Submitted to
Journal of the Mathematical Society of Japan

On the Mourre estimates for three-body Floquet Hamiltonians

By Tadayoshi ADACHI

(Received Dec. 13, 2023)

Abstract. In this paper, we consider the Floquet Hamiltonian K associated with a three-body Schrödinger operator with time-periodic pair potentials $H(t)$. By introducing a conjugate operator A for K in the standard Mourre theory, we prove the Mourre estimate for K. As by-products of the Mourre estimate for K, the minimal velocity estimates for the physical propagator $U(t,0)$ generated by $H(t)$ as well as the propagator $e^{-i\sigma K}$ generated by K can be obtained.

1. Introduction

In this paper, we consider a three-body quantum system with time-periodic pair interactions. Since we would like to introduce some notation in many body scattering theory, we denote the number of particles in the system by N for a while. Of course, we mainly consider the case where $N=3$. The system under consideration is governed by the following Schrödinger operator with time-periodic potentials

$$\tilde{H}(t) = \sum_{j=1}^{N} \left(-\frac{1}{2m_j}\Delta_j \right) + V(t), \quad V(t) = \sum_{1\le j<k\le N} V_{jk}(t, r_j - r_k) \tag{1.1}$$

acting on $L^2(\boldsymbol{R}^{d\times N})$, where m_j and $r_j \in \boldsymbol{R}^d$ are the mass and position vector of the j-th particle, respectively, $\Delta_j = \sum_{l=1}^d \partial_{r_{j,l}}^2$ is the Laplacian with respect to r_j, and $V_{jk}(t, r_j - r_k)$'s are pair potentials. We suppose that $V_{jk}(t,y)$'s are real-valued functions on $\boldsymbol{R}\times\boldsymbol{R}^d$ which are periodic in t with a period $T>0$:

$$V_{jk}(t+T, y) = V_{jk}(t,y), \quad (t,y) \in \boldsymbol{R}\times\boldsymbol{R}^d. \tag{1.2}$$

We would like to watch the motion of the system in the center-of-mass frame. To this end, we will introduce the following configuration spaces: We equip $\boldsymbol{R}^{d\times N}$ with the metric $r\cdot\tilde{r} = \sum_{j=1}^N m_j \langle r_j, \tilde{r}_j\rangle$; $r = (r_1,\ldots,r_N)$, $\tilde{r} = (\tilde{r}_1,\ldots,\tilde{r}_N) \in \boldsymbol{R}^{d\times N}$, where $\langle\cdot,\cdot\rangle$ is the standard inner product on $\boldsymbol{R}^d$. We will denote this $\boldsymbol{R}^{d\times N}$ by $\bar{X}$. We usually write $r\cdot r$ as r^2. We put $|r| = \sqrt{r^2}$. $\tilde{H}(t)$ can be written as

$$\tilde{H}(t) = -\frac{1}{2}\Delta_{\bar{X}} + V(t) = \frac{1}{2}(p_{\bar{X}})^2 + V(t)$$

2020 *Mathematics Subject Classification.* Primary 81U10; Secondary 81Q10.
Key Words and Phrases. Mourre estimate, three-body Floquet Hamiltonians, Minimal velocity estimates, AC Stark effect.

acting on $L^2(\bar{X})$, where $\Delta_{\bar{X}}$ is the Laplace-Beltrami operator on $\bar{X}$ and $p_{\bar{X}}$ is the velocity operator $p_{\bar{X}} = (p_1/m_1, \ldots, p_N/m_N) = (-i\nabla_{r_1}/m_1, \ldots, -i\nabla_{r_N}/m_N)$ on $\bar{X}$. From now on, the kinetic energies will be represented by the velocity operators, for simplicity. We define two subspaces X and $X_{\rm cm}$ of $\bar{X}$ as

$$X = \left\{ r \in \bar{X} \,\middle|\, \sum_{j=1}^{N} m_j r_j = 0 \right\}, \quad X_{\rm cm} = \left\{ r \in \bar{X} \,\middle|\, r_1 = \cdots = r_N = 0 \right\}.$$

Then X and $X_{\rm cm}$ are perpendicular to each other, and satisfy $\bar{X} = X \oplus X_{\rm cm}$. $\pi : \bar{X} \to X$ and $\pi_{\rm cm} : \bar{X} \to X_{\rm cm}$ denote the orthogonal projections onto X and $X_{\rm cm}$, respectively. We put $x = \pi r$ and $x_{\rm cm} = \pi_{\rm cm} r$ for $r \in \bar{X}$. Now we introduce the time-dependent Hamiltonian

$$H(t) = \frac{1}{2} p^2 + V(t) \tag{1.3}$$

acting on $\mathscr{H} = L^2(X)$. Then $\tilde{H}(t)$ is represented as

$$\tilde{H}(t) = H(t) \otimes \mathrm{Id} + \mathrm{Id} \otimes \left(\frac{1}{2} (p_{\rm cm})^2 \right)$$

on $L^2(\bar{X}) = \mathscr{H} \otimes L^2(X_{\rm cm})$. Here p and $p_{\rm cm}$ are the velocity operators on X and $X_{\rm cm}$, respectively. We would like to study some scattering problems for this Hamiltonian $H(t)$ with $N = 3$.

A non-empty subset of the set $\{1, \ldots, N\}$ is called a cluster. Let C_j, $1 \leq j \leq m$, be clusters. If $\cup_{1 \leq j \leq m} C_j = \{1, \ldots, N\}$ and $C_j \cap C_k = \emptyset$ for $1 \leq j < k \leq m$, $a = \{C_1, \ldots, C_m\}$ is called a cluster decomposition. $\#(a)$ denotes the number of clusters in a. Let $\mathscr{A}$ be the set of all cluster decompositions. Suppose a, $b \in \mathscr{A}$. If b is obtained as a refinement of a, that is, if each cluster in b is a subset of a cluster in a, we say $b \subset a$, and its negation is denoted by $b \not\subset a$. Any a is regarded as a refinement of itself. The one and N-cluster decompositions are denoted by $a_{\max}$ and $a_{\min}$, respectively. For the sake of brevity, we write

$$\mathscr{A}_0 = \mathscr{A} \setminus \{a_{\max}\}, \;\; \mathscr{A}^0 = \mathscr{A} \setminus \{a_{\min}\}, \;\; \mathscr{A}_0^0 = \mathscr{A} \setminus \{a_{\max}, a_{\min}\} = \mathscr{A}_0 \cap \mathscr{A}^0.$$

The pair (j, k) is identified with the $(N-1)$-cluster decomposition $\{(j,k), (1), \ldots, (\widehat{j}), \ldots, (\widehat{k}), \ldots, (N)\}$. If $N = 3$, then $\{(1,2),\, (1,3),\, (2,3)\}$ is the set of all two-cluster decompositions, and is equal to $\mathscr{A}_0^0$.

