains $\mathfrak{a}$, $\mathfrak{g}_{j_0}$ is contained in $\mathfrak{k}$. It follows that $\mathfrak{m}_{j_0} \cap \mathfrak{h}_2 \subset \mathfrak{h}_2^0$. Therefore, our assertion results from that already established for $\mathfrak{h}_1$ and $\mathfrak{h}_2'$.

Suppose from now on that neither $\mathfrak{h}_1$ nor $\mathfrak{h}_2$ contains $\mathfrak{a}$. If $\mathfrak{h}_1 \cap \mathfrak{h}_2$ is not contained in $\mathfrak{k}$, our assertion follows from that already established for $\mathfrak{h}_1'$ and $\mathfrak{h}_2'$. It remains to examine the situation where $\mathfrak{a} \subset \mathfrak{h}_1 + \mathfrak{h}_2$. Recall that $\mathfrak{g}_1 = \mathfrak{z}$ and $\mathfrak{a} = \mathfrak{g}_2$. So, let $\mathfrak{z} = \mathbb{R}Z$, $f(Z) = 1$, and $\mathfrak{g}_2 = \mathbb{R}Y + \mathbb{R}Z$ with $f(Y) = 0$.

First, suppose that $\mathfrak{a} \subset \mathfrak{h}_1^0 + \mathfrak{h}_2^0$. Write $Y = X_2 - X_1$ with $X_k \in \mathfrak{h}_k^0 (1 \leq k \leq 2)$. Let i_0 be the smallest index in $I_{\mathfrak{m}}^{\mathfrak{h}_2}$ such that $\mathfrak{h}_2 \cap \mathfrak{m}_{i_0} \not\subset \mathfrak{h}_2^0$. If there does not exist such an index i_0, we have nothing to do. It comes that $\mathfrak{h}_2^0 \cap \mathfrak{m}_{i_0} \subset \mathfrak{m}_{i_0 - 1}$ and then that there exists $V \in \mathfrak{g}_{i_0} \setminus \mathfrak{g}_{i_0 - 1}$ which is written as $V = T_2 - T_1$ with $T_k \in \mathfrak{h}_k \setminus \mathfrak{h}_k^0$ verifying

$[T_k, Y] = Z$ $(1 \leq k \leq 2)$ (cf. Lemma 4.4 in the sequel). Modifying the elements X_2, X_1 by elements of $\mathfrak{h}_1 \cap \mathfrak{h}_2$, we can choose X_2, X_1 in such a manner that $B_f(X_k, \mathfrak{g}_{i_0-1}) = \{0\}$. Write $[T_2, X_2] = \beta X_2 + w_2$ with $\beta \in \mathbb{R}$ and $w_2 \in \mathfrak{h}_2^0$ such that $B_f(T_1, w_2) = 0$. As $\mathfrak{n} = \mathfrak{g}_{j_0}$ is commutative, we have

$$\begin{aligned} 0 &= f([V, [T_2, X_2]]) = B_f(V, [T_2, X_2]) = -B_f(T_1, [T_2, X_2]) \\ &= -\beta B_f(T_1, X_2) = -\beta B_f(T_1, Y) = -\beta. \end{aligned}$$

Hence, $[T_2, X_2] = w_2$. Similarly, $[T_1, X_1] = w_1$ with $w_1 \in \mathfrak{h}_1^0$ verifying $B_f(T_2, w_1) = 0$. Then, we have

$$[V, X_2] = [T_2 - T_1, X_2] = [T_2, X_2] - [T_1, X_1 + Y] = w_2 - w_1 - Z$$

and consequently

$$\begin{aligned} 0 &= B_f(T_2, [V, X_2]) = B_f([T_2, V], X_2) + B_f(V, [T_2, X_2]) \\ &= B_f([T_2, V], X_2), \end{aligned}$$

what says that $[T_2, V]$ belongs to $\mathfrak{g}_{i_0-1}$.

Second, suppose that $\mathfrak{a} \not\subset \mathfrak{h}_1^0 + \mathfrak{h}_2^0$. Writing $Y = T_2 - T_1$ with $T_k \in \mathfrak{h}_k \setminus \mathfrak{h}_k^0$ $(1 \leq k \leq 2)$, we know that the Pukanszky condition allows us to choose T_k in such a manner that $[T_k, Y]$ belongs to $\mathfrak{g}_1$ (cf. Lemma 4.4).

We can treat similarly the case where $\dim \mathfrak{a} = 3$. Let i_0 be the smallest index in $I_{\mathfrak{m}}^{\mathfrak{h}_2}$ such that $\mathfrak{h}_2 \cap \mathfrak{m}_{i_0} \not\subset \mathfrak{h}_2^0$. Then, there exists $V \in \mathfrak{g}_{i_0} \setminus \mathfrak{g}_{i_0-1}$ which is written as $V = T_2 - T_1$ with $T_k \in \mathfrak{h}_k \setminus \mathfrak{h}_k^0$ verifying $[T_k, Y] = Z$ $(1 \leq k \leq 2)$ because of Lemma 4.4. Following the same way as above, we find $[T_k, V] \in \mathfrak{g}_{i_0-1}$.

Now, putting $M_j = \exp \mathfrak{m}_j$ for $1 \leq j \leq j_0$, let us show by induction on j that $H_2H_1 \cap M_j$ is a closed subset of M_j. When $j = 1$, it is trivial. Suppose that $H_2H_1 \cap M_{j-1}$ is closed in M_{j-1}. If $j \notin I_{\mathfrak{m}}^{\mathfrak{h}_2}$, it turns out that $H_2H_1 \cap M_j = H_2H_1 \cap M_{j-1}$. Suppose that $j \in I_{\mathfrak{m}}^{\mathfrak{h}_2}$. Take X_j in $\mathfrak{h}_2 \cap \mathfrak{m}_j$ outside of $\mathfrak{m}_{j-1}$ in the way that we have $[X_j, \mathfrak{g}_j] \subset \mathfrak{g}_{j-1}$. Let $X_j = V_j + X_j'$ with $V_j \in \mathfrak{g}_j \setminus \mathfrak{g}_{j-1}$ and $X_j' \in \mathfrak{h}_1$. Let now $\{x_i\}_{i=1}^{\infty}$ be a sequence in $H_2H_1 \cap M_j$, convergent in G. Write

$$x_i = \exp(t_i X_j) \cdot g_i \ (t_i \in \mathbb{R},\ g_i \in H_2H_1 \cap M_{j-1}).$$

Because

$$x_i = \exp(t_i(V_j + X_j')) \cdot g_i = \exp(t_i V_j) \cdot g_i'$$

with $g_i' \in M_{j-1}$, the sequence $\{t_i\}_{i=1}^{\infty}$ is convergent and consequently the sequence $g_i = \exp(-t_i X_j) \cdot x_i$ too. Say, $\lim_{i \to \infty} t_i = t_0 \in \mathbb{R}$ and $\lim_{i \to \infty} g_i = g_0 \in M_{j-1}$. Since $H_2H_1 \cap M_{j-1}$ is closed in M_{j-1}, we confirm that $g_0 \in H_2H_1$. After these observations, $\lim_{i \to \infty} x_i = \exp(t_i X_j) \cdot g_i$ is found in H_2H_1. We see in this way that $H_2H_1 \cap M_j$ is a closed subset of M_j.

Finally, $H_2H_1 \cap M_{j_0}$ is closed in M_{j_0}. By adding to the subalgebra $\mathfrak{m}_{j_0}$ a coexponential basis $\{W_1, \ldots, W_p\}$ in $\mathfrak{h}_2$ to $\mathfrak{h}_2 \cap \mathfrak{m}_{j_0}$, we conclude that

$$H_2H_1 = \exp(\mathbb{R}W_1)\cdots\exp(\mathbb{R}W_p)(H_2H_1 \cap M_{j_0})$$

is a closed subset in G. $\square$

4. Case of completely solvable Lie groups

Suppose here that $\mathfrak{g}$ is completely solvable, and we shall provide a proof of Claim 1.1. Our main result in this section is the following:

Theorem 4.1. *Let $G = \exp\mathfrak{g}$ be a completely solvable Lie group with Lie algebra $\mathfrak{g}$, $f \in \mathfrak{g}^*$ and $\mathfrak{h}_j \in S(f,\mathfrak{g})(j = 1, 2)$ two Pukanszky (real) polarizations of $\mathfrak{g}$ at $f \in \mathfrak{g}^*$. Put $H_j = \exp(\mathfrak{h}_j)(j = 1, 2)$. Then, the product H_2H_1 is closed in G.*

4.1. First preparations

Let $\mathfrak{n}$ be a nilpotent ideal of $\mathfrak{g}$ containing $[\mathfrak{g},\mathfrak{g}]$. Put $\mathfrak{n}_0 = \{0\}$ and let $\mathfrak{n}_1$ be an ideal of $\mathfrak{g}$ contained in $\mathfrak{n}$ and having the dimension 1. We denote by $c(\mathfrak{n}_1,\mathfrak{n}_0)$ the centralizer of $\mathfrak{n}_1$ in $\mathfrak{n}$, what evidently leads the equality $c(\mathfrak{n}_1,\mathfrak{n}_0) = \mathfrak{n}$. Take now an ideal $\mathfrak{n}_2$ of $\mathfrak{g}$ included in $\mathfrak{n}$, containing $\mathfrak{n}_1$ and having the dimension 2. If we put

$$c(\mathfrak{n}_2,\mathfrak{n}_0) = \{X \in \mathfrak{n}; [X,\mathfrak{n}_2] \subset \mathfrak{n}_0\},$$

then,

$$\{0\} = \mathfrak{n}_0 \subsetneq \mathfrak{n}_1 \subsetneq \mathfrak{n}_2 \subset c(\mathfrak{n}_2,\mathfrak{n}_0) \subset \mathfrak{n}, \quad \dim\big(\mathfrak{n}/c(\mathfrak{n}_2,\mathfrak{n}_0)\big) \leq 1.$$

If $\mathfrak{n}_2 \neq c(\mathfrak{n}_2,\mathfrak{n}_0)$, we take an ideal $\mathfrak{n}_3$ of $\mathfrak{g}$ such that

$$\mathfrak{n}_2 \subset \mathfrak{n}_3 \subset c(\mathfrak{n}_2,\mathfrak{n}_0), \quad \dim(\mathfrak{n}_3/\mathfrak{n}_2) = 1.$$

Next we consider the ideal

$$c(\mathfrak{n}_3,\mathfrak{n}_1) = \{X \in c(\mathfrak{n}_2,\mathfrak{n}_0) : [X,\mathfrak{n}_3] \subset \mathfrak{n}_1\}.$$

of $\mathfrak{g}$. Then,

$$\mathfrak{n}_1 \subsetneq \mathfrak{n}_2 \subsetneq \mathfrak{n}_3 \subset c(\mathfrak{n}_3,\mathfrak{n}_1) \subset c(\mathfrak{n}_2,\mathfrak{n}_0), \quad \dim\big(c(\mathfrak{n}_2,\mathfrak{n}_0)/c(\mathfrak{n}_3,\mathfrak{n}_1)\big) \leq 1.$$

If $\mathfrak{n}_3 \neq c(\mathfrak{n}_3,\mathfrak{n}_1)$, we take an ideal $\mathfrak{n}_4$ of $\mathfrak{g}$ such that

$$\mathfrak{n}_3 \subset \mathfrak{n}_4 \subset c(\mathfrak{n}_3,\mathfrak{n}_1), \quad \dim(\mathfrak{n}_4/\mathfrak{n}_3) = 1.$$

Next we consider the ideal

$$c(\mathfrak{n}_4,\mathfrak{n}_2) = \{X \in c(\mathfrak{n}_3,\mathfrak{n}_1); [X,\mathfrak{n}_4] \subset \mathfrak{n}_2\}$$

of $\mathfrak{g}$. Then,

$$\mathfrak{n}_1 \subsetneq \mathfrak{n}_2 \subsetneq \mathfrak{n}_3 \subsetneq \mathfrak{n}_4 \subset c(\mathfrak{n}_4, \mathfrak{n}_2) \subset c(\mathfrak{n}_3, \mathfrak{n}_1) \subset c(\mathfrak{n}_2, \mathfrak{n}_0) \subset \mathfrak{n}$$

and $\dim \big(c(\mathfrak{n}_3, \mathfrak{n}_1)/c(\mathfrak{n}_4, \mathfrak{n}_2)\big) \leq 1$.

