Maximal L_1-regularity for the linearized compressible Navier-Stokes equations

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Abstract

In this paper, we consider the linearized compressible Navier-Stokes equations with non-slip boundary conditions in the half space $\mathbb{R}^N_+$. We prove the generation of a continuous analytic semigroup associated with this compressible Stokes system with non-slip boundary conditions in the half space $\mathbb{R}^N_+$ and its L_1 in time maximal regularity. We choose the Besov space $\mathcal{H}^s_{q,r} = B^{s+1}_{q,r}(\mathbb{R}^N_+) \times B^s_{q,r}(\mathbb{R}^N_+)^N$ as an underlying space, where $1 < q < \infty$, $1 \le r < \infty$, and $-1 + 1/q < s < 1/q$. We prove the generation of a continuous analytic semigroup $\{T(t)\}_{t\ge 0}$ on $\mathcal{H}^s_{q,r}$, and show that its generator admits maximal L_1 regularity. Our approach is to prove the existence of the resolvent in $\mathcal{H}^s_{q,1}$ and some new estimates for the resolvent by using $B^{s+1}_{q,1}(\mathbb{R}^N_+) \times B^{s\pm\sigma}_{q,1}(\mathbb{R}^N_+)$ norms for some small $\sigma > 0$ satisfying the condition $-1 + 1/q < s - \sigma < s < s + \sigma < 1/q$.

1 Introduction

Let $\mathbb{R}^N_+ := \{x = (x', x_N) \in \mathbb{R}^N \mid x' \in \mathbb{R}^{N-1},\ x_N > 0\}$, $N \ge 2$, be the half space. In this paper, we consider the following linear system:

$$\begin{cases} \partial_t \rho + \gamma \operatorname{div} \mathbf{u} = 0 & \text{in } \mathbb{R}^N_+ \times (0,\infty), \\ \partial_t \mathbf{u} - \alpha \Delta \mathbf{u} - \beta \nabla \operatorname{div} \mathbf{u} + \gamma \nabla \rho = 0 & \text{in } \mathbb{R}^N_+ \times (0,\infty), \\ \mathbf{u} = 0 & \text{on } \partial\mathbb{R}^N_+ \times (0,\infty), \\ (\rho, \mathbf{u})(0,x) = (\rho_0, \mathbf{u}_0) & \text{in } \mathbb{R}^N_+. \end{cases} \tag{1.1}$$

Here, ρ and $\mathbf{u} = (u_1, \cdots, u_N)$ are unknown functions, while the initial datum $(\rho_0, \mathbf{u}_0)$ is assumed to be given. Moreover, the coefficients α, β, and γ are assumed to be constants such that $\alpha > 0$, $\alpha + \beta > 0$ and $\gamma > 0$. The aim of this paper is to show the generation of a continuous analytic semigroup associated with equations (1.1) and its L_1 in time maximal regularity property in some Besov spaces.

The system (1.1) is the linearized system of the compressible Navier-Stokes equations with homogeneous Dirichlet boundary conditions:

$$\begin{cases} \partial_t \varrho + \operatorname{div}(\varrho \mathbf{v}) = 0 & \text{in } \mathbb{R}^N_+ \times (0,\infty), \\ \varrho(\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{v}) - \mu \Delta \mathbf{v} - (\mu + \nu) \nabla \operatorname{div} \mathbf{v} + \nabla P(\varrho) = 0 & \text{in } \mathbb{R}^N_+ \times (0,\infty), \\ \mathbf{v} = 0 & \text{on } \partial\mathbb{R}^N_+ \times (0,\infty), \\ (\varrho, \mathbf{v})(0,x) = (\varrho_0, \mathbf{v}_0) & \text{in } \mathbb{R}^N_+, \end{cases} \tag{1.2}$$

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where ϱ and $\mathbf{v}$ describe the unknown density and the velocity field of the compressible viscous field, respectively, while the initial datum $(\varrho_0, \mathbf{v}_0)$ is a pair of given functions. The coefficients μ and ν are assumed to satisfy the ellipticity conditions $\mu > 0$ and $\mu + \nu > 0$. In addition, the pressure of the fluid P is a given smooth function with respect to ϱ, which is assumed to satisfy the stability condition $P'(\rho_*) > 0$. Here, ρ_* stands for the reference density that is a positive constant, and the initial density ϱ_0 is given as a perturbation from ρ_*. As discussed in [9, Sect. 8], the coefficients α, β, and γ are defined by $\alpha = \mu/\rho_*$, $\beta = \nu/\rho_*$, and $\gamma = \sqrt{P'(\rho_*)}$, respectively. Clearly, α, β, and γ satisfy the aforementioned given conditions.

There are a lot of results concerning the compressible Navier-Stokes equations (1.2). Let us briefly summarize the results. Mathematical studies on the compressible Navier-Stokes equations started with the uniqueness results in a bounded domain by Graffi [10], whose result is extended by Serrin [25] in the sense that there is no assumption on the equation of state of the fluid. In the studies [10] and [25], the fluid occupies a bounded domain surrounded by a smooth boundary. A local in time existence theorem in Hölder continuous spaces was first proved by Nash [24] and Itaya [12, 13], independently, for the whole space case. As for the boundary value problem case, Tani [34] proved a local in time existence theorem in a similar setting provided that a (bounded or unbounded) domain Ω has a smooth boundary. In Sobolev-Slobodetskii spaces, the local existence was shown by Solonnikov [31], see also the work due to Danchin [5] for an improvement of Solonnikov's result. Matsumura and Nishida [19] made a breakthrough in proving a unique global-in-time solution for the initial value problem of the compressible Navier-Stokes equations for the multidimensional case. More precisely, Matsumura and Nishida [19] investigated the system with heat-conductive effects in $\mathbb{R}^3$ and proved the global existence theorem with the aid of a local existence theorem together with a priori estimates for the solution. In particular, the a priori estimates were established by a combination of the linear spectral theory and the L_2-energy method. They also succeeded to prove the global existence result in the half space and exterior domains cases with sufficiently small given data, see [20]. We here mention that the rate of convergence (as $t \to \infty$) of the solution to the system, which is constructed in [20], is established in [16] for the half space case and [14] for the exterior domains cases provided that the initial data are close to the constant equilibrium state. We also refer to a recent work due to Shibata and Enomoto [28] as well as Shibata [26] for some refinement of [16] in the sense that the class of initial data $(\varrho_0 - \rho_*, \mathbf{v})$ may be weakened. Notice that the approach of Shibata and Enomoto [28] and Shibata [26] are completely different from Kawashita's argument [15], where Kawashita [15] also required less regularity on the initial conditions, in contrast to [19].

In the aforementioned works, the proof of the global existence theorem was mainly based on the L_2-energy method (excluded the contribution due to Shibata [26]), but another approach was established by Ströhmer [33]. His idea was to rewrite the system in Lagrangian coordinates, which is often said to be Lagrangian transformation. Thanks to this reformulation, the convection term in the density equation, namely $\varrho \cdot \nabla \mathbf{v}$, may be dropped off, so that the transformed system becomes the evolution equation of parabolic type, and he used the semigroup theory. On the basis of a different approach, Mucha and Zajączkowski [22] applied L_p-energy estimates to show the global existence theorem in the L_p in time and L_q in space framework. Recently, maximal L_p-regularity approach was developed by Enomoto and Shibata [9], which extended the result of Mucha and Zajączkowski [22] in the sense that it was allowed to construct global strong solution in the L_p in time and L_q in space framework. We emphasize that, on page 418 in [22], it was declared that there is no possibility to obtain a global existence theorem in the L_p-framework whenever we investigate the system in Eulerian coordinates, but this was wrong if the domain Ω is a bounded smooth domain. In fact, Kotschote [17] constructed global strong L_p-solutions in Eulerian coordinates, without making use of transformation to Lagrangian coordinates. For a list of relevant references of studies of the local or global existence theorem (for classical or strong solutions), the readers may consult [28, Section 2] and references therein.

Recall that the Jacobian of Lagrange transformation is given by $I + \int_0^t \nabla \mathbf{u}(\tau, \xi)\, \mathrm{d}\tau$, where $\mathbf{u}(\tau, \xi)$ stands for the velocity field of a fluid particle at time t which was located in ξ at initial time $t = 0$. Hence, to obtain the global existence theorem with the aid of Lagrangian transformation, it is always crucial to get a control of $\int_0^t \nabla \mathbf{u}(\tau, \xi)\, \mathrm{d}\tau$ in a suitable norm. In particular, it is necessary to find a small constant $c > 0$ such that

$$\left\| \int_0^t \nabla \mathbf{u}(\tau, \xi)\, \mathrm{d}\tau \right\|_{L_\infty} \leq c, \tag{1.3}$$

which ensures that Lagrangian transformation is invertible. If the estimate (1.3) is stemmed from an L_p in time estimate for $\mathbf{u}$ with $1 < p < \infty$, then we may only expect to have a t-dependent bound, if Ω is unbounded. Of course, as we mentioned before, it is still possible to prove the global existence theorem even if the constant c appearing in (1.3) depends on t, but the proof becomes more involved.

Recently, Danchin and Tolksdorf [7] proved maximal L_1-regularity estimate for $\mathbf{u}$, which implies that one may find a t-independent constant c such that (1.3) is valid. Here, they studied the system in the L_1 in time and $B^s_{p,1}$ in space framework, where p and s are taken such that $1 < p < \infty$ and $s = -1 + N/p$. Their function space is similar to the spaces used in [3, 4], but it was not necessary to consider homogeneous Besov spaces in [7], since it is well-known that homogeneous Besov spaces $\dot{B}^s_{p,1}(\Omega)$ coincide with inhomogeneous Besov spaces $B^s_{p,1}(\Omega)$ if $-1 + 1/p < s < 1/p$ and if the domain Ω is bounded of class C^1, see [6, Remark 2.2.1]. The essential assumption in [7] is that the fluid domain is bounded, which is required to prove their extension version of Da Prato-Grisvard theory [8].

We want to consider the viscous compressible fluid flow in general domains, which is described in (1.2) when the fluid domain is $\mathbb{R}^N_+$. The L_p-L_q, $1 < p, q < \infty$, maximal regularity theorem for (1.2) was constructed in the paper due to Enomoto-Shibata [9], but following this paper, we want to construct the L_1-$B^s_{p,1}$ maximal regularity theory for equations (1.2), where $1 < p < \infty$ and $-1 + 1/p < s < 1/p$ for the Stokes equations and $1 < p < \infty$ and $-1 + N/p \leq s < 1/p$ for the Navier-Stokes equations.

We want to study the viscous barotropic compressible fluid flow in an unbounded domain in an L_1 in time maximal regularity framework. Because, L_1 in time maximal regularity is the best framework to use the Lagrange transformation to solve the nonlinear problem. As a first step, in this paper we establish the L_1-$B^s_{p,1}(\mathbb{R}^N_+)$ maximal regularity theorem in the half space $\mathbb{R}^N_+$ with $1 < p < \infty$ and $-1 + 1/p < s < 1/p$ for equations (1.1), which is the model problem. The local well-posedness of the nonlinear problem (1.2) is treated in another paper [18]. Although there are several contributions toward this topic [7, 9, 17, 33], we intend to study the problem in the half space within inhomogeneous Besov spaces setting.

Before stating our main results, we introduce basical spaces in our paper as follows. Let $1 < q < \infty$, $-1 + 1/q < s < 1/q$, $1 \leq r < \infty$, $\mu \in \mathbb{R}$, and $\Omega \in \{\mathbb{R}^N, \mathbb{R}^N_+\}$. Let $B^\mu_{q,r}(\Omega)$ denote standard Besov spaces on Ω. Let

$$\begin{aligned}
\mathcal{H}^s_{q,r}(\Omega) &= B^{s+1}_{q,r}(\Omega) \times B^s_{q,r}(\Omega)^N, \\
\mathcal{D}^s_{q,r}(\mathbb{R}^N) &= B^{s+1}_{q,r}(\mathbb{R}^N) \times B^{s+2}_{q,r}(\mathbb{R}^N)^N, \\
\mathcal{D}^s_{q,r}(\mathbb{R}^N_+) &= \{(\rho, \mathbf{v}) \in B^{s+1}_{q,r}(\mathbb{R}^N_+) \times B^{s+2}_{q,r}(\mathbb{R}^N_+)^N \mid \mathbf{v}|_{\partial\mathbb{R}^N_+} = 0\}, \\
\|(f, \mathbf{g})\|_{\mathcal{H}^s_{q,r}(\Omega)} &= \|f\|_{B^{s+1}_{q,r}(\Omega)} + \|\mathbf{g}\|_{B^s_{q,r}(\Omega)}, \\
\|(f, \mathbf{g})\|_{\mathcal{D}^s_{q,r}(\Omega)} &= \|f\|_{B^{s+1}_{q,r}(\Omega)} + \|\mathbf{g}\|_{B^{s+2}_{q,r}(\Omega)}.
\end{aligned} \tag{1.4}$$

In addition, we introduce the operator $\mathcal{A}^s_{q,r}$ corresponding to equations (1.1) which is defined by setting

$$\mathcal{A}^s_{q,r}(\rho, \mathbf{v}) = (\gamma \operatorname{div} \mathbf{v}, -\alpha\Delta\mathbf{v} - \beta\nabla \operatorname{div} \mathbf{v} + \gamma\nabla\rho) \quad \text{for } (\rho, \mathbf{v}) \in \mathcal{D}^s_{q,r}(\mathbb{R}^N_+). \tag{1.5}$$

Using $\mathcal{A}^s_{q,r}$, equations (1.1) are written as

$$\partial_t(\rho, \mathbf{u}) + \mathcal{A}^s_{q,r}(\rho, \mathbf{u}) = (0, 0) \quad \text{for } t > 0, \quad (\rho, \mathbf{u})|_{t=0} = (\rho_0, \mathbf{u}_0) \in \mathcal{H}^s_{q,r}(\mathbb{R}^N_+) \tag{1.6}$$

for $(\rho, \mathbf{u})$ with

$$(\rho, \mathbf{u}) \in C^0[(0, \infty), \mathcal{H}^s_{q,r}(\mathbb{R}^N_+) \cap C^1((0, \infty), \mathcal{H}^s_{q,r}(\mathbb{R}^N_+)) \cap C^0((0, \infty), \mathcal{D}^s_{q,r}(\mathbb{R}^N_+)).$$

Our main results of this paper read as follows.

Theorem 1.1. *Let $1 < q < \infty$, $-1 + 1/q < s < 1/q$, and $1 \le r < \infty$. Then, the operator $\mathcal{A}^s_{q,r}$ generates a continuous analytic semigroup $\{T(t)\}_{t\ge0}$ on $\mathcal{H}^s_{q,r}(\mathbb{R}^N_+)$.*

Moreover, there exists a large $\omega_0 \ge 1$ such that, for any $\omega \ge \omega_0$ and $(\rho_0, \mathbf{u}_0) \in \mathcal{H}^s_{q,1}(\mathbb{R}^N_+)$,

$$\int_0^\infty e^{-\omega t}(\|\partial_t T(t)(\rho_0, \mathbf{u}_0)\|_{\mathcal{H}^s_{q,1}(\mathbb{R}^N_+)} + \|T(t)(\rho_0, \mathbf{u}_0)\|_{\mathcal{D}^s_{q,1}(\mathbb{R}^N_+)})dt \le C\|(\rho_0, \mathbf{u}_0)\|_{\mathcal{H}^s_{q,1}(\mathbb{R}^N_+)}.$$

To prove Theorem 1.1, we consider the following resolvent problem:

$$\begin{cases} \lambda\rho + \gamma \operatorname{div} \mathbf{u} = f & \text{in } \mathbb{R}^N_+, \\ \lambda\mathbf{u} - \alpha\Delta\mathbf{u} - \beta\nabla \operatorname{div} \mathbf{u} + \gamma\nabla\rho = \mathbf{g} & \text{in } \mathbb{R}^N_+, \\ \mathbf{u} = 0 & \text{on } \partial\mathbb{R}^N_+, \end{cases} \tag{1.7}$$

for $\lambda \in \Lambda_{\epsilon,\nu_0}$. Here, Λ_{ϵ,ν_0} is a subset of $\mathbb{C}$ defined as follows:

$$\begin{aligned} \Sigma_\epsilon &= \{\lambda \in \mathbb{C} \setminus \{0\} \mid |\arg\lambda| \le \pi - \epsilon, \}, \\ K_\epsilon &= \Big\{\lambda \in \mathbb{C} \mid \Big(\operatorname{Re}\lambda + \frac{\gamma^2}{\alpha+\beta} + \epsilon\Big)^2 + (\operatorname{Im}\lambda)^2 \ge \Big(\frac{\gamma^2}{\alpha+\beta} + \epsilon\Big)^2\Big\}, \\ \Lambda_{\epsilon,\nu_0} &= K_\epsilon \cap \Sigma_\epsilon \cap \{\lambda \in \mathbb{C} \mid |\lambda| \ge \nu_0\}. \end{aligned} \tag{1.8}$$

Remark 1.1. If one considers the inhomogeneous problem:

$$\partial_t(\rho, \mathbf{u}) + \mathcal{A}^s_{q,r}(\rho, \mathbf{u}) = (F, \mathbf{G}) \quad \text{for } t > 0, \quad (\rho, \mathbf{u})|_{t=0} = (0, 0) \in \mathcal{H}^s_{q,r}(\mathbb{R}^N_+),$$

then one may infer from Theorem 1.1 that there holds

$$\int_0^\infty e^{-\omega t}(\|\partial_t(\rho, \mathbf{u})\|_{\mathcal{H}^s_{q,1}(\mathbb{R}^N_+)} + \|\mathcal{A}^s_{q,1}(\rho, \mathbf{u})\|_{\mathcal{H}^s_{q,1}(\mathbb{R}^N_+)})dt \le C\int_0^\infty e^{-\omega t}\|(F, \mathbf{G})\|_{\mathcal{H}^s_{q,1}(\mathbb{R}^N_+)}dt.$$

This estimate follows from the Duhamel principle and the estimate for the semigroup, that is, by virtue of the Duhamel principle, the solution $(\rho, \mathbf{u})$ to the inhomogeneous problem is given by

$$(\rho, \mathbf{u}) = \int_0^t T(t-s)(F, \mathbf{G})(s)ds.$$

Then,

$$\|(\rho, \mathbf{u})\|_{\mathcal{D}^s_{q,1}(\mathbb{R}^N_+)} \le C\int_0^t \|T(t-s)(F, \mathbf{G})(s)\|_{\mathcal{D}^s_{q,1}(\mathbb{R}^N_+)}ds.$$

Therefore, by Fubini's theorem, change of variables (let $t-s=\ell$) and Theorem 1.1, we have

$$\begin{aligned}\int_0^\infty e^{-\omega t}\|(\rho,\mathbf{u})\|_{\mathcal{D}^s_{q,1}(\mathbb{R}^N_+)}dt &\le C\int_0^\infty(\int_0^t e^{-\omega t}\|T(t-s)(F,\mathbf{G})(s)\|_{\mathcal{D}^s_{q,1}(\mathbb{R}^N_+)}ds)dt\\ &\le C\int_0^\infty(\int_s^\infty e^{-\omega t}\|T(t-s)(F,\mathbf{G})(s)\|_{\mathcal{D}^s_{q,1}(\mathbb{R}^N_+)}dt)ds\\ &\le C\int_0^\infty e^{-\omega s}(\int_0^\infty e^{-\omega\ell}\|T(\ell)(F,\mathbf{G})(s)\|_{\mathcal{D}^s_{q,1}(\mathbb{R}^N_+)}d\ell)ds\\ &\le C\int_0^\infty e^{-\omega s}\|(F,\mathbf{G})(s)\|_{\mathcal{H}^s_{q,1}(\mathbb{R}^N_+)}ds.\end{aligned}$$

We first prove the maximal L_1 regularity for $\partial_t T(t)(\rho_0,\mathbf{u}_0)$, and then by (1.6), the L_1 estimate of $\mathcal{A}^s_{q,1}T(t)(\rho_0,\mathbf{u}_0)$ follows, which is

$$\|e^{-\omega t}\mathcal{A}^s_{q,1}T(\cdot)(\rho_0,\mathbf{u}_0)\|_{L_1((0,\infty),\mathcal{H}^s_{q,1}(\mathbb{R}^N_+))}\le C\|e^{-\omega t}\partial_t T(\cdot)(\rho_0,\mathbf{u}_0)\|_{L_1((0,\infty),\mathcal{H}^s_{q,1}(\mathbb{R}^N_+))}.$$

Thus, we have the standard maximal L_1 regularity, which read as

$$\int_0^\infty e^{-\omega t}(\|\partial_t T(t)(\rho_0,\mathbf{u}_0)\|_{\mathcal{H}^s_{q,1}(\mathbb{R}^N_+)}+\|\mathcal{A}^s_{q,1}T(t)(\rho_0,\mathbf{u}_0)\|_{\mathcal{H}^s_{q,1}(\mathbb{R}^N_+)})dt\le C\|(\rho_0,\mathbf{u}_0)\|_{\mathcal{H}^s_{q,1}(\mathbb{R}^N_+)}.$$

Theorem 1.1 may be proved by real interpolation theorem with the help of the following theorem.

Theorem 1.2. *Let $1<q<\infty$, $1\le r<\infty$, $-1+1/q<s<1/q$, and $\epsilon\in(0,\pi/2)$. Then, there exists a large constant $\omega>0$ such that for every $\lambda\in\Lambda_{\epsilon,\omega}$ and $(f,\mathbf{g})\in\mathcal{H}^s_{q,r}(\mathbb{R}^N_+)$, there exists a unique solution $(\rho,\mathbf{u})\in\mathcal{D}^s_{q,r}(\mathbb{R}^N_+)$ to (1.7) satisfying*

$$\|\lambda(\rho,\mathbf{u})\|_{\mathcal{H}^s_{q,r}(\mathbb{R}^N_+)}+\|\mathbf{u}\|_{B^{s+2}_{q,r}(\mathbb{R}^N_+)}\le C\|(f,\mathbf{g})\|_{\mathcal{H}^s_{q,r}(\mathbb{R}^N_+)}$$

Moreover, let σ be a small positive number such that $-1+1/q<s-\sigma<s<s+\sigma<1/q$. Then, there exist $\mathbf{u}_1$, $\mathbf{u}_2\in B^{s+2}_{q,r}(\mathbb{R}^N_+)^N$ such that $\mathbf{u}=\mathbf{u}_1+\mathbf{u}_2$ and for any $\lambda\in\Lambda_{\epsilon,\gamma}$ there hold

$$\begin{aligned}\|\mathbf{u}_1\|_{B^{s+2}_{q,r}(\mathbb{R}^N_+)}&\le C|\lambda|^{-\frac{\sigma}{2}}\|\mathbf{g}\|_{B^{s+\sigma}_{q,r}(\mathbb{R}^N_+)},\\ \|\partial_\lambda\mathbf{u}_1\|_{B^{s+2}_{q,r}(\mathbb{R}^N_+)}&\le C|\lambda|^{-(1-\frac{\sigma}{2})}\|\mathbf{g}\|_{B^{s-\sigma}_{q,r}(\mathbb{R}^N_+)}\end{aligned}$$

for any $\mathbf{g}\in C^\infty_0(\mathbb{R}^N_+)^N$ as well as

$$\begin{aligned}\|(\rho,\mathbf{u}_2)\|_{\mathcal{D}^s_{q,r}(\mathbb{R}^N_+)}&\le C|\lambda|^{-1}\|(f,\mathbf{g})\|_{\mathcal{H}^s_{q,r}(\mathbb{R}^N_+)},\\ \|\partial_\lambda(\rho,\mathbf{u}_2)\|_{\mathcal{D}^s_{q,r}(\mathbb{R}^N_+)}&\le C|\lambda|^{-2}\|(f,\mathbf{g})\|_{\mathcal{H}^s_{q,r}(\mathbb{R}^N_+)}\end{aligned}$$

for any $(f,\mathbf{g})\in\mathcal{H}^s_{q,r}(\mathbb{R}^N_+)$.

Remark 1.2. The conditions $1<q<\infty$, $1\le r<\infty$ and $-1+1/q<s<1/q$ assure that $C^\infty_0(\Omega)$ is a dense subset of $B^s_{q,r}(\Omega)$ for $\Omega\in\{\mathbb{R}^N,\mathbb{R}^N_+\}$. This fact is an important point for our analysis in this paper. For a proof of this fact, refer to [35, Theorems 2.9.3 and 2.10.3].

The rest of this paper is unfold as follows. In the next section, we recall the notation of functional spaces. Then, in Sect. 3, we prove boundedness properties of integral operators that will appear in the solution formula for (1.7) given in Sect. 4. Finally, in Section 5 Theorem 1.2 will be proved in the $\mathbb{R}^N$ case and in Sect. 6, Theorem 1.2 is proved in the half space case. Finally, in Sect. 7, we shall prove Theorem 1.1.

2 Preliminaries

2.1 Notation

Let us fix the symbols in this paper. Let $\mathbb{R}$, $\mathbb{N}$, and $\mathbb{C}$ be the set of all real, natural, complex numbers, respectively, while let $\mathbb{Z}$ be the set of all integers. Moreover, $\mathbb{K}$ stands for either $\mathbb{R}$ or $\mathbb{C}$. Set $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

For $N \in \mathbb{N}$ and a Banach space X, let $\mathcal{S}(\mathbb{R}^N; X)$ be the Schwartz class of X-valued rapidly decreasing functions on $\mathbb{R}^N$. We denote $\mathcal{S}'(\mathbb{R}^N; X)$ by the space of X-valued tempered distributions, which means the set of all continuous linear mappings from $\mathcal{S}(\mathbb{R}^N)$ to X. For $N \in \mathbb{N}$, we define the Fourier transform $f \mapsto \mathcal{F}[f]$ from $\mathcal{S}(\mathbb{R}^N; X)$ onto itself and its inverse as

$$\mathcal{F}[f](\xi) := \int_{\mathbb{R}^N} f(x) e^{-ix\cdot\xi}\, dx, \qquad \mathcal{F}_\xi^{-1}[g](x) := \frac{1}{(2\pi)^N} \int_{\mathbb{R}^N} g(\xi) e^{ix\cdot\xi}\, d\xi,$$

respectively. In addition, we define the partial Fourier transform $\mathcal{F}'[f(\,\cdot\,, x_N)] = \hat{f}(\xi', x_N)$ and partial inverse Fourier transform $\mathcal{F}_{\xi'}^{-1}$ by

$$\mathcal{F}'[f(\,\cdot\,, x_N)](\xi') := \hat{f}(\xi', x_N) = \int_{\mathbb{R}^{N-1}} f(x', x_N) e^{-ix'\cdot\xi'}\, dx',$$
$$\mathcal{F}_{\xi'}^{-1}[g(\,\cdot\,, x_N)](x') := \frac{1}{(2\pi)^{N-1}} \int_{\mathbb{R}^{N-1}} g(\xi', x_N) e^{ix'\cdot\xi'}\, d\xi',$$

where we have set $x' = (x_1, \cdots, x_{N-1}) \in \mathbb{R}^{N-1}$ and $\xi' = (\xi_1, \cdots, \xi_{N-1}) \in \mathbb{R}^{N-1}$. For $N \geq 2$, we set $(\mathbf{f}, \mathbf{g})_{\mathbb{R}^N_+} = \int_{\mathbb{R}^N_+} \mathbf{f}(x) \cdot \mathbf{g}(x)\, dx$ for N-vector functions $\mathbf{f}$ and $\mathbf{g}$ on $\mathbb{R}^N_+$, where we will write $(\mathbf{f}, \mathbf{g}) = (\mathbf{f}, \mathbf{g})_{\mathbb{R}^N_+}$ for short if there is no confusion. For a Banach space X, $\|\cdot\|_X$ denotes its norm. For Banach spaces X and Y, $X \times Y$ denotes the product of X and Y, that is $X \times Y = \{(x, y) \mid x \in X,\ y \in Y\}$, while $\|\cdot\|_{X\times Y}$ denotes its norm. $X \hookrightarrow Y$ means that X is continuously imbedded into Y, that is $X \subset Y$ and $\|x\|_Y \leq C\|x\|_X$ with some constant C. For any interpolation couple (X, Y) of Banach spaces X and Y, the operations $(X, Y) \to (X, Y)_{\theta,p}$ and $(X, Y) \to (X, Y)_{[\theta]}$ are called the real interpolation functor for each $\theta \in (0, 1)$ and $p \in [1, \infty]$ and the complex interpolation functor for each $\theta \in (0, 1)$, respectively. By $C > 0$ we will often denote a generic constant that does not depend on the quantities at stake. For differentiation with respect to space variables $x = (x_1, \ldots, x_N)$, $D^\delta f := \partial_x^\delta f = \partial^{|\delta|} f / \partial x_1^{\delta_1} \cdots \partial x_N^{\delta_N}$ for multi-index $\delta = (\delta_1, \ldots, \delta_N)$ with $|\delta| = \delta_1 + \cdots + \delta_N$. For the notational simplicity, we write $\nabla f = \{\partial_x^\delta f \mid |\delta| = 1\}$, $\nabla^2 f = \{\partial_x^\delta f \mid |\delta| = 2\}$, $\bar{\nabla} f = (f, \nabla f)$, and $\bar{\nabla}^2 f = (f, \nabla f, \nabla^2 f)$.

2.2 Function spaces on $\mathbb{R}^N$

Let us recall the definitions of Bessel potential spaces and inhomogeneous Besov spaces. In the following, let $s \in \mathbb{N}$ and $p \in (1, \infty)$. Bessel potential spaces $H_p^s(\mathbb{R}^N)$ are defined as the set of all $f \in \mathcal{S}'(\mathbb{R}^N)$ such that $\|f\|_{H_p^s(\mathbb{R}^N)} < \infty$, where the norm $\|\cdot\|_{H_p^s(\mathbb{R}^N)}$ is defined by

$$\|f\|_{H_p^s(\mathbb{R}^N)} := \left\| \mathcal{F}_\xi^{-1} \left[(1 + |\xi|^2)^{\frac{s}{2}} \mathcal{F}[f](\xi) \right] \right\|_{L_p(\mathbb{R}^N)}.$$

It is well-known that, if $s = m \in \mathbb{N}_0$, then $H_p^s(\mathbb{R}^N)$ coincides with the classical Sobolev space $W_p^m(\mathbb{R}^N)$, see, e.g., [1, Theorem 3.7].

To define inhomogeneous Besov spaces, we need to introduce Littlewood-Paley decomposition. Let $\phi \in \mathcal{S}(\mathbb{R}^N)$ with $\operatorname{supp}\phi = \{\xi \in \mathbb{R}^N \mid 1/2 \leq |\xi| \leq 2\}$ such that $\sum_{k\in\mathbb{Z}} \phi(2^{-k}\xi) = 1$ for all $\xi \in \mathbb{R}^N \setminus \{0\}$. Then, define

$$\phi_k := \mathcal{F}_\xi^{-1}[\phi(2^{-k}\xi)], \quad k \in \mathbb{Z}, \qquad \mathcal{F}[\psi] = 1 - \sum_{k\in\mathbb{N}} \phi(2^{-k}\xi). \tag{2.1}$$

For $1 \le p, q \le \infty$ and $s \in \mathbb{R}$ we denote

$$\|f\|_{B^s_{p,q}(\mathbb{R}^N)} := \begin{cases} \|\psi * f\|_{L_p(\mathbb{R}^N)} + \Big(\sum_{k\in\mathbb{N}} \Big(2^{sk} \|\phi_k * f\|_{L_p(\mathbb{R}^N)}\Big)^q \Big)^{1/q} & \text{if } 1 \le q < \infty, \\ \|\psi * f\|_{L_p(\mathbb{R}^N)} + \sup_{k\in\mathbb{N}} \Big(2^{sk} \|\phi_k * f\|_{L_p(\mathbb{R}^N)}\Big) & \text{if } q = \infty. \end{cases} \tag{2.2}$$

Here, $f * g$ means the convolution between f and g. Then inhomogeneous Besov spaces $B^s_{p,q}(\mathbb{R}^N)$ are defined as the sets of all $f \in \mathcal{S}'(\mathbb{R}^N)$ such that $\|f\|_{B^s_{p,q}(\mathbb{R}^N)} < \infty$.

It is well-known that $B^s_{p,q}(\mathbb{R}^N)$ may be *characterized* by means of real interpolation. In fact, for $-\infty < s_0 < s_1 < \infty$, $1 < p < \infty$, $1 \le q \le \infty$, and $0 < \theta < 1$, it follows that

$$B^{\theta s_0 + (1-\theta)s_1}_{p,q}(\mathbb{R}^N) = \big(H^{s_0}_p(\mathbb{R}^N), H^{s_1}_p(\mathbb{R}^N)\big)_{\theta,q},$$

cf. [23, Theorem 8], [36, Theorem 2.4.2].

2.3 Function spaces on $\mathbb{R}^N_+$

Let $\mathcal{D}'(\mathbb{R}^N_+)$ be the collection of all complex-valued distributions on $\mathbb{R}^N_+$. Let $s \in \mathbb{R}$, $p \in (1,\infty)$, and $q \in [1,\infty]$. Then for any $X \in \{H^s_p, B^s_{p,q}\}$, the space $X(\mathbb{R}^N_+)$ is the collection of all $f \in \mathcal{D}'(\mathbb{R}^N_+)$ such that there exists a function $g \in X(\mathbb{R}^N)$ with $g|_{\mathbb{R}^N_+} = f$. Moreover, the norm of $f \in X(\mathbb{R}^N_+)$ is given by

$$\|f\|_{X(\mathbb{R}^N_+)} = \inf \|g\|_{X(\mathbb{R}^N)},$$

where the infimum is taken over all $g \in X(\mathbb{R}^N)$ such that its restriction $g|_{\mathbb{R}^N_+}$ coincides in $\mathcal{D}'(\mathbb{R}^N_+)$ with f. We also define

$$X_0(\mathbb{R}^N_+) := \{f \in X(\mathbb{R}^N) \mid \operatorname{supp} f \subset \overline{\mathbb{R}^N_+}\}.$$

Clearly, we always have $X_0(\mathbb{R}^N_+) \hookrightarrow X(\mathbb{R}^N_+)$.

According to [35, Section 2.9], for $s \in \mathbb{R}$, $p \in (1,\infty)$, and $q \in [1,\infty)$, we have the following density result:

$$X_0(\mathbb{R}^N_+) = \overline{C^\infty_0(\mathbb{R}^N_+)}^{\|\cdot\|_{X(\mathbb{R}^N)}}.$$

Here, $X(\mathbb{R}^N_+)$ and $X_0(\mathbb{R}^N_+)$ may coincide if one restricts s such that $-1 + 1/p < s < 1/p$.

