Symplectic embeddings of balls into disk prequantization bundles

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Abstract

A prequantization bundle is a circle bundle over a symplectic surface with negative Euler class. A connection 1-form induces a natural contact form on such a bundle. The purpose of this paper is to study symplectic embeddings of balls into the associated disk bundles. To this end, we compute the ECH cobordism maps induced by these disk bundles. In addition, we compute the ECH spectrum of the prequantization bundles over the sphere and the torus.

1 Introduction and Main results

Let (X_1, ω_1) and (X_2, ω_2) be two compact symplectic manifolds of the same dimension. A symplectic embedding of (X_1, ω_1) into (X_2, ω_2) is a smooth embedding φ such that $\varphi^* \omega_2 = \omega_1$. When (X_1, ω_1) and (X_2, ω_2) are toric domains in $\mathbb{R}^4$, there are many results by M. Hutchings, D. Cristofaro-Gardiner, K. Choi, V. Frenkel, and V. G. B. Ramos [15, 17, 1, 4]. Beyond the toric cases, B. Ferreira and V. G. B. Ramos [9], along with A. Vicente [10], recently studied the embedding problems from balls to the unit disk cotangent bundle of the sphere or real projective space.

Let Y be a closed three-manifold equipped with a contact form λ such that $\lambda \wedge d\lambda > 0$. 0. To study the symplectic embedding phenomena in four dimension, M. Hutchings introduces a sequence of numerical invariants:

$$0 < c_1(Y, \lambda) \leq c_2(Y, \lambda) \leq \cdots \leq \infty.$$

associated to (Y, λ) which he calls the **ECH spectrum** [15]. The results mentioned in the first paragraph more or less rely on the computations of the ECH spectrum.

The purpose of this paper is to study symplectic embeddings a disjoint union of Darboux balls into a prequantization disk bundle. Furthermore, we give some computations of ECH spectrum for the prequantization bundles over the sphere or torus.

Roughly speaking, a prequantization bundle is a circle bundle over a symplectic surface with negative Euler class. A holomorphic curve in its symplecticization has certain S^1-symmetry due to the fibration structure. Based on this observation, J. Nelson and M. Weiler compute the embedded contact homology of the prequantization bundles [27] (based on D. Farriss PhD thesis [8]). Their computations play a crucial role in our proof.

The precise definition of the prequantization bundles is as follows. Let (Σ, ω_Σ) be a closed surface with a volume form such that $[\omega_\Sigma] \in H^2(\Sigma, \mathbb{R}) \cap H^2(\Sigma, \mathbb{Z})$ is integral. Let $\pi_E : E \to \Sigma$ be a complex line bundle with $c_1(E) = -[\omega_\Sigma]$. Then E is called a **prequantization line bundle**. Let $e := \langle c_1(E), [\Sigma] \rangle$ denote the degree of E.

Fix a Hermitian metric h and Hermitian connection 1-form A_∇ such that $\frac{i}{2\pi} F_{A_\nabla} = -\omega_\Sigma$. Let $\alpha_\nabla \in \Omega^1(E \setminus \Sigma, \mathbb{R})$ be the angular form. Under a unitary trivialization $U \times \mathbb{C}$, α_∇ is of the form $\frac{1}{2\pi}(d\theta - iA_\nabla|_U)$, where $d\theta$ is the angular form of $\mathbb{C}$ and $A_\nabla|_U$ is a $i\mathbb{R}$ valued 1-form. Therefore, we have $d\alpha_\nabla = \pi_E^* \omega_\Sigma$ over $E \setminus \Sigma$. A natural symplectic form on E is defined by

$$\Omega := \pi_E^* \omega_\Sigma + d(\rho^2 \alpha_\nabla),$$

where ρ is the radius coordinate of E defined by the metric h. Extend Ω over the zero section Σ by

$$d(\rho^2 \alpha_\nabla)|_{fiber} := (\text{area form of } \mathbb{C})/\pi \text{ and } d(\rho^2 \alpha_\nabla)(T\Sigma, \cdot) := 0.$$

Let $\pi : Y := \{\rho = 1\} \to \Sigma$ be the unit circle subbundle of E. Since

$$\Omega = 2\rho d\rho \wedge \alpha_\nabla + (\rho^2 + 1)d\alpha_\nabla$$

away from Σ, the Liouville vector field is $Z = \frac{1+\rho^2}{2\rho^2} \rho \partial_\rho$. Hence, Ω induces a contact form $\lambda = \Omega(Z, \cdot) = 2\alpha_\nabla$ on Y. The contact manifold (Y, λ) is called the **prequantization bundle** of (Σ, ω_Σ). Let $DE := \{\rho \le 1\}$ be the unit disk subbundle of E. Then (DE, Ω) forms a natural symplectic filling of (Y, λ).

Let $B^4(r) := \{(z_1, z_2) \in \mathbb{C}^2 \mid \pi|z_1|^2 + \pi|z_2|^2 \le r\}$ be a ball of radius $\sqrt{r/\pi}$ in $\mathbb{C}^2$. It is equipped with the standard symplectic form ω_{std} of $\mathbb{C}^2$. The **Gromov width** of a symplectic four-manifold (X, Ω_X) is defined by

$$c_{\text{Gr}}(X, \Omega_X) := \sup\{a \in \mathbb{R}_{\ge 0} \mid \exists \text{ embedding }, \varphi : B^4(a) \to X, \ \varphi^* \Omega_X = \omega_{\text{std}}\}.$$

Our main results are as follows.

Theorem 1. *Let $\varphi : \bigsqcup_{i=1}^k (B^4(r_i), \omega_{\text{std}}) \to (DE, \Omega)$ be a symplectic embedding. Then $\sum_{i=1}^k r_i \le k$. In particular, $c_{\text{Gr}}(DE, \Omega) \le 1$.*

We remark that for $k^2 < 2|e|$, the estimates in Theorem 1 are better than the volume constraint. Furthermore, they are also better than the constrains from the ECH spectrum (at least in the case that Σ is the sphere or the torus).

The proof of Theorem 1 relies on the computations of the ECH cobordism maps induced by (DE, Ω). From the construction, every fiber of $\pi : Y \to \Sigma$ is a Reeb orbit. Since the Reeb orbits with fixed length are not isolated, the contact form λ is degenerate. To define the ECH group and the cobordism map, we follow [27] to perform a perturbation on λ by a perfect Morse function $H : \Sigma \to \mathbb{R}$. Under this perturbation, the simple Reeb orbits with length less than L_ε correspond one-to-one to the critical points. Some suitable modifications also be made on Ω. The results are denoted by λ_ε and Ω_ε respectively. The details are given in Section 3.1.

Coefficient We use $\mathbb{F} := \mathbb{Z}/2\mathbb{Z}$-coefficient throughout this paper.

Theorem 2. *For any $0 < \varepsilon \ll 1$, let λ_ε and Ω_ε be the perturbations defined in Section 3.1, such that λ_ε is L_ε-nondegenerate. Fix $\Gamma \in \mathbb{Z}_{|e|}$. For any positive integer M such that $\Gamma = M \mod |e|$ and $2M < L_\epsilon$, there exists $A \in H_2(DE, Y, \mathbb{Z})$ such that $\partial A = \Gamma$ and the ECH cobordism map*

$$ECH^{L_\varepsilon}(DE, \Omega_\varepsilon, A) : ECH^{L_\varepsilon}(Y, \lambda_\varepsilon) \to \mathbb{F}$$

satisfies the following properties:

1. *$ECH^{L_\varepsilon}(DE, \Omega_\varepsilon, A)(e_-^M) = 1$, where e_- is the Reeb orbit corresponding to the minimum of H;*

2. *Let $\beta = e_+^{n_+} h_1^{n_1} \cdots h_{2g}^{n_{2g}} e_-^{n_-}$ be an ECH generator, where h_i are the Reeb orbits corresponding to the saddle points of H, and e_+ is the Reeb orbit corresponding to the maximum of H. Let $N = n_+ + n_1 + \cdots + n_{2g} + n_-$. If $N = M - d|e|$ for $d \geq 1$, then $ECH^{L_\varepsilon}(DE, \Omega_\varepsilon, A)([\beta]) = 0$.*

Using Nelson and Weiler's computations on ECH, spectrality property of ECH spectrum, and the finiteness of ECH spectrum, we obtain the precise values of ECH spectrum in the following special cases.

Theorem 3. *Suppose that Σ is the two-sphere. Then for any $k \geq 0$, the k-th value on the ECH spectrum of (Y, λ) is*

$$c_k(Y, \lambda) = 2d|e|,$$

where d is the unique nonnegative integer such that

$$2d + d|e|(d - 1) \leq 2k \leq 2d + d|e|(d + 1).$$

Theorem 4. *Suppose that Σ is the two-torus. Then we have the following statements.*

- *If $k \neq \frac{n(n+1)|e|}{2}$ for any $n \in \mathbb{Z}_{\geq 1}$, then*

$$c_k(Y, \lambda) = 2 \lfloor \sqrt{\frac{2k}{|e|} + \frac{1}{4}} + \frac{1}{2} \rfloor |e|,$$

  where $\lfloor x \rfloor$ denote the maximal integer that is less than or equal to x;

- $c_{|e|}(Y, \lambda) = 4|e|.$

Remark 1.1. *If we consider $Y = \{\rho = c\}$, then the induced contact form on Y is $\lambda_c = (1 + c^2)\alpha_\nabla = \frac{1+c^2}{2}\lambda$. By the conformality property of the ECH spectrum, i.e., $c_k(Y, \kappa\lambda) = \kappa c_k(Y, \lambda)$ for any positive constant κ (see Remark 3.2 of [15]), we have*

$$c_k(Y, \lambda_c) = \frac{1+c^2}{2} c_k(Y, \lambda).$$

Remark 1.2. *When Σ is the sphere, there is a standard method to compute the U map on $ECH^{L_\varepsilon}(Y, \lambda_\varepsilon)$ which can be used to compute $c_k(Y, \lambda_\varepsilon)$. The methods are given by Hutchings when he computes the ECH of the three-sphere (Proposition 4.1 of [16]). In the results of Ferreira and Ramos [9], they use these arguments to compute the ECH spectrum of the cotangent circle bundle of the sphere.*

We briefly sketch the argument in the case of prequantization bundles. By Nelson and Weiler's index computations (Proposition 3.5 of [27]), one can show that the holomorphic curves $\mathcal{C}$ contributing to the U map are holomorphic cylinders. The degree of these cylinders is either zero or one. If the degree of $\mathcal{C}$ is zero, then it corresponds to an index 2 Morse flow line on $\mathbb{S}^2$. If the degree of $\mathcal{C}$ is one, then under the natural holomorphic structure of E, $\mathcal{C}$ is a meromorphic section of E with poles and zeros. Combining these facts, one could show that

1. $U(e_+^i e_-^j) = (e_+^{i-1} e_-^{j+1})$ *(counting degree zero holomorphic cylinders);*

2. $U(e_-^j) = e_+^{j-|e|}$ *(counting degree one holomorphic cylinders).*

Here $e_\pm$ are defined in Section 3.1.

Remark 1.3. *The computations of ECH cobordism maps can be used as an auxiliary tool to study the U map on $ECH(Y, \lambda)$. Here is an example: Suppose that $e = -1$ and Σ is the torus. From the index computations of Nelson and Weiler and Lemma 3.13, we obtain*

$$U \begin{pmatrix} e_- e_+ \\ h_1 h_2 \end{pmatrix} = \begin{pmatrix} a & 1 \\ b & c \end{pmatrix} \begin{pmatrix} e_+ \\ e_-^2 \end{pmatrix},$$

where $a, b, c \in \mathbb{F}$ are some unknown numbers. By Theorem 2, we have $c = ECH^L(DE, \Omega_\varepsilon, A)(U[h_1 h_2])$. The methods in this paper can be used to show that the J-holomorphic currents $\mathcal{C}$ contributing to $ECH^L(DE, \Omega_\varepsilon, A)(U[h_1 h_2])$ are embedded J-holomorphic curves passing through a marked point. By Lemma 3.14, there exists a J-holomorphic plane $\mathcal{C}'$ passing through the same marked point. However, due to the homological reasons ($[\mathcal{C}] = Z_{h_1 h_2}$ and $[\mathcal{C}'] = Z_{e_+}$), these two curves should be disjoint. As a result, such a $\mathcal{C}$ cannot exist and $c = 0$. The advantage of the above argument is to avoid using the domain dependent almost complex structures as in [27] and all the methods are classical.

For computing the U map, Farris [8], Nelson and Weiler (Page 11 of [27]) propose a heuristic idea about this. They suggest that the curves contributing to the U map should be equivalent to the meromorphic multisections of E. We believe that to compute $ECH^L(DE, \Omega_\varepsilon, A) \circ U^k$ for a general class A is also the same as counting the meromorphic multisections of E. Because the cobordism maps are defined by counting curves in E, their perspective appears more natural in the cobordism setting.

Now we sketch the idea of the proofs of the main results.

