2-KNOTS WITH THE SAME KNOT GROUP BUT DIFFERENT KNOT QUANDLES

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Dedicated to Professor Seiichi Kamada on the occasion of his 60th birthday

Abstract. We give a first example of a pair of 2-knots that share the same knot group but have different knot quandles. In fact, we give infinitely many triples of twist spins that share the same knot group but have mutually different knot quandles. To this end, we prove that the type of the knot quandle of an n-twist spin of a non-trivial knot is equal to n. By using the latter result, we also complete the classification of the twist spins with finite knot quandles by distinguishing the 2-knots in Inoue's list.

1. Introduction

A 1-*knot* is a circle embedded in the 3-sphere S^3, a 2-*knot* is a 2-sphere embedded in the 4-sphere S^4, and they are collectively referred to as *knot*s in this paper, where we work in the smooth category. There are two related topological invariants of an oriented knot $\mathcal{K}$ defined by using information of an exterior of $\mathcal{K}$. One is the *knot group* $G(\mathcal{K})$, the fundamental group of the exterior, and the other is the *knot quandle* $Q(\mathcal{K})$ introduced in [17, 20]. While the underlying set of $G(\mathcal{K})$ is the set of homotopy classes of loops in the exterior, that of $Q(\mathcal{K})$ is the set of homotopy classes of paths in the exterior. In fact, $G(\mathcal{K})$ is constructed functorially from $Q(\mathcal{K})$, and hence the former is determined from the latter; see the second paragraph of Section 2. In this paper, we investigate difference between these two invariants.

Joyce [17] and Matveev [20] independently proved that the knot quandle of an oriented 1-knot is determined from the peripheral system [7] of the 1-knot, and vice versa. On the other hand, it is a classical result due to Fox [7] that the square knot and the granny knot share the same knot group but have different peripheral systems. Thus these two results imply that there exists a pair of 1-knots that share the same knot group but have different knot quandles; see also [33].

The purpose of this paper is to give a first example of a pair of such 2-knots by using an algebraic property, called the *type*, of the knot quandle. By applying Zeeman's twist-spinning construction [34] to torus knots, Gordon [9, 10] gave infinitely many triples of oriented 2-knots that share the same knot group but have mutually homotopy inequivalent complements. Hence

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it is natural to expect that Gordon's triples would be good candidates. In fact, we compute in Theorem 3.3 the type of the knot quandle for a twist spin, and show in the proof of Theorem 3.1 that each of Gordon's triples have mutually different knot quandles.

This paper is organized as follows. In Section 2, we review the basics of quandles including the type of a quandle, the knot quandle of an oriented knot and a generalized Alexander quandle. We compute the type of the knot quandle for a twist spin (Theorem 3.3) and prove our main result (Theorem 3.1) in Section 3. As an application of Theorem 3.3, we give a classification of all twist spins with finite knot quandles in Section 4. Appendix A is devoted to proving Proposition 3.2, which is a key proposition for the proof of Theorem 3.3. Finally, in Appendix B, we compute the types of the knot quandles for a certain class of 2-knots called branched twist spins including all twist spins.

2. Preliminaries

A *quandle* X [17, 20] is a non-empty set with a binary operation $*$ that satisfies the following conditions:

- For any $x \in X$, we have $x * x = x$.
- For any $y \in X$, the map $S_y : X \to X; x \mapsto x * y$ is bijective.
- For any $x, y, z \in X$, we have $(x * y) * z = (x * z) * (y * z)$.

For any $x, y \in X$ and $n \in \mathbb{Z}$, we denote $S_y^n(x)$ by $x *^n y$. The *type* of the quandle X is defined by $\min\{n \in \mathbb{Z}_{>0} \mid x *^n y = x \text{ for any } x, y \in X\}$, where $\min \emptyset = \infty$. We denote the type of X by $\mathrm{type}(X)$. The *associated group* of the quandle X is the group generated by the elements of X subject to the relations $x * y = y^{-1}xy$ for all $x, y \in X$. We denote the associated group of X by $\mathrm{As}(X)$. The associated group $\mathrm{As}(X)$ acts on X from the right by $x \cdot y := x * y$ for any $x, y \in X$. A quandle X is *connected* if the action of $\mathrm{As}(X)$ on X is transitive. A map $f : X \to Y$ between quandles is a (*quandle*) *isomorphism* if $f(x * y) = f(x) * f(y)$ for any $x, y \in X$ and f is a bijection. When there is an isomorphism $f : X \to Y$, we say that X and Y are (*quandle*) *isomorphic*.

