Ricci pinched compact submanifolds in space forms

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Abstract

We investigate the compact submanifolds in Riemannian space forms of nonnegative sectional curvature that satisfy a lower bound on the Ricci curvature, that bound depending solely on the length of the mean curvature vector of the immersion. While generalizing the results, we give a positive answer to a conjecture by H. Xu and J. Gu in (2013, Geom. Funct. Anal. 23). Our main accomplishment is the elimination of the need for the mean curvature vector field to be parallel.

Let $f\colon M^n \to \mathbb{Q}_c^{n+m}$ be an isometric immersion of a compact Riemannian manifold of dimension n into a simply-connected space form of sectional curvature c and substantial codimension m. Throughout this paper, the (not normalized) Ricci curvature of M^n is assumed to satisfy at any point the pinching condition

$$\mathrm{Ric}_M \geq (n-2)(c+H^2), \tag{$*$}$$

where H stands for the norm of the (normalized) mean curvature vector.

By the condition $(*)$ being satisfied with equality at $x \in M^n$ we mean that the inequality at that point is not strict, that is, there exists a unit vector $X \in T_xM$ such that $\mathrm{Ric}_M(X) = (n-2)(c+H^2)$. If otherwise, we call the inequality $(*)$ at $x \in M^n$ strict. Notice that if the inequality holds strictly at any given point, it will persist in its strict form after subjecting the submanifold to a sufficiently small smooth deformation.

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A complete classification of the submanifolds as above for $c > 0$ was obtained by N. Ejiri [3] in 1979 under the assumptions that f is a minimal immersion and that the manifold M^n is both, oriented and simply connected. H. Xu and J. Gu [10] in 2013 generalized Ejiri's result under the assumption that $c + H^2 > 0$ by only requiring the mean curvature vector to be parallel and the submanifold orientable. Additionally, they proved that when the assumption regarding the mean curvature is removed, but a strict inequality in $(*)$ holds at every point, the manifold must be homeomorphic to a sphere.

In the same paper, Xu and Gu put forward a conjecture, suggesting that their findings should hold true even when eliminating the condition on the mean curvature vector altogether. In this paper, we give affirmative confirmation to their conjecture. To achieve that result we do not require the manifold to be oriented. Furthermore, to reach the conclusion that the manifold is homeomorphic to a sphere we only ask for strict inequality in the condition $(*)$ at some point.

Theorem 1. *Let $f\colon M^n \to \mathbb{Q}_c^{n+m}$, $n \geq 4, c \geq 0$, be an isometric immersion of a compact Riemannian manifold that satisfies the condition $(*)$ at any point. Then either M^n is homeomorphic to $\mathbb{S}^n$ or one of the following holds:*

(i) *M^n is $\mathbb{S}^{n/2}(r/\sqrt{2}) \times \mathbb{S}^{n/2}(r/\sqrt{2})$, $r = 1/\sqrt{c + H^2}$, and $f = j \circ g$, where $g\colon M^n \to \mathbb{S}^{n+1}(r)$ is the standard embedding and $j\colon \mathbb{S}^{n+1}(r) \to \mathbb{Q}_c^{n+2}$ an umbilical inclusion.*

(ii) *M^n is the projective plane $\mathbb{CP}_r^2$ of constant holomorphic curvature $4r^2/3$ with $r = 1/\sqrt{c + H^2}$ and $f = j \circ g$, where g is the standard immersion of $\mathbb{CP}_r^2$ into $\mathbb{S}^7(r)$ and $j\colon \mathbb{S}^7(r) \to \mathbb{Q}_c^8$ an umbilical inclusion.*

Remarks: (i) In both cases the umbilical inclusion may be totally geodesic. (ii) If n is even and $c > 0$ the result generalizes Theorem 2 in [2] by means of a lower bound for the Ricci curvature.

When M^n possesses the topological structure of a sphere, the conjecture put forth by Xu and Gu proposes that it should not merely be topologically equivalent but diffeomorphic to a sphere. This holds true for dimensions $n = 5, 6, 12, 56, 61$ as in these cases, it is established that the differentiable structure is unique; as stated by Corollary 1.15 in [8].

1 The pinching condition

In this section, we analyze the relation between our pinching condition $(*)$ and the one for $c > 0$ due to Lawson and Simons in their seminal paper [5]. Their result was later strengthened by Elworthy and Rosenberg [4, p. 71] by only requiring the bound to be strict at some point of the submanifold. The case $c = 0$ was later considered by Xin [9].

Theorem 2. ([4],[5],[9]) *Let $f\colon M^n \to \mathbb{Q}_c^{n+m}$, $n \geq 4, c \geq 0$, be an isometric immersion of a compact manifold and p an integer such that $1 \leq p \leq n-1$. Assume that at any point $x \in M^n$ and for* any *orthonormal basis $\{e_1, \ldots, e_n\}$ of T_xM the second fundamental form $\alpha_f\colon TM \times TM \to N_fM$ satisfies*

$$\Theta_p = \sum_{i=1}^{p} \sum_{j=p+1}^{n} \big(2\|\alpha_f(e_i, e_j)\|^2 - \langle \alpha_f(e_i, e_i), \alpha_f(e_j, e_j)\rangle\big) \leq p(n-p)c. \quad (\#)$$

If the inequality $(\#)$ is strict at a point of M^n, then there are no stable p-currents and the homology groups satisfy $H_p(M^n; \mathbb{Z}) = H_{n-p}(M^n; \mathbb{Z}) = 0$.