Proposition 2.1. *Let $1 < p < \infty$, $1 \le q < \infty$, and $-1 + 1/p < s < 1/p$. Then $H^s_p(\mathbb{R}^N_+) = H^s_{p,0}(\mathbb{R}^N_+)$ as well as $B^s_{p,q}(\mathbb{R}^N_+) = B^s_{p,q,0}(\mathbb{R}^N_+)$.*

Finally, let us mention duality results. If one considers function spaces on $\mathbb{R}^N$, then it follows that $(H^s_p(\mathbb{R}^N))' = H^{-s}_{p'}(\mathbb{R}^N)$ and $(B^s_{p,q}(\mathbb{R}^N))' = B^{-s}_{p',q'}(\mathbb{R}^N)$ for all $s \in \mathbb{R}$, $p \in (1,\infty)$, and $q \in [1,\infty)$, where p' and q' stand for the Hölder conjugate of p and q, respectively. Indeed, these proofs may be found in [23, Sect. 6], [36, Theorem 2.11.2]. However, if one considers function spaces on $\mathbb{R}^N_+$, one has to pay attention to discuss the dual of function spaces due to the existence of the boundary $\partial\mathbb{R}^N_+$. Let us summarize the duality results and real interpolation functors for the half space case.

Proposition 2.2. *Let $p \in (1,\infty)$. Then the following assertions are valid.*

(1) *For $s \in \mathbb{R}$, there holds*

$$(H^s_{p,0}(\mathbb{R}^N_+))' = H^{-s}_{p'}(\mathbb{R}^N_+).$$

(2) *For $-\infty < s \le 1/p$, there holds*

$$(H^s_p(\mathbb{R}^N_+))' = H^{-s}_{p',0}(\mathbb{R}^N_+).$$

(3) *For* $-\infty < s_0 < s_1 < \infty$, $1 < p < \infty$, $1 \leq q \leq \infty$, *and* $0 < \theta < 1$, *there holds*

$$B^{\theta s_0+(1-\theta)s_1}_{p,q}(\mathbb{R}^N_+) = \left(H^{s_0}_p(\mathbb{R}^N_+), H^{s_1}_p(\mathbb{R}^N_+)\right)_{\theta,q}.$$

Proof. For proofs of (1) and (2), refer to [35, Section 2.10], and for a proof of (3), refer to [23, Theorem 8, Theorem 11], [35, Theorem 1.2.4]. □

2.4 Class of multipliers

Let U be a domain in $\mathbb{C}$. Let $m(\lambda, \xi')$ be a function defined on $U \times (\mathbb{R}^{N-1} \setminus \{0\})$ which is holomorphic in $\lambda \in U$ and infinitely many times differentiable with respect to $\xi' \in \mathbb{R}^{N-1} \setminus \{0\}$. If there exists a real number κ such that for any multi-index $\delta' \in \mathbb{N}_0^{N-1}$ and $(\lambda, \xi') \in \Sigma_{\epsilon,\lambda_0} \times (\mathbb{R}^{N-1} \setminus \{0\})$ there hold the estimate

$$\left|D^{\delta'}_{\xi'} m(\lambda, \xi')\right| \leq C_{\delta'} \left(|\lambda|^{1/2} + |\xi'|\right)^{\kappa - |\delta'|}$$

for some constant $C_{\delta'}$ depending on δ', then $m(\lambda, \xi')$ is called a multiplier of order κ with type $\mathbb{M}_\kappa(U)$.

Obviously, for any $m_i \in \mathbb{M}_{\kappa_i}(U)$ $(i = 1, 2)$, we see that $m_1 m_2 \in \mathbb{M}_{\kappa_1+\kappa_2}(U)$. Notice that $|\xi'|^2 \in \mathbb{M}_2(\mathbb{C})$ and $\xi_j \in \mathbb{M}_1(\mathbb{C})$, but any functions of $|\xi'|$ is usually not in $\mathbb{M}_\kappa(U)$ for any κ and U.

2.5 Interpolation of small ℓ^p spaces of vector-valued sequences

Let X be a Banach space, and $(a_\nu)_{\nu=-\infty}^{\infty}$ be a sequence in X. For $s \in \mathbb{R}$, the norm $\|\cdot\|_{\ell^s_q(X)}$ is defined by

$$\|(a_\nu)\|_{\ell^s_q(X)} = \begin{cases} \left(\displaystyle\sum_{\nu=-\infty}^{\infty} (2^{\nu s}\|a_\nu\|_X)^q\right)^{\frac{1}{q}} & (1 \leq q < \infty), \\ \displaystyle\sup_{\nu \in \mathbb{Z}} 2^{\nu s}\|a_\nu\|_X & (q = \infty), \end{cases}$$

where

$$\ell^s_q(X) = \{(a_\nu)_{\nu=-\infty}^{\infty} \mid \|a_\nu\|_{\ell^s_q(X)} < \infty\}.$$

Theorem 2.3. [2, Theorem 5.6.1]. *Assume that* $1 \leq q_0 \leq \infty$, $1 \leq q_1 \leq \infty$ *and that* $s_0 \neq s_1$. *Then we have, for all* $1 \leq q \leq \infty$

$$\left(\ell^{s_0}_{q_0}(X), \ell^{s_1}_{q_1}(X)\right)_{\theta,q} = \ell^s_q(X)$$

where $s = (1-\theta)s_0 + \theta s_1$.

3 Technical tools

We know the following three lemmas due to Enomoto-Shibata [9, Lemma 3.1].

Lemma 3.1. *Let* $0 < \epsilon < \pi/2$ *and* $\nu_0 > 0$. *Let* Σ_ϵ *and* Λ_{ϵ,ν_0} *be the sets defined in* (1.8). *Then, we have the following assertions.*

(1) *For any* $\lambda \in \Sigma_\epsilon$ *and* $\xi \in \mathbb{R}^N$, *there holds*

$$|\alpha^{-1}\lambda + |\xi|^2| \geq (\sin(\epsilon/2))(\alpha^{-1}|\lambda| + |\xi|^2).$$

(2) *Let* $p(\lambda) = (\alpha + \eta_\lambda)^{-1}\lambda$, *where* $\eta_\lambda = \beta + \gamma^2\lambda^{-1}$. *For any* $\nu_0 > 0$ *there exist constants* $\epsilon' \in (0, \pi/2)$ *and* $c_1 > 0$ *depending solely on* ϵ *and* ν_0 *such that for any* $\lambda \in \Lambda_{\epsilon,\nu_0}$ *and* $\xi \in \mathbb{R}^N$, *there hold*

$$|\arg p(\lambda)| \le \pi - \epsilon', \quad |p(\lambda) + |\xi|^2| \ge c_1(|\lambda| + |\xi|^2).$$

(3) *There exists a constant* $c_2 > 0$ *depending solely on* α, β *and* ϵ *such that for any* $\lambda \in \Sigma_\epsilon$ *there holds* $|\alpha + \eta_\lambda| \ge c_2$.

By Lemma 3.1, we have the following multiplier estimates which is used to estimate solution formulas in $\mathbb{R}^N$.

Lemma 3.2. *Let* $0 < \epsilon < \pi/2$, $\nu_0 > 0$ *and* $s \in \mathbb{R}$. *Let* Σ_ϵ *and* Λ_{ϵ,ν_0} *be the sets defined in* (1.8). *Then, for any* $\delta \in \mathbb{N}_0^N$ *there hold*

$$|D_\xi^\delta(\alpha^{-1}\lambda + |\xi|^2)^s| \le C_\delta(|\lambda|^{1/2} + |\xi|)^{s-|\delta|}$$

for any $(\lambda, \xi) \in \Sigma_\epsilon \times (\mathbb{R}^N/\{0\})$ *as well as*

$$|D_\xi^\delta(p(\lambda) + |\xi|^2)^s| \le C_\delta(|\lambda|^{1/2} + |\xi|)^{s-|\delta|}$$

for any $(\lambda, \xi) \in \Lambda_{\epsilon,\nu_0} \times (\mathbb{R}^N/\{0\})$, *where* $p(\lambda) = (\alpha + \eta_\lambda)^{-1}\lambda$ *and* $\eta_\lambda = \beta + \gamma^2\lambda^{-1}$.

Set

$$\begin{aligned} A &= \sqrt{p(\lambda) + |\xi'|^2}, \quad B = \sqrt{\alpha^{-1}\lambda + |\xi'|^2}, \quad K = (\alpha + \eta_\lambda)A + \alpha B, \\ \mathcal{M}(x_N) &= \frac{e^{-Ax_N} - e^{-Bx_N}}{A - B}. \end{aligned} \tag{3.1}$$

These symbols appear in the solution formula (4.7) below. We know the following multiplier's estimates.

Lemma 3.3. *Let* $0 < \epsilon < \pi/2$, $\nu_0 > 0$ *and* $s \in \mathbb{R}$. *Then, for any multi-index* $\delta' \in \mathbb{N}_0^{N-1}$ *there hold*

$$|D_{\xi'}^{\delta'} M^s| \le C_{\delta'}(|\lambda|^{1/2} + |\xi'|)^{s-|\delta'|}$$

for any $(\lambda, \xi') \in \Lambda_{\epsilon,\nu_0} \times (\mathbb{R}^{N-1}/\{0\})$, *where* $M \in \{A, B, K\}$.

Using Lemma 3.3 we have the following lemma.

Lemma 3.4. *Let* $0 < \epsilon < \pi/2$, $\nu_0 > 0$, $s \in \mathbb{R}$, *and* $x_N > 0$. *Then, for any multi-index* $\delta' \in \mathbb{N}_0^{N-1}$ *and* $\lambda \in \Lambda_{\epsilon,\nu_0}$, *there hold*

$$|D_{\xi'}^{\delta'} e^{-Mx_N}| \le C_{\delta'}(|\lambda|^{1/2} + |\xi'|)^{-|\delta'|} e^{-c(|\lambda|^{1/2}+|\xi'|)x_N}, \tag{3.2}$$

$$|D_{\xi'}^{\delta'}(B\mathcal{M}(x_N))| \le C_{\delta'}(|\lambda|^{1/2} + |\xi'|)^{-|\delta'|} e^{-c(|\lambda|^{1/2}+|\xi'|)x_N} \tag{3.3}$$

with some positive constant c, *where* $M \in \{A, B\}$.

Proof. For any $\theta \in [0, 1]$, by Bell's formula we have

$$\begin{aligned} &|D_{\xi'}^{\delta'} e^{-((1-\theta)A+\theta B)x_N}| \\ &\le C_{\delta'} \sum_{\ell=1}^{|\delta'|} x_N^\ell |e^{-((1-\theta)A+\theta B)x_N)}| \Big(\sum_{\substack{\delta'_1+\cdots+\delta'_\ell=\delta' \\ |\delta'_\ell|\ge 1}} |D^{\delta'_1}((1-\theta)A + \theta B)| \cdots |D^{\delta'_\ell}((1-\theta)A + \theta B)| \Big). \end{aligned}$$

Using Lemma 3.3, there exists a constant $c > 0$ such that

$$|e^{-((1-\theta)A+\theta B)x_N)}| \leq e^{-2c(|\lambda|^{1/2}+|\xi'|)x_N}.$$

Therefore, we have

$$|D^{\delta'}_{\xi'} e^{-((1-\theta)A+\theta B)x_N}| \leq C_{\delta'}(|\lambda|^{1/2}+|\xi'|)^{-|\delta'|}e^{-c(|\lambda|^{1/2}+|\xi'|)x_N}. \tag{3.4}$$

Therefore, setting $\theta = 0$ or $\theta = 1$, we have (3.2).

We write

$$B\mathcal{M}(x_N) = Bx_N \int_0^1 e^{-((1-\theta)A+\theta B)x_N)}\, \mathrm{d}\theta.$$

Applying (3.4) and Lemma 3.3 implies (3.3). This completes the proof of Lemma 3.4. □

In this section, we record the following proposition, which plays a crucial role in the proof of Theorem 1.2.

Proposition 3.5. *Let $1 < q < \infty$, $\epsilon \in (0, \pi/2)$, $\lambda_0 > 0$, and $\lambda \in \Lambda_{\epsilon,\lambda_0}$. Suppose that $m_0 \in \mathbb{M}_0$. Define the integral operators L_i, $i = 1, \cdots, 6$, by the formula:*

$$\begin{aligned}
L_1(\lambda)f &= \int_0^\infty \mathcal{F}^{-1}_{\xi'}\left[m_0(\lambda,\xi')B^3\mathcal{M}(x_N)\mathcal{M}(y_N)\hat{f}(\xi',y_N)\right](x')\,dy_N,\\
L_2(\lambda)f &= \int_0^\infty \mathcal{F}^{-1}_{\xi'}\left[m_0(\lambda,\xi')B^2\mathcal{M}(x_N)e^{-Ay_N}\hat{f}(\xi',y_N)\right](x')\,dy_N,\\
L_3(\lambda)f &= \int_0^\infty \mathcal{F}^{-1}_{\xi'}\left[m_0(\lambda,\xi')B^2\mathcal{M}(x_N)e^{-By_N}\hat{f}(\xi',y_N)\right](x')\,dy_N,\\
L_4(\lambda)f &= \int_0^\infty \mathcal{F}^{-1}_{\xi'}\left[m_0(\lambda,\xi')B^2e^{-Ax_N}\mathcal{M}(y_N)\hat{f}(\xi',y_N)\right](x')\,dy_N,\\
L_5(\lambda)f &= \int_0^\infty \mathcal{F}^{-1}_{\xi'}\left[m_0(\lambda,\xi')B^2e^{-Bx_N}\mathcal{M}(y_N)\hat{f}(\xi',y_N)\right](x')\,dy_N,\\
L_6(\lambda)f &= \int_0^\infty \mathcal{F}^{-1}_{\xi'}\left[m_0(\lambda,\xi')Be^{-Jx_N}e^{-Qy_N}\hat{f}(\xi',y_N)\right](x')\,dy_N,
\end{aligned}$$

respectively, where (J,Q) stands for an element of $\{(A,A),(A,B),(B,A),(B,B)\}$ in the formula of L_6. Then for every $f \in L_q(\mathbb{R}^N_+)$, it holds

$$\|L_i(\lambda)f\|_{L_q(\mathbb{R}^N_+)} \leq C_q\|f\|_{L_q(\mathbb{R}^N_+)} \quad (i = 1,2,3,4,5,6).$$

Proposition 3.6. *Let $1 < q < \infty$, $\epsilon \in (0, \pi/2)$, $\lambda_0 > 0$, and $\lambda \in \Lambda_{\epsilon,\lambda_0}$. Suppose that $m_0 \in \mathbb{M}_0$. Define the integral operators P_i, $i = 1, \cdots, 6$, by the formula:*

$$\begin{aligned}
P_1(\lambda)f &= \int_0^\infty \mathcal{F}^{-1}_{\xi'}\left[m_0(\lambda,\xi')B^2\partial_\lambda\left(B^3\mathcal{M}(x_N)\mathcal{M}(y_N)\right)\hat{f}(\xi',y_N)\right](x')\,dy_N,\\
P_2(\lambda)f &= \int_0^\infty \mathcal{F}^{-1}_{\xi'}\left[m_0(\lambda,\xi')B^2\partial_\lambda\left(B^2\mathcal{M}(x_N)e^{-Ay_N}\right)\hat{f}(\xi',y_N)\right](x')\,dy_N,\\
P_3(\lambda)f &= \int_0^\infty \mathcal{F}^{-1}_{\xi'}\left[m_0(\lambda,\xi')B^2\partial_\lambda\left(B^2\mathcal{M}(x_N)e^{-By_N}\right)\hat{f}(\xi',y_N)\right](x')\,dy_N,\\
P_4(\lambda)f &= \int_0^\infty \mathcal{F}^{-1}_{\xi'}\left[m_0(\lambda,\xi')B^2\partial_\lambda\left(B^2e^{-Ax_N}\mathcal{M}(y_N)\right)\hat{f}(\xi',y_N)\right](x')\,dy_N,\\
P_5(\lambda)f &= \int_0^\infty \mathcal{F}^{-1}_{\xi'}\left[m_0(\lambda,\xi')B^2\partial_\lambda\left(B^2e^{-Bx_N}\mathcal{M}(y_N)\right)\hat{f}(\xi',y_N)\right](x')\,dy_N,\\
P_6(\lambda)f &= \int_0^\infty \mathcal{F}^{-1}_{\xi'}\left[m_0(\lambda,\xi')B^2\partial_\lambda\left(Be^{-Jx_N}e^{-Qy_N}\right)\hat{f}(\xi',y_N)\right](x')\,dy_N,
\end{aligned}$$

respectively, where (J, Q) *stands for an element of* $\{(A, A), (A, B), (B, A), (B, B)\}$ *in the formula of* P_6. *Then for every* $f \in L_q(\mathbb{R}^N_+)$, *it holds*

$$\|P_i(\lambda)f\|_{L_q(\mathbb{R}^N_+)} \leq C_q\|f\|_{L_q(\mathbb{R}^N_+)} \quad (i = 1, 2, 3, 4, 5, 6).$$

To show Proposition 3.5, we need the following propositions.

Proposition 3.7. [32, A.3 p.271]. *Let* $1 < q < \infty$. *Define the integral operator* G *by the formula:*

$$Gf(x_N) = \int_0^\infty \frac{f(y_N)}{x_N + y_N}\, dy_N.$$

Then, for every $f \in L_q(0, \infty)$ *there exists a constant* A_q *such that*

$$\|Gf\|_{L_q((0,\infty))} \leq A_q\|f\|_{L_q((0,\infty))}.$$

Proposition 3.8. [29, Theorem 2.3]. *Let* X *be a Banach space,* σ *be a real number satisfying* $0 < \sigma \leq 1$, *and* m *be a nonnegative integer. Set* $\zeta = m + \sigma - (N - 1)$. *In addition, let* $\ell(\sigma)$ *be an integer part of* σ. *Suppose that a function* $f(\xi') \in C^{m+1+\ell(\sigma)}(\mathbb{R}^{N-1} \setminus \{0\}), X)$ *satisfies the following conditions:*

(1) For every $\delta' \in \mathbb{N}_0^{N-1}$ *satisfying* $|\delta'| \leq m$, *it holds* $D^{\delta'}_{\xi'} f(\xi') \in L_1(\mathbb{R}^{N-1}, X)$.

(2) For every $\delta' \in \mathbb{N}_0^{N-1}$ *satisfying* $|\delta'| \leq m + 1 + \ell(\sigma)$, *there exists a constant* $C_{\delta'}$ *such that* $\|D^{\delta'}_{\xi'} f(\xi')\|_X \leq C_{\delta'}|\xi'|^{\zeta-|\delta'|}$ *for all* $\xi' \in \mathbb{R}^{N-1} \setminus \{0\}$.

Then, there exists a constant $C_{N,\zeta}$ *depending on* N *and* ζ *such that*

$$\left\|\mathcal{F}^{-1}_{\xi'}[f](x')\right\|_X \leq C_{N,\zeta}\left(\max_{|\delta'|\leq m+1+\ell(\sigma)} C_\gamma\right)|x'|^{-N-1+\zeta}, \quad (x' \in \mathbb{R}^{N-1} \setminus \{0\}).$$

Proof of Proposition 3.5. Here, we only consider the estimate for $L_1(\lambda)f$, since the others may be proved in a similar way. First, we rewrite B as

$$B = \frac{B^2}{B} = \frac{\lambda^{1/2}}{B}\lambda^{1/2} + \frac{|\xi'|}{B}|\xi'|.$$

We set $m_0^1(\lambda, \xi') = m_0(\lambda, \xi')\lambda^{1/2}B^{-1}$ and $m_0^2(\lambda, \xi') = m_0(\lambda, \xi)|\xi'|B^{-1}$. Define

$$\begin{aligned} L_1^{(1)}(\lambda)f &:= \int_0^\infty \mathcal{F}^{-1}_{\xi'}\left[m_0^1(\lambda, \xi')\lambda^{1/2}B^2\mathcal{M}(x_N)\mathcal{M}(y_N)\hat{f}(\xi', y_N)\right](x')\, dy_N, \\ L_1^{(2)}(\lambda)f &:= \int_0^\infty \mathcal{F}^{-1}_{\xi'}\left[m_0^2(\lambda, \xi')|\xi'|B^2\mathcal{M}(x_N)\mathcal{M}(y_N)\hat{f}(\xi', y_N)\right](x')\, dy_N. \end{aligned}$$

Since for all $\delta' \in \mathbb{N}_0^{N-1}$ we have

$$\left|D^{\delta'}_{\xi'} m_0^1(\lambda, \xi')\right| \leq C_{\delta'}\left(|\lambda|^{1/2} + |\xi'|\right)^{-|\delta'|}, \quad \left|D^{\delta'}_{\xi'} m_0^2(\lambda, \xi')\right| \leq C_{\delta'}|\xi'|^{-|\delta'|} \tag{3.5}$$

as well as

$$\begin{aligned} \left|D^{\delta'}_{\xi'} B\mathcal{M}(x_N)\right| &\leq C_{\delta'}e^{-c(|\lambda|^{1/2}+|\xi'|)x_N}\left(|\lambda|^{1/2} + |\xi'|\right)^{-|\delta'|}, \\ \left|D^{\delta'}_{\xi'} B\mathcal{M}(y_N)\right| &\leq C_{\delta'}e^{-c(|\lambda|^{1/2}+|\xi'|)y_N}\left(|\lambda|^{1/2} + |\xi'|\right)^{-|\delta'|}, \\ \left|D^{\delta'}_{\xi'} B\right| &\leq C_{\delta'}\left(|\lambda|^{1/2} + |\xi'|\right)^{1-|\delta'|} \end{aligned}$$

as follows from Lemma 3.4, we see that

$$\left|D_{\xi'}^{\delta'}\left(m_0^1(\lambda,\xi')\lambda^{1/2}B^2\mathcal{M}(x_N)\mathcal{M}(y_N)\right)\right| \leq C_{\delta'}|\lambda|^{1/2}\left(|\lambda|^{1/2}+|\xi'|\right)^{-|\delta'|}e^{-(c/2)(|\lambda|^{1/2}+|\xi'|)(x_N+y_N)}. \tag{3.6}$$

By virtue of the identity:

$$e^{ix'\cdot\xi'} = \sum_{j=1}^{N-1}\frac{x_j}{i|x'|^2}\frac{\partial}{\partial\xi_j}e^{ix'\cdot\xi'},$$

we may write

$$\begin{aligned}
&\mathcal{F}_{\xi'}^{-1}\left[m_0^1(\lambda,\xi')\lambda^{1/2}B^2\mathcal{M}(x_N)\mathcal{M}(y_N)\right](x') \\
&= \frac{1}{(2\pi)^N}\int_{\mathbb{R}^{N-1}}\left(\sum_{|\delta'|=N}\left(\frac{x'}{i|x'|^2}\right)^{\delta'}D_{\xi'}^{\delta'}e^{ix'\cdot\xi'}\right)m_0^1(\lambda,\xi')\lambda^{1/2}B^2\mathcal{M}(x_N)\mathcal{M}(y_N)\,d\xi' \\
&= \frac{1}{(2\pi)^N}\sum_{|\delta'|=N}\left(\frac{-x'}{i|x'|^2}\right)^{\delta'}\int_{\mathbb{R}^{N-1}}e^{ix'\cdot\xi'}D_{\xi'}^{\delta'}\left(m_0^1(\lambda,\xi')\lambda^{1/2}B^2\mathcal{M}(x_N)\mathcal{M}(y_N)\right)\,d\xi'.
\end{aligned}$$

Hence, we obtain

$$\left|\mathcal{F}_{\xi'}^{-1}\left[m_0^1(\lambda,\xi')\lambda^{1/2}B^2\mathcal{M}(x_N)\mathcal{M}(y_N)\right](x')\right| \leq C|x'|^{-N}\int_{\mathbb{R}^{N-1}}|\lambda|^{1/2}\left(|\lambda|^{1/2}+|\xi'|\right)^{-N}\,d\xi'.$$

By changing of variables $\xi' = |\lambda|^{1/2}\eta'$, it follows that

$$\int_{\mathbb{R}^{N-1}}|\lambda|^{1/2}\left(|\lambda|^{1/2}+|\xi'|\right)^{-N}\,d\xi' = \int_{\mathbb{R}^{N-1}}\left(1+|\eta'|\right)^{-N}\,d\eta' < \infty.$$

Therefore, we have

$$\left|\mathcal{F}_{\xi'}^{-1}\left[m_0^1(\lambda,\xi')\lambda^{1/2}B^2\mathcal{M}(x_N)\mathcal{M}(y_N)\right](x')\right| \leq C|x'|^{-N}. \tag{3.7}$$

In addition, if we take $\delta' = 0$ in (3.6), it follows that

$$\begin{aligned}
&\left|\mathcal{F}_{\xi'}^{-1}\left[m_0^1(\lambda,\xi')\lambda^{1/2}B^2\mathcal{M}(x_N)\mathcal{M}(y_N)\right](x')\right| \\
&\leq C\int_{\mathbb{R}^{N-1}}|\lambda|^{1/2}e^{-(c/2)(|\lambda|^{1/2}+|\xi'|)(x_N+y_N)}\,d\xi' \\
&\leq \frac{C}{(x_N+y_N)^N}\int_{\mathbb{R}^{N-1}}\frac{|\lambda|^{1/2}}{\left(|\lambda|^{1/2}+|\xi'|\right)^N}\,d\xi' \\
&= \frac{C}{(x_N+y_N)^N}\int_{\mathbb{R}^{N-1}}\left(1+|\eta'|\right)^{-N}\,d\eta',
\end{aligned}$$

which together with (3.7) implies

$$\left|\mathcal{F}_{\xi'}^{-1}\left[m_0^1(\lambda,\xi')\lambda^{1/2}B^2\mathcal{M}(x_N)\mathcal{M}(y_N)\right](x')\right| \leq C\left(|x'|+x_N+y_N\right)^{-N}. \tag{3.8}$$

To shorten notation, set $\ell(x') := \mathcal{F}_{\xi'}^{-1}\left[m_0^1(\lambda,\xi')\lambda^{1/2}B^2\mathcal{M}(x_N)\mathcal{M}(y_N)\right](x')$. Then $L_1^{(1)}(\lambda)f$ may be bounded by

$$\begin{aligned}
\|L_1^{(1)}(\lambda)f\|_{L_q(\mathbb{R}^{N-1})} &\leq \int_0^\infty \|\ell * f(\,\cdot\,,y_N)\|_{L_q(\mathbb{R}^{N-1})}\,dy_N \\
&\leq \int_0^\infty \|\ell\|_{L_1(\mathbb{R}^{N-1})}\|f(\,\cdot\,,y_N)\|_{L_q(\mathbb{R}^{N-1})}\,dy_N.
\end{aligned}$$

Then (3.8) together with changing of variables $x' = (x_N + y_N)z'$ yields

$$\begin{aligned}\|\ell\|_{L_1(\mathbb{R}^{N-1})} &= \int_{\mathbb{R}^{N-1}} \frac{C}{(|x'| + x_N + y_N)^N}\,dx' \\ &= \frac{C}{x_N + y_N}\int_{\mathbb{R}^{N-1}} \frac{1}{(1+|z'|)^N}\,dz'.\end{aligned}$$

Thus, we observe

$$\|L_1^{(1)}(\lambda)f\|_{L_q(\mathbb{R}^{N-1})} \le C\int_0^\infty \frac{\|f(\,\cdot\,, y_N)\|_{L_q(\mathbb{R}^{N-1})}}{x_N + y_N}\,dy_N. \tag{3.9}$$

From Proposition 3.7 and (3.9), we have

$$\begin{aligned}&\|L_1^{(1)}(\lambda)f\|_{L_q(\mathbb{R}^N_+)} \\ &\quad\le C\left\|\int_0^\infty \frac{\|f(\,\cdot\,, y_N)\|_{L_q(\mathbb{R}^{N-1})}}{(x_N + y_N)}\,dy_N\right\|_{L_q((0,\infty))} \\ &\quad\le C_q\left\|G\|f\|_{L_q(\mathbb{R}^{N-1})}\right\|_{L_q((0,\infty))} \\ &\quad\le C_q\|f\|_{L_q(\mathbb{R}^N_+)}.\end{aligned}$$

It remains to establish the estimate for $L_1^{(2)}(\lambda)$. In a similar way as in (3.6), we obtain

$$\begin{aligned}&\left|D_{\xi'}^{\delta'}(B^2\mathcal{M}(x_N)\mathcal{M}(y_N))\right| \\ &\quad\le C_{\delta'}e^{-c(|\lambda|^{1/2}+|\xi'|)x_N}e^{-(c/2)(|\lambda|^{1/2}+|\xi'|)y_N}\left(|\lambda|^{1/2}+|\xi'|\right)^{-|\delta'|} \\ &\quad\le C_{\delta'}e^{-(c/2)(|\lambda|^{1/2}+|\xi'|)(x_N+y_N)}|\xi'|^{-|\delta'|}.\end{aligned}$$

Moreover, by $(3.5)_2$, we have

$$\left|D_{\xi'}^{\delta'}\left(m_0^2(\lambda,\xi')|\xi'|\right)\right| \le C_{\delta'}|\xi|^{1-|\delta'|},$$

which yields

$$\left|D_{\xi'}^{\delta'}\left(m_0^2(\lambda,\xi')|\xi'|B^2\mathcal{M}(x_N)\mathcal{M}(y_N)\right)\right| \le C_{\delta'}|\xi'|^{1-|\delta'|}e^{-(c/2)(|\lambda|^{1/2}+|\xi'|)(x_N+y_N)}. \tag{3.10}$$

By (3.10) and Proposition 3.8 we obtain

$$\left|\mathcal{F}_{\xi'}^{-1}\left[m_0^2(\lambda,\xi')|\xi'|B^2\mathcal{M}(x_N)\mathcal{M}(y_N)\right](x')\right| \le C|x'|^{-N}. \tag{3.11}$$

By (3.10) we also obtain

$$\begin{aligned}\left|\int_{\mathbb{R}^{N-1}} e^{ix'\cdot\xi'}m_0^2(\lambda,\xi')|\xi'|B^2\mathcal{M}(x_N)\mathcal{M}(y_N)\,d\xi'\right| &\le C\int_{\mathbb{R}^{N-1}}|\xi'|e^{-(c/2)|\xi'|(x_N+y_N)}\,d\xi' \\ &= \frac{C}{(x_N+y_N)^N}\int_{\mathbb{R}^{N-1}}|\eta'|e^{-(c/2)|\eta'|}\,d\eta',\end{aligned}$$

where we have replaced η' by $\xi'(x_N + y_N) = \eta'$. Thus, we have

$$\left|\mathcal{F}_{\xi'}^{-1}\left[m_0^2(\lambda,\xi')|\xi'|B^2\mathcal{M}(x_N)\mathcal{M}(y_N)\right](x')\right| \le \frac{C}{(x_N+y_N)^N}. \tag{3.12}$$

From (3.11) and (3.12), we deduce that

$$\left|\mathcal{F}^{-1}_{\xi'}\left[m_0^2(\lambda,\xi')|\xi'|B^2\mathcal{M}(x_N)\mathcal{M}(y_N))\right](x')\right| \leq \frac{C}{(|x'|+x_N+y_N)^N}.$$

In a similar way as in (3.9), we arrive at

$$\|L_1^{(2)}(\lambda)f\|_{L_q(\mathbb{R}^{N-1})} \leq C\int_0^\infty \frac{\|f(\,\cdot\,,y_N)\|_{L_q(\mathbb{R}^{N-1})}}{x_N+y_N}\,dy_N,$$

which together with Proposition 3.7 implies that $L_1^{(2)}$ is a bounded linear operator on $L_q(\mathbb{R}^N_+)$. The proof is complete. □

Proof of Proposition 3.6. Here, we only consider the estimate for $P_1(\lambda)f$ and $P_5(\lambda)f$, since the others may be proved in a similar way. First, by Taylor formula, we obtain

$$\mathcal{M}(x_N) = x_N\int_0^1 e^{-(A+\theta(B-A))x_N}\,d\theta.$$

Thus, we have

$$\partial_\lambda\mathcal{M}(x_N) = -x_N^2\int_0^1(\partial_\lambda A+(\partial_\lambda B-\partial_\lambda A)\theta)e^{-(A+\theta(B-A))x_N}\,d\theta.$$

Since we know that $\partial_\lambda B = 1/2\alpha B$, $\partial_\lambda A = p'(\lambda)/2A$, and $|p'(\lambda)| \leq C$ for $\lambda \in \Lambda_{\epsilon,\lambda_0}$, using Lemma 3.1 with $s=-1$, we see that for any $\delta' \in \mathbb{N}_0^{N-1}$

$$\begin{aligned}|D^{\delta'}_{\xi'}(B^3\partial_\lambda\mathcal{M}(x_N))| &\leq C_{\delta'}x_N^2(|\lambda|^{1/2}+|\xi'|)^{2-|\delta'|}e^{-2c(|\lambda|^{1/2}+|\xi'|)x_N}\\ &\leq C_{\delta'}(|\lambda|^{-1/2}+|\xi'|)^{-|\delta'|}e^{-c(|\lambda|^{1/2}+|\xi'|)x_N}\end{aligned} \tag{3.13}$$

with some positive constants $C_{\delta'}$ and c. Here and in the following c denotes a constant independent of δ'. Writing

$$\begin{aligned}B\partial_\lambda(B^3\mathcal{M}(x_N)\mathcal{M}(y_N)) &= 3B^3(\partial_\lambda B)\mathcal{M}(x_N)\mathcal{M}(y_N) + B^4(\partial_\lambda\mathcal{M}(x_N))\mathcal{M}(y_N)\\ &\quad + B^4\mathcal{M}(x_N)(\partial_\lambda\mathcal{M}(y_N)),\end{aligned}$$

and using Lemma 3.4 and (3.13), we see that for any $\delta' \in \mathbb{N}_0^{N-1}$

$$\left|D^{\delta'}_{\xi'}\left(B\partial_\lambda(B^3\mathcal{M}(x_N)\mathcal{M}(y_N))\right)\right| \leq C_{\delta'}(|\lambda|+|\xi'|)^{-|\delta'|}e^{-c(|\lambda|^{1/2}+|\xi'|)(x_N+y_N)}.$$

By the similar method as Proposition 3.5, we can derive $\|P_1(\lambda)f\|_{L_q(\mathbb{R}^N_+)} \leq C\|f\|_{L_q(\mathbb{R}^N_+)}$.