The ambient space of a knot is always assumed to be oriented, and a knot with a fixed orientation (of a circle or a 2-sphere) is called an *oriented knot*. Two oriented knots are said to be *equivalent* if there exists an orientation-preserving self-diffeomorphism of the ambient space carrying one to the other with respect to the orientations. Let $\mathcal{K}$ be an oriented n-knot for $n = 1, 2$. Let $N(\mathcal{K})$ be a tubular neighborhood of $\mathcal{K}$ and $E(\mathcal{K}) = S^{n+2} \setminus \mathrm{int} N(\mathcal{K})$ an exterior of $\mathcal{K}$. We fix a base point $p \in E(\mathcal{K})$. Let $Q(\mathcal{K}, p)$ be the set of homotopy classes of all paths in $E(\mathcal{K})$ from a point in $\partial E(\mathcal{K})$ to p. The set $Q(\mathcal{K}, p)$ is a quandle with an operation defined by $[\alpha] * [\beta] := [\alpha \cdot \beta^{-1} \cdot m_{\beta(0)} \cdot \beta]$, where $m_{\beta(0)}$ is a meridian loop starting from $\beta(0)$ and going along in the positive direction. We call $Q(\mathcal{K}, p)$ the *knot quandle* of $\mathcal{K}$. The isomorphism class of the knot quandle does not depend on the base point p. Thus, we denote the knot quandle simply by $Q(\mathcal{K})$. It is known that the knot quandle is an invariant for oriented knots, that is, if two oriented knots $\mathcal{K}$ and $\mathcal{K}'$ are equivalent, then their knot quandles $Q(\mathcal{K})$ and $Q(\mathcal{K}')$ are isomorphic; see [17, 20] and [6] for oriented 1-knots and 2-knots. We note that the associated group $\mathrm{As}(Q(\mathcal{K}))$ of $Q(\mathcal{K})$ is group isomorphic to the knot group $G(\mathcal{K}) := \pi_1(E(\mathcal{K}))$ and that $Q(\mathcal{K})$ is connected; see for example [25, Theorem 2.31] and [25, Lemma 2.27].

Let G be a group and $f : G \to G$ a group automorphism. We define the operation $*$ on G by $x * y := f(xy^{-1})y$. Then, $\mathrm{GAlex}(G, f) = (G, *)$ is a quandle, which is called the *generalized Alexander quandle*. Although it is difficult to compute the type of a quandle in general, the type of a generalized Alexander quandle can be computed as follows. This proposition holds the key to the proofs of the main results, Theorem 3.1 and 4.1.

Proposition 2.1. *We have* $\mathrm{type}(\mathrm{GAlex}(G, f)) = \mathrm{order}(f)$.

Proof. We first prove that $x *^i y = f^i(xy^{-1})y$ for any $x, y \in G$ and $i \in \mathbb{Z}_{\geq 0}$ by induction on i. The base case $i = 0$ is trivial, since $x *^0 y = x = f^0(xy^{-1})y$. By the induction hypothesis $x *^i y = f^i(xy^{-1})y$, it holds that

$$x *^{i+1} y = (x *^i y) * y = (f^i(xy^{-1})y) * y.$$

Then, by the definition of $\mathrm{GAlex}(G, f)$, we have

$$(f^i(xy^{-1})y) * y = f\big((f^i(xy^{-1})y)y^{-1}\big)y = f(f^i(xy^{-1}))y = f^{i+1}(xy^{-1})y,$$

and hence we have $x *^{i+1} y = f^{i+1}(xy^{-1})y$.

Let j be a positive integer. For any $x \in G$, $x *^j e$ is equal to $f^j(x)$ by the identity proved in the above, where e is the identity element of G. Thus, if $j < \mathrm{order}(f)$, there exists an element y such that $y *^j e$ is not equal to y. This implies that $\mathrm{type}(\mathrm{GAlex}(G, f)) \geq \mathrm{order}(f)$. In particular, if $\mathrm{order}(f) = \infty$, then we have $\mathrm{GAlex}(G, f) = \infty$. Hence, we assume that the order of f is finite.