Recall that a vector in the normal bundle $\eta \in N_fM(x)$ at $x \in M^n$ is named a *Dupin principal normal* of $f\colon M^n \to \mathbb{Q}_c^{n+m}$ if the vector subspace

$$E_\eta(x) = \{X \in T_xM \colon \alpha_f(X, Y) = \langle X, Y\rangle \eta \text{ for all } Y \in T_xM\}$$

is at least two dimensional. That dimension is called the *multiplicity* of η.

The proof of the following results is inspired by computations given by us in [2] and by Xu and Gu in [10].

Proposition 3. *Let $f\colon M^n \to \mathbb{Q}_c^{n+m}, n \geq 4$, be an isometric immersion satisfying the inequality $(*)$ at $x \in M^n$. Then the following assertions at $x \in M^n$ hold:*

(i) The inequality $(\#)$ is satisfied for any integer $2 \leq p \leq n/2$ and any orthonormal basis of T_xM. Moreover, if the inequality $(*)$ is strict then also $(\#)$ is strict for any integer $2 \leq p \leq n/2$.

(ii) Assume that equality holds in $(\#)$ for a certain integer $2 \leq p \leq n/2$ and a given orthonormal basis $\{e_j\}_{1 \leq j \leq n}$ of T_xM. Then the Ricci tensor satisfies

$$Ric_M(X) = (n-2)(c + H^2) \text{ for any unit } X \in T_xM.$$

Moreover, we have:

(a) If $n \geq 5$ then either $c = 0$ and f is totally geodesic or we have that $p = n/2$ and there are distinct Dupin principal normals η_1 and η_2 such that $E_{\eta_1} = span\{e_1, \ldots, e_p\}$ and $E_{\eta_2} = span\{e_{p+1}, \ldots, e_n\}$.

(b) If $n = 4$ and $p = 2$ there are normal vectors η_j, $j = 1, 2$, such that

$$\pi_{V_j} \circ A_\xi|_{V_j} = \langle \xi, \eta_j \rangle I \text{ for any } \xi \in N_f(x) \tag{1}$$

where $V_1 = span\{e_1, e_2\}$, $V_2 = span\{e_3, e_e\}$ and $\pi_{V_j} \colon T_xM \to V_j$ denotes the projection.

Proof. We first recall that the Gauss equation of $f \colon M^n \to \mathbb{Q}_c^{n+m}$ implies that the Ricci curvature for any unit vector $X \in T_xM$ satisfies

$$\mathrm{Ric}_M(X) = (n-1)c + \sum_{\alpha=1}^{m} (\mathrm{tr} A_\alpha) \langle A_\alpha X, X \rangle - \sum_{\alpha=1}^{m} \|A_\alpha X\|^2, \tag{2}$$

where the A_α, $1 \leq \alpha \leq m$, are the shape operators of f associated to an orthonormal basis $\{\xi_\alpha\}_{1 \leq \alpha \leq m}$ of the normal space $N_fM(x)$ of the submanifold.

From now on, we agree that $\{\xi_\alpha\}_{1 \leq \alpha \leq m}$ satisfies that the (normalized) mean curvature vector is $\mathcal{H}(x) = H(x)\xi_1$ when $H(x) \neq 0$. For a given orthonormal basis $\{e_j\}_{1 \leq j \leq n}$ of T_xM, we denote for simplicity $\alpha_{ij} = \alpha_f(e_i, e_j)$, $1 \leq i, j \leq n$. Then, we have

$$\begin{aligned}
\Theta_p =\ & 2 \sum_{i=1}^{p} \sum_{j=p+1}^{n} \|\alpha_{ij}\|^2 - n \sum_{i=1}^{p} \langle \alpha_{ii}, \mathcal{H} \rangle + \sum_{i,j=1}^{p} \langle \alpha_{ii}, \alpha_{jj} \rangle \\
= & \, 2 \sum_{i=1}^{p} \sum_{j=p+1}^{n} \sum_{\alpha} \langle A_\alpha e_i, e_j \rangle^2 - nH \sum_{i=1}^{p} \langle A_1 e_i, e_i \rangle + \sum_{\alpha} \Big(\sum_{i=1}^{p} \langle A_\alpha e_i, e_i \rangle \Big)^2 \\
\leq & \, 2 \sum_{i=1}^{p} \sum_{j=p+1}^{n} \sum_{\alpha} \langle A_\alpha e_i, e_j \rangle^2 - nH \sum_{i=1}^{p} \langle A_1 e_i, e_i \rangle + p \sum_{\alpha} \sum_{i=1}^{p} \langle A_\alpha e_i, e_i \rangle^2,
\end{aligned} \tag{3}$$

where the inequality part was obtained using the Cauchy-Schwarz inequality

$$\Big(\sum_{i=1}^{p} \langle A_\alpha e_i, e_i \rangle \Big)^2 \leq p \sum_{i=1}^{p} \langle A_\alpha e_i, e_i \rangle^2. \tag{4}$$

