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Sections of time-like twistor spaces with light-like or zero covariant derivatives

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Abstract. The conformal Gauss maps of time-like minimal surfaces in the Lorentz-Minkowski 3-space E^3_1 give sections of the time-like twistor spaces associated with the pull-back bundles such that the covariant derivatives are fully light-like, that is, these are either light-like or zero, and do not vanish at any point. For an oriented neutral $4n$-manifold $(\tilde{M}, h)$, if J is an h-reversing almost paracomplex structure of $\tilde{M}$ such that ∇J is locally given by the tensor product of a nowhere zero 1-form and an almost nilpotent structure related to J, then we will see that ∇J is valued in a light-like $2n$-dimensional distribution $\mathcal{D}$ such that $(\tilde{M}, h, \mathcal{D})$ is a Walker manifold and that the square norm $\|\nabla J\|^2$ of ∇J vanishes. We will obtain examples of h-reversing almost paracomplex structures of E^{4n}_{2n} as above. In addition, we will obtain all the pairs of h-reversing almost paracomplex structures of E^4_2 such that each pair gives sections of the two time-like twistor spaces with fully light-like covariant derivatives.

1. Introduction

Neutral metrics are already investigated in various situations. The space of oriented lines in the Euclidean 3-space E^3 admits a neutral Kähler structure ([**23**]). Analogous spaces are found in [**1**], [**28**]. See [**29**] for fibrations of E^3 by oriented lines. See [**21**] for almost paracomplex structures on neutral 4-manifolds. See [**16**] for anti-self-dual null-Kähler structures. The ultra-hyperbolic equation is a neutral analogue of the Laplace equation (see [**9**], [**13**], [**26**]), and related to tomography ([**22**]). See [**27**] for quantum field theories in neutral spaces.

Let E be an oriented vector bundle of rank 4. Let h be a positive-definite or neutral metric of E and ∇ an h-connection of E. An h-preserving complex structure I of E satisfies $\nabla I = 0$ if and only if the corresponding section Ω of one of the twistor or space-like twistor spaces associated with E is horizontal with respect to the connection $\hat{\nabla}$ induced by ∇. If E is the tangent bundle $T\tilde{M}$ of an oriented Riemannian or neutral 4-manifold $\tilde{M}$ and if h, ∇ are its metric and the Levi-Civita connection of h respectively, then I is an almost complex structure of $\tilde{M}$ and $\nabla I = 0$ just means that $(\tilde{M}, h, I)$ is a Kähler or neutral Kähler surface. If E is the pull-back bundle $F^*T\tilde{M}$ over a Riemann surface M by a space-like and conformal immersion $F : M \longrightarrow \tilde{M}$ with zero mean curvature vector, then $\nabla I = 0$ for a twistor lift Ω of F just means that F is isotropic (refer to [**19**], [**2**]). For the twistor spaces and the space-like twistor spaces, refer to [**18**], [**11**] respectively. Suppose that $\tilde{M}$ is neutral. Then, even if the square norm $\|\nabla I\|^2$ of ∇I vanishes, ∇I does not necessarily vanish. We say that $(\tilde{M}, h, I)$ is *isotropic Kähler* if

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$\| \nabla I \|^2 = 0$. See [**14**], [**20**] for examples of isotropic Kähler but non-Kähler 4-manifolds. See [**10**] for examples of isotropic Kähler but non-Kähler 6-manifolds.

Let E be as in the beginning of the previous paragraph. Let h, ∇ be a neutral metric and an h-connection of E respectively. Then an h-reversing paracomplex structure J of E satisfies $\nabla J = 0$ if and only if the corresponding section Ω of one of the time-like twistor spaces associated with E is horizontal with respect to $\hat{\nabla}$. If $E = T\tilde{M}$ for an oriented neutral 4-manifold $\tilde{M}$, then J is an almost paracomplex structure of $\tilde{M}$ and $\nabla J = 0$ just means that $(\tilde{M}, h, J)$ is a paraKähler surface. If $E = F^*T\tilde{M}$ for a time-like and conformal immersion $F : M \longrightarrow \tilde{M}$ of a Lorentz surface M into $\tilde{M}$ with zero mean curvature vector and if Ω is a twistor lift of F, then $\nabla J = 0$ implies that F is isotropic. Refer to [**24**], [**25**] for the time-like twistor spaces. Even if F is isotropic, ∇J does not necessarily vanish ([**2**]). It is possible that $\hat{\nabla}\Omega$ is either light-like or zero, and does not vanish at any point. In such a case, $\hat{\nabla}\Omega$ determines a light-like one-dimensional subspace of the fiber of $\bigwedge^2 E$ at each point of M and then we say that $\hat{\nabla}\Omega$ is *fully light-like*. Let F be the conformal Gauss map of a time-like minimal surface with nowhere zero curvature in the Lorentz-Minkowski 3-space E^3_1. Then $\tilde{M} = S^4_2$, and we see that F has zero mean curvature vector and isotropicity and that the covariant derivatives of the twistor lifts $\Omega_\pm$ are fully light-like. We will see that for a paracomplex structure $J = J_\varepsilon$ corresponding to Ω_ε ($\varepsilon \in \{+, -\}$), ∇J is locally represented as $\nabla J = \alpha \otimes N$ for a nowhere zero 1-form α and a nilpotent structure N related to J and therefore ∇J is valued in a light-like subbundle of the pull-back bundle of rank 2 (Theorem 5.2). We will define a nilpotent structure of an oriented vector bundle E of rank $4n$ with a neutral metric h. Then a nilpotent structure N gives a null structure on each fiber of E such that the image is a light-like $2n$-dimensional subspace and h is null-Hermitian with respect to N (see Section 3 and refer to [**17**]). We will characterize an h-reversing paracomplex structure J of E such that for an h-connection ∇ of E, ∇J is locally represented as $\nabla J = \alpha \otimes N$ for a 1-form α and a nilpotent structure N related to J (Theorem 4.1). See [**3**], [**8**] for nilpotent structures for $n = 1$.

Let $\tilde{M}$ be an oriented neutral $4n$-manifold. Then $\tilde{M}$ has a neutral metric h and h gives the Levi-Civita connection ∇. Let J be an h-reversing almost paracomplex structure of $\tilde{M}$ such that ∇J is locally represented as $\nabla J = \alpha \otimes N$ for a nowhere zero 1-form α and an almost nilpotent structure N related to J. Then ∇J is valued in a light-like $2n$-dimensional distribution $\mathcal{D}_J$ on $\tilde{M}$. We will see that $(\tilde{M}, h, \mathcal{D}_J)$ is a *Walker manifold*, that is, the covariant derivatives of local generators of $\mathcal{D}_J$ by any tangent vector of $\tilde{M}$ are contained in $\mathcal{D}_J$ (Theorem 6.1). This implies that $\mathcal{D}_J$ is involutive and therefore N is a null structure. See [**12**], [**30**] for Walker manifolds and see [**17**] for null structures. We will see that the square norm $\| \nabla J \|^2$ of ∇J vanishes, that is, $(\tilde{M}, h, J)$ is *isotropic paraKähler* (Theorem 7.1). See [**15**] for isotropic paraKähler manifolds. Suppose $n = 1$. Let J be an h-reversing almost paracomplex structure of $\tilde{M}$ such that the covariant derivative of the corresponding section of one of the time-like twistor spaces is fully light-like. Then ∇J is locally represented as above and therefore $(\tilde{M}, h, \mathcal{D}_J)$ is a Walker manifold. Let $\mathcal{D}$ be a light-like two-dimensional distribution on an oriented neutral 4-manifold $\tilde{M}$ such that $(\tilde{M}, h, \mathcal{D})$ is a Walker manifold. Then $\mathcal{D}$ is locally given by $\operatorname{Im} N$ for an almost nilpotent structure N. We will find an h-reversing almost paracomplex structure J of a neighborhood of each point of $\tilde{M}$ satisfying $\nabla J = \alpha \otimes N$ for a nowhere zero 1-form α and that N is related to J (Theorem 6.3). In particular, for an almost nilpotent structure N of $\tilde{M}$ satisfying $\nabla N = 0$, there exists J as above on a neighborhood of each point, since $(\tilde{M}, h, \mathcal{D})$ with $\mathcal{D} = \operatorname{Im} N$ is a Walker manifold.

Let E be an oriented vector bundle over $\boldsymbol{R}^m$ ($m \geq 2$) of rank $4n$. Let h be a neutral metric of E and ∇ a flat h-connection of E. We will find examples of h-reversing paracomplex structures of E such that for each J, ∇J is represented as $\nabla J = \alpha \otimes N$ for a nowhere zero 1-form α and a nilpotent structure N related to J (Example 8.1 $\sim$ Example 8.4). In particular, we will find examples of h-reversing almost paracomplex structures of E^{4n}_{2n} as above (Remark 8.6). Suppose $n = 1$. We will characterize sections of the time-like twistor spaces associated with E such that the covariant derivatives are fully light-like and we will see that the covariant derivative of a paracomplex structure J corresponding to such a section can be represented as $\nabla J = \alpha \otimes N$ for a nowhere zero 1-form α and a nilpotent structure N satisfying $\nabla N = 0$ (Proposition 9.1). In addition, we will obtain all the pairs of sections of the two time-like twistor spaces with fully light-like covariant derivatives (Theorem 9.3). Then we will find two types of such pairs (Remark 9.5). One type corresponds to the pair of the lifts of the conformal Gauss map of a time-like minimal surface in E^3_1 and the other type corresponds to the pair of the lifts of a time-like surface in a 4-dimensional neutral space form with zero mean curvature vector given in [**4**]. Based on these studies, we can obtain the results in the case of $E = TE^4_2$ (Corollary 9.2, Remark 9.4). Therefore we can find all the pairs of h-reversing almost paracomplex structures of E^4_2 such that each pair gives sections of the two time-like twistor spaces with fully light-like covariant derivatives.

2. Elements of $SO(2n, 2n)$ preserving oriented light-like $2n$-planes

Let V be an oriented $4n$-dimensional vector space and h_V a neutral metric of V. Let W be a light-like $2n$-dimensional subspace of V. Let $(e_1, \dots, e_{2n}, e_{2n+1}, \dots, e_{4n})$ be an ordered pseudo-orthonormal basis of V giving the orientation of V such that $e_1, \dots, e_{2n}$ (respectively, $e_{2n+1}, \dots, e_{4n}$) are space-like (respectively, time-like). Suppose that W is spanned by

$$\begin{aligned} &\xi_1 := e_1 - e_{2n+1}, && \dots, \quad \xi_n := e_n - e_{3n}, \\ &\xi_{n+1} := e_{n+1} + e_{3n+1}, && \dots, \quad \xi_{2n} := e_{2n} + e_{4n}. \end{aligned} \tag{2.1}$$

Let T be an automorphism of V which preserves h_V and the orientation of V. Then there exists an element A of $SO(2n, 2n)$ satisfying

$$(Te_1, \dots, Te_{2n}, Te_{2n+1}, \dots, Te_{4n}) = (e_1, \dots, e_{2n}, e_{2n+1}, \dots, e_{4n})A.$$

We represent A as

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ A_{41} & A_{42} & A_{43} & A_{44} \end{bmatrix},$$

where A_{ij} $(i, j = 1, 2, 3, 4)$ are $n \times n$ matrices. We call A_{ij} the (i, j)-*block* of A. We use not only A_{ij} but also $A_{(i,j)}$ in order to represent the (i, j)-block of A. We set

$$A^{\times} := \begin{bmatrix} A_{22} & -A_{21} & A_{42} & A_{41} \\ -A_{12} & A_{11} & A_{32} & A_{31} \\ A_{24} & A_{23} & A_{44} & -A_{43} \\ A_{14} & A_{13} & -A_{34} & A_{33} \end{bmatrix}.$$

Let $T^{\times}$ be an endomorphism of V such that $A^{\times}$ is the representation matrix with respect to $(e_1, \dots, e_{2n}, e_{2n+1}, \dots, e_{4n})$.