As for $P_5(\lambda)f$, writing $\partial_\lambda e^{-Ax_N} = -(\partial_\lambda A)x_Ne^{-Ax_N}$, using Lemma 3.4, we see that for any $\delta' \in \mathbb{N}_0^{N-1}$

$$|D^{\delta'}_{\xi'}B^2(\partial_\lambda e^{-A_N})| \leq Ce^{-c(|\lambda|^{1/2}+|\xi'|)x_N} \tag{3.14}$$

with some positive constants $C_{\delta'}$ and c. Writing

$$\begin{aligned}B\partial_\lambda(B^2e^{-Ax_N}\mathcal{M}(y_N)) &= 2B^2(\partial_\lambda B)e^{-Ax_N}\mathcal{M}(y_N) - B^3(\partial_\lambda e^{-AxN})\mathcal{M}(x_N)\\ &\quad + B^3e^{-Ax_N}(\partial_\lambda\mathcal{M}(y_N))\end{aligned}$$

using (3.13), (3.14) and Lemma 3.4, we see that for any $\delta' \in \mathbb{N}_0^{N-1}$,

$$\left|D^{\delta'}_{\xi'}\left(B\partial_\lambda(B^2e^{-Ax_N}\mathcal{M}(y_N))\right)\right| \leq C_{\delta'}\left(|\lambda|^{1/2}+|\xi'|\right)^{-|\delta'|}e^{-c(|\lambda|^{1/2}+|\xi'|)x_N}$$

with some positive constants $C_{\delta'}$ and c. By the similar method as Proposition 3.5 again, we can derive $\|P_5(\lambda)f\|_{L_q(\mathbb{R}^N_+)} \leq C\|f\|_{L_q(\mathbb{R}^N_+)}$. The proof is completed. □

4 Solution formula

In this section, we shall discuss solution formulas of equations (1.7). From the first equation in (1.7), we have $\rho = \lambda^{-1}(f - \gamma \operatorname{div} \mathbf{u})$, and inserting this formula into the second equation in (1.7) implies the complex Lamé equations

$$\lambda \mathbf{u} - \alpha \Delta \mathbf{u} - \eta_\lambda \nabla \operatorname{div} \mathbf{u} = \mathbf{g} - \gamma \lambda^{-1} \nabla f \quad \text{in } \mathbb{R}^N_+, \quad \mathbf{u}|_{\partial \mathbb{R}^N_+} = 0. \tag{4.1}$$

Here, we have set $\eta_\lambda = \beta + \gamma^2 \lambda^{-1}$.

If we find a solution $\mathbf{u}$ of equations (4.1) and if we set $\rho = \lambda^{-1}(f - \gamma \operatorname{div} \mathbf{u})$, then ρ and $\mathbf{u}$ are solutions of equations (1.7). Thus, in this section, we shall drive solution formulas of the complex Lamé equations

$$\lambda \mathbf{u} - \alpha \Delta \mathbf{u} - \eta_\lambda \nabla \operatorname{div} \mathbf{u} = \mathbf{g} \quad \text{in } \mathbb{R}^N_+, \quad \mathbf{u}|_{\partial \mathbb{R}^N_+} = 0. \tag{4.2}$$

4.1 Whole space case

For $\epsilon \in (0, \pi/2)$ and $\lambda_0 > 0$ let $\lambda \in \Lambda_{\epsilon,\lambda_0}$ be the resolvent parameter, where λ_0 is assumed to be sufficiently large if necessary. In this subsection, we derive the representation of the solution formula for the following model problem in $\mathbb{R}^N$:

$$\lambda \mathbf{u} - \alpha \Delta \mathbf{u} - \eta_\lambda \nabla \operatorname{div} \mathbf{u} = \mathbf{g} \quad \text{in } \mathbb{R}^N, \tag{4.3}$$

where $\mathbf{g} \in B^s_{q,1}(\mathbb{R}^N)^N$, with $1 < q < \infty$ and $-1 + 1/q < s < 1/q$. Applying the divergence to equation (4.3) yields

$$\lambda \operatorname{div} \mathbf{u} - (\alpha + \eta_\lambda) \Delta \operatorname{div} \mathbf{u} = \operatorname{div} \mathbf{g} \quad \text{in } \mathbb{R}^N. \tag{4.4}$$

Applying Fourier transform to (4.4) yields

$$(\lambda + (\alpha + \eta_\lambda)|\xi|^2) \mathcal{F}[\operatorname{div} \mathbf{u}](\xi) = i\xi \cdot \mathcal{F}[\mathbf{g}](\xi).$$

Applying Fourier transform to equation (4.3) yields

$$(\lambda + \alpha|\xi|^2)\hat{\mathbf{u}} - \eta_\lambda i\xi \mathcal{F}[\operatorname{div} \mathbf{u}] = \hat{\mathbf{g}}.$$

Thus,

$$\begin{aligned}
\hat{\mathbf{u}}(\xi) &= (\lambda + \alpha|\xi|^2)^{-1}(\hat{\mathbf{g}}(\xi) + \eta_\lambda i\xi(\lambda + (\alpha + \eta_\lambda)|\xi|^2)^{-1} i\xi \cdot \hat{\mathbf{g}}(\xi)) \\
&= \frac{\hat{\mathbf{g}}(\xi)}{\lambda + \alpha|\xi|^2} + \frac{\eta_\lambda (i\xi \otimes i\xi)\hat{\mathbf{g}}(\xi)}{(\lambda + \alpha|\xi|^2)(\lambda + (\alpha + \eta_\lambda)|\xi|^2)} \\
&= \frac{1}{\alpha} \frac{\hat{\mathbf{g}}(\xi)}{\alpha^{-1}\lambda + |\xi|^2} + \frac{\eta_\lambda}{\alpha(\alpha + \eta_\lambda)} \frac{(i\xi \otimes i\xi)\hat{\mathbf{g}}(\xi)}{(\alpha^{-1}\lambda + |\xi|^2)(p(\lambda) + |\xi|^2)},
\end{aligned}$$

where we have set

$$p(\lambda) = \frac{\lambda}{\alpha + \eta_\lambda} = \frac{\lambda^2}{(\alpha + \beta)\lambda + \gamma^2}.$$

Applying the Fourier inverse transform implies that

$$\mathbf{u} = \mathcal{F}^{-1}\left[\frac{\hat{\mathbf{g}}(\xi)}{\lambda + \alpha|\xi|^2}\right] - \frac{\beta\lambda + \gamma^2}{(\alpha + \beta)\lambda + \gamma^2} \mathcal{F}^{-1}\left[\frac{(\xi \otimes \xi)\hat{\mathbf{g}}(\xi)}{(\lambda + \alpha|\xi|^2)(p(\lambda) + |\xi|^2)}\right].$$

Thus, for the later use, we define an operator $\mathcal{S}_0(\lambda)$ by

$$\mathcal{S}^0(\lambda)\mathbf{g} = \mathcal{F}^{-1}\left[\frac{\hat{\mathbf{g}}(\xi)}{\lambda + \alpha|\xi|^2}\right] - \frac{\beta\lambda + \gamma^2}{(\alpha + \beta)\lambda + \gamma^2} \mathcal{F}^{-1}\left[\frac{(\xi \otimes \xi)\hat{\mathbf{g}}(\xi)}{(\lambda + \alpha|\xi|^2)(p(\lambda) + |\xi|^2)}\right], \tag{4.5}$$

which is a solution operator of equation (4.3).

4.2 Half space case

Let $\epsilon \in (0, \pi/2)$ and $\nu_0 > 0$. Let $\gamma_0 > 0$ be a large number such that $\Sigma_\epsilon + \gamma_0 \subset K_\epsilon \cap \Sigma_\epsilon \cap \{\lambda \in \mathbb{C} \mid |\lambda| \geq \nu_0\}$. In this subsection, we derive the representation of the solution formula for equations (4.2). To this end, we extend $\mathbf{g} = (g_1, \cdots, g_N)$ by

$$g_j^e(x) = \begin{cases} g_j(x) & \text{for } x_N > 0, \\ g_j(x', -x_N) & \text{for } x_N < 0, \end{cases} \quad g_N^o(x) = \begin{cases} g_N(x) & \text{for } x_N > 0, \\ -g_N(x', -x_N) & \text{for } x_N < 0. \end{cases}$$

Here and in the sequel j and k run from 1 through $N-1$. We now set $\mathbf{G} := (g_1^e, \cdots, g_{N-1}^e, g_N^o)$. Let $\mathbf{u}$ be a solution of equations (4.2) and let $\mathbf{w} = \mathbf{u} - \mathcal{S}^0(\lambda)\mathbf{G}$, and then $\mathbf{w}$ should satisfy the equations

$$\lambda \mathbf{w} - \alpha \Delta \mathbf{w} - \eta_\lambda \nabla \operatorname{div} \mathbf{w} = 0 \quad \text{in } \mathbb{R}^N_+, \quad \mathbf{w}|_{\partial \mathbb{R}^N_+} = -\mathcal{S}^0(\lambda)\mathbf{G}|_{\partial \mathbb{R}^N_+}. \tag{4.6}$$

In view of (4.5), we may have

$$\mathcal{S}^0(\lambda)\mathbf{G} = \mathcal{F}^{-1}\left[\frac{\hat{\mathbf{G}}(\xi)}{\lambda + \alpha|\xi|^2}\right] - \frac{\beta\lambda + \gamma^2}{(\alpha+\beta)\lambda + \gamma^2}\mathcal{F}^{-1}\left[\frac{(\xi \otimes \xi)\hat{\mathbf{G}}(\xi)}{(\lambda + \alpha|\xi|^2)(p(\lambda) + |\xi|^2)}\right].$$

Let $\mathbf{w} = (w_1, \ldots, w_N)$, and we shall investigate the formula of the partial Fourier transform $\mathcal{F}'[w_j](\xi', x_N)$ of w_j. Applying the partial Fourier transform $\mathcal{F}'$ to equations (4.6), we have the ordinary differential equations in x_N variable, which reads as

$$\begin{cases} (\lambda + \alpha|\xi'|^2)\mathcal{F}'[w_j](x_N) - \alpha\partial_N^2\mathcal{F}'[w_j](x_N) \\ \quad - \eta_\lambda i\xi_j(i\xi' \cdot \mathcal{F}'[\mathbf{w}'](x_N) + \partial_N \mathcal{F}'[w_N](x_N)) = 0, & \text{for } x_N > 0, \\ (\lambda + \alpha|\xi'|^2)\mathcal{F}'[w_N](x_N) - \alpha\partial_N^2\mathcal{F}'[w_N](x_N) \\ \quad - \eta_\lambda \partial_N(i\xi' \cdot \mathcal{F}'[\mathbf{w}'](x_N) + \partial_N \mathcal{F}'[w_N](x_N)) = 0, & \text{for } x_N > 0, \\ \mathcal{F}'[\mathbf{w}](0) = -\mathcal{F}'[\mathcal{S}^0(\lambda)\mathbf{G}](0). \end{cases}$$

Here, we have set $\mathcal{F}'[f](\xi', x_N) = \mathcal{F}'[f](x_N)$, $i\xi' \cdot \mathcal{F}'[\mathbf{w}'](x_N) = \sum_{j=1}^{N-1} i\xi_j \mathcal{F}'[w_j](x_N)$.

To obtain $\mathcal{F}'[w_j](\xi', x_N)$, first we derive the representation of $\mathcal{F}'[\mathcal{S}^0(\lambda)\mathbf{G}](0)$. Notice that

$$\begin{aligned} &\mathcal{F}'[\mathcal{S}^0(\lambda)\mathbf{G}](0) \\ &= \frac{1}{\alpha}\frac{1}{2\pi}\int_{\mathbb{R}} \frac{\hat{\mathbf{G}}(\xi)}{\lambda\alpha^{-1} + |\xi|^2}\, \mathrm{d}\xi_N - \frac{\beta\lambda + \gamma^2}{\alpha((\alpha+\beta)\lambda + \gamma^2)}\frac{1}{2\pi}\int_{\mathbb{R}} \frac{(\xi \otimes \xi)\hat{\mathbf{G}}(\xi)}{(\lambda\alpha^{-1} + |\xi|^2)(p(\lambda) + |\xi|^2)}\, \mathrm{d}\xi_N. \end{aligned}$$

Notice that $\alpha^{-1}\lambda + |\xi|^2 = (\xi_N + iB)(\xi_N - iB)$ and $p(\lambda) + |\xi|^2 = (\xi_N + iA)(\xi_N - iA)$. By the residue theorem in the theory of one complex variable, we have

$$\begin{aligned} h_j^{(1)} :&= \frac{1}{2\pi}\int_{\mathbb{R}} \frac{\hat{g}_j^e(\xi)}{\alpha^{-1}\lambda + |\xi|^2}\, \mathrm{d}\xi_N \\ &= i\int_0^\infty \mathcal{F}'[g_j](\xi', y_N)\Big(\frac{1}{2\pi i}\int_{\mathbb{R}} \frac{e^{-iy_N\xi_N} + e^{iy_N\xi_N}}{(\xi_N + iB)(\xi_N - iB)}\, d\xi_N\Big)\, dy_N \\ &= i\int_0^\infty \mathcal{F}'[g_j](\xi', y_N)\Big(-\frac{e^{-y_N B}}{-2iB} + \frac{e^{-y_N B}}{2iB}\Big)\, dy_N \\ &= \int_0^\infty \frac{e^{-y_N B}}{B}\mathcal{F}'[g_j](\xi', y_N)\, dy_N; \end{aligned}$$

$$h_N^{(1)} := \frac{1}{2\pi}\int_{\mathbb{R}} \frac{\hat{g}_N^o(\xi)}{\alpha^{-1}\lambda + |\xi|^2}\, \mathrm{d}\xi_N$$

$$
\begin{aligned}
&= i\int_0^\infty \mathcal{F}'[g_N](\xi', y_N)\Big(\frac{1}{2\pi i}\int_{\mathbb{R}} \frac{e^{-iy_N\xi_N} - e^{iy_N\xi_N}}{(\xi_N + iB)(\xi_N - iB)}\, d\xi_N\Big)\, dy_N \\
&= i\int_0^\infty \mathcal{F}'[g_N](\xi', y_N)\Big(-\frac{e^{-y_N B}}{-2iB} - \frac{e^{-y_N B}}{2iB}\Big)\, dy_N = 0.
\end{aligned}
$$

Likewise, for each $1 \leq j, k \leq N-1$, we also have

$$
\begin{aligned}
h^{(2)}_{jk} :&= \frac{1}{2\pi}\int_{\mathbb{R}} \frac{\xi_j\xi_k \hat{g}^e_k(\xi)}{(\alpha^{-1}\lambda + |\xi|^2)(p(\lambda) + |\xi|^2)}\, \mathrm{d}\xi_N \\
&= i\int_0^\infty \xi_j\xi_k \mathcal{F}'[g_k](\xi', y_N)\Big(\frac{1}{2\pi i}\int_{\mathbb{R}} \frac{e^{-iy_N\xi_N} + e^{iy_N\xi_N}}{(A^2 + \xi_N^2)(B^2 + \xi_N^2)}\, d\xi_N\Big)\, dy_N \\
&= i\int_0^\infty \mathcal{F}'[g_k](\xi', y_N)\xi_j\xi_k\Big(\frac{e^{-y_N B}}{(A^2 - B^2)iB} + \frac{e^{-y_N A}}{(B^2 - A^2)iA}\Big)\, dy_N \\
&= \int_0^\infty \Big(\frac{e^{-Ay_N}}{A} - \frac{e^{-By_N}}{B}\Big)\frac{\xi_j\xi_k}{B^2 - A^2}\mathcal{F}'[g_k](\xi', y_N)\, dy_N;
\end{aligned}
$$

while for each $1 \leq j \leq N-1$, we obtain

$$
\begin{aligned}
h^{(2)}_{jN} &= \frac{1}{2\pi}\int_{\mathbb{R}} \frac{\xi_j\xi_N \hat{g}^o_N(\xi)}{(\alpha^{-1}\lambda + |\xi|^2)(p(\lambda) + |\xi|^2)}\, \mathrm{d}\xi_N \\
&= i\int_0^\infty \mathcal{F}'[g_N](\xi', y_N)\xi_j\Big(\frac{1}{2\pi i}\int_{\mathbb{R}} \frac{\xi_N(e^{-iy_N\xi_N} - e^{iy_N\xi_N})}{(A^2 + \xi_N^2)(B^2 + \xi_N^2)}\, d\xi_N\Big)\, dy_N \\
&= i\int_0^\infty \mathcal{F}'[g_N](\xi', y_N)\xi_j\Big(-\frac{e^{-y_N B}(-iB)}{(A^2 - B^2)(-2iB)} - \frac{e^{-By_N}(iB)}{(A^2 - B^2)(2iB)} \\
&\qquad - \frac{e^{-Ay_N}(-iA)}{(B^2 - A^2)(-2iA)} - \frac{e^{-Ay_N}(iA)}{(B^2 - A^2)(2iA)}\Big)\, dy_N \\
&= i\int_0^\infty \frac{e^{-Ay_N} - e^{-By_N}}{A^2 - B^2}\xi_j \mathcal{F}'[g_N](\xi', y_N)\, dy_N.
\end{aligned}
$$

In addition, for the case $1 \leq j \leq N-1$ we see that

$$
\begin{aligned}
h^{(2)}_{Nk} &= \frac{1}{2\pi}\int_{\mathbb{R}} \frac{\xi_N\xi_k \hat{g}^e_k(\xi)}{(\alpha^{-1}\lambda + |\xi|^2)(p(\lambda) + |\xi|^2)}\, \mathrm{d}\xi_N \\
&= i\int_0^\infty \mathcal{F}'[g_k](\xi', y_N)\xi_k\Big(\frac{1}{2\pi i}\int_{\mathbb{R}} \frac{\xi_N(e^{-iy_N\xi_N} + e^{iy_N\xi_N})}{(A^2 + \xi_N^2)(B^2 + \xi_N^2)}\, d\xi_N\Big)\, dy_N \\
&= i\int_0^\infty \mathcal{F}'[g_k](\xi', y_N)\xi_k\Big(-\frac{e^{-y_N B}(-iB)}{(A^2 - B^2)(-2iB)} + \frac{e^{-By_N}(iB)}{(A^2 - B^2)(2iB)} \\
&\qquad - \frac{e^{-Ay_N}(-iA)}{(B^2 - A^2)(-2iA)} + \frac{e^{-Ay_N}(iA)}{(B^2 - A^2)(2iA)}\Big)\, dy_N \\
&= 0,
\end{aligned}
$$

as well as

$$
\begin{aligned}
h^{(2)}_{NN} &= \frac{1}{2\pi}\int_{\mathbb{R}} \frac{\xi_N^2 \hat{g}^o_N(\xi)}{(\alpha^{-1}\lambda + |\xi|^2)(p(\lambda) + |\xi|^2)}\, \mathrm{d}\xi_N \\
&= i\int_0^\infty \mathcal{F}'[g_N](\xi', y_N)\Big(\frac{1}{2\pi i}\int_{\mathbb{R}} \frac{\xi_N^2(e^{-iy_N\xi_N} - e^{iy_N\xi_N})}{(\xi_N^2 + A^2)(\xi_N^2 + B^2)}\Big)\, dy_N \\
&= i\int_0^\infty \mathcal{F}'[g_N](\xi', y_N)\Big(-\frac{e^{-y_N B}(-iB)^2}{(A^2 - B^2)(-2iB)} - \frac{e^{-By_N}(iB)^2}{(A^2 - B^2)(2iB)}
\end{aligned}
$$

$$
\begin{aligned}
&\qquad - \frac{e^{-Ay_N}(-iA)^2}{(B^2-A^2)(-2iA)} - \frac{e^{-Ay_N}(iA)^2}{(B^2-A^2)(2iA)}\Big)\, dy_N \\
&= 0.
\end{aligned}
$$

We write

$$
\Big(\frac{e^{-Ay_N}}{A} - \frac{e^{-By_N}}{B}\Big)\frac{1}{B^2-A^2} = -\frac{\mathcal{M}(y_N)}{A(A+B)} + \frac{e^{-By_N}}{AB(A+B)}, \quad \frac{e^{-Ay_N}-e^{-By_N}}{A^2-B^2} = \frac{\mathcal{M}(y_N)}{A+B}.
$$

Let h_j be the j-th component of $-\mathcal{F}'[\mathcal{S}^0(\lambda)\mathbf{G}](0)$, and then we have

$$
\begin{aligned}
h_j &:= -\frac{1}{\alpha}h_j^{(1)} + \frac{\beta\lambda+\gamma^2}{\alpha((\alpha+\beta)\lambda+\gamma^2)}\sum_{k=1}^N h_{jk}^{(2)} \\
&= -\frac{1}{\alpha B}\int_0^\infty \mathcal{F}'[g_j](\xi', y_N)e^{-y_N B}\, dy_N \\
&\quad - \frac{\beta\lambda+\gamma^2}{\alpha((\alpha+\beta)\lambda+\gamma^2)}\int_0^\infty \mathcal{M}(y_N)\frac{\xi_j}{A+B}\Big(\sum_{k=1}^{N-1}\frac{\xi_k}{A}\mathcal{F}'[g_k](\xi', y_N) - i\mathcal{F}'[g_N](\xi', y_N)\Big)\, dy_N \\
&\quad + \frac{\beta\lambda+\gamma^2}{\alpha((\alpha+\beta)\lambda+\gamma^2)}\int_0^\infty e^{-By_N}\frac{i\xi_j}{B(A+B)}\sum_{k=1}^{N-1}\frac{\xi_k}{A}\mathcal{F}'[g_k](\xi', y_N)\, dy_N,
\end{aligned}
$$

for $j = 1, \ldots, N-1$ and $h_N = 0$. According to [9, (4.9)], we have

$$
\mathcal{F}'[w_j](\xi', x_N) = h_j e^{-Bx_N} - \frac{i\xi_j\eta_\lambda}{K}\mathcal{M}(x_N)i\xi'\cdot h', \quad \mathcal{F}'[w_N](\xi', x_N) = \frac{A\eta_\lambda}{K}\mathcal{M}(x_N)i\xi'\cdot h',
$$

where $K = (\alpha+\eta_\lambda)A + \alpha B$ and $i\xi'\cdot h' = \sum_{j=1}^{N-1} i\xi_j h_j$.

We calculate the right hand side. For notational simplicity, we write $\mathcal{F}'[g_j](\xi', y_N) = \mathcal{F}'[g_j]$, namely, (ξ', y_N) is omitted. We have

$$
\begin{aligned}
h_j &= \frac{-1}{\alpha}\int_0^\infty \frac{e^{-y_N B}}{B}\mathcal{F}'[g_j]\, dy_N \\
&\quad - \frac{\beta\lambda+\gamma^2}{\alpha((\alpha+\beta)\lambda+\gamma^2)}\int_0^\infty \mathcal{M}(y_N)\Big(\sum_{k=1}^{N-1}\frac{\xi_j\xi_k}{(A+B)A}\mathcal{F}'[g_k] - \frac{i\xi_j}{A+B}\mathcal{F}'[g_N]\Big)\, dy_N \\
&\quad + \frac{\beta\lambda+\gamma^2}{\alpha((\alpha+\beta)\lambda+\gamma^2)}\int_0^\infty \frac{e^{-By_N}}{B}\sum_{k=1}^{N-1}\frac{i\xi_j\xi_k}{(A+B)A}\mathcal{F}'[g_k]\, dy_N; \\
i\xi'\cdot h' &= \frac{-1}{\alpha}\int_0^\infty \frac{e^{-By_N}}{B}\, i\xi'\cdot\mathcal{F}'[\mathbf{g}']e^{-y_N B}\, dy_N \\
&\quad + \frac{\beta\lambda+\gamma^2}{\alpha((\alpha+\beta)\lambda+\gamma^2)}\int_0^\infty \mathcal{M}(y_N)\Big(\sum_{k=1}^{N-1}\frac{|\xi'|^2\xi_k}{i(A+B)A}\mathcal{F}'[g_k] - \frac{|\xi'|^2}{A+B}\mathcal{F}'[g_N]\Big)\, dy_N \\
&\quad - \frac{\beta\lambda+\gamma^2}{\alpha((\alpha+\beta)\lambda+\gamma^2)}\int_0^\infty \frac{e^{-By_N}}{B}\sum_{k=1}^{N-1}\frac{|\xi'|^2\xi_k}{(A+B)A}\mathcal{F}'[g_k]\, dy_N.
\end{aligned}
$$

Thus, we have

$$
\mathcal{F}'[w_j](\xi', x_N) = -\int_0^\infty Be^{-(x_N+y_N)B}\frac{1}{\alpha B^2}\mathcal{F}'[g_j]\, dy_N
$$

$$
\begin{aligned}
&- \int_0^\infty B^2 e^{-Bx_N} \mathcal{M}(y_N) \frac{\beta\lambda+\gamma^2}{\alpha((\alpha+\beta)\lambda+\gamma^2)} (\sum_{k=1}^{N-1} \frac{\xi_j\xi_k}{(A+B)AB^2} \mathcal{F}'[g_k] - \frac{i\xi_j}{(A+B)B^2} \mathcal{F}'[g_N])\, dy_N \\
&+ \int_0^\infty B e^{-B(x_N+y_N)} \frac{\beta\lambda+\gamma^2}{\alpha((\alpha+\beta)\lambda+\gamma^2)} \sum_{k=1}^{N-1} \frac{i\xi_j\xi_k}{(A+B)AB^2} \mathcal{F}'[g_k]\, dy_N \\
&+ \int_0^\infty B^2 \mathcal{M}(x_N) e^{-By_N} \frac{i\xi_j\eta_\lambda}{K} \frac{1}{\alpha B^3} i\xi' \cdot \mathcal{F}'[\mathbf{g}']\, dy_N \\
&- \int_0^\infty B^3 \mathcal{M}(x_N)\mathcal{M}(y_N) \frac{\xi_j\eta_\lambda}{K} \frac{\beta\lambda+\gamma^2}{\alpha((\alpha+\beta)\lambda+\gamma^2)} (\sum_{k=1}^{N-1} \frac{|\xi'|^2\xi_k}{(A+B)AB^3} \mathcal{F}'[g_k] \\
&\qquad\qquad - \frac{i|\xi'|^2}{(A+B)B^3} \mathcal{F}'[g_N])\, dy_N \\
&+ \int_0^\infty B^2 \mathcal{M}(x_N) e^{-By_N} \frac{i\xi_j\eta_\lambda}{K} \frac{\beta\lambda+\gamma^2}{\alpha((\alpha+\beta)\lambda+\gamma^2)} \sum_{k=1}^{N-1} \frac{|\xi'|^2\xi_k}{(A+B)AB^3} \mathcal{F}'[g_k]\, dy_N, \\
&\mathcal{F}'[w_N](\xi', x_N) = - \int_0^\infty B^2 \mathcal{M}(x_N) e^{-By_N} \frac{A\eta_\lambda}{K} \frac{1}{\alpha B^3} i\xi' \cdot \mathcal{F}'[\mathbf{g}']\, dy_N \\
&+ \int_0^\infty B^3 \mathcal{M}(x_N)\mathcal{M}(y_N) \frac{A\eta_\lambda}{K} \frac{\beta\lambda+\gamma^2}{\alpha((\alpha+\beta)\lambda+\gamma^2)} (\sum_{k=1}^{N-1} \frac{|\xi'|^2\xi_k}{(A+B)AB^3} \mathcal{F}'[g_k] \\
&\qquad\qquad - \frac{|\xi'|^2}{(A+B)B^3} \mathcal{F}'[g_N])\, dy_N \\
&- \int_0^\infty B^2 \mathcal{M}(x_N) e^{-By_N} \frac{A\eta_\lambda}{K} \frac{\beta\lambda+\gamma^2}{\alpha((\alpha+\beta)\lambda+\gamma^2)} \sum_{k=1}^{N-1} \frac{|\xi'|^2\xi_k}{(A+B)AB^3} \mathcal{F}'[g_k]\, dy_N. \qquad (4.7)
\end{aligned}
$$

5 Estimates of solution operators in the whole space

In this section, we shall estimate the solution operator $\mathcal{S}^0(\lambda)$ defined in (4.5). To this end, we use the Fourier multiplier theorem of Mihlin-Hörmander type [11,21]. Let $m(\xi)$ be a complex-valued function defined on $\mathbb{R}^N \setminus \{0\}$ which satisfies the multiplier conditions:

$$
|D_\xi^\delta m(\xi)| \le C_\delta |\xi|^{-|\delta|} \qquad (5.1)
$$

for any multi-index $\delta \in \mathbb{N}_0^N$ with some constant C_δ depending on δ. We say that $m(\xi)$ is a multiplier. Then, the Fourier multiplier operator with kernel function $m(\xi)$ is defined by

$$
T_m f = \mathcal{F}_\xi^{-1}[m(\xi)\mathcal{F}[f](\xi)] = \frac{1}{(2\pi)^N} \int_{\mathbb{R}^N} e^{ix\cdot\xi} m(\xi)\mathcal{F}[f](\xi)\, \mathrm{d}\xi \quad \text{for } f \in \mathcal{S}(\mathbb{R}^N). \qquad (5.2)
$$

Then, we have the following theorem called the Fourier multiplier theorem.

Theorem 5.1. *Let $1 < q < \infty$ and $m(\xi)$ be a multiplier. Then, the Fourier multiplier T_m is an $L_q(\mathbb{R}^N)$ bounded operator, that is there exists a constant depending on q and N such that*

$$
\|T_m f\|_{L_q(\mathbb{R}^N)} \le C(\max_{|\delta|\le[N/2]+1} C_\delta)\|f\|_{L_q(\mathbb{R}^N)}.
$$

Here, $[N/2]$ denotes the integer part of $N/2$.

T_m is extended uniquely to an operator on $L_q(\mathbb{R}^N)$, which is also written by T_m.

To estimate solution operators, we use the following lemma.

Lemma 5.2. *Let $1 < q < \infty$, $1 \le r \le \infty$ and s, σ be two real numbers. Let $m(\xi)$ be a complex valued C^∞ function defined on $\mathbb{R}^N \setminus \{0\}$ satisfying* (5.1) *and let T_m be an operator defined by* (5.2). *Then, for any $f \in B^s_{q,r}(\mathbb{R}^N)$, there holds*

$$\|T_m f\|_{B^s_{q,r}(\mathbb{R}^N)} \le C_{s,q,r}(\max_{|\delta|\le[N/2]+1} C_\delta)\|f\|_{B^s_{q,r}(\mathbb{R}^N)}.$$

Here, C_α are constants appearing in (5.1).

Moreover, let

$$< D >^\sigma f = \mathcal{F}^{-1}[(1+|\xi|^2)^{\frac{\sigma}{2}}\mathcal{F}[f](\xi)] = \frac{1}{(2\pi)^N}\int_{\mathbb{R}^N} e^{ix\cdot\xi}(1+|\xi|^2)^{\frac{\sigma}{2}}\mathcal{F}[f](\xi)\,\mathrm{d}\xi.$$

Then,

$$\| < D >^\sigma f\|_{B^s_{q,r}(\mathbb{R}^N)} \le C\|f\|_{B^{s+\sigma}_{q,r}(\mathbb{R}^N)}. \tag{5.3}$$

Proof. Let ψ, ϕ, and ϕ_k be functions given in (2.1). Since m satisfies the condition (5.1), by Theorem 5.1 we have

$$\|\psi * T_m f\|_{L_q(\mathbb{R}^N)} = \|\mathcal{F}^{-1}_\xi[m(\xi)\psi(\xi)\mathcal{F}[f](\xi)]\|_{L_q(\mathbb{R}^N)} \le CD\|\mathcal{F}^{-1}_\xi[\psi(\xi)\mathcal{F}[f](\xi)]\|_{L_q(\mathbb{R}^N)},$$

where $D = \max_{|\delta|\le[N/2]+1} C_\delta$. Likewise, we have

$$\|\phi_k * T_m f\|_{L_q(\mathbb{R}^N)} = \|\mathcal{F}^{-1}_\xi[m(\xi)\phi_k(\xi)\mathcal{F}[f](\xi)]\|_{L_q(\mathbb{R}^N)} \le CD\|\mathcal{F}^{-1}_\xi[\phi_k(\xi)\mathcal{F}[f](\xi)]\|_{L_q(\mathbb{R}^N)}.$$

Thus, from the definition (2.2) we have

$$\|T_m f\|_{B^s_{q,r}(\mathbb{R}^N)} \le CD\|f\|_{B^s_{q,r}(\mathbb{R}^N)}.$$

To prove (5.3), we choose two $C^\infty_0(\mathbb{R}^N)$ functions $\tilde\phi$ and $\tilde\psi$ such that $\tilde\phi(\xi) = 1$ on $\operatorname{supp}\psi$, $\tilde\psi(\xi) = 1$ on $\operatorname{supp}\phi$, and $\tilde\phi$ vanishes outside of $\{\xi \in \mathbb{R}^N \mid 1/4 \le |\xi| \le 4\}$. We see that

$$|D^\delta_\xi((1+|\xi|^2)^{\frac{\sigma}{2}}\tilde\psi)| \le C_\delta|\xi|^{-|\delta|},$$
$$|D^\delta_\xi(2^{-\sigma k}((1+|\xi|^2)^{\frac{\sigma}{2}}\tilde\phi(2^{-k}\xi))| \le C_\delta|\xi|^{-|\delta|}$$

for any multi-index $\delta \in \mathbb{N}^N_0$. By Theorem 5.1, we have

$$\begin{aligned}\|\psi* < D >^\sigma f\|_{L_q(\mathbb{R}^N)} &= \|\mathcal{F}^{-1}[\mathcal{F}[\psi](\xi)(1+|\xi|^2)^{\frac{\sigma}{2}}\tilde\psi(\xi)\mathcal{F}[f](\xi)]\|_{L_q(\mathbb{R}^N)}\\ &\le C_\sigma\|\mathcal{F}^{-1}[\mathcal{F}[\psi](\xi)\mathcal{F}[f](\xi)]\|_{L_q(\mathbb{R}^N)} = C_\sigma\|\psi * f\|_{L_q(\mathbb{R}^N)}.\end{aligned}$$

Likewise,

$$\begin{aligned}2^{sk}\|\phi_k* < D >^\sigma f\|_{L_q(\mathbb{R}^N)} &= 2^{sk}2^{\sigma k}\|\mathcal{F}^{-1}[\mathcal{F}[\phi_k](\xi)(1+|\xi|^2)^{\frac{\sigma}{2}}2^{-\sigma k}\tilde\phi(2^{-k}\xi)\mathcal{F}[f](\xi)]\|_{L_q(\mathbb{R}^N)}\\ &\le C_\sigma 2^{(s+\sigma)k}\|\mathcal{F}^{-1}[\mathcal{F}[\phi_k](\xi)\mathcal{F}[f](\xi)]\|_{L_q(\mathbb{R}^N)} = C_\sigma 2^{(s+\sigma)k}\|\phi_k * f\|_{L_q(\mathbb{R}^N)}.\end{aligned}$$

Thus, from the definition (2.2) we have

$$\| < D >^\sigma f\|_{B^s_{q,r}(\mathbb{R}^N)} \le C_\sigma\|f\|_{B^{s+\sigma}_{q,r}(\mathbb{R}^N)}.$$

This completes the proof of Lemma 5.2. □

Now, we shall prove the following theorem which is a main result of this section.