Let $a \in \mathscr{A}$. We introduce two subspaces X^a and X_a of X:

$$X^a = \left\{ r \in X \,\middle|\, \sum_{j \in C} m_j r_j = 0 \;\text{ for each cluster } C \text{ in } a \right\},$$
$$X_a = \{ r \in X \,|\, r_j = r_k \;\text{ for each pair } (j,k) \subset a \}.$$

$\pi^a : X \to X^a$ and $\pi_a : X \to X_a$ denote the orthogonal projections onto X^a and X_a, respectively. We put $x^a = \pi^a x$ and $x_a = \pi_a x$ for $x \in X$. Since $X^{(j,k)}$ is identified with the configuration space for the relative position of j-th and k-th particles, one can put

$$V^{(j,k)}(t, x^{(j,k)}) = V_{jk}(t, r_j - r_k).$$

As for $\{X^a\}_{a \in \mathscr{A}}$, $X^a \subset X^b$ is equivalent to $a \subset b$. On the other hand, as for $\{X_a\}_{a \in \mathscr{A}}$, $X_a \subset X_b$ is equivalent to $a \supset b$. For $a,\, b \in \mathscr{A}$, $a \cup b$ stands for the smallest cluster decomposition $c \in \mathscr{A}$ with $a \subset c$ and $b \subset c$. Then we see that $X^a + X^b = X^{a \cup b}$ and $X_a \cap X_b = X_{a \cup b}$ hold. We now define the cluster Hamiltonian

$$H_a(t) = \frac{1}{2}p^2 + V^a(t), \quad V^a(t) = \sum_{(j,k) \subset a} V^{(j,k)}(t, x^{(j,k)}),$$

which governs the motion of the system broken into non-interacting clusters of particles. Then $H_a(t)$ is represented as

$$H_a(t) = H^a(t) \otimes \mathrm{Id} + \mathrm{Id} \otimes \left(\frac{1}{2}(p_a)^2 \right); \quad H^a(t) = \frac{1}{2}(p^a)^2 + V^a(t)$$

on $\mathscr{H} = \mathscr{H}^a \otimes \mathscr{H}_a = L^2(X^a) \otimes L^2(X_a)$, where p^a and p_a are the velocity operators on X^a and X_a, respectively. The intercluster potential $I_a(t)$ is given by

$$I_a(t, x) = V(t, x) - V^a(t, x) = \sum_{(j,k) \not\subset a} V^{(j,k)}(t, x^{(j,k)}).$$

Under some suitable conditions on $V_{jk}(t)$, the existence and uniqueness of the unitary propagator $U(t, s)$ generated by $H(t)$ can be guaranteed, even if $N \geq 3$ (see e.g. Yajima [**38, 39**]). In the study of the asymptotic behavior of $U(t, s)\phi$, $\phi \in \mathscr{H}$, as $t \to \pm\infty$, we will frequently utilize the so-called Floquet Hamiltonian K associated with $H(t)$ (see e.g. Howland [**19, 20**], Yajima [**37**]): Let $\boldsymbol{T} = \boldsymbol{R}/(T\boldsymbol{Z})$ be the torus. Set $\mathscr{K} = L^2(\boldsymbol{T}; \mathscr{H}) \cong L^2(\boldsymbol{T}) \otimes \mathscr{H}$, and introduce a strongly continuous one-parameter unitary group $\{\hat{U}(\sigma)\}_{\sigma \in \boldsymbol{R}}$ on $\mathscr{K}$ given by

$$(\hat{U}(\sigma)\Phi)(t) = U(t, t - \sigma)\Phi(t - \sigma) \tag{1.4}$$

for $\Phi \in \mathscr{K}$. By virtue of Stone's theorem, $\hat{U}(\sigma)$ is written as

$$\hat{U}(\sigma) = e^{-i\sigma K} \tag{1.5}$$

with a unique self-adjoint operator K on $\mathscr{K}$. K is called the Floquet Hamiltonian associated with $H(t)$, and is equal to the natural self-adjoint realization of $-i\partial_t + H(t)$. Here we denote by D_t the operator $-i\partial_t$ with domain $AC(\boldsymbol{T})$, which is the space of absolutely continuous functions on $\boldsymbol{T}$ with their derivatives being square integrable (following the notation in Reed-Simon [**30**]). As is well-known, D_t is self-adjoint on $L^2(\boldsymbol{T})$, and its spectrum $\sigma(D_t)$ is equal to $\omega\boldsymbol{Z}$ with $\omega = 2\pi/T$. For $a \in \mathscr{A}$, we also introduce the cluster Floquet Hamiltonian K_a associated with the cluster Hamiltonian $H_a(t)$ by $K_a = D_t + H_a(t)$ on $\mathscr{K}$. In particular, we have $K_{a_{\max}} = K$. We also write $K_{a_{\min}}$ as $K_0 = D_t + H_0$, which is the free Floquet Hamiltonian associated with the free Hamiltonian $H_0 = p^2/2$ acting on $\mathscr{H}$, for the sake of simplicity. Moreover, we introduce the subsystem Floquet Hamiltonian K^a associated with the subsystem Hamiltonian $H^a(t)$ by $K^a = D_t + H^a(t)$ on $\mathscr{K}^a = L^2(\boldsymbol{T}; \mathscr{H}^a) \cong L^2(\boldsymbol{T}) \otimes \mathscr{H}^a$. In particular, we see that $K^{a_{\max}} = K$, $K^{a_{\min}} = D_t$, and for $a \in \mathscr{A}$, K_a is represented as

$$K_a = K^a \otimes \mathrm{Id} + \mathrm{Id} \otimes \left(\frac{1}{2}(p_a)^2\right)$$

on $\mathscr{K} = \mathscr{K}^a \otimes \mathscr{H}_a = L^2(\boldsymbol{T}; \mathscr{H}^a) \otimes \mathscr{H}_a$.