PROPOSITION 2.1. *The following are mutually equivalent*:

(a) T *induces an endomorphism* Φ_T *of* W;

(b) $A_{(i,j)} + (-1)^j A_{(i,j+2)} = (-1)^i (A_{(i+2,j)} + (-1)^j A_{(i+2,j+2)})$ *for* $i, j = 1, 2$;

(c) $A_{(i,j)} + (-1)^{i-1} A_{(i+2,j)} = (-1)^{j-1} (A_{(i,j+2)} + (-1)^{i-1} A_{(i+2,j+2)})$ *for* $i, j = 1, 2$;

(d) $T^{\times}$ *induces an endomorphism* $\Phi_{T^{\times}}$ *of* W.

PROOF. We see that (a) and (b) are equivalent. We can rewrite (b) into (c) immediately. We see that (c) and (d) are equivalent. □

For $i, j = 1, 2$, we set

$$\begin{aligned} P_{ij} &:= A_{(i,j)} + (-1)^j A_{(i,j+2)}, \\ P^{\times}_{ij} &:= (-1)^{i'+j'} (A_{(i',j')} + (-1)^{i'-1} A_{(i'+2,j')}), \end{aligned}$$

where $\{i, i'\} = \{j, j'\} = \{1, 2\}$. We set

$$P := \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}, \quad P^{\times} := \begin{bmatrix} P^{\times}_{11} & P^{\times}_{12} \\ P^{\times}_{21} & P^{\times}_{22} \end{bmatrix}.$$

If T induces an endomorphism Φ_T of W, that is, if $T^{\times}$ induces an endomorphism $\Phi_{T^{\times}}$ of W, then P, $P^{\times}$ are the representation matrices of Φ_T, $\Phi_{T^{\times}}$ respectively with respect to $\xi_1, \dots, \xi_{2n}$. Let Λ_n be a $4n \times 4n$ matrix defined by

$$\Lambda_n := \begin{bmatrix} O_n & -I_n & O_n & I_n \\ I_n & O_n & I_n & O_n \\ O_n & I_n & O_n & -I_n \\ I_n & O_n & I_n & O_n \end{bmatrix},$$

where I_n is the $n \times n$ unit matrix and O_n is the $n \times n$ zero matrix. We obtain

PROPOSITION 2.2. *An automorphism* T *satisfies not only one of* (a)∼(d) *in Proposition 2.1 but also* $P = P^{\times}$ *if and only if* A *and* Λ_n *are commutative.*

REMARK 2.3. The condition $A\Lambda_n = \Lambda_n A$ just means that T is null-linear (see [**17**]).

Let G be a Lie subgroup of $SO(2n,2n)$ defined by $A\Lambda_n = \Lambda_n A$ for $A \in SO(2n,2n)$. Then each element of the Lie algebra of G is represented as

$$\begin{bmatrix} U_{11} & U_{12} & U_{11}+Z & -U_{12}+X \\ -{}^tU_{12} & U_{22} & -{}^tU_{12}+Y & -U_{22}+{}^tZ \\ -U_{11}+{}^tZ & -U_{12}+Y & -U_{11}-Z+{}^tZ & U_{12}-X-Y \\ -{}^tU_{12}+X & U_{22}+Z & -{}^tU_{12}+X+Y & -U_{22}-Z+{}^tZ \end{bmatrix},$$

where

$${}^tU_{11} = -U_{11}, \quad {}^tU_{22} = -U_{22}, \quad {}^tX = X, \quad {}^tY = Y.$$

Therefore we obtain $\dim G = 4n^2$. We will prove

Proposition 2.4. *Suppose that T satisfies one of* (a)∼(d) *in Proposition 2.1. Then*

(a) $|P||P^\times| = 1$;

(b) *if $P = P^\times$, that is, if $A \in G$, then $|P| = 1$, that is, $\xi := \xi_1 \wedge \cdots \wedge \xi_{2n}$ is invariant by $\Phi_T = \Phi_{T^\times}$.*

Proof. Since $A \in SO(2n,2n)$, we have $|A| = 1$. Using (b) in Proposition 2.1, we obtain

$$\begin{vmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ A_{41} & A_{42} & A_{43} & A_{44} \end{vmatrix}$$

$$= \begin{vmatrix} A_{11} & A_{12} & A_{13}-A_{11} & A_{14}+A_{12} \\ A_{21} & A_{22} & A_{23}-A_{21} & A_{24}+A_{22} \\ A_{31} & A_{32} & A_{33}-A_{31} & A_{34}+A_{32} \\ A_{41} & A_{42} & A_{43}-A_{41} & A_{44}+A_{42} \end{vmatrix}$$

$$= \begin{vmatrix} A_{11} & A_{12} & A_{13}-A_{11} & A_{14}+A_{12} \\ A_{21} & A_{22} & A_{23}-A_{21} & A_{24}+A_{22} \\ A_{31} & A_{32} & A_{11}-A_{13} & -A_{14}-A_{12} \\ A_{41} & A_{42} & A_{23}-A_{21} & A_{22}+A_{24} \end{vmatrix}$$

$$= \begin{vmatrix} A_{11}+A_{31} & A_{12}+A_{32} & O_n & O_n \\ A_{21}-A_{41} & A_{22}-A_{42} & O_n & O_n \\ A_{31} & A_{32} & A_{11}-A_{13} & -A_{14}-A_{12} \\ A_{41} & A_{42} & A_{23}-A_{21} & A_{22}+A_{24} \end{vmatrix}.$$

Therefore we obtain (a) in Proposition 2.4. We will prove (b) in Proposition 2.4 by induction as follows.
Part 1 Suppose $n = 1$. Then an element A of $SO(2,2)$ is represented as $A = BC$, where

$$B = \begin{bmatrix} b_1 & -b_2 & b_3 & b_4 \\ b_2 & b_1 & -b_4 & b_3 \\ b_3 & -b_4 & b_1 & b_2 \\ b_4 & b_3 & -b_2 & b_1 \end{bmatrix} \qquad \begin{pmatrix} b_1, b_2, b_3, b_4 \in \boldsymbol{R}, \\ b_1^2 + b_2^2 - b_3^2 - b_4^2 = 1 \end{pmatrix}$$

and

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c_{22} & c_{23} & c_{24} \\ 0 & c_{32} & c_{33} & c_{34} \\ 0 & c_{42} & c_{43} & c_{44} \end{bmatrix} \qquad ([c_{ij}] \in SO(1,2)).$$

Suppose $C = I_4$. Then $A = B$ satisfies (b) in Proposition 2.1, $P = P^\times$ and $|P| = 1$. Suppose $B = I_4$. Then $A = C$ satisfies (b) in Proposition 2.1 if and only if $[c_{ij}] \in SO(1,2)$ satisfies

$$\begin{bmatrix} c_{22} & c_{23} & c_{24} \\ c_{32} & c_{33} & c_{34} \\ c_{42} & c_{43} & c_{44} \end{bmatrix} = \begin{bmatrix} \varepsilon(\cosh t + e^{-t}c^2/2) & c & \varepsilon(\sinh t - e^{-t}c^2/2) \\ \varepsilon e^{-t}c & 1 & -\varepsilon e^{-t}c \\ \varepsilon(\sinh t + e^{-t}c^2/2) & c & \varepsilon(\cosh t - e^{-t}c^2/2) \end{bmatrix}, \tag{2.2}$$

where $\varepsilon = +1$ or -1, and c, $t \in \boldsymbol{R}$. Suppose that $[c_{ij}] \in SO(1,2)$ satisfies (2.2). Then we have

$$P = \begin{bmatrix} 1 & 0 \\ -c & \varepsilon e^t \end{bmatrix}, \quad P^\times = \begin{bmatrix} \varepsilon e^{-t} & 0 \\ -\varepsilon e^{-t}c & 1 \end{bmatrix}.$$

Therefore the following are mutually equivalent: (i) $\varepsilon = 1$ and $t = 0$; (ii) $P = P^\times$; (iii) $|P| = |P^\times| = 1$. In particular, if $P = P^\times$, then $|P| = 1$. In general, if $A = BC$ satisfies not only one of (a)$\sim$(d) in Proposition 2.1 but also $P = P^\times$, then noticing Proposition 2.2, we see that both B and C satisfy these conditions. Therefore we obtain $|P| = 1$ for T given by $A = BC$. Hence we obtain (b) in Proposition 2.4 in the case of $n = 1$.

Part 2 Suppose $n \geq 2$. We set

$$H := \left\{ \begin{bmatrix} A_{11} & -A_{21} & A_{31} & A_{41} \\ A_{21} & A_{11} & -A_{41} & A_{31} \\ A_{31} & -A_{41} & A_{11} & A_{21} \\ A_{41} & A_{31} & -A_{21} & A_{11} \end{bmatrix} \in SO(2n, 2n) \right\}. \tag{2.3}$$

Then H is a Lie subgroup of G with $\dim H = 2n^2 + n$. Since there exists a path in H from the unit element to each element of H, any element of H satisfies $|P| = 1$. For each element A of G, there exists an element B of H such that $C := AB$ satisfies

$$C_{11} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & * & \cdots & * \end{bmatrix}, \ C_{1j} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ * & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \cdots & * \end{bmatrix} \ (j = 2, 3, 4),$$

$$C_{i1} = \begin{bmatrix} 0 & * & \cdots & * \\ 0 & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & * & \cdots & * \end{bmatrix} \ (i = 2, 3, 4).$$

In addition, noticing $A\Lambda_n = \Lambda_n A$ and $A \in SO(2n, 2n)$, we see that C satisfies

$$C_{ii} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & * & \cdots & * \end{bmatrix} \ (i = 2, 3, 4), \quad C_{ij} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & * & \cdots & * \end{bmatrix} \ (i \neq j).$$

Therefore noticing Part 1, we see by induction that A satisfies $|P| = 1$. Hence we obtain (b) in Proposition 2.4 in the case of $n \geq 2$. □

REMARK 2.5. Suppose $n \geq 2$ and that $A \in SO(2n, 2n)$ satisfies one of (a)∼(d) in Proposition 2.1. Then even if $|P| = 1$, it is possible that $P \neq P^{\times}$. For example, if X, Y are distinct elements of $SO(n)$, then

$$\begin{bmatrix} X & O_n & O_n & O_n \\ O_n & Y & O_n & O_n \\ O_n & O_n & X & O_n \\ O_n & O_n & O_n & Y \end{bmatrix}$$

satisfies one of (a)∼(d) in Proposition 2.1 and $|P| = 1$ but does not satisfy $P \neq P^{\times}$.