Since $p \geq 2$ by assumption, then

$$\begin{aligned}
2\sum_{i=1}^{p}\sum_{j=p+1}^{n}\sum_{\alpha}\langle A_\alpha e_i, e_j\rangle^2 + p\sum_{i=1}^{p}\sum_{\alpha}\langle A_\alpha e_i, e_i\rangle^2 \\
\leq p\sum_{i=1}^{p}\sum_{j=p+1}^{n}\sum_{\alpha}\langle A_\alpha e_i, e_j\rangle^2 + p\sum_{i=1}^{p}\sum_{\alpha}\langle A_\alpha e_i, e_i\rangle^2 \\
\leq p\sum_{i=1}^{p}\sum_{\alpha}\|A_\alpha e_i\|^2 \qquad (5)
\end{aligned}$$

and thus (3) implies that

$$\Theta_p \leq p\sum_{i=1}^{p}\sum_{\alpha}\|A_\alpha e_i\|^2 - nH\sum_{i=1}^{p}\langle A_1 e_i, e_i\rangle.$$

Setting $\varphi = A_1 - HI$ and using (2), we obtain

$$\begin{aligned}
\Theta_p &\leq p\sum_{i=1}^{p}((n-1)c - \mathrm{Ric}_M(e_i)) + (p-1)nH\sum_{i=1}^{p}\langle A_1 e_i, e_i\rangle \\
&= p\sum_{i=1}^{p}((n-1)(c+H^2) - \mathrm{Ric}_M(e_i)) - p(n-p)H^2 \\
&+ (p-1)nH\sum_{i=1}^{p}\langle \varphi e_i, e_i\rangle. \qquad (6)
\end{aligned}$$

Then, we have

$$\begin{aligned}
\Theta_p \leq p^2\big((n-1)(c+H^2) - \mathrm{Ric}_M^{\min}(x)\big) - p(n-p)H^2 \\
+ (p-1)nH\sum_{i=1}^{p}\langle \varphi e_i, e_i\rangle. \qquad (7)
\end{aligned}$$

where $\mathrm{Ric}_M^{\min}(x) = \min\{\mathrm{Ric}_M(X)\colon X \in T_xM, \|X\| = 1\}$.

We claim that

$$(n-1)(c+H^2) \geq \mathrm{Ric}_M^{\min}(x) \qquad (8)$$

and that equality holds if f is umbilical at $x \in M^n$. Indeed, it follows from the Gauss equation that the scalar curvature τ is given by

$$\tau = n(n-1)c - S + n^2H^2, \qquad (9)$$

where S denotes the norm of the second fundamental form. Hence

$$S \leq n(n-1)c + n^2 H^2 - n\mathrm{Ric}_M^{\min}(x). \tag{10}$$

Therefore, we have

$$(n-1)(c+H^2) - \mathrm{Ric}_M^{\min}(x) \geq \frac{1}{n}(S - nH^2) = \frac{1}{n}\|\phi\|^2,$$

where $\phi = \alpha - \langle\, ,\,\rangle\mathcal{H}$ is the traceless second fundamental form, and the claim now follows.

From (8) and having that $p \leq n/2$, then

$$p^2\big((n-1)(c+H^2)-\mathrm{Ric}_M^{\min}(x)\big) \leq p(n-p)\big((n-1)(c+H^2)-\mathrm{Ric}_M^{\min}(x)\big). \tag{11}$$

Therefore, it follows from (7) the estimate

$$\Theta_p \leq p(n-p)\big((n-1)(c+H^2)-\mathrm{Ric}_M^{\min}(x)-H^2\big)+(p-1)nH\sum_{i=1}^{p}\langle \varphi e_i, e_i\rangle. \tag{12}$$

Next, we provide a second estimate of

$$\Theta_p = \sum_\alpha \Big(2\sum_{i=1}^{p}\sum_{j=p+1}^{n}\langle A_\alpha e_i, e_j\rangle^2 - \sum_{i=1}^{p}\langle A_\alpha e_i, e_i\rangle \sum_{j=p+1}^{n}\langle A_\alpha e_j, e_j\rangle\Big)$$

Decomposing

$$\begin{aligned}&\sum_{i=1}^{p}\langle A_\alpha e_i, e_i\rangle \sum_{j=p+1}^{n}\langle A_\alpha e_j, e_j\rangle\\ =&\frac{n-p}{n}\sum_{i=1}^{p}\langle A_\alpha e_i, e_i\rangle \sum_{j=p+1}^{n}\langle A_\alpha e_j, e_j\rangle + \frac{p}{n}\sum_{i=1}^{p}\langle A_\alpha e_i, e_i\rangle \sum_{j=p+1}^{n}\langle A_\alpha e_j, e_j\rangle,\end{aligned}$$

we have

$$\begin{aligned}\Theta_p =& \sum_\alpha \Big(2\sum_{i=1}^{p}\sum_{j=p+1}^{n}\langle A_\alpha e_i, e_j\rangle^2 - \frac{n-p}{n}\mathrm{tr}\, A_\alpha \sum_{i=1}^{p}\langle A_\alpha e_i, e_i\rangle\\ &+ \frac{n-p}{n}\big(\sum_{i=1}^{p}\langle A_\alpha e_i, e_i\rangle\big)^2 - \frac{p}{n}\mathrm{tr}\, A_\alpha \sum_{j=p+1}^{n}\langle A_\alpha e_j, e_j\rangle + \frac{p}{n}\big(\sum_{j=p+1}^{n}\langle A_\alpha e_j, e_j\rangle\big)^2\Big).\end{aligned}$$