We continue the same process until we have an index k_0 such that

$$\mathfrak{n}_{k_0} = c(\mathfrak{n}_{k_0}, \mathfrak{n}_{k_0-2}).$$

By this way, we arrive to a sequence of ideals of $\mathfrak{g}$:

$$\begin{aligned}\{0\} = \mathfrak{n}_0 \subsetneq \mathfrak{n}_1 \subsetneq \mathfrak{n}_2 \subsetneq \cdots \subsetneq \mathfrak{n}_{k_0} &= c(\mathfrak{n}_{k_0}, \mathfrak{n}_{k_0-2}) \\ &\subset c(\mathfrak{n}_{k_0-1}, \mathfrak{n}_{k_0-3}) \subset \cdots \subset c(\mathfrak{n}_2, \mathfrak{n}_0) \subset \mathfrak{n}.\end{aligned}$$

Denote this sequence made by different terms by $\mathcal{S}$:

$$\{0\} = \mathfrak{n}_0 \subset \mathfrak{n}_1 \subset \cdots \subset \mathfrak{n}_{m-1} \subset \mathfrak{n}_m = \mathfrak{n},\ \dim(\mathfrak{n}_j/\mathfrak{n}_{j-1}) = 1\ (1 \leq j \leq m).$$

For $1 \leq j \leq m$, we set $\mathfrak{m}_j = \mathfrak{n}_j + \mathfrak{h}_1$ and

$$I^{\mathfrak{h}_2} = \{1 \leq j \leq m; \mathfrak{h}_2 \cap \mathfrak{m}_j \neq \mathfrak{h}_2 \cap \mathfrak{m}_{j-1}\}.$$

Now, we propose the following claim:

Claim 4.2. *Let $j \in I^{\mathfrak{h}_2}$. There exists X_j in $\mathfrak{h}_2 \cap \mathfrak{m}_j$ outside of $\mathfrak{h}_2 \cap \mathfrak{m}_{j-1}$ such that we have $[X_j, \mathfrak{n}_j] \subset \mathfrak{n}_{j-1}$.*

Then the following is immediate from the proof of Proposition 3.2:

Proposition 4.3. *If Claim 4.2 holds, then so does Claim 1.1, so a proof of Theorem 4.1.*

Proof of Theorem 4.1. We shall draw a proof through different steps, here is the first one:

4.2. Step 1: An induction procedure

We proceed by induction on $\dim \mathfrak{g}$. If $\mathfrak{h}_2 = \mathfrak{h}_1 = \mathfrak{g}(f) = \mathfrak{g}$, $H_2H_1 = H_1 = H_2 = G$. If f vanishes on an ideal $\mathfrak{a} \neq \{0\}$ of $\mathfrak{g}$, $\mathfrak{a} \subset \mathfrak{g}(f) \subset \mathfrak{h}_1 \cap \mathfrak{h}_2$ and we can pass to the quotient $\mathfrak{g}/\mathfrak{a}$ to which applies the induction hypothesis. Hence, we can suppose that the center $\mathfrak{z}$ satisfies $\dim \mathfrak{z} \leq 1$, and that f does not vanish on $\mathfrak{z}$ when $\dim \mathfrak{z} = 1$. Let $\mathfrak{a}$ be a minimal non-central ideal of $\mathfrak{g}$ and $\widetilde{\mathfrak{g}} = \{X \in \mathfrak{g}; B_f(X, \mathfrak{a}) = \{0\}\}$ which is a proper subalgebra of $\mathfrak{g}$. If $\mathfrak{a}$ is a minimal ideal of $\mathfrak{g}$, then the Pukanszky condition implies $\mathfrak{h}_j (j = 1, 2)$ are contained in $\widetilde{\mathfrak{g}}$ (cf. [**4**]) and the result commes immediately from the induction hypothesis applied to $\widetilde{G} = \exp(\widetilde{\mathfrak{g}})$.

Suppose hereafter that $\mathfrak{a}$ is not a minimal ideal of $\mathfrak{g}$. This means $\mathfrak{n}_1 = \dim \mathfrak{z} = 1, \dim \mathfrak{a} = 2$ and $\mathfrak{z} \subset \mathfrak{a}$. Let $\mathfrak{z} = \mathbb{R}Z, \mathfrak{a} = \mathbb{R}Y \oplus \mathfrak{z}$ such that $f(Z) = 1$ and $f(Y) = 0$. Note that the intersection of $\widetilde{\mathfrak{g}}$ with $\mathfrak{n}$ is nothing but $c(\mathfrak{n}_2, \mathfrak{n}_0) = c(\mathfrak{n}_2, \{0\})$.

Now let $\mathfrak{a} \not\subset \mathfrak{h}_1 \cap \mathfrak{h}_2$ and put $\mathfrak{h}_i^0 = \mathfrak{h}_i \cap \widetilde{\mathfrak{g}},\ \mathfrak{h}_i' = \mathfrak{h}_i^0 + \mathfrak{a}$ for $1 \leq i \leq 2$. It is well known that $\mathfrak{h}_1'$ is a polarization at f verifying the Pukanszky condition (cf. [**4**]). Let us introduce

$$\mathfrak{g}_0 = \{X \in \mathfrak{g}; [X, \mathfrak{a}] \subset \mathfrak{n}_1\}.$$

Case 1. $\mathfrak{h}_1 \subset \widetilde{\mathfrak{g}}$ and $\mathfrak{h}_2 \not\subset \widetilde{\mathfrak{g}}$. It matters only that $j = m$, otherwise we can apply the induction hypothesis to $\mathfrak{h}_1, \mathfrak{h}_2'$ contained in $\widetilde{\mathfrak{g}}$. If $\mathfrak{h}_2 \subset \mathfrak{g}_0$, then $[\mathfrak{h}_2, \mathfrak{n}] \subset \mathfrak{n} \cap \widetilde{\mathfrak{g}}$ and hence $[\mathfrak{h}_2, \mathfrak{n}_j] \subset \mathfrak{n}_{j-1}$, from which comes the result. If $\mathfrak{h}_2 \not\subset \mathfrak{g}_0$, the Pukanszky condition implies (cf. [**4**]) $\mathfrak{h}_2^0 \not\subset \mathfrak{g}_0$ and hence there is a nilpotent element in $\mathfrak{h}_2 \setminus \mathfrak{h}_2^0$ which serves as a coexponential basis in $\mathfrak{g}$ to $\widetilde{\mathfrak{g}}$.

Case 2. $\mathfrak{h}_1 \not\subset \widetilde{\mathfrak{g}}$ and $\mathfrak{h}_2 \subset \widetilde{\mathfrak{g}}$. Just as in the case 1, we are led to the situation where $j = m$. If $\mathfrak{h}_1 \subset \mathfrak{g}_0$, $[\mathfrak{h}_1 + \mathfrak{n}, \mathfrak{n}] \subset \mathfrak{n}_{j-1}$ and hence $[\mathfrak{m}_j, \mathfrak{n}_j] \subset \mathfrak{n}_{j-1}$, further $[\mathfrak{h}_2 \cap \mathfrak{m}_j, \mathfrak{n}_j] \subset \mathfrak{n}_{j-1}$. If $\mathfrak{h}_1 \not\subset \mathfrak{g}_0$, the Pukanszky condition demands just as in Case 1 that $\mathfrak{h}_1 \cap \mathfrak{n}_j \neq \mathfrak{h}_1 \cap \mathfrak{n}_{j-1}$. Namely, $\mathfrak{m}_j = \mathfrak{m}_{j-1}$ and hence $j \notin I^{\mathfrak{h}_2}$.

Case 3. $\mathfrak{h}_1 \not\subset \widetilde{\mathfrak{g}}$ and $\mathfrak{h}_2 \not\subset \widetilde{\mathfrak{g}}$. First, let $\mathfrak{a} \not\subset \mathfrak{h}_1 + \mathfrak{h}_2$. Then, there exists $0 \neq X \in \mathfrak{h}_1 \cap \mathfrak{h}_2$ outside of $\widetilde{\mathfrak{g}}$. If $j < m$, modifying by X the concerned elements of $\mathfrak{h}_1, \mathfrak{h}_2$, we immediately see that

$$\mathfrak{h}_2' \cap (\mathfrak{n}_j + \mathfrak{h}_1') \neq \mathfrak{h}_2' \cap (\mathfrak{n}_{j-1} + \mathfrak{h}_1').$$

If $j = m$, we can proceed as in the previous cases. Indeed, if $\mathfrak{h}_2 \not\subset \mathfrak{g}_0$, we can take in $[\mathfrak{h}_2, X]$ a coexponential basis in $\mathfrak{g}$ to $\widetilde{\mathfrak{g}}$. Let $\mathfrak{h}_2 \subset \mathfrak{g}_0$. If $\mathfrak{h}_1 \subset \mathfrak{g}_0$, we can apply the induction hypothesis to $\mathfrak{g}_0$. If $\mathfrak{h}_1 \not\subset \mathfrak{g}_0$, we have that $\mathfrak{h}_1 \cap \mathfrak{n}_m \neq \mathfrak{h}_1 \cap \mathfrak{n}_{m-1}$. Namely, $\mathfrak{m}_m = \mathfrak{m}_{m-1}$ and hence $m \notin I^{\mathfrak{h}_2}$.

Suppose in what follows that $\mathfrak{a} \subset \mathfrak{h}_1 + \mathfrak{h}_2$. First of all let $\mathfrak{a} \not\subset \mathfrak{h}_1^0 + \mathfrak{h}_2^0$. Write $Y = T_2 - T_1$ with $T_i \in \mathfrak{h}_i \backslash \mathfrak{h}_i^0$ for $1 \leq i \leq 2$. We know that $T_i \in \mathfrak{g}_0$ $(1 \leq i \leq 2)$ from the Pukanszky condition (cf. Lemma 4.4 below). By replacing X by T_1, T_2 we can make the same argument as in the precedent situation.