REMARK 2.6. We set

$$\tilde{G} := \{A \in GL(4n, \boldsymbol{R}) \mid A\Lambda_n = \Lambda_n A\}.$$

Then $\tilde{G}$ is a Lie subgroup of $GL(4n, \boldsymbol{R})$ with $\dim \tilde{G} = 8n^2$ and G is a Lie subgroup of $\tilde{G}$. Referring to the proof of (a) in Proposition 2.4, we see that $\tilde{G}$ is contained in the connected component $GL_0(4n, \boldsymbol{R})$ of $GL(4n, \boldsymbol{R})$ with the unit element. Since each $A \in \tilde{G}$ gives an automorphism of W, we obtain a homomorphism ψ from $\tilde{G}$ to $GL(2n, \boldsymbol{R})$. Since $\dim \operatorname{Ker} \psi = 4n^2$, ψ is a surjective homomorphism from the connected component of $\tilde{G}$ with the unit element onto $GL_0(2n, \boldsymbol{R})$. In addition, if A is an element of $\tilde{G}$ given by

$$A = \begin{bmatrix} I_n & O_n & O_n & O_n \\ O_n & I_n & O_n & -2Z_n \\ -2Z_n & O_n & I_{n,-} & O_n \\ O_n & O_n & O_n & I_{n,-} \end{bmatrix},$$

where

$$I_{1,-} := -1, \quad I_{n,-} := \begin{bmatrix} -1 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix} \ (n \geq 2) \tag{2.4}$$

and

$$Z_1 := 1, \quad Z_n := \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \ (n \geq 2),$$

then P satisfies $P_{11} = I_n$, $P_{22} = I_{n,-}$ and $P_{12} = P_{21} = O_n$. Therefore $\psi : \tilde{G} \longrightarrow GL(2n, \boldsymbol{R})$ is surjective. We see from (b) in Proposition 2.4 that $\operatorname{Im} \psi|_G$ is contained in $SL(2n, \boldsymbol{R})$, and since $\dim \operatorname{Ker} \psi|_G = 2n^2 - n$, $\psi|_G : G \longrightarrow SL(2n, \boldsymbol{R})$ is not surjective if $n \geq 2$.

3. Nilpotent structures

Let M be a manifold and E an oriented vector bundle over M of rank $4n$. Let h be a neutral metric of E. Let N be a section of $\operatorname{End} E$. We say that N is a *nilpotent structure* of E if on a neighborhood of each point of M, there exists an ordered pseudo-orthonormal local frame field $e = (e_1, \dots, e_{2n}, e_{2n+1}, \dots, e_{4n})$ of E satisfying $Ne = e\Lambda_n$ and that $e_1, \dots, e_{2n}$ (respectively, $e_{2n+1}, \dots, e_{4n}$) are space-like (respectively, time-like). Let N be a nilpotent structure of E. Then we have

(i) $\operatorname{Im} N = \operatorname{Ker} N$,

(ii) $\pi_N := \operatorname{Im} N = \operatorname{Ker} N$ is a subbundle of E of rank $2n$ such that each fiber is light-like,

(iii) $h(\phi, N\phi) = 0$ for any local section ϕ of E.

In particular, N gives a null structure on each fiber of E and h is null-Hermitian with respect to N (see [**17**]).

Let N be a section of $\operatorname{End} E$ satisfying (i), (ii), (iii) in the previous paragraph. Then there exist local sections $\xi_1, \dots, \xi_{2n}$ of π_N which form a local frame field of π_N and local sections $\xi'_1, \dots, \xi'_{2n}$ of E satisfying

(i) $h(\xi'_i, \xi'_j) = 0$ for $i, j = 1, \dots, 2n$,

(ii) $h(\xi_i, \xi'_j) = 0$ for $i, j = 1, \dots, 2n$ with $i \neq j$,

(iii) $N(\xi'_i) = \xi_{n+i}$, $N(\xi'_{n+i}) = \xi_i$ for $i = 1, \dots, n$,

(iv) for $i = 1, \dots, n$, $h(\xi_i, \xi'_i) = 1$, that is, $h(\xi_{n+i}, \xi'_{n+i}) = -1$.

For $i = 1, \dots, n$, we set

$$e_i := \frac{1}{2}(2\xi'_i + \xi_i), \qquad e_{n+i} := \frac{1}{2}(-2\xi'_{n+i} + \xi_{n+i}),$$
$$e_{2n+i} := \frac{1}{2}(2\xi'_i - \xi_i), \quad e_{3n+i} := \frac{1}{2}(2\xi'_{n+i} + \xi_{n+i}).$$

Then $(e_1, \dots, e_{2n}, e_{2n+1}, \dots, e_{4n})$ is an ordered pseudo-orthonormal local frame field of E such that $e_1, \dots, e_{2n}$ (respectively, $e_{2n+1}, \dots, e_{4n}$) are space-like (respectively, time-like). From $N(\xi'_i) = \xi_{n+i}$, $N(\xi'_{n+i}) = \xi_i$, we see that N is a nilpotent structure of E. Let S be an element of $GL(4n, \boldsymbol{R})$ satisfying

$$(e_1, \dots, e_{4n}) = (\xi'_1, \dots, \xi'_{2n}, \xi_1, \dots, \xi_{2n})S.$$

Then $\det S > 0$. Therefore $(\xi'_1, \dots, \xi'_{2n}, \xi_1, \dots, \xi_{2n})$ gives the orientation of E if and only if $(e_1, \dots, e_{4n})$ gives the orientation of E.

Let N be a nilpotent structure of E. For $\varepsilon \in \{+, -\}$, we set

$$I'_{4n,\varepsilon} := \begin{bmatrix} I_n & O_n & O_n & O_n \\ O_n & I_n & O_n & O_n \\ O_n & O_n & I_n & O_n \\ O_n & O_n & O_n & I_{n,\varepsilon} \end{bmatrix},$$

where $I_{n,+} := I_n$ and $I_{n,-}$ is as in (2.4). For $\varepsilon \in \{+, -\}$, we say that N is an *ε-nilpotent structure* if we can choose an ordered pseudo-orthonormal local frame field e of E on a neighborhood of each point of M giving the orientation of E and satisfying $NeI'_{4n,\varepsilon} = eI'_{4n,\varepsilon}\Lambda_n$. Let N be an ε-nilpotent structure of E. Then such a frame field as e for N is called an *admissible frame field* of N. For an admissible frame field e of N,

$$\begin{aligned} &\xi_1 := e_1 - e_{2n+1}, & &\xi_i := e_i - e_{2n+i}, \\ &\xi_{n+1} := e_{n+1} + \varepsilon e_{3n+1}, & &\xi_{n+i} := e_{n+i} + e_{3n+i} \end{aligned} \quad (i = 2, \dots, n) \tag{3.1}$$

form a local frame field of π_N ((3.1) with $\varepsilon = +$ coincides with (2.1)). Let $f = (f_1, \dots, f_{2n}, f_{2n+1}, \dots, f_{4n})$ be an ordered pseudo-orthonormal local frame field of E giving the orientation. Then f is an admissible frame field of N if and only if for each admissible frame field e of N, an $SO(2n, 2n)$-valued function A on the intersection of the domains of e and f given by $fI'_{4n,\varepsilon} = eI'_{4n,\varepsilon}A$ is valued in G. Therefore we see by (b) in Proposition 2.4 that for $\xi_1, \dots, \xi_{2n}$ as in (3.1), $\xi = \xi_1 \wedge \dots \wedge \xi_{2n}$ does not depend on the choice of an admissible frame field of N and this means that N gives a section ξ of $\bigwedge^{2n} E$.

Let ξ be a section of $\bigwedge^{2n} E$ and suppose that ξ is locally represented as $\xi = \xi_1 \wedge \dots \wedge \xi_{2n}$ for local light-like sections $\xi_1, \dots, \xi_{2n}$ of E defined on a neighborhood U of each point of M such that at each point of U, they span a light-like $2n$-dimensional subspace of the fiber of E. Then $\xi_1, \dots, \xi_{2n}$ are represented as in (3.1) for $\varepsilon \in \{+, -\}$ and an ordered pseudo-orthonormal local frame field $e = (e_1, \dots, e_{2n}, e_{2n+1}, \dots, e_{4n})$ of E giving the orientation of E. Suppose that n is even. Then noticing that there exists $A \in SO(2n, 2n)$ satisfying one of (a)$\sim$(d) in Proposition 2.1, $|P| = 1$ and $P \neq P^{\times}$

(Remark 2.5), we see that ξ gives plural ε-nilpotent structures of the restriction of E on a neighborhood of each point of M. Suppose that n is odd. Then ξ gives at least one ε-nilpotent structure as above and we see in particular that $n = 1$ if and only if ξ determines a unique ε-nilpotent structure of E. See [**3**], [**8**] for the case of $n = 1$.

In the rest of this section, suppose $n = 1$. Let $\hat{h}$ be the metric of $\bigwedge^2 E$ induced by h. Then $\hat{h}$ has signature $(2, 4)$. Noticing the double covering $SO_0(2,2) \longrightarrow SO_0(1,2) \times SO_0(1,2)$, we see that $\bigwedge^2 E$ is represented as the direct sum of its two orientable subbundles $\bigwedge^2_+ E$, $\bigwedge^2_- E$ of rank 3. We see that $\bigwedge^2_+ E$ is orthogonal to $\bigwedge^2_- E$ with respect to $\hat{h}$ and that the restriction of $\hat{h}$ on each of $\bigwedge^2_+ E$, $\bigwedge^2_- E$ has signature $(1, 2)$. Let (e_1, e_2, e_3, e_4) be an ordered pseudo-orthonormal local frame field of E giving the orientation of E. We set

$$\begin{aligned} \Omega_{\pm,1} &:= \frac{1}{\sqrt{2}}(e_1 \wedge e_2 \pm e_3 \wedge e_4), \\ \Omega_{\pm,2} &:= \frac{1}{\sqrt{2}}(e_1 \wedge e_3 \pm e_4 \wedge e_2), \\ \Omega_{\pm,3} &:= \frac{1}{\sqrt{2}}(e_1 \wedge e_4 \pm e_2 \wedge e_3). \end{aligned} \tag{3.2}$$

Then $\Omega_{\pm,1}$ are space-like and $\Omega_{\pm,2}$, $\Omega_{\pm,3}$ are time-like, and we can suppose that $\Omega_{-,1}$, $\Omega_{+,2}$, $\Omega_{+,3}$ (respectively, $\Omega_{+,1}$, $\Omega_{-,2}$, $\Omega_{-,3}$) form a pseudo-orthonormal local frame field of $\bigwedge^2_+ E$ (respectively, $\bigwedge^2_- E$). The *light-like twistor spaces* associated with E are fiber bundles

$$U_0\left(\bigwedge\nolimits^2_\pm E\right) := \left\{ \theta \in \bigwedge\nolimits^2_\pm E \setminus \{0\} \;\middle|\; \hat{h}(\theta, \theta) = 0 \right\}$$

in $\bigwedge^2_\pm E$ respectively. We can refer to [**3**], [**8**] for the light-like twistor spaces. Let Ω be a section of $U_0\left(\bigwedge^2_\varepsilon E\right)$ ($\varepsilon \in \{+, -\}$). Then we can find (e_1, e_2, e_3, e_4) as above satisfying

$$\Omega = \Omega_{-\varepsilon,1} + \varepsilon \Omega_{\varepsilon,3}.$$

Therefore Ω is locally represented as $\Omega = (1/\sqrt{2})\xi_1 \wedge \xi_2$. Then Ω gives a unique ε-nilpotent structure N of E. It satisfies

$$(Ne_1, Ne_2, Ne_3, N\varepsilon e_4) = (e_1, e_2, e_3, \varepsilon e_4)\Lambda_1, \tag{3.3}$$

that is, (e_1, e_2, e_3, e_4) is an admissible frame field of N. Notice that in (3.3), ε is put before e_4, while in the corresponding equation in [**3**], ε is put before e_3. Each ε-nilpotent structure of E gives a unique section of $U_0\left(\bigwedge^2_\varepsilon E\right)$ and therefore there exists a one-to-one correspondence between the set of sections of $U_0\left(\bigwedge^2_\varepsilon E\right)$ and the set of ε-nilpotent structures of E.