Using the Cauchy-Schwarz inequality, we obtain

$$
\begin{aligned}
\Theta_p \leq & \sum_\alpha \Big(2\sum_{i=1}^p \sum_{j=p+1}^n \langle A_\alpha e_i, e_j\rangle^2 - \frac{n-p}{n}\mathrm{tr}\, A_\alpha \sum_{i=1}^p \langle A_\alpha e_i, e_i\rangle \\
& + \ \frac{p(n-p)}{n}\sum_{i=1}^p \langle A_\alpha e_i, e_i\rangle^2 - \frac{p}{n}\mathrm{tr}\, A_\alpha \sum_{j=p+1}^n \langle A_\alpha e_j, e_j\rangle \\
& + \ \frac{p(n-p)}{n} \sum_{j=p+1}^n \langle A_\alpha e_j, e_j\rangle^2\Big) \\
\leq & \ \frac{p(n-p)}{n}S - (n-p)H\sum_{i=1}^p \langle A_1 e_i, e_i\rangle - pH\sum_{j=p+1}^n \langle A_1 e_j, e_j\rangle \\
= & \ \frac{p(n-p)}{n}S - pnH^2 - (n-2p)H\sum_{i=1}^p \langle A_1 e_i, e_i\rangle \\
= & \ \frac{p(n-p)}{n}S - 2p(n-p)H^2 - (n-2p)H\sum_{i=1}^p \langle \varphi e_i, e_i\rangle.
\end{aligned}
$$

Then using (10) we obtain

$$
\Theta_p \leq p(n-p)\big((n-1)(c+H^2) - \mathrm{Ric}_M^{\min}(x) - H^2\big) - (n-2p)H\sum_{i=1}^p \langle \varphi e_i, e_i\rangle. \quad (13)
$$

Finally, by computing $(n-2p)\times(12)+n(p-1)\times(13)$ and using $(*)$ it follows that

$$
\Theta_p \leq p(n-p)\Big((n-1)(c+H^2) - \mathrm{Ric}_M^{\min}(x) - H^2\Big) \leq p(n-p)c,
$$

and $(\#)$ has been proved. Clearly, if the inequality $(*)$ is strict then also (14) becomes strict, and this completes the proof of part (i).

We now prove part (ii). Thus, we assume that equality holds in $(\#)$ for a certain integer $2 \leq p \leq n/2$ and a given orthonormal basis $\{e_j\}_{1\leq j\leq n}$ of T_xM. Then, all the inequalities from (3) to (7) as well as the ones from (11) to (14) become equalities. In particular, from (4) we obtain that

$$
\langle A_\alpha e_i, e_i\rangle = \rho_\alpha \ \text{for all } 1 \leq i \leq p, \ 1 \leq \alpha \leq m. \quad (14)
$$

From (5) it follows that

$$(p-2)\langle A_\alpha e_i, e_j\rangle = 0 \ \text{for all} \ 1 \le i \le p,\ p+1 \le j \le n,\ 1 \le \alpha \le m, \tag{15}$$

and

$$\langle A_\alpha e_i, e_{i'}\rangle = 0 \ \text{for all} \ 1 \le i \ne i' \le p,\ 1 \le \alpha \le m. \tag{16}$$

We obtain from (11) that

$$(p-n/2)\big(\mathrm{Ric}_M^{\min}(x) - (n-1)(c+H^2)\big) = 0. \tag{17}$$

From (6) and (7) we have $\mathrm{Ric}_M(e_i) = \mathrm{Ric}_M^{\min}(x)$ and then (14) gives

$$\mathrm{Ric}_M(e_i) = \mathrm{Ric}_M^{\min}(x) = (n-2)(c+H^2) \ \text{for all} \ 1 \le i \le p. \tag{18}$$

At first suppose that $p \ne n/2$. Then (17) implies that equality holds in the inequality (8), and we have seen that this gives that f is umbilical at x. At umbilical points, it now follows from (18) that $\mathrm{Ric}_M(x) = (n-1)(c+H^2)$, that $c = 0$ and f is totally geodesic.