Idea of the proof   The methods of computing $ECH^L(DE, \Omega_\varepsilon, A)([\alpha_k])$ are more or less the same as [3]. Choose an almost complex structure J such that the fiber of E over the minimum of H is J-holomorphic. For some fixed $A \in H_2(DE, Y, \mathbb{Z})$, we show that a cover of the fiber is the only holomorphic current with $I = 0$ with relative class A. The result then follows from the correspondence between solutions to the Seiberg-Witten equations and holomorphic curves ([12]). The same argument can also be used to compute $ECH^{L_\varepsilon}(DE, \Omega_\varepsilon, A)(U^k[e_+^k])$. The non-vanishing of $ECH^{L_\varepsilon}(DE, \Omega_\varepsilon, A)(U^k[e_+^k])$ yields J-holomorphic curves passing through the centers of $\varphi(\sqcup_{i=1}^k (B^4(r_i), \omega_{\mathrm{std}}))$. Then the standard monotonicity lemma gives the result of Theorem 1.

When the base manifold is the sphere or the torus, thanks to the computations from P. Ozsváth, Z. Szabó [28], and K. Park [29], the existence of U sequences is known. Combining these results with the "volume property" of the ECH spectrum [2], we obtain the finiteness of $c_k(Y, \lambda)$. Using spectrality, we derive Theorem 3 and the first item of Theorem 4. To obtain the second item of Theorem 4, we also need to compute $\langle U^k e_+^k, e_-^k \rangle$.

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2 Preliminaries

In this section, we give a quick review of the embedded contact homology (abbreviated as ECH). For more details, please refer to [16].

Let Y be a closed, contact 3-dimensional manifold equipped with a nondegenerate contact form λ. The contact structure of Y is denoted by $\xi := \ker \lambda$. The **Reeb vector field** R of (Y, λ) is characterized by conditions $\lambda(R) = 1$ and $d\lambda(R, \cdot) = 0$. A **Reeb orbit** is a smooth map $\gamma : \mathbb{R}_\tau / T\mathbb{Z} \to Y$ satisfying the ODE $\partial_\tau \gamma = R \circ \gamma$ for some $T > 0$.

The linearized Reeb flow for time T defines a symplectic linear map

$$P_\gamma : (\xi_{\gamma(0)}, d\lambda) \to (\xi_{\gamma(T)}, d\lambda).$$

The Reeb orbit γ is called **nondegenerate** if 1 is not an eigenvalue of P_γ. A nondegenerate Reeb orbit γ is called **elliptic** if the eigenvalues of P_γ are on the unit circle, **positive hyperbolic** if the eigenvalues are real positive numbers, and **negative hyperbolic** if the eigenvalues are real negative numbers. Given $L \in \mathbb{R}$, the contact form λ is called L-**nondegenerate** if all Reeb orbits with action less than L are nondegenerate. We call λ nondegenerate if all Reeb orbits are nondegenerate.

An **orbit set** $\alpha = \{(\alpha_i, m_i)\}$ is a finite set of Reeb orbits together with multiplicities, where α_i are distinct, nondegenerate, irreducible embedded Reeb orbits and m_i are positive integers. An orbit set α is called an **ECH generator** if $m_i = 1$ whenever α_i is a hyperbolic orbit. Define the **action** of an orbit set α by

$$\mathcal{A}_\lambda(\alpha) := \int_\alpha \lambda = \sum_i m_i \int_{\alpha_i} \lambda.$$

Note that the action $\int_\gamma \lambda$ is just the period T in the definition of Reeb orbits.

Notation. In the rest of the paper, we often write an orbit set using multiplicative notation $\alpha = \Pi_i \alpha_i^{m_i}$ instead.

The following definition will be used in the computations of the cobordism maps. In fact, the elliptic Reeb orbits considered in our cases are one of the following two types of Reeb orbits.

Definition 2.1. *(see [17] Definition 4.1) Fix $L > 0$. Let γ be an embedded elliptic orbit with action $\mathcal{A}_\lambda(\gamma) < L$.*

- γ is called *L-positive elliptic* if the rotation number θ is in $(0, \frac{\mathcal{A}_\lambda(\gamma)}{L})$ mod 1.

- γ is called *L-negative elliptic* if the rotation number θ is in $(-\frac{\mathcal{A}_\lambda(\gamma)}{L}, 0)$ mod 1.

The ECH index   Fix $\Gamma \in H_1(Y, \mathbb{Z})$. Given orbit sets $\alpha = \Pi_i \alpha_i^{m_i}$ and $\beta = \Pi_j \beta_j^{n_j}$ on Y with $[\alpha] = [\beta] = \Gamma$, let $H_2(Y, \alpha, \beta)$ be the set of 2-chains Z such that $\partial Z = \sum_i m_i \alpha_i - \sum_j n_j \beta_j$, modulo boundaries of 3-chains. An element in $H_2(Y, \alpha, \beta)$ is called **a relative homology class**. Note that the set $H_2(Y, \alpha, \beta)$ is an affine space over $H_2(Y, \mathbb{Z})$.

Given $Z \in H_2(Y, \alpha, \beta)$ and trivializations τ of $\xi|_\alpha$ and $\xi|_\beta$, the ECH index is defined by

$$I(\alpha, \beta, Z) := c_\tau(Z) + Q_\tau(Z) + \sum_i \sum_{p=1}^{m_i} CZ_\tau(\alpha_i^p) - \sum_j \sum_{q=1}^{n_j} CZ_\tau(\beta_j^q),$$

where $c_\tau(Z)$ and $Q_\tau(Z)$ are respectively the relative Chern number and the relative self-intersection number (see [16] and [14]), and CZ_τ is the Conley-Zehnder index. The ECH index I only depends on orbit sets α, β and the relative homology class Z.

J-Holomorphic currents   An almost complex structure on $(\mathbb{R} \times Y, d(e^s \lambda))$ is called **admissible** if J is $\mathbb{R}$-invariant, $J(\partial_s) = R$, $J(\xi) = \xi$ and $J|_\xi$ is $d\lambda$-compatible. We denote set of admissible almost complex structures by $\mathcal{J}(Y, \lambda)$.

A *J-holomorphic current* from α to β is a formal sum $\mathcal{C} = \sum_a d_a C_a$, where C_a are distinct irreducible simple holomorphic curves, the d_a are positive integers, $\mathcal{C}$ is asymptotic to α as a current as $s \to \infty$ and is asymptotic to β as a current as $s \to -\infty$. Fix $Z \in H_2(Y, \alpha, \beta)$. Let $\mathcal{M}^J(\alpha, \beta, Z)$ denote the moduli space of holomorphic currents with relative homology class Z.

Let C be J-holomorphic curve in $\mathbb{R} \times Y$ whose positive ends are asymptotic to $\alpha = \Pi \alpha_i^{m_i}$ and negative ends are asymptotic to $\beta = \Pi_j \beta_j^{n_j}$. For each i, let k_i denotes the number of ends of C at α_i, and let $\{p_{ia}\}_{a=1}^{k_i}$ denote their multiplicities. Likewise, for each j, let l_j denote the number of ends of u at β_j, and let $\{q_{jb}\}_{b=1}^{l_j}$ denote their multiplicities. The set of numbers $\{p_{ia}\}_{a=1}^{k_i}$ modulo order is called the **partition** of C at α_i. The **Fredholm index** of u is defined by

$$\mathrm{ind}(C) := -\chi(C) + 2c_\tau(\xi) + \sum_i \sum_{a=1}^{k_i} CZ_\tau(\alpha_i^{p_{ia}}) - \sum_j \sum_{b=1}^{l_j} CZ_\tau(\beta_j^{q_{jb}}).$$

By [13, 14] or Section 3.4 of [16], if C is a simple holomorphic curve, then the ECH index and Fredholm index satisfy the following **ECH index inequality**

$$I(C) \geq \mathrm{ind}(C) + 2\delta(C), \tag{2.1}$$

where $\delta(C)$ is the count of singularities of C with positive integer weights. Moreover, the equality holds only if C is embedded and C satisfies the **ECH partition conditions**. The ECH partition conditions mean that the partition of C at each α_i satisfies a certain special form. The general definition of such a form is quite complicated. Here we only present the examples that will be considered in our proof. Suppose that C has no negative ends and has positive ends at covers of an Reeb orbit γ with total multiplicities m. If C satisfies the ECH partition conditions, then the partition at γ is

- $(1, ..., 1)$ if γ is L-positive elliptic and m satisfies $\mathcal{A}_\lambda(\gamma^m) < L$;

- (m) if γ is L-negative elliptic and m satisfies $\mathcal{A}_\lambda(\gamma^m) < L$;

- $(1, ..., 1)$ γ is positive hyperbolic.

ECH group Fix a class $\Gamma \in H_1(Y, \mathbb{Z})$. The chain group $ECC(Y, \lambda, \Gamma)$ is a free module generated by the ECH generators with homology class Γ. Fix a generic $J \in \mathcal{J}(Y, \lambda)$. The differential is defined by

$$\langle \partial \alpha, \beta \rangle := \sum_{Z \in H_2(Y, \alpha, \beta), I(Z)=1} \# \left(\mathcal{M}^J(\alpha, \beta, Z) / \mathbb{R} \right).$$

Then $ECH(Y, \lambda, \Gamma)$ is the homology of this chain complex $(ECC(Y, \lambda, \Gamma), \partial)$.

Given $L > 0$, define $ECC^L(Y, \lambda, \Gamma)$ to be a submodule generated by the ECH generators with $\mathcal{A}_\lambda < L$. Note that the differential ∂ decreases the action. Therefore, $ECC^L(Y, \lambda, \Gamma)$ is a subcomplex and its homology is well defined, denoted by $ECH^L(Y, \lambda, \Gamma)$. The group $ECH^L(Y, \lambda, \Gamma)$ is called **filtered ECH**. The inclusion induces a homomorphism

$$i_L : ECH^L(Y, \lambda, \Gamma) \rightarrow ECH(Y, \lambda, \Gamma).$$

ECH spectrum ECH also equips with a homomorphism

$$U : ECH(Y, \lambda, \Gamma) \rightarrow ECH(Y, \lambda, \Gamma),$$

called the **U map**. It is defined by counting $I = 2$ holomorphic currents passing through a fixed point (see (3.18)).

Suppose that $c_1(\xi)$ is torsion. Then $I(\alpha, \beta, Z)$ is independent of the choice of Z for null-homologous orbit sets α and β. Hence, we write it as $I(\alpha, \beta)$. For any orbit set α with $[\alpha] = 0$, define its **grading** by

$$\mathrm{gr}(\alpha) := I(\alpha, \emptyset). \tag{2.2}$$

This gives a $\mathbb{Z}$ grading on $ECH(Y, \lambda, 0)$. Note that the U map is a degree -2 map with respect to this grading.

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Remark 2.1. *In the case that Y is a prequantization bundle, by Lemma 3.11 of [27], $c_1(\xi)$ is torsion. Therefore, the grading (2.2) on $ECH_*(Y, \lambda, 0)$ is well defined.*

There is a canonical element $[\emptyset] \in ECH(Y, \lambda, 0)$ which is represented by the empty orbit set. The class $[\emptyset]$ is called the **contact invariant.** We remark that $[\emptyset] \neq 0$ if (Y, λ) admits a symplectic filling. So the contact invariant of the prequantization bundle $\pi : Y \to \Sigma$ is nonzero.

Assume that λ is nondegenerate. For $k \in \mathbb{Z}_{\geq 1}$, the **$k$-th value on ECH spectrum** is defined by

$$c_k(Y, \lambda) := \inf\{L \in \mathbb{R} | \exists \sigma \in ECH^L(Y, \lambda, 0) \text{ such that } U^k(\sigma) = [\emptyset]\}.$$

If λ is degenerate, define $c_k(Y, \lambda) := \lim_{n \to \infty} c_k(Y, f_n \lambda)$, where $f_n : Y \to \mathbb{R}_{>0}$ is a sequence of smooth functions such that $f_n \lambda$ is nondegenerate and f_n converges to 1 in C^0 topology.

Cobordism maps A (strong) symplectic cobordism from (Y_+, λ_+) to (Y_-, λ_-) is a symplectic manifold (X, Ω_X) such that $\partial X = Y_+ \bigsqcup (-Y_-)$ and $\Omega_X(V_\pm, \cdot) = \lambda_\pm$ along $Y_\pm$, where $V_\pm$ are Liouville vector fields defined near $Y_\pm$. Using $V_\pm$, we can identify collar neighborhoods of $Y_\pm$ symplectically as $((-\delta, 0] \times Y_+, d(e^s \lambda_+))$ and $([0, \delta) \times Y_-, d(e^s \lambda_-))$. We glue (X, Ω_X) with cylindrical ends $([0, \infty) \times Y_+, d(e^s \lambda_+))$ and $(-\infty, 0] \times Y_-, d(e^s \lambda_-))$ along $Y_\pm$ to obtain a symplectic manifold $(\widehat{X}, \Omega_X)$, which we call the symplectic completion (X, Ω_X). The ECH index, Fredholm index and holomorphic currents can be defined similarly in the cobordism setting (see [14]). Furthermore, the ECH index inequality (2.1) still holds.