Theorem 5.3. *Let $1 < q < \infty$, $1 \le r \le \infty$, $-1 + 1/q < s < 1/q$, and $\epsilon \in (0, \pi/2)$. Let $\mathcal{S}^0(\lambda)$ be the operator defined in* (4.5)*. Then, there exists a large constant $\omega_0 > 0$ such that for any $\lambda \in \Sigma_{\epsilon,\omega_0}$ and $\mathbf{g} \in B^s_{q,r}(\mathbb{R}^N)^N$, there hold*

$$\|(\lambda, \lambda^{1/2}\nabla, \nabla^2)\mathcal{S}^0(\lambda)\mathbf{g}\|_{B^s_{q,r}(\mathbb{R}^N)} \le C\|\mathbf{g}\|_{B^s_{q,r}(\mathbb{R}^N)}, \tag{5.4}$$

$$\|(\lambda, \lambda^{1/2}\nabla, \nabla^2)\partial_\lambda\mathcal{S}^0(\lambda)\mathbf{g}\|_{B^s_{q,r}(\mathbb{R}^N)} \le C|\lambda|^{-1}\|\mathbf{g}\|_{B^s_{q,r}(\mathbb{R}^N)}. \tag{5.5}$$

Moreover, let $\sigma > 0$ be a small number such that $-1+1/q < s-\sigma < s < s+\sigma < 1/q$. Then, there exist a large number $\omega_1 \ge \omega_0$ and two operators $\mathcal{T}^0_1(\lambda)$ and $\mathcal{T}^0_2(\lambda)$ which are holomorphic on $\Lambda_{\epsilon,\omega_1}$ such that $\mathcal{S}^0(\lambda) = \mathcal{T}^0_1(\lambda) + \mathcal{T}^0_1(\lambda)$ and for any $\mathbf{g} \in C^\infty_0(\mathbb{R}^N)$ and $\lambda \in \Lambda_{\epsilon,\omega_1}$, there hold

$$\|(\lambda^{1/2}\nabla, \nabla^2)\mathcal{T}^0_1(\lambda)\mathbf{g}\|_{B^s_{q,r}(\mathbb{R}^N)} \le C|\lambda|^{-\sigma/2}\|\mathbf{g}\|_{B^{s+\sigma}_{q,r}(\mathbb{R}^N)}, \tag{5.6}$$

$$\|(\lambda^{1/2}\nabla, \nabla^2)\partial_\lambda\mathcal{T}^0_1(\lambda)\mathbf{g}\|_{B^s_{q,r}(\mathbb{R}^N)} \le C|\lambda|^{-(1-\frac{\sigma}{2})}\|\mathbf{g}\|_{B^{s-\sigma}_{q,r}(\mathbb{R}^N)} \tag{5.7}$$

as well as for any $\lambda \in \Lambda_{\epsilon,\omega_1}$ and $\mathbf{g} \in B^s_{q,1}(\mathbb{R}^N)$, there hold

$$\|(\lambda^{1/2}\nabla, \nabla^2)\mathcal{T}^0_2(\lambda)\mathbf{g}\|_{B^s_{q,r}(\mathbb{R}^N)} \le C|\lambda|^{-\sigma/2}\|\mathbf{g}\|_{B^s_{q,r}(\mathbb{R}^N)}, \tag{5.8}$$

$$\|(\lambda^{1/2}\nabla, \nabla^2)\partial_\lambda\mathcal{T}^0_2(\lambda)\mathbf{g}\|_{B^s_{q,r}(\mathbb{R}^N)} \le C|\lambda|^{-(1-\frac{\sigma}{2})}\|\mathbf{g}\|_{B^s_{q,r}(\mathbb{R}^N)}. \tag{5.9}$$

Proof. To prove the theorem, we divide $\mathcal{S}^0(\lambda)$ as

$$\mathcal{S}^0(\lambda) = \frac{1}{\alpha}\mathcal{S}^0_1(\lambda) + \frac{\eta_\lambda}{\alpha(\alpha+\eta_\lambda)}\mathcal{S}^0_2(\lambda),$$

where we have defined $\mathcal{S}^0_j(\lambda)$ $(j = 1, 2)$ by

$$\mathcal{S}^0_1(\lambda)\mathbf{g} = \frac{1}{\alpha}\mathcal{F}^{-1}_\xi\Big[\frac{\mathcal{F}[\mathbf{g}](\xi)}{\lambda\alpha^{-1} + |\xi|^2}\Big],$$
$$\mathcal{S}^0_2(\lambda)\mathbf{g} = \mathcal{F}^{-1}_\xi\Big[\frac{(i\xi\otimes i\xi)\mathcal{F}[\mathbf{g}](\xi))}{(\lambda\alpha^{-1} + |\xi|^2)(p(\lambda) + |\xi|^2)}\Big].$$

By Lemma 3.2 with $s = -1$ we have for any multi-index $\delta \in \mathbb{N}^N_0$ there exists a constant C_δ such that

$$\Big|D^\delta_\xi\frac{(\lambda, \lambda^{1/2}i\xi, (i\xi)^2)}{\lambda\alpha^{-1} + |\xi|^2}\Big| \le C_\delta|\xi|^{-|\delta|},$$
$$\Big|D^\delta_\xi\frac{(i\xi\otimes i\xi)(\lambda, \lambda^{1/2}i\xi, (i\xi)^2)}{(p(\lambda) + |\xi|^2)^{-1})(\lambda\alpha^{-1} + |\xi|^2)}\Big| \le C_\delta|\xi|^{-|\delta|}.$$

Here and in the following, we denote $i\xi = (i\xi_1, \dots, i\xi_N)$ $(N-vector)$, and $(i\xi)^2 = (i\xi_j i\xi_k \mid j,k = 1, \dots, N)$ $(N^2-vector)$. In particular, $i\xi$ and $(i\xi)^2$ are corresponding to ∇ and ∇^2 through the Fourier transform. By Lemma 5.2, we have

$$\begin{aligned}\|(\lambda, \lambda^{1/2}\nabla, \nabla^2)\mathcal{S}^0_1(\lambda)\mathbf{g}\|_{B^s_{q,r}(\mathbb{R}^N)} &= \Big\|\mathcal{F}^{-1}_\xi\Big[\frac{(\lambda, \lambda^{1/2}i\xi, (i\xi)^2)\mathcal{F}[\mathbf{g}](\xi)}{\lambda\alpha^{-1} + |\xi|^2}\Big]\Big\|_{B^s_{q,r}(\mathbb{R}^N)}\\ &\le C\|\mathbf{g}\|_{B^s_{q,r}(\mathbb{R}^N)},\\ \|(\lambda, \lambda^{1/2}\nabla, \nabla^2)\mathcal{S}^0_2(\lambda)\mathbf{g}\|_{B^s_{q,r}(\mathbb{R}^N)} &= \Big\|\mathcal{F}^{-1}_\xi\Big[\frac{(i\xi\otimes i\xi)(\lambda, \lambda^{1/2}i\xi, (i\xi)^2)\mathcal{F}[\mathbf{g}](\xi)}{(p(\lambda) + |\xi|^2)^{-1})(\lambda\alpha^{-1} + |\xi|^2)}\Big]\Big\|_{B^s_{q,r}(\mathbb{R}^N)} \\ &\le C\|\mathbf{g}\|_{B^s_{q,r}(\mathbb{R}^N)}.\end{aligned} \tag{5.10}$$

Note that

$$\Big|\frac{\eta_\lambda}{\alpha(\alpha+\eta_\lambda)}\Big| \leq C$$

for any $\lambda \in \Sigma_\epsilon + \omega_1$ as follows from Lemma 3.1 (3). Combining these estimates gives (5.4).

Now, we estimate $\partial_\lambda \mathcal{S}^0(\lambda)$. Noting that

$$\partial_\lambda\Big(\frac{\eta_\lambda}{\alpha(\alpha+\eta_\lambda)}\Big) = -\frac{\gamma^2\lambda^{-2}}{(\alpha+\eta_\lambda)^2}, \quad \partial_\lambda p(\lambda) = -\frac{\alpha+\beta+2\gamma^2\lambda^{-1}}{(\alpha+\eta_\lambda)^2},$$

we have

$$\partial_\lambda \mathcal{S}^0(\lambda)\mathbf{g} = \frac{1}{\alpha}\partial_\lambda \mathcal{S}_1^0(\lambda)\mathbf{g} - \frac{\gamma^2\lambda^{-2}}{(\alpha+\eta_\lambda)^2}\mathcal{S}_2^0(\lambda)\mathbf{g} + \frac{\eta_\lambda}{\alpha(\alpha+\eta_\lambda)}\partial_\lambda \mathcal{S}_2^0(\lambda)\mathbf{g}.$$

Notice that

$$\begin{aligned}
&\partial_\lambda \mathcal{S}_1^0(\lambda)\mathbf{g} = -\frac{1}{\alpha^2}\mathcal{F}_\xi^{-1}\Big[\frac{\mathcal{F}[\mathbf{g}](\xi)}{(\lambda\alpha^{-1}+|\xi|^2)^2}\Big],\\
&\partial_\lambda \mathcal{S}_2^0(\lambda)\mathbf{g} = -\partial_\lambda p(\lambda)\mathcal{F}_\xi^{-1}\Big[\frac{\mathcal{F}[\mathbf{g}](\xi)}{(p(\lambda)+|\xi|^2)^2}\Big] - \frac{1}{\alpha^2}\mathcal{F}_\xi^{-1}\Big[\frac{(i\xi\otimes i\xi)\mathcal{F}[\mathbf{g}](\xi)}{(p(\lambda)+|\xi|^2)(\lambda\alpha^{-1}+|\xi|^2)^2}\Big].
\end{aligned}$$

By Lemma 3.2 with $s=-2$, we have

$$\begin{aligned}
\Big|D_\xi^\delta \frac{\lambda(\lambda, \lambda^{1/2}i\xi, (i\xi)^2)}{(\lambda\alpha^{-1}+|\xi|^2)^2}\Big| &\leq C_\delta|\xi|^{-|\delta|},\\
\Big|D_\xi^\delta \frac{(i\xi\otimes i\xi)\lambda(\lambda, \lambda^{1/2}i\xi, (i\xi)^2)}{(p(\lambda)+|\xi|^2)^{-1})^2(\lambda\alpha^{-1}+|\xi|^2)}\Big| &\leq C_\delta|\xi|^{-|\delta|},\\
\Big|D_\xi^\delta \frac{(i\xi\otimes i\xi)\lambda(\lambda, \lambda^{1/2}i\xi, (i\xi)^2)}{(p(\lambda)+|\xi|^2)(\lambda\alpha^{-1}+|\xi|^2)^2}\Big| &\leq C_\delta|\xi|^{-|\delta|}.
\end{aligned}$$

Thus, by Lemma 5.2, we have

$$\|\lambda(\lambda, \lambda^{1/2}\nabla, \nabla^2)\partial_\lambda \mathcal{S}_\ell^0(\lambda)\mathbf{g}\|_{B^s_{q,1}(\mathbb{R}^N)} \leq C\|\mathbf{g}\|_{B^s_{q,r}(\mathbb{R}^N)} \quad \text{for } \ell=1,2. \tag{5.11}$$

Moreover, by Lemma 3.1 (3), we have

$$\Big|\frac{\gamma^2\lambda^{-2}}{(\alpha+\eta_\lambda)^2}\Big| \leq C|\lambda|^{-2}$$

for any $\lambda \in \Lambda_{\epsilon,\lambda_0}$. Thus, by (5.11) we have

$$\begin{aligned}
\|\lambda(\lambda, \lambda^{1/2}\nabla, \nabla^2)\frac{\gamma^2\lambda^{-2}}{(\alpha+\eta_\lambda)}\mathcal{S}_2^0(\lambda)\mathbf{g}\|_{B^s_{q,r}(\mathbb{R}^N)} &\leq C|\lambda|^{-1}\|(\lambda, \lambda^{1/2}\nabla, \nabla^2)\mathcal{S}_2^0(\lambda)\mathbf{g}\|_{B^s_{q,r}(\mathbb{R}^N)}\\
&\leq C|\lambda|^{-1}\|\mathbf{g}\|_{B^s_{q,r}(\mathbb{R}^N)}.
\end{aligned}$$

Combining these estimates gives (5.5).

Now, we shall prove (5.6)-(5.9). To this end, we write

$$\frac{\eta_\lambda}{\alpha(\alpha+\eta_\lambda)} = \frac{\eta_\lambda}{\alpha(\alpha+\beta)}(1+\gamma^2(\alpha+\beta)^{-1}\lambda^{-1})^{-1} = \frac{\beta+\gamma^2\lambda^{-1}}{\alpha(\alpha+\beta)}\sum_{\ell=0}^\infty\Big(-\frac{\gamma^2}{(\alpha+\beta)\lambda}\Big)^\ell.$$

Thus, choosing $\lambda_0>0$ in such a way that $\frac{\gamma^2}{(\alpha+\beta)\lambda_0}<1$, we have

$$\frac{\eta_\lambda}{\alpha(\alpha+\eta_\lambda)} = \frac{\beta}{\alpha(\alpha+\beta)} + \lambda^{-1}\zeta(\lambda^{-1}),$$

where $\zeta(\tau)$ be a C^∞ function defined for $|\tau| < \tau_0 := (\alpha+\beta)\gamma^{-2}$. Thus, we set

$$\mathcal{T}_1^0(\lambda) = \mathcal{S}_1^0(\lambda) + \frac{\beta}{\alpha(\alpha+\beta)}\mathcal{S}_2^0(\lambda), \quad \mathcal{T}_2^0(\lambda) = \lambda^{-1}\zeta(\lambda^{-1})\mathcal{S}_2^0(\lambda).$$

Obviously, $\mathcal{S}^0(\lambda) = \mathcal{T}_1^0(\lambda) + \mathcal{T}_2^0(\lambda)$. By (5.10), we have

$$\|(\lambda, \lambda^{1/2}\nabla, \nabla^2)\mathcal{T}_2^0(\lambda)\mathbf{g}\|_{B^s_{q,r}(\mathbb{R}^N)} \leq C|\lambda|^{-1}\|\mathbf{g}\|_{B^s_{q,r}(\mathbb{R}^N)} \leq C\lambda_0^{-1+\frac{\sigma}{2}}|\lambda|^{-\frac{\sigma}{2}}\|\mathbf{g}\|_{B^s_{q,r}(\mathbb{R}^N)}.$$

We write

$$\partial_\lambda \mathcal{T}_2^0(\lambda)\mathbf{g} = \{\partial_\lambda(\lambda^{-1}\zeta(\lambda^{-1})\}\mathcal{S}_2^0(\lambda)\mathbf{g} + \lambda^{-1}\zeta(\lambda^{-1})\partial_\lambda\mathcal{S}_2^0(\lambda)$$

and then, applying (5.10) and (5.11) gives

$$\|(\lambda, \lambda^{1/2}\nabla, \nabla^2)\partial_\lambda\mathcal{T}_2^0(\lambda)\mathbf{g}\|_{B^s_{q,r}(\mathbb{R}^N)} \leq C|\lambda|^{-1}\|\mathbf{g}\|_{B^s_{q,r}(\mathbb{R}^N)} \leq C\lambda_0^{-\frac{\sigma}{2}}|\lambda|^{-(1-\frac{\sigma}{2})}\|\mathbf{g}\|_{B^s_{q,r}(\mathbb{R}^N)}.$$

Combining these two estimates gives (5.8) and (5.9).

Finally, we shall prove (5.6) and (5.7). Let $\mathbf{g} \in C_0^\infty(\mathbb{R}^N)$. Writing

$$\lambda^{\frac{\sigma}{2}}(\lambda^{1/2}\nabla, \nabla^2)\mathcal{S}_1^0(\lambda)\mathbf{g} = \mathcal{F}_\xi^{-1}\Big[\frac{\lambda^{\frac{\sigma}{2}}(\lambda^{1/2}(i\xi), (i\xi)^2)}{(\lambda\alpha^{-1}+|\xi|^2)(1+|\xi|^2)^{\frac{\sigma}{2}}}(1+|\xi|^2)^{\frac{\sigma}{2}}\mathcal{F}[\mathbf{g}](\xi)\Big],$$
$$\lambda^{\frac{\sigma}{2}}(\lambda^{1/2}\nabla, \nabla^2)\mathcal{S}_1^0(\lambda)\mathbf{g} = \mathcal{F}_\xi^{-1}\Big[\frac{\lambda^{\frac{\sigma}{2}}(\lambda^{1/2}(i\xi), (i\xi)^2)(i\xi\otimes i\xi)}{(p(\lambda)+|\xi|^2)(\lambda\alpha^{-1}+|\xi|^2)(1+|\xi|^2)^{\frac{\sigma}{2}}}(1+|\xi|^2)^{\frac{\sigma}{2}}\mathcal{F}[\mathbf{g}](\xi)\Big]$$

and observing that

$$\Big|D_\xi^\delta\frac{\lambda^{\frac{\sigma}{2}}(\lambda^{1/2}(i\xi), (i\xi)^2)}{(\lambda\alpha^{-1}+|\xi|^2)(1+|\xi|^2)^{\frac{\sigma}{2}}}\Big| \leq C_\delta|\xi|^{-|\delta|},$$
$$\Big|D_\xi^\delta\frac{\lambda^{\frac{\sigma}{2}}(\lambda^{1/2}(i\xi), (i\xi)^2)(i\xi\otimes i\xi)}{(p(\lambda)+|\xi|^2)(\lambda\alpha^{-1}+|\xi|^2)(1+|\xi|^2)^{\frac{\sigma}{2}}}\Big| \leq C_\delta|\xi|^{-|\delta|}$$

for any multi-index $\delta \in \mathbb{N}_0^N$, by Lemma 5.2 we have

$$\|\lambda^{\frac{\sigma}{2}}(\lambda^{1/2}\nabla, \nabla^2)\mathcal{S}_1^0(\lambda)\mathbf{g}\|_{B^s_{q,r}(\mathbb{R}^N)} \leq C\|\mathcal{F}_\xi^{-1}[(1+|\xi|^2)^{\frac{\sigma}{2}}\mathcal{F}[\mathbf{g}]]\|_{B^s_{q,r}(\mathbb{R}^N)} \leq C\|\mathbf{g}\|_{B^{s+\sigma}_{q,r}(\mathbb{R}^N)},$$
$$\|\lambda^{\frac{\sigma}{2}}(\lambda^{1/2}\nabla, \nabla^2)\mathcal{S}_2^0(\lambda)\mathbf{g}\|_{B^s_{q,r}(\mathbb{R}^N)} \leq C\|\mathcal{F}_\xi^{-1}[(1+|\xi|^2)^{\frac{\sigma}{2}}\mathcal{F}[\mathbf{g}]]\|_{B^s_{q,r}(\mathbb{R}^N)} \leq C\|\mathbf{g}\|_{B^{s+\sigma}_{q,r}(\mathbb{R}^N)}.$$

Combining these two estimates gives (5.6).

Write

$$\partial_\lambda\mathcal{T}_1^0(\lambda)\mathbf{g} = \partial_\lambda\mathcal{S}_1^0(\lambda)\mathbf{g} + \frac{\beta}{\alpha(\alpha+\beta)}\partial_\lambda\mathcal{S}_2^0(\lambda)\mathbf{g}.$$

Writing

$$\begin{aligned}&\lambda^{1-\frac{\sigma}{2}}(\lambda, \lambda^{1/2}\nabla, \nabla^2)\partial_\lambda\mathcal{S}_1^0(\lambda)\mathbf{g}\\ &\quad= -\frac{2}{\alpha^2}\mathcal{F}_\xi^{-1}\Big[\frac{\lambda^{1-\frac{\sigma}{2}}(\lambda, \lambda^{1/2}(i\xi), (i\xi)^2)(1+|\xi|^2)^{\frac{\sigma}{2}}}{(\lambda\alpha^{-1}+|\xi|^2)^2}(1+|\xi|^2)^{-\frac{\sigma}{2}}\mathcal{F}[\mathbf{g}](\xi)\Big],\end{aligned}$$

and observing that

$$\Big|D_\xi^\delta\frac{\lambda^{1-\frac{\sigma}{2}}(\lambda, \lambda^{1/2}i\xi, (i\xi)^2)(1+|\xi|^2)^{\frac{\sigma}{2}}}{(\lambda\alpha^{-1}+|\xi|^2)^2}\Big| \leq C|\xi|^{-|\delta|}$$

for any multi-index $\delta \in \mathbb{N}_0^N$, by Lemma 5.2 we have

$$\|\lambda^{1-\frac{\sigma}{2}}(\lambda, \lambda^{1/2}\nabla, \nabla^2)\partial_\lambda\mathcal{S}_1^0(\lambda)\mathbf{g}\|_{B^s_{q,r}(\mathbb{R}^N)} \leq C\|\mathcal{F}_\xi^{-1}[(1+|\xi|^2)^{-\frac{\sigma}{2}}\mathcal{F}[\mathbf{g}](\xi)]\|_{B^s_{q,r}(\mathbb{R}^N)}$$

$$\leq C\|\mathbf{g}\|_{B^{s-\sigma}_{q,r}(\mathbb{R}^N)}.$$

Writing

$$\begin{aligned}
&\lambda^{1-\frac{\sigma}{2}}(\lambda, \lambda^{1/2}\nabla, \nabla^2)\mathcal{S}^0_2(\lambda)\mathbf{g} \\
&\quad = -\partial_\lambda p(\lambda)\mathcal{F}^{-1}_\xi\Big[\frac{\lambda^{1-\frac{\sigma}{2}}(\lambda, \lambda^{1/2}(i\xi), (i\xi)^2)(i\xi\otimes i\xi)(1+|\xi|^2)^{\frac{\sigma}{2}}}{(p(\lambda)+|\xi|^2)^2(\lambda\alpha^{-1}+|\xi|^2)}(1+|\xi|^2)^{-\frac{\sigma}{2}}\mathcal{F}[\mathbf{g}](\xi)\Big] \\
&\quad - \frac{1}{\alpha}\mathcal{F}^{-1}_\xi\Big[\frac{\lambda^{1-\frac{\sigma}{2}}(\lambda, \lambda^{1/2}(i\xi), (i\xi)^2)(i\xi\otimes i\xi)(1+|\xi|^2)^{\frac{\sigma}{2}}}{(p(\lambda)+|\xi|^2)(\lambda\alpha^{-1}+|\xi|^2)^2}(1+|\xi|^2)^{-\frac{\sigma}{2}}\mathcal{F}[\mathbf{g}](\xi)\Big],
\end{aligned}$$

and observing that

$$\Big|D^\delta_\xi\frac{\lambda^{1-\frac{\sigma}{2}}(\lambda, \lambda^{1/2}i\xi, (i\xi)^2)(i\xi\otimes i\xi)(1+|\xi|^2)^{\frac{\sigma}{2}}}{(p(\lambda)+|\xi|^2)^{2-\ell}(\lambda\alpha^{-1}+|\xi|^2)^{1+\ell}}\Big| \leq C|\xi|^{-|\delta|} \quad \text{for } \ell=0,1$$

for any multi-index $\delta\in\mathbb{N}^N_0$, by Lemma 5.2 we have

$$\begin{aligned}
\|\lambda^{1-\frac{\sigma}{2}}(\lambda, \lambda^{1/2}\nabla, \nabla^2)\partial_\lambda\mathcal{S}^0_2(\lambda)\mathbf{g}\|_{B^s_{q,r}(\mathbb{R}^N)} &\leq C\|\mathcal{F}^{-1}_\xi[(1+|\xi|^2)^{-\frac{\sigma}{2}}\mathcal{F}[\mathbf{g}](\xi)]\|_{B^s_{q,r}(\mathbb{R}^N)} \\
&\leq C\|\mathbf{g}\|_{B^{s-\sigma}_{q,r}(\mathbb{R}^N)}.
\end{aligned}$$

Combining these two estimates gives (5.7), which completes the proof of Theorem 5.3. □

6 Estimates of solution formulas of complex Lamé equations

Let $\mathcal{S}^b(\lambda) = (\mathcal{S}^b_1(\lambda), \ldots, \mathcal{S}^b_N(\lambda))$ be the solution operator corresponding to equations (4.3) defined by

$$\mathcal{S}^b_J(\lambda)\mathbf{g} = w_J$$

for $J=1,\ldots,N$, where the partial Fourier transform $\mathcal{F}'[w_J]$ of w_J are defined by (4.7). In this section, we shall estimate $\mathcal{S}^b_J(\lambda)$. Namely, we shall prove the following theorem.

Theorem 6.1. *Let $1<q<\infty$, $1\leq r\leq\infty$, $-1+1/q<s<1/q$, $\epsilon\in(0,\pi/2)$ and $\lambda_0>0$. Then, for any $\lambda\in\Lambda_{\epsilon,\lambda_0}$ and $\mathbf{g}\in B^s_{q,r}(\mathbb{R}^N_+)^N$, there hold*

$$\begin{aligned}
\|(\lambda, \lambda^{1/2}\nabla, \nabla^2)\mathcal{S}^b(\lambda)\mathbf{g}\|_{B^s_{q,r}(\mathbb{R}^N_+)} &\leq C\|\mathbf{g}\|_{B^s_{q,r}(\mathbb{R}^N_+)}, \\
\|(\lambda, \lambda^{1/2}\nabla, \nabla^2)\partial_\lambda\mathcal{S}^b(\lambda)\mathbf{g}\|_{B^s_{q,r}(\mathbb{R}^N_+)} &\leq C|\lambda|^{-1}\|\mathbf{g}\|_{B^s_{q,r}(\mathbb{R}^N_+)}.
\end{aligned}$$

Moreover, let $\sigma>0$ be a small number such that $-1+1/q<s-\sigma<s<s+\sigma<1/q$. Then, there exist a large number $\lambda_0>0$ and two operators $\mathcal{T}^b_1(\lambda)$ and $\mathcal{T}^b_2(\lambda)$ which are holomorphic on $\Lambda_{\epsilon,\lambda_0}$ such that $\mathcal{S}^b(\lambda)=\mathcal{T}^b_1(\lambda)+\mathcal{T}^b_2(\lambda)$ and for any $\mathbf{g}\in C^\infty_0(\mathbb{R}^N_+)^N$, there hold

$$\|(\lambda, \lambda^{1/2}\bar\nabla, \bar\nabla^2)\mathcal{T}^b_1(\lambda)\mathbf{g}\|_{B^s_{q,r}(\mathbb{R}^N_+)} \leq C|\lambda|^{-\sigma/2}\|\mathbf{g}\|_{B^{s+\sigma}_{q,r}(\mathbb{R}^N_+)}, \tag{6.1}$$

$$\|(\lambda, \lambda^{1/2}\bar\nabla, \nabla^2)\partial_\lambda\mathcal{T}^b_1(\lambda)\mathbf{g}\|_{B^s_{q,r}(\mathbb{R}^N_+)} \leq C|\lambda|^{-(1-\frac{\sigma}{2})}\|\mathbf{g}\|_{B^{s-\sigma}_{q,r}(\mathbb{R}^N_+)}, \tag{6.2}$$

$$\|(\lambda, \lambda^{1/2}\bar\nabla, \bar\nabla^2)\mathcal{T}^b_2(\lambda)\mathbf{g}\|_{B^s_{q,r}(\mathbb{R}^N_+)} \leq C|\lambda|^{-1}\|\mathbf{g}\|_{B^s_{q,r}(\mathbb{R}^N_+)}, \tag{6.3}$$

$$\|(\lambda, \lambda^{1/2}\bar\nabla, \bar\nabla^2)\partial_\lambda\mathcal{T}^b_2(\lambda)\mathbf{g}\|_{B^s_{q,r}(\mathbb{R}^N_+)} \leq C|\lambda|^{-2}\|\mathbf{g}\|_{B^s_{q,r}(\mathbb{R}^N_+)}. \tag{6.4}$$

To prove Theorem 6.1, the argument based on interpolation theory due to Shibata [27] and Shibata and Watanabe [30] play an important role. We quote this in the following subsection.

6.1 Spectral analysis based on interpolation theory

Below, we assume that $1 < q < \infty$ and $-1 + 1/q < s < 1/q$, $\epsilon \in (0, \pi/2)$, and $\omega > 0$. Let $T(\lambda)$ be an operator valued holomorphic function acting on $f \in C_0^\infty(\mathbb{R}^N_+)$ defined for $\lambda \in \Lambda_{\epsilon,\omega}$. In this subsection, we shall show our strategy how to obtain the estimates of $T(\lambda)$ as an operator from one Besov space into another Besov space. The following argument is due to Shibata [27], see also Shibata and Watanabe [30]. Since this gives one of main ideas, for the convenience of readers we record arguments there.

We consider two operator valued holomorphic functions $T_i(\lambda)$ defined on $\Lambda_{\epsilon,\lambda_0}$ acting on $f \in C_0^\infty(\mathbb{R}^N_+)$. We denote the dual operator of $T_i(\lambda)$ by $T_i(\lambda)^*$, namely, $T_i(\lambda)^*$ satisfies the equality:

$$(T_i(\lambda)f, \varphi) = (f, T_i(\lambda)^*\varphi) \quad (i = 1, 2)$$

for any f, $\varphi \in C_0^\infty(\mathbb{R}^N_+)$. Here, $(f, g) = \int_{\mathbb{R}^N_+} f(x)g(x)\,dx$. Namely, we do not take the complex conjugate.

We consider the following two cases.

Assumption 6.2. *Let $1 < q < \infty$, $\epsilon \in (0, \pi/2)$, and $\omega > 0$. We assume that the starting evaluations hold as follows:*

For any $f \in C_0^\infty(\mathbb{R}^N_+)$ and $\lambda \in \Lambda_{\epsilon,\omega}$, the following estimates hold:

$$\|T_1(\lambda)f\|_{H^1_q(\mathbb{R}^N_+)} \leq C\|f\|_{H^1_q(\mathbb{R}^N_+)}, \tag{6.5}$$

$$\|T_1(\lambda)f\|_{L_q(\mathbb{R}^N_+)} \leq C\|f\|_{L_q(\mathbb{R}^N_+)}, \tag{6.6}$$

$$\|T_1(\lambda)f\|_{L_q(\mathbb{R}^N_+)} \leq C|\lambda|^{-1/2}\|f\|_{H^1_q(\mathbb{R}^N_+)}, \tag{6.7}$$

$$\|T_1(\lambda)^*f\|_{L_{q'}(\mathbb{R}^N_+)} \leq C\|f\|_{L_{q'}(\mathbb{R}^N_+)}, \tag{6.8}$$

$$\|T_1(\lambda)^*f\|_{H^1_{q'}(\mathbb{R}^N_+)} \leq C\|f\|_{H^1_{q'}(\mathbb{R}^N_+)}, \tag{6.9}$$

$$\|T_1(\lambda)^*f\|_{L_{q'}(\mathbb{R}^N_+)} \leq C|\lambda|^{-1/2}\|f\|_{H^1_{q'}(\mathbb{R}^N_+)}. \tag{6.10}$$

Assumption 6.3. *Let $1 < q < \infty$, $\epsilon \in (0, \pi/2)$, and $\omega > 0$. We assume that the starting evaluations hold as follows:*

For any $f \in C_0^\infty(\mathbb{R}^N_+)$ and $\lambda \in \Lambda_{\epsilon,\omega}$, the following estimates hold:

$$\|T_2(\lambda)f\|_{H^1_q(\mathbb{R}^N_+)} \leq C|\lambda|^{-1}\|f\|_{H^1_q(\mathbb{R}^N_+)}, \tag{6.11}$$

$$\|T_2(\lambda)f\|_{L_q(\mathbb{R}^N_+)} \leq C|\lambda|^{-1}\|f\|_{L_q(\mathbb{R}^N_+)}, \tag{6.12}$$

$$\|T_2(\lambda)f\|_{H^1_q(\mathbb{R}^N_+)} \leq C|\lambda|^{-1/2}\|f\|_{L_q(\mathbb{R}^N_+)}, \tag{6.13}$$

$$\|T_2(\lambda)^*f\|_{L_{q'}(\mathbb{R}^N_+)} \leq C|\lambda|^{-1}\|f\|_{L_{q'}(\mathbb{R}^N_+)}, \tag{6.14}$$

$$\|T_2(\lambda)^*f\|_{H^1_{q'}(\mathbb{R}^N_+)} \leq C|\lambda|^{-1}\|f\|_{H^1_{q'}(\mathbb{R}^N_+)}, \tag{6.15}$$

$$\|T_2(\lambda)^*f\|_{H^1_{q'}(\mathbb{R}^N_+)} \leq C|\lambda|^{-1/2}\|f\|_{L_{q'}(\mathbb{R}^N_+)}. \tag{6.16}$$

Then, we have the following theorems.

Theorem 6.4. *Let $1 < q < \infty$, $1 \leq r \leq \infty$, $-1 + 1/q < s < 1/q$, $\epsilon \in (0, \pi/2)$, and $\omega > 0$. Let $\sigma > 0$ be a small number such that $-1 + 1/q < s - \sigma < s < s + \sigma < 1/q$. Assume that Assumptions 6.2 and 6.3 hold. Then, for any $f \in C_0^\infty(\mathbb{R}^N_+)$ and $\lambda \in \Lambda_{\epsilon,\omega}$, there hold*

$$\|T_1(\lambda)f\|_{B^s_{q,r}(\mathbb{R}^N_+)} \leq C\|f\|_{B^s_{q,r}(\mathbb{R}^N_+)},$$

$$\|T_1(\lambda)f\|_{B^s_{q,r}(\mathbb{R}^N_+)} \le C|\lambda|^{-\frac{\sigma}{2}}\|f\|_{B^{s+\sigma}_{q,r}(\mathbb{R}^N_+)},$$
$$\|T_2(\lambda)f\|_{B^s_{q,r}(\mathbb{R}^N_+)} \le C|\lambda|^{-1}\|f\|_{B^s_{q,r}(\mathbb{R}^N_+)},$$
$$\|T_2(\lambda)f\|_{B^s_{q,r}(\mathbb{R}^N_+)} \le C|\lambda|^{-(1-\frac{\sigma}{2})}\|f\|_{B^{s-\sigma}_{q,r}(\mathbb{R}^N_+)}$$

for some constant C.

We divide the proof into the case $0 < s < 1/q$ and the case $-1 + 1/q < s < 0$. The $s = 0$ case follows from the real interpolation between $B^{-\nu_1}_{q,r}$ and $B^{\nu_2}_{q,r}$ for some small $\nu_i > 0$ $(i = 1, 2)$.

