A refinement of bounded cohomology

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Dedicated to Jacek Świątkowski on the occasion of his 60th birthday

Abstract. We introduce a refinement of bounded cohomology and prove that the suitable comparison homomorphisms vanish for an amenable group. We investigate in this context Thompson's group F and provide further evidence towards its amenability. We show that the space of 1-bounded cocycles of degree two is essentially as big as the space of Lipschitz functions on the underlying group. We also explain that such classes define metrics on the group.

1. Introduction

Although the cohomology of a space is a homotopy invariant, it sometimes captures geometric information. For example, existence of a Kähler metric on a $2n$-dimensional manifold M implies that the dimension of the odd degree cohomology of M is even [13]. Another impressive example is simplicial volume, which is a cohomological counterpart of the minimal volume of a Riemannian manifold. Here, however, one has to put more structure on the space of cochains. Namely, consider those that are bounded functions on the spaces of singular chains. This has been done by Gromov in [14] where he introduced the *bounded cohomology*. Since then it became a standard tool in various branches of mathematics like group theory, differential geometry, dynamical systems and others [21]. Gromov observed that the bounded cohomology of a space depends only on its fundamental group which, in principle, reduced the theory to group cohomology.

It is well known that the bounded cohomology of an amenable group is trivial [14, Section 3]. If one looks closely at the proof of this fact [9, 19], it is clear that the standard averaging argument yields more. Not only bounded cocycles become coboundaries but also cocycles that are bounded after fixing a number of variables do so. This observation (presented in the proof of Theorem 2.1 below) led us to defining semibounded cohomology which is the main subject of this paper.

We investigate the properties of semibounded cohomology in relation to other geometric properties of spaces or groups. For example, we show that a cohomology class of a group is hyperbolic if and only if it admits a 1-bounded representative; see Section 5 for definitions and more details.

It is well known that the kernel of the comparison map $H^2_b(\Gamma;\mathbf{R}) \to H^2(\Gamma;\mathbf{R})$ consists of quasimorphisms and is often an infinite dimensional space. This holds, for example, for non-elementary hyperbolic groups, mapping class group which are not virtually abelian and many others [2, 7]. Quasimorphisms on a group Γ form a function theory tightly related to the geometry of the stable commutator length on the commutator subgroup $[\Gamma, \Gamma]$; see [6] for more details. In this spirit we show in Section 4 that the kernel of the comparison map between 1-bounded degree 2 cohomology and the standard cohomology consist of functions which we call abstractly Lipschitz. These are functions for which there exists a length on Γ with respect to which the function is Lipschitz.

A similar concept to 1-bounded cocycles was defined by Neumann and Reeves [23] for 2-cocycles and by Frigerio and Sisto [10] for any degree under the name *weakly bounded* cocycles. Both papers consider cocycles that take finite set of values after fixing all but the first argument.

Weakly bounded 2-cocycles were used to characterise central extensions of finitely generated groups that are quasi-isometric with trivial extensions. It was a folklore that this holds if the defining cocycle for an extension is bounded. Necessary and sufficient condition in terms of ℓ^∞-cohomology (see [20]) was given by Kleiner and Leeb [18, Proposition 8.3]. The characterisation in terms of weakly bounded cocycles is due to Frigerio and Sisto [10, Corollary 2.5]. The connection between weakly bounded cocycles and ℓ^∞-cohomology is a direct analogy to our Theorem 3.3.

It is a longstanding open problem (popularised by R. Geoghegan around 1979) to determine whether Thopson's group F is amenable [1]. It is a recent result of Monod that the bounded cohomology of F vanishes [22] and in Section 7 we apply our refined theory to test the amenability of F. Unfortunately, we prove that no class in the standard cohomology of F can be represented by a 1-bounded cocycle. This shows that the comparison homomorphisms between semibounded and the standard cohomology are trivial and, as in the case of Monod's vanishing result, provides inconclusive answer as to whether F is amenable. The remaining line of attack within our framework is to investigate the comparison homomorphisms between various semibounded cohomology of F. However, they seem to be very hard to compute.

Throughout the paper when we consider group cohomology we use nonhomogeneous notation for chains [4] with exceptions in Section 5.

2. Semibounded cohomology

Let Γ be a discrete group. An n-cochain $\omega\colon \Gamma^n \to \mathbf{R}$ is called ***p*-bounded** if for fixed $g_{p+1}, \dots, g_n \in \Gamma$ the function

$$\Gamma^p \ni g_1, \dots, g_p \mapsto \omega(g_1, \dots, g_p, g_{p+1}, \dots, g_n)$$

is bounded. For example, an n-bounded cochain is bounded in the usual sense. It is straightforward to check that $C^*_{(p)}(\Gamma; \mathbf{R}) \subseteq C^*(\Gamma; \mathbf{R})$ consisting of p-bounded

[1] A short historical account is given here: `https://mathoverflow.net/questions/55214/does-the-amenability-problem-for-thompsons-group-f-predate-1980`

cochains is preserved by the codifferential and hence it is a subcomplex. Indeed, if $\alpha \in C^n_{(p)}(\Gamma;\mathbf{R})$ is a p-bounded n-chain and $g_{p+1},\dots,g_{n+1} \in \Gamma$ are fixed elements then the following computation shows that $\delta\alpha$ is p-bounded.