Fix a relative class $A \in H_2(X, \partial X, \mathbb{Z})$ such that $\partial A = \Gamma_+ - \Gamma_-$. Suppose that $\lambda_\pm$ are $\max\{L, L + \rho(A)\}$-nondegenerate. Here $\rho : H_2(X, \partial X, \mathbb{Z}) \to \mathbb{R}$ is a homomorphism defined by $\rho(A) := \int_A \Omega_X - \int_{\Gamma_+} \lambda_+ + \int_{\Gamma_-} \lambda_-$. Hutchings and Taubes define a canonical homomorphism

$$ECH^L(X, \Omega_X, A) : ECH^L(Y_+, \lambda_+, \Gamma_+) \to ECH^{L + \rho(A)}(Y_-, \lambda_-, \Gamma_-)$$

which is called a **cobordism map** [18]. If $\lambda_\pm$ are nondegenerate, then we can take $L \to \infty$ and get a cobordism map on the whole ECH groups

$$ECH(X, \Omega_X, A) : ECH(Y_+, \lambda_+, \Gamma_+) \to ECH(Y_-, \lambda_-, \Gamma_-).$$

The cobordism map $ECH^L(X, \Omega_X, A)$ is defined by counting the gauge classes of solutions to the Seiberg-Witten equations. We will not provide any details about the Seiberg-Witten theory here, and we refer readers to the book of P. Kronheimer and T. Mrowka [20]. Hutchings and Taubes show that $ECH^L(X, \Omega_X, A)$ satisfies the **holomorphic curve axioms** (see Theorem 1.9 of [18]). Roughly speaking, it means that if

the cobordism map is nonvanishing, then there exists a holomorphic current. In some special cases, C. Gerig enhances the holomorphic curves axioms. He shows that there is a one-to-one correspondence between the holomorphic currents and the gauge classes of solutions to the Seiberg-Witten equations [11, 12]. In other words, the cobordism map is actually defined by counting holomorphic curves. We will show that this is the case in our situations.

3 Computations of the cobordism maps

In this section, we prove the Theorem 2.

3.1 Perturbations

Before we go ahead, we need to clarify the perturbations made on the contact form λ and the symplectic form Ω.

Morse-Bott perturbations Note that the contact form λ is Morse-Bott. The Reeb orbits are iterations of the fibers of $\pi : Y \to \Sigma$. Following Farris, Nelson and Weiler's approach [8, 27], we perturb the contact form by a perfect Morse function $H : \Sigma \to \mathbb{R}$. More precisely, define

$$\lambda_\varepsilon := (1 + \varepsilon \pi^* H)\lambda,$$

where $0 < \varepsilon \ll 1$ is a small fixed number.

Let e_-, e_+ and $\{h_i\}_{i=1}^{2g}$ denote the fiber over the minimum, maximum and saddle points of H respectively. These are simple Reeb orbits of λ_ε. Moreover, $e_\pm$ are elliptic orbits and $\{h_i\}_{i=1}^{2g}$ are positive hyperbolic orbits.

For any $0 < \varepsilon \ll 1$, there exists a constant L_ε such that λ_ε is L_ε-nondegenerate and the covers of $e_\pm$, and $\{h_i\}_{i=1}^{2g}$ are the only Reeb orbits of (Y, λ_ε) with action less than L_ε. According to the computations in Lemma 3.9 of [27], e_+ is L_ε-positive and e_- is L_ε-negative. We remark that $L_\varepsilon \to \infty$ as $\varepsilon \to 0$. For an orbit set $\alpha = e_+^{m_+} h_1^{m_1} \cdots h_{2g}^{m_{2g}} e_-^{m_-}$, its action is given by

$$\mathcal{A}_{\lambda_\varepsilon}(\alpha) = 2M + \varepsilon \left(m_+ \pi^* H(e_+) + m_- \pi^* H(e_-) + \sum_{i=1}^{2g} m_i \pi^* H(h_i) \right),$$

where $M = m_- + m_+ + \sum_{i=1}^{2g} m_i$.

By Lemma 3.7 in [27], we have $H_1(Y, \mathbb{Z}) = \mathbb{Z}^{2g} \oplus \mathbb{Z}_{|e|}$. The homology class of each fiber of $\pi : Y \to \Sigma$ is 1 mod $|e|$ in the $\mathbb{Z}_{|e|}$ summand of $H_1(Y, \mathbb{Z})$. Therefore, $M = d|e| + \Gamma$ for some integer $d \geq 0$ and $\Gamma = [\alpha] \in H_1(Y, \mathbb{Z})$.

Symplectic completion of (DE, Ω) We regard E as a symplectic completion of (DE, Ω) by the following way. Define $r := \frac{1}{2}(1 + \rho^2)$. Then we have a symplectomorphism

$$(E \setminus \Sigma, \Omega) \cong (\mathbb{R}_{r > \frac{1}{2}} \times Y, d(r\lambda)). \tag{3.3}$$

Sometimes we identify the conical end $(\mathbb{R}_r \times Y, d(r\lambda))$ with the cylindrical end $(\mathbb{R}_s \times Y, d(e^s \lambda))$ via changing coordinate $r = e^s$.

Let $\varepsilon(r)$ be a nondecreasing cutoff function such that $\varepsilon(r) = \varepsilon \ll 1$ when $r \geq \frac{4}{5}$ and $\varepsilon(r) = 0$ when $r \leq \frac{3}{4}$. Define $\lambda_{\varepsilon(r)} := (1 + \varepsilon(r)\pi^* H)\lambda$. Under the identification (3.3), we define

$$\Omega_\varepsilon = \begin{cases} \Omega, & r < \frac{3}{4} \\ d(r\lambda_{\varepsilon(r)}), & r \geq \frac{3}{4}. \end{cases} \tag{3.4}$$

Note that $\Omega_\varepsilon = d(e^s \lambda_\varepsilon)$ over the cylindrical end.

Almost complex structures in E An Ω_ε-compatible almost complex structure J is **cobordism admissible** if $J = J_+$ for some $J_+ \in \mathcal{J}(Y, \lambda_\varepsilon)$ over the cylindrical end. We choose J such that

J.1 In the neighbourhood $U_z \times \mathbb{C}_w$ of e_-. $J(r\partial_r) = f(r)\partial_\theta$ and $J(\partial_z) = i\partial_z$ along the fiber $\{0\} \times \mathbb{C}$, where $f(r)$ is a positive function such that $f(r) = 1$ when $r \geq e^{-\epsilon}$ and $f(r) = \frac{r}{2r-1}$ when r is close to $\frac{1}{2}$. The latter assumption on f is equivalent to $J(\rho\partial_\rho) = \partial_\theta$. This implies that J is well defined on the whole $\{0\} \times \mathbb{C}$.

J.2 The zero section Σ is J-holomorphic.

The choice of J satisfying **J.1** and **J.2** is always feasible. The advantage of **J.2** is that $\mathcal{C} \cdot \Sigma \geq 0$ for any holomorphic current $\mathcal{C}$ without closed components. We use this property in Lemma 3.5 and Lemma 3.6.

By **J.1**, $C_{e_-} = \{0\} \times \mathbb{C}$ is holomorphic because $J(TC_{e_-}) \subset TC_{e_-}$. Moreover, C_{e_-} is asymptotic to e_- under the identification (3.3). We remark that C_{e_-} is Fredholm regular for any J. This follows directly from C. Wendl's automatic transversality theorem [36] and the index computation in Lemma 3.2. The fiber C_{e_-} plays the same role as the horizontal section in [3].

An almost complex structure J is called **generic** if all simple holomorphic curves are Fredholm regular except for the closed holomorphic curves. We assume that J is generic, unless stated otherwise.

3.2   Moduli space of holomorphic currents

Computing the ECH index The first task of studying the holomorphic currents is to compute their index. The computations here follow the similar argument in [3],

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where the author computes ECH index of the relative homology classes in an elementary Lefschetz fibration. This approach coincides with Nelson and Weiler's methods (see Remark 3.1).

Let $\alpha = e_+^{m_+} h_1^{m_1} \cdots h_{2g}^{m_{2g}} e_-^{m_-}$ be an orbit set. Let $M := m_+ + m_1 + \cdots + m_{2g} + m_-$. Let $H_2(DE, \alpha)$ denote the set of relative homology classes. It is an affine space over $H_2(DE, \mathbb{Z})$. Let $C_{e_\pm}$ and C_{h_i} denote the fibers over the critical points corresponding to $e_\pm$ and h_i. For α, we define a relative homology class Z_α represented by

$$m_+ C_{e_+} + \sum_{i=1}^{2g} m_i C_{h_i} + m_- C_{e_-}.$$

Since $H_2(DE, \mathbb{Z}) \cong H_2(\Sigma, \mathbb{Z})$ is generated by $[\Sigma]$, any other relative homology class can be written as $Z_\alpha + d[\Sigma]$ for some integer d.

Fix a relative homology class $A \in H_2(DE, Y, \mathbb{Z})$. Consider a subset $\{Z \in H_2(DE, \alpha) : [Z] = A\}$. Note that it is an affine space of $\ker j_*$, where j_* is the map in the following exact sequence

$$\ldots \to H_2(Y, \mathbb{Z}) \xrightarrow{i_*} H_2(DE, \mathbb{Z}) \xrightarrow{j_*} H_2(DE, Y, \mathbb{Z}) \xrightarrow{\partial_*} H_1(Y, \mathbb{Z}) \to \ldots.$$

By Lemma 3.7 in [27], $H_2(Y, \mathbb{Z})$ is spanned by the classes represented by $\pi^{-1}(\eta)$, where η is a simple closed curve in Σ. Note that $\pi^{-1}(\eta)$ is the boundary of the surface $\pi_E^{-1}(\eta) \cap DE$ in DE. Then, i_* is the zero map and $\ker j_* = 0$. Therefore, a relative class $A \in H_2(DE, Y, \mathbb{Z})$ and an orbit set α determine a unique relative homology class $Z_{\alpha, A} \in H_2(DE, \alpha)$.

For a relative homology class $Z \in H_2(DE, \alpha)$, we obtain the following formula for the ECH index.

Lemma 3.1. *Given an orbit set $\alpha = e_+^{m_+} h_1^{m_1} \cdots h_{2g}^{m_{2g}} e_-^{m_-}$, the ECH index of a relative class $Z_\alpha + d[\Sigma] \in H_2(DE, \alpha)$ is*

$$I(Z_\alpha + d[\Sigma]) = M + m_+ - m_- + 2dM + d^2 e + de + d\chi(\Sigma).$$

Proof. Let p be a critical point of H, $\gamma_p = \pi^{-1}(p)$ and $C_p = \pi_E^{-1}(p)$. We fix a constant trivialization as in [27]. More precisely, a trivialization of $T_p\Sigma$ can be lifted to a trivialization of $\xi|_{\gamma_p(t)}$. Under this trivialization, by Lemma 3.9 of [27], we have

$$CZ_\tau(\gamma_p^k) = \mathrm{ind}_p H - 1,$$

where $\mathrm{ind}_p H$ is the Morse index at p.

Regard the connection A_∇ as a map $A_\nabla : TE \to E$. Then it induces a splitting $TE = TE^{hor} \oplus TE^{vert}$, where $TE^{hor} := \ker A_\nabla$ and $TE^{vert} := \ker d\pi_E$. The trivialization of $T_p\Sigma$ also can be lifted to $T^{hor}E|_{C_p}$. In particular, $c_\tau(T^{hor}E|_{C_p}) = 0$. Note that

$T^{hor}E|_{C_p}$ can be identified with the normal bundle N_{C_p} of C_p. By the definition of Q_τ, we obtain $Q_\tau(C_p) = c_\tau(N_{C_p}) = 0$. The section $r\partial_r - i\partial_\theta$ can be extended to a section ψ of $T^{vert}E|_{C_p} = TC_p$. We choose such an extension such that $\psi = \rho\partial_\rho - i\partial_\theta$ near $r = \frac{1}{2}$ and is non-vanishing except at the zero section. Hence, the relative Chern number is

$$c_\tau(T^{vert}E|_{C_p}) = \#\psi^{-1}(0) = \#(\rho\partial_\rho - i\partial_\theta)^{-1}(0) = 1.$$

Therefore, we have

$$I(C_p) = c_\tau(TE|_{C_p}) + Q_\tau(C_p) + CZ_\tau(\gamma_p) = \mathrm{ind}_p H.$$

In other words, $I(C_{e_+}) = 2$, $I(C_{h_i}) = 1$ and $I(C_{e_-}) = 0$. For $p \neq q$, the fibers C_p and C_q are obviously disjoint. Hence, $Q_\tau(C_p, C_q) = 0$. Therefore, the formula for $I(Z_\alpha)$ follows from the facts that the relative Chern number is additivity and Q_τ is quadratic.

By the definition of the ECH index, we have

$$I(Z_\alpha + d[\Sigma]) = I(Z_\alpha) + 2dZ_\alpha \cdot \Sigma + I(d\Sigma) = M - m_- + m_+ + 2dM + I(d\Sigma). \quad (3.5)$$

Under the identification $T^{hor}E = \pi_E^* T\Sigma$ and $T^{vert}E = \pi_E^* E$, we have

$$\langle c_1(TE), \Sigma \rangle = \langle c_1(T^{hor}E), \Sigma \rangle + \langle c_1(T^{vert}E), \Sigma \rangle = \chi(\Sigma) + e.$$

The self-intersection number of Σ is

$$\Sigma \cdot \Sigma = \#s^{-1}(0) = \langle c_1(E), \Sigma \rangle = e,$$

where s is a generic section of E. In sum, we have

$$I(d\Sigma) = d\chi(\Sigma) + de + d^2 e. \quad (3.6)$$

The statement of the lemma follows from (3.5) and (3.6). $\qquad\square$

Remark 3.1. *Let $\alpha = e_+^{m_+} h_1^{m_1}, ..., h_{2g}^{m_{2g}} e_-^{m_-}$ and $\beta = e_+^{n_+} h_1^{n_1}, ..., h_{2g}^{n_{2g}} e_-^{n_-}$. In Proposition 3.5 of [27], Nelson and Weiler show that*

$$I(\alpha, \beta) = d\chi(\Sigma) + d^2|e| + 2dN + m_+ - m_- + n_- - n_+, \quad (3.7)$$

where M and N are total multiplicities of α and β respectively, and $d = (M - N)/|e|$. Our computations here is equivalent to (3.7) in the following sense.