If $H(t)$ is strictly time-dependent, the lack of energy conservation becomes a barrier in the study of the asymptotic behavior of $U(t, s)\phi$. Howland [**19**] proposed the stationary scattering theory for time-dependent Hamiltonians, by introducing a new Hamiltonian $-i\partial_t + H(t)$ acting on $L^2(\boldsymbol{R}; \mathscr{H})$. As mentioned above, its formulation was the quantum analogue to the procedure in the classical mechanics in order to *recover* the conservation of energy. Yajima [**37**] applied this Howland method to the two-body quantum system with a time-periodic short-range potential, and studied the problem of the asymptotic completeness of the wave operators

$$W^{\pm}(s) = \operatorname*{s-lim}_{t \to \pm\infty} U(t, s)^* e^{-i(t-s)H_0}$$

(see also Howland [**20**]). In fact, he proved the asymptotic completeness of

$$\mathscr{W}^{\pm} = \operatorname*{s-lim}_{\sigma \to \pm\infty} e^{i\sigma K} e^{-i\sigma K_0}$$

firstly, and deduced that of $W^{\pm}(s)$ from this result. Such a method is called the Howland-Yajima method. This method, together with the Faddeev method, was applied to the three-body case under some *very short-range* conditions by Korotyaev [**23**] and Nakamura [**28**] later. However, even in the case where $N = 3$, the problem of the asymptotic completeness under a general short-range condition on pair potentials $V^{(j,k)}(t, x^{(j,k)})$ has not been solved yet, unlike in the case of time-independent many body Schrödinger operators (see e.g. Sigal-Soffer [**31**], Graf [**15**], Yafaev [**36**], Dereziński [**11**], Dereziński-Gérard [**12**], and so on), as far as we know. Thus it is worth continuing the study of the spectral and scattering theory for N-body Floquet Hamiltonians, even in the case where $N = 3$.

In this paper, we would like to propose the definition of a conjugate operator for K with $N = 3$ in the standard Mourre theory. We will impose the following *well-regulated* condition $(V_{\mathrm{WR}})_\rho$ on V with $\rho > 0$:

$(V_{\mathrm{WR}})_\rho$ $V_{jk}(t, y)$, $(j, k) \in \mathscr{A}$, belongs to $C^2(\boldsymbol{R} \times \boldsymbol{R}^d; \boldsymbol{R})$, is T-periodic in t, and satisfies the decaying conditions

$$\sup_{t \in \boldsymbol{R}} |(\partial_t^m \partial_y^\alpha V_{jk})(t, y)| \le C_{m,\alpha} \langle y \rangle^{-\rho-(m+|\alpha|)}, \quad 0 \le m + |\alpha| \le 2. \tag{1.6}$$

Here $\langle y \rangle = (1 + y^2)^{1/2}$. In Adachi-Kiyose [**5**], the condition (1.6) was imposed on the regular parts of pair potentials. First we recall known results in the case where $N = 2$ for reference. Yokoyama [**41**] introduced the self-adjoint operator

$$\bar{A}_{0,1} = \frac{1}{2}\{x \cdot p\langle p \rangle^{-2} + \langle p \rangle^{-2} p \cdot x\}; \quad \langle p \rangle^{-2} = (1 + p^2)^{-1} \tag{1.7}$$

on $\mathscr{K}$ as a conjugate operator for K. For the sake of brevity, we will use the notation $\operatorname{Re} T$ for an operator on $\mathscr{K}$ in this paper, which is defined by $\operatorname{Re} T = (T + T^*)/2$. Then $\bar{A}_{0,1}$ can be written as $\operatorname{Re}(x \cdot p\langle p \rangle^{-2})$. Roughly speaking, $\bar{A}_{0,1}$ is defined by multiplying the generator of dilations

$$\hat{A}_0 = \mathrm{Re}\,(x \cdot p) \tag{1.8}$$

and the resolvent $\langle p\rangle^{-2}$ of p^2. He established the following Mourre estimate under some suitable conditions on V, by using the commutation relation $i[K_0, \bar{A}_{0,1}] = p^2\langle p\rangle^{-2} = p^2(1+p^2)^{-1}$ by simple calculation. Put

$$d_0(\lambda) = \mathrm{dist}(\lambda, \omega\boldsymbol{Z}), \quad d_1(\lambda) = \mathrm{dist}(\lambda, \omega\boldsymbol{Z} \cap (-\infty, \lambda])$$

for $\lambda \in \boldsymbol{R}$. Here we note that $\omega\boldsymbol{Z} = \sigma(D_t)$ is equal to $\Theta = \sigma(K^{a_{\min}})$, which is the threshold set for K with $N = 2$. Suppose $\lambda_0 \in \boldsymbol{R} \setminus \omega\boldsymbol{Z}$, that is, λ_0 is a non-threshold energy, and $0 < \delta < \mathrm{dist}(\lambda_0, \omega\boldsymbol{Z}) = d_0(\lambda_0)$. Then, for any $f_{\lambda_0,\delta} \in C_0^\infty(\boldsymbol{R}; \boldsymbol{R})$ supported in $[\lambda_0-\delta, \lambda_0+\delta]$, the Mourre estimate

$$f_{\lambda_0,\delta}(K)i[K, \bar{A}_{0,1}]f_{\lambda_0,\delta}(K) \geq \frac{2(d_1(\lambda_0)-\delta)}{1+2(d_1(\lambda_0)-\delta)}f_{\lambda_0,\delta}(K)^2 + C_1 \tag{1.9}$$

holds with some compact operator C_1 on $\mathscr{K}$. This estimate (1.9) is slightly better than the one obtained in [**41**]

$$f_{\lambda_0,\delta}(K)i[K, \bar{A}_{0,1}]f_{\lambda_0,\delta}(K) \geq \frac{2(d_0(\lambda_0)-\delta)}{1+2(d_0(\lambda_0)-\delta)}f_{\lambda_0,\delta}(K)^2 + C_1'$$

with some compact operator C_1' on $\mathscr{K}$, since $d_0(\lambda_0) \leq d_1(\lambda_0)$. Here we note that the positive constant of the Mourre estimate (1.9) depends on λ_0 strictly but the conjugate operator $\bar{A}_{0,1}$ is independent of λ_0. However, its extension to the case where $N \geq 3$ has not been obtained yet, as far as we know (see also Møller-Skibsted [**27**]). Recently, Adachi-Kiyose [**5**] proposed an alternative conjugate operator for K with $N = 2$ at a non-threshold energy λ_0: Let $\lambda_0 \in \boldsymbol{R}\setminus\omega\boldsymbol{Z}$. Then there exists a unique $n_{\lambda_0} \in \boldsymbol{Z}$ such that $\lambda_0 \in I_{n_{\lambda_0}}$. Take δ as $0 < \delta < \mathrm{dist}(\lambda_0, \omega\boldsymbol{Z}) = d_0(\lambda_0)$. Since $\lambda_0 - \delta \in I_{n_{\lambda_0}}$, it is obvious that $\lambda_0 - \delta \in \boldsymbol{R} \setminus \omega\boldsymbol{Z} = \rho(D_t)$. Then we introduce the self-adjoint operator

$$A_{\lambda_0,\delta} = R_{-D_t,\lambda_0-\delta} \otimes \hat{A}_0; \quad R_{-D_t,\lambda_0-\delta} = (\lambda_0 - \delta - D_t)^{-1} \tag{1.10}$$

on $\mathscr{K} \cong L^2(\boldsymbol{T}) \otimes \mathscr{H}$, by multiplying $\hat{A}_0$ and the resolvent $R_{-D_t,\lambda_0-\delta}$ of D_t. Here we note that $R_{-D_t,\lambda_0-\delta}$ is bounded and self-adjoint, and $A_{\lambda_0,\delta}$ satisfies the commutation relation

$$i[K_0, A_{\lambda_0,\delta}] = p^2 R_{-D_t,\lambda_0-\delta} = 2(K_0 - D_t)(\lambda_0 - \delta - D_t)^{-1}.$$