$$\begin{aligned}\delta\alpha(g_1,\dots,g_{n+1}) = \alpha(g_2,\dots,g_{p+1},g_{p+2},\dots,g_{n+1})\\ +\sum_{k=1}^{p}(-1)^k\alpha(g_1,\dots,g_kg_{k+1},\dots,g_{p+1},g_{p+2},\dots,g_{n+1})\\ +\sum_{k=p+1}^{n}(-1)^k\alpha(g_1,\dots,g_p,g_{p+1},\dots,g_kg_{k+1},\dots,g_{n+1})\\ +(-1)^{n+1}\alpha(g_1,\dots,g_p,g_{p+1},\dots,g_n).\end{aligned}$$

The cohomology of the complex $C^*_{(p)}(\Gamma;\mathbf{R})$ will be denoted by $H^n_{(p)}(\Gamma;\mathbf{R})$. The inclusions $C^*_{(p)}(\Gamma;\mathbf{R}) \subseteq C^*_{(p-1)}(\Gamma;\mathbf{R})$ induce maps in cohomology

$$H^n_{(n)}(\Gamma;\mathbf{R}) \to H^n_{(n-1)}(\Gamma;\mathbf{R}) \to \cdots \to H^n_{(1)}(\Gamma;\mathbf{R}) \to H^n_{(0)}(\Gamma;\mathbf{R}) = H^n(\Gamma;\mathbf{R}),$$

called **comparison homomorphisms**. Notice that $H^n_{(n)}(\Gamma;\mathbf{R}) = H^n_b(\Gamma;\mathbf{R})$. In what follows, we will refer to $H^n(\Gamma;\mathbf{R})$ as the standard cohomology.

Recall that a group Γ is called **amenable** if it admits a left-invariant **mean**. That is, a linear functional $\ell^\infty\Gamma \to \mathbf{R}$, on the left module of bounded functions on Γ, such that

(1) $\int F(gh)\mathfrak{m}(g) = \int F(g)\mathfrak{m}(g)$ for all $h \in \Gamma$ and all $F \in \ell^\infty\Gamma$;
(2) $\int 1\mathfrak{m}(g) = 1$;
(3) if $F \geq 0$ almost everywhere then $\int F(g)\mathfrak{m}(g) \geq 0$.

It is well known [14, Section 3.0] that if a group Γ is amenable then its bounded cohomology vanishes in positive degrees. In particular, the comparison homomorphisms between bounded and standard cohomology are trivial. We generalise this result to the p-bounded cohomology.

Theorem 2.1. *If Γ is an amenable group then the comparison homomorphisms*

$$H^n_{(p)}(\Gamma;\mathbf{R}) \to H^n_{(p-1)}(\Gamma;\mathbf{R})$$

vanish for all $n > 0$ and $p = 1, 2, \dots, n$.

Proof. The argument follows the proof of the fact that bounded cohomology of an amenable group vanishes in positive degree. Let $\mathfrak{m}$ be a right-invariant mean on Γ. Define a map $M\colon C^{n+1}_{(p)}(\Gamma;\mathbf{R}) \to C^n_{(p-1)}(\Gamma;\mathbf{R})$ by

$$M\omega(g_1,\dots,g_n) = \int \omega(h,g_1,\dots,g_n)\,\mathfrak{m}(h).$$

To see that $M\omega$ is $(p-1)$-bounded fix $g_p,\dots,g_n$ and observe that the function $h \mapsto \omega(h,g_1,\dots,g_{p-1},g_p,\dots,g_n)$ is bounded uniformly for any choice of $g_1,\dots,g_{p-1}$. This shows $(p-1)$-boundedness of $M\omega$.

Then

$$
\begin{aligned}
M\delta\omega(g_1,\dots,g_n) &= \int \delta\omega(h,g_1,\dots,g_n)\,\mathfrak{m}(h)\\
&= \int \omega(g_1,\dots,g_n)\mathfrak{m}(h)\\
&\quad - \int \omega(hg_1,g_2,\dots,g_n)\,\mathfrak{m}(h) + \int \omega(h,g_1g_2,\dots,g_n)\,\mathfrak{m}(h) - \dots\\
&\quad \dots \pm \int \omega(h,g_1,\dots,g_{n-1})\,\mathfrak{m}(h)\\
&\overset{!}{=} \omega(g_1,\dots,g_n)\\
&\quad - \int \omega(h,g_2,\dots,g_n)\,\mathfrak{m}(h) + \int \omega(h,g_1g_2,\dots,g_n)\,\mathfrak{m}(h) - \dots\\
&\quad \dots \pm \int \omega(h,g_1,\dots,g_{n-1})\,\mathfrak{m}(h)\\
&= \omega(g_1,\dots,g_n) - M\omega(g_2,\dots,g_n) + M\omega(g_1g_2,\dots,g_n) - \dots\\
&\quad \dots \pm M\omega(g_1,\dots,g_{n-1})\\
&= \omega(g_1,\dots,g_n) - \delta M\omega(g_1,\dots,g_n),
\end{aligned}
$$

where, in the marked equality, we used the invariance of the mean in the second term. We obtain that

$$M\delta\omega + \delta M\omega = \omega = i(\omega),$$

where $i\colon C^n_{(p)}(\Gamma;\mathbf{R}) \longrightarrow C^n_{(p-1)}(\Gamma;\mathbf{R})$ is the inclusion of p-bounded cochains into $(p-1)$-bounded ones. Consequently, the map M is a homotopy for the comparison map. □