One can use Lemma 3.1 and the additivity of ECH index to recover the index formula $I(\alpha, \beta)$ as follows. Let $Z_\alpha + d_\alpha[\Sigma] \in H_2(DE, \alpha)$, $Z_\beta + d_\beta[\Sigma] \in H_2(DE, \beta)$ and $Z \in H_2(Y, \alpha, \beta)$ be relative homology classes such that $Z_\alpha + d_\alpha[\Sigma] = Z\#(Z_\beta + d_\beta[\Sigma])$. By a similar argument as in Proposition 4.4 of [27] (see (3.15) for the details in the current setting), one can show that $d_\alpha - d_\beta = (M - N)/|e|$. Then, by direct computations, we have

$$I(\alpha, \beta) = I(Z) = I(Z_\alpha + d_\alpha[\Sigma]) - I(Z_\beta + d_\beta[\Sigma]). \quad (3.8)$$

Following from Lemma 3.1 and (3.8), we get the formula (3.7) for $I(\alpha, \beta)$.

Conversely, we also can deduce Lemma 3.1 easily by applying Nelson and Weiler's index computations as follows. Suppose that $[\alpha] = 0$ and $M = k|e|$. Let $Z_0 = Z_\alpha + k[\Sigma]$. Note that $Z_0 \cdot [\Sigma] = 0$. We can represent Z_0 by a surface S_0 which is disjoint from Σ. Under the identification (3.3), S_0 represents a relative homology class in $H_2(Y, \alpha, \emptyset)$. In particular, we have $I(Z_0) = I(S_0) = I(\alpha, \emptyset)$. Then one can apply the index ambiguity formula (3.5) to drive the result for a general $Z = Z_0 + d[\Sigma]$.

In the next lemma, we use the computations in Lemma 3.1 to drive a formula for the Fredholm index.

Lemma 3.2. *Let C be a holomorphic curve with relative class $[C] = Z_\alpha + d[\Sigma]$. Then the Fredholm index of C is*

$$\text{ind}(C) = 2g(C) - 2 + h(C) + 2e_+(C) + 2M + 2d\chi(\Sigma) + 2de, \qquad (3.9)$$

where $h(C)$ is the number of ends at hyperbolic orbits and $e_+(C)$ is the number of ends at covers of e_+.

Proof. The proof follows directly from definition and the calculations in Lemma 3.1. □

J-Holomorphic currents without closed components We first study the case that the J-holomorphic current $\mathcal{C}$ contains no closed components. Also, we assume that the holomorphic curves are asymptotic to orbit sets with $\mathcal{A}_{\lambda_\varepsilon} < L_\varepsilon$, unless stated otherwise.

Fix a relative homology class $Z \in H_2(DE, \alpha)$. Let $\mathcal{M}^J(\alpha, Z)$ denote the set of J-holomorphic currents in E. We denote the moduli space of **broken J-holomorphic currents** in E by $\overline{\mathcal{M}^J(\alpha, Z)}$ (Definition 1.6 of [18]). An element in $\overline{\mathcal{M}^J(\alpha, Z)}$ is a chain of holomorphic currents $\{\mathcal{C}^0, ..., \mathcal{C}^N\}$ such that

1. $\mathcal{C}^0 \in \mathcal{M}^J(\alpha_0, Z_0)$ and $\mathcal{C}^i \in \mathcal{M}^J(\alpha_i, \beta_i)$ for $i \geq 1$;

2. $\alpha_i = \beta_{i+1}$ for $1 \leq i \leq N - 1$, and $\alpha^N = \alpha$;

3. $Z = \sum_{i=0}^{N} [\mathcal{C}^i]$.

By Lemma 6.8 of [18], a sequence of J-holomorphic currents in $\mathcal{M}^J(\alpha, Z)$ converges to a broken J-holomorphic current in $\overline{\mathcal{M}^J(\alpha, Z)}$ in the current sense (see Section 9 of [13] for details).

For a J-holomorphic curve in E, we define its degree which is an analog of Definition 4.1 in [27]. Let $\mathcal{C} \in \mathcal{M}^J(\alpha, Z)$ be a J-holomorphic current represented by a J-holomorphic map $u : \dot{F} \to E$, where $\dot{F} = F - P$, F is a closed Riemann surface

(possibly disconnected) and P is the set of punctures. Since $\pi_E \circ u$ maps the punctures to the critical points of H, we extend $\pi_E \circ u$ to a map $\pi_E \circ u : F \to \Sigma$. Then, we have a well-defined degree $\deg(\pi_E \circ u)$. Define $\deg(\mathcal{C}) := \deg(\pi_E \circ u)$. It is called the **degree** of $\mathcal{C}$. Alternatively, we can define $\deg(\mathcal{C})$ to be the unique integer d such that $[\mathcal{C}] = Z_\alpha + d[\Sigma]$. To understand why this is valid, consider the following computation: $\int_{\mathcal{C}} \pi_E^* \omega_\Sigma = \int_{\pi_E(\mathcal{C})} \omega_\Sigma = \deg(\mathcal{C})|e|$. On the other hand, $\int_{\mathcal{C}} \pi_E^* \omega_\Sigma = \int_{Z_\alpha + d[\Sigma]} \pi_E^* \omega_\Sigma = d|e|$. Thus, we conclude that $d = \deg(\mathcal{C})$. The degree of a broken holomorphic curve is defined in the same way.

Lemma 3.3. *For a generic almost complex structure J, let $\mathcal{C}$ be a J-holomorphic current without closed components. Then, we have $\deg(\mathcal{C}) \geq 0$.*

Proof. Write $\mathcal{C} = \sum_a d_a C_a$. Since $\deg(\mathcal{C}) = \sum_a d_a \deg(C_a)$, it suffices to prove the conclusion for an irreducible simple holomorphic curve C with at least one end.

Let $d = \deg(C)$. Assume that $d \leq -1$. Then, Lemma 3.1 implies that

$$I(C) \leq |d|(1 - |d|)|e| + d\chi(\Sigma) \leq d\chi(\Sigma).$$

If $\chi(\Sigma) = 2$, then $I(C) \leq -2$. If $\chi(\Sigma) \leq 0$, then $\mathrm{ind}(C) \geq 2d\chi(\Sigma) + 2de > d\chi(\Sigma)$ by Lemma 3.2.

In both cases, they violate the ECH inequality $I(C) \geq \mathrm{ind}(C) \geq 0$. $\square$

From now on, we shall assume that $\deg(C) \geq 0$ due to the above lemma.

To deal with the J-holomorphic currents with multiply covered components, we need the following self-intersection number that appears in the ECH index inequality.

Definition 3.4 (Definition 4.7 [17])**.** *For two simple J-holomorphic curves C, C' which are asymptotic positively to an orbit set α with action less than L, define an integer $C \star C'$ as follows.*

- *If C and C' are distinct, then $C \star C'$ is the algebraic count of intersections of C and C'. By intersection positivity, we have $C \star C' \geq 0$. The equality holds if and only if C and C' are disjont.*

- *If C and C' are the same curve, then define*

$$2C \star C = 2g(C) - 2 + h(C) + \mathrm{ind}(C) + 2e_L(C) + 4\delta(C), \qquad (3.10)$$

where $e_L(C)$ is the total multiplicity of all elliptic orbits in α that are L-negative, and $\delta(C)$ is the count of singularities of C with positive integer weights. $\delta(C) \geq 0$ and equality holds if and only if C is embedded.

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Let $\mathcal{C} = \sum_a d_a C_a$ and $\mathcal{C}' = \sum_b d'_b C'_b$ be two J-holomorphic currents. By Proposition 4.8 of [17], we have

$$I(\mathcal{C} + \mathcal{C}') \geq I(\mathcal{C}) + I(\mathcal{C}') + 2\mathcal{C} \star \mathcal{C}', \tag{3.11}$$

where $\mathcal{C} \star \mathcal{C}' = \sum_a \sum_b d_a d'_b C_a \star C'_b$.

If the self-intersection number (3.10) is nonnegative, then the ECH index inequality (2.1) and (3.11) imply that the ECH index is nonnegative. However, this is not always the case. See Section 5.5 of [16] for the explanations. In our setting, using the index computations in Lemma 3.1 and Lemma 3.2, we show that if $C \star C < 0$, then its ECH index is at least 2. As a result, the ECH indices of the holomorphic currents in E are nonnegative.

Lemma 3.5. *Let C be an irreducible simple holomorphic curve with at least one end. If $C \star C < 0$, then $I(kC) \geq 2$ for any $k \geq 1$. In particular, if $\mathcal{C}$ is a J-holomorphic current without closed component, then $I(\mathcal{C}) \geq 0$.*

Proof. Assume that $C \star C < 0$. By Lemma 3.2, we know that $h(C) + \mathrm{ind}(C)$ is a nonnegative even integer. By the definition (3.10), we have

$$g(C) = h(C) = \mathrm{ind}(C) = e_L(C) = \delta(C) = 0.$$

The condition $h(C) = e_L(C) = 0$ forces $M = m_+$.

By Lemma 3.2, $\mathrm{ind}(C) = 0$ implies that

$$M = d|e| - d\chi(\Sigma) + 1 - e_+(C). \tag{3.12}$$

Write the relative homology class of C as $Z_{e_+^M} + d[\Sigma]$, then $[kC] = Z_{e_+^{kM}} + dk[\Sigma]$. Note that $C \cdot \Sigma = M + de \geq 0$ by intersection positivity; then $M \geq d|e|$. If $\chi(\Sigma) > 0$, then

$$I(kC) = 2kM + d^2k^2e + dke + dk\chi(\Sigma) + 2dk^2M$$
$$\geq 2kM + dk\chi(\Sigma) + k^2d^2|e| - dk|e| \geq 2kM \geq 2.$$

If $\chi(\Sigma) \leq 0$, then by Lemma 3.2 and Equation (3.12), we have

$$I(kC) = 2kM + d^2k^2e + dke + dk\chi(\Sigma) + 2dk^2M$$
$$= kM + k(1 + d|e| - d\chi(\Sigma) - e_+(C)) + d^2k^2e + dke + dk\chi(\Sigma)$$
$$+ dk^2M + dk^2(d|e| - d\chi(\Sigma) + 1 - e_+(C))$$
$$= k(M - e_+(C)) + k + dk^2(M - e_+(C)) - d^2k^2\chi(\Sigma) + dk^2 \geq k \geq 1.$$

Note that $I(kC)$ is even. Hence, we get $I(kC) \geq 2$.

Write $\mathcal{C} = \sum_a d_a C_a + \sum_b d'_b C'_b$ such that $C_a \star C_a \geq 0$ and $C'_b \star C'_b < 0$. By Inequality (3.11), we have

$$
\begin{aligned}
I(\mathcal{C}) &\geq \sum_a I(d_a C_a) + \sum_b I(d'_b C'_b) + 2 \sum_{a,b} d_a d'_b C_a \star C'_b + 2 \sum_{a \neq a'} d_a d_{a'} C_a \star C_{a'} + 2 \sum_{b \neq b'} d'_b d'_{b'} C'_b \star C'_{b'} \\
&\geq \sum_a d_a I(C_a) + \sum_a d_a (d_a - 1) C_a \star C_a + \sum_b I(d'_b C'_b) + 2 \sum_{a,b} d_a d'_b C_a \star C'_b \\
&\quad + 2 \sum_{a \neq a'} d_a d_{a'} C_a \star C_{a'} + 2 \sum_{b \neq b'} d'_b d'_{b'} C'_b \star C'_{b'}.
\end{aligned}
$$

$$(3.13)$$

By ECH index inequality (2.1), $\sum_a d_a I(C_a) \geq 0$. The intersection positivity implies that the last three terms are nonnegative. $C_a \star C_a \geq 0$ by assumption and $I(d'_b C'_b) \geq 2$. In sum, $I(\mathcal{C}) \geq 0$. $\qquad \square$

A simple J-holomorphic curve C is called a **special J-holomorphic plane** if it has $I(C) = \mathrm{ind}(C) = 0$, and is an embedded plane whose positive end is asymptotic to e_- with multiplicity 1. This is a counterpart of the Definition 3.15 in [11].

Lemma 3.6. *Assume that C is not closed. If $I(C) = \mathrm{ind}(C) = C \star C = 0$, then C is a special J-holomorphic plane.*

Proof. By Definition (3.10), $C \star C = 0$ forces $\delta(C) = 0$, i.e, C is embedded. It is easy to check that C satisfies one of the following properties:

1. $h(C) = e_L(C) = 0$ and $g(C) = 1$;

2. $h(C) = 2$ and $g(C) = e_L(C) = 0$;

3. $h(C) = g(C) = 0$ and $e_L(C) = 1$.

Let $d = \deg(C)$. By Lemma 3.1 and Lemma 3.2, we have

$$2I(C) - \mathrm{ind}(C) = 2m_+ - 2m_- + 4dM - 2d^2|e| + 2 - 2g(C) - h(C) - 2e_+(C). \quad (3.14)$$

Since $I(C) = \mathrm{ind}(C)$, the ECH partition condition implies that $e_+(C) = m_+$. Also, $e_L(C) = 0$ is equivalent to $m_- = 0$. In the first two cases, we have

$$0 = 4dM - 2d^2|e| \geq 2dM \geq 0.$$

The last step comes from the positivity intersection of holomorphic curves $C \cdot \Sigma = M - d|e| \geq 0$. Hence, we have either $M = 0$ or $d = 0$. If $d = 0$, then the formula in Lemma 3.1 still implies that $M = 0$. We get contradiction since we have assumed that C is not closed.