Then the Mourre estimate

$$f_{\lambda_0,\delta}(K)i[K, A_{\lambda_0,\delta}]f_{\lambda_0,\delta}(K) \geq 2f_{\lambda_0,\delta}(K)^2 + C_{f_{\lambda_0,\delta}} \tag{1.11}$$

holds with some compact operator $C_{f_{\lambda_0,\delta}}$ on $\mathscr{K}$. Here we note that the positive constant of the Mourre estimate (1.11) is independent of λ_0 but the conjugate operator $A_{\lambda_0,\delta}$ depends on λ_0 strictly. Its extension to the case where $N \geq 3$ has not been obtained generally yet, except in the case where all the pair potentials are independent of t.

The aim of this paper is that we will introduce a conjugate operator for K with $N = 3$. As is pointed out by Møller-Skibsted [**27**], it is important in obtaining the Mourre estimates for time-independent many body Schrödinger operators that the generator of dilations $\hat{A}_0$ in (1.8) can be decomposed into the sum

$$(\hat{A}_0)^a \otimes \mathrm{Id} + \mathrm{Id} \otimes (\hat{A}_0)_a$$

acting on $\mathscr{H} \cong \mathscr{H}^a \otimes \mathscr{H}_a$, for $a \in \mathscr{A}$, where

$$\begin{aligned} (\hat{A}_0)^a &= \frac{1}{2}(x^a \cdot p^a + p^a \cdot x^a) = \mathrm{Re}\,(x^a \cdot p^a), \\ (\hat{A}_0)_a &= \frac{1}{2}(x_a \cdot p_a + p_a \cdot x_a) = \mathrm{Re}\,(x_a \cdot p_a). \end{aligned} \tag{1.12}$$

Unfortunately, the conjugate operator $\bar{A}_{0,1}$ in (1.7) does not have such a property. This seems one of the reasons why its extension to the case where $N \geq 3$ has not been given yet. On the other hand, the conjugate operator $A_{\lambda_0,\delta}$ in (1.10) can be decomposed into the sum

$$R_{-D_t,\lambda_0-\delta} \otimes \{(\hat{A}_0)^a \otimes \mathrm{Id} + \mathrm{Id} \otimes (\hat{A}_0)_a\}$$

acting on $\mathscr{K} \cong L^2(\boldsymbol{T}) \otimes \mathscr{H}^a \otimes \mathscr{H}_a$, for $a \in \mathscr{A}$. If $N = 3$ and $a \in \mathscr{A}$ is a pair, that is, $a \in \mathscr{A}_0^0 = \mathscr{A} \setminus \{a_{\max}, a_{\min}\}$, then one can recognize the operator $R_{-D_t,\lambda_0-\delta} \otimes (\hat{A}_0)^a$ as a conjugate operator for $K^a = D_t + H^a(t)$ acting on $\mathscr{K}^a = L^2(\boldsymbol{T}; \mathscr{H}^a) \cong L^2(\boldsymbol{T}) \otimes \mathscr{H}^a$, by virtue of a result of Adachi-Kiyose [**5**]. However, we cannot interpret the operator $R_{-D_t,\lambda_0-\delta} \otimes (\hat{A}_0)_a$ as a conjugate operator for the intercluster Hamiltonian $(p_a)^2/2$ acting on $\mathscr{H}_a$, unfortunately. The operator $R_{-D_t,\lambda_0-\delta} \otimes (\hat{A}_0)_a$ can be a conjugate operator for the Floquet Hamiltonian $D_t + (p_a)^2/2$ acting on $L^2(\boldsymbol{T}; \mathscr{H}_a) \cong L^2(\boldsymbol{T}) \otimes \mathscr{H}_a$ associated with $(p_a)^2/2$. But, since $K_a = K^a \otimes \mathrm{Id} + \mathrm{Id} \otimes (p_a)^2/2$, what we need here is not a conjugate operator for $D_t + (p_a)^2/2$ but that just for $(p_a)^2/2$. We think that this is one of the reasons why any extension of $A_{\lambda_0,\delta}$ to the case where $N \geq 3$ has not been given yet.

In order to overcome the difficulty mentioned above, roughly speaking, we will recognize the operator

$$\bar{A}_{a,1} = \mathrm{Re}\,(x_a \cdot p_a \langle p_a \rangle^{-2}); \quad \langle p_a \rangle^{-2} = (1 + (p_a)^2)^{-1}$$

acting on $\mathscr{H}_a$ as a conjugate operator for $(p_a)^2/2$, and the sum

$$R_{-D_t,\lambda_0-\delta} \otimes (\hat{A}_0)^a \otimes \mathrm{Id} + \mathrm{Id} \otimes \mathrm{Id} \otimes \bar{A}_{a,1}$$

as a conjugate operator A_a for $K_a = D_t + H_a(t)$ acting on $\mathscr{K} \cong L^2(\boldsymbol{T}) \otimes \mathscr{H}^a \otimes \mathscr{H}_a$. After introducing A_a's, we will glue these together by using a partition of unity of X. This is our strategy of introducing a conjugate operator A for K with $N = 3$. Obtaining a conjugate operator for K with $N = 3$ is the first step for the definition of conjugate operators for K with $N \geq 4$. In our strategy, for example, to define a conjugate operator for K with $N = 4$, we need conjugate operators for not only two-body subsystem Floquet Hamiltonians but also three-body subsystem Floquet Hamiltonians. The latter has not been obtained until now.