4. The covariant derivatives of paracomplex structures

Let E, h be as in the beginning of the previous section. Let ∇ be an h-connection of E, i.e., a connection of E satisfying $\nabla h = 0$. Let J be a section of $\mathrm{End}\, E$. For $\varepsilon \in \{+, -\}$, we say that J is an *ε-paracomplex structure* of E if J satisfies

(i) J is a paracomplex structure of E,

(ii) J is h-reversing, that is, $J^*h = -h$,

(iii) there exists an ordered pseudo-orthonormal local frame field $e = (e_1, \dots e_{2n}, e_{2n+1}, \dots, e_{4n})$ of E on a neighborhood of each point of M giving the orientation and satisfying

$$\begin{aligned}&(Je_1, \dots, Je_{2n}, Je_{2n+1}, \dots, Je_{4n})I'_{4n,\varepsilon} \\ &= (e_1, \dots, e_{2n}, e_{2n+1}, \dots, e_{4n})I'_{4n,\varepsilon} \begin{bmatrix} O_n & O_n & I_n & O_n \\ O_n & O_n & O_n & -I_n \\ I_n & O_n & O_n & O_n \\ O_n & -I_n & O_n & O_n \end{bmatrix}.\end{aligned} \tag{4.1}$$

Let J be an ε-paracomplex structure of E. Then such a frame field as e is called an *admissible frame field* of J. For an admissible frame field e of J, let $\omega = [\omega^i_j]$ be the connection form of ∇ with respect to e: $\nabla e = e\omega$. Then we have

$$\omega^i_j = \begin{cases} -\omega^j_i & ((i,j) \in (\Sigma_{1\sim 2n} \times \Sigma_{1\sim 2n}) \cup (\Sigma_{2n+1\sim 4n} \times \Sigma_{2n+1\sim 4n})), \\ \omega^j_i & ((i,j) \in (\Sigma_{1\sim 2n} \times \Sigma_{2n+1\sim 4n}) \cup (\Sigma_{2n+1\sim 4n} \times \Sigma_{1\sim 2n})), \end{cases} \tag{4.2}$$

where $\Sigma_{1\sim 2n} := \{1, \dots, 2n\}$, $\Sigma_{2n+1\sim 4n} := \{2n+1, \dots, 4n\}$. For $\mu \in \{+, -\}$, we set

$$e(\mu) := (e_1, \dots, e_{2n}, \mu e_{2n+1}, \dots, \mu e_{4n}).$$

Suppose that ∇J is locally represented as the tensor product of a 1-form α and an ε-nilpotent structure N so that $e(\mu)$ is an admissible frame field of N. In the following, we say that such an ε-nilpotent structure as N is *related to J (by (e, μ))*. Then using $(\nabla J)(e_k) = \nabla(Je_k) - J(\nabla e_k)$, we obtain

$$\begin{aligned}&\mu(\omega^{n+i}_j + (\varepsilon 1)^{\delta_{i1}} \omega^{3n+i}_{2n+j}) = \omega^{n+i}_{2n+j} + (\varepsilon 1)^{\delta_{i1}} \omega^{3n+i}_j = \alpha \delta^i_j, \\ &\omega^i_j - \omega^{2n+i}_{2n+j} = \omega^i_{2n+j} - \omega^{2n+i}_j = 0, \\ &\omega^{n+i}_{n+j} - (\varepsilon 1)^{\delta_{i1}+\delta_{j1}} \omega^{3n+i}_{3n+j} = \omega^{n+i}_{3n+j} - (\varepsilon 1)^{\delta_{i1}+\delta_{j1}} \omega^{3n+i}_{n+j} = 0\end{aligned} \tag{4.3}$$

for $i, j \in \{1, \dots, n\}$. Conversely, if we suppose (4.3), then ∇J is locally represented as above. Therefore we obtain

THEOREM 4.1. *Let J be an ε-paracomplex structure of E for $\varepsilon \in \{+, -\}$ and e an admissible frame field of J. Then for $\mu \in \{+, -\}$, the following are equivalent*:

(a) *∇J is locally represented by the tensor product of a 1-form α and an ε-nilpotent structure N related to J by (e, μ)*;

(b) *the connection form* $\omega = [\omega^i_j]$ *of* ∇ *with respect to* e *satisfies* (4.3).

REMARK 4.2. The main object of study in the present paper is an ε-paracomplex structure J of E such that ∇J is locally represented by the tensor product of a 1-form and an ε-nilpotent structure related to J. In [**6**], the author studies nilpotent structures of an oriented vector bundle E of rank $4n$ with a neutral metric h and an h-connection ∇. For a Lie subgroup K of $SO(2n, 2n)$, K-nilpotent structures of (E, h, ∇) are defined. For each K-nilpotent structure, a principal K-bundle P is constructed, by choosing special admissible frame fields. In addition, ∇ gives a connection in P, so that the connection form of ∇ with respect to such an admissible frame field is valued in the Lie algebra of K. A G-nilpotent structure N of (E, h, ∇) is characterized by $\nabla N = 0$ ([**6**]). If there exist a complex structure I and paracomplex structures J_1, J_2 such that h, ∇, I, J_1, J_2 form a neutral hyperKähler structure of E, then there exist H-nilpotent structures of (E, h, ∇) ([**6**]), where H is as in (2.3). In addition, if there exists an H-nilpotent structure of (E, h, ∇), then N has the dual H-nilpotent structure $N^\times$, and h, ∇, $I := (1/2)(N + N^\times)$, $J_1 := -IJ_2$, $J_2 := (1/2)(N - N^\times)$ form a neutral hyperKähler structure of E ([**6**]).

In the following, suppose $n = 1$. Let $\hat{\nabla}$ be the connection of $\bigwedge^2 E$ induced by ∇. Then $\hat{\nabla}$ is an $\hat{h}$-connection and it gives connections of $\bigwedge^2_+ E$, $\bigwedge^2_- E$. Fiber bundles

$$U_-\left(\textstyle\bigwedge^2_\pm E\right) := \left\{\theta \in \textstyle\bigwedge^2_\pm E \;\middle|\; \hat{h}(\theta, \theta) = -1\right\}$$

in $\bigwedge^2_\pm E$ respectively are the time-like twistor spaces associated with E. There exists a one-to-one correspondence between the set of sections of $U_-\left(\bigwedge^2_\varepsilon E\right)$ and the set of ε-paracomplex structures of E. Let Ω be a section of $U_-\left(\bigwedge^2_\varepsilon E\right)$ for $\varepsilon \in \{+, -\}$. Then we can find $e = (e_1, e_2, e_3, e_4)$ satisfying $\Omega = \Omega_{\varepsilon,2}$. Let $\omega = [\omega^i_j]$ be the connection form of ∇ with respect to e. Then we have

$$\omega^i_j = \begin{cases} 0 & (i = j), \\ -\omega^j_i & (\{i, j\} \in \{\{1, 2\}, \{3, 4\}\}), \\ \omega^j_i & (\{i, j\} \in \{\{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}\}). \end{cases} \tag{4.4}$$

Suppose that the covariant derivative of Ω is fully light-like. Then for $\mu = +$ or $-$, we obtain

$$\omega^3_2 + \varepsilon\omega^4_1 = \mu(\omega^4_3 + \varepsilon\omega^2_1) \tag{4.5}$$

and $\hat{\nabla}\Omega = \alpha \otimes \Omega_0$, where

$$\alpha := \omega^3_2 + \varepsilon\omega^4_1, \quad \Omega_0 := \Omega_{-\varepsilon,1} + \mu\Omega_{\varepsilon,3}.$$

We have $\alpha \neq 0$ and Ω_0 is a local section of a light-like twistor space $U_0\left(\bigwedge^2_\varepsilon E\right)$. Let J be an ε-paracomplex structure of E corresponding to Ω. Then we obtain $Je_1 = e_3$, $Je_2 = -\varepsilon e_4$ and therefore e is an admissible frame field of J. In addition, we obtain

$$\nabla J = \alpha \otimes (\theta^1 \otimes \xi_2 - \theta^2 \otimes \xi_1), \tag{4.6}$$

where

$$\begin{aligned} \xi_1 &:= e_1 - \varepsilon\mu e_3, \quad \xi_2 := e_2 + \mu e_4, \\ \theta^1 &:= e^1 + \varepsilon\mu e^3, \quad \theta^2 := e^2 - \mu e^4 \end{aligned} \tag{4.7}$$

and (e^1, e^2, e^3, e^4) is the dual frame field of (e_1, e_2, e_3, e_4). We see that $N := \theta^1 \otimes \xi_2 - \theta^2 \otimes \xi_1$ is an ε-nilpotent structure corresponding to a local light-like section Ω_0 and related to J by $(e, \varepsilon\mu)$. Noticing that $e = (e_1, e_2, e_3, e_4)$ satisfying $\Omega = \Omega_{\varepsilon,2}$ is not unique, we see that Ω_0 and N are not uniquely determined by Ω. However, Ω_0 is uniquely determined by Ω up to a nowhere zero function. Therefore E has a light-like subbundle of rank 2 given by Ω and locally generated by ξ_1, ξ_2. Hence we obtain

Proposition 4.3. *Let Ω be a section of $U_-\left(\bigwedge^2_\varepsilon E\right)$ for $\varepsilon \in \{+, -\}$ such that $\hat{\nabla}\Omega$ is fully light-like. Then $\hat{\nabla}\Omega$ is locally represented as $\hat{\nabla}\Omega = \alpha \otimes \Omega_0$ for a nowhere zero 1-form α and a light-like section Ω_0. In addition, if J is an ε-paracomplex structure of E corresponding to Ω, then ∇J is locally represented as $\nabla J = \alpha \otimes N$, where N is an ε-nilpotent structure corresponding to Ω_0 and related to J.*

In the next section, we will see that the lifts of the conformal Gauss maps of time-like minimal surfaces in E^3_1 are examples of sections of the time-like twistor spaces with fully light-like covariant derivatives.