In the last case, $m_- = 1$. By Equation (3.14), then we still get $d = 0$. The formula in Lemma 3.1 and $I(C) = 0$ imply that $m_+ = 0$. Hence, C is a J-holomorphic plane with one end at e_-, i.e., it is a special J-holomorphic plane. $\qquad\square$

Lemma 3.7. *Let $\mathcal{C} \in \mathcal{M}^J(\alpha, Z)$ be a J-holomorphic current with $I(\mathcal{C}) = i$, where $i = 0$ or 1. If $i = 1$, we also assume that α is an ECH generator. Suppose that $\mathcal{C}$ has no closed component. Then $\mathcal{C} = \mathcal{C}_{emb} \cup \mathcal{C}_{spec}$, where $\mathcal{C}_{emb}$ is embedded with $I(\mathcal{C}_{emb}) = \mathrm{ind}(\mathcal{C}_{emb}) = i$ and $\mathcal{C}_{spec}$ consists of special J-holomorphic planes.*

Proof. Write $\mathcal{C} = \sum_a d_a C_a + \sum_b d'_b C'_b$ as in the proof of Lemma 3.5, with the properties that $C_a \star C_a \geq 0$ and $C'_b \star C'_b < 0$. By Lemma 3.5 and Inequality (3.13), we must have $d'_b = 0$ because $I(d'_b C'_b) \geq 2$.

In the case that $I(\mathcal{C}) = 0$, we have $I(C_a) = 0$ for any a. Also, $d_a = 1$ unless $C_a \star C_a = 0$. The ECH index inequality implies that $\mathrm{ind}(C_a) = 0$ and $\delta(C_a) = 0$ as well. If $d_a > 1$, then $C_a \star C_a = 0$. By Lemma 3.6, C_a is a special J-holomorphic plane.

In the case that $I(\mathcal{C}) = 1$, then we have $I(C_a) \leq 1$. If $I(C_a) = 0$ for all a, the ECH index inequality and Lemma 3.2 implies that C_a has even ends at hyperbolic orbits. Since α is an ECH generator, we know that α contains even distinct simple hyperbolic orbits. By Lemma 3.1, $I(\mathcal{C}) = 0 \mod 2$, we get a contradiction. Therefore, there exists C_{a_0} with $I(C_{a_0}) = \mathrm{ind}(C_{a_0}) = 1$. The Inequality (3.13) implies that such a_0 is unique and $d_{a_0} = 1$. For any other a, we also have $I(C_a) = \mathrm{ind}(C_a) = \delta(C_a) = 0$. Moreover, $d_a = 1$ unless $C_a \star C_a = 0$.

In both cases, $\mathcal{C}$ is a union of embedded curves and covers of special J-holomorphic planes.

$\qquad\square$

Closed holomorphic curves   Now we begin to consider the holomorphic currents that contain closed J-holomorphic curves. We first need to figure out what kind of closed J-holomorphic curves could exist in E.

Lemma 3.8. *The zero section Σ is the unique simple closed J-holomorphic curve in E.*

Proof. Suppose we have a simple closed holomorphic curve C which is different from Σ. Since $H_2(DE, \mathbb{Z})$ is generated by $[\Sigma]$, we must have $[C] = k[\Sigma]$. By energy reason, we have $k \geq 1$. However, $C \cdot \Sigma = k[\Sigma] \cdot [\Sigma] = ke < 0$, contradicts with the intersection positivity of holomorphic curves. $\qquad\square$

Lemma 3.9. *Let $\mathcal{C} \in \overline{\mathcal{M}^J(\alpha, Z_\alpha)}$ be a broken J-holomorphic current . Then $\mathcal{C}$ contains no closed components. Moreover, each level of $\mathcal{C}$ has degree 0.*

Proof. By Lemma 3.8, we can write $\mathcal{C} = \mathcal{C}_\diamond + k\Sigma$, where $\mathcal{C}_\diamond$ has no closed components and $k \geq 0$.

Write $\mathcal{C}_\diamond = \{\mathcal{C}^0, ..., \mathcal{C}^N\}$, where $\mathcal{C}^0 \in \mathcal{M}^J(\alpha_0)$, $\mathcal{C}^i \in \mathcal{M}^J(\alpha_i, \alpha_{i-1})$ and $\alpha^N = \alpha$. We claim that the degree is additive, i.e., $\deg(\mathcal{C}_\diamond) = \sum_{i=0}^N \deg(\mathcal{C}_i)$, where $\deg(\mathcal{C}_i)(i \geq 1)$ is the degree defined in Definition 4.1 of [27]. Proposition 4.4 of [27] implies that $\deg(\mathcal{C}_i) \geq 0$ for $1 \leq i \leq N$. By Lemma 3.3, we have $\deg(\mathcal{C}^0) \geq 0$. Let M be the total multiplicity of α. Then $\mathcal{C} \cdot \Sigma = M + \deg(\mathcal{C}_\diamond)e + ke = Z_\alpha \cdot [\Sigma] = M$. Thus, we have $k = \deg(\mathcal{C}_\diamond) = \deg(\mathcal{C}^i) = 0$. In other words, $\mathcal{C}$ has no closed components, and each level of $\mathcal{C}$ has degree 0.

To prove the claim, the argument here is the same as Proposition 4.4 in [27]. Let S be a representative of $[\mathcal{C}_\diamond] = Z_\alpha + \deg(\mathcal{C}_\diamond)[\Sigma]$. We have the following energies:

$$\int_{S \cap DE} \Omega_\varepsilon + \int_{S \cap \mathbb{R}_{s \geq 0} \times Y} d\lambda_\varepsilon = M + \varepsilon \pi^* H(\alpha) + \deg(\mathcal{C}_\diamond)|e|$$

$$\int_{\mathcal{C}^0 \cap DE} \Omega_\varepsilon + \int_{\mathcal{C}^0 \cap \mathbb{R}_{s \geq 0} \times Y} d\lambda_\varepsilon = M_{\alpha_0} + \varepsilon \pi^* H(\alpha_0) + \deg(\mathcal{C}^0)|e| \qquad (3.15)$$

$$\int_{\mathcal{C}^i \cap \mathbb{R}_s \times Y} d\lambda_\varepsilon = 2(M_{\alpha_i} - M_{\alpha_{i-1}}) + \varepsilon \left(\pi^* H(\alpha_i) - \pi^* H(\alpha_{i-1}) \right),$$

where M_{α_i} is the total multiplicity of α_i, $\pi^* H(\alpha_i)$ is short for $m_+^i \pi^* H(e_+) + m_-^i \pi^* H(e_-) + \sum_{j=1}^{2g} m_j^i \pi^* H(h_i)$ and $m_\pm^i$, m_j^i are multiplicities of $e_\pm, h_j$ in α_i. Since the energy only depends on the relative homology class, we have

$$\int_{S \cap DE} \Omega_\varepsilon + \int_{S \cap \mathbb{R}_{s \geq 0} \times Y} d\lambda_\varepsilon = \int_{\mathcal{C}^0 \cap DE} \Omega_\varepsilon + \int_{\mathcal{C}^0 \cap \mathbb{R}_{s \geq 0} \times Y} d\lambda_\varepsilon + \sum_{i=1}^N \int_{\mathcal{C}^i \cap \mathbb{R}_s \times Y} d\lambda_\varepsilon.$$

By Equations (3.15), we have

$$\deg(\mathcal{C}_n)|e| = \deg(\mathcal{C}^0)|e| + \sum_{i=1}^N (M_{\alpha_i} - M_{\alpha_{i-1}}) = \sum_{i=0}^N \deg(\mathcal{C}^i)|e|.$$

The second equality follows from $\deg(\mathcal{C}^i)|e| = M_{\alpha_i} - M_{\alpha_{i-1}}$ (Proposition 4.4 of [27]). $\qquad \square$

Remark 3.2. *The argument in Lemma 3.9 still works for a general generic cobordism admissible almost complex structure, because any closed J-holomorphic current has homology class $k[\Sigma]$ for $k \geq 1$. Also, the proof of Lemma 3.3 does not use the conditions* **J.1** *and* **J.2**.

Lemma 3.10. *The moduli space $\mathcal{M}^J(e_-^M, Z_{e_-^M})$ is a finite set.*

Proof. Let $\mathcal{C}_\infty = \{\mathcal{C}^0, ..., \mathcal{C}^N\}$ be a broken J-holomorphic curve which which is a limit from a sequence of holomorphic curves $\{\mathcal{C}_n\}_{n=1}^\infty$ in $\mathcal{M}^J(e_-^M, Z_{e_-^M})$. By Lemma 3.9, $\mathcal{C}^0$ has no closed components.

Then the rest of the proof is the same as Proposition 3.13 in [11]. Here we just sketch the proof. Because the J-holomorphic currents in E has nonnegative ECH index, $I(\mathcal{C}_\infty) = I(\mathcal{C}_n) = 0$ implies that $I(\mathcal{C}^i) = 0$ for $0 \le i \le N$. As a result, $\mathcal{C}^i$ are branched covers of trivial cylinders. The branched covers of trivial cylinders have nonnegative Fredholm index (see Exercise 3.14 of [16]). By Lemma 3.8, $\mathcal{C}^0$ also has nonnegative Fredholm index. Since Fredholm index is additive, we have $\mathrm{ind}(\mathcal{C}^i) = 0$ for $0 \le i \le N$. These branched covers of trivial cylinders can be ruled out by the ECH partition conditions (see Exercise 3.14 of [16])). Therefore, $\mathcal{M}^J(e_-^M, Z_{e_-^M})$ is compact. The transversality is guaranteed by the automatic transversality theorem [36].

$\square$

Uniqueness We show that MC_{e_-} is the unique J-holomorphic current in the moduli space $\mathcal{M}^J(e_-^M, Z_{e_-^M})$. The energy constraint argument in [3] does not work in the current situation. We use the argument as in Lemma 3.8 instead.

To this end, we need to apply R. Siefring's intersection theory for punctured J-holomorphic curves. In [30], Siefring defines a intersection pairing $C \bullet C'$ for punctured holomorphic curves, where C and C' are simple holomorphic curves. Here we do not use the precise definition of $\bullet$, we only need to know the following facts:

F.1 The intersection pairing $C \bullet C'$ is invariant under homotopy as cylindrical asymptotic maps.

F.2 (Theorem 2.2 of [30]) If C and C' are distinct, then $C \bullet C' \ge 0$.

F.3 (Theorem 2.3 of [30]) In the case that $C = C'$, we have the following adjunction formula:

$$C \bullet C = 2(\delta(C) + \delta_\infty(C)) + \frac{1}{2}(2g(C) - 2 + \mathrm{ind}(C) + \#P_{even}) + (\bar{\sigma}(C) - \#P),$$

where P denote the set of punctures, P_{even} is the set of punctures which are asymptotic to Reeb orbits with even Conley-Zehnder index, $\delta_\infty(C)$ is an algebraic count of "hidden" singularities at the infinity, and $\bar{\sigma}(C)$ is the spectral covering number. We refer readers to [30] for the precise definition of $\delta_\infty(C)$ and $\bar{\sigma}(C)$. According to the definition, if all the ends of C are asymptotic to distinct simple orbits, then $\delta_\infty(C)$ and $\bar{\sigma}(C) - \#P$ vanish.

Lemma 3.11. *The moduli space $\mathcal{M}^J(e_-^M, Z_{e_-^M})$ only consists of one element.*

Proof. Note that $MC_{e_-} \in \mathcal{M}^J(e_-^M, Z_{e_-^M})$. Moreover, $I(MC_{e_-}) = I(Z_{e_-^M}) = 0$. Thus, the moduli space is nonempty.

Let $\mathcal{C} = \sum_a d_a C_a \in \mathcal{M}^J(e_-^M, Z_{e_-^M})$. By Lemma 3.9, $\mathcal{C}$ has no closed components, and $\deg(C_a) = 0$ for any a. By (3.13) and (2.1), we have $I(C_a) = \mathrm{ind}(C_a) = \delta(C_a) = 0$

for every a. Lemma 3.2 forces $M_a = 1$, $g(C_a) = 0$ and $h(C_a) = e_+(C_a) = 0$. In sum, C_a are special J-holomorphic planes.

By our choice of J, the fiber C_{e_-} is a holomorphic plane with $I(C_{e_-}) = \text{ind}(C_{e_-}) = 0$. By **F.3**, we have $C_{e_-} \bullet C_{e_-} = -1$. If there exists another special J-holomorphic plane C_a other than C_{e_-}, then $C_{e_-} \bullet C_a \geq 0$ by **F.2**. Note that C_a is homotopy to C_{e_-} as a asymptotic cylindrical map because $\pi_E(C_{e_-} - C_a)$ is trivial in $\pi_2(\Sigma)$. Therefore, **F.1** implies that $0 \leq C_a \bullet C_{e_-} = C_{e_-} \bullet C_{e_-} = -1$. We get a contradiction. $\qquad\square$

(L, δ)-**flat approximation**   Fix an admissible almost complex structure $J \in \mathcal{J}(Y, \lambda_\varepsilon)$. To ensure that the ECH generators are in a one-to-one correspondence with the solutions to the Seiberg-Witten equations, we need to perturb (λ_ε, J) such that it has certain standard forms ((2-11) of [31]) in δ-neighborhoods of Reeb orbits with action less than L. Moreover, the modifications do not change the ECH chain complex. The result of the perturbation of (λ_ε, J) is called a (L, δ)-**flat approximation**, and it was introduced by Taubes [31]. The (L, δ)-flat approximation closes to the original one in C^0 topology.