Remark 2.2. Notice that we only prove the vanishing of the comparison maps. We don't know whether p-bounded cohomology of an amenable group vanishes for $p \neq n$. ◊

Remark 2.3. One could use another definition of p-boundedness. Namely, an n-cochain $\omega\colon \Gamma^n \longrightarrow \mathbf{R}$ is called **left-p-bounded** if for fixed $g_1,\dots,g_{n-p} \in \Gamma$ the function

$$\Gamma^p \ni g_{n-p+1},\dots,g_n \mapsto \omega(g_1,\dots,g_{p-n},g_{p-n+1},\dots,g_n)$$

is bounded. Such definition (of 1-bounded 2-cocycle) appears in [11, Section 4] under the name of **semibounded** cocycle, where the authors use it to prove Polterovich's theorem stating that on the group of symplectic diffeomorphisms of a symplectically hyperbolic manifold all its elements are undistorted [11, Theorem 5.2].

Nevertheless, the involution

$$\widehat{\omega}(g_1,\dots,g_n) = \omega(g_n^{-1},\dots,g_1^{-1})$$

is a chain map that interchanges p-bounded and left-p-bounded cocycles (with trivial coefficients), so both notions define isomorphic objects. ◊

3. The 1-bounded cohomology

Let $\ell^\infty\Gamma$ denote the left module of bounded functions on Γ with the action defined by $(hF)(g) = F(gh)$.

Lemma 3.1. *The map* $\Lambda_0\colon C^n_{(1)}(\Gamma;\mathbf{R}) \to C^{n-1}(\Gamma;\ell^\infty\Gamma)$ *given by the formula*

$$\Lambda_0\omega(g_1,\dots,g_n)(h) = \omega(h,g_1,\dots,g_n)$$

is a homotopy, i.e. $\partial\Lambda_0\omega + \Lambda_0\partial\omega = \omega$.

Proof.

$$\begin{aligned}
\Lambda_0\partial\omega(g_0,\dots,g_n)(h) &= \partial\omega(h,g_0,\dots,g_n)\\
&= \omega(g_0,\dots,g_n)\\
&\quad -\omega(hg_0,\dots,g_n) + \omega(h,g_0g_1,\dots,g_n) \pm \dots\\
&= \omega(g_0,\dots,g_n)\\
&\quad -\Lambda_0\omega(g_1,\dots,g_n)(hg_0) + \Lambda_0\omega(g_0g_1,\dots,g_n)(h) \pm \dots\\
&= \omega(g_0,\dots,g_n) - \partial\Lambda_0\omega(g_0,\dots,g_n)(h).
\end{aligned}$$

□

Let $\Lambda\colon C^n_{(1)}(\Gamma;\mathbf{R}) \to C^{n-1}(\Gamma;\ell^\infty\Gamma/\mathbf{R})$ be the composition of Λ_0 followed by the map $C^{n-1}(\Gamma;\ell^\infty\Gamma) \to C^{n-1}(\Gamma;\ell^\infty\Gamma/\mathbf{R})$ induced by the quotient of the coefficients. The map Λ is surjective and anticommutes with the codifferential. Its kernel consists of cocycles that do not depend on the first variable. We call them **1-constant** and denote by $C^n_{[1]}(\Gamma,\mathbf{R})$.

Lemma 3.2. *The complex of 1-constant cochains with restricted codifferential is acyclic.*

Proof. We need to show that ∂ preserves 1-constant cochains. That is, if $\Lambda\omega = 0$ then $\Lambda\partial\omega = 0$. This is equivalent to showing that if for any $g_1',\dots,g_{n-1}' \in \Gamma$ the function $\Lambda_0\omega(g_1',\dots,g_{n-1}')$ is constant then so is the function $\Lambda_0\partial\omega(g_1,\dots,g_n)$ for any $g_1,\dots,g_{n+1} \in \Gamma$. According to Lemma 3.1, we have $\Lambda_0\partial\omega = \omega - \partial\Lambda_0\omega$. Evaluating both sides on $g_1,\dots,g_n$ we get that both terms on the right hand side are constant, which proves that 1-constant cochains form a subcomplex.

The map H defined by

$$H\omega(g_2,\dots,g_n) = \omega(1,1,g_3,\dots,g_n)$$

is a homotopy proving acyclicity. Indeed, we have

$$H\partial\omega(g_1,\dots,g_n) + \partial H\omega(g_1,\dots,g_n) = \omega(1,g_2,\dots,g_n).$$

Since $h \mapsto \omega(h,g_2,\dots,g_n)$ is a constant function, the right hand side of the above equality is equal to $\omega(g_1,\dots,g_n)$. □

Consider the following short exact sequence

$$0 \to C^n_{[1]}(\Gamma, \mathbf{R}) \to C^n_{(1)}(\Gamma; \mathbf{R}) \to C^{n-1}(\Gamma; \ell^\infty\Gamma/\mathbf{R}) \to 0$$

and the induced long exact sequence in cohomology. As a corollary, we deduce the following theorem.