Finally, let us treat the essential case where $\mathfrak{a} \subset \mathfrak{h}_1^0 + \mathfrak{h}_2^0$. Let j_0 be the smallest index such that there exists $V \in \mathfrak{n}_{j_0} \backslash \mathfrak{n}_{j_0-1}$ which is written $V = T_2 - T_1$ with $T_i \in \mathfrak{h}_i \backslash \mathfrak{h}_i^0$ $(1 \leq i \leq 2)$. Then $(\mathfrak{h}_1^0 + \mathfrak{h}_2^0) \cap \mathfrak{n}_{j_0} \subset \mathfrak{n}_{j_0-1}$. If $j_0 = m$, we can repeat the arguments utilized in the case 1 or 2. Thus, let $j_0 < m$, namely $V \in \widetilde{\mathfrak{g}}$. When we write $Y = X_2 - X_1$ with $X_i \in \mathfrak{h}_i^0$ $(1 \leq i \leq 2)$, we have $\mathfrak{h}_1' \cap \mathfrak{h}_2 = \mathbb{R}X_2 + (\mathfrak{h}_1 \cap \mathfrak{h}_2)$ and $\mathfrak{h}_1 \cap \mathfrak{h}_2' = \mathbb{R}X_1 + (\mathfrak{h}_1 \cap \mathfrak{h}_2)$. Since $(\mathfrak{h}_1 + \mathfrak{h}_2) \cap \mathfrak{n}_{j_0-1} \subset \mathfrak{h}_1^0 + \mathfrak{h}_2^0$, we ascertain

$$B_f(X_2, \mathfrak{n}_{j_0-1}) = B_f(X_1, \mathfrak{n}_{j_0-1}) = \{0\}$$

by modifying X_i by the elements of $\mathfrak{h}_1 \cap \mathfrak{h}_2$ if necessary. In fact, let $\mathfrak{n}_k = \mathbb{R}V_k + \mathfrak{n}_{k-1}$ with $k \leq j_0 - 1$. If $V_k \notin \mathfrak{h}_1 + \mathfrak{h}_2$, there exists $x_k \in \mathfrak{h}_1 \cap \mathfrak{h}_2$ such that $B_f(x_k, V_k) \neq 0$. Modifying X_j by x_k, we can assume $B_f(X_j, V_k) = 0$. If $V_k \in \mathfrak{h}_1 + \mathfrak{h}_2$, then $V_k = x_2 - x_1$ with $x_j \in \mathfrak{h}_j^0$. This means $B_f(X_1, x_2) = B_f(X_2, x_1) = 0$ and hence $B_f(X_2, V_k) = B_f(X_1, V_k) = 0$. Moreover, we can suppose that $f(X_1) = f(X_2) = 0$ by modifying X_i by Z if necessary. Let us normalize T_i or Y so that $B_f(T_i, Y) = 1$. This leads to

$$B_f(T_1, X_2) = 1, \ B_f(T_2, X_1) = B_f(V, X_1) = B_f(V, X_2) = -1.$$

Lemma 4.4. *Let us assume more generally that $\mathfrak{g}$ is exponential here. We can take T_j in $\mathfrak{g}_0$, for $1 \leq j \leq 2$.*

PROOF. Suppose for a while that $\mathfrak{g}$ is completely solvable and that $T_j \notin \mathfrak{g}_0$. Then, from the Pukanszky condition there exist (cf. [**4**]) $T'_j \in \mathfrak{h}_j^0$ outside $\mathfrak{g}_0$ and hence two elements $0 \neq v_j = [T'_j, T_j] \in (\mathfrak{h}_j \cap \mathfrak{n}) \backslash \mathfrak{h}_j^0$ $(j = 1, 2)$ and let k_0 be the smallest index satisfying $a = v_2 - v_1 \in \mathfrak{n}_{k_0}$ by the use of such nilpotent elements v_1, v_2 in $\mathfrak{h}_j \setminus \mathfrak{h}_j^0$. This brings that $j_0 \leq k_0 \leq m-1$. If $k_0 = j_0$, we can choose $T_j = v_j \in \mathfrak{n} \subset \mathfrak{g}_0$. Suppose hence $j_0 < k_0$. This being done, from the induction hypothesis there exists $S_j \in \mathfrak{h}_j^0$ $(j = 1, 2)$ such that

$$b = S_2 - S_1 \in \mathfrak{n}_{k_0} \backslash \mathfrak{n}_{k_0 - 1}, \quad [S_j, \mathfrak{n}_{k_0}] \subset \mathfrak{n}_{k_0 - 1}.$$

This says that

$$[S_2, a] = [S_2, v_2] - [S_2, v_1] \in \mathfrak{n}_{k_0 - 1},$$

namely,

$$\begin{aligned} [S_2, a] &= [S_2, v_2] - [S_1 + b, v_1] \\ &= [S_2, v_2] - [S_1, v_1] - [b, v_1] \in \mathfrak{n}_{k_0 - 1}. \end{aligned}$$

Consequently,

$$[S_2, v_2] - [S_1, v_1] \in \mathfrak{n}_{k_0 - 1},$$

which contradicts the choice of k_0 unless $S_j \in \mathfrak{g}_0$ $(j = 1, 2)$.

By considering an appropriate multiple of S_j, we can suppose that

$$b + a = (S_2 + v_2) - (S_1 + v_1) \in \mathfrak{n}_{k_0 - 1}.$$

Then,

$$\begin{aligned} \mathfrak{n}_{k_0 - 1} \ni [T_2, b + a] &= [T_2, S_2 + v_2] - [T_2, S_1 + v_1] \\ &= [T_2, S_2 + v_2] - [T_1 + V, S_1 + v_1] \\ &= [T_2, S_2 + v_2] - [T_1, S_1 + v_1] - [V, S_1 + v_1]. \end{aligned}$$

Since $[V, S_1 + v_1] \in \mathfrak{n}_{j_0}$ and $j_0 \leq k_0 - 1$, we have

$$[T_2, S_2 + v_2] - [T_1, S_1 + v_1] \in \mathfrak{n}_{k_0 - 1}.$$

All taken into account, if $T_j \notin \mathfrak{g}_0$, this contradicts the choice of k_0 because $[T_j, S_j + v_j] \notin \mathfrak{h}_j^0$ $(j = 1, 2)$. In sum, $T_j \in \mathfrak{g}_0$.

Now suppose that $\mathfrak{g}$ is exponential. Let $k_0 = j_0$. If $[T_j, \mathfrak{n}_{j_0}] \not\subset \mathfrak{n}_{j_0 - 1}$, the equation

$$[T_2, a] = [T_2, v_2] - [T_1, v_1] - [V, v_1]$$

shows that we can take T_j in $\mathfrak{h}_j \cap \mathfrak{n}$. Thus, we can assume $j_0 < k_0$ and we can repeat the above arguments done for the completely solvable case. $\square$

Remark 4.5. In the proof of Lemma 4.4, if $j_0 = k_0$, we establish our assertion by taking $T_j = v_j (j = 1, 2)$. Besides, if $j_0 < k_0$, it is obligatory that $T_j \in \mathfrak{g}_0 (j = 1, 2)$. So, when we write an element A of $\mathfrak{n}_i, i \leq j_0$, as $A = a_2 - a_1$ with $a_i \in \mathfrak{h}_i (i = 1, 2)$, we can always suppose that $a_i \in \mathfrak{g}_0 (i = 1, 2)$. For example, this is the case for X_i introduced above. (See the proof of the next lemma.)

Taking into account $B_f(T_1, X_2) = 1$ and $B_f(T_2, X_1) = -1$, write

$$[T_1, X_1] = \beta X_1 + w_1,\ [T_2, X_2] = \gamma X_2 + w_2$$

with $w_j \in \mathfrak{h}_j^0 (j = 1, 2)$ verifying

$$B_f(T_1, w_2) = B_f(T_2, w_1) = 0.$$

Then, it is immediate that

$$\begin{aligned}
[V, X_1] = [V, X_2] &= [T_2 - T_1, X_2] = [T_2, X_2] - [T_1, X_2] \\
&= [T_2, X_2] - [T_1, X_1 + Y] = \gamma X_2 + w_2 - (\beta X_1 + w_1) - Z \\
&= (\gamma - \beta) X_2 + w_2 - w_1 + \beta Y - Z \\
&= (\gamma - \beta) X_1 + w_2 - w_1 + \gamma Y - Z.
\end{aligned} \tag{4.1}$$

Let us calculate further

$$B_f(T_2, [V, X_2]) = f([T_2, [V, X_2]]) = (\gamma - \beta) f([T_2, X_2]) + \beta f(Z) = \beta.$$

On the other hand,

$$\begin{aligned}
&B_f(T_2, [V, X_2]) = B_f([T_2, V], X_2) + B_f(V, [T_2, X_2]) \\
=&B_f([T_2, V], X_2) + B_f(V, \gamma X_2 + w_2) = B_f([T_2, V], X_2) - \gamma.
\end{aligned}$$

Consequently,

$$[T_1, V] = [T_2, V] = -(\beta + \gamma) V + W \tag{4.2}$$

with a certain element $W \in \mathfrak{n}_{j_0 - 1}$. So, the task consists in showing that $\beta + \gamma = 0$, which will close the proof thanks to Proposition 4.3. Note that this aim is immediately attained if both $\mathfrak{h}_1$ and $\mathfrak{h}_2$ are abelian.

4.3. Step 2: Toward $\beta + \gamma = 0$

Let us first compute:

$$\begin{aligned}
&[T_2, [V, X_2]] = (\gamma - \beta)[T_2, X_2] + [T_2, w_2] - [T_2, w_1] + \beta Z \\
=& -(\beta + \gamma)[V, X_2] + [W, X_2] + \gamma [V, X_2] + [V, w_2] \\
=& -\beta \{(\gamma - \beta) X_2 + w_2 - w_1 + \beta Y - Z\} + [W, X_2] + [V, w_2].
\end{aligned}$$

Hence,

$$\begin{aligned}&(\gamma-\beta)[T_2,X_2]+[T_2,w_2]-[T_2,w_1] \\ &=-\beta(\gamma-\beta)X_2-\beta w_2+\beta w_1-\beta^2 Y+[W,X_2]+[V,w_2].\end{aligned} \tag{4.3}$$

Thus, in particular $f([W,X_2])=0$.

The three equations (4.1)-(4.3) are our poor tools.

As we may replace V by any element of $\mathfrak{g}_{j_0}$ which is found in $\mathfrak{h}_1+\mathfrak{h}_2$ but outside of $\mathfrak{h}_1^0+\mathfrak{h}_2^0$, think for a while about the possibility to modify V so that we would have $\gamma=0$ for instance. Let $U=R_2-R_1\in\mathfrak{n}_{j_0-1}$ with $R_j\in\mathfrak{h}_j^0$ $(j=1,2)$. Let us compute γ', the new γ, for $V+\alpha U$. We find that

$$\gamma'=B_f(T_1+\alpha R_1,[T_2+\alpha R_2,X_2]).$$

Namely,

$$\begin{aligned}\gamma' =&B_f(R_1,[R_2,X_2])\alpha^2+\big(B_f(R_1,[T_2,X_2]) \\ &+B_f(T_1,[R_2,X_2])\big)\alpha+\gamma\end{aligned} \tag{4.4}$$

and similarly

$$\begin{aligned}-\beta' =&B_f(R_2,[R_1,X_1])\alpha^2+\big(B_f(R_2,[T_1,X_1]) \\ &+B_f(T_2,[R_1,X_1])\big)\alpha-\beta.\end{aligned} \tag{4.5}$$

Thus,

$$\begin{aligned}&\beta'+\gamma'=\beta+\gamma \\ &+\big(B_f(R_1,w_2)+B_f(T_1,[R_2,X_2])-B_f(R_2,w_1)-B_f(T_2,[R_1,X_1])\big)\alpha\end{aligned}$$

and we may choose $\alpha\in\mathbb{R}$ in such a manner that

$$\beta'+\gamma'=0$$

provided that

$$B_f(R_1,w_2)+B_f(T_1,[R_2,X_2])-B_f(R_2,w_1)-B_f(T_2,[R_1,X_1])\neq 0.$$

We can hence suppose that

$$B_f(R_1,w_2)+B_f(T_1,[R_2,X_2])-B_f(R_2,w_1)-B_f(T_2,[R_1,X_1])=0.$$

When we make the changes

$$T_j\mapsto kT_j,\ X_j\mapsto k^{-1}X_j\ (k\in\mathbb{R}\setminus\{0\},\ j=1,2)$$

in the expressions of (4.4) and (4.5), we check that

$$B_f(R_1,[R_2,X_2])=B_f(R_2,[R_1,X_1])=0,$$

otherwise we can choose the sign of $B_f(R_1, [R_2, X_2])$ by multiplying it by k^{-1} so that we would have a real α such that $\gamma' = 0$ for example. Consequently,

$$\begin{aligned} B_f(R_1, [T_2, X_2]) + B_f(T_1, [R_2, X_2]) \\ = B_f(R_2, [T_1, X_1]) + B_f(T_2, [R_1, X_1]) = 0, \end{aligned} \tag{4.6}$$

otherwise we may arrive to the situation $\beta + \gamma = 0$ by a suitable choice of α. The last equation (4.6) is written again

$$B_f(R_1, w_2) + B_f(T_1, [R_2, X_2]) = B_f(R_2, w_1) + B_f(T_2, [R_1, X_1]) = 0.$$