Now we will give the precise definition of A. Without loss of generality, we may assume that an energy λ_0 belongs to the interval $[0, \omega)$, because the spectrum $\sigma(K)$ of K is ω-periodic (see Proposition 3.4), as is well-known. As for a conjugate operator for the free Floquet Hamiltonian

$K_0 = D_t + p^2/2$, we introduce

$$\tilde{A}_0 = \langle D_t \rangle_{2\omega}^{-1/2} \hat{A}_0 \langle D_t \rangle_{2\omega}^{-1/2} = \mathrm{Re}\,(x \cdot p \langle D_t \rangle_{2\omega}^{-1}) \tag{1.13}$$

acting on $\mathscr{K}$, where $\langle D_t \rangle_{2\omega} = ((2\omega)^2 + (D_t)^2)^{1/2}$. In this work, we have found that one can utilize the positive weight $\langle D_t \rangle_{2\omega}^{-1}$ in place of the signed weight $R_{-D_t, \lambda_0 - \delta}$ in the definition of a conjugate operator $\tilde{A}_0$ for K_0. We think that $\langle D_t \rangle_{2\omega}^{-1}$ is more suitable for $\tilde{A}_0$ than $R_{-D_t, \lambda_0 - \delta}$. For instance, $i[K_0, \tilde{A}_0] = \langle D_t \rangle_{2\omega}^{-1} p^2 = \langle D_t \rangle_{2\omega}^{-1/2} p^2 \langle D_t \rangle_{2\omega}^{-1/2}$ holds (see Lemma 4.1 in §4), which makes its non-negativ- ity available. This is a big advantage of $\langle D_t \rangle_{2\omega}^{-1}$. As for the detailed arguments, see §4 and §5. The term $(2\omega)^2$ in the definition of $\langle D_t \rangle_{2\omega}$ may be replaced by 1, that is, in (1.13), $\langle D_t \rangle_{2\omega}^{-1}$ can be replaced by $\langle D_t \rangle^{-1} = (1 + (D_t)^2)^{-1/2}$. However, for the sake of simplifying the proof of the Mourre estimate in the case where $N = 3$, we will adopt not $\langle D_t \rangle^{-1}$ but $\langle D_t \rangle_{2\omega}^{-1}$ as the positive weight for $\tilde{A}_0$ in this paper. For $a \in \mathscr{A}_0^0$, we also introduce

$$\begin{aligned}
(\tilde{A}_0)^a &= \langle D_t \rangle_{2\omega}^{-1/2} (\hat{A}_0)^a \langle D_t \rangle_{2\omega}^{-1/2} = \mathrm{Re}\,(x^a \cdot p^a \langle D_t \rangle_{2\omega}^{-1}), \\
(\tilde{A}_0)_a &= \langle D_t \rangle_{2\omega}^{-1/2} (\hat{A}_0)_a \langle D_t \rangle_{2\omega}^{-1/2} = \mathrm{Re}\,(x_a \cdot p_a \langle D_t \rangle_{2\omega}^{-1}), \\
\bar{A}_a &= \mathrm{Re}\,(x_a \cdot p_a \langle p_a \rangle^{-2}); \quad \langle p_a \rangle^{-2} = (1 + (p_a)^2)^{-1}.
\end{aligned} \tag{1.14}$$

As will be seen in §4, $(\tilde{A}_0)^a$ can be recognized as a conjugate operator for the two-body subsystem Floquet Hamiltonian K^a on $\mathscr{K}^a$. $\tilde{A}_0$, $(\tilde{A}_0)^a$'s and $\bar{A}_a$'s are all self-adjoint. For the sake of glueing these together, we have mainly three choices of partitions of unity of X; the reducing partition of unity (see e.g. Cycon-Froese-Kirsch-Simon [**10**]), which was used in Froese-Herbst [**13**] for the sake of showing the Mourre estimate for time-independent N-body Schrödinger operators, the Graf partition of unity (see e.g. Graf [**15**], Dereziński [**11**], Dereziński-Gérard [**12**] and Gérard-Łaba [**14**]), and the Yafaev partition of unity (see e.g. Yafaev [**36**], Hunziker-Sigal [**21**] and Gérard-Łaba [**14**]). In this paper, we utilize the Yafaev partition of unity $\{\tilde{J}_a\}_{a \in \mathscr{A}} \subset C^\infty(X; \boldsymbol{R})$ of X such that $\sum_{a \in \mathscr{A}} \tilde{J}_a(x)^2 \equiv 1$. In §2, we will give its definition and state its useful properties. By using $\{\tilde{J}_a\}_{a \in \mathscr{A}}$, we will introduce

$$\begin{aligned}
A &= \sum_{a \in \mathscr{A} \setminus \mathscr{A}_0^0} \tilde{J}_a(x) \tilde{A}_0 \tilde{J}_a(x) + \sum_{a \in \mathscr{A}_0^0} \tilde{J}_a(x) (\tilde{A}_0)^a \tilde{J}_a(x) + L_0 \sum_{a \in \mathscr{A}_0^0} \tilde{J}_a(x_a) \bar{A}_a \tilde{J}_a(x_a) \\
&= \tilde{A}_0 + \sum_{a \in \mathscr{A}_0^0} \tilde{J}_a(x) \{-(\tilde{A}_0)_a\} \tilde{J}_a(x) + L_0 \sum_{a \in \mathscr{A}_0^0} \tilde{J}_a(x_a) \bar{A}_a \tilde{J}_a(x_a),
\end{aligned} \tag{1.15}$$

with sufficiently large $L_0 > 0$. As will be seen in §2, the second representation of A can be derived directly from the first one, by $\sum_{a \in \mathscr{A}} \tilde{J}_a(x)^2 \equiv 1$. In our dealing with the errors like $i[K_0, \tilde{J}_a(x)](\tilde{A}_0)_a \tilde{J}_a(x)$ and $i[K_0, \tilde{J}_a(x_a)] \bar{A}_a \tilde{J}_a(x_a)$, which come from the *overlap width* of glueing $\tilde{A}_0$, $(\tilde{A}_0)^a$'s and $\bar{A}_a$'s by some partition of unity of X, the Yafaev partition of unity is much handier than the other two ones: In our analysis, we give importance to that $\tilde{J}_a(x)$ with $a \in \mathscr{A}_0$ is homogeneous of degree 0 outside some compact neighborhood of the origin 0 of X. Unfortunately, the Graf one does not satisfy the property, unlike the reducing one and the Yafaev one. Moreover, as for the Yafaev partition of unity, by introducing the so-called intercluster distance $|x|_a = \min_{b \in \mathscr{A}_0, b \not\subset a} |x^b|$ for $a \in \mathscr{A}_0$ (see (2.3) in §2), the estimate $|x|_a \geq c_a |x|$ holds for $x \in \operatorname{supp} \tilde{J}_a$ with some $c_a > 0$ (see Lemma 2.3 in §2). This property is important in our analysis. For instance, one can utilize the decay estimates with respect to $\langle x\rangle$, which are stated in Corollary 2.4 in §2. By taking account of these, we adopt the Yafaev one for the sake of glueing $\tilde{A}_0$, $(\tilde{A}_0)^a$'s and $\bar{A}_a$'s. The factor $\tilde{J}_a(x_a)$ of the last term in (1.15) cannot be replaced by $\tilde{J}_a(x)$, because we need the K_0-boundedness of $i[K_0, A]$. We also need the large parameter L_0 for the sake of dealing with the errors mentioned above. As for the detailed arguments, see §5.