Lemma 6.5. *Assume that Assumption 6.2 above holds. Let q, ϵ, and ω be the same as in Assumption 6.2. Let $1 \le r \le \infty$. Let $0 < s < 1/q$ and let $\sigma > 0$ be numbers such that $0 < s + \sigma < 1/q$. Then, for any $\lambda \in \Lambda_{\epsilon,\omega}$ and $f \in C^\infty_0(\mathbb{R}^N_+)$, there hold*

$$\|T_1(\lambda)f\|_{B^s_{q,r}(\mathbb{R}^N_+)} \le C\|f\|_{B^s_{q,r}(\mathbb{R}^N_+)}, \tag{6.17}$$
$$\|T_1(\lambda)f\|_{B^s_{q,r}(\mathbb{R}^N_+)} \le C|\lambda|^{-\frac{\sigma}{2}}\|f\|_{B^{s+\sigma}_{q,r}(\mathbb{R}^N_+)}. \tag{6.18}$$

Proof. Below, we always assume that $f \in C^\infty_0(\mathbb{R}^N_+)$ and $\lambda \in \Lambda_{\epsilon,\omega}$. Choose μ and μ' in such a way that $0 < s < s + \sigma < \mu' < \mu < 1/q$. Estimates (6.5), (6.6), and (6.7) are interpolated with complex interpolation method to obtain

$$\|T_1(\lambda)f\|_{L_q(\mathbb{R}^N_+)} \le C\|f\|_{L_q(\mathbb{R}^N_+)}, \tag{6.19}$$
$$\|T_1(\lambda)f\|_{H^\mu_q(\mathbb{R}^N_+)} \le C\|f\|_{H^\mu_q(\mathbb{R}^N_+)}, \tag{6.20}$$
$$\|T_1(\lambda)f\|_{H^{\mu'}_q(\mathbb{R}^N_+)} \le C\|f\|_{H^{\mu'}_q(\mathbb{R}^N_+)}, \tag{6.21}$$
$$\|T_1(\lambda)f\|_{L_q(\mathbb{R}^N_+)} \le C|\lambda|^{-\mu/2}\|f\|_{H^\mu_q(\mathbb{R}^N_+)}. \tag{6.22}$$

By interpolating (6.19) and (6.20) with real interpolation method,

$$\|T_1(\lambda)f\|_{B^s_{q,r}(\mathbb{R}^N_+)} \le C\|f\|_{B^s_{q,r}(\mathbb{R}^N_+)}. \tag{6.23}$$

Choosing $\theta = s/\mu'$ and setting $A = \mu(1 - s/\mu')$, by (6.21) and (6.22) with real interpolation method,

$$\|T_1(\lambda)f\|_{B^s_{q,r}(\mathbb{R}^N_+)} \le C|\lambda|^{-\frac{A}{2}}\|f\|_{B^{s+A}_{q,r}(\mathbb{R}^N_+)}. \tag{6.24}$$

Now, we choose μ and μ' in such a way that $s < s + \sigma < s + A$, that is, we choose μ and μ' in such a way that $\sigma/\mu + s/\mu' < 1$ and $s + \sigma < \mu' < \mu < 1/q$. Thus, choosing $\theta \in (0, 1)$ in such a way that $s + \sigma = (1 - \theta)s + \theta(s + A)$, that is, $\theta = \sigma/A$, by (6.23) and (6.24) we have

$$\|T_1(\lambda)f\|_{B^s_{q,r}(\mathbb{R}^N_+)} \le C|\lambda|^{-\frac{\sigma}{2}}\|f\|_{B^{s+\sigma}_{q,r}(\mathbb{R}^N_+)}.$$

Therefore, we have (6.17) and (6.18). This completes the proof of Lemma 6.5. □

Lemma 6.6. *Assume that Assumption 6.2 above holds. Let q, ϵ, and ω be the same as in Assumption 6.2. Let $1 \le r \le \infty$. Let $-1 + 1/q < s < 0$ and let $\sigma > 0$ be a number such that $-1 + 1/q < s + \sigma < 0$. Then, for any $\lambda \in \Lambda_{\epsilon,\omega}$ and $f \in C^\infty_0(\mathbb{R}^N_+)$, there hold*

$$\|T_1(\lambda)f\|_{B^s_{q,r}(\mathbb{R}^N_+)} \le C\|f\|_{B^s_{q,r}(\mathbb{R}^N_+)}, \tag{6.25}$$
$$\|T_1(\lambda)f\|_{B^s_{q,r}(\mathbb{R}^N_+)} \le C|\lambda|^{-\frac{\sigma}{2}}\|f\|_{B^{s+\sigma}_{q,r}(\mathbb{R}^N_+)}. \tag{6.26}$$

Proof. Since $-1+1/q<s<0$, we have $0<|s|<1-1/q=1/q'$. Let μ, μ' and σ be positive numbers such that

$$0<\mu'<|s|-\sigma<|s|<\mu<1/q'. \tag{6.27}$$

Using the complex interpolation method, by (6.8), (6.9), and (6.10), we have

$$\begin{aligned}
\|T_1(\lambda)^*\varphi\|_{L_{q'}(\mathbb{R}^N_+)} &\le C\|\varphi\|_{L_{q'}(\mathbb{R}^N_+)},\\
\|T_1(\lambda)^*\varphi\|_{H^{\mu}_{q'}(\mathbb{R}^N_+)} &\le C\|\varphi\|_{H^{\mu}_{q'}(\mathbb{R}^N_+)},\\
\|T_1(\lambda)^*\varphi\|_{H^{\mu'}_{q'}(\mathbb{R}^N_+)} &\le C\|\varphi\|_{H^{\mu'}_{q'}(\mathbb{R}^N_+)},\\
\|T_1(\lambda)^*\varphi\|_{L_{q'}(\mathbb{R}^N_+)} &\le C|\lambda|^{-\mu/2}\|\varphi\|_{H^{\mu}_{q'}(\mathbb{R}^N_+)}.
\end{aligned}$$

By the duality argument, we have

$$\|T_1(\lambda)f\|_{L_q(\mathbb{R}^N_+)} \le C\|f\|_{L_q(\mathbb{R}^N_+)}, \tag{6.28}$$

$$\|T_1(\lambda)f\|_{H^{-\mu}_q(\mathbb{R}^N_+)} \le C\|f\|_{H^{-\mu}_q(\mathbb{R}^N_+)}, \tag{6.29}$$

$$\|T_1(\lambda)f\|_{H^{-\mu'}_q(\mathbb{R}^N_+)} \le C\|f\|_{H^{-\mu'}_q(\mathbb{R}^N_+)}, \tag{6.30}$$

$$\|T_1(\lambda)f\|_{H^{-\mu}_q(\mathbb{R}^N_+)} \le C|\lambda|^{-\mu/2}\|f\|_{L_q(\mathbb{R}^N_+)}. \tag{6.31}$$

In fact, note that $H^{-\mu}_q(\mathbb{R}^N_+)=(H^{\mu}_{q',0}(\mathbb{R}^N_+))'$. For any f and $\varphi\in C^\infty_0(\mathbb{R}^N_+)$, by the dual argument we have

$$\begin{aligned}
|(T_1(\lambda)f,\varphi)| &= |(f,T_1(\lambda)^*\varphi)|\\
&\le \|f\|_{H^{-\mu}_q(\mathbb{R}^N_+)}\|T_1(\lambda)^*\varphi\|_{H^{\mu}_{q'}(\mathbb{R}^N_+)}\\
&\le \|f\|_{H^{-\mu}_q(\mathbb{R}^N_+)}C\|\varphi\|_{H^{\mu}_{q'}(\mathbb{R}^N_+)},
\end{aligned}$$

which implies (6.29). Likewise, we have (6.30) and (6.28). And also,

$$\begin{aligned}
|(T_1(\lambda)f,\varphi)| &= |(f,T_1(\lambda)^*\varphi)|\\
&\le \|f\|_{L_q(\mathbb{R}^N_+)}\|T_1(\lambda)^*\varphi\|_{L_{q'}(\mathbb{R}^N_+)}\\
&\le \|f\|_{L_q(\mathbb{R}^N_+)}C|\lambda|^{-\mu/2}\|\varphi\|_{H^{\mu}_{q'}(\mathbb{R}^N_+)},
\end{aligned}$$

which implies (6.31).

Now, we shall prove (6.25) and (6.26). Combining (6.28) and (6.29) with real interpolation method, we have

$$\|T_1(\lambda)f\|_{B^{-|s|}_{q,r}(\mathbb{R}^N_+)} \le C\|f\|_{B^{-|s|}_{q,r}(\mathbb{R}^N_+)}, \tag{6.32}$$

which shows (6.25).

Next, recall that $0<\mu'<|s|-\sigma<|s|<\mu<1/q'$ as follows from (6.27). Choose $\theta\in(0,1)$ in such a way that $-|s|=-\mu(1-\theta)-\mu'\theta$, that is $\theta=\dfrac{\mu-|s|}{\mu-\mu'}$. Combining (6.30) and (6.31) with real interpolation method which implies that

$$\|T_1(\lambda)f\|_{B^{-|s|}_{q,r}(\mathbb{R}^N_+)} \le C|\lambda|^{-\frac{\mu}{2}(1-\theta)}\|f\|_{B^{(-\mu')\theta}_{q,r}}.$$

Therefore, we have

$$\|T_1(\lambda)f\|_{B^{-|s|}_{q,1}(\mathbb{R}^N_+)} \le C|\lambda|^{-\frac{\mu}{2}\frac{|s|-\mu'}{\mu-\mu'}}\|f\|_{B_{q,r}^{-\frac{\mu'(\mu-|s|)}{\mu-\mu'}}}. \tag{6.33}$$

Since $0 < \mu' < |s| - \sigma$ and $0 < \mu - |s| < \mu - \mu'$, we have

$$-|s| < -|s| + \sigma < -\frac{\mu'(\mu - |s|)}{\mu - \mu'}.$$

Choose $\theta \in (0, 1)$ in such a way that

$$-|s| + \sigma = (1 - \theta)(-|s|) + \theta(-\frac{\mu'(\mu - |s|)}{\mu - \mu'})$$

Combining (6.32) and (6.33) with real interpolation method implies that

$$\|T_1(\lambda)f\|_{B^{-|s|}_{q,r}(\mathbb{R}^N_+)} \leq C|\lambda|^{-\frac{\mu}{2}\frac{|s|-\mu'}{\mu-\mu'}\theta}\|f\|_{B^{-|s|+\sigma}_{q,r}(\mathbb{R}^N_+)}.$$

Inserting $\theta = \dfrac{(\mu - \mu')\sigma}{\mu(|s| - \mu')}$, we have

$$\|T_1(\lambda)f\|_{B^{-|s|}_{q,r}(\mathbb{R}^N_+)} \leq C\|\lambda|^{-\frac{\sigma}{2}}\|f\|_{B^{-|s|+\sigma}_{q,r}(\mathbb{R}^N_+)},$$

which shows (6.26). □

Lemma 6.7. *Assume that Assumption 6.3 holds. Let q, ϵ, and ω be the same as in Assumption 6.3. Let $1 \leq r \leq \infty$. Let $0 < s < 1/q$ and let $\sigma > 0$ be numbers such that $0 < s - \sigma < 1/q$. Then, for any $\lambda \in \Lambda_{\epsilon,\omega}$ and $f \in C^\infty_0(\mathbb{R}^N_+)$, there hold*

$$\|T_2(\lambda)f\|_{B^s_{q,r}(\mathbb{R}^N_+)} \leq C|\lambda|^{-1}\|f\|_{B^s_{q,r}(\mathbb{R}^N_+)}, \tag{6.34}$$

$$\|T_2(\lambda)f\|_{B^s_{q,r}(\mathbb{R}^N_+)} \leq C|\lambda|^{-(1-\frac{\sigma}{2})}\|f\|_{B^{s-\sigma}_{q,r}(\mathbb{R}^N_+)}. \tag{6.35}$$

Proof. Let μ be a number such that $0 < s < s + \sigma < \mu < 1/q$. Combining (6.11) and (6.12), and (6.11) and (6.13) with complex interpolation method, implies that

$$\|T_2(\lambda)f\|_{L_q(\mathbb{R}^N_+)} \leq C|\lambda|^{-1}\|f\|_{L_q(\mathbb{R}^N_+)}, \tag{6.36}$$

$$\|T_2(\lambda)f\|_{H^\mu_q(\mathbb{R}^N_+)} \leq C|\lambda|^{-1}\|f\|_{H^\mu_q(\mathbb{R}^N_+)}, \tag{6.37}$$

$$\|T_2(\lambda)f\|_{H^\mu_q(\mathbb{R}^N_+)} \leq C|\lambda|^{-(1-\frac{\mu}{2})}\|f\|_{L_q(\mathbb{R}^N_+)}. \tag{6.38}$$

Combining (6.36) and (6.37) with real interpolation method yields

$$\|T_2(\lambda)f\|_{B^s_{q,r}(\mathbb{R}^N_+)} \leq C|\lambda|^{-1}\|f\|_{B^s_{q,r}(\mathbb{R}^N_+)}, \tag{6.39}$$

which shows (6.34).

Now, choosing μ' and θ in such a way that $0 < \mu' < \mu$ and $\theta = \mu'/\mu \in (0, 1)$ and combining (6.37) and (6.38) with complex interpolation, we have

$$\|T_2(\lambda)f\|_{H^\mu_q(\mathbb{R}^N_+)} \leq C|\lambda|^{-(1-(1/2)(\mu-\mu'))}\|f\|_{H^{\mu'}_q(\mathbb{R}^N_+)}, \tag{6.40}$$

as follows from $\theta + (1 - \mu/2)(1 - \theta) = 1 - (\mu/2)(1 - \theta) = 1 - \frac{\mu}{2}(1 - \frac{\mu'}{\mu}) = 1 - (1/2)(\mu - \mu')$.

Next, we will combine (6.36) and (6.40) with real interpolation method for $s = \theta\mu$, Namely, we choose $\theta = s/\mu \in (0, 1)$ and so $\theta\mu' = (\mu'/\mu)s$, and so

$$(1 - (1/2)(\mu - \mu'))\theta + (1 - \theta) = 1 - \frac{\theta}{2}(\mu - \mu') = (1 - \frac{s}{2\mu}(\mu - \mu')).$$

Thus, we have

$$\|T_2(\lambda)f\|_{B^s_{q,r}(\mathbb{R}^N_+)} \leq C|\lambda|^{-(1-\frac{s}{2\mu}(\mu-\mu'))}\|f\|_{B^{\frac{\mu'}{\mu}s}_{q,r}(\mathbb{R}^N_+)} \tag{6.41}$$

Finally, we will combine (6.39) and (6.41) with real interpolation method. We choose $0 < \mu' < \mu$ in such a way that $(\mu'/\mu)s < s - \sigma < s$, that is $0 < \mu' < (1 - \dfrac{\sigma}{s})\mu$. And, we choose $\theta \in (0,1)$ in such a way that $s - \sigma = (1-\theta)s + \theta(\mu'/\mu)s$, that is $\theta = \sigma/A$ with $A = s(1 - \mu'/\mu)$. In this case, we have

$$(1-\theta) + \theta(1 - \frac{s}{2\mu}(\mu - \mu')) = 1 - \frac{s}{2}(1 - \frac{\mu'}{\mu})\theta = 1 - \frac{s}{2}\frac{A}{s}\frac{\sigma}{A} = 1 - \frac{\sigma}{2}.$$

Thus, by (6.39) and (6.41), we have

$$\|T_2(\lambda)f\|_{B^s_{q,r}(\mathbb{R}^N_+)} \leq C|\lambda|^{-(1-\frac{\sigma}{2})}\|f\|_{B^{s-\sigma}_{q,r}(\mathbb{R}^N_+)},$$

which shows (6.35). Therefore, we have proved Lemma 6.7. □

Lemma 6.8. *Assume that Assumption 6.3 holds. Let q, ϵ, and ω be the same as in Assumption 6.3. Let $1 \leq r \leq \infty$. Let $-1 + 1/q < s < 0$ and let $\sigma > 0$ be numbers such that $-1 + 1/q < s - \sigma < 0$. Then, for any $\lambda \in \Lambda_{\epsilon,\omega}$ and $f \in C^\infty_0(\mathbb{R}^N_+)$, there hold*

$$\|T_2(\lambda)f\|_{B^s_{q,r}(\mathbb{R}^N_+)} \leq C|\lambda|^{-1}\|f\|_{B^s_{q,r}(\mathbb{R}^N_+)}, \tag{6.42}$$

$$\|T_2(\lambda)f\|_{B^s_{q,r}(\mathbb{R}^N_+)} \leq C|\lambda|^{-(1-\frac{\sigma}{2})}\|f\|_{B^{s-\sigma}_{q,r}(\mathbb{R}^N_+)}. \tag{6.43}$$

Proof. Combining (6.14), (6.15) and (6.16) with complex interpolation method for $|s| < \mu, \mu' < 1 - 1/q = 1/q'$, we have

$$\|T_2(\lambda)^*\varphi\|_{L_{q'}(\mathbb{R}^N_+)} \leq C|\lambda|^{-1}\|\varphi\|_{L_{q'}(\mathbb{R}^N_+)},$$
$$\|T_2(\lambda)^*\varphi\|_{H^\mu_{q'}(\mathbb{R}^N_+)} \leq C|\lambda|^{-1}\|\varphi\|_{H^\mu_{q'}(\mathbb{R}^N_+)},$$
$$\|T_2(\lambda)^*\varphi\|_{H^{\mu'}_{q'}(\mathbb{R}^N_+)} \leq C|\lambda|^{-1}\|\varphi\|_{H^{\mu'}_{q'}(\mathbb{R}^N_+)},$$
$$\|T_2(\lambda)^*\varphi\|_{H^\mu_{q'}(\mathbb{R}^N_+)} \leq C\||\lambda|^{-(1-\frac{\mu}{2})}\|\varphi\|_{L_{q'}(\mathbb{R}^N_+)}.$$

Thus, by the duality argument, we have

$$\|T_2(\lambda)f\|_{L_q(\mathbb{R}^N_+)} \leq C|\lambda|^{-1}\|f\|_{L_q(\mathbb{R}^N_+)}, \tag{6.44}$$

$$\|T_2(\lambda)f\|_{H^{-\mu}_q(\mathbb{R}^N_+)} \leq C|\lambda|^{-1}\|f\|_{H^{-\mu}_q(\mathbb{R}^N_+)}, \tag{6.45}$$

$$\|T_2(\lambda)f\|_{H^{-\mu'}_q(\mathbb{R}^N_+)} \leq C|\lambda|^{-1}\|f\|_{H^{-\mu'}_q(\mathbb{R}^N_+)}, \tag{6.46}$$

$$\|T_2(\lambda)f\|_{L_q(\mathbb{R}^N_+)} \leq C|\lambda|^{-(1-\frac{\mu}{2})}\|f\|_{H^{-\mu}_q(\mathbb{R}^N_+)}. \tag{6.47}$$

Noting that $-1 + 1/q < -\mu < -|s| < 0$ and combining (6.44) and (6.45) with real interpolation method implies

$$\|T_2(\lambda)f\|_{B^{-|s|}_{q,1}(\mathbb{R}^N_+)} \leq C|\lambda|^{-1}\|f\|_{B^{-|s|}_{q,1}(\mathbb{R}^N_+)}, \tag{6.48}$$

which shows (6.42).

Choosing $\theta \in (0,1)$ in such a way that $|s| = \mu'\theta$ and combining (6.46) and (6.47) with real interpolation method, we have

$$\|T_2(\lambda)f\|_{B^{-|s|}_{q,r}(\mathbb{R}^N_+)} \leq C|\lambda|^{-a}\|f\|_{B^c_{q,r}(\mathbb{R}^N_+)}.$$

Here,

$$
\begin{aligned}
a &= -\theta - (1-\theta)(1-\frac{\mu}{2}) = -1 + \frac{\mu}{2}(1-\frac{|s|}{\mu'}), \\
c &= -\mu'\theta - \mu(1-\theta) = -\mu'\frac{|s|}{\mu'} - \mu(1-\frac{|s|}{\mu'}) = -|s| - \mu(1-\frac{|s|}{\mu'}) = -(|s| + \mu(1-\frac{|s|}{\mu'})).
\end{aligned}
$$

Thus, we have obtained

$$
\|T_2(\lambda)f\|_{B^{-|s|}_{q,r}(\mathbb{R}^N_+)} \le C|\lambda|^{-(1-\frac{\mu}{2}(1-\frac{|s|}{\mu'}))}\|f\|_{B^{-(|s|+\mu(1-\frac{|s|}{\mu'}))}_{q,r}(\mathbb{R}^N_+)}. \tag{6.49}
$$

Now, we choose $\mu' \in (0,1)$ in such a way that

$$
-|s| > -|s| - \sigma > -|s| - \mu(1-\frac{|s|}{\mu'}),
$$

that is

$$
\frac{\mu|s|}{\mu-\sigma} < \mu' < 1 - 1/q. \tag{6.50}
$$

Since $\sigma > 0$ may be chosen so small that $\mu/(\mu-\sigma)$ is very close to 1, we can choose μ' in such a way that $|s| < \mu'$ and (6.50) holds.

We choose $\theta \in (0,1)$ in such a way that

$$
-|s| - \sigma = -|s|\theta - (|s| + \mu(1-\frac{|s|}{\mu'}))(1-\theta).
$$

Combining (6.48) and (6.49) with real interpolation method implies that

$$
\|T_2(\lambda)f\|_{B^{-|s|}_{q,r}(\mathbb{R}^N_+)} \le C|\lambda|^{-d}\|f\|_{B^{-|s|-\sigma}_{q,r}(\mathbb{R}^N_+)},
$$

where

$$
d = \theta + (1-\theta)(1-\frac{\mu}{2}(1-\frac{|s|}{\mu'})) = 1 - \frac{\sigma}{2}.
$$

Thus, we have

$$
\|T_2(\lambda)f\|_{B^{-|s|}_{q,r}(\mathbb{R}^N_+)} \le C|\lambda|^{-(1-\frac{\sigma}{2})}\|f\|_{B^{-|s|-\sigma}_{q,r}(\mathbb{R}^N_+)}.
$$

Namely, we have (6.43), which completes the proof of Lemma 6.8. □

The end of the proof of Theorem 6.4. In view of Lemmas 6.5–6.8, it suffices to prove the case $s = 0$. Let $0 < \sigma < 1/q$ and let ν_1 and ν_2 be positive numbers such that $-1+1/q < -\nu_1 < -\nu_1 + \sigma < 0 < \nu_2 < \sigma + \nu_2 < 1/q$. By Lemmas 6.5 and 6.6, we have

$$
\|T_1(\lambda)f\|_{B^{-\nu_1}_{q,r}(\mathbb{R}^N_+)} \le C\|f\|_{B^{-\nu_1}_{q,r}(\mathbb{R}^N_+)}, \quad \|T_1(\lambda)f\|_{B^{\nu_2}_{q,r}(\mathbb{R}^N_+)} \le C\|f\|_{B^{\nu_2}_{q,r}(\mathbb{R}^N_+)}.
$$

Let θ be a number $\in (0,1)$ such that $0 = (1-\theta)(-\nu_1) + \theta\nu_2$. Since

$$
B^0_{q,r}(\mathbb{R}^N_+) = (B^{-\nu_1}_{q,r}(\mathbb{R}^N_+), B^{\nu_2}_{q,r}(\mathbb{R}^N_+))_{\theta,r},
$$

by real interpolation we have

$$
\|T_1(\lambda)f\|_{B^0_{q,r}(\mathbb{R}^N_+)} \le C\|f\|_{B^0_{q,r}(\mathbb{R}^N_+)}.
$$

Moreover, by Lemmas 6.5 and 6.6, we have

$$\|T_1(\lambda)f\|_{B^{-\nu_1}_{q,r}(\mathbb{R}^N_+)} \le C|\lambda|^{-\frac{\sigma}{2}}\|f\|_{B^{-\nu_1+\sigma}_{q,r}(\mathbb{R}^N_+)}, \quad \|T_1(\lambda)f\|_{B^{\nu_2}_{q,r}(\mathbb{R}^N_+)} \le C|\lambda|^{-\frac{\sigma}{2}}\|f\|_{B^{\nu_2+\sigma}_{q,r}(\mathbb{R}^N_+)}.$$

Since $B^\sigma_{q,r}(\mathbb{R}^N_+) = (B^{-\nu_1+\sigma}_{q,r}(\mathbb{R}^N_+), B^{\nu_2+\sigma}_{q,r}(\mathbb{R}^N_+))_{\theta,r}$, by real interpolation, we have

$$\|T_1(\lambda)f\|_{B^0_{q,r}(\mathbb{R}^N_+)} \le C|\lambda|^{-\frac{\sigma}{2}}\|f\|_{B^\sigma_{q,r}(\mathbb{R}^N_+)}.$$

Analogously, we have

$$\begin{aligned}\|T_2(\lambda)f\|_{B^0_{q,r}(\mathbb{R}^N_+)} &\le C|\lambda|^{-1}\|f\|_{B^0_{q,r}(\mathbb{R}^N_+)},\\ \|T_2(\lambda)f\|_{B^0_{q,r}(\mathbb{R}^N_+)} &\le C|\lambda|^{-(1-\frac{\sigma}{2})}\|f\|_{B^{-\sigma}_{q,r}(\mathbb{R}^N_+)}.\end{aligned}$$

This completes the proof of Theorem 6.4. □

6.2 A proof of Theorem 6.1.

In this subsection, we shall prove Theorem 6.1. First, we divide $\dfrac{\eta\lambda}{K}$ and $\dfrac{\eta\lambda}{K}\dfrac{\beta\lambda+\gamma^2}{(\alpha+\beta)\lambda+\gamma^2}$ appearing in (4.7). To this end, we start with the following lemma.

Lemma 6.9. *Let $\epsilon \in (0, \pi/2)$ and $\lambda_0 > 0$. Set $K_1 = (\alpha+\beta)A + \alpha B$. Then, there exists a constant $c_3 > 0$ depending on α, β, ϵ and ϵ' appearing in Lemma 3.1 such that for any $\lambda \in \Lambda_{\epsilon,\lambda_0}$ and $\xi' \in \mathbb{R}^{N-1}\setminus\{0\}$, there holds*

$$|K_1| \ge c_3(|\lambda|^{1/2} + |\xi'|). \tag{6.51}$$

Moreover, for any multi-index $\delta' \in \mathbb{N}_0^{N-1}$, $\lambda \in \Lambda_{\epsilon,\lambda_0}$ and $\xi' \in \mathbb{R}^{N-1}\setminus\{0\}$ there holds

$$\begin{aligned}|D^{\delta'}_{\xi'}K_1^{-1}| &\le C_{\delta'}(|\lambda|^{1/2}+|\xi'|)^{-1-|\delta'|},\\ |D^{\delta'}_{\xi'}\partial_\lambda K_1^{-1}| &\le C_{\delta'}(|\lambda|^{1/2}+|\xi'|)^{-3-|\delta'|}.\end{aligned} \tag{6.52}$$

Proof. By Lemma 3.1, we have

$$|\arg(\alpha+\beta)A| \le \frac{\pi-\epsilon'}{2}, \quad |\arg\alpha B| \le \frac{\pi-\epsilon}{2}.$$

Moreover, we see that

$$(\alpha+\beta)|A| \ge \frac{\alpha+\beta}{\sqrt{2}}\sqrt{c_1}(|\lambda|^{1/2}+|\xi'|), \quad \alpha|B| \ge \frac{\alpha}{\sqrt{2}}\sqrt{\sin(\epsilon/2)}(|\lambda|^{1/2}\alpha^{-1/2}+|\xi'|).$$

From geometric interpretation of the sum of complex numbers we see that (6.51) holds with

$$c_3 = (\sin\frac{\min(\sigma,\epsilon)}{2})\min(\frac{(\alpha+\beta)}{\sqrt{2}}\sqrt{c_1}, \frac{\alpha}{\sqrt{2}}\sqrt{\sin(\epsilon/2)}\min(\alpha^{-1/2},1))$$

By Bell's formula,

$$|D^{\delta'}_{\xi'}K_1^{-1}| \le C_{\delta'}\sum_{\ell=1}^{|\delta'|}|K_1|^{-(\ell+1)}\sum_{\substack{|\delta'_1|+\cdots+|\delta'_\ell|=|\delta'|\\ |\delta'_i|\ge 1}}|D^{\delta'_1}_{\xi'}K_1|\cdots|D^{\delta'_\ell}_{\xi'}K_1|. \tag{6.53}$$

By Lemma 3.2 with $s = 1$, we have

$$|D_{\xi'}^{\delta'_1} K_1| \cdots |D_{\xi'}^{\delta'_\ell} K_1| \le C_{\delta'}(|\lambda|^{1/2} + |\xi'|)^{\ell - |\delta'|},$$

which, combined with (6.53), implies (6.52).

Writing $\partial_\lambda K_1 = (\alpha+\beta)\partial_\lambda A + \alpha\partial_\lambda B$, by Lemma 3.3 we have

$$|D_{\xi'}^{\delta'}(\partial_\lambda K_1)| \le C_{\delta'}(|\lambda|^{1/2} + |\xi'|)^{-1-|\delta'|}.$$

Since $\partial_\lambda K_1^{-1} = -(\partial_\lambda K_1)K_1^{-2}$, (6.52) follows from $\partial_\lambda K_1 \in \mathbb{M}_{-1}$ and $K_1^{-1} \in \mathbb{M}_{-1}$. This completes the proof of Lemma 6.9. □

Recall that $K = (\alpha + \eta_\lambda)A + \alpha B = K_1 + \gamma^2\lambda^{-1}A$ (cf. (3.1)). In particular,

$$|A| \le \max(c_2^{-1/2}, 1)(|\lambda|^{1/2} + |\xi'|).$$

By (6.51) $|\gamma^2 A K_1^{-1}| \le \gamma^2 \max(c_2^{-1/2}, 1)c_3^{-1}$. Setting $\lambda_1 = 2\gamma^2 \max(c_2^{-1/2}, 1)c_3^{-1}$, we have

$$|\gamma^2 A K_1^{-1}\lambda^{-1}| \le 1/2 \tag{6.54}$$

for any $\lambda \in \Lambda_{\epsilon,\lambda_1}$ and $\xi' \in \mathbb{R}^{N-1}\setminus\{0\}$. Thus, we have

$$K^{-1} = \frac{1}{K_1} - \frac{1}{K_1}\frac{\gamma^2 A K_1^{-1}\lambda^{-1}}{1 + \gamma^2 A K_1^{-1}\lambda^{-1}}.$$

From this observation, it follows that

$$\frac{\eta_\lambda}{K} = \frac{\beta}{K_1} - \frac{1}{\lambda}\frac{\beta}{K_1}\frac{\gamma^2 A K_1^{-1}}{1 + \gamma^2 A K_1^{-1}\lambda^{-1}} + \frac{\gamma^2}{\lambda}\frac{\beta}{K}.$$

Thus, setting

$$K_2(\lambda) = -\frac{\beta}{K_1}\frac{\gamma^2 A K_1^{-1}}{1 + \gamma^2 A K_1^{-1}\lambda^{-1}} + \frac{\gamma^2\beta}{K}, \tag{6.55}$$

we may write

$$\frac{\eta_\lambda}{K} = \frac{\beta}{K_1} + \lambda^{-1}K_2(\lambda).$$

Lemma 6.10. *Let $\epsilon \in (0, \pi/2)$ and let λ_1 be a positive number defined in* (6.54)*. Let K_2 be the function defined in* (6.55)*. Then, for any $\lambda \in \Lambda_{\epsilon,\lambda_1}$, $\xi' \in \mathbb{R}^{N-1}\setminus\{0\}$, and multi-index $\delta' \in \mathbb{N}_0^{N-1}$, there hold*

$$|D_{\xi'}^{\delta'} K_2(\lambda)| \le C_{\delta'}(|\lambda|^{1/2} + |\xi'|)^{-1-|\delta'|}, \tag{6.56}$$

$$|D_{\xi'}^{\delta'}(\partial_\lambda K_2(\lambda))| \le C_{\delta'}(|\lambda|^{1/2} + |\xi'|)^{-1-|\delta'|}|\lambda|^{-1}, \tag{6.57}$$

with some constant $C_{\delta'}$.

Proof. In what follows, we assume that $\lambda \in \Lambda_{\epsilon,\lambda_1}$ and $\xi' \in \mathbb{R}^{N-1}\setminus\{0\}$. By (6.52) and Lemma 3.2, we have

$$|D_{\xi'}^{\delta'}(A K_1^{-1})| \le C_{\delta'}(|\lambda|^{1/2} + |\xi'|)^{-|\delta'|} \tag{6.58}$$

for any multi-index $\delta' \in \mathbb{N}_0^{N-1}$. We may assume that $|1 + \gamma^2 A K_1^{-1}\lambda^{-1}|^{-1} \le 2$ from (6.54), and so by Bell's formula, (6.54), and (6.58)

$$|D_{\xi'}^{\delta'}(1 + \gamma^2 A K_1^{-1}\lambda^{-1})^{-1}|$$

$$\leq C_{\delta'} \sum_{\ell=1}^{|\delta'|} |1+\gamma^2 AK_1^{-1}\lambda^{-1}|^{-(\ell+1)} \sum_{\substack{\delta'_1+\cdots+\delta'_\ell=\delta' \\ |\delta'_i|\geq 1}} |D_{\xi'}^{\delta'_1}(\gamma^2 AK_1^{-1}\lambda^{-1})|\cdots|D_{\xi'}^{\delta'_\ell}(\gamma^2 AK_1^{-1}\lambda^{-1})|$$

$$\leq C_{\delta'}\{\sum_{\ell=1}^{|\delta'|}(\gamma^2|\lambda|^{-1})^\ell 2^{\ell+1}\}(|\lambda|^{1/2}+|\xi'|)^{-|\delta'|} \leq C_{\delta'}\frac{4\gamma^2|\lambda|^{-1}}{(1-2\gamma^2|\lambda|^{-1})}(|\lambda|^{1/2}+|\xi'|)^{-|\delta'|} \tag{6.59}$$

provided that $2\gamma^2|\lambda|^{-1}<1$. Combining Lemma 3.3 with $M=K$, (6.52), (6.58) and (6.59) gives (6.56).