Theorem 3.3. *The map induced by* Λ*,*

$$H^n_{(1)}(\Gamma; \mathbf{R}) \to H^{n-1}(\Gamma; \ell^\infty\Gamma/\mathbf{R})$$

is an isomorphism. Its composition with the Bockstein homomorphism

$$H^{n-1}(\Gamma; \ell^\infty\Gamma/\mathbf{R}) \to H^n(\Gamma; \mathbf{R})$$

corresponding to the short exact sequence of coefficients $\mathbf{R} \to \ell^\infty\Gamma \to \ell^\infty\Gamma/\mathbf{R}$ *is equal to the comparison map* $H^n_{(1)}(\Gamma; \mathbf{R}) \to H^n(\Gamma; \mathbf{R})$*.*

Proof. Let $\omega \in C^n_{(1)}(\Gamma; \mathbf{R})$ be a cocycle. Then $\Lambda_0(\omega) \in C^{n-1}(\Gamma; \ell^\infty\Gamma)$ is the lift of $\Lambda(\omega) \in C^{n-1}(\Gamma; \ell^\infty\Gamma/\mathbf{R})$. By definition [16, Section 3.E], the Bockstein homomorphism sends $[\Lambda(\omega)]$ to $[\delta\Lambda_0(\omega)]$. Since ω is a cocycle, it follows from Lemma 3.1 that $\delta\Lambda_0(\omega) = \omega - \Lambda_0(\delta\omega) = \omega$, which proves the statement. □

Notice that a group Γ is amenable if and only if the short exact sequence of coefficients $0 \to \mathbf{R} \to \ell^\infty\Gamma \to \ell^\infty\Gamma/\mathbf{R} \to 0$ splits. Indeed, if $s\colon \ell^\infty\Gamma/\mathbf{R} \to \ell^\infty\Gamma$ is a section then $F \mapsto F - s(F\cdot\mathbf{R})$ is a mean. Conversely, if there exists a mean then $s[F] = F - \int F(g)\mathfrak{m}(g)$ is a well defined section. In such a case the Bockstein map is trivial in accordance with Theorem 2.1.

4. 1-bounded 2-cocycles

In this section we discuss two constructions of pseudometrics on a group defined by 1-bounded 2-cocycles (in what follows, we will abuse terminology and refer to them as metrics). The first metric is defined by cocycles that are trivial in the standard cohomology. The second construction is defined for arbitrary cocycles.

We start with the first construction. Recall, that any bounded 2-cocycle in the kernel of the comparison homomorphism $H^2_b(\Gamma; \mathbf{R}) \to H^2(\Gamma; \mathbf{R})$ is a codifferential of a quasimorphism, *i.e.*, a function $\phi\colon \Gamma \to \mathbf{R}$ such that $\sup_{g,h\in\Gamma} |\phi(g) - \phi(gh) + \phi(h)| < \infty$. Thus the kernel of the comparison map can be identified with the space of quasimorphisms divided by the sum of the space of homomorphisms (annihilated by the codifferential) and the space of bounded functions (which correspond to coboundaries). See Calegari [6, Chapter 2] for a survey.

Similarly, any 1-bounded 2-cocycle in the kernel of the comparison homomorphism $H^2_{(1)}(\Gamma; \mathbf{R}) \to H^2(\Gamma; \mathbf{R})$ is a codifferential of a function $\phi\colon \Gamma \to \mathbf{R}$ such that for each fixed $h \in \Gamma$ we have $\sup_{g\in\Gamma} |\phi(g) - \phi(gh) + \phi(h)| < \infty$. That is,

$$\|h\phi - \phi\|_0 := \sup_{g\in\Gamma} |\phi(gh) - \phi(g)| < \infty.$$

We call such functions **abstractly Lipschitz**. The above argument can be then summed up as follows.

Proposition 4.1. *The kernel of the comparison map* $H^2_{(1)}(\Gamma;\mathbf{R}) \to H^2(\Gamma;\mathbf{R})$ *is isomorphic to the quotient on abstractly Lipschitz functions divided by the sum of the space of homomorphisms and the space of bounded functions.* □

On the other hand, given abstractly Lipschitz function ϕ, the map $\Phi\colon \Gamma \to \ell^\infty\Gamma$ given by $\Phi(g) = g\phi - \phi$ is a cocycle. Indeed, we have

$$\begin{aligned}\partial\Phi(g,h) &= g\Phi(h) - \Phi(gh) + \Phi(g)\\ &= g\left(h\phi - \phi\right) - gh\phi + \phi + g\phi - \phi\\ &= gh\phi - g\phi - gh\phi + g\phi = 0.\end{aligned}$$

Taking the class of Φ in $H^1(\Gamma;\ell^\infty\Gamma/\mathbf{R}) \cong H^2_{(1)}(\Gamma;\mathbf{R})$ corresponds to taking class of ϕ modulo homomorphisms (constant cocycles) and bounded functions (coboundaries).