Next, as

$$\begin{aligned} B_f(T_2, [R_1, X_1]) &= B_f([T_2, R_1], X_1) + B_f(R_1, [T_2, X_1]) \\ &= B_f(R_1, [V, X_1]) = B_f(R_1, w_2), \end{aligned}$$

we have

$$B_f(U, w_1 - w_2) = B_f(R_2 - R_1, w_1 - w_2) = B_f(R_1, w_2) + B_f(R_2, w_1) = 0.$$

Lemma 4.6. *$R_j \in \mathfrak{g}_0$ for $1 \leq j \leq 2$.*

Proof. If $R_j \notin \mathfrak{g}_0$, we find

$$[R_2, T_2] - [R_1, T_1] = [R_2, V] - [T_1, U]$$

in $\mathfrak{n}_{j_0}$. We can take this relation instead of $T_2 - T_1 = V$, if $[R_2, V] \notin \mathfrak{n}_{j_0-1}$. Otherwise, $[R_2, V] \in \mathfrak{n}_{j_0-1}$ and

$$[R_j, T_j] \notin \mathfrak{h}_j^0 \ (j = 1, 2)$$

if $R_j \notin \mathfrak{g}_0$, the above relation contradicts the choice of j_0. □

Now let us start the following computations. Put

$$\begin{aligned} B_f(T_1, [T_1, w_2]) = p, \ B_f(T_2, [T_2, w_1]) = q, \\ B_f(T_1, [T_2, w_1]) = r, \ B_f(T_2, [T_1, w_2]) = s. \end{aligned} \tag{4.7}$$

and let us begin our calculations. From (4.3),

$$[T_1, w_2] - [T_2, w_1] = (\beta^2 - \gamma^2)X_2 + \beta w_1 - \gamma w_2 - \beta^2 Y + [W, X_2]. \tag{4.8}$$

Since

$$\begin{aligned} B_f(T_2, [W, X_2]) = B_f(W, [T_2, X_2]) = B_f(W, w_2) \\ B_f(T_1, [W, X_1]) = B_f(W, [T_1, X_1]) = B_f(W, w_1), \end{aligned}$$

it follows from (4.8) that

$$p - r = B_f(W, w_1) - \gamma^2,\ q - s = \beta^2 - B_f(W, w_2). \tag{4.9}$$

Furthermore,

$$r = B_f(T_2, [T_1, w_1]) + B_f(W, w_1),$$

$$s = B_f(T_1, [T_2, w_2]) - B_f(W, w_2). \tag{4.10}$$

We emphasize

$$B_f(W, X_1) = B_f(W, X_2) = 0. \tag{4.11}$$

Suppose first $r \neq 0$, $s \neq 0$ and put

$$\mathfrak{m}_i = \{X \in \mathfrak{h}_i; B_f(T_j, X) = B_f(T_i, [T_j, X]) = 0\}$$

for $1 \leq i \neq j \leq 2$. Then, as vector spaces,

$$\mathfrak{h}_i = \mathbb{R}X_i \oplus \mathbb{R}w_i \oplus \mathfrak{m}_i\ (1 \leq i \leq 2).$$

Let

$$[T_1, w_1] = (B_f(W, w_1) - r)X_1 + bw_1 + u,\ u \in \mathfrak{m}_1, \tag{4.12}$$

$$[T_2, w_2] = (s + B_f(W, w_2))X_2 + dw_2 + v,\ v \in \mathfrak{m}_2.$$

Further, let us consider the subspaces

$$\begin{aligned}\mathfrak{w}_i = \{X \in \mathfrak{g}; B_f(T_1, X) &= B_f(T_2, X) \\ &= B_f(T_i, [T_j, X]) = B_f(T_i, [T_i, X]) = 0\}\end{aligned}$$

for $1 \leq i \neq j \leq 2$. Then, supposing first $pq \neq 0$, we have

$$\mathfrak{g} = \mathbb{R}X_2 \oplus \mathbb{R}X_1 \oplus \mathbb{R}w_1 \oplus \mathbb{R}w_2 \oplus \mathfrak{w}_i.$$

We immediately see that $\{X_1, X_2, w_1, w_2\}$ are linearly independent if $pq \neq 0$, what will turn out little important in the future situation where $sr = 0$ or $pq = 0$. In fact, if $r = 0$ or $s = 0$, we can take as $\mathfrak{m}_i$ or $\mathfrak{w}_i (i = 1, 2)$ a subspace of $\mathfrak{h}_i$ or $\mathfrak{g}$ complementary to $\mathbb{R}X_i \oplus \mathbb{R}w_i$ or $\mathbb{R}X_1 \oplus \mathbb{R}X_2 \oplus \mathbb{R}w_1 \oplus \mathbb{R}w_2$. Let

$$\begin{aligned}[][T_1, w_2] &= pX_2 + aw_2 - sX_1 + ew_1 + m,\ m \in \mathfrak{w}_1, \\ [T_2, w_1] &= rX_2 + cw_2 - qX_1 + hw_1 + n,\ n \in \mathfrak{w}_2.\end{aligned} \tag{4.13}$$

From these, the relations (4.9), (4.13) give us

$$\begin{aligned}
&[W, X_1] = [W, X_2] \\
=&[T_1, w_2] - [T_2, w_1] + (\gamma^2 - \beta^2)X_2 + \gamma w_2 - \beta w_1 + \beta^2 Y \\
=&(p - r + \gamma^2)X_2 + (a - c + \gamma)w_2 + (q - s - \beta^2)X_1 \\
&+ (e - h - \beta)w_1 + m - n \\
=&(a - c + \gamma)w_2 + (e - h - \beta)w_1 + B_f(W, w_2)Y \\
&+ (B_f(W, w_1) - B_f(W, w_2))X_2 + m - n.
\end{aligned} \tag{4.14}$$

We further calculate:

$$[T_2, [T_1, w_1]] = (B_f(W, w_1) - r)[T_2, X_1] + b[T_2, w_1] + [T_2, u]. \tag{4.15}$$

Taking the value $B_f(T_1, \cdot)$ of both members, we have

$$B_f(T_1, [T_2, [T_1, w_1]]) = (B_f(W, w_1) - r)\gamma + br(+B_f(T_1, [T_2, u]) \text{ if } r = 0).$$

On the other hand,

$$\begin{aligned}
B_f(T_1, [T_2, [T_1, w_1]]) =&B_f(T_1, [(\beta + \gamma)V - W, w_1]) \\
&+ B_f(T_1, [T_1, rX_2 + cw_2 - qX_1 + hw_1 + n]) \\
=&(\beta + \gamma)r - B_f(T_1, [W, w_1]) + cp + B_f(T_1, [T_1, n]).
\end{aligned}$$

Hence

$$\begin{aligned}
&B_f(T_1, [W, w_1]) = (r - B_f(W, w_1))\gamma + (\beta + \gamma - b)r \\
&+cp + B_f(T_1, [T_1, n])(-B_f(T_1, [T_2, u]) \text{ if } r = 0).
\end{aligned} \tag{4.16}$$

Taking the value $B_f(T_2, \cdot)$ of both members of equation (4.15), we have

$$B_f(T_2, [T_2, [T_1, w_1]]) = bq + B_f(T_2, [T_2, u]).$$

On the other hand,

$$\begin{aligned}
&B_f(T_2, [T_2, [T_1, w_1]]) = B_f(T_2, [(\beta + \gamma)V - W, w_1]) \\
&+B_f(T_2, [T_1, rX_2 + cw_2 - qX_1 + hw_1 + n]) \\
&=(\beta + \gamma)(q + B_f(W, w_1) - r) - B_f(T_2, [W, w_1]) \\
&+\beta(q - r) + cs + h(r - B_f(W, w_1))(+B_f(T_2, [T_1, n]) \text{ if } s = 0).
\end{aligned}$$

Hence,

$$\begin{aligned}
&B_f(T_2, [W, w_1]) = cs + (2\beta + \gamma - b)q + (h - 2\beta - \gamma)r \\
&+(\beta + \gamma - h)B_f(W, w_1) - B_f(T_2, [T_2, u]) \\
&(+B_f(T_2, [T_1, n]) \text{ if } s = 0).
\end{aligned} \tag{4.17}$$

Now,

$$[T_1,[T_2,w_2]]=(s+B_f(W,w_2))[T_1,X_2]+d[T_1,w_2]+[T_1,v]. \tag{4.18}$$

Taking the value $B_f(T_1,\cdot)$ of both members of this relation, we see

$$B_f(T_1,[T_1,[T_2,w_2]])=dp+B_f(T_1,[T_1,v]).$$

On the other hand,

$$\begin{aligned}&B_f(T_1,[T_1,[T_2,w_2]])=B_f(T_1,[-(\beta+\gamma)V+W,w_2])\\&+B_f(T_1,[T_2,pX_2+aw_2-sX_1+ew_1+m])\\&=-(\beta+\gamma)(s+B_f(W,w_2)-p)+B_f(T_1,[W,w_2])\\&+\gamma(p-s)+a(s+B_f(W,w_2))+er(+B_f(T_1,[T_2,m])\text{ if }r=0).\end{aligned}$$

Thus,

$$\begin{aligned}&B_f(T_1,[W,w_2])=(d-\beta-2\gamma)p+(\beta+2\gamma-a)s-er\\&+(\beta+\gamma-a)B_f(W,w_2)+B_f(T_1,[T_1,v])\\&(-B_f(T_1,[T_2,m])\text{ if }r=0).\end{aligned} \tag{4.19}$$

Next, we take the value $B_f(T_2,\cdot)$ of both members of the equation (4.18).
On one hand,

$$B_f(T_2,[T_1,[T_2,w_2]])=-\beta(s+B_f(W,w_2))+ds(+B_f(T_2,[T_1,v])\text{ if }s=0).$$

On the other hand,

$$\begin{aligned}B_f(T_2,[T_1,[T_2,w_2]])=&B_f(T_2,[-(\beta+\gamma)V+W,w_2])\\&+B_f(T_2,[T_2,pX_2+aw_2-sX_1+ew_1+m])\\=&(\beta+\gamma)s+B_f(T_2,[W,w_2])+eq+B_f(T_2,[T_2,m]).\end{aligned}$$

Therefore,

$$\begin{aligned}B_f(T_2,[W,w_2])=&(d-2\beta-\gamma)s-eq-\beta B_f(W,w_2)-B_f(T_2,[T_2,m])\\&(+B_f(T_2,[T_1,v])\text{ if }s=0).\end{aligned} \tag{4.20}$$

Now,

$$\begin{aligned}[V,[V,X_2]]=&(\gamma-\beta)\{(\gamma-\beta)X_2+w_2-w_1+\beta Y-Z\}\\&+[V,w_2-w_1].\end{aligned} \tag{4.21}$$