Let J be an ε-paracomplex structure of E for $\varepsilon \in \{+, -\}$. Suppose that the covariant derivative of a section Ω of $U_-\left(\bigwedge^2_\varepsilon E\right)$ corresponding to J is fully light-like. Then whether we can find an ordered pseudo-orthonormal local frame field $e = (e_1, e_2, e_3, e_4)$ of E as above for Ω such that Ω_0 is horizontal is determined by J. Referring to [**3**], we see that Ω_0 is horizontal if and only if the corresponding nilpotent structure N satisfies $\nabla N = 0$. The following proposition will be used in Section 9.

Proposition 4.4. *The following two conditions are equivalent to each other:*

(a) *there exists an ordered pseudo-orthonormal local frame field e of E as above for Ω such that Ω_0 is horizontal*;

(b) *there exists an ordered pseudo-orthonormal local frame field e of E as above for Ω such that the connection form $[\omega^i_j]$ of ∇ with respect to e satisfies $d(\omega^4_2 - \varepsilon\omega^3_1) = 0$.*

Proof. Suppose (a). Then we see that Ω_0 is horizontal for an ordered pseudo-orthonormal local frame field e. Using (4.5), we obtain

$$\sqrt{2}\hat{\nabla}\Omega_0 = \hat{\nabla}(\xi_1 \wedge \xi_2) = \mu(\omega^4_2 - \varepsilon\omega^3_1) \otimes \xi_1 \wedge \xi_2. \tag{4.8}$$

Therefore $\hat{\nabla}\Omega_0 = 0$ means $\omega^4_2 - \varepsilon\omega^3_1 = 0$. Hence e is a suitable frame field for Condition (b). Suppose (b) and let e be as in (b). Then there exists a function f satisfying $\omega^4_2 - \varepsilon\omega^3_1 = -df$. Using this and (4.8), we obtain $\hat{\nabla}(e^{\mu f}\xi_1 \wedge \xi_2) = 0$. We set

$$e'_1 := e_1,\ \ e'_2 := (\cosh f)e_2 + (\sinh f)e_4,$$
$$e'_3 := e_3,\ \ e'_4 := (\sinh f)e_2 + (\cosh f)e_4.$$

Then $e' = (e'_1, e'_2, e'_3, e'_4)$ is an ordered pseudo-orthonormal local frame field satisfying

(i) e'_1, e'_2 are space-like and e'_3, e'_4 are time-like,

(ii) e' gives the orientation of E,

(iii) $\Omega = \Omega'_{\varepsilon,2}$, where $\Omega'_{\pm,2}$ are defined for e' as in (3.2),

(iv) Ω_0 for e' is horizontal.

Hence $e' = (e'_1, e'_2, e'_3, e'_4)$ is a suitable frame field for Condition (a). □

5. The conformal Gauss maps of time-like minimal surfaces in E^3_1

Let M be a Lorentz surface. Let $\iota : M \longrightarrow E^3_1$ be a time-like and conformal immersion of M into E^3_1. Let $\langle\ ,\ \rangle$ be the metric of E^5_2 and set

$$L := \{x = (x^1, x^2, x^3, x^4, x^5) \in E^5_2 \setminus \{0\} \mid \langle x, x\rangle = 0\}.$$

We identify E^3_1 with $L \cap \{x^5 = x^1 + 1\}$ and therefore we consider ι to be an L-valued function. Suppose that ι is minimal. Let γ be the unit normal vector field of ι determined by the orientations of M and E^3_1. Then γ is a map from M into S^4_2. Let g^M be the induced metric of M by ι and K^M the curvature of g^M. Let $\mathrm{Reg}\,(\iota)$ be the set of nonzero points of K^M. Then the restriction of γ on $\mathrm{Reg}\,(\iota)$ is a time-like and conformal immersion which induces a Lorentz metric g given by $g = -K^M g^M$. We call $\gamma : M \longrightarrow S^4_2$ the *conformal Gauss map* of ι. We can refer to [**7**] for the conformal Gauss maps of time-like minimal surfaces in E^3_1.

Let $w = u + jv$ be a local paracomplex coordinate of $\mathrm{Reg}\,(\iota)$. Then we represent g^M as $g^M = e^{2\lambda} dw d\overline{w}$ and we see that there exist functions l, m satisfying

$$\iota_{uu} = \lambda_u \iota_u + \lambda_v \iota_v + l\gamma, \quad \iota_{uv} = \lambda_v \iota_u + \lambda_u \iota_v + m\gamma \tag{5.1}$$

and $\iota_{vv} = \iota_{uu}$. By (5.1), we obtain

$$(\gamma_u\ \gamma_v) = (\iota_u\ \iota_v)\frac{1}{e^{2\lambda}}\begin{bmatrix} -l & -m \\ m & l \end{bmatrix}. \tag{5.2}$$

Therefore we can consider ι to be a light-like normal vector field of $F := \gamma|_{\mathrm{Reg}\,(\iota)}$. Since $K^M \neq 0$ on $\mathrm{Reg}\,(\iota)$, we have $l^2 \neq m^2$. Let A_ι be the shape operator of F with respect to ι. Then from (5.2), we obtain

$$\begin{aligned} &\left(A_\iota\left(\frac{\partial}{\partial u}\right)\ A_\iota\left(\frac{\partial}{\partial v}\right)\right) \\ &= \left(dF\left(\frac{\partial}{\partial u}\right)\ dF\left(\frac{\partial}{\partial v}\right)\right)\frac{e^{2\lambda}}{l^2 - m^2}\begin{bmatrix} l & m \\ -m & -l \end{bmatrix}. \end{aligned} \tag{5.3}$$

Let ν be a light-like normal vector field of F satisfying $\langle \nu, \iota\rangle = -1$.

Proposition 5.1. *The shape operator A_ν of F with respect to ν vanishes.*

Proof. We can prove Proposition 5.1, referring to [**2**]. Since ι is minimal, a paracomplex quartic differential $\tilde{Q}$ defined on M vanishes. This means that a paracomplex quartic differential Q defined on $\mathrm{Reg}\,(\iota)$ and given by F vanishes. Therefore ν is contained in a constant direction in E^5_2. This means that A_ν vanishes. □

From (5.3) and Proposition 5.1, we see that F has zero mean curvature vector. This is also seen from a fact that a time-like minimal immersion ι is of Willmore type ([**2**]).

Suppose $K^M < 0$, i.e., $l^2 > m^2$. Then F_u is space-like and F_v is time-like. We set

$$e_1 := \frac{1}{\sqrt{-K^M e^\lambda}} F_u, \quad e_3 := \frac{1}{\sqrt{-K^M e^\lambda}} F_v.$$

Let e_2, e_4 be normal vector fields of F satisfying

$$\langle e_2, e_2\rangle = 1,\ \langle e_4, e_4\rangle = -1,\ \iota = \frac{1}{\sqrt{2}}(e_4 - e_2),\ \nu = \frac{1}{\sqrt{2}}(e_4 + e_2).$$

Then $e = (e_1, e_2, e_3, e_4)$ is an ordered pseudo-orthonormal local frame field of the pull-back bundle $E := F^*TS^4_2$ by $F : \mathrm{Reg}\,(\iota) \longrightarrow S^4_2$ and we can suppose that $e = (e_1, e_2, e_3, e_4)$ gives the orientation of S^4_2. Let (e^1, e^3) be the dual frame field of an ordered pseudo-orthonormal local frame field (e_1, e_3) of the tangent bundle TM. Let h, ∇ be the metric and the Levi-Civita connection of S^4_2. Then they naturally give a metric and an h-connection of E and these are also denoted by h, ∇. Let $\omega = [\omega^i_j]$ be the connection form of ∇ with respect to e. Then we obtain

$$\omega^2_1 = \omega^4_1 = -(\tilde{l}e^1 + \tilde{m}e^3), \quad \omega^3_2 = \omega^4_3 = -(\tilde{m}e^1 + \tilde{l}e^3) \tag{5.4}$$

with

$$\tilde{l} := \frac{le^{2\lambda}}{\sqrt{2}(l^2 - m^2)}, \quad \tilde{m} := \frac{me^{2\lambda}}{\sqrt{2}(l^2 - m^2)}.$$

Let $\hat{\nabla}$ be the connection of $\bigwedge^2 E$ induced by ∇. Then by (5.4), we obtain $\hat{\nabla}\Omega_{\varepsilon,2} = \alpha \otimes \Omega_0$ for $\varepsilon \in \{+, -\}$, where

$$\alpha := -(\varepsilon\tilde{l} + \tilde{m})(e^1 + \varepsilon e^3), \quad \Omega_0 := \Omega_{-\varepsilon,1} + \Omega_{\varepsilon,3}$$

and $\Omega_{\pm,1}$, $\Omega_{\pm,2}$, $\Omega_{\pm,3}$ are as in (3.2). We have $\alpha \neq 0$. Let $\hat{F}_\pm$ be the lifts of F. Then $\hat{F}_\pm$ are sections of $U_-\left(\bigwedge^2_\pm E\right)$ and locally represented as $\hat{F}_\pm = \Omega_{\pm,2}$ respectively. Let J_ε be an ε-paracomplex structure of E corresponding to $\hat{F}_\varepsilon$. Then we have $J_\varepsilon e_1 = e_3$, $J_\varepsilon e_2 = -\varepsilon e_4$. For $J = J_\varepsilon$, we obtain (4.6) with (4.7) and $\mu = +$. We see from (4.6) for $J = J_\varepsilon$ that ∇J_ε is valued in a light-like subbundle of $E = F^*TS^4_2$ of rank 2 which is locally generated by $e_1 - \varepsilon e_3$, $e_2 + e_4$. Hence we obtain

Theorem 5.2. *The covariant derivatives of the lifts $\hat{F}_{\pm}$ are fully light-like. In addition, for an ε-paracomplex structure J_ε corresponding to $\hat{F}_\varepsilon$, ∇J_ε is locally represented as $\nabla J_\varepsilon = \alpha \otimes N$, where $\alpha := -(\varepsilon\tilde{l} + \tilde{m})\theta^1$, $N := \theta^1 \otimes \xi_2 - \theta^2 \otimes \xi_1$ with (4.7) and $\mu = +$.*

Let M be a Lorentz surface and $F : M \longrightarrow S^4_2$ a time-like and conformal immersion with zero mean curvature vector. According to [**4**], if the covariant derivatives of the lifts $\hat{F}_{\pm}$ are fully light-like, then F has one of the following two properties:

(i) the shape operator of F with respect to a light-like normal vector field vanishes;

(ii) the shape operator of F with respect to any normal vector field is light-like or zero.