For any $x, y \in G$, we have

$$x *^{\mathrm{order}(f)} y = f^{\mathrm{order}(f)}(xy^{-1})y = xy^{-1}y = x,$$

which implies that $\mathrm{type}(\mathrm{GAlex}(G, f)) \leq \mathrm{order}(f)$. By the above discussion, we see that $\mathrm{order}(f) = \mathrm{type}(\mathrm{GAlex}(G, f))$. □

3. Knot quandles of twist spins

In this section, we discuss the knot quandle of twist spins. The purpose of this section is to show the following theorem:

Theorem 3.1. *There exist infinitely many triples $\{F_1, F_2, F_3\}$ of oriented 2-knots such that*

1. *the groups $G(F_1), G(F_2)$ and $G(F_3)$ are mutually isomorphic, but*
2. *no two of the quandles $Q(F_1), Q(F_2)$ and $Q(F_2)$ are isomorphic.*

Moreover, the isomorphism class of $Q(F_i)$ is independent of the orientation of F_i for each i.

For an oriented knot $\mathcal{K}$, we denote by $-\mathcal{K}$ (or $\mathcal{K}!$, respectively) the oriented knot obtained from an oriented knot $\mathcal{K}$ by inverting the orientation (or taking the mirror image) of $\mathcal{K}$. The oriented knot $\mathcal{K}$ is said to be *invertible* (resp. (+)-*amphicheiral*) if $\mathcal{K}$ is equivalent to $-\mathcal{K}$ (resp. $\mathcal{K}!$). Since it is known in [17, 20] that $Q(-\mathcal{K}!)$ is isomorphic to $Q(\mathcal{K})$, if $\mathcal{K}$ is invertible or (+)-amphicheiral then the isomorphism class of $Q(\mathcal{K})$ is independent of the orientation of $\mathcal{K}$.

Let k be an oriented 1-knot in S^3. For an integer n, Zeeman [34] defined the oriented 2-knot $\tau^n(k)$, which is called the *n-twist spun k* or the *n-twist spin of k*. We note that the orientation of the 1-knot k naturally induces that of the 2-knot $\tau^n(k)$; see also [19] and [30, Section 5] for orientation conventions. It was shown in [34] that the n-twist spun k is a fibered 2-knot whose fiber is the once punctured M_k^n for $n > 0$, where M_k^n is the n-fold cyclic branched covering space of S^3 branched along k. In particular, the 1-twist spun k is trivial for any oriented 1-knot k, and the n-twist spin of the trivial 1-knot is also trivial for any $n > 0$. Litherland [19] showed that, for any oriented 1-knot k and any integer n, the $(-n)$-twist spun knot $\tau^{-n}(k)$ is equivalent to $\tau^n(-k!)$. Hence, we consider the case where n is greater than one and k is non-trivial, because we are interested in fibered 2-knots.

Let φ be the group automorphism of $\pi_1(M_k^n)$ induced by the monodromy of the fibered 2-knot complement $S^4\backslash\tau^n(k)$. We note that the monodromy of $S^4\backslash\tau^n(k)$ coincides with the canonical generator of the covering transformation group of M_k^n. Then we have the following proposition, whose proof will be given in Appendix A:

Proposition 3.2. *Let φ be the group automorphism of $\pi_1(M_k^n)$ induced by the monodromy of $S^4\backslash\tau^n(k)$. Then the order of φ is equal to n.*

Inoue [15, Theorem 3.1] showed that the knot quandle $Q(\tau^n(k))$ of $\tau^n(k)$ is isomorphic to $\mathrm{GAlex}(\pi_1(M_k^n), \varphi)$. Thus, we have:

Theorem 3.3. *The type of $Q(\tau^n(k))$ is equal to n.*

Proof. It follows from Proposition 2.1 and Proposition 3.2 that we have $\mathrm{type}(Q(\tau^n(k))) = \mathrm{type}\left(\mathrm{GAlex}(\pi_1(M_k^n), \varphi)\right) = \mathrm{order}(\varphi) = n$. □