Hereafter, let $2p = n$. Then equality also holds in (#) for the reordered orthonormal basis $\{e_{p+1}, \dots, e_n, e_1, \dots, e_p\}$ of T_xM. Thus, we also have

$$\langle A_\alpha e_j, e_j\rangle = \mu_\alpha \ \text{for all} \ p+1 \le j \le n,\ 1 \le \alpha \le m, \tag{19}$$

$$\langle A_\alpha e_j, e_{j'}\rangle = 0 \ \text{for all} \ p+1 \le j \ne j' \le n,\ 1 \le \alpha \le m, \tag{20}$$

and

$$\mathrm{Ric}_M(e_j) = \mathrm{Ric}_M^{\min}(x) = (n-2)(c+H^2) \ \text{for all} \ p+1 \le j \le n. \tag{21}$$

Hence, we obtain from (18) and (21) that the Ricci tensor satisfies

$$\mathrm{Ric}_M(X) = (n-2)(c+H^2) \ \text{for any unit} \ X \in T_xM. \tag{22}$$

In particular, if $n \ge 6$ then it follows from (14), (15), (16), (19) and (20) that $\eta_1 = \sum_\alpha \rho_\alpha \xi_\alpha$ and $\eta_2 = \sum_\alpha \mu_\alpha \xi_\alpha$ are Dupin principal normals with

$$E_{\eta_1} = \mathrm{span}\{e_1, \dots, e_p\} \ \text{and} \ E_{\eta_2} = \mathrm{span}\{e_{p+1}, \dots, e_n\}.$$

If $\eta_1 = \eta_2$, then f is umbilical at x and equality holds in (8). This combined with (22) yields $c = 0$ and that f is totally geodesic at x. Otherwise η_1 and η_2 are distinct Dupin principal normals, and this concludes the proof of part (a).

Finally, if $n = 4$ then for any $\xi \in N_f(x)$ we have (1) where $\eta_1 = \sum_\alpha \rho_\alpha \xi_\alpha$ and $\eta_2 = \sum_\alpha \mu_\alpha \xi_\alpha$, and part (ii) has also been proved. $\square$

2 The proof of Theorem 1

We first establish a topological result necessary for proving the theorem.

Lemma 4. *Let $f\colon M^n \to \mathbb{Q}_c^{n+m}$, $n \geq 4, c \geq 0$, be an isometric immersion of a compact manifold satisfying*

$$Ric_M \geq \frac{n(n-1)}{n+2}(c+H^2) \tag{23}$$

with strict inequality at some point. Then $\pi_1(M^n) = 0$ and $H_{n-1}(M^n, \mathbb{Z}) = 0$.

Proof. From (9) and (23) we obtain that $\phi = \alpha - \langle\, , \rangle \mathcal{H}$ satisfies

$$\|\phi\|^2 \leq \frac{2n(n-1)}{n+2}(c+H^2). \tag{24}$$

Let $\{e_i\}_{1\leq i\leq n}$ be an orthonormal tangent basis and let $\{\xi_\alpha\}_{1\leq\alpha\leq m}$ be an orthonormal normal basis at $x \in M^n$. Using (2) we have

$$\begin{aligned}
&\sum_{j=2}^{n}\big(2\|\alpha_{1j}\|^2 - \langle\alpha_{11}, \alpha_{jj}\rangle\big)\\
&= 2\sum_\alpha\sum_{j=2}^{n}\langle A_\alpha e_1, e_j\rangle^2 - \sum_\alpha\langle A_\alpha e_1, e_1\rangle\sum_{j=2}^{n}\langle A_\alpha e_j, e_j\rangle\\
&= \sum_\alpha\sum_{j=2}^{n}\langle A_\alpha e_1, e_j\rangle^2 - \sum_\alpha \operatorname{tr} A_\alpha\langle A_\alpha e_1, e_1\rangle + \sum_\alpha\|A_\alpha e_1\|^2\\
&= \sum_{j=2}^{n}\|\phi(e_1, e_j)\|^2 + (n-1)c - \mathrm{Ric}_M(e_1).
\end{aligned}$$

This inequality together with (23) and (24) give

$$\begin{aligned}
\sum_{j=2}^{n}\big(2\|\alpha_{1j}\|^2 - \langle\alpha_{11}, \alpha_{jj}\rangle\big) \leq\ &\frac{1}{2}\|\phi\|^2 + (n-1)c - \mathrm{Ric}_M(e_1)\\
\leq (n-1)c + \frac{n(n-1)}{n+2}&(c+H^2) - \mathrm{Ric}_M(e_1) \leq (n-1)c.
\end{aligned}$$

Clearly, if (23) is a strict inequality at a point then also is the above. Then by Theorem 2 there are no stable 1-currents on M^n and thus $H_1(M^n, \mathbb{Z}) =$

$H_{n-1}(M^n, \mathbb{Z}) = 0$. Since in each nontrivial free homotopy class there is a length minimizing curve, we conclude that $\pi_1(M^n) = 0$. $\square$

Proof of Theorem 1: According to part (i) of Proposition 3 the inequality (#) is satisfied at any point of M^n for any $2 \leq p \leq n/2$ and any orthonormal tangent basis at that point. Since $n(n-1)/(n+2) < (n-2)$, then for $c > 0$ we have that

$$\mathrm{Ric}_M \geq (n-2)(c + H^2) > \frac{n(n-1)}{n+2}(c + H^2). \tag{25}$$

If $c = 0$, then compactness implies that there exists a point $x \in M^n$ where $H(x) \neq 0$, and (25) holds at that point. Now Lemma 4 yields that M^n is simply connected and that $H_{n-1}(M^n, \mathbb{Z}) = 0$.