Suppose that (λ, J) is (L, δ)-flat, then we have a canonical isomorphism (Theorem 4.2 of [31])

$$\Psi : ECC_*^L(Y, \lambda, \Gamma) \to CM_L^{-*}(Y, \lambda, \mathfrak{s}_\Gamma) \tag{3.16}$$

between the ECH chain complex and the Seiberg-Witten chain complex, where $CM_L^{-*}(Y, \lambda, \mathfrak{s}_\Gamma)$ is the Seiberg-Witten chain complex defined in Section 2.3 of [18], and $\mathfrak{s}_\Gamma$ is the $Spin^c$ structure such that $c_1(\mathfrak{s}_\Gamma) = c_1(\xi) + 2\,\text{PD}(\Gamma)$.

To apply the correspondence between solutions to Seiberg-Witten equations and J-holomorphic curves, we need to modify (Ω_ε, J) such that the induced contact form and almost complex structure are $(L\text{-}\delta)$-flat over the cylindrical end. Fix $\epsilon > 0$ and $\delta > 0$. By the same construction as in Section 6.3 of [18], we can define a new symplectic form Ω_δ and an almost complex structure J_δ such that

C.1 $\Omega_\delta = d(e^s \lambda_\delta)$ on the cylindrical end $[0, \infty) \times Y$. J_δ is a cobordism admissible almost complex structure with respect to Ω_δ. Furthermore, $(\lambda_\delta, J_\delta)$ is a $(L\text{-}\delta)$-flat approximation of (λ_ε, J).

C.2 $(\Omega_\delta, J_\delta) = (\Omega_\varepsilon, J)$ on the region $s \leq -\epsilon$.

C.3 J_δ satisfies estimates $|J_\delta - J|_{C^0} \leq c_0\delta$ and $|J_\delta - J|_{C^1} \leq c_0$ for some δ-independent constant c_0.

The following lemma tell us that the conclusion in Lemma 3.11 is still true if we replace (Ω_ε, J) by $(\Omega_\delta, J_\delta)$.

Lemma 3.12. *For sufficiently small $\delta > 0$, $\mathcal{M}^{J_\delta}(e_-^M, Z_{e_-^M})$ only consists of one element.*

Proof. From the proof of Lemma 3.11, we know that $\mathcal{M}^{J_\delta}(e_-^M, Z_{e_-^M})$ only consists of unbranched cover of special J_δ-holomorphic plane. Moreover, the special J_δ-holomorphic plane is unique provided that it exists.

It suffices to show that the special J_δ-holomorphic plane exists for small δ. By (2-11) of [31], $J_\delta = J$ along $\mathbb{R}_{\geq 0} \times e_-$. Since $J_\delta = J$ for $s \leq -\epsilon$, C_{e_-} is J_δ-holomorphic except the part in the small region $[-\epsilon, 0] \times U$, where U is a δ-neighborhood of e_-. C_{e_-} is nearly J_δ-holomorphic in the sense that it satisfies (A-15) in [31]. Therefore, we can apply the same argument in Appendix A of [31] to deform C_{e_-} so that it becomes J_δ-holomorphic. We sketch the main argument as follows.

Let N_C denote the normal bundle of C_{e_-} and $e_C : N_C \to E$ denote an exponential map. Let η be a section of N_C which is asymptotic to zero as $s \to \infty$. Then $e_C(\eta)$ is J_δ-holomorphic if and only if η satisfies the following equation:

$$D_C\eta + \mathfrak{p}_1\eta + \mathfrak{p}_2\nabla_C\eta + \mathfrak{R}_0\eta + \mathfrak{R}_1\nabla_C\eta + \mathfrak{p}_0 = 0. \tag{3.17}$$

Here $\mathfrak{p}_0 = \bar{\partial}_{J_\delta}C_{e_-}$ satisfying $|\mathfrak{p}_0|_{L^2} \leq c_0\delta$ due to the discussion in the last paragraph and the estimates of J_δ. The operators $\mathfrak{p}_1$, $\mathfrak{p}_2$ come from the linearization of $(J_\delta - J)\circ de_C(\eta)\circ j$ and they satisfy estimates $|\mathfrak{p}_1| \leq c_0|J - J_\delta|_{C^1} \leq c_0$ and $|\mathfrak{p}_2| \leq c_0|J - J_\delta|_{C^0} \leq c_0\delta$. $\mathfrak{R}_0$ and $\mathfrak{R}_1$ are high order terms and they satisfy (A-17) in [31].

If the operator $D_C + \mathfrak{p}_1 + \mathfrak{p}_2\nabla_C : L_1^2(N_C) \to L^2(N_C \otimes T^{0,1}C)$ is invertible, then we can apply the contraction mapping argument in Lemma A.3 of [31] to find a solution η satisfying (3.17). Then, $\mathcal{M}^{J_\delta}(e_-^M, Z_{e_-^M})$ is nonempty and it contains exactly one element.

Since $|\mathfrak{p}_2| \leq c_0\delta$, it suffices to prove that $D_C + \mathfrak{p}_1$ is invertible. Note that $D_C + \mathfrak{p}_1$ is a CR-operator which is asymptotic to the operator $\frac{i}{2}\frac{d}{dt} + \frac{\theta}{2}$ with odd Conley-Zehnder. Because the $(L\text{-}\delta)$-approximation does not change the rotation number of the Reeb orbit (see (2-11) of [31]), we still have $\mathrm{ind}(D_C + \mathfrak{p}_1) = 0$. By the automatic transversality theorem [36], $D_C + \mathfrak{p}_1$ is invertible. $\square$

Proof of Theorem 1. Let $A \in H_2(DE, Y, \mathbb{Z})$ be the relative class represented by $[Z_{e_-^M}]$. Recall that $Z_{e_-^M}$ is the unique relative homology class Z in $H_2(DE, e_-^M)$ such that $[Z] = A$.

Let $(\Omega_\delta, J_\delta)$ be the pair satisfying **C.1**, **C.2** and **C.3**. Then we have a bijection (3.10) between the ECH generators and the gauge classes of the Seiberg-Witten solutions. Let $\mathfrak{c}_{e_-^M} = \Psi(e_-^M)$. The ECH cobordism map is defined by (Definition 5.9 of [18])

$$ECC^L(DE, \Omega_\varepsilon, A)(e_-^M) = \#\mathfrak{M}(\mathfrak{c}_{e_-^M}, \mathfrak{s}_A),$$

where $\mathfrak{M}(\mathfrak{c}_{e^M_-}, \mathfrak{s}_A)$ is the moduli space of solutions to the Seiberg-Witten equations on E which are asymptotic to $\mathfrak{c}_{e^M_-}$ (see (4.15) of [18]), $\mathfrak{s}_A$ is the $Spin^c$ structure such that $c_1(\mathfrak{s}_A) = c_1(K_{DE}^{-1}) + 2\,\mathrm{PD}_{DE}(A)$ and K_{DE}^{-1} is the canonical line bundle. By Theorem 4.2 of [12] and Lemma 3.11, we have

$$\#\mathfrak{M}(\mathfrak{c}_{e^M_-}, \mathfrak{s}_A) = \#\mathcal{M}^{J_\delta}(e^M_-, Z_{e^M_-}) = 1.$$

By Lemma 4.6 and Corollary 6.2 of [27], the differential vanishes on $ECC^{L_\varepsilon}(Y, \lambda_\varepsilon)$. Then e^M_- is a cycle, and we have $ECH^L(DE, \Omega_\varepsilon, A)([e^M_-]) = 1$.

Let $\beta = e_+^{n_+} h_1^{n_1}, ..., h_{2g}^{n_{2g}} e_-^{n_-}$ be an ECH generator with $N = M - d|e|$ for $d \geq 1$, where $N = n_+ \sum_{i=1}^{2g} n_i + n_-$. To see $ECH^L(DE, \Omega_\varepsilon, A)([\beta]) = 0$, by the holomorphic curve axioms (see Theorem 1.9 of [18]), it suffices to show that the moduli space $\overline{\mathcal{M}^J(\beta, Z_{\beta,A})}$ is empty, where $Z_{\beta,A} \in H_2(DE, \beta)$ is the unique relative homology class determined by A. Let $\mathcal{C} = \mathcal{C}_\diamond + k\Sigma$ be a holomorphic current in this moduli space, where $\mathcal{C}_\diamond$ has no closed component and $k \geq 0$. Then

$$\mathcal{C} \cdot \Sigma = M - |e| + \deg(\mathcal{C}_\diamond)e + ke = A \cdot \Sigma = Z_{e^M_-} \cdot \Sigma = M.$$

Then $\deg(\mathcal{C}_\diamond) + k = -d \leq -1$. This contradicts with Lemma 3.3 (also see Remark 3.2). $\qquad\square$

3.3 Proof of Theorem 1

To prove Theorem 1 and Theorem 4, we first need to compute the U map for some ECH generators. The computations are parallel to Lemma 4.6 of [27].

Let $\mathbf{z} = \{z_1, ..., z_k\}$ be k distinct marked points in Y away from the Reeb orbits. Let $\mathcal{M}_i^J(\alpha, \beta)_\mathbf{z}$ denote the moduli space of ECH index-i J-holomorphic currents passing through the marked points $\{0\} \times \mathbf{z} \in \mathbb{R} \times Y$. By the same argument as in Lemma 2.6 of [19], $\mathcal{M}_{2k}^J(\alpha, \beta)_\mathbf{z}$ is a finite set for a generic almost complex structure. In the case that $k = 1$, the counting of this moduli space is used to define the U map. By the similar argument in Proposition 3.25 of [11], we can define the U^k at the chain level by

$$U^k \alpha := \sum_\beta \#\mathcal{M}_{2k}^J(\alpha, \beta)_\mathbf{z} \beta \tag{3.18}$$

Lemma 3.13. *Fix a positive integer k, and let J be a generic admissible almost complex structure. Then the moduli space $\mathcal{M}_{2k}^J(e+^k, e_-^k)_\mathbf{z}$ consists of exactly one element, which is made up of k distinct index-2 holomorphic cylinders passing through the marked point $\mathbf{z}$. As a result, we have $\langle U^k e_+^k, e_-^k \rangle = 1$.*

Proof. Let $\mathcal{C} = \sum_a d_a C_a$ be a holomorphic current in $\mathcal{M}_{2k}^J(e_+^k, e_-^k)_\mathbf{z}$. Note that $I(\mathcal{C}) = I(e_+^k, e_-^k) = 2k$ by by the index formula (3.7).

Since J is generic, if C_a passes through $l_a \geq 0$ marked points, then $\mathrm{ind}(C_a) \geq 2l_a$. By Hutchings's ECH inequality (see Theorem 4.15 and Theorem 5.1 of [14] for example), we have

$$
\begin{aligned}
2k = I(\mathcal{C}) &\geq \sum_a d_a I(C_a) + 2 \sum_{a \neq a'} d_a d_{a'} C_a \star C_{a'} \\
&\geq \sum_a d_a \left(\mathrm{ind}(C_a) + 2\delta(C_a) \right) \\
&\geq \sum_{a, l_a \geq 1} 2l_a + \sum_{a, l_a \geq 1} 2(d_a - 1)l_a + \sum_a 2d_a \delta(C_a).
\end{aligned}
$$

Since $\sum_a l_a = k$, we must have $d_a = 1$, $I(C_a) = \mathrm{ind}(C_a) = 2l_a$ provided that $l_a \geq 1$ and C_a is a trivial cylinder provided that $l_a = 0$. Furthermore, $C_a \star C_{a'} = 0$, i.e., C_a are pairwise disjoint. By Proposition 4.4 of [27], the degree of C_a are nonnegative and additivity. Hence, we must have $\deg(C_a) = 0$ because their sum is $\deg(\mathcal{C}) = (k - k)/|e| = 0$. Consequently, we have $C_a \in \mathcal{M}^J(e_+^{m_a}, e_-^{m_a})$.

By Nelson and Weiler's index formula 3.7 and the ECH partition condition, we have

$$
\begin{aligned}
2l_a &= \mathrm{ind}(C_a) = 2g(C_a) - 2 + 4m_a, \\
2l_a &= I(C_a) = 2m_a.
\end{aligned}
\tag{3.19}
$$

Hence, we have $g(C_a) = 0$ and $l_a = m_a = 1$, i.e., C_a is a holomorphic cylinder from e_+ to e_- passing a marked point. Consequently, there are k holomorphic cylinders and there are no trivial cylinders. By Proposition 4.7 of [27], there is a bijection between $\mathcal{M}_{2k}^J(e_+^k, e_-^k)_{\mathbf{z}}$ and the moduli space of Morse flow lines passing through the marked points. For each marked point, there is exactly one index 2 Morse flow line passing through it. Therefore, $\mathcal{M}_{2k}^J(e_+^k, e_-^k)_{\mathbf{z}}$ contains exactly one element and $< U^k e_+^k, e_-^k >= \#\mathcal{M}_{2k}^J(e_+^k, e_-^k)_{\mathbf{z}} = 1$. $\qquad \square$

We have the following parallel version of Lemma 3.13 for J-holomorphic currents in E with points constraint. Let $\mathcal{M}^J(\alpha, Z)_{\mathbf{z}}$ denote the moduli space of J-holomorphic currents in E passing through the marked points $\mathbf{z}$.