The name *abstractly Lipschitz* is motivated by the following observation.

Proposition 4.2. *Let* ϕ *be a function on a group* Γ*. The following are equivalent*

(1) ϕ *is abstractly Lipschitz,*
(2) *there exists a length function* $|\cdot|$ *on* Γ*, such that* ϕ *is Lipschitz with respect to* $|\cdot|$*.*

Proof. Assume (1). Then ϕ is Lipschitz with respect to

$$|g|_\phi = \|g\phi - \phi\|_0.$$

Assume (2). Then $\|g\phi - \phi\|_0 \le |g|$. □

Notice that if Γ is finitely generated then abstractly Lipschitz functions are Lipschitz with respect to the word norm associated with a finite generating set.

The second construction goes as follows. Let $\omega \in C^2_{(1)}(\Gamma;\mathbf{R})$ be an arbitrary 1-bounded 2-cocycle. For a function $\psi \in \ell^\infty\Gamma$ define $|\psi|_{\mathrm{osc}} = \sup_{h,h'\in\Gamma} |\psi(h) - \psi(h')|$ and define $\|g\|_\omega = |\omega(g,\cdot)|_{\mathrm{osc}}$.

Proposition 4.3. *The function* $\|\cdot\|_\omega$ *is a well defined metric on* Γ*. Moreover, if* η *is a bounded 1-cochain, then* $\|\cdot\|_\omega$ *and* $\|\cdot\|_{\omega+\partial\eta}$ *are within finite (at most* $2|\eta|_{\mathrm{osc}}$*) distance.*

Proof. By cocycle identity

$$\begin{aligned}\omega(g_1g_2,h) - \omega(g_1g_2,h') &= \omega(g_2,h) + \omega(g_1,g_2h) - \omega(g_1,g_2)\\ &\quad - \omega(g_2,h') - \omega(g_1,g_2h') + \omega(g_1,g_2)\\ &= \omega(g_2,h) - \omega(g_2,h') + \omega(g_1,g_2h) - \omega(g_1,g_2h').\end{aligned}$$

Taking supremum of the absolute value of the above over all h and h' we get the triangle identity.

To show that $\|\cdot\|_\omega$ is symmetric first write

$$0 = \delta\omega(1,1,h) = \omega(1,h) - \omega(1,h) + \omega(1,h) - \omega(1,1),$$

thus $\omega(1,h)$ does not depend oh h. Then

$$0 = \delta\omega(g^{-1},g,h) = \omega(g,h) - \omega(1,h) + \omega(g^{-1},gh) - \omega(g^{-1},g),$$

hence

$$\omega(g,h) + \omega(g^{-1},gh) = \omega(1,1) + \omega(g^{-1},g) = \omega(g,k) + \omega(g^{-1},gk),$$

or, equivalently

$$\omega(g,h) - \omega(g,k) = \omega(g^{-1},gk) - \omega(g^{-1},gh).$$

Taking the supremum on both sides we derive $\|g\|_\omega = \|g^{-1}\|_\omega$.
The last statement is obvious. □

One can rephrase the above statement by saying that every element of $H^2_{(1)}(\Gamma,\mathbf{R})$ defines a class of metrics on Γ which are within finite distance from each other.

In [11, Theorem 5.2] we essentially prove that a certain 1-bounded 2-cocycle $\mathfrak{G}$ defines a metric on the group of Hamiltonian diffeomorphisms $\mathrm{Ham}(X,\sigma)$ of a symplectically hyperbolic manifold (X,σ) and prove that each element of this group is undistorted with respect to $\|\cdot\|_{\mathfrak{G}}$. Notice, that in [11] the function $\|\cdot\|_{\mathfrak{G}}$ was defined slightly differently and did not satisfy triangle inequality (cf. [11, Lemma 4.1]).

Let us compare the two constructions. Given a 1-bounded 2-cocycle in the kernel of the comparison map, we defined two norms $\|\cdot\|_{\delta\phi}$ and $|\cdot|_\phi$. Since

$$\begin{aligned}\delta\phi(g,h) - \delta\phi(g,h') &= \phi(g) - \phi(gh) + \phi(h) - \phi(g) + \phi(gh') - \phi(h')\\ &= (g\phi - \phi)(h') - (g\phi - \phi)(h),\end{aligned}$$

we see that $\|g\|_{\delta\phi} = |g\phi - \phi|_{\mathrm{osc}} \le 2|g|_\phi$.

5. Hyperbolic classes

Let $p\colon \widetilde{X} \to X$ be the universal cover of a CW-complex X and let $\Gamma = \pi_1 X$. Let $C^*(\widetilde{X})$ and $C^*(X)$ denote CW-cochains on $\widetilde{X}$ and X, respectively. The cochains are considered with trivial real coefficients. We say that a cochain $\alpha \in C^n(\widetilde{X})$ is **tamed** by $\nu \in C^n(\widetilde{X})$ if for every $g \in \Gamma$ and every cell $\triangle$ in $\widetilde{X}$

$$|(g\alpha - \alpha)(\triangle)| \le \nu(\triangle).$$

Clearly, a cochain is equivariant (it is a lift of a cochain from $C^*(X)$) if and only if it is tamed by a cochain vanishing identically.