Now we state the main results of this paper.

THEOREM 1.1. *Suppose $N = 3$. Assume V satisfies $(V_{\mathrm{WR}})_\rho$ with some $\rho > 0$. Put*

$$\Theta = \bigcup_{a\in\mathscr{A}_0} \sigma_{\mathrm{pp}}(K^a), \quad \widehat{\Theta} = \bigcup_{a\in\mathscr{A}} \sigma_{\mathrm{pp}}(K^a) = \Theta \cup \sigma_{\mathrm{pp}}(K).$$

Let $\lambda_0 \in [0, \omega) \setminus \Theta$ and $\epsilon > 0$. Then there exist the Yafaev partitions of unity $\{\tilde{J}_a\}_{a\in\mathscr{A}}$'s and sufficiently large $L_0 > 0$ in the definition (1.15) *of A such that for A defined by using these, the following hold:*
(1) *Put*

$$d_0(\lambda) = \mathrm{dist}(\lambda, \Theta), \quad d_1(\lambda) = \mathrm{dist}(\lambda, \Theta \cap (-\infty, \lambda])$$

for $\lambda \in \boldsymbol{R}$. Note $d_0(\lambda) \le d_1(\lambda)$. Then there exists a small $\delta_{\epsilon,0} > 0$ such that $\delta_{\epsilon,0} \le d_0(\lambda_0)/2$ and the following holds: Let $0 < \delta \le \delta_{\epsilon,0}$. Then, for any $f_{\lambda_0,\delta} \in C_0^\infty(\boldsymbol{R};\boldsymbol{R})$ supported in $[\lambda_0 - \delta, \lambda_0 + \delta]$,

$$f_{\lambda_0,\delta}(K) i[K, A] f_{\lambda_0,\delta}(K) \ge \frac{d_1(\lambda_0) - \delta - \epsilon}{\omega} f_{\lambda_0,\delta}(K)^2 + C_0 \tag{1.16}$$

holds with some compact operator C_0 on $\mathscr{K}$.

Hence, for any $\hat{\delta}$ such that $0 < \hat{\delta} < \delta$, $\sigma_{\mathrm{pp}}(K) \cap (\lambda_0 - \hat{\delta}, \lambda_0 + \hat{\delta})$ is finite, and the eigenvalues of K in $(\lambda_0 - \hat{\delta}, \lambda_0 + \hat{\delta})$ are of finite multiplicity.
(2) *Suppose $\lambda_0 \in [0, \omega) \setminus \widehat{\Theta}$. Take ϵ as $0 < 2\epsilon < d_1(\lambda_0) - d_0(\lambda_0)/2$. Then there exists a small $\delta_{\epsilon,1} > 0$ such that $\delta_{\epsilon,1} \le \delta_{\epsilon,0} \le d_0(\lambda_0)/2$, and for any $f_{\lambda_0,\delta} \in C_0^\infty(\boldsymbol{R};\boldsymbol{R})$ supported in $[\lambda_0 - \delta, \lambda_0 + \delta]$ with $0 < \delta \le \delta_{\epsilon,1}$,*

$$f_{\lambda_0,\delta}(K) i[K, A] f_{\lambda_0,\delta}(K) \ge \frac{d_1(\lambda_0) - \delta - 2\epsilon}{\omega} f_{\lambda_0,\delta}(K)^2 \tag{1.17}$$

holds. Suppose $s > 1/2$ and $0 < \hat{\delta} < \delta_{\epsilon,1}$. Then

$$\sup_{\substack{\mathrm{Re}\, z\in[\lambda_0-\hat{\delta},\lambda_0+\hat{\delta}] \\ \mathrm{Im}\, z\neq 0}} \|\langle A\rangle^{-s}(K - z)^{-1}\langle A\rangle^{-s}\|_{\mathscr{B}(\mathscr{K})} < \infty \tag{1.18}$$

holds. Moreover, $\langle A\rangle^{-s}(K - z)^{-1}\langle A\rangle^{-s}$ is a $\mathscr{B}(\mathscr{K})$-valued $\theta(s)$-Hölder continuous function on $z \in S_{\lambda_0,\hat{\delta},\pm}$ with some $0 < \theta(s) < 1$, where $S_{\lambda_0,\hat{\delta},\pm} = \{\zeta \in \boldsymbol{C} \mid \mathrm{Re}\,\zeta \in [\lambda_0 - \hat{\delta}, \lambda_0 + \hat{\delta}],\ 0 < \pm\mathrm{Im}\,\zeta \le 1\}$. And, there exist the norm limits

$$\langle A\rangle^{-s}(K - (\lambda \pm i0))^{-1}\langle A\rangle^{-s} = \lim_{\varepsilon\to+0} \langle A\rangle^{-s}(K - (\lambda \pm i\varepsilon))^{-1}\langle A\rangle^{-s}$$

in $\mathscr{B}(\mathscr{K})$ for any $\lambda \in [\lambda_0 - \hat{\delta}, \lambda_0 + \hat{\delta}]$. $\langle A\rangle^{-s}(K - (\lambda \pm i0))^{-1}\langle A\rangle^{-s}$ are also $\theta(s)$-Hölder continuous in λ.

If $\lambda_0 \in [0, \omega) \cap \Theta$, then a weaker estimate like

$$f_{\lambda_0,\delta}(K) i[K, A] f_{\lambda_0,\delta}(K) \geq \frac{-\epsilon}{\omega} f_{\lambda_0,\delta}(K)^2 + C_0$$

holds with some compact operator C_0 on $\mathscr{K}$. When we will introduce a conjugate operator for the Floquet Hamiltonian K with $N = 4$, not only (1.16) but also this estimate will be needed.