Corollary 3.4. *Let k, k' be non-trivial oriented 1-knots and n, n' integers greater than 1. If $Q(\tau^n(k))$ is isomorphic to $Q(\tau^{n'}(k'))$, then we have $n = n'$.*

Proof. Since the type of quandles is an invariant of quandles, it holds that

$$n = \mathrm{type}(Q(\tau^n(k))) = \mathrm{type}(Q(\tau^{n'}(k'))) = n'$$

by Theorem 3.3 and the assumption. □

Proof of Theorem 3.1. Let p, q and r be coprime integers. Gordon [9] showed that $G(\tau^r(t_{p,q}))$ is isomorphic to $\pi_1(M^r_{t_{p,q}}) \times \mathbb{Z}$, where $t_{p,q}$ is the (p,q)-torus knot. It is known in [21] that $M^p_{t_{q,r}}, M^q_{t_{r,p}}$ and $M^r_{t_{p,q}}$ are homeomorphic, which implies that $G(\tau^p(t_{q,r})), G(\tau^q(t_{r,p}))$ and $G(\tau^r(t_{p,q}))$ are mutually group isomorphic. Thus, putting $F_1 := \tau^p(t_{q,r}), F_2 := \tau^q(t_{r,p})$ and

$F_3 := \tau^r(t_{p,q})$, we see that the 2-knots F_1, F_2 and F_3 satisfy the condition (1). By Theorem 3.3, we have $\mathrm{type}(Q(\tau^p(t_{q,r}))) = p$, $\mathrm{type}(Q(\tau^q(t_{r,p}))) = q$ and $\mathrm{type}(Q(\tau^r(t_{p,q}))) = r$. This implies that the 2-knots F_1, F_2 and F_3 also satisfy the condition (2). Since any torus knot in S^3 is invertible, any twist spun torus knot in S^4 is (+)-amphicheiral by [19]; see also Subsection 4.2. Since the isomorphism class of the knot quandle of a (+)-amphicheiral 2-knot is independent of its orientation, the last assertion holds. Varying a triple of comprime integers, we obtain infinitely many such triples of 2-knots. □

Remark 3.5. Although the square knot k and the granny knot k' share the same knot group but have different knot quandles, it is known that $\tau^0(k)$ and $\tau^0(k')$ are equivalent [11, 28], and hence they share the same knot quandle. Moreover it follows that $\tau^n(k)$ and $\tau^n(k')$ share the same knot quandle for any integer n, even if we do not know whether they are equivalent or not.[1]

4. Classification of twist spins whose knot quandles are finite

In this section, we complete the classification of the twist spins with the finite knot quandles by distinguishing the oriented 2-knots in Inoue's list [15] of such 2-knots. Let k be an oriented 1-knot and n an integer greater than 1. Since the knot quandle $Q(\tau^n(k))$ is isomorphic to the generalized Alexander quandle $\mathrm{GAlex}(\pi_1(M_k^n), \varphi)$, the knot quandle $Q(\tau^n(k))$ is finite if and only if the fundamental group $\pi_1(M_k^n)$ is finite. Then Inoue [15, Theorem 4.1] proved that the knot quandle $Q(\tau^n(k))$ is finite if and only if the pair (n, k) belongs to one of the following six sets; see also the list[2] after the proof of [15, Theorem 4.1].

- $S_1 = \{(2, k) \mid k$: a non-trivial 2-bridge knot$\}$.
- $S_2 = \{(2, k) \mid k$: a Montesinos knot $M(b; (2, \beta_1), (3, \beta_2), (3, \beta_3))\}$.
- $S_3 = \{(2, k) \mid k$: a Montesinos knot $M(b; (2, \beta_1), (3, \beta_2), (5, \beta_3))\}$.
- $S_4 = \{(3, k) \mid k$: the torus knot $t_{2,3}$ or the torus knot $t_{2,5}\}$.
- $S_5 = \{(4, k) \mid k$: the torus knot $t_{2,3}\}$.
- $S_6 = \{(5, k) \mid k$: the torus knot $t_{2,3}\}$.