We say that a cocycle $\alpha \in C^n(X)$ is **hyperbolic** if $p^*\alpha$ can be expressed as $\delta\beta$ with β tamed by some cochain lifted from X. A class in $H^n(X;\mathbf{R})$ is **hyperbolic** if it has a hyperbolic representative.

The notion of a hyperbolic class is inspired by Gromov [15] who considered differential forms α with the property that the lift $p^*\alpha$ to the universal cover has a primitive that is bounded with respect to the induced Riemannian metric (see also the recent paper of Ascari and Milizia [1] for new developments). By integrating such classes over cells we get that they yield hyperbolic classes in our sense. Indeed, if the n-skeleton of X is finite, then any homogeneous cochain takes only finitely many values. Thus, $a \in H^n(X;\mathbf{R})$ is hyperbolic if one can choose $\beta \in C^{n-1}(\widetilde{X};\mathbf{R})$ with $[\partial\beta] = p^*a$, and $M \in \mathbf{R}$, such that for any cell $\triangle$ in $\widetilde{X}$, $|\beta(\triangle)| < M$.

Hyperbolicity as defined above depends on the cellular structure if X is not compact. We need it so we can speak of hyperbolic classes in $\mathrm{B}\Gamma$. Clearly, hyperbolicity is preserved by cellular maps.

Proposition 5.1. *Hyperbolicity does not change under subdivisions of a given CW-structure.*

Proof. Let X' be a subdivision of X. Wherever we write that a cell in X' divides a cell in X we implicitly assume that they are of the same dimension.

Maps inducing the natural isomorphism on $H^*(X;\mathbf{R})$ and $H^*(X';\mathbf{R})$ are induced by $q\colon C^*(X;\mathbf{R}) \to C^*(X';\mathbf{R})$ and $b\colon C^*(X';\mathbf{R}) \to C^*(X;\mathbf{R})$ defined as $q\alpha(\triangle') = \frac{1}{n_\triangle}\alpha(\triangle)$, where $\triangle$ is the cell which $\triangle'$ divides and $n_\triangle$ denote the number of cells in X' dividing $\triangle$, and $b\alpha'(\triangle) = \sum \alpha'(\triangle')$, where the sum runs over all cells $\triangle'$ in X' dividing $\triangle$. We denote corresponding maps for $\widetilde{X}$ and its induced subdivision $\widetilde{X}'$ by the same letters. We observe, in particular, that they commute with p^*.

If $\alpha = p^*\partial\beta$ and β is tamed by ν then $q\alpha = p^*\partial q\beta$ and $q\beta$ is tamed by $q\nu$. Similarly, if $\alpha' = p^*\partial\beta'$ and β' is tamed by ν' then $b\alpha' = p^*\partial b\beta'$ and $b\beta'$ is tamed by $b\nu'$. □

The aim of this section is to prove the following connection between hyperbolic and 1-bounded classes.

Theorem 5.2. *Let X be a finite connected CW–complex with $\pi_1(X) = \Gamma$ and let $c\colon X \to \mathrm{B}\Gamma$ be the classifying map. Then a pullback of a 1-bounded class is hyperbolic. Moreover, if the universal cover of X is $(n-1)$-connected and $a \in H^n(X;\mathbf{R})$ is a hyperbolic class, then there exists a 1-bounded cocycle $\omega \in C^n_{(1)}(\mathrm{B}\Gamma;\mathbf{R})$ such that $c^*[\omega] = a$.*

Remark 5.3. We do not know wheather one can replace the assumption on higher connectivity of $\widetilde{X}$ by assuming that a lies in the image of c. ◊

Until the end of this section we use the homogeneous notation, since we need to talk about chains on $\mathrm{E}\Gamma$.

We say that a $(n-1)$-cochain λ **controls** an n-cochain ω if for all $h, g_1, \ldots, g_n \in \Gamma$

$$|\omega(h, g_1, \ldots, g_n) - \omega(1, g_1, \ldots, g_n)| < \lambda(g_1, \ldots, g_n). \tag{5.1}$$

By definition, a homogeneous cochain ω is 1-bounded if it is controlled by a homogeneous cochain.

It is straightforward to verify that the map $H\colon C^n(E\Gamma;\mathbf{R}) \to C^{n-1}(E\Gamma;\mathbf{R})$ given by the formula $(H\omega)(g_1,\dots,g_n) = \omega(1,g_1,\dots,g_n)$ is a homotopy satisfying $\partial H\omega + H\partial\omega = \omega$ (see [3, Page 18]).