As was already seen by Proposition 5.1, the conformal Gauss maps of time-like minimal surfaces in E^3_1 satisfy (i). We can find a characterization of a time-like and conformal immersion $F : M \longrightarrow S^4_2$ with zero mean curvature vector and (ii) in terms of the Gauss-Codazzi-Ricci equations ([**4**]). The immersion F satisfies either (i) or (ii) if and only if the curvature K of the induced metric on M by F is identically equal to 1 ([**5**]). In addition, $K \equiv 1$ if and only if not only the paracomplex quartic differential Q is null or zero but also the normal connection $\nabla^{\perp}$ of F is flat ([**5**]). Notice that Q is null or zero if and only if at least one of the covariant derivatives of $\hat{F}_{\pm}$ is light-like or zero ([**5**]) and that $K \equiv 1$ means that $\nabla^{\perp}$ is flat, while the converse is not necessarily true ([**5**]).

6. Walker 4*n*-manifolds

Let M be an oriented neutral $4n$-manifold and h its metric. Let ∇ be the Levi-Civita connection of h. Let $\mathcal{D}$ be a light-like $2n$-dimensional distribution on M. We say that $(M, h, \mathcal{D})$ is a *Walker manifold* if the covariant derivatives of local generators of $\mathcal{D}$ by any tangent vector of M with respect to ∇ are contained in $\mathcal{D}$. See [**12**], [**30**] for Walker manifolds. If $(M, h, \mathcal{D})$ is a Walker manifold, then $\mathcal{D}$ is involutive. In the case of $n = 1$, $\mathcal{D}$ is involutive if and only if the covariant derivatives of local generators of $\mathcal{D}$ by any tangent vector in $\mathcal{D}$ are contained in $\mathcal{D}$ (see [**3**]).

Let J be an almost ε-paracomplex structure of M for $\varepsilon \in \{+, -\}$. Suppose that ∇J satisfies (a) in Theorem 4.1 for a nowhere zero 1-form α. Then ∇J gives a light-like subbundle of *TM* of rank $2n$, i.e., a light-like $2n$-dimensional distribution $\mathcal{D}_J$ on M.

Theorem 6.1. *Let J be as above. Then $(M, h, \mathcal{D}_J)$ is a Walker manifold.*

Proof. Let $e = (e_1, \dots e_{2n}, e_{2n+1}, \dots, e_{4n})$ be an admissible frame field of J such that ∇J is locally represented as $\nabla J = \alpha \otimes N$ for an almost ε-nilpotent structure N related to J by (e, μ). Then we have (4.3). Let $\xi_1, \dots, \xi_{2n}$ be as in (3.1) for $e(\mu)$. For $i, j \in \{1, \dots, n\}$, we obtain

$$\begin{aligned}
&h(\nabla\xi_i, \xi_j) = \omega^j_i - \mu\omega^j_{2n+i} + \mu\omega^{2n+j}_i - \omega^{2n+j}_{2n+i},\\
&h(\nabla\xi_i, \xi_{n+j}) = -h(\xi_i, \nabla\xi_{n+j})\\
&= \omega^{n+j}_i - \mu\omega^{n+j}_{2n+i} - \mu(\varepsilon 1)^{\delta_{j1}}\omega^{3n+j}_i + (\varepsilon 1)^{\delta_{j1}}\omega^{3n+j}_{2n+i},\\
&h(\nabla\xi_{n+i}, \xi_{n+j})\\
&= \omega^{n+j}_{n+i} + \mu(\varepsilon 1)^{\delta_{i1}}\omega^{n+j}_{3n+i} - \mu(\varepsilon 1)^{\delta_{j1}}\omega^{3n+j}_{n+i} - (\varepsilon 1)^{\delta_{i1}+\delta_{j1}}\omega^{3n+j}_{3n+i}.
\end{aligned} \tag{6.1}$$

Applying (4.3) to (6.1), we obtain $h(\nabla\xi_i, \xi_j) = 0$ for $i, j = 1, \ldots, 2n$. Therefore $(M, h, \mathcal{D}_J)$ is a Walker manifold. □

REMARK 6.2. Suppose $n = 1$. Let J be an almost ε-paracomplex structure of M such that the covariant derivative of the corresponding section of $U_-\left(\bigwedge^2_\varepsilon TM\right)$ is fully light-like. Then noticing Proposition 4.3, we see from Theorem 6.1 that $(M, h, \mathcal{D}_J)$ is a Walker manifold.

Suppose $n = 1$. Let $\mathcal{D}$ be a light-like two-dimensional distribution on M. Then $\mathcal{D}$ is locally generated by light-like vector fields ξ_1, ξ_2 such that $(1/\sqrt{2})\xi_1 \wedge \xi_2$ is a local section of $U_0\left(\bigwedge^2_\varepsilon TM\right)$ for $\varepsilon = +$ or $-$. Therefore there exists an almost ε-nilpotent structure N of a neighborhood of each point of M such that $\mathcal{D}$ is locally given by π_N. We will prove

THEOREM 6.3. *Suppose $n = 1$. Let $\mathcal{D}$ be a light-like two-dimensional distribution on M such that $(M, h, \mathcal{D})$ is a Walker manifold. Let N be an almost ε-nilpotent structure as above. Then there exists an almost ε-paracomplex structure J of a neighborhood of the point satisfying*

(a) $\nabla J = \alpha \otimes N$ *for a nowhere zero* 1*-form* α,

(b) N *is related to* J.

PROOF. Let $\mathcal{D}$, N be as in Theorem 6.3. Suppose that N corresponds to a local section Ω_0 of $U_0\left(\bigwedge^2_+ TM\right)$. Let (e_1, e_2, e_3, e_4) be an admissible frame field of N. Then $\Omega_0 = \Omega_{-,1} + \Omega_{+,3}$ and $\mathcal{D}$ is locally generated by $\xi_1 = e_1 - e_3$, $\xi_2 = e_2 + e_4$. Since $(M, h, \mathcal{D})$ is a Walker manifold, we have $h(\nabla\xi_1, \xi_2) = 0$ and this is rewritten into $\omega^3_2 + \omega^4_1 = \omega^4_3 + \omega^2_1$. In addition, we can suppose $\omega^3_2 + \omega^4_1 \neq 0$, rechoosing (e_1, e_2, e_3, e_4) if necessary: if we set

$$(\tilde{e}_1, \tilde{e}_2, \tilde{e}_3, \tilde{e}_4) = (e_1, e_2, e_3, e_4) \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{f^2+2}{2} & f & -\frac{f^2}{2} \\ 0 & f & 1 & -f \\ 0 & \frac{f^2}{2} & f & -\frac{f^2-2}{2} \end{bmatrix}$$

for a function f, then

(i) $(\tilde{e}_1, \tilde{e}_2, \tilde{e}_3, \tilde{e}_4)$ is an ordered pseudo-orthonormal local frame field of M giving the orientation of M,

(ii) $(\tilde{e}_1, \tilde{e}_2, \tilde{e}_3, \tilde{e}_4)$ is an admissible frame field of N,

(iii) for the connection form $[\tilde{\omega}^i_j]$ of ∇ with respect to $(\tilde{e}_1, \tilde{e}_2, \tilde{e}_3, \tilde{e}_4)$,

$$\tilde{\omega}^3_2 + \tilde{\omega}^4_1 = df - \frac{f^2 - 2}{2}(\omega^3_2 + \omega^4_1) + \frac{f^2}{2}(\omega^4_3 + \omega^2_1) + f(\omega^4_2 - \omega^3_1).$$

Therefore the covariant derivative of $\Omega = \Omega_{+,2}$ is fully light-like. Let J be an almost $+$-paracomplex structure corresponding to Ω. Then J satisfies (a), (b) in Theorem 6.3. In the case where Ω_0 is a local section of $U_0\left(\bigwedge^2_- TM\right)$, we can obtain the same result. □

Remark 6.4. Let N be an almost ε-nilpotent structure of M satisfying $\nabla N = 0$. Then $(M, h, \mathcal{D})$ is a Walker manifold for a light-like two-dimensional distribution $\mathcal{D}$ on M given by π_N. Therefore there exists an almost ε-paracomplex structure J on a neighborhood of each point of M satisfying (a), (b) in Theorem 6.3.

7. The square norm of the covariant derivative of an h-reversing almost paracomplex structure

Let M, h, ∇ be as in the beginning of the previous section. Let J be an h-reversing almost paracomplex structure of M. Then the square norm $\| \nabla J \|^2$ of ∇J is defined by

$$\| \nabla J \|^2 := \sum_{i,j=1}^{4n} \delta_i \delta_j h((\nabla_{e_i} J)(e_j), (\nabla_{e_i} J)(e_j)),$$

where $e = (e_1, \dots, e_{2n}, e_{2n+1}, \dots, e_{4n})$ is an ordered pseudo-orthonormal local frame field of M giving the orientation and $\delta_k = 1$, $\delta_{2n+k} = -1$ $(k = 1, \dots, 2n)$. The square norm $\| \nabla J \|^2$ does not depend on the choice of e. Let Ω^* be a 2-form on M defined by $\Omega^*(X, Y) := h(X, JY)$ for tangent vectors X, Y of M. Then the square norm $\| \nabla\Omega^* \|^2$ of $\nabla\Omega^*$ is defined by

$$\| \nabla\Omega^* \|^2 := \sum_{i,j,k=1}^{4n} \delta_i \delta_j \delta_k ((\nabla_{e_i} \Omega^*)(e_j, e_k))^2. \tag{7.1}$$

Noticing

$$h((\nabla_{e_i} J)(e_j), e_k) = -(\nabla_{e_i} \Omega^*)(e_j, e_k),$$

we obtain $\| \nabla J \|^2 = \| \nabla\Omega^* \|^2$.

Suppose that J is an almost ε-paracomplex structure of M for $\varepsilon \in \{+, -\}$. Let e be an admissible frame field of J. Then we have

$$\Omega^* = \sum_{i=1}^{n} (e^i \wedge e^{2n+i} - (\varepsilon 1)^{\delta_{i1}} e^{n+i} \wedge e^{3n+i}),$$

where $(e^1, \dots, e^{2n}, e^{2n+1}, \dots, e^{4n})$ is the dual frame field of e and $\varepsilon 1$ denotes 1 or -1 according to $\varepsilon = +$ or $-$. Suppose that ∇J satisfies (a) in Theorem 4.1. Then we have (4.3). Therefore we see

(i) $((\nabla_{e_i} \Omega^*)(e_j, e_k))^2 = \alpha(e_i)^2$ if

$$\{j, k\} \in \{\{l, n+l\}, \{l, 3n+l\}, \{n+l, 2n+l\}, \{2n+l, 3n+l\}\} \tag{7.2}$$

for $l \in \{1, \dots, n\}$,

(ii) $((\nabla_{e_i} \Omega^*)(e_j, e_k))^2 = 0$ if $\{j, k\}$ does not satisfy (7.2) for any $l \in \{1, \dots, n\}$.

Therefore noticing

$$\delta_j\delta_k = \begin{cases} 1 & (\{j,k\} \in \bigcup_{l=1}^{n}\{\{l, n+l\}, \{2n+l, 3n+l\}\}), \\ -1 & (\{j,k\} \in \bigcup_{l=1}^{n}\{\{l, 3n+l\}, \{n+l, 2n+l\}\}), \end{cases}$$

we see by (7.1) that $\| \nabla\Omega^* \|^2 = 0$. This means $\| \nabla J \|^2 = 0$.