We need to distinguish two cases:

Case I. Suppose first that

$$H_p(M^n; \mathbb{Z}) = 0 = H_{n-p}(M^n; \mathbb{Z}) \text{ for all } 2 \leq p \leq n/2, \tag{26}$$

which by Theorem 2 is necessarily the case if $(*)$ is strict at some point of M^n. Hence M^n is a simply connected homology sphere and it follows from the Hurewicz isomorphism theorem that M^n is a homotopy sphere. Then the resolution of the generalized Poincaré conjecture gives that M^n is homeomorphic to $\mathbb{S}^n$.

Case II. Suppose now that (26) does not hold. Consider the nonempty set

$$P = \{2 \leq p \leq n/2 : H_p(M^n; \mathbb{Z}) \neq 0 \;\text{ or }\; H_{n-p}(M^n; \mathbb{Z}) \neq 0\}$$

and set $k = \max P$. Hence $H_k(M^n; \mathbb{Z}) \neq 0$ or $H_{n-k}(M^n; \mathbb{Z}) \neq 0$. Then, by Theorem 2, there exists at any point $x \in M^n$ an orthonormal tangent basis such that equality holds in (#) for $p = k$. Moreover, at any point $x \in M^n$ we have from part (ii) of Proposition 3 that

$$\mathrm{Ric}_M(X) = (n-2)(c + H^2) \text{ for any unit } X \in T_xM.$$

Then the manifold M^n is Einstein and H is constant.

If $H = 0$ then $c > 0$. Since $\mathrm{Ric}_M = (n-2)c$, it follows from the Theorem of Ejiri in [3] that the submanifold is as in parts (i) or (ii) of the statement.

We assume hereafter that $H > 0$. We have to distinguish two cases according to the dimension of the submanifold.

Case $n \geq 5$. Part (ii) of Proposition 3 yields $k = n/2$ and that there are smooth Dupin principal normal vector fields η_1 and η_2 of multiplicity k and corresponding smooth distributions E_1 and E_2. Let $\{X_\ell\}_{1\leq\ell\leq n}$ be a smooth local orthonormal frame satisfying that $E_{\eta_1} = \operatorname{span}\{X_1, \ldots, X_k\}$ and $E_{\eta_2} = \operatorname{span}\{X_{k+1}, \ldots, X_n\}$. Then $\alpha_f(X_i, X_i) = \eta_1$ if $1 \leq i \leq k$ and $\alpha_f(X_j, X_j) = \eta_2$ if $k+1 \leq j \leq n$.

If follows from (2) that

$$Ric_M(X) = (n-1)c\|X\|^2 + n\langle \mathcal{H}, \alpha_f(X,X)\rangle - III(X) \text{ for any } X \in \mathfrak{X}(M),$$

where $III(X) = \sum_{\ell=1}^n \|\alpha_f(X, X_\ell)\|^2$ is the third fundamental form of f. Since we have that $\mathcal{H} = (\eta_1 + \eta_2)/2$, then

$$4H^2 = \|\eta_1\|^2 + \|\eta_2\|^2 + 2\langle \eta_1, \eta_2\rangle. \tag{27}$$

Moreover, we have for $1 \leq i \leq k$ that

$$III(X_i) = \sum_{\ell=1}^n \|\alpha(X_\ell, X_i)\|^2 = \|\eta_1\|^2$$

and

$$\begin{aligned}(n-2)(c+H^2) = Ric_M(X_i) &= (n-1)c + n\langle \mathcal{H}, \alpha(X_i, X_i)\rangle - III(X_i)\\ &= (n-1)c + k\langle \eta_1 + \eta_2, \eta_1\rangle - \|\eta_1\|^2.\end{aligned}$$

Thus

$$(n-2)(c+H^2) = (n-1)c + (k-1)\|\eta_1\|^2 + k\langle \eta_1, \eta_2\rangle. \tag{28}$$

Arguing similarly for $k+1 \leq j \leq n$, we obtain

$$(n-2)(c+H^2) = (n-1)c + (k-1)\|\eta_2\|^2 + k\langle \eta_1, \eta_2\rangle. \tag{29}$$