To prove (6.57), we write

$$\partial_\lambda K_2(\lambda) = -\beta\partial_\lambda K_1^{-1}\frac{\gamma^2 AK_1^{-1}}{1+\gamma^2 AK_1^{-1}\lambda^{-1}} + \frac{\beta}{K_1}\frac{\gamma^2\lambda^{-1}\partial_\lambda(AK_1^{-1})\gamma^2 AK_1^{-1}}{(1+\gamma^2 AK_1^{-1}\lambda^{-1})^2} - \frac{\gamma^2\partial_\lambda(AK_1^{-1})}{1+\gamma^2 AK_1^{-1}\lambda^{-1}} - \partial_\lambda K^{-1}.$$

By (6.52), we have $\partial_\lambda K_1^{-1}\in\mathbb{M}_{-3}$. Since $\partial_\lambda K=(\alpha+\beta+\gamma^2\lambda^{-1})\partial_\lambda A+\beta\partial_\lambda B-\gamma^2\lambda^{-2}A$, by Lemma 3.3, we have

$$|D_{\xi'}^{\delta'}(\partial_\lambda K)| \leq C_{\delta'}\{(|\lambda|^{1/2}+|\xi'|)^{-1-|\delta'|}+|\lambda|^{-2}(|\lambda|^{1/2}+|\xi'|)^{1-|\delta'|}\}. \tag{6.60}$$

Writing $\partial_\lambda K^{-1}=-K^{-2}\partial_\lambda K$ and using Lemma 3.3 and (6.60), we have

$$|D_{\xi'}^{\delta'}(\partial_\lambda K^{-1})| \leq C_{\delta'}|\lambda|^{-1}(|\lambda|^{1/2}+|\xi'|)^{-1-|\delta'|}. \tag{6.61}$$

Writing $\partial_\lambda(AK_1^{-1})=(\partial_\lambda A)K_1^{-1}-AK_1^{-2}\partial_\lambda K_1$, by (6.59), Lemma 3.3 and (6.52) we have

$$|D_{\xi'}^{\delta'}\partial_\lambda(AK_1^{-1})| \leq C_{\delta'}(|\lambda|^{1/2}+|\xi'|)^{-2-|\delta'|}. \tag{6.62}$$

Thus, by (6.52), (6.59), (6.64), (6.52), (6.61), and (6.62), we have (6.57). This completes the proof of Lemma 6.10. □

We write

$$\begin{aligned} &\frac{\beta\lambda+\gamma^2}{(\alpha+\beta)\lambda+\gamma^2} = \frac{\beta}{\alpha+\beta}+\frac{q(\lambda)}{\lambda}, \quad \frac{\eta_\lambda}{K} = \frac{\beta}{K_1}+\frac{1}{\lambda}K_2(\lambda), \\ &\frac{\eta_\lambda}{K}\frac{\beta\lambda+\gamma^2}{(\alpha+\beta)\lambda+\gamma^2} = \frac{\beta^2}{\alpha+\beta}\frac{1}{K_1}+\frac{1}{\lambda}K_3(\lambda) \end{aligned} \tag{6.63}$$

where

$$\begin{aligned} q(\lambda) &= \frac{\alpha\gamma^2\lambda}{(\alpha+\beta)((\alpha+\beta)\lambda+\gamma^2)}, \\ K_3(\lambda) &= \frac{\alpha\beta\gamma^2}{(\alpha+\beta)K_1}\frac{\lambda}{(\alpha+\beta)\lambda+\gamma^2}+K_2(\lambda)\frac{\beta\lambda+\gamma^2}{(\alpha+\beta)\lambda+\gamma^2}. \end{aligned} \tag{6.64}$$

Notice that $|(\alpha+\beta)\lambda+\gamma^2|\geq(1/2)(\alpha+\beta)|\lambda|$ for $|\lambda|\geq 2\gamma^2(\alpha+\beta)^{-1}$.

Corollary 6.11. *Let $\epsilon\in(0,\pi/2)$ and let λ_1 be a positive number defined in* (6.54)*. Set $\lambda_2=\max(\lambda_1, 2\gamma^2/(\alpha+\beta))$. Let K_3 be the function defined in* (6.64)*. Then, for any $\lambda\in\Lambda_{\epsilon,\lambda_2}$, $\xi'\in\mathbb{R}^{N-1}\setminus\{0\}$, and multi-index $\delta'\in\mathbb{N}_0^{N-1}$, there hold*

$$\begin{aligned} |D_{\xi'}^{\delta'}K_3(\lambda)| &\leq C_{\delta'}(|\lambda|^{1/2}+|\xi'|)^{-1-|\delta'|}, \\ |D_{\xi'}^{\delta'}(\partial_\lambda K_3(\lambda))| &\leq C_{\delta'}(|\lambda|^{1/2}+|\xi'|)^{-1-|\delta'|}|\lambda|^{-1}, \end{aligned}$$

with some constant $C_{\delta'}$.

Applying the decompositions (6.63) to the formulas in (4.7), we define $\mathcal{T}^b_{1J}(\lambda)$ and $\mathcal{T}^b_{2J}(\lambda)$ by

$$
\begin{aligned}
\mathcal{T}^b_{1j}(\lambda)\mathbf{g} &= \int_0^\infty \mathcal{F}^{-1}_{\xi'}\Big[Be^{-(x_N+y_N)B}\frac{1}{\alpha B^2}\mathcal{F}[g_j](\xi',y_N)\Big](x')\,dy_N \\
&+ \int_0^\infty \mathcal{F}^{-1}_{\xi'}\Big[B^2e^{-Bx_N}\mathcal{M}(y_N)\frac{\beta}{\alpha+\beta}\Big(\sum_{k=1}^{N-1}\frac{i\xi_j\xi_k}{(A+B)AB^2}\mathcal{F}'[g_k](\xi',y_N) \\
&\qquad - \frac{i\xi_j}{(A+B)B^2}\mathcal{F}'[g_N](\xi',y_N)\Big)\Big](x')\,dy_N \\
&- \int_0^\infty \mathcal{F}^{-1}_{\xi'}\Big[Be^{-B(x_N+y_N)}\frac{\beta}{\alpha+\beta}\sum_{k=1}^{N-1}\frac{i\xi_j\xi_k}{AB^2(A+B)}\mathcal{F}'[g_k](\xi',y_N)\Big](x')\,dy_N \\
&- \int_0^\infty \mathcal{F}^{-1}_{\xi'}\Big[B^2\mathcal{M}(x_N)e^{-By_N}\sum_{k=1}^{N-1}\frac{\beta i\xi_j i\xi_k}{\alpha K_1B^3}\mathcal{F}[\mathbf{g}_k](\xi',y_N)\Big](x')\,dy_N \\
&+ \int_0^\infty \mathcal{F}^{-1}_{\xi'}\Big[B^3\mathcal{M}(x_N)\mathcal{M}(y_N)\frac{\beta^2 i\xi_j}{(\alpha+\beta)K_1}\Big(\sum_{k=1}^{N-1}\frac{|\xi'|^2\xi_k}{(A+B)AB^3}\mathcal{F}'[g_k](\xi',y_N) \\
&\qquad - \frac{|\xi'|^2}{(A+B)B^3}\mathcal{F}'[g_N](\xi',y_N)\Big)\Big](x')\,dy_N \\
&- \int_0^\infty \mathcal{F}^{-1}_{\xi'}\Big[B^2\mathcal{M}(x_N)e^{-By_N}\frac{\beta^2 i\xi_j}{(\alpha+\beta)K_1}\sum_{k=1}^{N-1}\frac{|\xi'|^2\xi_k}{AB^3(A+B)}\mathcal{F}'[g_k](\xi',y_N)\Big](x')\,dy_N, \\
\mathcal{T}^b_{1N}(\lambda)\mathbf{g} &= \int_0^\infty \mathcal{F}^{-1}_{\xi'}\Big[B^2\mathcal{M}(x_N)e^{-By_N}\sum_{k=1}^{N-1}\frac{\beta A i\xi_k}{\alpha K_1B^3}\mathcal{F}[\mathbf{g}_k](\xi',y_N)\Big](x')\,dy_N \\
&- \int_0^\infty \mathcal{F}^{-1}_{\xi'}\Big[B^3\mathcal{M}(x_N)\mathcal{M}(y_N)\frac{\beta^2 A}{(\alpha+\beta)K_1}\Big(\sum_{k=1}^{N-1}\frac{|\xi'|^2\xi_k}{(A+B)AB^3}\mathcal{F}'[g_k](\xi',y_N) \\
&\qquad - \frac{|\xi'|^2}{(A+B)B^3}\mathcal{F}'[g_N](\xi',y_N)\Big](x')\,dy_N \\
&+ \int_0^\infty \mathcal{F}^{-1}_{\xi'}\Big[B^2\mathcal{M}(x_N)e^{-By_N}\frac{\beta^2 A}{(\alpha+\beta)K_1}\sum_{k=1}^{N-1}\frac{|\xi'|^2\xi_k}{AB^3(A+B)}\mathcal{F}'[g_k](\xi',y_N)\Big](x')\,dy_N, \\
\mathcal{T}^b_{2j}&(\lambda)\mathbf{g} \\
&= \int_0^\infty \mathcal{F}^{-1}_{\xi'}\Big[B^2e^{-Bx_N}\mathcal{M}(y_N)\frac{q(\lambda)}{\lambda}\Big(\sum_{k=1}^{N-1}\frac{i\xi_j\xi_k}{(A+B)AB^2}\mathcal{F}'[g_k](\xi',y_N) \\
&\qquad - \frac{i\xi_j}{(A+B)B^2}\mathcal{F}'[g_N](\xi',y_N)\Big)\Big](x')\,dy_N \\
&- \int_0^\infty \mathcal{F}^{-1}_{\xi'}\Big[Be^{-B(x_N+y_N)}\frac{q(\lambda)}{\lambda}\sum_{k=1}^{N-1}\frac{i\xi_j\xi_k}{AB^2(A+B)}\mathcal{F}'[g_k](\xi',y_N)\Big](x')\,dy_N \\
&- \int_0^\infty \mathcal{F}^{-1}_{\xi'}\Big[B^2\mathcal{M}(x_N)e^{-By_N}\frac{K_2(\lambda)}{\lambda}\frac{i\xi_j}{\alpha B^3}i\xi'\cdot\mathcal{F}[\mathbf{g}'](\xi',y_N)\Big](x')\,dy_N \\
&+ \int_0^\infty \mathcal{F}^{-1}_{\xi'}\Big[B^3\mathcal{M}(x_N)\mathcal{M}(y_N)\Big(\sum_{k=1}^{N-1}\frac{K_3(\lambda)}{\lambda}\frac{i\xi_j\xi_k|\xi'|^2}{(A+B)AB^3}\mathcal{F}'[g_k](\xi',y_N) \\
&\qquad - \frac{K_3(\lambda)}{\lambda}\frac{i\xi_j|\xi'|^2}{(A+B)B^3}\mathcal{F}'[g_N](\xi',y_N)\Big)\Big](x')\,dy_N
\end{aligned}
$$

$$
\begin{aligned}
&- \int_0^\infty \mathcal{F}^{-1}_{\xi'}\Big[B^2\mathcal{M}(x_N)e^{-By_N}\sum_{k=1}^{N-1}\frac{K_3(\lambda)}{\lambda}\frac{i\xi_j|\xi'|^2\xi_k}{AB^3(A+B)}\mathcal{F}'[g_k](\xi', y_N)\Big](x')\,dy_N,\\
\mathcal{T}^b_{2N}(\lambda)\mathbf{g} &= \int_0^\infty \mathcal{F}^{-1}_{\xi'}\Big[B^2\mathcal{M}(x_N)e^{-By_N}\sum_{k=1}^{N-1}\frac{K_2(\lambda)}{\lambda}\frac{i\xi_k A}{\alpha B^3}\mathcal{F}[\mathbf{g}_k](\xi', y_N)\Big](x')\,dy_N\\
&- \int_0^\infty \mathcal{F}^{-1}_{\xi'}\Big[B^3\mathcal{M}(x_N)\mathcal{M}(y_N)(\sum_{k=1}^{N-1}\frac{K_3(\lambda)}{\lambda}\frac{|\xi'|^2\xi_k}{(A+B)B^3}\mathcal{F}'[g_k](\xi', y_N)\\
&\qquad\qquad - \frac{K_3(\lambda)}{\lambda}\frac{|\xi'|^2A}{(A+B)B^3}\mathcal{F}'[g_N](\xi', y_N)\Big](x')\,dy_N\\
&+ \int_0^\infty \mathcal{F}^{-1}_{\xi'}\Big[B^2\mathcal{M}(x_N)e^{-By_N}\sum_{k=1}^{N-1}\frac{K_3(\lambda)}{\lambda}\frac{|\xi'|^2\xi_k}{B^3(A+B)}\mathcal{F}'[g_k](\xi', y_N)\Big](x')\,dy_N.
\end{aligned}
$$

And then, we have

$$\mathcal{S}^b(\lambda)\mathbf{g} = \mathcal{T}^b_1(\lambda)\mathbf{g} + \mathcal{T}^b_2(\lambda)\mathbf{g}.$$

where we have set $\mathcal{T}^b_i(\lambda) = (\mathcal{T}^b_{i1}(\lambda), \ldots, \mathcal{T}^b_{iN}(\lambda))$ $(i = 1, 2)$.

To estimate $\mathcal{T}^b_1(\lambda) = (\mathcal{T}^b_{11}(\lambda), \ldots, \mathcal{T}^b_{1N}(\lambda))$, we introduce the multiplier class $\mathbb{N}_k$ defined by

$$\mathbb{N}_k = \{m(\lambda, \xi') \in \mathbb{M}_k(\Lambda_{\epsilon,\lambda_0}) \mid \partial_\lambda m(\lambda, \xi') \in \mathbb{M}_{k-2}(\Lambda_{\epsilon,\lambda_0})\}.$$

By Lemmas 3.3 and 6.9, all the following symbols appearing in the definition of $\mathcal{T}^b_{1J}(\lambda)$ $(J = 1, \ldots, N-1, N)$:

$$
\begin{gathered}
\frac{1}{B^2},\quad \frac{i\xi_j 1\xi_k}{(A+B)AB^2},\quad \frac{i\xi_j}{(A+B)B^2},\quad \frac{i\xi_j i\xi_k}{K_1B^3},\quad \frac{i\xi_j|\xi'|^2\xi_k}{K_1(A+B)AB^3},\\
\frac{i\xi_j|\xi'|^2}{K_1(A+B)B^3},\quad \frac{Ai\xi_k}{K_1B^3},\quad \frac{A|\xi'|^2\xi_k}{K_1(A+B)AB^3},\quad \frac{A|\xi'|^2}{K_1(A+B)B^3}
\end{gathered}
$$

belong to $\mathbb{N}_{-2}$. To represent $\mathcal{T}^b_1(\lambda)$ in a little bit simple way, we define symbols $\mathcal{P}_i$ $(i = 1, 2, 3, 4)$ by

$$
\begin{aligned}
\mathcal{P}_1(x_N, y_N) &= Be^{-B(x_N+y_N)}, \quad \mathcal{P}_2(x_N, y_N) = B^2e^{-Bx_N}\mathcal{M}(y_N),\\
\mathcal{P}_3(x_N, y_N) &= B^2\mathcal{M}(x_N)e^{-By_N}, \quad \mathcal{P}_4(x_N, y_N) = B^3\mathcal{M}(x_N)\mathcal{M}(y_N).
\end{aligned}
$$

Then, we may assert that there exist four $N \times N$ matrices of $\mathbb{N}_{-2}$ symbols $\mathbf{T}^{b,0}_{1j}(\lambda, \xi')$ such that $\mathcal{T}^b_1(\lambda)$ is represented by

$$\mathcal{T}^b_1(\lambda)\mathbf{g} = \int_0^\infty \mathcal{F}^{-1}_{\xi'}\Big[\Big(\sum_{j=1}^4 \mathcal{P}_j(x_N, y_N)\mathbf{T}^{b,0}_{1j}(\lambda, \xi')\Big)\mathcal{F}'[\mathbf{g}](\xi', y_N)\Big](x')\,dy_N. \tag{6.65}$$

First, we shall prove the L_q-L_q estimate. Below, we write $\nabla' = (\partial_1, \ldots, \partial_{N-1})$, $\nabla'' = (\partial_j\partial_k \mid j, k = 1, \ldots, N-1)$, and $\nabla''' = (\partial^\delta \mid |\delta| = 3)$. Corresponding symbols are written by $\xi' = (\xi_1, \ldots, \xi_{N-1})$, $(i\xi')^2 = (i\xi_j i\xi_k \mid j, k = 1, \ldots, N-1)$, and $(\xi')^3 = (i\xi_j i\xi_k i\xi_\ell \mid j, k, \ell = 1, \ldots, N-1)$. Using the formulas:

$$\partial^\ell_N\mathcal{M}(x_N) = (-1)^\ell(A^\ell\mathcal{M}(x_N) + \frac{A^\ell - B^\ell}{A-B}e^{-Bx_N}) \quad (\ell \geq 1),$$

we write

$$\partial^\ell_N\mathcal{T}^b_1(\lambda)\mathbf{g} = (-1)^\ell\int_0^\infty \mathcal{F}^{-1}_{\xi'}\Big[\Big(\sum_{j=1}^4 \mathcal{P}_j(x_N, y_N)\mathbf{T}^{b,\ell}_{1j}(\lambda, \xi')\Big)\mathcal{F}'[\mathbf{g}](\xi', y_N)\Big](x')\,dy_N \tag{6.66}$$

for $\ell = 1, 2$, where we have set

$$\mathbf{T}^{b,\ell}_{1k}(\lambda, \xi') = B^\ell \mathbf{T}^{b,0}_{1k}(\lambda, \xi') + \frac{A^\ell - B^\ell}{A - B} B\mathbf{T}^{b,0}_{1k+2}, \quad \mathbf{T}^{b,\ell}_{1m}(\lambda, \xi') = A^\ell \mathbf{T}^{b,0}_{1m}(\lambda, \xi')$$

for $k = 1, 2$ and $m = 3, 4$. We see that $\mathbf{T}^{b,\ell}_{1j}(\lambda, \xi') \in \mathbb{N}_{-2+\ell}$ for $\ell = 1, 2, 3$. Then, for $(\lambda, \lambda^{1/2}\nabla, \nabla^2)$, using (6.65) and (6.66), we may write

$$\begin{aligned}
&(\lambda, \lambda^{1/2}\nabla', \nabla'')\mathcal{T}^b_1(\lambda)\mathbf{g} \\
&\quad = \int_0^\infty \mathcal{F}^{-1}_{\xi'}\Big[\Big(\sum_{j=1}^4 \mathcal{P}_j(x_N, y_N)(\lambda, \lambda^{1/2} i\xi', (i\xi')^2)\mathbf{T}^{b,0}_{1j}(\lambda, \xi')\Big)\mathcal{F}'[\mathbf{g}](\xi', y_N)\Big](x')\,dy_N, \\
&(\lambda^{1/2}, \nabla')\partial_N \mathcal{T}^b_1(\lambda)\mathbf{g} \\
&\quad = (-1)\int_0^\infty \mathcal{F}^{-1}_{\xi'}\Big[\Big(\sum_{j=1}^4 \mathcal{P}_j(x_N, y_N)(\lambda^{1/2}, i\xi')\mathbf{T}^{b,1}_{1j}(\lambda, \xi')\Big)\mathcal{F}'[\mathbf{g}](\xi', y_N)\Big](x')\,dy_N, \\
&\partial_N^2 \mathcal{T}^b_1(\lambda)\mathbf{g} = \int_0^\infty \mathcal{F}^{-1}_{\xi'}\Big[\Big(\sum_{j=1}^4 \mathcal{P}_j(x_N, y_N)\mathbf{T}^{b,2}_{1j}(\lambda, \xi')\Big)\mathcal{F}'[\mathbf{g}](\xi', y_N)\Big](x')\,dy_N.
\end{aligned}$$

Since $(\lambda, \lambda^{1/2}i\xi', (i\xi')^2)\mathbf{T}^b_{1j}(\lambda, \xi')$, $(\lambda^{1/2}, i\xi')\mathbf{T}^{b,1}_{1j}$ and $\mathbf{T}^{b,2}_{1j}(\lambda, \xi')$ belong to $\mathbb{M}_0$, by Proposition 3.5, we have

$$\|(\lambda, \lambda^{1/2}\nabla, \nabla^2)\mathcal{T}^b_1(\lambda)\mathbf{g}\|_{L_q(\mathbb{R}^N_+)} \le C\|\mathbf{g}\|_{L_q(\mathbb{R}^N_+)}. \tag{6.67}$$

Next, we consider H^1_q-H^1_q estimate. To this end, we use the formulas:

$$\begin{aligned}
&\mathcal{P}_1(x_N, y_N) = -B^{-1}\partial_{y_N}\mathcal{P}(x_N, y_N), \quad \mathcal{P}_2(x_N, y_N) = -A^{-1}\partial_{y_N}(\mathcal{P}_2(x_N, y_N) - \mathcal{P}_1(x_N, y_N)), \\
&\mathcal{P}_3(x_N, y_N) = -B^{-1}\mathcal{P}_3(x_N, y_N), \quad \mathcal{P}_4(x_N, y_N) = A^{-1}\partial_{y_N}(\mathcal{P}_4(x_N, y_N) - \mathcal{P}_3(x_N, y_N)),
\end{aligned}$$

which follows from

$$e^{-Bt_N} = \frac{1}{B}\partial_N e^{-B(y_N)}, \quad \mathcal{M}(y_N) = \partial_N\Big\{\frac{-1}{A}\mathcal{M}(y_N) + \frac{1}{AB}e^{-By_N}\Big\}.$$

Since $\mathbf{g} \in C^\infty_0(\mathbb{R}^N_+)^N$, by integration by parts, we rewrite the formulas in (6.65) and (6.66) as follows:

$$\partial^\ell_N \mathcal{T}^b_1(\lambda)\mathbf{g} = \int_0^\infty \mathcal{F}^{-1}_{\xi'}\Big[\Big(\sum_{j=1}^4 \mathcal{P}_j(x_N, y_N)\tilde{\mathbf{T}}^{b,\ell}_{1j}(\lambda, \xi')\Big)\mathcal{F}'[\partial_N\mathbf{g}](\xi', y_N)\Big](x')\,dy_N. \tag{6.68}$$

Here, we have set

$$\begin{aligned}
\tilde{\mathbf{T}}^{b,\ell}_{11} &= B^{-1}\mathbf{T}^{b,\ell}_{11} - A^{-1}\mathbf{T}^{b,\ell}_{12}, \quad \tilde{\mathbf{T}}^{b,\ell}_{12} = A^{-1}\mathbf{T}^{b,2}_{12}, \\
\tilde{\mathbf{T}}^{b,\ell}_{13} &= B^{-1}\mathbf{T}^{b,\ell}_{13} + A^{-1}\mathbf{T}^{b,\ell}_{14}, \quad \tilde{\mathbf{T}}^{b,\ell}_{14} = -A^{-1}\mathbf{T}^{b,\ell}_{14}.
\end{aligned}$$

Since $\mathbf{T}^{b,\ell}_{1j} \in \mathbb{N}_{-2+\ell}$, we see that $\tilde{\mathbf{T}}^{b,\ell}_{1j} \in \mathbb{N}_{-3+\ell}$ for $\ell = 0, 1, 2, 3$.

Using (6.68), we may write

$$\begin{aligned}
&\nabla'(\lambda, \lambda^{1/2}\nabla', \nabla'')\mathcal{T}^{b,0}_1(\lambda)\mathbf{g} \\
&\quad = \int_0^\infty \mathcal{F}^{-1}_{\xi'}\Big[\Big(\sum_{j=1}^4 \mathcal{P}_j(x_N, y_N) i\xi'(\lambda, \lambda^{1/2}i\xi', (i\xi')^2)\tilde{\mathbf{T}}^{b,0}_{1j}(\lambda, \xi')\Big)\mathcal{F}'[\partial_N\mathbf{g}](\xi', y_N)\Big](x')\,dy_N,
\end{aligned}$$

$$
\begin{aligned}
&\partial_N(\lambda, \lambda^{1/2}\nabla', \nabla'')\mathcal{T}_1^{b,0}(\lambda)\mathbf{g} \\
&= \int_0^\infty \mathcal{F}_{\xi'}^{-1}\Big[\Big(\sum_{j=1}^4 \mathcal{P}_j(x_N, y_N)(\lambda, \lambda^{1/2}i\xi', (i\xi')^2)\tilde{\mathbf{T}}_{1j}^{b,1}(\lambda, \xi')\Big)\mathcal{F}'[\partial_N\mathbf{g}](\xi', y_N)\Big](x')\,dy_N, \\
&\nabla'(\lambda^{1/2}, \nabla')\partial_N\mathcal{T}_1^b(\lambda)\mathbf{g} \\
&= -\int_0^\infty \mathcal{F}_{\xi'}^{-1}\Big[\Big(\sum_{j=1}^4 \mathcal{P}_j(x_N, y_N)i\xi'(\lambda^{1/2}, i\xi')\tilde{\mathbf{T}}_{1j}^{b,1}(\lambda, \xi')\Big)\mathcal{F}'[\partial_N\mathbf{g}](\xi', y_N)\Big](x')\,dy_N, \\
&\partial_N(\lambda^{1/2}, \nabla')\partial_N\mathcal{T}_1^b(\lambda)\mathbf{g} \\
&= -\int_0^\infty \mathcal{F}_{\xi'}^{-1}\Big[\Big(\sum_{j=1}^4 \mathcal{P}_j(x_N, y_N)(\lambda^{1/2}, i\xi')\tilde{\mathbf{T}}_{1j}^{b,2}(\lambda, \xi')\Big)\mathcal{F}'[\partial_N\mathbf{g}](\xi', y_N)\Big](x')\,dy_N, \\
&\nabla'\partial_N^2\mathcal{T}_1^b(\lambda)\mathbf{g} = \int_0^\infty \mathcal{F}_{\xi'}^{-1}\Big[\Big(\sum_{j=1}^4 \mathcal{P}_j(x_N, y_N)i\xi'\tilde{\mathbf{T}}_{1j}^{b,2}(\lambda, \xi')\Big)\mathcal{F}'[\partial_N\mathbf{g}](\xi', y_N)\Big](x')\,dy_N, \\
&\partial_N\partial_N^2\mathcal{T}_1^b(\lambda)\mathbf{g} = \int_0^\infty \mathcal{F}_{\xi'}^{-1}\Big[\Big(\sum_{j=1}^4 \mathcal{P}_j(x_N, y_N)\tilde{\mathbf{T}}_{1j}^{b,3}(\lambda, \xi')\Big)\mathcal{F}'[\partial_N\mathbf{g}](\xi', y_N)\Big](x')\,dy_N.
\end{aligned}
$$

Since the following symbols:

$$
\begin{gathered}
i\xi'(\lambda, \lambda^{1/2}i\xi', (i\xi')^2)\tilde{\mathbf{T}}_{1j}^{b,0}(\lambda, \xi'), \quad (\lambda, \lambda^{1/2}i\xi', (i\xi')^2)\tilde{\mathbf{T}}_{1j}^{b,1}(\lambda, \xi'), \quad i\xi'(\lambda^{1/2}, i\xi')\tilde{\mathbf{T}}_{1j}^{b,1}(\lambda, \xi'), \\
(\lambda^{1/2}, i\xi')\tilde{\mathbf{T}}_{1j}^{b,2}(\lambda, \xi'), \quad i\xi'\tilde{\mathbf{T}}_{1j}^{b,2}(\lambda, \xi'), \quad \tilde{\mathbf{T}}_{1j}^{b,3}(\lambda, \xi')
\end{gathered}
$$

belong to $\mathbb{M}_0(\Lambda_{\epsilon,\lambda_0})$, by Proposition 3.5, we have

$$
\|\nabla(\lambda, \lambda^{1/2}\nabla, \nabla^2)\mathcal{T}_1^b(\lambda)\mathbf{g}\|_{L_q(\mathbb{R}^N_+)} \le C\|\partial_N\mathbf{g}\|_{L_q(\mathbb{R}^N_+)} \le C\|\nabla\mathbf{g}\|_{L_q(\mathbb{R}^N_+)} \tag{6.69}
$$

for any $\lambda \in \Lambda_{\epsilon,\lambda_0}$ and $\mathbf{g} \in C_0^\infty(\mathbb{R}^N_+)^N$.

We also have

$$
\begin{aligned}
&\lambda^{1/2}(\lambda, \lambda^{1/2}\nabla', \nabla'')\mathcal{T}_1^{b,0}(\lambda)\mathbf{g} \\
&= \int_0^\infty \mathcal{F}_{\xi'}^{-1}\Big[\Big(\sum_{j=1}^4 \mathcal{P}_j(x_N, y_N)\lambda^{1/2}(\lambda, \lambda^{1/2}i\xi', (i\xi')^2)\tilde{\mathbf{T}}_{1j}^{b,0}(\lambda, \xi')\Big)\mathcal{F}'[\partial_N\mathbf{g}](\xi', y_N)\Big](x')\,dy_N, \\
&\lambda^{1/2}(\lambda^{1/2}, \nabla')\partial_N\mathcal{T}_1^b(\lambda)\mathbf{g} \\
&= -\int_0^\infty \mathcal{F}_{\xi'}^{-1}\Big[\Big(\sum_{j=1}^4 \mathcal{P}_j(x_N, y_N)\lambda^{1/2}(\lambda^{1/2}, i\xi')\tilde{\mathbf{T}}_{1j}^{b,1}(\lambda, \xi')\Big)\mathcal{F}'[\partial_N\mathbf{g}](\xi', y_N)\Big](x')\,dy_N, \\
&\lambda^{1/2}\partial_N^2\mathcal{T}_1^b(\lambda)\mathbf{g} = \int_0^\infty \mathcal{F}_{\xi'}^{-1}\Big[\Big(\sum_{j=1}^4 \mathcal{P}_j(x_N, y_N)\lambda^{1/2}\tilde{\mathbf{T}}_{1j}^{b,2}(\lambda, \xi')\Big)\mathcal{F}'[\partial_N\mathbf{g}](\xi', y_N)\Big](x')\,dy_N.
\end{aligned}
$$

Since the following symbols:

$$
\lambda^{1/2}(\lambda, \lambda^{1/2}i\xi', (i\xi')^2)\tilde{\mathbf{T}}_{1j}^{b,0}(\lambda, \xi'), \quad \lambda^{1/2}(\lambda^{1/2}, i\xi')\tilde{\mathbf{T}}_{1j}^{b,1}(\lambda, \xi'), \quad \lambda^{1/2}\tilde{\mathbf{T}}_{1j}^{b,2}(\lambda, \xi')
$$

belong to $\mathbb{M}_0(\Lambda_{\epsilon,\lambda_0})$, by Proposition 3.5, we have

$$
\|(\lambda, \lambda^{1/2}\nabla, \nabla^2)\mathcal{T}_1^b(\lambda)\mathbf{g}\|_{L_q(\mathbb{R}^N_+)} \le C|\lambda|^{-1/2}\|\nabla\mathbf{g}\|_{L_q(\mathbb{R}^N_+)} \tag{6.70}
$$

for any $\lambda \in \Lambda_{\epsilon,\lambda_0}$ and $\mathbf{g} \in C_0^\infty(\mathbb{R}^N_+)^N$.

Now, we consider the dual operator $\partial_N^\ell \mathcal{T}_1^b(\lambda)^*$ of $\partial_N^\ell \mathcal{T}_1^b(\lambda)$ acting on $\mathbf{h} \in C_0^\infty(\mathbb{R}^N_+)^N$, which satisfies the equality: $|(\partial_N^\ell \mathcal{T}_1^b(\lambda)^*\mathbf{g}, \mathbf{h})| = |(\mathbf{g}, \mathcal{T}_1^b(\lambda)^*\mathbf{h})|$. In fact, from (6.65) and (6.66) by Fubini's theorem and Plancherel's theorem, we have

$$
\begin{aligned}
&(-1)^\ell(\partial_N^\ell \mathcal{T}_1^b(\lambda)\mathbf{g}, \mathbf{h}) \\
&= \int_0^\infty \int_{\mathbb{R}^{N-1}} \Big(\int_0^\infty \mathcal{F}^{-1}_{\xi'}\Big[\Big(\sum_{j=1}^4 \mathcal{P}_j(x_N, y_N)\mathbf{T}^{b,\ell}_{1j}(\lambda, \xi')\Big)\mathcal{F}'[\mathbf{g}](\xi', y_N)\Big](x')\,dy_N\Big)\mathbf{h}(x', x_N)\,dx'dx_N \\
&= \int_0^\infty \int_0^\infty \Big(\sum_{j=1}^4 \mathcal{P}_j(x_N, y_N)\mathbf{T}^{b,\ell}_{1j}(\lambda, \xi')\Big)\mathcal{F}'[\mathbf{g}](\xi', y_N)\mathcal{F}^{-1}_{\xi'}[\mathbf{h}](\xi', x_N)\,d\xi'\Big)\,dy_N dx_N \\
&= \int_0^\infty \int_{\mathbb{R}^{N-s}} \mathbf{g}(y', y_N)\Big(\int_0^\infty \mathcal{F}'\Big[\Big(\sum_{j=1}^4 \mathcal{P}_j(x_N, y_N)\mathbf{T}^{b,\ell}_{1j}(\lambda, \xi')\Big)\mathcal{F}^{-1}_{\xi'}[\mathbf{h}](\xi', x_N)\Big](y')\,dx_N\Big)\,dy'dy_N,
\end{aligned}
$$

which yields

$$
\partial_N^\ell \mathcal{T}_1^b(\lambda)^*\mathbf{h} = \int_0^\infty \mathcal{F}'\Big[\Big(\sum_{j=1}^4 \mathcal{P}_j(x_N, y_N)\mathbf{T}^{b,\ell}_{1j}(\lambda, \xi')\Big)\mathcal{F}^{-1}_{\xi'}[\mathbf{h}](\xi', x_N)\Big](y')\,dx_N. \tag{6.71}
$$

Namely, $\partial_N^\ell \mathcal{T}_1^b(\lambda)^*\mathbf{h}$ is obtained by exchanging $\mathcal{F}^{-1}_{\xi'}$ and $\mathcal{F}'$ in the representation of $\partial_N^\ell \mathcal{T}_1^b(\lambda)$. Thus, employing the completely same argument as in proving (6.67), (6.69), and (6.70), we have

$$
\begin{aligned}
&\|(\lambda, \lambda^{1/2}\nabla, \nabla^2)\mathcal{T}_1^b(\lambda)^*\mathbf{h}\|_{L_{q'}(\mathbb{R}^N_+)} \leq C\|\mathbf{h}\|_{L_{q'}(\mathbb{R}^N_+)}, \\
&\|(\lambda, \lambda^{1/2}\nabla, \nabla^2)\mathcal{T}_1^b(\lambda)^*\mathbf{h}\|_{H^1_{q'}(\mathbb{R}^N_+)} \leq C\|\mathbf{h}\|_{H^1_{q'}(\mathbb{R}^N_+)}, \\
&\|(\lambda, \lambda^{1/2}\nabla, \nabla^2)\mathcal{T}_1^b(\lambda)^*\mathbf{h}\|_{L_{q'}(\mathbb{R}^N_+)} \leq C|\lambda|^{-1/2}\|\mathbf{h}\|_{H^1_{q'}(\mathbb{R}^N_+)}.
\end{aligned} \tag{6.72}
$$

From (6.67), (6.69), (6.70), and (6.72) it follows that $\mathbf{T}_1^b(\lambda)$ satisfies Assumption 6.2, and so by Theorem 6.4 we have obtained

$$
\begin{aligned}
&\|(\lambda, \lambda^{1/2}\nabla, \nabla^2)\mathcal{T}_1^b(\lambda)\mathbf{g}\|_{B^s_{q,r}(\mathbb{R}^N_+)} \leq C\|\mathbf{g}\|_{B^s_{q,r}(\mathbb{R}^N_+)}, \\
&\|(\lambda, \lambda^{1/2}\nabla, \nabla^2)\mathcal{T}_1^b(\lambda)\mathbf{g}\|_{B^s_{q,r}(\mathbb{R}^N_+)} \leq C|\lambda|^{-\frac{\sigma}{2}}\|\mathbf{g}\|_{B^{s+\sigma}_{q,r}(\mathbb{R}^N_+)}
\end{aligned}
$$

for any $\lambda \in \Lambda_{\epsilon,\lambda_0}$ and $\mathbf{g} \in C_0^\infty(\mathbb{R}^N_+)^N$. In particular, we have obtained (6.1).