Hence we obtain

THEOREM 7.1. *Let J be an almost ε-paracomplex structure of M such that ∇J is locally represented as in* (a) *in Theorem 4.1. Then $\|\nabla J\|^2 = 0$, that is, (M, h) equipped with J is isotropic paraKähler.*

8. Frame fields of neutral vector bundles of rank $4n$ with flat metric connections

Let E be an oriented vector bundle over $\boldsymbol{R}^m$ of rank $4n$ ($m \geq 2$, $n \geq 1$). Let h be a neutral metric of E and ∇ an h-connection of E. Suppose that ∇ is flat, that is, the curvature tensor R of ∇ vanishes.

Let J be a $+$-paracomplex structure of E. Then there exists an ordered pseudo-orthonormal frame field $e = (e_1, \dots, e_{2n}, e_{2n+1}, \dots, e_{4n})$ of E giving the orientation and satisfying (4.1) with $\varepsilon = +$. Let $\omega = [\omega^i_j]$ be the connection form of ∇ with respect to e. Then we have (4.2). Suppose that ∇J is nowhere zero and satisfies (a) in Theorem 4.1 for $\varepsilon = +$. Then we have (4.3) for a nowhere zero 1-form α on $\boldsymbol{R}^m$. Suppose $\mu = +$. Let D_{ij} be the (i, j)-block of ω ($i, j \in \{1, 2, 3, 4\}$). Then we have

$$\omega = \begin{bmatrix} D_{11} & D_{12} & D_{13} & D_{14} \\ D_{21} & D_{22} & D_{23} & D_{24} \\ D_{31} & D_{32} & D_{33} & D_{34} \\ D_{41} & D_{42} & D_{43} & D_{44} \end{bmatrix}.$$

From (4.2), we obtain

$$^tD_{ij} = \begin{cases} -D_{ji} & (i, j = 1, 2 \text{ or } i, j = 3, 4), \\ D_{ji} & (i = 1, 2, j = 3, 4 \text{ or } i = 3, 4, j = 1, 2). \end{cases} \tag{8.1}$$

From (4.3), we obtain

$$\begin{aligned} &D_{11} = D_{33}, \quad D_{13} = D_{31}, \quad D_{22} = D_{44}, \quad D_{24} = D_{42}, \\ &D_{21} + D_{43} = D_{41} + D_{23} = \alpha I_n. \end{aligned} \tag{8.2}$$

Let Ψ_{ij} be the (i, j)-block of $\omega \wedge \omega$ ($i, j \in \{1, 2, 3, 4\}$). Suppose

$$D_{11} = D_{22}, \quad D_{13} = D_{24}. \tag{8.3}$$

Then from (8.2) and (8.3), we obtain $\Psi_{21} + \Psi_{43} = O_n$, $\Psi_{41} + \Psi_{23} = O_n$. Since R vanishes, $d\omega + \omega \wedge \omega$ vanishes. Therefore we obtain $d(D_{21} + D_{43}) = O_n$, $d(D_{41} + D_{23}) = O_n$. These mean $d\alpha = 0$. Therefore a function f on $\boldsymbol{R}^m$ satisfies $df = \alpha$. Suppose $D_{21} = D_{43}$, $D_{41} = D_{23}$. Then we see by (8.1) that ω is represented as in the form of

$$\omega = \begin{bmatrix} D_{11} & -\frac{1}{2}df I_n & D_{31} & \frac{1}{2}df I_n \\ \frac{1}{2}df I_n & D_{11} & \frac{1}{2}df I_n & D_{31} \\ D_{31} & \frac{1}{2}df I_n & D_{11} & -\frac{1}{2}df I_n \\ \frac{1}{2}df I_n & D_{31} & \frac{1}{2}df I_n & D_{11} \end{bmatrix}, \tag{8.4}$$

where D_{11}, D_{31} satisfy ${}^tD_{11} = -D_{11}$, ${}^tD_{31} = D_{31}$ and

$$\begin{aligned} &dD_{11} + D_{11} \wedge D_{11} + D_{31} \wedge D_{31} = O_n, \\ &dD_{31} + D_{31} \wedge D_{11} + D_{11} \wedge D_{31} = O_n. \end{aligned} \tag{8.5}$$

Let D_{11}, D_{31} be $n \times n$ matrices such that each component is a 1-form on $\boldsymbol{R}^m$. Suppose that D_{11}, D_{31} satisfy ${}^tD_{11} = -D_{11}$, ${}^tD_{31} = D_{31}$ and (8.5). Let f be a function on $\boldsymbol{R}^m$ such that df is nowhere zero. Then a $4n \times 4n$ matrix ω defined by (8.4) satisfies (4.2) and (4.3) with $\alpha = df$ and $\varepsilon = \mu = +$, and $d\omega + \omega \wedge \omega$ vanishes. Therefore, noticing $R = 0$, we see that there exists an ordered frame field $e = (e_1, \dots, e_{2n}, e_{2n+1}, \dots, e_{4n})$ of E such that ω is a connection form of ∇ with respect to e. Such a frame field is uniquely determined by an initial value at a point. If e satisfies

$$h(e_i, e_j) = \begin{cases} 1 & (i = j = 1, \dots, 2n), \\ -1 & (i = j = 2n+1, \dots, 4n), \\ 0 & (i \neq j) \end{cases} \tag{8.6}$$

at a point of $\boldsymbol{R}^m$, then noticing that ∇ is an h-connection, we see that e satisfies (8.6) on $\boldsymbol{R}^m$. Therefore there exists an ordered pseudo-orthonormal frame field e of E giving the orientation of E and satisfying $\nabla e = e\omega$. Let J be a $+$-paracomplex structure of E satisfying (4.1) for $\varepsilon = +$ and e. Then ∇J is nowhere zero and satisfies (a) in Theorem 4.1 with $\alpha = df$ for $\varepsilon = \mu = +$.

Example 8.1. We set $D_{11} := O_n$ and $D_{31} := d\phi C_0$, where ϕ is a function on $\boldsymbol{R}^m$ and C_0 is a constant symmetric matrix. Then D_{11}, D_{31} satisfy ${}^tD_{11} = -D_{11}$, ${}^tD_{31} = D_{31}$ and (8.5).

Example 8.2. We set $D_{11} := O_n$ and

$$D_{31} := \begin{bmatrix} df_1 & df_2 & 0 & \cdots & 0 \\ df_2 & df_1 & df_2 & \ddots & \vdots \\ 0 & df_2 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & df_2 \\ 0 & \cdots & 0 & df_2 & df_1 \end{bmatrix}$$

for functions f_1, f_2 on $\boldsymbol{R}^m$. Then D_{11}, D_{31} satisfy ${}^tD_{11} = -D_{11}$, ${}^tD_{31} = D_{31}$ and (8.5).

Example 8.3. We set $D_{31} := O_n$ and $D_{11} := d\psi C_0$, where ψ is a function on $\boldsymbol{R}^m$ and C_0 is a constant skew-symmetric matrix. Then D_{11}, D_{31} satisfy ${}^tD_{11} = -D_{11}$, ${}^tD_{31} = D_{31}$ and (8.5).

EXAMPLE 8.4. Suppose $n = 4p$ for $p \in \boldsymbol{N}$. We set

$$D_{i1} := \begin{bmatrix} C_{i1} & C_{i2} & O_4 & \cdots & O_4 \\ C_{i2} & C_{i1} & C_{i2} & \ddots & \vdots \\ O_4 & C_{i2} & \ddots & \ddots & O_4 \\ \vdots & \ddots & \ddots & \ddots & C_{i2} \\ O_4 & \cdots & O_4 & C_{i2} & C_{i1} \end{bmatrix} \quad (i = 1, 3),$$

where

$$C_{1k} := \begin{bmatrix} 0 & -da_k & 0 & -db_k \\ da_k & 0 & db_k & 0 \\ 0 & -db_k & 0 & -da_k \\ db_k & 0 & da_k & 0 \end{bmatrix}, \quad C_{3k} := \begin{bmatrix} 0 & da_k & 0 & db_k \\ da_k & 0 & db_k & 0 \\ 0 & db_k & 0 & da_k \\ db_k & 0 & da_k & 0 \end{bmatrix}$$

and a_k, b_k are functions on $\boldsymbol{R}^m$ $(k = 1, 2)$. Then D_{11}, D_{31} satisfy ${}^tD_{11} = -D_{11}$, ${}^tD_{31} = D_{31}$, and noticing

$$C_{ik} \wedge C_{jk} = O_4, \quad C_{i1} \wedge C_{j2} + C_{i2} \wedge C_{j1} = O_4 \quad (i, j = 1, 3, k = 1, 2),$$

we obtain $D_{i1} \wedge D_{j1} = O_n$ $(i, j = 1, 3)$. Therefore D_{11}, D_{31} satisfy (8.5).

REMARK 8.5. For any of the above examples, ω is exact and represented as $\omega = dx$ for a function x on $\boldsymbol{R}^m$ valued in the Lie algebra of $SO(2n, 2n)$. Therefore a suitable frame field e is given by $\overline{e}\exp(x)$ for an ordered pseudo-orthonormal parallel frame field $\overline{e}$ of E giving the orientation of E.

If we suppose $\mu = -$, then we have similar discussions and examples. In addition, we can have similar discussions and examples of $-$-paracomplex structures.

REMARK 8.6. Suppose $m = 4n$ and that E is the tangent bundle TE^{4n}_{2n} of E^{4n}_{2n}. Let h be the metric of E^{4n}_{2n} and ∇ the Levi-Civita connection of h. Then the curvature tensor R of ∇ vanishes. Therefore we can find examples of almost ε-paracomplex structures of E^{4n}_{2n} such that the covariant derivatives are nowhere zero and satisfy (a) in Theorem 4.1.