4.1. Classification up to weak equivalence.

We say that two oriented knots $\mathcal{K}$ and $\mathcal{K}'$ are *weakly equivalent* if at least one of $\mathcal{K}$, $-\mathcal{K}$, $\mathcal{K}!$ and $-\mathcal{K}!$ is equivalent to $\mathcal{K}'$. We classify twist spins whose knot quandles are finite up to weak equivalence for oriented 2-knots.

Theorem 4.1. *Let k, k' be non-trivial oriented 1-knots and n, n' integers greater than 1. Suppose that $Q(\tau^n(k))$ and $Q(\tau^{n'}(k'))$ are finite. Then $\tau^n(k)$ is weakly equivalent to $\tau^{n'}(k')$ if and only if $n = n'$ and k is weakly equivalent to k'.*

[1] We can check that $\tau^n(k)$ and $\tau^n(k')$ are not equivalent for $n = 2, 3$ by using quandle cocycle invariants [3].

[2] His list contains a pair of an integer n and a 1-link k with two or more components such that $\pi_1(M_k^n)$ is finite. Here we exclude such pairs.

Proof. It is obvious that $\tau^n(k)$ and $\tau^{n'}(k')$ are weakly equivalent if $n = n'$ and k and k' are weakly equivalent. We discuss the converse. Suppose that $\tau^n(k)$ and $\tau^{n'}(k')$, whose knot quandles are finite, are weakly equivalent. Then $Q(\tau^n(k))$ and $Q(\tau^{n'}(k'))$ are isomorphic, since $\tau^n(k)$ and $\tau^{n'}(k')$ are (+)-amphicheiral by Proposition 4.3, which is proved in Subsection 4.2.

Case 1. $(n, k) \in S_1 \cup S_2 \cup S_3$.

By Corollary 3.4, we have $n' = n = 2$. Since $Q(\tau^{n'}(k'))$ is finite, the pair (n', k') is also an element of $S_1 \cup S_2 \cup S_3$. By [16], we see that k and k' are weakly equivalent.

Case 2. $(n, k) \in S_4$.

By Corollary 3.4, we have $n' = n = 3$. Since $Q(\tau^{n'}(k'))$ is finite, the pair (n', k') is also an element of S_4, and hence k and k' are weakly equivalent to $t_{2,3}$ or $t_{2,5}$. It is known that

$$|Q(\tau^3(t_{2,3}))| = |\pi_1(M^3_{t_{2,3}})| = 8 \quad \text{and} \quad |Q(\tau^3(t_{2,5}))| = |\pi_1(M^3_{t_{2,5}})| = 120,$$

which implies that $\tau^3(t_{2,3})$ and $\tau^3(t_{2,5})$ are not weakly equivalent. Thus, we see that k and k' are weakly equivalent.

Case 3. $(n, k) \in S_5 \cup S_6$.

By Corollary 3.4, we have $n = n' \in \{4, 5\}$. Since $Q(\tau^{n'}(k'))$ is finite, the pair (n', k') is also an element of $S_5 \cup S_6$, and hence k and k' are weakly equivalent to $t_{2,3}$. □

Remark 4.2. We note that most parts of Theorem 4.1 follows from

(i) the classical work due to Gordon [10, Section 3] that classifies $\tau^2(t_{3,5})$, $\tau^3(t_{5,2})$ and $\tau^5(t_{2,3})$,
(ii) the classical work due to Plotnick [27, Theorem 6.3] that classifies all 2-twist spun 2-bridge knots, and
(iii) the recent works by Kataoka [18], Jang–Kataoka–Miyakoshi [16] and Miyakoshi [22].

In fact, Kataoka [18] classified the twist spins corresponding to the elements in $S_1 \cup S_2 \cup S_3$, which consists of all 2-twist spun spherical Montesinos knots, by computing the orders of the knot quandles, and using dihedral group representations of the knot groups (and the classification by Plotnick [27] mentioned above). Then Jang–Kataoka–Miyakoshi [16] simplified its proof by using 3-colorings rather than dihedral group representations. With these in mind, Miyakoshi [22] almost proved our Theorem 4.1 by further computing the orders of the knot quandles for the twist spins corresponding to the elements in $S_4 \cup S_5 \cup S_6$, and using 3-colorings (and the classification by Gordon [9] mentioned above). The only remaining part was whether $\tau^2(t_{3,4})$ is equivalent to $\tau^4(t_{2,3})$ or not, where the torus knot $t_{3,4}$ is nothing but the Montesinos knot $M(0; (2, -1), (3, 1), (3, 1))$. Note that, for both 2-knots, the order of the knot quandle is 24 and the number of 3-colorings is 9.