Hence, on one hand

$$\begin{aligned}&B_f(T_2,[T_2,[V,[V,X_2]]])\\=&B_f(T_2,[T_2,(\gamma-\beta)\{(\gamma-\beta)X_2+w_2-w_1\}+[V,w_2-w_1]])\\=&(\beta-\gamma)B_f(T_2,[T_2,w_1])+B_f(T_2,[T_2,[V,w_2-w_1]])\end{aligned}$$

$$\begin{aligned}
&=(\beta-\gamma)q+B_f(T_2,[T_2,sX_1-ew_1-m+qX_1-hw_1-n\\
&+(B_f(W,w_1)-r)X_1+bw_1+u]) \qquad (4.22)\\
&=(\beta-\gamma)q+(b-e-h)q-B_f(T_2,[T_2,m+n])+B_f(T_2,[T_2,u])\\
&=(\beta-\gamma+b-e-h)q-B_f(T_2,[T_2,m+n])+B_f(T_2,[T_2,u]).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&B_f(T_2,[T_2,[V,[V,X_2]]])\\
&=B_f(T_2,[-(\beta+\gamma)V+W,(\gamma-\beta)X_2+w_2-w_1])\\
&+B_f(T_2,[V,[-(\beta+\gamma)V+W,X_2]])\\
&+B_f(T_2,[V,[V,\gamma X_2+w_2]])\\
&=(\beta^2-\gamma^2)\beta-(\beta+\gamma)B_f(T_2,[V,w_2-w_1])\\
&+(\gamma-\beta)B_f(W,w_2)+B_f(T_2,[W,w_2-w_1])\\
&-(\beta+\gamma)B_f(T_2,[V,(\gamma-\beta)X_2+w_2-w_1])\\
&+B_f(T_2,[V,(a-c+\gamma)w_2+(e-h-\beta)w_1\\
&+(B_f(W.w_1)-B_f(W,w_2))X_2+m-n])\\
&+\gamma B_f(T_2,[V,(\gamma-\beta)X_2+w_2-w_1])\\
&+B_f(T_2,[V,(s+B_f(W,w_2))X_2+dw_2+v\\
&-pX_2-aw_2+sX_1-ew_1-m])\\
&=\beta(\beta^2-\gamma^2)+(\beta+\gamma)(s+q+B_f(W,w_1)-r)\\
&+(\gamma-\beta)B_f(W,w_2)+B_f(T_2,[W,w_2-w_1])-(\gamma^2-\beta^2)\beta\\
&-(\beta+\gamma)B_f(T_2,[V,w_2-w_1])-(a-c+\gamma)s\\
&+(e-h-\beta)(q+B_f(W,w_1)-r)\\
&+\beta(B_f(W,w_1)-B_f(W,w_2))\\
&+B_f(T_2,[V,m-n])+\beta\gamma(\gamma-\beta)+\gamma B_f(T_2,[V,w_2-w_1])\\
&+\beta(s+B_f(W,w_2))-ds+B_f(T_2,[V,v])+\beta(s-p)+as\\
&-e(q+B_f(W,w_1)-r)-B_f(T_2,[V,m])\\
&=2\beta(\beta^2-\gamma^2)+(2\beta+\gamma)(s+q-r+B_f(W,w_1))\\
&+(\gamma-\beta)B_f(W,w_2)+B_f(T_2,[W,w_2-w_1])-(a-c+\gamma)s\\
&+(e-h-\beta)(q+B_f(W,w_1)-r)\\
&+\beta(B_f(W,w_1)-B_f(W,w_2)) \qquad (4.23)\\
&+B_f(T_2,[V,m-n])+\beta\gamma(\gamma-\beta)+\beta(s+B_f(W,w_2))\\
&+(a-d)s+B_f(T_2,[V,v])+\beta(s-p)\\
&+e(r-B_f(W,w_1)-q)-B_f(T_2,[V,m]).
\end{aligned}$$

4.4. Step 3: $(2\beta + 3\gamma)q = 0$.

Putting in equality the results of the two last calculations, we have

$$\begin{aligned}
&B_f(T_2,[W,w_2-w_1])\\
=&2\beta(\gamma^2-\beta^2)+(2\beta+\gamma)(r-s-q-B_f(W,w_1))\\
&+(\beta-\gamma)B_f(W,w_2)+(a-c+\gamma)s\\
&+(\beta+h-e)(q+B_f(W,w_1)-r)-\beta B_f(W,w_1)\\
&-B_f(T_2,[V,m-n])+\beta\gamma(\beta-\gamma)\\
&+(d-a-\beta)s-B_f(T_2,[V,v])+\beta(p-s)\\
&+e(q+B_f(W,w_1)-r)+B_f(T_2,[V,m])\\
&+(\beta+b-\gamma-e-h)q-B_f(T_2,[T_2,m+n])+B_f(T_2,[T_2,u]).
\end{aligned} \tag{4.24}$$

On the other hand, equations (4.17) and (4.20) become

$$\begin{aligned}
&B_f(T_2,[W,w_2-w_1])\\
=&(d-2\beta-\gamma)s-\beta B_f(W,w_2)-eq+B_f(T_2,[T_1,v])\\
&-B_f(T_2,[T_2,m])-cs-(2\beta+\gamma-b)q-(h-2\beta-\gamma)r\\
&-(\beta+\gamma-h)B_f(W,w_1)+B_f(T_2,[T_2,u])-B_f(T_2,[T_1,n]).
\end{aligned}$$

Now, we have from (4.21):

$$\begin{aligned}
&B_f(T_1,[T_2,[V,[V,X_2]]])=B_f(-(\beta+\gamma)V+W,\\
&(\gamma-\beta)((\gamma-\beta)X_2+w_2-w_1)+[V,w_2-w_1])\\
&+B_f(T_2,[-(\beta+\gamma)V+W,(\gamma-\beta)X_2+w_2-w_1])\\
&+B_f(T_2,[V,[-(\beta+\gamma)V+W,X_2]])+B_f(T_2,[V,[V,\beta X_1+w_1]])\\
&=(\beta+\gamma)(\beta-\gamma)^2-(\beta+\gamma)B_f(V,[V,w_2-w_1])\\
&+(\gamma-\beta)B_f(W,w_2-w_1)+B_f(W,[V,w_2-w_1])-\beta(\gamma^2-\beta^2)\\
&-(\beta+\gamma)B_f(T_2,[V,w_2-w_1])+(\gamma-\beta)B_f(T_2,[W,X_2])\\
&+B_f(T_2,[W,w_2-w_1])-(\beta+\gamma)B_f(T_2,[V,(\gamma-\beta)X_2+w_2-w_1])\\
&+B_f(T_2,[V,[W,X_2]])+\beta B_f(T_2,[V,(\gamma-\beta)X_2+w_2-w_1])\\
&+B_f(T_2,[V,[V,w_1]])\\
&=(\beta+\gamma)(\gamma-\beta)^2-(\beta+\gamma)B_f(V,[V,w_2-w_1])\\
&+(\gamma-\beta)B_f(W,w_2-w_1)+B_f(W,[V,w_2-w_1])+\beta(\beta^2-\gamma^2)\\
&-(\beta+2\gamma)B_f(T_2,[V,w_2-w_1])+(\gamma-\beta)B_f(W,w_2)\\
&+B_f(T_2,[W,w_2-w_1])+\beta(\beta^2-\gamma^2)+B_f(T_2,[V,(a-c+\gamma)w_2\\
&+(e-h-\beta)w_1+(B_f(W,w_1)-B_f(W,w_2))X_2+m-n])\\
&+\beta^2(\gamma-\beta)+B_f(T_2,[V,rX_2+cw_2-qX_1+hw_1+n\\
&-(B_f(W,w_1)-r)X_1-bw_1-u])
\end{aligned}$$

$$
\begin{aligned}
&=(\beta+\gamma)(\gamma-\beta)^2+(\beta+\gamma)B_f(T_1,[V,w_2-w_1])\\
&-(2\beta+3\gamma)B_f(T_2,[V,w_2-w_1])+(\gamma-\beta)B_f(W,w_2)\\
&+(\beta-\gamma)B_f(W,w_1)+B_f(W,[T_2,w_2-w_1])-B_f(W,[T_1,w_2-w_1])\\
&+\beta(\beta^2-\gamma^2)+(\gamma-\beta)B_f(W,w_2)+B_f(T_2,[W,w_2-w_1])\\
&+\beta(\beta^2-\gamma^2)-(a-c+\gamma)s+(e-h-\beta)(q+B_f(W,w_1)-r)\\
&+\beta(B_f(W,w_1)-B_f(W,w_2))+B_f(T_2,[V,m-n])+\beta^2(\gamma-\beta)\\
&+\beta(r-q+r-B_f(W,w_1))-cs+(h-b)(q+B_f(W,w_1)-r)\\
&+B_f(T_2,[V,n-u])\\
&=(\beta+\gamma)(\gamma-\beta)^2+(\beta+\gamma)(s+B_f(W,w_2)-p-r)\\
&-(2\beta+3\gamma)(q-s+r-B_f(W,w_1))+(\gamma-\beta)B_f(W,w_2)\\
&+(\beta-\gamma)B_f(W,w_1)+B_f(W,dw_2+v-cw_2-hw_1-n)\\
&-B_f(W,aw_2+ew_1+m-bw_1-u)+\beta(\beta^2-\gamma^2)\\
&+(\gamma-\beta)B_f(W,w_2)+B_f(T_2,[W,w_1-w_2])+\beta(\beta^2-\gamma^2)\\
&-(a-c+\gamma)s+(e-h-\beta)(q+B_f(W,w_1)-r)\\
&+\beta(B_f(W,w_1)-B_f(W,w_2))+B_f(T_2,[V,m-n])+\beta^2(\gamma-\beta)\\
&+\beta(2r-q-B_f(W,w_1))-cs+(h-b)(q+B_f(W,w_1)-r)\\
&+B_f(T_2,[V,n-u]).
\end{aligned}
$$

From (4.21), we also get

$$
\begin{aligned}
&\quad B_f(T_1,[T_2,[V,[V,X_2]]])\\
&=B_f(T_1,[T_2,(\gamma-\beta)\{(\gamma-\beta)X_2+w_2-w_1\}+[V,w_2-w_1]])\\
&=(\gamma-\beta)^2\gamma+(\gamma-\beta)B_f(T_1,[T_2,w_2-w_1])\\
&+B_f(T_1,[T_2,[V,w_2-w_1]])\\
&=(\gamma-\beta)^2\gamma+(\gamma-\beta)(s+B_f(W,w_2)-r)\\
&+B_f(T_1,[T_2,(s+B_f(W,w_2))X_2+dw_2+v-pX_2\\
&-aw_2+sX_1-ew_1-m-rX_2-cw_2+qX_1-hw_1-n\\
&+(B_f(W,w_1)-r)X_1+bw_1+u])\\
&=(\gamma-\beta)^2\gamma+(\gamma-\beta)(s+B_f(W,w_2)-r)\\
&+(s+B_f(W,w_2))(\gamma+d-a-c)\\
&+(B_f(W,w_1)-r+s-p-r+q)\gamma\\
&+(b-e-h)r+B_f(T_1,[T_2,u-m-n+v]).
\end{aligned}
\tag{4.25}
$$

Comparing these two computations, we find

$$
\begin{aligned}
&B_f(T_2,[W,w_2-w_1])=\beta\gamma^2-\beta(\gamma-\beta)^2-\beta^2\gamma\\
&-\beta(\beta^2-\gamma^2)+(\beta+\gamma)(p+r-s-B_f(W,w_2))
\end{aligned}
$$

$$\begin{aligned}
&+(2\beta+3\gamma)(q-s+r-B_f(W,w_1))+(\gamma-\beta)B_f(W,w_1)\\
&-(\gamma-2\beta)B_f(W,w_2)-B_f(W,dw_2+v-n)\\
&+B_f(W,ew_1+m-u)+(h+\beta-e)(q+B_f(W,w_1)-r)\\
&+\beta(q-2r)+(\gamma-c)s+(b-h)q-B_f(T_2,[V,m])\\
&+B_f(T_2,[V,u])+(\gamma-\beta)(s-r)\\
&+(s+B_f(W,w_2))(\gamma+d)+(B_f(W,w_1)-2r-p+s+q)\gamma\\
&-er+B_f(T_1,[T_2,u-m-n+v]) \qquad (4.26)\\
&=\beta(\gamma^2+\beta\gamma-2\beta^2)-(4\beta+c-d)s+\beta p\\
&+(\beta+\gamma-h)r+(4\beta+4\gamma+b-e)q+(\beta-\gamma)B_f(W,w_2)\\
&-(2\beta+\gamma-h)B_f(W,w_1)-B_f(T_2,[T_2,m])+B_f(T_2,[T_2,u])\\
&+B_f(T_2,[T_1,v])-B_f(T_2,[T_1,n]).
\end{aligned}$$