Lemma 3.14. *Let J be a generic cobordism admissible almost complex structure (not necessarily satisfying the conditions **J.1** and **J.2**) and $\mathbf{z} = \{z_1, ..., z_k\} \in \mathrm{Int}(DE)$. Then there is a k disjoint union of embedded J-holomorphic planes in $\mathcal{M}^J(e_+^k, Z_{e_+^k})_{\mathbf{z}}$.*

Proof. Let $\beta = e_+^{n_+} h_1^{n_1}, ..., h_{2g}^{n_{2g}} e_-^{n_-}$. Suppose that $< U^k e_+^k, \beta >= 1$. By definition of the U map, there exists a J-holomorphic current $\mathcal{C} \in \mathcal{M}_{2k}^J(e_+^k, \beta)_{\mathbf{z}}$. By Proposition 4.4 of [27], the degree of $\mathcal{C}$ is $\deg(\mathcal{C}) = \frac{k-N}{|e|} \geq 0$, where $N = n_+ + \sum_{i=1}^{2g} n_i + n_-$. Then $N = k - \deg(\mathcal{C})|e| \leq k$. If $N = k$, then $\beta = e_-^k$ by the index formula (3.7).

Let $A = [Z_{e_-^k}] \in H_2(DE, Y, \mathbb{Z})$. By Lemma 3.13, Theorem 2 and the above discussion, we have

$$ECH^L(DE, \Omega_\varepsilon, A)(U^k[e_+^k]) = ECH^L(DE, \Omega_\varepsilon, A)([e_-^k]) = 1.$$

Note that $Z_{e_+^k, A} = Z_{e_+^k}$. By the holomorphic curve axiom [18], we have a broken J-holomorphic current $\mathcal{C} = \{\mathcal{C}^0, ..., \mathcal{C}^N\} \in \overline{\mathcal{M}^J(e_+^k, Z_{e_+^k})_{\mathbf{z}}}$, where $\mathcal{C}^0 \in \mathcal{M}^J(\alpha_0)_{\mathbf{z}}$, $\mathcal{C}^i \in \mathcal{M}^J(\alpha_i, \alpha_{i-1})$ $(1 \le i \le N)$, and $\alpha_N = e_+^k$. By Lemma 3.9 and Remark 3.2, there is no closed components and the degree of each level is zero.

Write $\mathcal{C}^0 = \sum_a d_a C_a$. By Lemma 3.1, $I(d_a C_a) \ge 0$ because its degree is zero. Since J is generic, $\text{ind}(C_a) \ge 2l_a$ provided that C_a passes through $l_a \ge 0$ marked points. By definition, $C_a \star C_a \ge 0$ if C_a passes at least one marked point. By (3.11) and (2.1), we have

$$2k = I(\mathcal{C}) = \sum_{i=0}^N I(\mathcal{C}^i) \ge I(\mathcal{C}^0)$$

$$\ge \sum_{a, l_a \ge 1} d_a I(C_a) + \sum_{a, l_a = 0} I(d_a C_a) + 2\sum_{a \ne a'} C_a \star C_{a'} \qquad (3.20)$$

$$\ge 2k + \sum_{a, l_a \ge 1} 2(d_a - 1)l_a + 2\sum_{a, l_a \ge 1} \delta(C_a).$$

As a result, if $l_a \ge 1$, then $d_a = 1$. Moreover, $I(\mathcal{C}^0) = 2k$ and $I(\mathcal{C}^i) = 0$ for $i \ge 1$. By Proposition 3.7 of [16], we have $\mathcal{C}^i = k(\mathbb{R} \times e_+)$ for $i \ge 1$ and $\alpha_0 = e_+^k$. The inequality (3.20) also implies that $\{C_a\}$ are pairwise disjoint, and C_a is embedding provided that $l_a \ge 1$.

Let $d_a C_a$ be a component in the summation $\sum_{a, l_a = 0} I(d_a C_a)$. The degree of C_a is zero and the ends of C_a are asymptotic to covers of e_+. By Lemma 3.1, we have $I(d_a C_a) \ge 2$. This contradicts (3.20). Therefore, each component of $\mathcal{C}^0$ passes at least one marked point. For each C_a, by Lemma 3.1 and Lemma 3.2, the ECH index and Fredholm index of C_a satisfy (3.19). Then we obtain $l_a = m_a = 1$ and $g(C_a) = 0$. Thus, C_a is a J-holomorphic plane and it passes exactly one marked point.

In sum, $\mathcal{C}^0 = \sum_{a=1}^k C_a$ is a k disjoint union of embedded J-holomorphic planes in $\mathcal{M}^J(e_+^k, Z_{e_+^k})_{\mathbf{z}}$. $\qquad \square$

Remark 3.3. *By the correspondence between holomorphic curves and the solutions to the Seiberg-Witten equations (see Theorem 4.2 of [12]), we should have $\#\mathcal{M}^J(e_+^k, Z_{e_+^k})_{\mathbf{z}} = 1$. But for our purpose, we do not need this stronger result.*

Proof of Theorem 1. Suppose that we have a symplectic embedding

$$\varphi : (\sqcup_{i=1}^k(B^4(r_i), \omega_{\text{std}}) \hookrightarrow (DE, \Omega).$$

Let $\mathbf{z} = \{z_1, ..., z_k\} \in DE$ be the image of the center of the balls. Since $\Omega_\varepsilon = \Omega$ except in a small collar neighbourhood of Y. For sufficiently small $\varepsilon > 0$, we can choose such a collar neighbourhood such that it is disjoint from the image of φ. Thus, we regard φ as a symplectic embedding of $(\sqcup_{i=1}^k (B(r_i), \omega_{\mathrm{std}})$ into (DE, Ω_ε).

Let J_0 be the standard almost complex structure on $\sqcup_{i=1}^k B^4(r_i)$. Extend $\varphi_*(J_0)$ to be a generic cobordism admissible complex structure J. By Lemma 3.14, there is a k disjoint union of embedded J-holomorphic planes $\mathcal{C}$ passing through $\mathbf{z}$. By the standard monotonicity lemma (see Theorem 5.2.1 of [37]) and (3.15), we obtain

$$\sum_{i=1}^k r_i \leq \int_{\mathcal{C} \cap \varphi(\sqcup_{i=1}^k (B^4(r_i))} \Omega_\varepsilon \leq \int_{\mathcal{C} \cap DE} \Omega_\varepsilon + \int_{\mathcal{C} \cap (\mathbb{R}_{s \geq 0} \times Y)} d\lambda_\varepsilon \leq k + O(\varepsilon).$$

Consequently, we get the conclusion of Theorem 1 by taking $\varepsilon \to 0$. $\qquad\square$

4 Proof of Theorem 3 and Theorem 4

To prove Theorem 3 and Theorem 4, we need the following spectrality of the ECH spectrum.

Lemma 4.1. *Suppose that (Y, λ) is L_0-nondegenerate and $c_k(Y, \lambda) < L_0$. Then there exists a null homologous ECH generator α such that $c_k(Y, \lambda) = \mathcal{A}_\lambda(\alpha)$ and $\mathrm{gr}(\alpha) = 2k$.*

Proof. Let $\sigma \in ECH_{2k}(Y, \lambda, 0)$ with $U^k \sigma = [\emptyset]$. Define

$$c_\sigma(Y, \lambda) := \inf\{L | \sigma \in \text{ image of } i_L : ECH_{2k}^L(Y, \lambda, 0) \to ECH_{2k}(Y, \lambda, 0)\}.$$

By definition, we have $c_k(Y, \lambda) = \min\{c_\sigma(Y, \lambda) | \sigma \in ECH_{2k}(Y, \lambda, 0), U^k \sigma = [\emptyset]\}$.

Assume that $c_\sigma(Y, \lambda) < L_0$. By definition, we can rewrite it as

$$c_\sigma(Y, \lambda) := \inf\{\mathcal{A}_\lambda(x) | x = \sum a_i \alpha_i \in ECC_{2k}^{L_0}(Y, \lambda, 0), \partial x = 0, \text{ and } i_{L_0}([x]_{L_0}) = \sigma\},$$
$$(4.21)$$

where $\mathcal{A}_\lambda(x) := \max\{\mathcal{A}_\lambda(\alpha_i) | a_i \neq 0\}$ and $[x]_{L_0}$ denotes the class in $ECH_{2k}^{L_0}(Y, \lambda, 0)$ represented by x. Since λ is L_0-nondegenerate, there are only finitely many ECH generators with $\mathcal{A}_\lambda < L_0$. We can find α such that $c_\sigma(Y, \lambda) = \mathcal{A}_\lambda(\alpha)$ and $\mathrm{gr}(\alpha) = 2k$. Take σ to the class such that $c_k(Y, \lambda) = c_\sigma(Y, \lambda)$; then we obtain conclusion of the lemma.

$\qquad\square$

4.1 Sphere case

In this subsection, we assume that $\Sigma = \mathbb{S}^2$. It is well known that the diffeomorphism type of Y is the lens space $L(|e|, 1)$. The ECH group of Y (as an $\mathbb{F}$ module) has

been computed by Nelson and Weiler (Example 1.3) [27]. But we still need to know the U module structure of $ECH(Y, \lambda_\varepsilon, 0)$ by using Taubes's isomorphism "ECH=HM" [31, 32, 33, 34, 35] and the computations of P. Kronheimer, T. Mrowka, P. Ozsváth and Z. Szabó in [21].

Proposition 4.2. *The ECH of the lens space $Y \cong L(|e|, 1)$ is*

$$ECH_*(Y, \lambda, 0) = \begin{cases} \mathbb{F}, & * = 2k \text{ and } k \geq 0, \\ 0, & else, \end{cases} \tag{4.22}$$

where the $\mathbb{Z}$ grading is defined by (2.2). Moreover, $U : ECH_{2k}(Y, \lambda, 0) \to ECH_{2k-2}(Y, \lambda, 0)$ is an isomorphism for $k \in \mathbb{Z}_{\geq 1}$. Also, $ECH_0(Y, \lambda, 0)$ is spanned by $[\emptyset]$.

Proof. The isomorphism (4.22) is just the sphere case of Theorem 1.1 in [27]. It remains to show that the U map is an isomorphism.

By Taubes's series papers [31, 32, 33, 34, 35], we have a canonical isomorphism $ECH_*(Y, \lambda, 0) \cong \widehat{HM}^{-*}(Y, \mathfrak{s}_\xi)$ as an U-module, where $\widehat{HM}^{-*}(Y, \mathfrak{s}_\xi)$ is the hat-version of the Seiberg-Witten Floer cohomology. Since $L(|e|, 1)$ admits a metric with positive scalar curvature, by Proposition 2.2 and Corollary 2.12 of [21], we have the $\mathbb{F}[U]$ module isomorphism

$$\widehat{HM}(Y, \mathfrak{s}_\xi) \cong \mathbb{F}[U^{-1}, U]/\mathbb{F}[U].$$

Therefore, U is an isomorphism when the grading is at least two. $\qquad\square$

Remark 4.1. *We remark that the computation of $ECH(L(|e|, 1), \lambda, \Gamma)$ holds similarly for nonzero homology class Γ (see Corollary 3.4 of [21]).*

Proof of Theorem 3. By Proposition 4.2, we have a sequence $\sigma_k \in ECH_{2k}(Y, \lambda, 0)$ such that $U^k \sigma_k = [\emptyset]$ and $U\sigma_k = \sigma_{k-1}$. By the "volume property" of ECH spectrum (Theorem 1.3 of [2]) which is discovered by D. Cristofaro-Gardiner, M. Hutchings and V.G.B. Ramos, we know that $c_k(Y, \lambda)$ is finite for each k.

For each k, since $c_k(Y, \lambda)$ is finite, there is a constant $L_k > 0$ such that $c_k(Y, \lambda) < L_k$. Then for $0 < \varepsilon \ll 1$, we still have $c_k(Y, \lambda_\varepsilon) < L_k$. Recall that $\lim_{\varepsilon \to \infty} L_\varepsilon = \infty$. Thus, $L_\varepsilon > L_k$ for sufficiently small ε. Since λ_ε is L_ε-nondegenerate, by Lemma 4.1, we have an orbit sets $\alpha = e_+^{m_+} e_-^{m_-}$ such that $c_k(Y, \lambda_\varepsilon) = \mathcal{A}_{\lambda_\varepsilon}(\alpha)$, $\mathrm{gr}(\alpha) = 2k$ and $m_- + m_+ = 0$ mod $|e|$. The last condition is equivalent to $[\alpha] = 0$.

In [27], Nelson and Weiler show that there is a bijection between the nonnegative integers k and the pairs (m_-, m_+) satisfying $m_- + m_+ = 0 \mod |e|$. Therefore, the unique ECH generator $e_+^{m_+} e_-^{m_-} \in ECC_{2k}^{L_\varepsilon}(Y, \lambda_\varepsilon, 0)$ is characterized by the following relation

$$\begin{aligned} & m_- + m_+ = d|e| \text{ for some positive integer } d, \\ & 2k = \mathrm{gr}(e_-^{m_-} e_+^{m_+}) = 2d + d^2|e| + m_+ - m_-. \end{aligned} \tag{4.23}$$

By (4.23), we get

$$2d + d|e|(d-1) \leq 2k \leq 2d + d|e|(d+1). \tag{4.24}$$

It is easy to show that the positive integer d satisfying (4.24) is unique. Conversely, for d satisfying (4.24), we have unique solution (m_+, m_-) to (4.23).