Lemma 5.4. *Assume that $\omega \in C^n(E\Gamma;\mathbf{R})$ is closed, Γ-invariant and controlled by λ. Then $H\omega$ is tamed by λ.*

Proof.

$$\begin{aligned}|(gH\omega - H\omega)(g_1\dots,g_n)| &= |H\omega(g_1g,\dots,g_ng) - H\omega(g_1,\dots,g_n)|\\ &= |\omega(1,g_1g,\dots,g_ng) - \omega(1,g_1,\dots,g_n)|\\ &= |\omega(g^{-1},g_1,\dots,g_n) - \omega(1,g_1,\dots,g_n)|\\ &\le \lambda(g_1,\dots,g_n).\end{aligned}$$

□

Given $\eta \in C^{n-1}(E\Gamma)$ define $\bar\eta \in C^{n-1}(E\Gamma)^\Gamma$ by the formula

$$\bar\eta(g_1,\dots,g_n) = \eta(1,g_2g_1^{-1},\dots,g_ng_1^{-1}).$$

Lemma 5.5. *Assume that $\eta \in C^{n-1}(E\Gamma;\mathbf{R})$ is tamed by ν and $\partial\eta$ is Γ-invariant then $\partial(\eta-\bar\eta)$ is controlled by ν.*

Proof. Since $\partial(\eta-\bar\eta)$ is Γ-invariant,

$$\begin{aligned}\partial(\eta-\bar\eta)(h,g_1,\dots,g_n)eta &= \partial(\eta-\bar\eta)(1,g_1h^{-1},\dots,g_nh^{-1})\\ &= \eta(g_1h^{-1},\dots,g_nh^{-1}) - \eta(1,g_2h^{-1},\dots,g_nh^{-1}) \pm\cdots\\ &\quad - \eta(1,g_2g_1^{-1}\dots,g_ng_1^{-1}) + \eta(1,g_2h^{-1},\dots,g_nh^{-1}) \mp\cdots\\ &= \eta(g_1h^{-1},\dots,g_nh^{-1}) - \eta(1,g_2g_1^{-1},\dots,g_ng_1^{-1})\\ &= (h^{-1}\eta - g_1^{-1}\eta)(g_1,\dots,g_n).\end{aligned}$$

Thus, since, by assumption, η is tamed by ν,

$$\begin{aligned}|\partial(\eta-\bar\eta)(h,g_1,\dots,g_n) - \partial(\eta-\bar\eta)(1,g_1,\dots,g_n)| &= |(h^{-1}\eta-\eta)(g_1,\dots,g_n)|\\ &\le \nu(g_1,\dots,g_n).\end{aligned}$$

□

Corollary 5.6. *Let $a \in H^n(B\Gamma;\mathbf{R})$. Then a is hyperbolic if and only if a has 1-bounded representative.*

Proof of Theorem 5.2. Let ω be a 1-bounded cocycle. Then $[\omega]$ is hyperbolic, so, by the above corollary, is $c^*[\omega]$.

Let $a \in H^n(X;\mathbf{R})$ be hyperbolic. Taking $\widetilde{Y}$, an Γ-invariant subdivision of the k-skeleton of the simplicial complex $E\Gamma$, we may build an equivariant cellular map $\widetilde{f}\colon \widetilde{Y} \to \widetilde{X}$. Let $f\colon Y = \widetilde{Y}/\Gamma \to X$ be the corresponding map of the quotients. Let $\alpha \in C^k(X;\mathbf{R})$ be a hyperbolic cocycle and let $p^*(\alpha) = \partial\beta$, where β is a tamed cochain. Define $\eta = b(\widetilde{f}^*\beta)$. By Proposition 5.1, η is tamed. By Lemma 5.5 $\partial(\eta-\overline{\eta})$ is controlled.

The equivariant cocycle $\delta(\eta - \overline{\eta})$ is a lift of a cocycle $\omega \in C^k(\mathrm{B}\Gamma; \mathbf{R})$. Being controlled for $\delta(\eta - \overline{\eta})$ translates to 1-boudedness of ω; see comment after (5.1). Since $\overline{\eta}$ is a lift of a cocycle from $\mathrm{B}\Gamma$ we see that $[\omega] = f^*[\alpha]$. Since $c \circ f$ is homotopic to the inclusion $Y \to \mathrm{B}\Gamma$, it follows that $c^*[\omega] = [\alpha]$ which finishes the proof. □

6. Hyperbolic classes and amenability

It immediately follows from Theorems 2.1 and 5.2 that any hyperbolic class on a CW–complex X as above is trivial if $\pi_1(X)$ is amenable. However, a more direct proof yields the following slightly stronger observation originally due to Brunnbauer Kotschick [5, Theorem 3.2] who used an isoperimetric characterisation of manifolds with amenable fundamental group due to Brooks. A different argument was used in [17] to show that the fundamental group of a closed symplectically hyperbolic manifold cannot be amenable.