It follows from (28) and (29) that $\|\eta_1\| = \|\eta_2\|$, and hence (27) becomes

$$2H^2 = \|\eta_1\|^2 + \langle \eta_1, \eta_2\rangle. \tag{30}$$

Combining (28) with (30) gives

$$c + \langle \eta_1, \eta_2\rangle = 0. \tag{31}$$

Then, we conclude from (30) that

$$\|\eta_1\|^2 = \|\eta_2\|^2 = 2H^2 + c. \tag{32}$$

The Codazzi equation for f is easily seen to yield

$$\langle \nabla_X Y, Z\rangle(\eta_i - \eta_j) = \langle X, Y\rangle \nabla^\perp_Z \eta_i \text{ if } i \neq j \tag{33}$$

for all $X, Y \in E_i, Z \in E_j$. Using (31) and (32) then (33) gives

$$2\langle \nabla_X Y, Z\rangle H^2 = \langle X, Y\rangle\langle \nabla^\perp_Z \eta_i, \eta_i\rangle = 0 \text{ for all } X, Y \in E_i \text{ and } Z \in E_j,\ i \neq j,$$

that is, the distributions E_1 and E_2 are totally geodesic. Being simply connected, it is well-known that M^n is a Riemannian product $M_1^k \times M_2^k$ (cf. Theorem 8.2 in [1]) such that $TM_j^k = E_j$, $j = 1, 2$. Since the second fundamental form of f is adapted to the distributions E_1 and E_2, then Theorem 8.4 and Corollary 8.6 in [1] imply that the submanifold is an extrinsic product of isometric immersions each of which is totally umbilical. Hence, if $c = 0$ then the submanifold is a torus $\mathbb{S}^{n/2}(r/\sqrt{2}) \times \mathbb{S}^{n/2}(r/\sqrt{2})$ in a sphere $\mathbb{S}^{n+1}(r) \subset \mathbb{R}^{n+2}$. If $c > 0$, then the submanifold is a torus $\mathbb{S}^{n/2}(r/\sqrt{2}) \times \mathbb{S}^{n/2}(r/\sqrt{2})$ in a sphere $\mathbb{S}^{n+1}(r) \subset \mathbb{S}^{n+2}(1/\sqrt{c})$.

Case $n = 4$. We have $k = 2$ and $H_2(M^4; \mathbb{Z}) \neq 0$. Since $\mathrm{Ric}_M = 2(c + H^2)$ then $\tau = 8(c + H^2)$ and (9) gives $S = 4c + 8H^2$. Thus equality at any point of the submanifold holds in the pinching condition (1) in [6]. Since $\|\phi\|^2 = S - 4H^2$, it then follows from Proposition 16 in [6] that the Bochner operator $\mathcal{B}^{[2]}\colon \Omega^2(M^4) \to \Omega^2(M^4)$, a certain symmetric endomorphism of the bundle of 2-forms $\Omega^2(M^4)$, satisfies for any $\omega \in \Omega^2(M^4)$ the inequality

$$\langle \mathcal{B}^{[2]}\omega, \omega\rangle \geq (4c + 8H^2 - S)\|\omega\|^2 = 0. \tag{34}$$

We claim that there exists a nonzero 2-form for which equality holds in (34) at any point. By the Bochner-Weitzenböck formula the Laplacian of any 2-form $\omega \in \Omega^2(M^4)$ is given by

$$\Delta\omega = \nabla^*\nabla\omega + \mathcal{B}^{[2]}\omega,$$

where $\nabla^*\nabla$ is the rough Laplacian. From this we obtain

$$\langle \Delta\omega, \omega\rangle = \|\nabla\omega\|^2 + \langle \mathcal{B}^{[2]}\omega, \omega\rangle + \frac{1}{2}\,\Delta\|\omega\|^2. \tag{35}$$

If ω is an harmonic 2-form, it follows from the maximum principle, (34) and (35) and it is parallel. Then, for any harmonic 2-form we have that (34) holds as an equality at any point. On the other hand, the universal coefficient theorem of cohomology yields that the torsion subgroups of $H_1(M^4;\mathbb{Z})$ and $H^2(M^4;\mathbb{Z})$ are isomorphic (cf. [7, p. 244 Corollary 4]). Since M^4 is simply connected, we have that $H_1(M^n;\mathbb{Z})=0$ and thus $H^2(M^4;\mathbb{Z})$ is torsion free. Then the Poincaré duality yields that also $H_2(M^4;\mathbb{Z})$ is torsion free. Hence, $0\neq H_2(M^4;\mathbb{Z})=\mathbb{Z}^{\beta_2(M)}$. Thus M^4 supports a nonzero parallel harmonic 2-form, and the claim has been proved.

From the claim and Proposition 16 in [6] it follows that the shape operator $A_\xi(x)$ at any $x\in M^4$ and for any $0\neq\xi\in N_fM(x)$ has at most two distinct eigenvalues with multiplicity 2. We choose an orthonormal normal basis $\{\xi_\alpha\}_{1\leq\alpha\leq m}$ at $x\in M^4$ such that the mean curvature vector is $\mathcal{H}(x)=H\xi_1$. By part (ii) of Proposition 3 there exists an orthonormal basis $\{e_i\}_{1\leq i\leq 4}$ of T_xM such that the corresponding shape operators $A_\alpha, 1\leq\alpha\leq m$, are as

$$\begin{cases} A_\alpha e_1=\rho_\alpha e_1+\kappa_\alpha e_3+\lambda_\alpha e_4\\ A_\alpha e_2=\rho_\alpha e_2+\mu_\alpha e_3+\nu_\alpha e_4\\ A_\alpha e_3=\kappa_\alpha e_1+\mu_\alpha e_2+\sigma_\alpha e_3\\ A_\alpha e_4=\lambda_\alpha e_1+\nu_\alpha e_2+\sigma_\alpha e_4,\end{cases}\tag{36}$$

where

$$\rho_1+\sigma_1=2H \ \text{ and } \ \rho_\alpha+\sigma_\alpha=0 \ \text{ for any } \ 2\leq\alpha\leq m.\tag{37}$$