Combining this with (4.24), we find

$$\beta(\gamma^2+\beta\gamma-2\beta^2)+(4\beta+6\gamma)q=2\beta(\gamma^2-\beta^2)+\beta\gamma(\beta-\gamma).$$

In sum,

$$(2\beta+3\gamma)q=0. \qquad (4.27)$$

4.5. Step 4: $\gamma^2(\gamma-\beta)=0$.

From (4.21) we get:

$$\begin{aligned}
&B_f(T_1,[T_2,[V,[V,X_2]]])\\
&=B_f(T_1,[T_2,(\gamma-\beta)\{(\gamma-\beta)X_2+w_2-w_1\}+[V,w_2-w_1]])\\
&=(\gamma-\beta)^2\gamma+(\gamma-\beta)B_f(T_1,[T_2,w_2-w_1])\\
&+B_f(T_1,[T_2,[V,w_2-w_1]]) \qquad (4.28)\\
&=(\gamma-\beta)^2\gamma+(\gamma-\beta)(s+B_f(W,w_2)-r)\\
&+B_f(T_1,[T_2,(s+B_f(W,w_2))X_2+dw_2+v-pX_2\\
&-aw_2+sX_1-ew_1-m-rX_2-cw_2+qX_1-hw_1-n\\
&+(B_f(W,w_1)-r)X_1+bw_1+u])\\
&=(\gamma-\beta)^2\gamma+(\gamma-\beta)(s+B_f(W,w_2)-r)\\
&+(s+B_f(W,w_2))(\gamma+d-a-c)\\
&+(B_f(W,w_1)-r+s-p-r+q)\gamma\\
&+(b-e-h)r+B_f(T_1,[T_2,u-m-n+v]).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&B_f(T_1,[T_2,[V,[V,X_2]]])\\
&=B_f(T_1,[-(\beta+\gamma)V+W,(\gamma-\beta)X_2+w_2-w_1])
\end{aligned}$$

$$\begin{aligned}
&+B_f(T_1,[V,[-(\beta+\gamma)V+W,X_2]])+B_f(T_1,[V,[V,\gamma X_2+w_2]])\\
=&\gamma(\beta^2-\gamma^2)-(\beta+\gamma)B_f(T_1,[V,w_2-w_1])+(\gamma-\beta)B_f(W,w_1)\\
&+B_f(T_1,[W,w_2-w_1])-(\beta+\gamma)B_f(T_1,[V,(\gamma-\beta)X_2+w_2-w_1])\\
&+B_f(T_1,[V,(a-c+\gamma)w_2+(e-h-\beta)w_1\\
&+(B_f(W,w_1)-B_f(W,w_2))X_2+m-n])\\
&+\gamma B_f(T_1,[V,(\gamma-\beta)X_2+w_2-w_1])\\
&+B_f(T_1,[V,(s+B_f(W,w_2))X_2+dw_2+v\\
&-pX_2-aw_2+sX_1-ew_1-m])\\
=&(\beta^2-\gamma^2)\gamma+(\gamma-\beta)B_f(W,w_1)-2(\beta+\gamma)B_f(T_1,[V,w_2-w_1])\\
&+B_f(T_1,[W,w_2-w_1])-\gamma(\gamma^2-\beta^2)\\
&+(a-c+\gamma)(s+B_f(W,w_2)-p)+(e-h-\beta)r+\gamma(B_f(W,w_1)\\
&-B_f(W,w_2))+B_f(T_1,[V,m-n])+\gamma^2(\gamma-\beta)\\
&+\gamma(s+B_f(W,w_2)-p-r)+\gamma(s+B_f(W,w_2))\\
&+\gamma(s-p)+(d-a)(s+B_f(W,w_2)-p)-er+B_f(T_1,[V,v-m])\\
=&(\beta^2-\gamma^2)\gamma+(\gamma-\beta)B_f(W,w_1)\\
&-(2\beta+\gamma)(s+B_f(W,w_2)-p-r)+B_f(T_1,[W,w_2-w_1])\\
&-\gamma(\gamma^2-\beta^2)+(c-d)(p-s-B_f(W,w_2))+2\gamma(s-p)\\
&-(\beta+h)r+\gamma B_f(W,w_1)+B_f(T_1,[V,m-n])+\gamma^2(\gamma-\beta)\\
&+\gamma(s+B_f(W,w_2))+B_f(T_1,[V,v-m]).
\end{aligned}$$

Equaling these two expressions, we have

$$\begin{aligned}
&B_f(T_1,[W,w_2-w_1])=\gamma(\gamma^2-\beta^2)+(\beta-\gamma)B_f(W,w_1)\\
&+(2\beta+\gamma)(s+B_f(W,w_2)-p-r)+\gamma(\gamma^2-\beta^2)-cp\\
&+\beta r-\gamma B_f(W,w_1)+B_f(T_1,[V,n])\\
&+\gamma^2(\beta-\gamma)+dp+B_f(T_1,[T_1,v])+(\gamma-\beta)^2\gamma\\
&+(\gamma-\beta)(s+B_f(W,w_2)-r)-a(s+B_f(W,w_2))-\gamma(s-p)\\
&+(B_f(W,w_1)-2r+q)\gamma+(b-e)r+B_f(T_1,[T_2,u-m-n]).
\end{aligned}\tag{4.29}$$

On the other side, the equations (4.16) and (4.19) give

$$\begin{aligned}
&B_f(T_1,[W,w_2])-B_f(T_1,[W,w_1])\\
=&(\beta+2\gamma)(s-p)+(\beta+\gamma-a)B_f(W,w_2)\\
&-er+dp+B_f(T_1,[T_1,v])-B_f(T_1,[T_2,m])-as-(\beta+2\gamma)r\\
&+\gamma B_f(W,w_1)+br-cp-B_f(T_1,[T_1,n])+B_f(T_1,[T_2,u]).
\end{aligned}\tag{4.30}$$

Equaling once again these two expressions, we have

$$2\gamma(\gamma^2-\beta^2)+(\beta-2\gamma)B_f(W,w_1)-\beta p-2\gamma r$$

$$+\gamma(\gamma-\beta)^2+\gamma B_f(W,w_2)+\gamma q=-2rp-\beta r+\gamma s$$

Taking into account the equations (4.9), we get

$$\gamma^2(\gamma-\beta)=0. \tag{4.31}$$

4.6. Step 5: $\beta(2s+p+\beta\gamma)=0$.

We need to continue our calculations:

$$\begin{aligned}
&B_f(T_2,[T_1,[V,[V,X_2]]])\\
=&B_f(T_2,[T_1,(\gamma-\beta)\{(\gamma-\beta)X_2+w_2-w_1\}])\\
+&B_f(T_2,[T_1,[V,w_2-w_1]])\\
=&-\beta(\gamma-\beta)^2+(\gamma-\beta)(s+B_f(W,w_1)-r)\\
+&B_f(T_2,[T_1,(s+B_f(W,w_2))X_2+dw_2+v\\
-&pX_2-aw_2+sX_1-ew_1-m-rX_2-cw_2\\
+&qX_1-hw_1-n+(B_f(W,w_1)-r)X_1+bw_1+u])\\
=&-\beta(\gamma-\beta)^2+(\gamma-\beta)(s+B_f(W,w_1)-r)\\
-&\beta(s+B_f(W,w_2)-p+s-r+q+B_f(W,w_1)-r)\\
+&(d-a-c)s+(b-e-h)(r-B_f(W,w_1))\\
+&B_f(T_2,[T_1,v-m-n+u]).
\end{aligned} \tag{4.32}$$

On the other hand,

$$\begin{aligned}
&B_f(T_2,[T_1,[V,[V,X_2]]])\\
=&B_f(T_2,[-(\beta+\gamma)V+W,(\gamma-\beta)X_2+w_2-w_1])\\
+&B_f(T_2,[V,[-(\beta+\gamma)V+W,X_2]])+B_f(T_2,[V,[V,\beta X_1+w_1]])\\
=&\beta(\beta^2-\gamma^2)-(\beta+\gamma)B_f(T_2,[V,w_2-w_1])+(\gamma-\beta)B_f(W,w_2)\\
+&B_f(T_2,[W,w_2-w_1])-(\beta+\gamma)B_f(T_2,[V,(\gamma-\beta)X_2+w_2-w_1])\\
+&B_f(T_2,[V,(a-c+\gamma)w_2+(e-h-\beta)w_1\\
+&(B_f(W,w_1)-B_f(W,w_2))X_2+m-n])\\
+&\beta B_f(T_2,[V,(\gamma-\beta)X_2+w_2-w_1])\\
+&B_f(T_2,[V,rX_2+cw_2-qX_1+hw_1+n\\
-&(B_f(W,w_1)-r)X_1-bw_1-u])\\
=&\beta(\beta^2-\gamma^2)+(\beta+2\gamma)(s+q+B_f(W,w_1)-r)\\
+&(\gamma-\beta)B_f(W,w_2)+B_f(T_2,[W,w_2-w_1])-\beta(\gamma^2-\beta^2)\\
-&(a-c+\gamma)s+(e-h-\beta)(q+B_f(W,w_1)-r)\\
+&\beta(B_f(W,w_1)-B_f(W,w_2))+B_f(T_2,[V,m-n])+\beta^2(\gamma-\beta)\\
-&\beta(s+q+B_f(W,w_1)-r)-cs+(h-b)(q+B_f(W,w_1)-r)\\
+&B_f(T_2,[V,n-u]).
\end{aligned}$$

Everything as before we find from these calculations

$$
\begin{aligned}
&B_f(T_2,[W,w_2-w_1])\\
=&2\beta(\gamma^2-\beta^2)-\gamma(q-r)-(\beta+\gamma)(s+B_f(W,w_1))\\
&+\gamma(r-s-q-B_f(W,w_1))-B_f(T_2,[V,m-n])+\gamma s\\
&-eq+\beta^2(\beta-\gamma)+bq-B_f(T_2,[V,n-u])-\beta(\gamma-\beta)^2\\
&+(\gamma-\beta)(s-r)-\beta(s-p-r)+(d-c)s\\
&+h(B_f(W,w_1)-r)+B_f(T_2,[T_1,v-m-n+u])\\
&+(\gamma-\beta)(B_f(W,w_1)-B_f(W,w_2)).
\end{aligned}
\tag{4.33}
$$

While we have

$$
\begin{aligned}
&B_f(T_2,[W,w_2-w_1])\\
=&-(2\beta+\gamma)s-\beta B_f(W,w_2)+ds-eq+B_f(T_2,[T_1,v])\\
&-B_f(T_2,[T_2,m])-cs-(2\beta+\gamma-b)q-(h-2\beta-\gamma)r\\
&-(\beta+\gamma-h)B_f(W,w_1)+B_f(T_2,[T_2,u])-B_f(T_2,[T_1,n]).
\end{aligned}
$$