Now, we consider $\partial_\lambda \mathbf{T}_1^b(\lambda)$, which is represented by

$$
\begin{aligned}
\partial_\lambda \mathcal{T}_1^b(\lambda)\mathbf{g} &= \int_0^\infty \mathcal{F}^{-1}_{\xi'}\Big[\Big(\sum_{j=1}^4 B^2(\partial_\lambda \mathcal{P}_j(x_N, y_N))B^{-2}\mathbf{T}^{b,0}_{1j}(\lambda, \xi')\Big)\mathcal{F}'[\mathbf{g}](\xi', y_N)\Big](x')\,dy_N \\
&\quad + \int_0^\infty \mathcal{F}^{-1}_{\xi'}\Big[\Big(\sum_{j=1}^4 \mathcal{P}_j(x_N, y_N)(\partial_\lambda \mathbf{T}^{b,0}_{1j}(\lambda, \xi'))\Big)\mathcal{F}'[\mathbf{g}](\xi', y_N)\Big](x')\,dy_N
\end{aligned}
$$

as follows from (6.65). Moreover, from (6.66) and (6.68), we have

$$
\begin{aligned}
\partial_N^\ell \partial_\lambda \mathcal{T}_1^b(\lambda)\mathbf{g} &= (-1)^\ell \int_0^\infty \mathcal{F}^{-1}_{\xi'}\Big[\Big(\sum_{j=1}^4 B^2(\partial_\lambda \mathcal{P}_j(x_N, y_N))B^{-2}\mathbf{T}^{b,\ell}_{1j}(\lambda, \xi')\Big)\mathcal{F}'[\mathbf{g}](\xi', y_N)\Big](x')\,dy_N \\
&\quad + (-1)^\ell \int_0^\infty \mathcal{F}^{-1}_{\xi'}\Big[\Big(\sum_{j=1}^4 \mathcal{P}_j(x_N, y_N))(\partial_\lambda \mathbf{T}^{b,\ell}_{1j}(\lambda, \xi'))\Big)\mathcal{F}'[\mathbf{g}](\xi', y_N)\Big](x')\,dy_N;
\end{aligned}
$$

$$\partial_N^\ell \partial_\lambda \mathcal{T}_1^b(\lambda)\mathbf{g} = \int_0^\infty \mathcal{F}_{\xi'}^{-1}\Big[\Big(\sum_{j=1}^4 B^2(\partial_\lambda \mathcal{P}_j(x_N, y_N))B^{-2}\tilde{\mathbf{T}}_{1j}^{b,\ell}(\lambda,\xi')\Big)\mathcal{F}'[\partial_N \mathbf{g}](\xi', y_N)\Big](x')\,dy_N$$
$$+ \int_0^\infty \mathcal{F}_{\xi'}^{-1}\Big[\Big(\sum_{j=1}^4 \mathcal{P}_j(x_N, y_N)(\partial_\lambda \tilde{\mathbf{T}}_{1j}^{b,\ell}(\lambda,\xi'))\Big)\mathcal{F}'[\partial_N \mathbf{g}](\xi', y_N)\Big](x')\,dy_N.$$

If we write

$$\lambda\partial_N^\ell \partial_\lambda \mathcal{T}_1^b(\lambda)\mathbf{g} = (-1)^\ell \int_0^\infty \mathcal{F}_{\xi'}^{-1}\Big[\Big(\sum_{j=1}^4 B^2(\partial_\lambda \mathcal{P}_j(x_N, y_N))\lambda B^{-2}\mathbf{T}_{1j}^{b,\ell}(\lambda,\xi')\Big)\mathcal{F}'[\mathbf{g}](\xi', y_N)\Big](x')\,dy_N$$
$$+ (-1)^\ell \int_0^\infty \mathcal{F}_{\xi'}^{-1}\Big[\Big(\sum_{j=1}^4 \mathcal{P}_j(x_N, y_N))\lambda(\partial_\lambda \mathbf{T}_{1j}^{b,\ell}(\lambda,\xi'))\Big)\mathcal{F}'[\mathbf{g}](\xi', y_N)\Big](x')\,dy_N;$$
$$\lambda\partial_N^\ell \partial_\lambda \mathcal{T}_1^b(\lambda)\mathbf{g} = \int_0^\infty \mathcal{F}_{\xi'}^{-1}\Big[\Big(\sum_{j=1}^4 B^2(\partial_\lambda \mathcal{P}_j(x_N, y_N))\lambda B^{-2}\tilde{\mathbf{T}}_{1j}^{b,\ell}(\lambda,\xi')\Big)\mathcal{F}'[\partial_N \mathbf{g}](\xi', y_N)\Big](x')\,dy_N$$
$$+ \int_0^\infty \mathcal{F}_{\xi'}^{-1}\Big[\Big(\sum_{j=1}^4 \mathcal{P}_j(x_N, y_N)\lambda(\partial_\lambda \tilde{\mathbf{T}}_{1j}^{b,\ell}(\lambda,\xi'))\Big)\mathcal{F}'[\partial_N \mathbf{g}](\xi', y_N)\Big](x')\,dy_N$$

for $\ell = 0, \ldots, 3$, then using the facts that $\lambda B^{-2}\mathbf{T}_{1j}^{b,\ell}(\lambda,\xi') \in \mathbb{M}_{-2+\ell}(\Lambda_{\epsilon,\lambda_0})$, $\lambda(\partial_\lambda \mathbf{T}_{1j}^{b,\ell}(\lambda,\xi')) \in \mathbb{M}_{-2+\ell}(\Lambda_{\epsilon,\lambda_0})$, $\lambda B^{-2}\tilde{\mathbf{T}}_{1j}^{b,\ell}(\lambda,\xi') \in \mathbb{M}_{-3+\ell}(\Lambda_{\epsilon,\lambda_0})$, and $\lambda(\partial_\lambda \tilde{\mathbf{T}}_{1j}^{b,\ell}(\lambda,\xi')) \in \mathbb{M}_{-3+\ell}(\Lambda_{\epsilon,\lambda_0})$ for $\ell = 0, 1, 2, 3$ and employing the same argument as in the proof of (6.67), (6.69), (6.70), and (6.72), by Propositions 3.5 and 3.6, we have

$$\begin{aligned}\|(\lambda, \lambda^{1/2}\nabla, \nabla^2)\partial_\lambda \mathcal{T}_1^b(\lambda)\mathbf{g}\|_{L_q(\mathbb{R}^N_+)} &\le C|\lambda|^{-1}\|\mathbf{g}\|_{L_q(\mathbb{R}^N_+)},\\ \|(\lambda, \lambda^{1/2}\nabla, \nabla^2)\partial_\lambda \mathcal{T}_1^b(\lambda)\mathbf{g}\|_{H^1_q(\mathbb{R}^N_+)} &\le C|\lambda|^{-1}\|\mathbf{g}\|_{H^1_q(\mathbb{R}^N_+)}.\end{aligned} \tag{6.73}$$

Moreover, writing

$$\lambda^{1/2}\partial_N^\ell \partial_\lambda \mathcal{T}_1^b(\lambda)\mathbf{g}$$
$$= (-1)^\ell \int_0^\infty \mathcal{F}_{\xi'}^{-1}\Big[\Big(\sum_{j=1}^4 B^2(\partial_\lambda \mathcal{P}_j(x_N, y_N))\lambda^{1/2} B^{-2}\mathbf{T}_{1j}^{b,\ell}(\lambda,\xi')\Big)\mathcal{F}'[\mathbf{g}](\xi', y_N)\Big](x')\,dy_N$$
$$+ (-1)^\ell \int_0^\infty \mathcal{F}_{\xi'}^{-1}\Big[\Big(\sum_{j=1}^4 \mathcal{P}_j(x_N, y_N))\lambda^{1/2}(\partial_\lambda \mathbf{T}_{1j}^{b,\ell}(\lambda,\xi'))\Big)\mathcal{F}'[\mathbf{g}](\xi', y_N)\Big](x')\,dy_N$$

and using the facts that $\lambda^{1/2}B^{-2}\mathbf{T}_{1j}^{b,\ell}(\lambda,\xi') \in \mathbb{M}_{-3+\ell}(\Lambda_{\epsilon,\lambda_0})$ and $\lambda^{1/2}(\partial_\lambda \mathbf{T}_{1j}^{b,\ell}(\lambda,\xi')) \in \mathbb{N}_{-3+\ell}(\Lambda_{\epsilon,\lambda_0})$ for $\ell = 0, 1, 2, 3$, by Propositions 3.5 and 3.6, we have

$$\|(\lambda, \lambda^{1/2}\nabla, \nabla^2)\partial_\lambda \mathcal{T}_1^b(\lambda)\mathbf{g}\|_{H^1_q(\mathbb{R}^N_+)} \le C|\lambda|^{-1/2}\|\mathbf{g}\|_{L_q(\mathbb{R}^N_+)}, \tag{6.74}$$

for any $\lambda \in \Lambda_{\epsilon,\lambda_0}$ and $\mathbf{g} \in C_0^\infty(\mathbb{R}^N_+)^N$.

Employing the same argument as in the proof of (6.71), we see that the dual operators $\partial_N^\ell \partial_\lambda \mathcal{T}_1^b(\lambda)^*$ of $\partial_N^\ell \partial_\lambda \mathcal{T}_1^b(\lambda)$ are defined by

$$\partial_N^\ell \partial_\lambda \mathcal{T}_1^b(\lambda)^*\mathbf{h} = \int_0^\infty \mathcal{F}'\Big[\Big(\sum_{j=1}^4 B^2(\partial_\lambda \mathcal{P}_j(x_N, y_N))B^{-2}\mathbf{T}_{1j}^{b,\ell}(\lambda,\xi')\Big)\mathcal{F}_{\xi'}^{-1}[\mathbf{h}](\xi', y_N)\Big](x')\,dy_N$$

$$+\int_0^\infty \mathcal{F}'\Big[\Big(\sum_{j=1}^4 \mathcal{P}_j(x_N, y_N))(\partial_\lambda \mathbf{T}_{1j}^{b,\ell}(\lambda, \xi'))\Big)\mathcal{F}_{\xi'}^{-1}[\mathbf{h}](\xi', y_N)\Big](x')\,dy_N.$$

Employing the same argument as in the proof of (6.68), we have

$$\begin{aligned}\partial_N^\ell \partial_\lambda \mathcal{T}_1^b(\lambda)^*\mathbf{h} &= \int_0^\infty \mathcal{F}'\Big[\Big(\sum_{j=1}^4 B^2(\partial_\lambda \mathcal{P}_j(x_N, y_N))B^{-2}\tilde{\mathbf{T}}_{1j}^{b,\ell}(\lambda, \xi')\Big)\mathcal{F}_{\xi'}^{-1}[\partial_N\mathbf{h}](\xi', y_N)\Big](x')\,dy_N \\ &+ \int_0^\infty \mathcal{F}'\Big[\Big(\sum_{j=1}^4 \mathcal{P}_j(x_N, y_N)(\partial_\lambda \tilde{\mathbf{T}}_{1j}^{b,\ell}(\lambda, \xi'))\Big)\mathcal{F}_{\xi'}^{-1}[\partial_N\mathbf{h}](\xi', y_N)\Big](x')\,dy_N.\end{aligned}$$

Thus, we have

$$\begin{aligned}\|(\lambda, \lambda^{1/2}\nabla, \nabla^2)\partial_\lambda \mathcal{T}_1^b(\lambda)^*\mathbf{h}\|_{L_{q'}(\mathbb{R}^N_+)} &\leq C|\lambda|^{-1}\|\mathbf{h}\|_{L_{q'}(\mathbb{R}^N_+)}, \\ \|(\lambda, \lambda^{1/2}\nabla, \nabla^2)\partial_\lambda \mathcal{T}_1^b(\lambda)^*\mathbf{h}\|_{H^1_{q'}(\mathbb{R}^N_+)} &\leq C|\lambda|^{-1}\|\mathbf{h}\|_{H^1_{q'}(\mathbb{R}^N_+)}, \\ \|(\lambda, \lambda^{1/2}\nabla, \nabla^2)\partial_\lambda \mathcal{T}_1^b(\lambda)^*\mathbf{h}\|_{H^1_{q'}(\mathbb{R}^N_+)} &\leq C|\lambda|^{-1/2}\|\mathbf{h}\|_{L_{q'}(\mathbb{R}^N_+)}.\end{aligned} \tag{6.75}$$

From (6.73), (6.74), and (6.75) it follows that $\partial_\lambda \mathcal{T}_1^b(\lambda)$ satisfies Assumption 6.3, and so by Theorem 6.4, we have

$$\begin{aligned}\|\partial_\lambda \mathcal{T}_1^b(\lambda)\mathbf{g}\|_{B^s_{q,r}(\mathbb{R}^N_+)} &\leq C|\lambda|^{-1}\|\mathbf{g}\|_{B^s_{q,r}(\mathbb{R}^N_+)}, \\ \|\partial_\lambda \mathcal{T}_1^b(\lambda)\mathbf{g}\|_{B^s_{q,r}(\mathbb{R}^N_+)} &\leq C|\lambda|^{-(1-\frac{\sigma}{2})}\|\mathbf{g}\|_{B^{s-\sigma}_{q,r}(\mathbb{R}^N_+)}\end{aligned}$$

for any $\lambda \in \Lambda_{\epsilon,\lambda_0}$ and $\mathbf{g} \in C_0^\infty(\mathbb{R}^N_+)^N$. In particular, we have (6.2).

Now, we consider $\mathcal{T}_2^b(\lambda)$ and we shall prove (6.3) and (6.4). To this end, we introduce the class of multipliers $\mathbb{N}_k^d$ defined by

$$\begin{aligned}\mathbb{N}_k^d = \big\{m(\lambda, \xi') \in \mathbb{M}_k(\Lambda_{\epsilon,\lambda_0}) \mid & \text{ there hold} \\ & |D_{\xi'}^{\delta'} m(\lambda, \xi')| \leq C|\lambda|^{-1}(|\lambda|^{1/2} + |\xi'|)^{k-|\delta'|} \\ & |D_{\xi'}^{\delta'}(\partial_\lambda m(\lambda, \xi'))| \leq C|\lambda|^{-2}(|\lambda|^{1/2} + |\xi'|)^{k-|\delta'|} \\ & \text{for any multi-index } \delta' \in \mathbb{N}_0^{N-1},\ \lambda \in \Lambda_{\epsilon,\lambda_0} \text{ and } \xi' \in \mathbb{R}^{N-1}\big\}.\end{aligned}$$

For $m_1(\lambda, \xi') \in \mathbb{N}_k$ and $m_2(\lambda, \xi') \in \mathbb{N}_\ell^d$, we have $m_1(\lambda, \xi')m_2(\lambda, \xi') \in \mathbb{N}_{k+\ell}^d$. For $m(\lambda, \xi') \in \mathbb{N}_{-2}$, we have $q(\lambda)\lambda^{-1}m(\lambda, \xi') \in \mathbb{N}_{-2}^d$. From Lemma 6.10 and Corollary 6.11, it follows that $K_2(\lambda)\lambda^{-1} \in \mathbb{N}_{-1}^d$, $K_3(\lambda)\lambda^{-1} \in \mathbb{N}_{-1}^d$, and so $K_2(\lambda)\lambda^{-1}m(\lambda, \xi') \in \mathbb{N}_{-2}^d$ and $K_3(\lambda)\lambda^{-1}m(\lambda, \xi') \in \mathbb{N}_{-2}^d$ for $m(\lambda, \xi') \in \mathbb{N}_{-1}$. From these observations, we see that all the following symbols appearing in the definition of $\mathcal{T}_{2J}^b(\lambda)$:

$$\frac{q(\lambda)}{\lambda}\frac{i\xi_j i\xi_k}{(A+B)AB^2}, \quad \frac{q(\lambda)}{\lambda}\frac{i\xi_j}{(A+B)B^2}, \quad \frac{K_2(\lambda)}{\lambda}\frac{i\xi_j}{B^3}, \quad \frac{K_3(\lambda)}{\lambda}\frac{i\xi_j i\xi_k|\xi'|^2}{(A+B)AB^3},$$
$$\frac{K_3(\lambda)}{\lambda}\frac{i\xi_j|\xi'|^2}{(A+B)B^3}, \quad \frac{K_2(\lambda)}{\lambda}\frac{i\xi_k A}{B^3}, \quad \frac{K_3(\lambda)}{\lambda}\frac{|\xi'|^2 A}{(A+B)B^3}$$

belong to $\mathbb{N}_{-2}^d$. Thus, we may assert that there exist four $N \times N$ matrices of $\mathbb{N}_{-2}^d$ symbols $\mathbf{T}_{2j}^{b,0}$ ($j = 1, 2, 3, 4$) such that $\mathcal{T}_2^b(\lambda)$ is represented by

$$\mathcal{T}_2^b(\lambda)\mathbf{g} = \int_0^\infty \mathcal{F}_{\xi'}^{-1}\Big[\Big(\sum_{j=1}^4 \mathcal{P}_j(x_N, y_N)\mathbf{T}_{2j}^{b,0}(\lambda, \xi')\Big)\mathcal{F}'[\mathbf{g}](\xi', y_N)\Big](x')\,dy_N. \tag{6.76}$$

Employing the same arguments as in (6.66) and (6.68), we have

$$\partial_N^\ell \mathcal{T}_2^b(\lambda)\mathbf{g} = (-1)^\ell \int_0^\infty \mathcal{F}_{\xi'}^{-1}\Big[\Big(\sum_{j=1}^4 \mathcal{P}_j(x_N, y_N)\mathbf{T}_{2j}^{b,\ell}(\lambda,\xi')\Big)\mathcal{F}'[\mathbf{g}](\xi', y_N)\Big](x')\,dy_N \tag{6.77}$$

for $\ell = 1, 2$, where we have set

$$\mathbf{T}_{2k}^{b,\ell}(\lambda,\xi') = B^\ell \mathbf{T}_{1k}^b(\lambda,\xi') + \frac{A^\ell - B^\ell}{A - B} B\mathbf{T}_{2k+2}^b, \quad \mathbf{T}_{2\ell}^{b,\ell}(\lambda,\xi') = A^\ell \mathbf{T}_{2\ell}^b(\lambda,\xi')$$

for $k = 1, 2$ and $\ell = 3, 4$, and

$$\partial_N^\ell \mathcal{T}_2^b(\lambda)\mathbf{g} = \int_0^\infty \mathcal{F}_{\xi'}^{-1}\Big[\Big(\sum_{j=1}^4 \mathcal{P}_j(x_N, y_N)\tilde{\mathbf{T}}_{2j}^{b,\ell}(\lambda,\xi')\Big)\mathcal{F}'[\partial_N\mathbf{g}](\xi', y_N)\Big](x')\,dy_N. \tag{6.78}$$

Here, we have set

$$\begin{aligned}
\tilde{\mathbf{T}}_{21}^{b,\ell} &= B^{-1}\mathbf{T}_{21}^{b,\ell} - A^{-1}\mathbf{T}_{22}^{b,\ell}, \quad \tilde{\mathbf{T}}_{22}^{b,\ell} = A^{-1}\mathbf{T}_{22}^{b,2}, \\
\tilde{\mathbf{T}}_{23}^{b,\ell} &= B^{-1}\mathbf{T}_{23}^{b,\ell} + A^{-1}\mathbf{T}_{24}^{b,\ell}, \quad \tilde{\mathbf{T}}_{24}^{b,\ell} = -A^{-1}\mathbf{T}_{24}^{b,\ell}.
\end{aligned}$$

Since $\mathbf{T}_{2k}^{b,\ell}(\lambda,\xi') \in \mathbb{N}_{-2+\ell}^d$ ($\ell = 0, 1, 2, 3$), applying Proposition 3.5 to the formulas in (6.77) yields

$$\begin{aligned}
\|(\lambda, \lambda^{1/2}\nabla', \nabla'')\mathcal{T}_2^b(\lambda)\mathbf{g}\|_{L_q(\mathbb{R}_+^N)} &\leq C|\lambda|^{-1}\|\mathbf{g}\|_{L_q(\mathbb{R}_+^N)}, \\
\|(\lambda^{1/2}, \nabla')\partial_N\mathcal{T}_2^b(\lambda)\mathbf{g}\|_{L_q(\mathbb{R}_+^N)} &\leq C|\lambda|^{-1}\|\mathbf{g}\|_{L_q(\mathbb{R}_+^N)}, \\
\|\partial_N^2\mathcal{T}_2^b(\lambda)\mathbf{g}\|_{L_q(\mathbb{R}_+^N)} &\leq C|\lambda|^{-1}\|\mathbf{g}\|_{L_q(\mathbb{R}_+^N)}.
\end{aligned} \tag{6.79}$$

Since $\tilde{\mathbf{T}}_{2k}^{b,\ell}(\lambda,\xi') \in \mathbb{N}_{-3+\ell}^d$ ($\ell = 0, 1, 2, 3$), applying Proposition 3.5 to the formulas in (6.78) yields

$$\begin{aligned}
\|\nabla'(\lambda, \lambda^{1/2}\nabla', \nabla'')\mathcal{T}_2^b(\lambda)\mathbf{g}\|_{L_q(\mathbb{R}_+^N)} &\leq C|\lambda|^{-1}\|\partial_N\mathbf{g}\|_{L_q(\mathbb{R}_+^N)}, \\
\|\partial_N(\lambda, \lambda^{1/2}\nabla', \nabla'')\mathcal{T}_2^b(\lambda)\mathbf{g}\|_{L_q(\mathbb{R}_+^N)} &\leq C|\lambda|^{-1}\|\partial_N\mathbf{g}\|_{L_q(\mathbb{R}_+^N)}, \\
\|\nabla'(\lambda^{1/2}, \nabla')\partial_N\mathcal{T}_2^b(\lambda)\mathbf{g}\|_{L_q(\mathbb{R}_+^N)} &\leq C|\lambda|^{-1}\|\partial_N\mathbf{g}\|_{L_q(\mathbb{R}_+^N)}, \\
\|\partial_N(\lambda^{1/2}, \nabla')\partial_N\mathcal{T}_2^b(\lambda)\mathbf{g}\|_{L_q(\mathbb{R}_+^N)} &\leq C|\lambda|^{-1}\|\partial_N\mathbf{g}\|_{L_q(\mathbb{R}_+^N)}, \\
\|\nabla'\partial_N^2\mathcal{T}_2^b(\lambda)\mathbf{g}\|_{L_q(\mathbb{R}_+^N)} &\leq C|\lambda|^{-1}\|\partial_N\mathbf{g}\|_{L_q(\mathbb{R}_+^N)}, \\
\|\partial_N^3\mathcal{T}_2^b(\lambda)\mathbf{g}\|_{L_q(\mathbb{R}_+^N)} &\leq C|\lambda|^{-1}\|\partial_N\mathbf{g}\|_{L_q(\mathbb{R}_+^N)}.
\end{aligned} \tag{6.80}$$

Combining these estimates yields

$$\|(\lambda, \lambda^{1/2}\nabla, \nabla^2)\mathcal{T}_2^b(\lambda)\mathbf{g}\|_{L_q(\mathbb{R}_+^N)} \leq C|\lambda|^{-1}\|\mathbf{g}\|_{L_q(\mathbb{R}_+^N)}, \tag{6.81}$$

$$\|(\lambda, \lambda^{1/2}\nabla, \nabla^2)\mathcal{T}_2^b(\lambda)\mathbf{g}\|_{H_q^1(\mathbb{R}_+^N)} \leq C|\lambda|^{-1}\|\mathbf{g}\|_{H_q^1(\mathbb{R}_+^N)}. \tag{6.82}$$

When $0 < s < 1/q$, applying real interpolation to (6.81) and (6.82) yields

$$\|(\lambda, \lambda^{1/2}\nabla, \nabla^2)\mathcal{T}_2^b(\lambda)\mathbf{g}\|_{B_{q,r}^s(\mathbb{R}_+^N)} \leq C|\lambda|^{-1}\|\mathbf{g}\|_{B_{q,r}^s(\mathbb{R}_+^N)}. \tag{6.83}$$

The dual operator $(\lambda, \lambda^{1/2}\nabla, \nabla^2)\mathcal{T}_2^b(\lambda)^*$ of $(\lambda, \lambda^{1/2}\nabla, \nabla^2)\mathcal{T}_2^b(\lambda)$ is obtained by exchanging $\mathcal{F}_{\xi'}^{-1}$ and $\mathcal{F}'$ in (6.76) and (6.77). Thus, employing the same argument as in (6.81) and (6.82), we have

$$\|((\lambda, \lambda^{1/2}\nabla, \nabla^2)\mathcal{T}_2^b(\lambda)^*\mathbf{h}\|_{L_{q'}(\mathbb{R}_+^N)} \leq C|\lambda|^{-1}\|\mathbf{h}\|_{L_{q'}(\mathbb{R}_+^N)},$$

$$\|((\lambda, \lambda^{1/2}\nabla, \nabla^2)\mathcal{T}_2^b(\lambda))^*\mathbf{h}\|_{H^1_{q'}(\mathbb{R}^N_+)} \leq C|\lambda|^{-1}\|\mathbf{h}\|_{H^1_{q'}(\mathbb{R}^N_+)}$$

for any $\lambda \in \Lambda_{\epsilon,\lambda_0}$ and $\mathbf{h} \in C_0^\infty(\mathbb{R}^N_+)^N$. Thus, by duality argument, we have

$$\|(\lambda, \lambda^{1/2}\nabla, \nabla^2)\mathcal{T}_2^b(\lambda)\mathbf{g}\|_{L_q(\mathbb{R}^N_+)} \leq C|\lambda|^{-1}\|\mathbf{g}\|_{L_q(\mathbb{R}^N_+)}, \tag{6.84}$$

$$\|(\lambda, \lambda^{1/2}\nabla, \nabla^2)\mathcal{T}_2^b(\lambda)\mathbf{g}\|_{H^{-1}_q(\mathbb{R}^N_+)} \leq C|\lambda|^{-1}\|\mathbf{g}\|_{H^{-1}_q(\mathbb{R}^N_+)}. \tag{6.85}$$

Applying real interpolation (6.84) and (6.85) yields

$$\|(\lambda, \lambda^{1/2}\nabla, \nabla^2)\mathcal{T}_2^b(\lambda)\mathbf{g}\|_{B^s_{q,r}(\mathbb{R}^N_+)} \leq C|\lambda|^{-1}\|\mathbf{g}\|_{B^s_{q,r}(\mathbb{R}^N_+)}, \tag{6.86}$$

provided that $-1+1/q < s < 0$. Finally, interpolating (6.83) and (6.86) yields

$$\|(\lambda, \lambda^{1/2}\nabla, \nabla^2)\mathcal{T}_2^b(\lambda)\mathbf{g}\|_{B^0_{q,r}(\mathbb{R}^N_+)} \leq C|\lambda|^{-1}\|\mathbf{g}\|_{B^0_{q,r}(\mathbb{R}^N_+)}. \tag{6.87}$$

Thus, we have obtained (6.3).

Now, we consider $\partial_\lambda \mathbf{T}_2^b(\lambda)$, which is represented by

$$\begin{aligned}\partial_\lambda \mathcal{T}_2^b(\lambda)\mathbf{g} &= \int_0^\infty \mathcal{F}^{-1}_{\xi'}\Big[\Big(\sum_{j=1}^4 B^2(\partial_\lambda \mathcal{P}_j(x_N,y_N))B^{-2}\mathbf{T}^{b,0}_{2j}(\lambda,\xi')\Big)\mathcal{F}'[\mathbf{g}](\xi',y_N)\Big](x')\,dy_N\\ &\quad+\int_0^\infty \mathcal{F}^{-1}_{\xi'}\Big[\Big(\sum_{j=1}^4 \mathcal{P}_j(x_N,y_N)(\partial_\lambda \mathbf{T}^{b,0}_{2j}(\lambda,\xi'))\Big)\mathcal{F}'[\mathbf{g}](\xi',y_N)\Big](x')\,dy_N\end{aligned}$$

as follows from (6.76). Moreover, from (6.66) and (6.68), we have

$$\begin{aligned}\partial_N^\ell\partial_\lambda \mathcal{T}_2^b(\lambda)\mathbf{g} &= (-1)^\ell\int_0^\infty \mathcal{F}^{-1}_{\xi'}\Big[\Big(\sum_{j=1}^4 B^2(\partial_\lambda \mathcal{P}_j(x_N,y_N))B^{-2}\mathbf{T}^{b,\ell}_{2j}(\lambda,\xi')\Big)\mathcal{F}'[\mathbf{g}](\xi',y_N)\Big](x')\,dy_N\\ &\quad+(-1)^\ell\int_0^\infty \mathcal{F}^{-1}_{\xi'}\Big[\Big(\sum_{j=1}^4 \mathcal{P}_j(x_N,y_N))(\partial_\lambda \mathbf{T}^{b,\ell}_{2j}(\lambda,\xi'))\Big)\mathcal{F}'[\mathbf{g}](\xi',y_N)\Big](x')\,dy_N;\\ \partial_N^\ell\partial_\lambda \mathcal{T}_2^b(\lambda)\mathbf{g} &= \int_0^\infty \mathcal{F}^{-1}_{\xi'}\Big[\Big(\sum_{j=1}^4 B^2(\partial_\lambda \mathcal{P}_j(x_N,y_N))B^{-2}\tilde{\mathbf{T}}^{b,\ell}_{2j}(\lambda,\xi')\Big)\mathcal{F}'[\partial_N\mathbf{g}](\xi',y_N)\Big](x')\,dy_N\\ &\quad+\int_0^\infty \mathcal{F}^{-1}_{\xi'}\Big[\Big(\sum_{j=1}^4 \mathcal{P}_j(x_N,y_N)(\partial_\lambda \tilde{\mathbf{T}}^{b,\ell}_{2j}(\lambda,\xi'))\Big)\mathcal{F}'[\partial_N\mathbf{g}](\xi',y_N)\Big](x')\,dy_N.\end{aligned} \tag{6.88}$$

If we write

$$\begin{aligned}\lambda\partial_N^\ell\partial_\lambda \mathcal{T}_2^b(\lambda)\mathbf{g} &= (-1)^\ell\int_0^\infty \mathcal{F}^{-1}_{\xi'}\Big[\Big(\sum_{j=1}^4 B^2(\partial_\lambda \mathcal{P}_j(x_N,y_N))\lambda B^{-2}\mathbf{T}^{b,\ell}_{2j}(\lambda,\xi')\Big)\mathcal{F}'[\mathbf{g}](\xi',y_N)\Big](x')\,dy_N\\ &\quad+(-1)^\ell\int_0^\infty \mathcal{F}^{-1}_{\xi'}\Big[\Big(\sum_{j=1}^4 \mathcal{P}_j(x_N,y_N))\lambda(\partial_\lambda \mathbf{T}^{b,\ell}_{2j}(\lambda,\xi'))\Big)\mathcal{F}'[\mathbf{g}](\xi',y_N)\Big](x')\,dy_N;\\ \lambda\partial_N^\ell\partial_\lambda \mathcal{T}_2^b(\lambda)\mathbf{g} &= \int_0^\infty \mathcal{F}^{-1}_{\xi'}\Big[\Big(\sum_{j=1}^4 B^2(\partial_\lambda \mathcal{P}_j(x_N,y_N))\lambda B^{-2}\tilde{\mathbf{T}}^{b,\ell}_{2j}(\lambda,\xi')\Big)\mathcal{F}'[\partial_N\mathbf{g}](\xi',y_N)\Big](x')\,dy_N\\ &\quad+\int_0^\infty \mathcal{F}^{-1}_{\xi'}\Big[\Big(\sum_{j=1}^4 \mathcal{P}_j(x_N,y_N)\lambda(\partial_\lambda \tilde{\mathbf{T}}^{b,\ell}_{2j}(\lambda,\xi'))\Big)\mathcal{F}'[\partial_N\mathbf{g}](\xi',y_N)\Big](x')\,dy_N\end{aligned} \tag{6.89}$$

for $\ell = 0, \ldots, 3$, then using the facts that $\lambda B^{-2}\mathbf{T}^{b,\ell}_{2j}(\lambda, \xi') \in \mathbb{N}^d_{-2+\ell}$, $\lambda(\partial_\lambda \mathbf{T}^{b,\ell}_{2j}(\lambda, \xi')) \in \mathbb{N}^d_{-2+\ell}$, $\lambda B^{-2}\tilde{\mathbf{T}}^{b,\ell}_{2j}(\lambda, \xi') \in \mathbb{N}^d_{-3+\ell}$, and $\lambda(\partial_\lambda \tilde{\mathbf{T}}^{b,\ell}_{2j}(\lambda, \xi')) \in \mathbb{N}^d_{-3+\ell}$ for $\ell = 0, 1, 2, 3$ and employing the same argument as in the proof of (6.79) and (6.80), by Propositions 3.5 and 3.6, we have

$$\|(\lambda, \lambda^{1/2}\nabla, \nabla^2)\partial_\lambda \mathcal{T}^b_2(\lambda)\mathbf{g}\|_{L_q(\mathbb{R}^N_+)} \leq C|\lambda|^{-2}\|\mathbf{g}\|_{L_q(\mathbb{R}^N_+)}, \tag{6.90}$$

$$\|(\lambda, \lambda^{1/2}\nabla, \nabla^2)\partial_\lambda \mathcal{T}^b_2(\lambda)\mathbf{g}\|_{H^1_q(\mathbb{R}^N_+)} \leq C|\lambda|^{-2}\|\mathbf{g}\|_{H^1_q(\mathbb{R}^N_+)}. \tag{6.91}$$

If $0 < s < 1/q$, interpolating (6.90) and (6.91) yields

$$\|(\lambda, \lambda^{1/2}\nabla, \nabla^2)\partial_\lambda \mathcal{T}^b_2(\lambda)\mathbf{g}\|_{B^s_{q,r}(\mathbb{R}^N_+)} \leq C|\lambda|^{-2}\|\mathbf{g}\|_{B^s_{q,r}(\mathbb{R}^N_+)}. \tag{6.92}$$