Now since each shape operator has at most two distinct eigenvalues with multiplicity 2, it follows easily using (36) that

$$\nu_\alpha=\pm\kappa_\alpha \ \text{ and } \ \mu_\alpha=\mp\lambda_\alpha \ \text{ for any } \ 2\leq\alpha\leq m.\tag{38}$$

Since M^4 is Einstein, then $\mathrm{Ric}_M(e_i,e_j)=0, i=1,2,\ j=3,4$. Using that

$$\mathrm{Ric}_M(X,Y)=(n-1)c\langle X,Y\rangle+\sum_{\alpha=1}^m(\mathrm{tr}A_\alpha)\langle A_\alpha X,Y\rangle-\sum_{\alpha=1}^m\langle A_\alpha^2X,Y\rangle,$$

together with (36) and (37) the above yields that $\{e_i\}_{1\leq i\leq 4}$ diagonalizes A_1. Moreover, from $\mathrm{Ric}_M(e_j)=2(c+H^2), 1\leq j\leq 4$, (2) and (36) we obtain that

$$2H^2-c=\begin{cases}4H\rho_1-\sum_{\alpha\geq1}\rho_\alpha^2-\|\alpha_{13}\|^2-\|\alpha_{14}\|^2\\ 4H\rho_1-\sum_{\alpha\geq1}\rho_\alpha^2-\|\alpha_{23}\|^2-\|\alpha_{24}\|^2\\ 4H\sigma_1-\sum_{\alpha\geq1}\rho_\alpha^2-\|\alpha_{13}\|^2-\|\alpha_{23}\|^2\\ 4H\sigma_1-\sum_{\alpha\geq1}\rho_\alpha^2-\|\alpha_{14}\|^2-\|\alpha_{24}\|^2.\end{cases}$$

This implies that $\|\alpha_{13}\| = \|\alpha_{24}\|, \|\alpha_{23}\| = \|\alpha_{14}\|$ and then that $\rho_1 = \sigma_1 = H$, namely, the submanifold is pseudo-umbilical.

Being the submanifold pseudo-umbilical, we have from (36) and (38) that

$$\alpha_{11} = \alpha_{22},\ \alpha_{33} = \alpha_{44},\ \alpha_{12} = \alpha_{34} = 0,\ \alpha_{13} = \pm\alpha_{24},\ \alpha_{23} = \mp\alpha_{14}. \tag{39}$$

Thus the vector subspace $N_1(x) \subset N_f M(x)$ spanned by the second fundamental form at $x \in M^n$ satisfies $\dim N_1(x) \leq 4$ at any $x \in M^n$.

We claim that the mean curvature vector field is parallel in the normal bundle. Let U be an open subset of M^n where the subspaces $N_1(x)$ have constant dimension r, $1 \leq r \leq 4$, and hence $N_1|_U$ is a smooth vector subbundle of the normal bundle. Then let $\{e_i\}_{1\leq i\leq 4}$ be a local smooth orthonormal frame with respect to which the second fundamental form is as in (39). From the Codazzi equation

$$(\nabla^\perp_{e_i}\alpha)(e_j, e_k) = (\nabla^\perp_{e_j}\alpha)(e_i, e_k),$$

we obtain that $\nabla^\perp_{e_i}\alpha(e_j, e_k) \in N_1|_U$ for any $1 \leq i, j, k \leq 4$. Hence $N_1|_U$ is a parallel subbundle of the normal bundle and, consequently, $f|_U$ reduces its codimension, that is, it is a composition $f|_U = i \circ g$ where $i\colon \mathbb{Q}_c^{4+r} \to \mathbb{Q}_c^{4+m}$ is a totally geodesic inclusion and g is an isometric immersion into $\mathbb{Q}_c^{4+r}$.

Since the submanifold is pseudo-umbilical, from the Codazzi equation

$$(\nabla_X A_1)Y - (\nabla_Y A_1)X = A_{\nabla^\perp_X\xi_1}Y - A_{\nabla^\perp_Y\xi_1}X,\ \text{ for all } X, Y \in \mathfrak{X}(M),$$

we obtain that $\nabla^\perp_X\xi_1 \in (N_1|_U)^\perp$ for any $X \in \mathfrak{X}(M)$. Hence the mean curvature vector field is parallel in the normal bundle of f along any open subset where the dimension of the first normal space is constant. By continuity this is the case globally. Thus, it is an elementary fact that the submanifold decomposes as $f = j \circ g$, where $g\colon M^n \to \mathbb{Q}_{\tilde c}^{4+p}$ is a minimal submanifold and $j\colon \mathbb{Q}_{\tilde c}^{4+p} \to \mathbb{Q}_c^{4+m}$ is totally umbilical with $\tilde c = c + H^2$. It now follows from the result of Ejiri [3] that the submanifold is as in parts (i) or (ii) of the statement of the theorem. $\square$

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