COROLLARY 1.2. *Assume V satisfies $(V_{\mathrm{WR}})_\rho$ with some $\rho > 0$. Then the following hold:*
(1) The eigenvalues of K in $\boldsymbol{R} \setminus \Theta$ can accumulate only at Θ. Moreover, $\widehat{\Theta}$ is a countable closed set.
(2) Let I be a compact interval in $\boldsymbol{R} \setminus \widehat{\Theta}$. Suppose $1/2 < s \leq 1$. Then

$$\sup_{\substack{\mathrm{Re}\, z \in I \\ \mathrm{Im}\, z \neq 0}} \|\langle x\rangle^{-s}(K - z)^{-1}\langle x\rangle^{-s}\|_{\mathscr{B}(\mathscr{K})} < \infty \tag{1.19}$$

holds. Moreover, $\langle x\rangle^{-s}(K - z)^{-1}\langle x\rangle^{-s}$ is a $\mathscr{B}(\mathscr{K})$-valued $\theta(s)$-Hölder continuous function on $z \in S_{I,\pm}$, where $S_{I,\pm} = \left\{\zeta \in \boldsymbol{C} \,\middle|\, \mathrm{Re}\,\zeta \in I,\ 0 < \pm \mathrm{Im}\,\zeta \leq 1\right\}$. And, there exist the norm limits

$$\langle x\rangle^{-s}(K - (\lambda \pm i0))^{-1}\langle x\rangle^{-s} = \lim_{\varepsilon \to +0} \langle x\rangle^{-s}(K - (\lambda \pm i\varepsilon))^{-1}\langle x\rangle^{-s}$$

in $\mathscr{B}(\mathscr{K})$ for $\lambda \in I$. $\langle x\rangle^{-s}(K - (\lambda \pm i0))^{-1}\langle x\rangle^{-s}$ are also $\theta(s)$-Hölder continuous in λ.

In order to obtain Corollary 1.2, we use the argument of Perry-Sigal-Simon [**29**], and the boundedness of $A(K - \lambda_0 - i)^{-1}\langle x\rangle^{-1}$, which can be given by that $\langle D_t\rangle^{-1}(K - \lambda_0 - i)^{-1}\langle p\rangle^2$ is bounded (see Proposition 3.5 in §3). By virtue of this, one can show that $A(K - \lambda_0 - i)^{-1}\langle p\rangle\langle x\rangle^{-1}$ and $A(K - \lambda_0 - i)^{-1}\langle D_t\rangle^{1/2}\langle x\rangle^{-1}$ are also bounded. Then the limiting absorption principle like

$$\sup_{\substack{\mathrm{Re}\, z \in I \\ \mathrm{Im}\, z \neq 0}} \|\langle x\rangle^{-s}\mathcal{D}^{s'}(K - z)^{-1}\mathcal{D}^{s'}\langle x\rangle^{-s}\|_{\mathscr{B}(\mathscr{K})} < \infty$$

with $s > 1/2$ and some $s' > 0$ may be expected, where $\mathcal{D} = \langle p\rangle + \langle D_t\rangle^{1/2}$ is equivalent to $\mathscr{D}^{1/2} = (\langle p\rangle^4 + \langle D_t\rangle^2)^{1/4}$ as weights, which was introduced in Kuwabara-Yajima [**24**] for the sake of obtaining a refined limiting absorption principle for K with $N = 2$ in the Besov space setting. In fact, the result of [**24**] yields the estimate

$$\sup_{\substack{\mathrm{Re}\, z \in I \\ \mathrm{Im}\, z \neq 0}} \|\langle x\rangle^{-s}\mathscr{D}^{1/2}(K - z)^{-1}\langle x\rangle^{-s}\|_{\mathscr{B}(\mathscr{K})} < \infty$$

with $s > 1/2$. By virtue of complex interpolation, one can also obtain a refined limiting absorption principle

$$\sup_{\substack{\mathrm{Re}\, z \in I \\ \mathrm{Im}\, z \neq 0}} \|\langle x\rangle^{-s}\mathscr{D}^{1/4}(K - z)^{-1}\mathscr{D}^{1/4}\langle x\rangle^{-s}\|_{\mathscr{B}(\mathscr{K})} < \infty$$

with $s > 1/2$. But this has not been given by our analysis yet. It is caused by the unboundedness of $(K - \lambda_0 - i)^{-1} \langle p \rangle \langle x \rangle^{-1}$ and $(K - \lambda_0 - i)^{-1} \langle D_t \rangle^{1/2} \langle x \rangle^{-1}$. As for general N-body Floquet Hamiltonians, a refined limiting absorption principle for K

$$\sup_{\substack{\mathrm{Re}\, z \in I \\ \mathrm{Im}\, z \neq 0}} \| \langle x \rangle^{-s} \langle p \rangle^{r} (K - z)^{-1} \langle p \rangle^{r} \langle x \rangle^{-s} \|_{\mathscr{B}(\mathscr{K})} < \infty$$

with $0 \le r < 1/2 < s \le 1$ was obtained by Møller-Skibsted [**27**]. They used an extended Mourre theory due to Skibsted [**34**], and took a conjugate operator for K in the extended Mourre theory as $\hat{A}_0$. However, we would like to stick to find an option of a conjugate operator for K not in an extended but in the standard Mourre theory.

As is well-known, the limiting absorption principle (1.19) yields the local K-smoothness of $\langle x \rangle^{-s}$ with $s > 1/2$

$$\int_{-\infty}^{\infty} \| \langle x \rangle^{-s} e^{-i\sigma K} f_{\lambda_0,\delta}(K) \varPhi \|_{\mathscr{K}}^2 \, d\sigma \le C \| \varPhi \|_{\mathscr{K}}^2 \tag{1.20}$$

for $\lambda_0 \in [0, \omega) \setminus \widehat{\Theta}$. (1.20) was already obtained by [**27**], even if $N > 3$. However, (1.20) is not enough for the proof of the asymptotic completeness in the case where $N \ge 3$, unlike in the case where $N = 2$. For instance, when K is a time-independent many body Schrödinger operator $p^2/2 + V$ with short-range pair interactions, instead of the propagation estimate (1.20) with $\mathscr{K} = L^2(X)$, the so-called minimal velocity estimate

$$\int_{1}^{\infty} \left\| F\left(\frac{|x|}{\sigma} \le \tilde{c}_{\lambda_0} \right) e^{-i\sigma K} f_{\lambda_0,\delta}(K) \varPhi \right\|_{\mathscr{K}}^2 \frac{d\sigma}{\sigma} \le C \| \varPhi \|_{\mathscr{K}}^2$$

with $\tilde{c}_{\lambda_0} > 0$ and $\mathscr{K} = L^2(X)$ has been used as a key propagation estimate in the proof of the asymptotic completeness (see e.g. Graf [**15**]). Here we used the following convention for smooth cut-off functions F with $0 \le F \le 1$: For sufficiently small $\delta > 0$, we define