4.2. **Classification up to equivalence.** We classify twist spins whose knot quandles are finite up to equivalence for oriented 2-knots. Litherland [19] showed that, for an oriented 1-knot k and an integer n, $\tau^n(-k)$ is equivalent to $(\tau^n(k))!$ and $\tau^n(k!)$ is equivalent to $-(\tau^n(k))$. Hence, if the oriented 1-knot k is invertible (resp. (+)-amphicheiral), then the oriented 2-knot $\tau^n(k)$ is (+)-amphicheiral (resp. invertible). Although it is not known whether or not the converse holds in general, it holds for twist spins whose knot quandles are finite. More precisely, we can show the following:

Proposition 4.3. *Suppose that $Q(\tau^n(k))$ is finite for a non-trivial oriented 1-knot k and an integer n greater than 1.*

(1) *k is invertible and $\tau^n(k)$ is (+)-amphicheiral.*
(2) *k is (+)-amphicheiral if and only if $\tau^n(k)$ is invertible.*

Proof. We prove the assertion (1). Since $Q(\tau^n(k))$ is finite, it follows from Subsection 4.1 that the 1-knot k is a 2-bridge knot or a Montesinos knot of type $(b; (2, \beta_1), (3, \beta_2), (\alpha_3, \beta_3))$, where $\alpha_3 = 3$ or 5, which implies that the oriented 1-knot k is invertible; see [2, Proposition 12.5 and Theorem 12.42]. Hence, by [19], the oriented 2-knot $\tau^n(k)$ is (+)-amphicheiral.

We prove the assertion (2). Since $Q(\tau^n(k))$ is finite, the pair (n, k) is an element of $S_1 \cup \cdots \cup S_6$. Gordon [13, Theorem 1(1)] showed that, when the pair (n, k) is an element of S_1, the oriented 1-knot k is (+)-amphicheiral if and only if the oriented 2-knot $\tau^n(k)$ is invertible. Now suppose that (n, k) belongs to $S_4 \cup S_5 \cup S_6$. Then the oriented 1-knot k is a torus knot and so the oriented 2-knot $\tau^n(k)$ is not invertible by Gordon [13, Theorem 1(2)]. Since torus knots are not (+)-amphicheiral, this implies the assertion (2) for $(n, k) \in S_4 \cup S_5 \cup S_6$.

Hence, it is sufficient to consider the case where the pair (n, k) is an element in $S_2 \cup S_3$. In this case, we have that $n = 2$ and the 1-knot k is a Montesinos knot of type $(b; (2, \beta_1), (3, \beta_2), (\alpha_3, \beta_3))$, where $\alpha_3 = 3$ or 5. Since the oriented 1-knot k is not (+)-amphicheiral by [2, Proposition 12.41], all need is to prove that the oriented 2-knot $\tau^2(k)$ is not invertible. To this end, note that the fiber M_k^2 of the fibered 2-knot $\tau^2(k)$ has a finite non-abelian fundamental group. Thus the fiber M_k^2 does not admit an orientation reversing self homotopy equivalence; see the proof of [24, Theorem 8.2]. Hence, by Gordon's result [13, Proposition 3], the oriented 2-knot $\tau^2(k)$ is not invertible. □

Combining Theorem 4.1 with Proposition 4.3, we have the following:

Theorem 4.4. *Let k, k' be non-trivial oriented 1-knots and n, n' integers greater than 1. Suppose that $Q(\tau^n(k))$ and $Q(\tau^{n'}(k'))$ are finite. Then $\tau^n(k)$ is equivalent to $\tau^{n'}(k')$ if and only if $n = n'$ and k is equivalent to k'.*

Appendix A. Proof of Proposition 3.2

In this appendix, we will show Proposition 3.2. For a non-trivial oriented 1-knot k in S^3 and an integer n greater than 1, let M_k^n be the n-fold cyclic branched covering space of S^3 branched along k, and φ the group automorphism of $\pi_1(M_k^n)$ induced by the monodromy of the complement $S^4 \backslash \tau^n(k)$ of the fibered 2-knot $\tau^n(k)$. Then, by [4, Proposition A.11.] and [27, Lemma 2.3], we have the following:

Lemma A.1. *If the universal covering space of M_k^n is homeomorphic to $\mathbb{R}^3$, then the order of φ is equals to n.*

Using the lemma above, we give a proof of Proposition 3.2, in which we call a 1-knot in S^3 a knot for simplicity.