9. Frame fields of neutral vector bundles of rank 4 with flat metric connections

In the following, suppose $n = 1$. Let $e = (e_1, e_2, e_3, e_4)$ be an ordered pseudo-orthonormal frame field of E giving the orientation of E such that $\hat{\nabla}\Omega_{\varepsilon,2}$ $(\varepsilon \in \{+, -\})$ is fully light-like. Let $\omega = [\omega^i_j]$ be the connection form of ∇ with respect to e. Then we have (4.4). Since R vanishes, $d\omega + \omega \wedge \omega$ vanishes. Since $\hat{\nabla}\Omega_{\varepsilon,2}$ is fully light-like, we have (4.5) for $\mu = \mu_\varepsilon \in \{+, -\}$. We set $\psi = [\psi^i_j] := \omega \wedge \omega$. Then from (4.5), we obtain $\psi^4_2 = \varepsilon\psi^3_1$. Since $d\omega + \psi$ vanishes, we obtain $d(\omega^4_2 - \varepsilon\omega^3_1) = 0$. Therefore noticing Proposition 4.4, we can choose e so that Ω_0 for $\Omega = \Omega_{\varepsilon,2}$ is horizontal. If e is such a frame field, then referring to the proof of Proposition 4.4, we obtain $\omega^4_2 - \varepsilon\omega^3_1 = 0$. However, in the following, we suppose only $d(\omega_2^4-\varepsilon\omega_1^3)=0$. Then there exists a function f^ε on $\boldsymbol{R}^m$ satisfying $\omega_2^4-\varepsilon\omega_1^3=-df^\varepsilon$. Applying this to ψ and noticing that $d\omega+\psi$ vanishes, we obtain

$$d(\omega_2^3+\varepsilon\omega_1^4)=\mu df^\varepsilon\wedge(\omega_2^3+\varepsilon\omega_1^4),$$

i.e., $d(e^{-\mu f^\varepsilon}(\omega_2^3+\varepsilon\omega_1^4))=0$. This means that there exists a function g^ε on $\boldsymbol{R}^m$ satisfying $dg^\varepsilon\neq 0$ and

$$dg^\varepsilon=e^{-\mu f^\varepsilon}(\omega_2^3+\varepsilon\omega_1^4)=\mu e^{-\mu f^\varepsilon}(\omega_3^4+\varepsilon\omega_1^2).$$

Therefore we obtain

$$\begin{aligned}&\omega_1^4=\varepsilon e^{\mu f^\varepsilon}dg^\varepsilon-\varepsilon\omega_2^3,\quad \omega_2^4=\varepsilon\omega_1^3-df^\varepsilon,\\ &\omega_3^4=\mu e^{\mu f^\varepsilon}dg^\varepsilon-\varepsilon\omega_1^2.\end{aligned}\tag{9.1}$$

Then the condition that $d\omega+\omega\wedge\omega$ vanishes is rewritten into a system of three equations as follows:

$$\begin{aligned}d\omega_1^2=&2\omega_1^3\wedge\omega_2^3-e^{\mu f^\varepsilon}\omega_1^3\wedge dg^\varepsilon\\ &+\varepsilon\omega_2^3\wedge df^\varepsilon+\varepsilon e^{\mu f^\varepsilon}df^\varepsilon\wedge dg^\varepsilon,\\ d\omega_1^3=&2\omega_1^2\wedge\omega_2^3-e^{\mu f^\varepsilon}\omega_1^2\wedge dg^\varepsilon+\varepsilon\mu e^{\mu f^\varepsilon}\omega_2^3\wedge dg^\varepsilon,\\ d\omega_2^3=&-2\omega_1^2\wedge\omega_1^3+\varepsilon\omega_1^2\wedge df^\varepsilon\\ &-\varepsilon\mu e^{\mu f^\varepsilon}\omega_1^3\wedge dg^\varepsilon+\mu e^{\mu f^\varepsilon}df^\varepsilon\wedge dg^\varepsilon.\end{aligned}\tag{9.2}$$

Let f^ε, g^ε be functions on $\boldsymbol{R}^m$ with $dg^\varepsilon\neq 0$. Let ω_1^2, ω_1^3, ω_2^3 be 1-forms on $\boldsymbol{R}^m$ satisfying (9.2). Let ω_1^4, ω_2^4, ω_3^4 be as in (9.1) and ω a 4×4 matrix such that the (i,j)-component is given by ω_j^i with (4.4). Then $d\omega+\omega\wedge\omega$ vanishes. Therefore, noticing $R=0$, we see that there exists an ordered pseudo-orthonormal frame field $e=(e_1,e_2,e_3,e_4)$ of E giving the orientation of E and satisfying $\nabla e=e\omega$. Then for $\Omega_{\varepsilon,2}$ as in (3.2), we obtain $\hat{\nabla}\Omega_{\varepsilon,2}=\alpha\otimes\Omega_0$, where

$$\alpha:=e^{\mu f^\varepsilon}dg^\varepsilon,\quad \Omega_0:=\Omega_{-\varepsilon,1}+\mu\Omega_{\varepsilon,3}.\tag{9.3}$$

Since $dg^\varepsilon\neq 0$, we have $\alpha\neq 0$. Since $d(\omega_2^4-\varepsilon\omega_1^3)=0$, we see from Proposition 4.4 that Ω_0 can be horizontal for a suitable frame field. Let J_ε be an ε-paracomplex structure of E corresponding to $\Omega_{\varepsilon,2}$. Then we have $J_\varepsilon e_1=e_3$, $J_\varepsilon e_2=-\varepsilon e_4$, and $J=J_\varepsilon$ satisfies (4.6) with (4.7) and (9.3).

Hence we obtain

Proposition 9.1. *The following two conditions are equivalent*:

(a) *the covariant derivative of* $\Omega_{\varepsilon,2}$ *is fully light-like*;

(b) *the connection form* $\omega=[\omega_j^i]$ *of* ∇ *with respect to the frame field* $e=(e_1,e_2,e_3,e_4)$ *satisfies* (9.1), (9.2) *and* $dg^\varepsilon\neq 0$ *for functions* f^ε, g^ε *on* $\boldsymbol{R}^m$ *and* $\mu\in\{+,-\}$.

In addition, if one of (a), (b) *holds, then* $\hat{\nabla}\Omega_{\varepsilon,2}$ *is represented as* $\hat{\nabla}\Omega_{\varepsilon,2} = \alpha \otimes \Omega_0$*, where* α*,* Ω_0 *are as in* (9.3) *and* Ω_0 *can be horizontal for a suitable frame field.*

Suppose $m = 4$ and that E is the tangent bundle TE^4_2 of E^4_2. Let h be the metric of E^4_2 and ∇ the Levi-Civita connection of h. Then the curvature tensor R of ∇ vanishes. Therefore, referring to the above discussions, we obtain

Corollary 9.2. *For* $E = TE^4_2$*,* (a), (b) *in Proposition 9.1 are equivalent to each other. In addition, if one of* (a), (b) *in Proposition 9.1 holds, then the remaining statements in Proposition 9.1 hold.*

Suppose that there exist sections $\Omega_\pm$ of $U_-\left(\bigwedge^2_\pm E\right)$ respectively such that the covariant derivatives are fully light-like. Then noticing the double covering $SO_0(2,2) \longrightarrow SO_0(1,2) \times SO_0(1,2)$, we see that there exists an ordered pseudo-orthonormal frame field $e = (e_1, e_2, e_3, e_4)$ of E such that both of $\hat{\nabla}\Omega_{\pm,2}$ are fully light-like. Let $\omega = [\omega^i_j]$ be the connection form of ∇ with respect to e. Then we have (4.5) for $\mu = \mu_\varepsilon$, $\varepsilon = +, -$. Therefore we have the second relation in (9.1) for $\varepsilon = +, -$. Therefore we have

$$\omega^3_1 = \frac{1}{2}(df^+ - df^-), \quad \omega^4_2 = -\frac{1}{2}(df^+ + df^-). \tag{9.4}$$

Noticing (4.5), we obtain either $\omega^4_1 = \mu\omega^2_1$, $\omega^4_3 = \mu\omega^3_2$ or $\omega^3_2 = \mu\omega^2_1$, $\omega^4_3 = \mu\omega^4_1$ for $\mu \in \{+, -\}$. Suppose $\omega^4_1 = \mu\omega^2_1$ and $\omega^4_3 = \mu\omega^3_2$. Then by (9.1), we obtain

$$\begin{aligned} \omega^2_1 &= \frac{\mu}{2}\left(e^{\mu f^+} dg^+ - e^{\mu f^-} dg^-\right), \\ \omega^3_2 &= \frac{1}{2}\left(e^{\mu f^+} dg^+ + e^{\mu f^-} dg^-\right) \end{aligned} \tag{9.5}$$

and

$$\begin{aligned} \omega^4_1 &= \frac{1}{2}\left(e^{\mu f^+} dg^+ - e^{\mu f^-} dg^-\right), \\ \omega^4_3 &= \frac{\mu}{2}\left(e^{\mu f^+} dg^+ + e^{\mu f^-} dg^-\right). \end{aligned} \tag{9.6}$$

Suppose $\omega^3_2 = \mu\omega^2_1$ and $\omega^4_3 = \mu\omega^4_1$. Then by (9.1), we obtain

$$\begin{aligned} \omega^2_1 &= \frac{\mu}{2}\left(e^{\mu f^+} dg^+ + e^{-\mu f^-} dg^-\right), \\ \omega^3_2 &= \frac{1}{2}\left(e^{\mu f^+} dg^+ + e^{-\mu f^-} dg^-\right) \end{aligned} \tag{9.7}$$

and

$$\begin{aligned} \omega^4_1 &= \frac{1}{2}\left(e^{\mu f^+} dg^+ - e^{-\mu f^-} dg^-\right), \\ \omega^4_3 &= \frac{\mu}{2}\left(e^{\mu f^+} dg^+ - e^{-\mu f^-} dg^-\right). \end{aligned} \tag{9.8}$$

Let $f^\pm$, $g^\pm$ be functions on $\boldsymbol{R}^m$ with $dg^\pm \neq 0$ and ω^3_1, ω^4_2 1-forms on $\boldsymbol{R}^m$ given by

(9.4). Let ω_1^2, ω_2^3, ω_1^4, ω_3^4 be 1-forms on $\boldsymbol{R}^m$ given by either (9.5), (9.6) or (9.7), (9.8). Then ω_1^2, ω_1^3, ω_2^3, ω_1^4, ω_2^4, ω_3^4 satisfy (9.1), (9.2) for $\varepsilon = +, -$ and $\mu = \mu_\varepsilon$. Let ω be a 4×4 matrix such that the (i, j)-component is given by ω_j^i with (4.4). Then there exists an ordered pseudo-orthonormal frame field $e = (e_1, e_2, e_3, e_4)$ of E with $\nabla e = e\omega$ giving the orientation of E and both of $\hat{\nabla}\Omega_{\pm,2}$ are fully light-like.

Hence we obtain

Theorem 9.3. *The following two conditions are equivalent*:

(a) *the covariant derivatives of both of* $\Omega_{\pm,2}$ *are fully light-like*;

(b) *the connection form* $\omega = [\omega_j^i]$ *of* ∇ *with respect to the frame field* $e = (e_1, e_2, e_3, e_4)$ *satisfies* (9.4) *and either* (9.5), (9.6) *or* (9.7), (9.8) *for functions* $f^\pm$, $g^\pm$ *on* $\boldsymbol{R}^m$ *with* $dg^\pm \neq 0$.

Remark 9.4. Based on Theorem 9.3, we see that each pair of almost $\pm$-paracomplex structures of E_2^4 giving sections $\Omega_{\pm,2}$ of the two time-like twistor spaces as in (a) in Theorem 9.3 is given by functions $f^\pm$, $g^\pm$ on E_2^4 with $dg^\pm \neq 0$.

Remark 9.5. Suppose that the covariant derivatives of $\Omega_{\pm,2}$ are fully light-like. Then as was already seen, we have either $\omega_1^4 = \mu\omega_1^2$, $\omega_3^4 = \mu\omega_2^3$ or $\omega_2^3 = \mu\omega_1^2$, $\omega_3^4 = \mu\omega_1^4$ for $\mu \in \{+, -\}$. The former (respectively, latter) case corresponds to (i) (respectively, (ii)) in the last paragraph of Section 5.

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