The equality of these two expressions gives us

$$
\begin{aligned}
&\beta(\gamma^2+\beta\gamma-2\beta^2)\\
&+(2\beta-\gamma)(q+B_f(W,w_2)-s)+\beta s+\beta(p-B_f(W,w_1))=0.
\end{aligned}
$$

Taking into account the equations (4.9),

$$
\beta(s+r)=0. \tag{4.34}
$$

Further we compute

$$
\begin{aligned}
&B_f(T_2,[T_1,[V,[V,X_2]]])\\
=&B_f((\beta+\gamma)V-W,(\gamma-\beta)((\gamma-\beta)X_2+w_2-w_1)+[V,w_2-w_1])\\
&+B_f(T_1,[-(\beta+\gamma)V+W,(\gamma-\beta)X_2+w_2-w_1])\\
&+B_f(T_1,[V,[-(\beta+\gamma)V+W,X_2]])+B_f(T_1,[V,[V,\gamma X_2+w_2]])\\
=&-(\gamma-\beta)^2(\beta+\gamma)+(\beta+\gamma)B_f(V,[V,w_2-w_1])+(\beta-\gamma)B_f(W,w_2)\\
&+(\gamma-\beta)B_f(W,w_1)-B_f(W,[V,w_2-w_1])+(\beta^2-\gamma^2)\gamma\\
&-(\beta+\gamma)B_f(T_1,[V,w_2-w_1])+(\gamma-\beta)B_f(T_1,[W,X_2])\\
&+B_f(T_1,[W,w_2-w_1])-(\beta+\gamma)B_f(T_1,[V,(\gamma-\beta)X_2+w_2-w_1])\\
&+B_f(T_1,[V,[W,X_2]])+\gamma B_f(T_1,[V,(\gamma-\beta)X_2+w_2-w_1])\\
&+B_f(T_1,[V,[V,w_2]])\\
=&-(\beta+\gamma)(\gamma-\beta)^2-(3\beta+2\gamma)B_f(T_1,[V,w_2-w_1])\\
&+(\beta+\gamma)B_f(T_2,[V,w_2-w_1])+(\beta-\gamma)B_f(W,w_2)+(\gamma-\beta)B_f(W,w_1)
\end{aligned}
$$

$$
\begin{aligned}
&-B_f(W,[V,w_2-w_1])+(\beta^2-\gamma^2)\gamma+(\gamma-\beta)B_f(W,w_1)\\
&+B_f(T_1,[W,w_2-w_1])-\beta\gamma(\gamma-\beta)+B_f(T_1,[V,[W,X_2]])\\
&+B_f(T_1,[V,[V,w_2]])\\
&=-(\beta+\gamma)(\gamma-\beta)^2-(3\beta+2\gamma)(s+B_f(W,w_2)-p-r)\\
&+(\beta+\gamma)B_f(T_2,[V,w_2-w_1])+(\beta-\gamma)B_f(W,w_2)+2(\gamma-\beta)B_f(W,w_1)\\
&-B_f(W,dw_2+v-aw_2-ew_1-cw_2-hw_1-m-n+bw_1+u)\\
&+(\beta^2-\gamma^2)\gamma+B_f(T_1,[W,w_2-w_1])-\beta\gamma(\gamma-\beta)\\
&+(a-c+\gamma)(s+B_f(W,w_2)-p)+(e-h-\beta)r\\
&+\gamma(B_f(W,w_1)-B_f(W,w_2))+B_f(T_1,[V,m-n])\\
&+\gamma(s+B_f(W,w_2)-p+s)+(d-a)(s+B_f(W,w_2)-p)\\
&-er+B_f(T_1,[V,v-m]).
\end{aligned}
$$

Comparing this with (4.32),

$$
\begin{aligned}
&B_f(T_1,[W,w_2-w_1])=(\gamma-\beta)(2\beta^2+2\gamma^2+3\beta\gamma)\\
&+(\beta+\gamma-a)s+(d-c-2\beta)p+(b-e-4\gamma)r+\gamma q\\
&+(\beta+2\gamma-a)B_f(W,w_2)+(\beta-\gamma)B_f(W,w_1)+B_f(T_1,[T_1,v])\\
&-B_f(T_1,[T_2,m])-B_f(T_1,[T_1,n])+B_f(T_1,[T_2,u]).
\end{aligned}
\tag{4.35}
$$

Now the equations (4.29), (4.35) give us

$$
\begin{aligned}
&(\gamma-\beta)(2\beta^2+2\gamma^2+3\beta\gamma)+(\beta+\gamma-a)s+(d-c-2\beta)p\\
&+(b-e-4\gamma)r+\gamma q+(\beta+2\gamma-a)B_f(W,w_2)+(\beta-\gamma)B_f(W,w_1)\\
&+B_f(T_1,[T_1,v])-B_f(T_1,[T_2,m])-B_f(T_1,[T_1,n])+B_f(T_1,[T_2,u])\\
&=(d-\beta-2\gamma)p+(\beta+2\gamma-a)s-er+(\beta+\gamma-a)B_f(W,w_2)\\
&+B_f(T_1,[T_1,v])-B_f(T_1,[T_2,m])-(\gamma-B_f(W,w_1))\gamma-(\beta+\gamma-b)r\\
&-cp-B_f(T_1,[T_1,n])+B_f(T_1,[T_2,u]).
\end{aligned}
$$

Namely,

$$
\begin{aligned}
&(\gamma-\beta)(2\beta^2+2\gamma^2+3\beta\gamma)+(\beta-2\gamma)(r-p+B_f(W,w_1))\\
&+\gamma(q-s+B_f(W,w_2))=0.
\end{aligned}
$$

From (4.9) this becomes

$$2\beta(\gamma-\beta)(\beta+\gamma)=0.$$

Hence, if $\beta+\gamma\neq 0$,

$$\beta(\gamma-\beta)=0. \tag{4.36}$$

Here, substituting (4.33) into (4.23) and comparing the result with (4.22), we get

$$\begin{aligned}
&2\beta(\beta^2-\gamma^2)-(2\beta+\gamma)(r-B_f(W,w_1)-s-q)+(\gamma-\beta)B_f(W,w_2)\\
&+2\beta(\gamma^2-\beta^2)-\gamma(q-r)-(\beta+\gamma)(s+B_f(W,w_1))\\
&+\gamma(r-s-q-B_f(W,w_1))-B_f(T_2,[V,m-n])+\gamma s-eq+\beta^2(\beta-\gamma)\\
&+bq-B_f(T_2,[V,n-u])-\beta(\gamma-\beta)^2+(\gamma-\beta)(s-r)-\beta(s-p-r)\\
&+(d-c)s+h(B_f(W,w_1)-r)+B_f(T_2,[T_1,v-m-n+u])\\
&+(\gamma-\beta)(B_f(W,w_1)-B_f(W,w_2))+(c-a-\gamma)s\\
&+(e-h-\beta)(q+B_f(W,w_1)-r)+\beta(B_f(W,w_1)-B_f(W,w_2))\\
&+B_f(T_2,[V,m-n])+\beta\gamma(\gamma-\beta)+\beta(B_f(W,w_2)+s)\\
&-ds+B_f(T_2,[V,v])+\beta(s-p)+as\\
&-e(q+B_f(W,w_1)-r)-B_f(T_2,[V,m])\\
&=(\beta-\gamma+b-e-h)q-B_f(T_2,[T_2,m+n])+B_f(T_2,[T_2,u]).
\end{aligned}$$

Thus,

$$2\beta s+\beta(r+B_f(W,w_1))+\beta^2(\beta-\gamma)-\beta(\gamma-\beta)^2=0.$$

Taking into account the equations (4.9) again,

$$2\beta s+\beta(p+\gamma^2)-\beta\gamma^2+\beta^2\gamma=0.$$

In sum,

$$\beta(2s+p+\beta\gamma)=0. \tag{4.37}$$

4.7. Step 6: $\beta+\gamma=0$.

We add further two calculations.

$$\begin{aligned}
&B_f(T_1,[T_1,[V,[V,X_2]]])\\
&=B_f(T_1,[-(\beta+\gamma)V+W,(\gamma-\beta)X_2+w_2-w_1])\\
&+B_f(T_1,[V,[T_1,(\gamma-\beta)X_2+w_2-w_1]])\\
&=-(\beta+\gamma)B_f(-(\beta+\gamma)V+W,(\gamma-\beta)X_2+w_2-w_1)\\
&-(\beta+\gamma)B_f(V,[T_1,(\gamma-\beta)X_2+w_2-w_1])\\
&+B_f(T_1,[W,(\gamma-\beta)X_2+w_2-w_1])+(\gamma-\beta)B_f(T_1,[V,\beta X_1+w_1])\\
&+B_f(T_1,[V,pX_2+aw_2-sX_1+ew_1+m\\
&-(B_f(W,w_1)-r)X_1-bw_1-u])\\
&=-(\beta+\gamma)(\gamma^2-\beta^2)-(\beta+\gamma)(B_f(W,w_2)-B_f(W,w_1))\\
&-\beta(\beta^2-\gamma^2)-(\beta+\gamma)(s-p+B_f(W,w_1)-r)+(\gamma-\beta)B_f(W,w_1)\\
&+B_f(T_1,[W,w_2-w_1])+\beta\gamma(\gamma-\beta)-(\gamma-\beta)r\\
&+\gamma(p-s+r-B_f(W,w_1))+a(s+B_f(W,w_2)-p)+(e-b)r\\
&+B_f(T_1,[V,m])-B_f(T_1,[T_2,u])
\end{aligned}$$

$$=(\gamma-\beta)p+dp+B_f(T_1,[T_1,v])-(a+c)p$$
$$-B_f(T_1,[T_1,m])-B_f(T_1,[T_1,n]).$$

Therefore,

$$\begin{aligned}
&B_f(T_1,[W,w_2-w_1])\\
=&\gamma(\gamma^2-\beta^2)+(\beta+\gamma)B_f(W,w_2)-(\beta+\gamma)(p+r)\\
&+(\gamma-\beta)p+dp-(a+c)p+B_f(T_1,[T_1,v])-B_f(T_1,[T_1,m])\\
&-B_f(T_1,[T_1,n])+\beta\gamma(\beta-\gamma)+(\gamma-\beta)r\\
&+\gamma(s+B_f(W,w_1)-p-r)+a(p-s-B_f(W,w_2))+(b-e)r\\
&-B_f(T_1,[V,m])+B_f(T_1,[T_2,u]).
\end{aligned}\tag{4.38}$$

The equality of two equations (4.29) and (4.38) brings

$$\begin{aligned}
&\beta\gamma(\beta-\gamma)-\beta r+\gamma(s+B_f(W,w_1))\\
=&(\beta-\gamma)B_f(W,w_1)+(2\beta+\gamma)s+\gamma(\gamma^2-\beta^2)+\beta r\\
&+\gamma^2(\beta-\gamma)+(\gamma-\beta)^2\gamma+(\gamma-\beta)s-\gamma(s-p)+(q-2r)\gamma
\end{aligned}$$