Proof of Proposition 3.2. If $\mathrm{order}(\varphi) = 1$, the quandle $\mathrm{GAlex}(\pi_1(M_k^n), \varphi)$ is trivial, that is, $x * y = x$ for any $x, y \in \pi_1(M_k^n)$. Since $\mathrm{GAlex}(\pi_1(M_k^n), \varphi)$ is isomorphic to the knot quandle $Q(\tau^n(k))$, the quandle $\mathrm{GAlex}(\pi_1(M_k^n), \varphi)$ is connected. Since the cardinality of the connected trivial quandle is 1, the group $\pi_1(M_k^n)$ must be trivial. Then, by the Smith conjecture [23], the knot k must be the unknot. Hence, we have $\mathrm{order}(\varphi) \neq 1$.

Since φ is induced by the canonical generator of the covering transformation group of M_k^n, we see that φ^n is the identity map, and hence n is divisible by $\mathrm{order}(\varphi)$. Then, if n is a prime number, it follows from $\mathrm{order}(\varphi) \neq 1$ that $\mathrm{order}(\varphi) = n$. Hence, we assume that n is a composite number.

If k is the composite knot of k_1 and k_2, then the fundamental group $\pi_1(M_k^n)$ is the free product of $\pi_1(M_{k_1}^n)$ and $\pi_1(M_{k_2}^n)$. For $i = 1, 2$, let φ_i be the group automorphism induced by the canonical generator of the covering transformation group of $M_{k_i}^n$, and η_i the injective group homomorphism from $\pi_1(M_{k_i}^n)$ to $\pi_1(M_k^n)$. It holds that the restriction $\varphi|_{\pi_1(M_{k_i}^n)}$ coincides with the group homomorphism $\eta_i \circ \varphi_i : \pi_1(M_{k_i}^n) \to \pi_1(M_k^n)$ for $i \in \{1, 2\}$. Thus, if $\mathrm{order}(\varphi_1) = \mathrm{order}(\varphi_2) = n$, we see that $\mathrm{order}(\varphi) = n$. Hence it is sufficient to consider the case where k is a prime knot, that is, k is a torus knot, a hyperbolic knot or a satellite knot. We note that, by an argument similar to the proof of [29, Theorem 2], the branched covering space M_k^n of the prime knot k is irreducible. This fact will be used in Case 1 and Case 3 below.

<u>Case 1.</u> The knot k is a torus knot.

Since $S^3 \backslash k$ is a Seifert fibered space, M_k^n is also a Seifert fibered space. Thus, the universal covering space of M_k^n is homeomorphic to either $S^3, S^2 \times \mathbb{R}$ or $\mathbb{R}^3$ (see [31, Lemma 3.1]).

- Suppose that the universal covering space of M_k^n is $S^2 \times \mathbb{R}$. Then $\pi_2(M_k^n)$ is non-trivial. By the sphere theorem [26], M_k^n is not irreducible. This contradicts the fact that M_k^n is irreducible as mentioned above.
- Suppose that universal covering space of M_k^n is S^3. Since n is a composite number, we see that $k = t_{2,3}$ and $n = 4$. Then we have $\mathrm{order}(\varphi) \mid 4$. Moreover, it follows from $\mathrm{order}(\varphi) \neq 1$ that $\mathrm{order}(\varphi) = 2$ or 4. It is known that $Q(\tau^4(t_{2,3}))$ has the following presentation (see [30]):

$$\langle a, b \mid (b * a) * b = a, a *^4 b = a \rangle.$$

If order(φ) = 2, then type($Q(\tau^4(t_{2,3}))$) = 2 by Proposition 2.1. Hence, the following is also a presentation of $Q(\tau^4(t_{2,3}))$:

$$\langle a, b \mid (b * a) * b = a, a *^4 b = a, a *^2 b = a \rangle.$$

Using Tietze's moves, we see that $Q(\tau^4(t_{2,3}))$ has the following presentation:

$$\langle a, b \mid (b * a) * b = a, a *^2 b = a \rangle.$$

In fact, it follows from the relator $a *^2 b = a$ that

$$a *^4 b = (a *^2 b) *^2 b = a *^2 b = a.$$

This implies that $Q(\tau^4(t_{2,3}))$ is isomorphic to $Q(\tau^2(t_{2,3}))$. On the other hand, by Inoue's result, we have

$$|Q(\tau^4(t_{2,3}))| = |\pi_1(M^4_{t_{2,3}})| = 24 \text{ and } |Q(\tau^2(t_{2,3}))| = |\pi_1(M^2_{t_{2,3}})| = 3.$$

This is a contradiction. Hence we have order(φ) = 4.

- Suppose that the universal covering space of M^n_k is $\mathbb{R}^3$. By Lemma A.1, we have order(φ) = n.

Case 2. The knot k is a hyperbolic knot.

Since n is a composite number, n is greater than 3. By [5, Corollary 1.26], which is a consequence of the orbifold theorem (cf. [1]), M^n_k is a hyperbolic manifold, and therefore the universal covering space of M^n_k is homeomorphic to $\mathbb{R}^3$. By Lemma A.1, we have order(φ) = n.

Case 3. The knot k is a satellite knot.

Since the complement $S^3 \setminus k$ of k is irreducible by sphere theorem, the assumption implies that the knot k is sufficiently large in the sense of [12, Section 2]. Hence, we see that M^n_k is sufficiently large by [12, Theorem 1], where we use the obvious fact that S^3 does not contain a non-separating 2-sphere. Recall here that M^n_k is also irreducible as metioned above. Hence, by Waldhausen's result [32, Remark in Section 8], the universal covering space of M^n_k is homeomorphic to $\mathbb{R}^3$. By Lemma A.1, we have order(φ) = n. □

Appendix B. Branched twist spin

In this appendix, we compute the types of the knot quandles for a certain class of oriented 2-knots called *branched twist spin*s [14, Subsection 16.3] including all twist spins. We also give an alternative proof of [8, Theorem 1.1] quoted below as Theorem B.2, which followed from [8, Theorem 4.1] on knot group representations for branched twist spins.

Let $\tau^{n,s}(k)$ be the branched twist spin obtained from an oriented 1-knot k and a pair of coprime positive integers n and s with $n > 1$. Since the branched twist spin $\tau^{n,s}(k)$ is the branch set of the s-fold cyclic branched covering space of S^4 branched along the twist spin $\tau^n(k)$, it is a fibered 2-knot whose fiber is the once punctured n-fold cyclic branched covering space of S^3 branched along k and whose monodromy is the s-times composite of the canonical generator of the covering transformation group of

M^n_k. We note that $\tau^{n,1}(k)$ is nothing but $\tau^n(k)$. By [15, Theorem 3.1], the knot quandle $Q(\tau^{n,s}(k))$ is isomorphic to the generalized Alexander quandle $\mathrm{GAlex}(\pi_1(M^n_k), \varphi^s)$, where φ is the group automorphism of $\pi_1(M^n_k)$ as before. Then, for a non-trivial oriented 1-knot k, we have the following:

Theorem B.1. *The type of $Q(\tau^{n,s}(k))$ is equal to n.*

Proof. It follows from Proposition 3.2 and the coprimeness of n and s that $\mathrm{order}(\varphi^s) = \mathrm{order}(\varphi) = n$. Hence, by Proposition 2.1, the type of $Q(\tau^{n,s}(k))$ is equal to n. □

This theorem immediately implies the following, which was first proved by Fukuda [8] using dihedral group representations of the knot groups.

Theorem B.2 ([8, Theorem 1.1])**.** *Let k_1 and k_2 be non-trivial oriented 1-knots, and $\tau^{n_1,s_1}(k_1)$ and $\tau^{n_2,s_2}(k_2)$ be branched twist spins. If n_1 and n_2 are different, then $\tau^{n_1,s_1}(k_1)$ and $\tau^{n_2,s_2}(k_2)$ are not equivalent.*

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