To consider the case where $-1+1/q < s < 0$, we consider the dual operator $(\lambda, \lambda^{1/2}\nabla, \nabla^2)\partial_\lambda \mathcal{T}^b_2(\lambda)^*$ of $(\lambda, \lambda^{1/2}\nabla, \nabla^2)\partial_\lambda \mathcal{T}^b_2(\lambda)$, which is obtained by exchanging $\mathcal{F}^{-1}_{\xi'}$ and $\mathcal{F}'$ in (6.88). Then, from (6.89) we have

$$\begin{aligned}
\lambda(\partial^\ell_N \partial_\lambda \mathcal{T}^b_2(\lambda))^*\mathbf{h} &= (-1)^\ell \int_0^\infty \mathcal{F}'\Big[\Big(\sum_{j=1}^4 B^2(\partial_\lambda \mathcal{P}_j(x_N, y_N))\lambda B^{-2}\mathbf{T}^{b,\ell}_{2j}(\lambda, \xi')\Big)\mathcal{F}^{-1}_{\xi'}[\mathbf{h}](\xi', y_N)\Big](x')\,dy_N \\
&\quad + (-1)^\ell \int_0^\infty \mathcal{F}'\Big[\Big(\sum_{j=1}^4 \mathcal{P}_j(x_N, y_N))\lambda(\partial_\lambda \mathbf{T}^{b,\ell}_{2j}(\lambda, \xi'))\Big)\mathcal{F}^{-1}_{\xi'}[\mathbf{h}](\xi', y_N)\Big](x')\,dy_N; \\
\lambda(\partial^\ell_N \partial_\lambda \mathcal{T}^b_2(\lambda))^*\mathbf{h} &= \int_0^\infty \mathcal{F}'\Big[\Big(\sum_{j=1}^4 B^2(\partial_\lambda \mathcal{P}_j(x_N, y_N))\lambda B^{-2}\tilde{\mathbf{T}}^{b,\ell}_{2j}(\lambda, \xi')\Big)\mathcal{F}^{-1}_{\xi'}[\partial_N \mathbf{g}](\xi', y_N)\Big](x')\,dy_N \\
&\quad + \int_0^\infty \mathcal{F}'\Big[\Big(\sum_{j=1}^4 \mathcal{P}_j(x_N, y_N)\lambda(\partial_\lambda \tilde{\mathbf{T}}^{b,\ell}_{2j}(\lambda, \xi'))\Big)\mathcal{F}^{-1}_{\xi'}[\partial_N \mathbf{g}](\xi', y_N)\Big](x')\,dy_N.
\end{aligned}$$

Since

$$\begin{aligned}
&\lambda B^{-2}\mathbf{T}^{b,\ell}_{2j}(\lambda, \xi') \in \mathbb{N}^d_{-2+\ell}, \quad \lambda(\partial_\lambda \mathbf{T}^{b,\ell}_{2j}(\lambda, \xi')) \in \mathbb{N}^d_{-2+\ell}, \\
&\lambda B^{-2}\tilde{\mathbf{T}}^{b,\ell}_{2j}(\lambda, \xi') \in \mathbb{N}^d_{-3+\ell}, \quad \lambda(\partial_\lambda \tilde{\mathbf{T}}^{b,\ell}_{2j}(\lambda, \xi')) \in \mathbb{N}^d_{-3+\ell}
\end{aligned}$$

for $\ell = 0, 1, 2, 3$, employing the same argument as in the proof of (6.79) and (6.80), by Propositions 3.5 and 3.6, we have

$$\begin{aligned}
\|((\lambda, \lambda^{1/2}\nabla, \nabla^2)\partial_\lambda \mathcal{T}^b_2(\lambda))^*\mathbf{h}\|_{L_{q'}(\mathbb{R}^N_+)} &\leq C|\lambda|^{-2}\|\mathbf{h}\|_{L_{q'}(\mathbb{R}^N_+)}, \\
\|((\lambda, \lambda^{1/2}\nabla, \nabla^2)\partial_\lambda \mathcal{T}^b_2(\lambda))^*\mathbf{h}\|_{H^1_q(\mathbb{R}^N_+)} &\leq C|\lambda|^{-2}\|\mathbf{h}\|_{H^1_{q'}(\mathbb{R}^N_+)}.
\end{aligned}$$

By duality argument, we have

$$\begin{aligned}
\|(\lambda, \lambda^{1/2}\nabla, \nabla^2)\partial_\lambda \mathcal{T}^b_2(\lambda)\mathbf{g}\|_{L_q(\mathbb{R}^N_+)} &\leq C|\lambda|^{-2}\|\mathbf{g}\|_{L_q(\mathbb{R}^N_+)}, \\
\|(\lambda, \lambda^{1/2}\nabla, \nabla^2)\partial_\lambda \mathcal{T}^b_2(\lambda)\mathbf{g}\|_{H^{-1}_q(\mathbb{R}^N_+)} &\leq C|\lambda|^{-2}\|\mathbf{g}\|_{H^{-1}_q(\mathbb{R}^N_+)}.
\end{aligned}$$

Thus, by real interpolation, we have

$$\||(\lambda, \lambda^{1/2}\nabla, \nabla^2)\partial_\lambda \mathcal{T}^b_2(\lambda)\mathbf{g}\|_{B^s_{q,r}(\mathbb{R}^N_+)} \leq C|\lambda|^{-2}\|\mathbf{g}\|_{B^s_{q,r}(\mathbb{R}^N_+)} \tag{6.93}$$

provided that $-1 + 1/q < s < 0$. Combining (6.92) and (6.93) yields

$$\|(\lambda, \lambda^{1/2}\nabla, \nabla^2)\partial_\lambda \mathcal{T}^b_2(\lambda)\mathbf{g}\|_{B^0_{q,r}(\mathbb{R}^N_+)} \leq C|\lambda|^{-2}\|\mathbf{g}\|_{B^0_{q,r}(\mathbb{R}^N_+)} \tag{6.94}$$

Therefore, from (6.83), (6.86), (6.87), (6.92), (6.93), and (6.94), we have obtained (6.3) and (6.4). This completes the proof of Theorem 6.1.

7 Proof of Main Results

In this section, first of all we construct solution operators of equations:

$$\begin{cases} \lambda\rho + \gamma \operatorname{div} \mathbf{u} = f & \text{in } \mathbb{R}^N_+, \\ \lambda\mathbf{u} - \alpha\Delta\mathbf{u} - \beta\nabla \operatorname{div} \mathbf{u} + \gamma\nabla\rho = \mathbf{g} & \text{in } \mathbb{R}^N_+, \\ \mathbf{u} = 0 & \text{on } \partial\mathbb{R}^N_+. \end{cases} \tag{7.1}$$

First, from the first equation in (7.1), we set $\rho = \lambda^{-1}(f - \gamma \operatorname{div} \mathbf{u})$, and inserting this formula into the second equation in (7.1), we have the complex Lamé equation:

$$\lambda\mathbf{u} - \alpha\Delta\mathbf{u} - \eta_\lambda\nabla \operatorname{div} \mathbf{u} = \mathbf{g} - \gamma\lambda^{-1}\nabla f \quad \text{in } \mathbb{R}^N_+, \quad \mathbf{u}|_{\partial\mathbb{R}^N_+} = 0. \tag{7.2}$$

From Theorems 5.3 and 6.1, we have

$$\mathbf{u} = \mathcal{S}^0(\lambda)(\mathbf{g} - \gamma\lambda^{-1}\nabla f) - \mathcal{S}^b(\lambda)(\mathbf{g} - \gamma\lambda^{-1}\nabla f).$$

Thus, defining ρ by

$$\rho = \lambda^{-1}(f - \gamma \operatorname{div} \mathbf{u}) = \lambda^{-1} f - \gamma\lambda^{-1} \operatorname{div}(\mathcal{S}^0(\lambda)(\mathbf{g} - \gamma\lambda^{-1}\nabla f) - \mathcal{S}^b(\lambda)(\mathbf{g} - \gamma\lambda^{-1}\nabla f)),$$

we see that $\mathbf{u}$ and ρ are solutions of equations (7.1). In view of Theorems 5.3 and 6.1, we decompose $\mathbf{u}$ as

$$\mathbf{u} = \mathcal{T}^0_1(\lambda)\mathbf{g} - \mathcal{T}^b_1(\lambda)\mathbf{g} + \mathcal{T}^0_2(\lambda)\mathbf{g} - \mathcal{T}^b_2(\lambda)\mathbf{g} - \gamma\lambda^{-1}\mathcal{S}^0(\lambda)\nabla f + \gamma\lambda^{-1}\mathcal{S}^b(\lambda)\nabla f.$$

Summing up, there exist solution operators $\mathcal{S}(\lambda)$, $\mathcal{S}^1(\lambda)$, $\mathcal{S}^2(\lambda)$ such that $\mathbf{u} = \mathcal{S}(\lambda)(f, \mathbf{g})$, $\rho = \mathcal{R}(\lambda)(f, \nabla\mathbf{g})$, and

$$\begin{aligned} \mathcal{S}^1(\lambda)\mathbf{g} &= \mathcal{T}^0_1(\lambda)\mathbf{g} - \mathcal{T}^b_1(\lambda)\mathbf{g}, \\ \mathcal{S}^2(\lambda)(f, \mathbf{g}) &= \mathcal{T}^0_2(\lambda)\mathbf{g} - \mathcal{T}^b_2(\lambda)\mathbf{g} - \gamma\lambda^{-1}\mathcal{S}^0(\lambda)\nabla f + \gamma\lambda^{-1}\mathcal{S}^b(\lambda)\nabla f, \\ \mathcal{S}(\lambda)(f, \mathbf{g}) &= \mathcal{S}^1(\lambda)\mathbf{g} + \mathcal{S}^2(\lambda)(f, \mathbf{g}), \\ \mathcal{R}(\lambda)(f, \mathbf{g}) &= \lambda^{-1} f - \gamma\lambda^{-1} \operatorname{div} \mathcal{S}^0(\lambda)\mathbf{g} + \gamma^2\lambda^{-2} \operatorname{div} \mathcal{S}^0(\lambda)\nabla f \\ &\quad - \gamma\lambda^{-1} \operatorname{div} \mathcal{S}^b(\lambda)\mathbf{g} + \gamma^2\lambda^{-2} \operatorname{div} \mathcal{S}^b(\lambda)\nabla f. \end{aligned} \tag{7.3}$$

We see easily that

$$\begin{aligned} \mathcal{S}(\lambda) &\in \operatorname{Hol}(\Lambda_{\epsilon,\lambda_0}, \mathcal{L}(B^{s+1}_{q,r}(\mathbb{R}^N_+) \times B^s_{q,r}(\mathbb{R}^N_+), B^{s+2}_{q,r}(\mathbb{R}^N_+))), \\ \mathcal{S}^1(\lambda) &\in \operatorname{Hol}(\Lambda_{\epsilon,\lambda_0}, \mathcal{L}(B^s_{q,r}(\mathbb{R}^N_+), B^{s+2}_{q,r}(\mathbb{R}^N_+))), \\ \mathcal{S}^2(\lambda) &\in \operatorname{Hol}(\Lambda_{\epsilon,\lambda_0}, \mathcal{L}(B^{s+1}_{q,r}(\mathbb{R}^N_+) \times B^s_{q,r}(\mathbb{R}^N_+), B^{s+2}_{q,r}(\mathbb{R}^N_+))), \\ \mathcal{R}(\lambda) &\in \operatorname{Hol}(\Lambda_{\epsilon,\lambda_0}, \mathcal{L}(B^{s+1}_{q,r}(\mathbb{R}^N_+) \times B^s_{q,r}(\mathbb{R}^N_+), B^{s+1}_{q,r}(\mathbb{R}^N_+))). \end{aligned}$$

Moreover, by Theorems 5.3 and 6.1, we see the following theorem.

Theorem 7.1. *Let $1 < q < \infty$, $1 \le r < \infty$, $-1 + 1/q < s < 1/q$, and $\epsilon \in (0, \pi/2)$. Then, there exists a large number $\lambda_0 > 0$ such that for any $\lambda \in \Lambda_{\epsilon,\lambda_0}$, $f \in B^{s+1}_{q,r}(\mathbb{R}^N_+)$, and $\mathbf{g} \in C^\infty_0(\mathbb{R}^N_+)^N$, there hold*

$$\begin{aligned} \|(\lambda, \lambda^{1/2}\nabla, \nabla^2)\mathcal{S}(\lambda)(f, \mathbf{g})\|_{B^s_{q,r}(\mathbb{R}^N_+)} &\le C\|(f, \mathbf{g})\|_{\mathcal{H}^s_{q,r}(\mathbb{R}^N_+)}, \\ \|(\lambda^{1/2}\nabla, \nabla^2)\mathcal{S}^1(\lambda)\mathbf{g}\|_{B^s_{q,r}(\mathbb{R}^N_+)} &\le C|\lambda|^{-\frac{\sigma}{2}}\|\mathbf{g}\|_{B^{s+\sigma}_{q,r}(\mathbb{R}^N_+)}, \end{aligned}$$

$$
\begin{aligned}
\|(\lambda^{1/2}\nabla, \nabla^2)\partial_\lambda \mathcal{S}^1(\lambda)\mathbf{g}\|_{B^s_{q,r}(\mathbb{R}^N_+)} &\le C|\lambda|^{-(1-\frac{\sigma}{2})}\|\mathbf{g}\|_{B^{s-\sigma}_{q,r}(\mathbb{R}^N_+)},\\
\|(\lambda^{1/2}\nabla, \nabla^2)\mathcal{S}^2(\lambda)(f,\mathbf{g})\|_{B^s_{q,r}(\mathbb{R}^N_+)} &\le C|\lambda|^{-1}\|(f,\mathbf{g})\|_{\mathcal{H}^s_{q,r}(\mathbb{R}^N_+)},\\
\|(\lambda^{1/2}\nabla, \nabla^2)\partial_\lambda \mathcal{S}^2(\lambda)(f,\mathbf{g})\|_{B^s_{q,r}(\mathbb{R}^N_+)} &\le C|\lambda|^{-2}\|(f,\mathbf{g})\|_{\mathcal{H}^s_{q,r}(\mathbb{R}^N_+)},\\
\|\mathcal{R}(\lambda)f\|_{B^{s+1}_{q,r}(\mathbb{R}^N_+)} &\le C|\lambda|^{-1}\|(f,\mathbf{g})\|_{\mathcal{H}^s_{q,r}(\mathbb{R}^N_+)},\\
\|\partial_\lambda\mathcal{R}(\lambda)f\|_{B^{s+1}_{q,r}(\mathbb{R}^N_+)} &\le C|\lambda|^{-2}\|(f,\mathbf{g})\|_{\mathcal{H}^s_{q,r}(\mathbb{R}^N_+)}.
\end{aligned}
$$

Theorem 1.2 follows from Theorem 7.1 immediately.

Now, we consider an initial value problem:

$$
\begin{cases}
\partial_t \Pi + \gamma \operatorname{div}\mathbf{U} = 0 & \text{in } \mathbb{R}^N_+ \times \mathbb{R}_+,\\
\partial_t \mathbf{U} - \alpha\Delta\mathbf{U} - \beta\nabla\operatorname{div}\mathbf{U} + \gamma\nabla\Pi = 0 & \text{in } \mathbb{R}^N_+ \times \mathbb{R}_+,\\
\mathbf{U} = 0 & \text{on } \partial\mathbb{R}^N_+ \times \mathbb{R}_+,\\
(\Pi, \mathbf{U})|_{t=0} = (\Pi_0, \mathbf{U}_0) & \text{in } \mathbb{R}^N_+.
\end{cases}
\tag{7.4}
$$

To formulate problem (7.4) in the semigroup setting, we introduce spaces $\mathcal{H}^s_{q,r}(\mathbb{R}^N_+)$, $\mathcal{D}^s_{q,r}(\mathbb{R}^N_+)$ and an operator $\mathcal{A}_{q,r}$ defined in (1.4) and (1.5), respectively. Then, as was seen in (1.6), equations (7.4) are written as

$$
\partial_t(\Pi, \mathbf{U}) + \mathcal{A}^s_{q,r}(\Pi, \mathbf{U}) = (0,0) \quad \text{for } t > 0, \quad (\Pi, \mathbf{U})|_{t=0} = (\Pi_0, \mathbf{U}_0) \in \mathcal{H}^s_{q,r}.
$$

And, the corresponding resolvent problem (7.1) is written as

$$
\lambda(\rho, \mathbf{u}) + \mathcal{A}^s_{q,r}(\rho, \mathbf{u}) = (f, \mathbf{g})
$$

for $(f, \mathbf{g}) \in \mathcal{H}^s_{q,r}(\mathbb{R}^N_+)$ and $(\rho, \mathbf{u}) \in \mathcal{D}^s_{q,r}(\mathbb{R}^N_+)$. From Theorem 7.1 it follows that the resolvent operator $(\lambda + \mathcal{A}^s_{q,r})^{-1}$ exists for any $\lambda \in \Lambda_{\epsilon,\lambda_0}$ for sufficient large $\lambda_0 > 0$. In fact, $(\lambda + \mathcal{A}^s_{q,r})^{-1}(f, \mathbf{g}) = (\mathcal{R}(\lambda), \mathcal{S}(\lambda))(f, \mathbf{g})$ for $(f, \mathbf{g}) \in \mathcal{H}^s_{q,r}(\mathbb{R}^N_+)$. Thus, the resolvent estimate: $\|\lambda(\lambda + \mathcal{A}^s_{q,r})^{-1}\|_{\mathcal{L}(\mathcal{H}^s_{q,r})} \le C$ holds for any $\lambda \in \Lambda_{\epsilon,\lambda_0}$.

From these observations, by theory of C_0 analytic semigroup ([37]), there exists a C_0 analytic semigroup $\{T(t)\}_{t\ge0}$ associated with (7.4) and $(\Pi, \mathbf{U}) = T(t)(\Pi_0, \mathbf{U}_0)$ is a unique solution of (7.4), which satisfies the regularity condition:

$$
(\Pi, \mathbf{U}) \in C^0([0,\infty), \mathcal{H}^s_{q,r}(\mathbb{R}^N_+)) \cap C^0((0,\infty), \mathcal{D}^s_{q,r}(\mathbb{R}^N_+)) \cap C^1((0,\infty), \mathcal{H}^s_{q,r}(\mathbb{R}^N_+))
$$

as well as

$$
\lim_{t\to0} \|(\Pi(\cdot,t), \mathbf{U}(\cdot,t)) - (\Pi_0, \mathbf{U}_0)\|_{\mathcal{H}^s_{q,r}(\mathbb{R}^N_+)} = (0,0).
$$

Finally, we shall show the following theorem about the maximal L_1 regularity of $\{T(t)\}_{t\ge0}$. Obviously, combining the results about continuous analytic semigroup theory mentioned above and the following theorem completes the proof of Theorem 1.1.

Theorem 7.2. *Let $1 < q < \infty$ and $-1 + 1/q < s < 1/q$. Then, there exists $\omega > 0$ such that for any $(\Pi_0, \mathbf{U}_0) \in \mathcal{H}^s_{q,1}(\mathbb{R}^N_+)$, there holds*

$$
\int_0^\infty e^{-\omega t}(\|\partial_t T(t)(\Pi_0, \mathbf{U}_0)\|_{\mathcal{H}^s_{q,1}(\mathbb{R}^N_+)} + \|T(t)(\Pi_0, \mathbf{U}_0)\|_{\mathcal{D}^s_{q,1}(\mathbb{R}^N_+)})\,dt \le C\|(\Pi_0, \mathbf{U}_0)\|_{\mathcal{H}^s_{q,1}(\mathbb{R}^N_+)}.
$$

In the sequel, we shall prove Theorem 7.2. We start with the following lemma.

Proposition 7.3. *Let X_0 and X_1 be Banach spaces which are an interpolation couple, and Y be another Banach space. Assume that $0 < \sigma_0, \sigma_1, \theta < 1$ satisfy $1 = (1-\theta)(1-\sigma_0) + \theta(1+\sigma_1)$. Let $\omega \geq 0$. For $t > 0$ let $T(t)\colon Y \to X_0 + X_1$ be a bounded linear operator such that*

$$\begin{aligned} \|T(t)f\|_Y &\leq Ce^{\omega t} t^{-1+\sigma_0} \|f\|_{X_0}, \qquad f \in X_0, \\ \|T(t)f\|_Y &\leq Ce^{\omega t} t^{-1-\sigma_1} \|f\|_{X_1}, \qquad f \in X_1. \end{aligned} \tag{7.5}$$

Then, there holds

$$\int_0^\infty e^{-\omega t} \|T(t)f\|_Y \, \mathrm{d}t \leq C \|f\|_{(X_0, X_1)_{\theta,1}}$$

with a constant $C > 0$ independent of ω.

Proof. The proof is based on real interpolation. For $k \in \mathbb{Z}$ set

$$b_k(f) := \sup_{t \in [2^k, 2^{k+1}]} e^{-\omega t} \|T(t)f\|_Y.$$

We observe that

$$\int_0^\infty e^{-\omega t} \|T(t)f\|_Y \, \mathrm{d}t = \sum_{k \in \mathbb{Z}} \int_{2^k}^{2^{k+1}} e^{-\omega t} \|T(t)f\|_Y \, \mathrm{d}t \leq \sum_{k \in \mathbb{Z}} 2^k b_k(f). \tag{7.6}$$

Then we infer from the assumptions (7.5) that

$$\begin{aligned} b_k(f) &\leq C \sup_{t \in [2^k, 2^{k+1}]} t^{-1+\sigma_0} \|f\|_{X_0} \leq C 2^{-k(1-\sigma_0)} \|f\|_{X_0}, \qquad f \in X_0, \\ b_k(f) &\leq C \sup_{t \in [2^k, 2^{k+1}]} t^{-1-\sigma_1} \|f\|_{X_1} \leq C 2^{-k(1+\sigma_1)} \|f\|_{X_1}, \qquad f \in X_1. \end{aligned}$$

Namely, there hold

$$\begin{aligned} \|(b_k)_{k \in \mathbb{Z}}\|_{\ell_\infty^{1-\sigma_0}(\mathbb{Z})} &\leq C \|f\|_{X_0}, \qquad f \in X_0, \\ \|(b_k)_{k \in \mathbb{Z}}\|_{\ell_\infty^{1+\sigma_1}(\mathbb{Z})} &\leq C \|f\|_{X_1}, \qquad f \in X_1. \end{aligned}$$

Since $(\ell_\infty^{1-\sigma_0}(\mathbb{Z}), \ell_\infty^{1+\sigma_1}(\mathbb{Z}))_{\theta,1} = \ell_1^1(\mathbb{Z})$ due to [2, Thm. 5.6.1], it follows that

$$\sum_{k \in \mathbb{Z}} 2^k b_k(f) = \|b_k(f)_{k \in \mathbb{Z}}\|_{\ell_1^1(\mathbb{Z})} \leq C \|f\|_{(X_0, X_1)_{\theta,1}}. \tag{7.7}$$

Thus, the desired estimates follows from (7.6) and (7.7). □

A Proof of Theorem 7.2. Let $\omega > 0$ be a large number such that $\Sigma_\epsilon + \omega \subset \Lambda_{\epsilon, \lambda_0}$. Let Γ be a contour in $\mathbb{C}$ defined by $\Gamma = \Gamma_+ \cup \Gamma_-$ with

$$\Gamma_\pm = \{\lambda = re^{\pm(\pi - \epsilon)} \mid r \in (0, \infty)\}.$$

As was well-known in theory of C_0 analytic semigroup (cf. [37]), we have

$$T(t)(\Pi_0, \mathbf{U}_0) = \frac{1}{2\pi i} \int_{\Gamma + \omega} (\mathcal{S}(\lambda), \mathcal{R}(\lambda))(\Pi_0, \mathbf{U}_0) \, d\lambda \quad \text{for } t > 0.$$

To show the L_1 integrability of $T(t)$, we use Theorem 7.1. According to the formulas in (7.3), we divide $T(t)$ into the following three parts:

$$T_1(t)\mathbf{U}_0 = \frac{1}{2\pi i} \int_{\Gamma + \omega} \mathcal{S}^1(\lambda) \mathbf{U}_0 \, \mathrm{d}\lambda, \tag{7.8}$$

$$T_2(t)(\Pi_0, \mathbf{U}_0) = \frac{1}{2\pi i} \int_{\Gamma+\omega} \mathcal{S}^2(\lambda)(\Pi_0, \mathbf{U}_0)\, \mathrm{d}\lambda, \tag{7.9}$$

$$T_3(t)(\Pi_0, \mathbf{U}_0) = \frac{1}{2\pi i} \int_{\Gamma+\omega} \mathcal{R}(\lambda)(\Pi_0, \mathbf{U}_0)\, \mathrm{d}\lambda. \tag{7.10}$$

We have $T(t)(\Pi_0, \mathbf{U}_0) = (T_3(t)(\Pi_0, \mathbf{U}_0), T_1(t)\mathbf{U}_0 + T_2(t)(\Pi_0, \mathbf{U}_0))$.

We first show that

$$\int_0^\infty e^{-\omega t} \|T_1(t)\mathbf{U}_0\|_{B^{s+2}_{q,1}(\mathbb{R}^N_+)}\, \mathrm{d}t \le C\|\mathbf{U}_0\|_{B^s_{q,1}(\mathbb{R}^N_+)}. \tag{7.11}$$

To this end, in view of Proposition 7.3, we first prove that for every $t > 0$ and $\mathbf{U}_0 \in C^\infty_0(\mathbb{R}^N_+)^N$ there hold

$$\|T_1(t)\mathbf{U}_0\|_{B^{s+2}_{q,1}(\mathbb{R}^N_+)} \le Ce^{\omega t} t^{-1+\frac{\sigma}{2}} \|\mathbf{U}_0\|_{B^{s+\sigma}_{q,1}(\mathbb{R}^N_+)}, \tag{7.12}$$

$$\|T_1(t)\mathbf{U}_0\|_{B^{s+2}_{q,1}(\mathbb{R}^N_+)} \le Ce^{\omega t} t^{-1-\frac{\sigma}{2}} \|\mathbf{U}_0\|_{B^{s-\sigma}_{q,1}(\mathbb{R}^N_+)}. \tag{7.13}$$

Notice that $\lambda = \omega + re^{\pm i(\pi-\epsilon)}$ for $\lambda \in \Gamma_\pm + \omega$, and thus $|e^{\lambda t}| = e^{\omega t} e^{\cos(\pi-\epsilon) rt} = e^{\omega t} e^{-rt\cos\epsilon}$ for $\lambda \in \Gamma_\pm + \omega$. Since $\|\mathcal{S}^1(\lambda)\mathbf{U}\|_{B^{s+2}_{q,1}(\mathbb{R}^N_+)} \le C|\lambda|^{-\frac{\sigma}{2}} \|\mathbf{U}\|_{B^{s+\sigma}_{q,1}(\mathbb{R}^N_+)}$ as follows from Theorem 7.1, using (7.8), for $t > 0$ we have

$$\begin{aligned}
\|T_1(t)\mathbf{U}_0\|_{B^{s+2}_{q,1}(\mathbb{R}^N_+)} &\le Ce^{\omega t} \int_0^\infty e^{-rt\cos\epsilon} r^{-\sigma/2}\, \mathrm{d}r \|\mathbf{U}_0\|_{B^{s+\sigma}_{q,1}(\mathbb{R}^N_+)} \\
&= Ce^{\omega t} t^{-1+\frac{\sigma}{2}} \int_0^\infty e^{-\ell\cos\epsilon} \ell^{-\sigma/2}\, \mathrm{d}\ell\, \|\mathbf{U}_0\|_{B^{s+\sigma}_{q,1}(\mathbb{R}^N_+)},
\end{aligned}$$

which yields (7.12). To prove (7.13), by integration by parts we write

$$T_1(t)\mathbf{U}_0 = -\frac{1}{2\pi i t} \int_{\Gamma+\omega} e^{\lambda t} \partial_\lambda \mathcal{S}^1(\lambda)\mathbf{U}_0\, \mathrm{d}\lambda.$$

Since $\|\partial_\lambda \mathcal{S}^1(\lambda)\mathbf{U}_0\|_{B^{s+2}_{q,1}(\mathbb{R}^N_+)} \le C|\lambda|^{-(1-\frac{\sigma}{2})} \|\mathbf{U}_0\|_{B^{s-\sigma}_{q,1}(\mathbb{R}^N_+)}$ as follows from Theorem 7.1, we have

$$\begin{aligned}
\|T_1(t)\mathbf{U}_0\|_{B^{s+2}_{q,1}(\mathbb{R}^N_+)} &\le Ct^{-1} e^{\omega t} \int_0^\infty e^{-rt\cos\epsilon} r^{-(1-\frac{\sigma}{2})}\, \mathrm{d}r\, \|\mathbf{U}_0\|_{B^{s-\sigma}_{q,1}(\mathbb{R}^N_+)} \\
&= Ce^{\omega t} t^{-1-\frac{\sigma}{2}} \int_0^\infty e^{-\ell\cos\epsilon} \ell^{-1+\frac{\sigma}{2}}\, \mathrm{d}\ell\, \|\mathbf{U}_0\|_{B^{s-\sigma}_{q,1}(\mathbb{R}^N_+)},
\end{aligned}$$

which yields (7.13). Choosing $\theta = 1/2$ in Proposition 7.3 and using the fact that

$$(B^{s+\sigma}_{q,1}(\mathbb{R}^N_+), B^{s-\sigma}_{q,1}(\mathbb{R}^N_+))_{1/2,1} = B^s_{q,1}(\mathbb{R}^N_+),$$

by Proposition 7.3, we have (7.11) for $\mathbf{U}_0 \in C^\infty_0(\mathbb{R}^N_+)^N$. But, since $C^\infty_0(\mathbb{R}^N_+)^N$ is dense in $B^s_{q,r}(\mathbb{R}^N_+)^N$, the estimate (7.11) holds for any $\mathbf{U}_0 \in B^s_{q,1}(\mathbb{R}^N_+)^N$.

We now show that

$$\begin{aligned}
&\int_0^\infty e^{-\omega t} \|T_2(t)(\Pi_0, \mathbf{U}_0)\|_{B^{s+2}_{q,1}}\, \mathrm{d}t \le C\|(\Pi_0, \mathbf{U}_0)\|_{\mathcal{H}^s_{q,1}(\mathbb{R}^N_+)}, \\
&\int_0^\infty e^{-\omega t} \|T_3(t)(\Pi_0, \mathbf{U}_0)\|_{B^{s+1}_{q,1}}\, \mathrm{d}t \le C\|(\Pi_0, \mathbf{U}_0)\|_{\mathcal{H}^s_{q,1}(\mathbb{R}^N_+)}.
\end{aligned} \tag{7.14}$$

In fact, using Theorem 7.1 and $|\lambda| \ge \lambda_0$, we have

$$\|(\lambda^{1/2}\nabla, \nabla^2)\mathcal{S}^2(\lambda)(f, \mathbf{g})\|_{B^s_{q,r}(\mathbb{R}^N_+)} \le C|\lambda|^{-1} \|(f, \mathbf{g})\|_{\mathcal{H}^s_{q,r}(\mathbb{R}^N_+)}$$

$$
\begin{aligned}
&\leq C\lambda_0^{-(1-\frac{\sigma}{2})}|\lambda|^{-\frac{\sigma}{2}}\|(f,\mathbf{g})\|_{\mathcal{H}^s_{q,r}(\mathbb{R}^N_+)}, \\
\|(\lambda^{1/2}\nabla,\nabla^2)\partial_\lambda\mathcal{S}^2(\lambda)(f,\mathbf{g})\|_{B^s_{q,r}(\mathbb{R}^N_+)} &\leq C|\lambda|^{-2}\|(f,\mathbf{g})\|_{\mathcal{H}^s_{q,r}(\mathbb{R}^N_+)} \\
&\leq C\lambda_0^{-(1+\frac{\sigma}{2})}|\lambda|^{-(1-\frac{\sigma}{2})}\|(f,\mathbf{g})\|_{\mathcal{H}^s_{q,r}(\mathbb{R}^N_+)}, \\
\|\mathcal{R}(\lambda)f\|_{B^{s+1}_{q,r}(\mathbb{R}^N_+)} &\leq C|\lambda|^{-1}\|(f,\mathbf{g})\|_{\mathcal{H}^s_{q,r}(\mathbb{R}^N_+)} \\
&\leq C\lambda_0^{-(1-\frac{\sigma}{2})}|\lambda|^{-\frac{\sigma}{2}}\|(f,\mathbf{g})\|_{\mathcal{H}^s_{q,r}(\mathbb{R}^N_+)}, \\
\|\partial_\lambda\mathcal{R}(\lambda)f\|_{B^{s+1}_{q,r}(\mathbb{R}^N_+)} &\leq C|\lambda|^{-2}\|(f,\mathbf{g})\|_{\mathcal{H}^s_{q,r}(\mathbb{R}^N_+)} \\
&\leq C\lambda_0^{-(1+\frac{\sigma}{2})}|\lambda|^{-(1-\frac{\sigma}{2})}\|(f,\mathbf{g})\|_{\mathcal{H}^s_{q,r}(\mathbb{R}^N_+)}
\end{aligned}
$$

for any $\lambda \in \Sigma_\epsilon + \omega$ and $(f,\mathbf{g}) \in \mathcal{H}^s_{q,1}$. In view of (7.9) and (7.10), employing the same argument as in the proof of (7.12) and (7.13), we have

$$
\begin{aligned}
\|T_2(t)(\Pi_0,\mathbf{U}_0)\|_{B^{s+2}_{q,1}(\mathbb{R}^N_+)} &\leq C\lambda_0^{-\frac{\sigma}{2}}e^{\omega t}t^{-1+\frac{\sigma}{2}}\|(\Pi_0,\mathbf{U}_0)\|_{\mathcal{H}^s_{q,1}}, \\
\|T_2(t)(\Pi_0,\mathbf{U}_0)\|_{B^{s+2}_{q,1}(\mathbb{R}^N_+)} &\leq C\lambda_0^{-(1+\frac{\sigma}{2})}e^{\omega t}t^{-1-\frac{\sigma}{2}}\|(\Pi_0,\mathbf{U}_0)\|_{\mathcal{H}^s_{q,1}}, \\
\|T_3(t)(\Pi_0,\mathbf{U}_0)\|_{B^{s+1}_{q,1}(\mathbb{R}^N_+)} &\leq C\lambda_0^{-\frac{\sigma}{2}}e^{\omega t}t^{-1+\frac{\sigma}{2}}\|(\Pi_0,\mathbf{U}_0)\|_{\mathcal{H}^s_{q,1}}, \\
\|T_3(t)(\Pi_0,\mathbf{U}_0)\|_{B^{s+1}_{q,1}(\mathbb{R}^N_+)} &\leq C\lambda_0^{-(1+\frac{\sigma}{2})}e^{\omega t}t^{-1-\frac{\sigma}{2}}\|(\Pi_0,\mathbf{U}_0)\|_{\mathcal{H}^s_{q,1}}.
\end{aligned}
$$

Thus, using Proposition 7.3 and noting that $(\mathcal{H}^s_{q,1},\mathcal{H}^s_{q,1})_{1/2,1} = \mathcal{H}^s_{q,1} = B^{s+1}_{q,1}(\mathbb{R}^N_+) \times B^s_{q,1}(\mathbb{R}^N_+)^N$, we have (7.14). This completes the proof of Theorem 7.2. □

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