Therefore,

$$c_k(Y, \lambda) = \lim_{\varepsilon \to 0} c_k(Y, \lambda_\varepsilon) = \mathcal{A}_{\lambda_\varepsilon}(e_+^{m_+} e_-^{m_-}) = 2d|e|,$$

where d is the positive integer satisfying (4.24).

□

4.2　Torus case

By definition, $c_k(Y, \lambda) = \infty$ if we cannot find $\sigma \in ECH(Y, \lambda, 0)$ such that $U^k \sigma = [\emptyset]$. Using computations of P. Ozsváth, Z. Szabó [28], and K. Park [29], the existence of such the classes can be guaranteed.

Lemma 4.3. *There exists a sequence of classes $\sigma_k \in ECH(Y, \lambda, 0)$ such that $U(\sigma_k) = \sigma_{k-1}$ and $U^k(\sigma_k) = [\emptyset]$.*

Proof. By the results of V. Colin, P. Ghiggini and K. Honda [5, 6, 7], we know that $ECH(Y, \Gamma)$ is isomorphic to the Heegaard Floer homology $HF^+(-Y, \mathfrak{s}_\Gamma)$ as an $\mathbb{F}[U]$-module.

One can see this isomorphism alternatively by Taubes's isomorphism "ECH=SWF" [31, 32, 33, 34, 35], and also by C. Kutluhan, Y-J. Lee, and CH. Taubess isomorphism "SWF=HF" [22, 23, 24, 25, 26]. Here "SWF" stands for the Seiberg-Witten Floer homology and "HF" for $HF^+(-Y, \mathfrak{s}_\Gamma)$.

Thanks to Theorem 5.6 of [28] and Theorem 4.1.1 of [29], we have

$$HF^+(-Y, \mathfrak{s}_0) \cong \mathbb{F}[[U] \oplus \mathbb{F}[[U] \oplus \mathbb{F}[[U] \oplus \mathbb{F}[[U]$$

where $\mathbb{F}[[U] := \mathbb{F}[U^{-1}, U]/U\mathbb{F}[U]$. Therefore, $\sigma_k := U^{-k}([\emptyset])$ are well defined. Since U is degree -2 and $\mathrm{gr}([\emptyset]) = 0$, we have $\mathrm{gr}(\sigma_k) = 2k$.

□

Proof of Theorem 4. As in the proof of Theorem 3, Lemma 4.3 and Theorem 1.3 of [2] imply that $c_k(Y, \lambda_\varepsilon)$ is finite for each k. Then for each k and $0 < \varepsilon \ll 1$, we have $c_k(Y, \lambda_\varepsilon) < L_k < L_\varepsilon$.

By Lemma 4.1, we have an ECH generator α such that $c_k(Y, \lambda_\varepsilon) = \mathcal{A}_{\lambda_\varepsilon}(\alpha)$ and $\mathrm{gr}(\alpha) = 2k$. The ECH generator $\alpha = e_-^{m_-} h_1^{m_1} h_2^{m_2} e_+^{m_+}$ satisfying $\mathrm{gr}(\alpha) = 2k$ and $[\alpha] = 0$ is equivalent to

$$\mathrm{gr}(\alpha) = d^2|e| + m_+ - m_- = 2k$$
$$m_+ + m_1 + m_2 + m_- = d|e| \text{ and } m_1, m_2 \in \{0, 1\} \tag{4.25}$$

for some $d \in \mathbb{Z}_{\geq 1}$. From the relations (4.25), it is easy check that d satisfy

$$d(d-1)|e| \leq 2k \text{ and } d(d+1)|e| \geq 2k.$$

Solve the inequality $d(d-1) \leq 2k/|e| \leq d(d+1)$; we get

$$\sqrt{\frac{2k}{|e|} + \frac{1}{4}} - \frac{1}{2} \leq d \leq \sqrt{\frac{2k}{|e|} + \frac{1}{4}} + \frac{1}{2}.$$

Let $d_{\max} := \sqrt{\frac{2k}{|e|} + \frac{1}{4}} + \frac{1}{2}$ and $d_{\min} := \sqrt{\frac{2k}{|e|} + \frac{1}{4}} - \frac{1}{2}$. Since d is an integer, we have $\lceil d_{\min} \rceil \leq d \leq \lfloor d_{\max} \rfloor$, where $\lceil d_{\min} \rceil$ is the minimal integer that is greater than or equal to $d_{\min}$, and $\lfloor d_{\max} \rfloor$ is the maximal integer that is no more than $d_{\max}$.

Assume that $k \neq \frac{n(n+1)|e|}{2}$ for any $n \in \mathbb{N}$. Then $d_{\min}$ is not an integer. Hence, we have $d_{\min} = \lfloor d_{\min} \rfloor + r$ for some $0 < r < 1$. Note that $d_{\max} = d_{\min} + 1$. Then

$$\lfloor d_{\max} \rfloor = \lfloor \lfloor d_{\min} \rfloor + r + 1 \rfloor = \lfloor d_{\min} \rfloor + 1 = \lceil d_{\min} \rceil.$$

As a result, $d = \lfloor d_{\max} \rfloor = \lceil d_{\min} \rceil$. Therefore,

$$c_k(Y, \lambda) = \lim_{\varepsilon \to 0} c_k(Y, \lambda_\varepsilon) = \lim_{\varepsilon \to 0} 2\lfloor d_{\max} \rfloor |e| + O(\varepsilon) = 2\lfloor d_{\max} \rfloor |e|.$$

Now we consider the case that $k = |e|$. Let $\sigma \in ECH_{2|e|}(Y, \lambda_\varepsilon, 0)$ such that $U^{|e|}\sigma = [\emptyset]$. By the index formula (3.7), $e_+^{d|e|}$ and $e_-^{(d+1)|e|}$ are the only ECH generators with grading $d(d+1)|e|$. By Lemma 3.13, the map $U^{|e|}$ is of the form

$$U^{|e|} \begin{pmatrix} e_+^{|e|} \\ e_-^{2|e|} \end{pmatrix} = \begin{pmatrix} a & 1 \\ b & c \end{pmatrix} \begin{pmatrix} \emptyset \\ e_-^{|e|} \end{pmatrix},$$

By the computation in Lemma 4.3, we know that $U^{|e|} : ECH_{2|e|}(Y, \lambda, 0) \to ECH_0(Y, \lambda, 0)$ is an isomorphism. Thus, we have $b + ac = 1$. If $b = 0$, then $a = c = 1$. If $b = 1$, then $a = 0$ or $c = 0$. In either cases, we have $U(e_+^{|e|} + e_-^{2|e|}) = \emptyset$ or $U(e_-^{2|e|}) = \emptyset$. By (4.21), we have

$$c_{|e|}(Y, \lambda) = \lim_{\varepsilon \to 0} c_{|e|}(Y, \lambda_\varepsilon) = \lim_{\varepsilon \to 0} \mathcal{A}_{\lambda_\varepsilon}(e_-^{2|e|}) = 4|e|.$$

$\square$

References

[1] D. Cristofaro-Gardiner, Symplectic embeddings from concave toric domains into convex ones. **112** (2) 199-232. *J. Diff. Geom.* (2019).

[2] D. Cristofaro-Gardiner, M. Hutchings, V.G.B. Ramos, The asymptotics of ECH capacities. *Invent. math.* **199**, 187–214 (2015).

[3] G. Chen, Cobordism maps on periodic Floer homology induced by elementary Lefschetz fibrations. *Topology Appl.* **302** (2021).

[4] K. Choi, D. Cristofaro-Gardiner, D. Frenkel, M. Hutchings, and V. G. B. Ramos, Symplectic embeddings into four-dimensional concave toric domains. *Journal of Topology* **7**, 4 (2014), 1054-1076.

[5] V. Colin, P. Ghiggini and K. Honda, The equivalence of Heegaard Floer homology and embedded contact homology via open book decompositions I, arXiv:1208.1074.

[6] V. Colin, P. Ghiggini and K. Honda, The equivalence of Heegaard Floer homology and embedded contact homology via open book decompositions II, ArXiv:1208.1077.

[7] V. Colin, P. Ghiggini and K. Honda, The equivalence of Heegaard Floer homology and embedded contact homology III: from hat to plus, ArXiv:1208.1526.

[8] D. Farris, The embedded contact homology of nontrivial circle bundles over Riemann surfaces, PhD Thesis UC Berkeley, 2011., 41 pp.

[9] B. Ferreira, V. G. B. Ramos and A. Vicente, Symplectic embeddings into disk cotangent bundles. *Journal of Fixed Point Theory and Applications* **24**, 3 (2022), 1-31.

[10] B. Ferreira, V. G. B. Ramos and A. Vicente, Gromov width of the disk cotangent bundle of spheres of revolution, arXiv:2301.08528 (2023)

[11] C. Gerig, Taming the pseudoholomorphic beasts in $\mathbb{R} \times S^1 \times S^2$, *Geom. Topol.* **24** (2020) 1791-1839.

[12] C. Gerig, Seiberg-Witten and Gromov invariants for self-dual harmonic 2-forms, *Geom. Topol.* **26** (2022) 3307-3365.

[13] M. Hutchings, An index inequality for embedded pseudoholomorphic curves in symplectizations, *J. Eur. Math. Soc.* **4** (2002) 313–361.

[14] M. Hutchings, The embedded contact homology index revisited, New perspectives and challenges in symplectic field theory, 263–297, CRM Proc. Lecture Notes 49, *Amer. Math. Soc.*, 2009.

[15] M. Hutchings, Quantitative embedded contact homology, *J. Diff. Geom.* **88** (2011), 231-266.

[16] M. Hutchings, Lecture notes on embedded contact homology, Contact and Symplectic Topology, Bolyai Society Mathematical Studies, vol. 26, Springer, 2014, 389–484.

[17] M. Hutchings, Beyond ECH capacities, *Geom. Topol.* **20** (2016) 1085–1126.

[18] M. Hutchings and C. H. Taubes, Proof of the Arnold chord conjecture in three dimensions II, *Geom. Topol.* **17** (2003), 2601–2688.

[19] M. Hutchings and C. H. Taubes, The Weinstein conjecture for stable Hamiltonian structures, *Geom. Topol.* **13** (2009), 901–941.

[20] P. Kronheimer, and T. Mrowka, Monopoles and three-manifolds, New Math. Monogr. 10, Cambridge Univ. Press (2007).

[21] P. Kronheimer, T. Mrowka, P. Ozsváth and Z. Szabó, Monopoles and lens space surgeries, *Annals of Mathematics*, **165** (2007), no. 2, 457-546.

[22] C. Kutluhan, Y-J. Lee, and CH. Taubes, HF= HM, I: Heegaard Floer homology and Seiberg-Witten Floer homology, *Geom. Topol.* **24** (2020) 2829–2854

[23] C. Kutluhan, Y-J. Lee, and CH. Taubes, HF= HM, II: Reeb orbits and holomorphic curves for the ech/Heegaard-Floer correspondence, *Geom. Topol.* **24** (2020) 2855–3012

[24] C. Kutluhan, Y-J. Lee, and CH. Taubes, HF =HM, III: Holomorphic curves and the differential for the ech/Heegaard Floer correspondence, *Geom. Topol.* **24** (2020) 3013–3218

[25] C. Kutluhan, Y-J. Lee, and CH. Taubes, HF = HM, IV: The Seiberg-Witten Floer homology and ech correspondence, *Geom. Topol.* **24** (2020) 3219–3469

[26] C. Kutluhan, Y-J Lee, and CH. Taubes, HF = HM, V: Seiberg-Witten Floer homology and handle addition, *Geom. Topol.* **24** (2020) 3471–3748

[27] J. Nelson, and M. Weiler, Embedded contact homology of prequantization bundles, arXiv:2007.13883 (2020).

[28] P. Ozsváth and Z. Szabó, Knot Floer homology and integer surgeries, *Algebr. Geom. Topol.* **8**(1) (2008), 101153.

[29] K. Park, Some computations and applications of Heegaard Floer correction terms, https://d.lib.msu.edu/etd/2543.

[30] R. Siefring, Intersection theory of punctured pseudoholomorphic curves, *Geom. Topol.* **15** (2011), 2351-2457.

[31] C. H. Taubes, Embedded contact homology and Seiberg–Witten Floer cohomology I, *Geom. Topol.* **14** (2010), 2497-2581.

[32] C. H. Taubes, Enbedded contact homology and Seiberg-Witten Floer homology II, *Geom. Topol.* **14** (2010), 2583-2720.

[33] C. H. Taubes, Enbedded contact homology and Seiberg-Witten Floer homology III, *Geom. Topol.* **14** (2010), 2721-2817.

[34] C. H. Taubes, Enbedded contact homology and Seiberg-Witten Floer homology IV, *Geom. Topol.* **14** (2010), 2819-2960.

[35] C. H. Taubes, Enbedded contact homology and Seiberg-Witten Floer homology V, *Geom. Topol.* **14** (2010), 2961-3000.

[36] C. Wendl, Automatic Transversality and Orbifolds of Punctured Holomorphic Curves in Dimension Four, *Comment. Math. Helv.* **85** (2010), 347-407.

[37] C. Wendl, Lectures on holomorphic curves in symplectic and contact geometry, arXiv:1011.1690 (2010).

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