Proposition 6.1. *Let X be a finite complex and let α be a hyperbolic cocycle on X. If the fundamental group of X is amenable, then α is a coboundary.*

Proof. Let $p^*\alpha = \delta\beta$ with β bounded cochain on $\widetilde{X}$. By averaging this equation with respect to a mean on the deck transformation group $\pi_1(X)$ we find β which is $\pi_1(X)$–invariant, thus β is a lift of a cocycle β' on X and $\alpha = \delta\beta'$ which proves the claim. □

7. Test case: Thompson's group *F*

It is a stimulating open question to determine whether Thompson's group F is amenable. Since it is a split extension

$$[F, F] \to F \to \mathbf{Z}^2,$$

its amenability is equivalent to the amenability of its commutator subgroup. Notice that the commutator subgroup $[F, F]$ is boundedly acyclic due to a recent result of Campagnolo, Fournier-Facio, Lodha and Moraschini [8, Corollary 1.14]. The cohomology ring $H^*([F, F]; \mathbf{R})$ is isomorphic to a polynomial ring $\mathbf{R}[u]$, where $\deg(u) = 2$. The explicit cocycle α (defined in (7.1) below) representing u has been given by Ghys and Sergiescu [12, Corollaire 4.5]. Let $\mathbf{Z}^2 \to [F, F]$ be an injective homomorphism generated by two commuting nontrivial elements. It is easy to choose such elements so that the pull-back of u is nontrivial in $H^2(\mathbf{Z}^2; \mathbf{R})$ and, since $\mathbf{Z}^2$ is amenable, it can't be represented by a 1-bounded cocycle. This is explained in the first step of the proof of the following general observation which provides further evidence for a possible amenability of F.

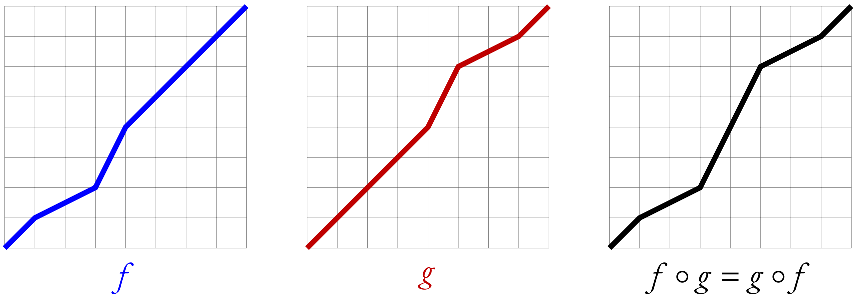
Let $\alpha\colon F' \times F' \to \mathbf{R}$ be a cocycle defined by the following formula

$$\alpha(f, g) = \sum_x \det \begin{pmatrix} \log_2 g'_L(x) & \log_2 (f \circ g)'_L(x) \\ \log_2 g'_R(x) & \log_2 (f \circ g)'_R(x) \end{pmatrix}, \tag{7.1}$$

where the summation is over the breakpoints. The subscripts L and R refer to the left and right derivatives. This cocycle represents a generator of the cohomology ring $H^*(F';\mathbf{R}) = \mathbf{R}[u]$.

Proposition 7.1. *For every positive integer $n \in \mathbf{N}$ there exists an injective homomorphism $\psi\colon \mathbf{Z}^{2n} \to [F,F]$ such that $\psi^*(u^n) \neq 0$ in $H^{2n}(\mathbf{Z}^{2n};\mathbf{R})$. Consequently, u^n cannot be represented by a 1-bounded cocycle.*

Proof. Let $F' = [F,F]$ denote the commutator subgroup of Thompson's group F and let $f, g \in F'$ be elements defined by the following pictures.



Let $\psi\colon \mathbf{Z}^2 \to F'$ be an injective homomorphism defined by

$$\psi(e_1) = f \text{ and } \psi(e_2) = g,$$

where $e_1 = (1,0)$ and $e_2 = (0,1)$. It is a direct computation that $\alpha(f,g) = 1$ and that $\alpha(g,f) = -1$ (only the breaking point $(1/2, 1/2)$ contributes). Since $\zeta_2 = (e_1|e_2) - (e_2|e_1)$ is a cycle representing nontrivial class in $H_2(\mathbf{Z}^2;\mathbf{R})$ we get that $\langle \psi^*\alpha, \zeta_2\rangle = 2$ and hence $\psi^*(u) \neq 0$ in $H^2(\mathbf{Z}^2;\mathbf{R})$. Since $\mathbf{Z}^2$ is amenable, u cannot be represented by a 1-bounded cocycle. We use here the bar notation for non-homogeneous chains (see Brown [3, Section II.3]).

Consider $\psi\colon \mathbf{Z}^{2n} \to F'$ generated by

$$f_1, g_1, \ldots, f_n, g_n,$$

where f_i and g_i are supported on $\left[\frac{i-1}{n}, \frac{i}{n}\right]$ and are rescaled copies of f and g discussed above. Observe that

$$\begin{aligned} &\psi^*\alpha(f_i, g_i) = 2 \\ &\psi^*\alpha(f_i, g_j) = \psi^*\alpha(f_i, f_j) = \psi^*\alpha(g_i, g_j) = 0 \qquad \text{for } i \neq j. \end{aligned}$$

It follows that cocycle $\psi^*\alpha\colon \mathbf{Z}^{2n} \times \mathbf{Z}^{2n} \to \mathbf{R}$ is a symplectic form and hence its n-th power represents a non-trivial cohomology class in $H^{2n}(\mathbf{Z}^{2n};\mathbf{R})$. That is, $[\psi^*\alpha]^n = \psi^*[\alpha]^n = \psi^*(u^n) \neq 0$. Since $\mathbf{Z}^{2n}$ is amenable, u^n cannot be represented by a 1-bounded cocycle. ◻

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