By means of (4.9), this reduces to

$$\beta^2\gamma-\beta\gamma^2+\gamma^3=\beta(s+p+r)+\gamma(q-p).\tag{4.39}$$

Finally,

$$\begin{aligned}
&B_f(T_1,[T_2,[V,[V,X_2]]])\\
=&B_f(T_1,[-(\beta+\gamma)V+W,(\gamma-\beta)X_2+w_2-w_1])\\
&+B_f(T_1,[V,[T_2,(\gamma-\beta)X_2+w_2-w_1]])\\
=&-(\beta+\gamma)B_f(T_1,[V,(\gamma-\beta)X_2+w_2-w_1])\\
&+B_f(T_1,[W,(\gamma-\beta)X_2+w_2-w_1])\\
&+(\gamma-\beta)B_f(T_1,[V,\gamma X_2+w_2])+B_f(T_1,[V,[T_2,w_2-w_1]])\\
=&-(\beta+\gamma)B_f(-(\beta+\gamma)V+W,(\gamma-\beta)X_2+w_2-w_1)\\
&-(\beta+\gamma)B_f(V,[T_1,(\gamma-\beta)X_2+w_2-w_1])+(\gamma-\beta)B_f(W,w_1)\\
&+B_f(T_1,[W,w_2-w_1])+\gamma^2(\gamma-\beta)+(\gamma-\beta)(s+B_f(W,w_2)-p)\\
&+B_f(T_1,[V,(s+B_f(W,w_2))X_2+dw_2+v\\
&-rX_2-cw_2+qX_1-hw_1-n])\\
=&(\beta+\gamma)(\beta^2-\gamma^2)-(\beta+\gamma)(B_f(W,w_2)-B_f(W,w_1))+\beta(\gamma^2-\beta^2)\\
&-(\beta+\gamma)(s-p+B_f(W,w_1)-r)+(\gamma-\beta)B_f(W,w_1)\\
&+B_f(T_1,[W,w_2-w_1])+\gamma^2(\gamma-\beta)+(\gamma-\beta)(s+B_f(W,w_2)-p)\\
&+\gamma(s+B_f(W,w_2)-r+q)+(d-c)(s+B_f(W,w_2)-p)-hr
\end{aligned}$$

$$
\begin{aligned}
&-B_f(T_1,[V,n])+B_f(T_1,[V,v])\\
&=(\gamma-\beta)^2\gamma+(\gamma-\beta)(s+B_f(W,w_2)-p)\\
&+(s+B_f(W,w_2))(\gamma+d-a-c)+(B_f(W,w_1)-r+s-r+q)\gamma\\
&+(b-e-h)r+B_f(T_1,[T_2,u-m-n+v]),
\end{aligned}
$$

taking (4.23) into account. Thus,

$$
\begin{aligned}
&B_f(T_1,[W,w_2-w_1])=(\beta+\gamma)(\gamma^2-\beta^2)\\
&+(\beta+\gamma)(B_f(W,w_2)-B_f(W,w_1))+\beta(\beta^2-\gamma^2)\\
&+(\beta+\gamma)(s-p+B_f(W,w_1)-r)+(\beta-\gamma)B_f(W,w_1)\\
&-\gamma^2(\gamma-\beta)+(\beta-\gamma)(s+B_f(W,w_2)-p)\\
&-\gamma(s+B_f(W,w_2)-r+q)+hr+B_f(T_1,[V,n])\\
&+(c-d)(s+B_f(W,w_2)-p)-B_f(T_1,[V,v])+(\gamma-\beta)^2\gamma\\
&+(\gamma-\beta)(s+B_f(W,w_2)-r)+(s+B_f(W,w_2))(\gamma+d-a-c)\\
&+(B_f(W,w_1)-2r+s-p+q)\gamma+(b-e-h)r\\
&+B_f(T_1,[T_2,u-m-n+v]).
\end{aligned}
\tag{4.40}
$$

If $(\beta,\gamma)\neq(0,0)$, then equations (4.31), (4.36) imply $\beta=\gamma$. Then, equation (4.27) implies $q=0$ and finally (4.34), (4.39) say $\beta=\gamma=0$. In this way, we arrive to the desired result $\beta+\gamma=0$. From the result $[T_2,V]\in\mathfrak{n}_{j_0-1}$, we can argue just as in the proof of Proposition 3.2 to show that the limit point of a convergent sequence $\{x_i\}_{i=1}^{\infty}$ in $H_2H_1\cap M_j$ belongs to H_2H_1. □

5. The general case

We now proceed in parallel for the general exponential case. Employing the same induction procedure, we are led to the case where the center $\mathfrak{z}$ of $\mathfrak{g}$ is of dimension 1 and f does not vanish there. This being assumed, let $\mathfrak{n}$ be a nilpotent ideal of $\mathfrak{g}$ containing $[\mathfrak{g},\mathfrak{g}]$ and $\mathfrak{a}$ a minimal non-central ideal of $\mathfrak{g}$ contained in $\mathfrak{n}$. Then,

$$\{0\}=\mathfrak{g}_0\subset\mathfrak{a}\subset\mathfrak{n}\cap\mathfrak{a}^f\subset\mathfrak{n}\subset\mathfrak{g}$$

is a sequence of ideals of $\mathfrak{g}$. Cutting this sequence, we obtain a good sequence

$$\{0\}=\mathfrak{g}_0\subset\mathfrak{g}_1\subset\cdots\subset\mathfrak{g}_{n-1}\subset\mathfrak{g}_n=\mathfrak{g},\ \dim(\mathfrak{g}_j/\mathfrak{g}_{j-1})=1(1\leq j\leq n),$$

where $\mathfrak{n}=\mathfrak{g}_{j_0}$, of subalgebras of $\mathfrak{g}$. Just as Proposition 3.2, in order to pass from M_{j-1} to M_j for $j\leq j_0$ the essential case is one where there exist $Y\in\mathfrak{a}$ outside of $\mathfrak{a}\cap\mathfrak{z}$ which is written $Y=X_2-X_1$ with $X_i\in\mathfrak{h}_i^0(i=1,2)$ and $V\in\mathfrak{g}_j\setminus\mathfrak{g}_{j-1}$ which is written $V=T_2-T_1$ with $T_i\in\mathfrak{h}_i(i=1,2)$ where $[T_2,Y]=[T_1,Y]=Z,\mathfrak{z}=\mathbb{R}Z,f(Z)=1$ and $f(Y)=0$ as before. We choose such an index j minimal. In these circumstances we can see that the root of the adjoint action of T_i on V vanishes even if $\mathfrak{g}_j$ or $\mathfrak{g}_{j-1}$ is not an ideal of $\mathfrak{g}$. Indeed, let us place ourselves in this situation and put

$$[T_j, W] \equiv \lambda W + \delta V$$

modulo the precedent ideal of the Jordan-Hölder sequence of $\mathfrak{g}$ in question. We are obliged to follow the route made above. We go to notice the necessary changes in the calculations already done above in the completely solvable case.

We have

$$[T_1, w_2] - [T_2, w_1] = (\beta^2 - \gamma^2)X_2 + \beta w_1 - \gamma w_2 - \beta^2 Y + [W, X_2].$$

and

$$\begin{gathered}[W, X_1] = [W, X_2] = (a - c + \gamma)w_2 + (e - h - \beta)w_1 \\ +(B_f(W, w_2) - \delta)Y + (B_f(W, w_1) - B_f(W, w_2))X_2 + m - n.\end{gathered}$$

Since

$$B_f(T_j, [W, X_j]) = B_f([T_j, W], X_j) + B_f(W, [T_j, X_j]) = -\delta + B_f(W, w_j)$$

for $j = 1, 2$, it follows that

$$p - r = B_f(W, w_1) - \gamma^2 - \delta, q - s = \beta^2 + \delta - B_f(W, w_2).$$

Repeating with these changes all the calculations done in the completely solvable case, we remark that equations (4.27) - (4.39) all hold without change. So, we arrive to the result $\beta + \gamma = 0$ even in the exponential case.

Our algebra $\mathfrak{g}$ being exponential, the result $\beta + \gamma = 0$ signifies that the two roots of the action $\mathrm{ad}(T_i)(i = 1, 2)$ at the level of $\mathfrak{g}_j$ are zeros and this allows us to apply the same reasoning as before about a convergent sequence in H_2H_1. There is a possibility that we would have the same situation at other level, but we can treat it in parallel. Namely, when $\mathfrak{a} = \mathbb{R}Y + \mathbb{R}Y' + \mathfrak{z}$, we repeat the same arguments for Y' and at other levels we apply the induction hypothesis applied to $(\mathfrak{h}_1', \mathfrak{h}_2')$ contained in the proper subalgebra $\widetilde{\mathfrak{g}}$.

It is in this way that we achieve the proof of Claim 1.1, and verify the following:

Theorem 5.1. *Let $G = \exp \mathfrak{g}$ be an exponential solvable Lie group with Lie algebra $\mathfrak{g}$. Let $f \in \mathfrak{g}^*$ and $\mathfrak{h}_j$ $(j = 1, 2)$ two polarizations verifying the Pukanszky condition of $\mathfrak{g}$ at f. Put $H_j = \exp(\mathfrak{h}_j)$ for $j = 1, 2$. Then the product set H_2H_1 is a closed set of G.*

As explained in the introduction, the followings are straightforward:

Corollary 5.2. *(cf. [**3**]) Let $G = \exp \mathfrak{g}$ be an exponential solvable Lie group with Lie algebra $\mathfrak{g}$. Let $f \in \mathfrak{g}^*$ and let $\mathfrak{h}_j$ $(j = 1, 2)$ be two polarizations verifying the Pukanszky condition of $\mathfrak{g}$ at f. Put $H_j = \exp(\mathfrak{h}_j)$ and $\pi_j = ind_{H_j}^G \chi_f$ for $j = 1, 2$, where χ_f denotes the unitary character of H_j defined by $\chi_f(\exp X) = e^{if(X)}$ $(X \in \mathfrak{h}_j)$. Let $\mathcal{H}_{\pi_1}$ be the Hilbert space of π_1. Then the integral T_{21} as defined in formula (1.1) by*

$$(T_{21}\varphi)(g) := T_{\mathfrak{h}_2\mathfrak{h}_1}\varphi)(g) = \oint_{H_2/(H_2\cap H_1)} \varphi(gh)\chi_f(h)\Delta_{H_2,G}^{-1/2}(h)d\nu(h),$$

converges for continuous functions $\varphi \in \mathcal{H}_{\pi_1}$ with compact supports modulo H_1 and extends into an intertwining operator from π_1 to π_2.

For given three lagrangian subspaces for the bilinear form B_f, let $\tau(\mathfrak{h}_3, \mathfrak{h}_2, \mathfrak{h}_1)$ designates the Maslov index for the triple $(\mathfrak{h}_3, \mathfrak{h}_2, \mathfrak{h}_1)$ (cf. [**1**]). The following is a direct consequence from Corollary 5.2 and [**1**]. For more details, the readers could consult the reference [**7**] for the treatment of the nilpotent contexts, and also [**8**], [**9**].

Corollary 5.3. *(cf. [2]) Let $G = \exp \mathfrak{g}$ be an exponential solvable Lie group with Lie algebra $\mathfrak{g}$. Let $f \in \mathfrak{g}^*$ and let $\mathfrak{h}_j$ $(j = 1, 2, 3)$ be three polarizations verifying the Pukanszky condition of $\mathfrak{g}$ at f. Put $H_j = \exp(\mathfrak{h}_j)$ and $\pi_j = ind_{H_j}^G \chi_f$ for $j = 1, 2, 3$, where χ_f denotes the unitary character of H_j defined by $\chi_f(\exp X) = e^{if(X)}$ $(X \in \mathfrak{h}_j)$. For $1 \leq i, j \leq 3$, we normalize the intertwining operators T_{ij} as defined in Corollary 5.2, in order to obtain an isometry I_{ij} which intertwines π_j and π_i. Then, we have the composition formula*

$$I_{13} \circ I_{32} \circ I_{21} = e^{\frac{i\pi\tau(\mathfrak{h}_3,\mathfrak{h}_2,\mathfrak{h}_1)}{4}} Id.$$

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Ali Baklouti
Département de Mathématiques,
Faculté des Sciences de Sfax,
Route de Soukra, 3038 Sfax, Tunisie
E-mail: Ali.Baklouti@usf.tn

Hidenori Fujiwara
6-13-27-402, Minamisho,
Sawara-ku, Fukuoka, 814-0031 Japan
E-mail: fujiwara6913@yahoo.co.jp