$$\begin{aligned} F(s \le d) &= 1 \quad \text{for} \quad s \le d - \delta, \quad = 0 \quad \text{for} \quad s \ge d, \\ F(s \ge d) &= 1 \quad \text{for} \quad s \ge d + \delta, \quad = 0 \quad \text{for} \quad s \le d, \end{aligned}$$

and $F(d_1 \le s \le d_2) = F(s \ge d_1)\, F(s \le d_2)$. The above minimal velocity estimate can be obtained by the Mourre estimate for $K = p^2/2 + V$. At the present stage, we have gotten the Mourre estimate (1.17) for the Floquet Hamiltonian $K = D_t + p^2/2 + V(t)$. Hence, it can be expected that the Mourre estimate (1.17) will yield the minimal velocity estimate for $e^{-i\sigma K}$. In fact, there is an abstract theory for getting the minimal velocity estimate from the Mourre estimate, which was initiated by Sigal-Soffer (see e.g. [**31, 32**]). For the detail, see e.g. Sect. 4.4 "Minimal velocity estimates" of Gérard-Łaba [**14**]. By virtue of the abstract theory, one can obtain the following minimal velocity estimate for $e^{-i\sigma K}$:

THEOREM 1.3. *Suppose* $N = 3$. *Assume* V *satisfies* $(V_{\mathrm{WR}})_\rho$ *with some* $\rho > 0$. *Let* $\lambda_0 \in [0, \omega) \setminus \widehat{\Theta}$. *Put*

$$\begin{aligned} &B=(1+\tilde{B}_0+\tilde{B}_1)^{1/2}; \quad \tilde{B}_0=\langle D_t\rangle^{-1/2}Q_0(x)\langle D_t\rangle^{-1/2},\\ &\tilde{B}_1=\sum_{a\in\mathscr{A}_0^0}\tilde{B}_{1,a}, \quad \tilde{B}_{1,a}=\langle p_a\rangle^{-1}Q_{1,a}(x_a)\langle p_a\rangle^{-1}, \end{aligned} \tag{1.21}$$

where

$$\begin{aligned} &Q_0(x)=\sum_{a\in\mathscr{A}\setminus\mathscr{A}_0^0}x^2\tilde{J}_a(x)^2+\sum_{a\in\mathscr{A}_0^0}(x^a)^2\tilde{J}_a(x)^2=x^2-\sum_{a\in\mathscr{A}_0^0}(x_a)^2\tilde{J}_a(x)^2,\\ &Q_{1,a}(x_a)=(x_a)^2\tilde{J}_a(x_a)^2, \end{aligned}$$

with $a\in\mathscr{A}_0^0$. *Then there exists an* $\varepsilon_0(\lambda_0)=\varepsilon_0(\lambda_0;\epsilon,\delta)>0$*, which is determined by the positive constant* $(d_1(\lambda_0)-\delta-2\epsilon)/\omega$ *in the Mourre estimate* (1.17) *such that*

$$\int_1^\infty\left\|F\left(\frac{B}{\sigma}\le\varepsilon_0(\lambda_0)\right)e^{-i\sigma K}f_{\lambda_0,\delta}(K)\Phi\right\|_{\mathscr{K}}^2\frac{d\sigma}{\sigma}\le C\|\Phi\|_{\mathscr{K}}^2, \tag{1.22}$$

$$\operatorname*{s-lim}_{\sigma\to\infty}F\left(\frac{B}{\sigma}\le\varepsilon_0(\lambda_0)\right)e^{-i\sigma K}f_{\lambda_0,\delta}(K)=0 \tag{1.23}$$

hold. In particular, these yield

$$\int_1^\infty\left\|F\left(\frac{Q(x)^{1/2}}{\sigma}\le\frac{\varepsilon_0(\lambda_0)}{2}\right)e^{-i\sigma K}f_{\lambda_0,\delta_{\epsilon,1}}(K)\Phi\right\|_{\mathscr{K}}^2\frac{d\sigma}{\sigma}\le C\|\Phi\|_{\mathscr{K}}^2, \tag{1.24}$$

$$\operatorname*{s-lim}_{\sigma\to\infty}F\left(\frac{Q(x)^{1/2}}{\sigma}\le\frac{\varepsilon_0(\lambda_0)}{2}\right)e^{-i\sigma K}f_{\lambda_0,\delta_{\epsilon,1}}(K)=0. \tag{1.25}$$

Here

$$Q(x)=Q_0(x)+\sum_{a\in\mathscr{A}_0^0}Q_{1,a}(x_a)=x^2+\sum_{a\in\mathscr{A}_0^0}(x_a)^2\{\tilde{J}_a(x_a)^2-\tilde{J}_a(x)^2\}.$$

Here we note that

$$B^2=1+\tilde{B}_0+\tilde{B}_1\le 1+Q_0(x)+\sum_{a\in\mathscr{A}_0^0}Q_{1,a}(x_a)=1+Q(x)$$

holds, in deriving (1.24) from (1.22).

By using the arguments of Yajima-Kitada [**40**] and Møller-Skibsted [**27**], one can translate (1.24) into the minimal velocity estimate for the physical propagator $U(t,s)$ generated by $H(t)$:

COROLLARY 1.4. *Suppose that the hypotheses of Theorem 1.3 are satisfied. Let* $g_{\lambda_0,\delta}$ *be the function on* $\{z\in\boldsymbol{C}\mid |z|=1\}$ *such that* $g_{\lambda_0,\delta}(e^{-i2\pi\lambda/\omega})=f_{\lambda_0,\delta}(\lambda)$ *for* $\lambda\in[0,\omega)$*. Then*

$$\int_1^\infty\left\|F\left(\frac{Q(x)^{1/2}}{t}\le\frac{\varepsilon_0(\lambda_0)}{2}\right)U(t,0)g_{\lambda_0,\delta}(U(T,0))\phi\right\|_{\mathscr{H}}^2\frac{dt}{t}\le C\|\phi\|_{\mathscr{H}}^2, \tag{1.26}$$

$$\operatorname*{s-lim}_{t\to\infty}F\left(\frac{Q(x)^{1/2}}{t}\le\frac{\varepsilon_0(\lambda_0)}{2}\right)U(t,0)g_{\lambda_0,\delta}(U(T,0))=0 \tag{1.27}$$

hold, where $U(T,0)$ is the Floquet operator associated with $H(t)$.

One can utilize the above results in the study of quantum systems of N particles under the so-called AC Stark effect with $N = 3$: We suppose that $\mathscr{E}(t) \in C^0(\boldsymbol{R}; \boldsymbol{R}^d)$, $\mathscr{E}(t)$ has a period $T > 0$, that is, $\mathscr{E}(t+T) = \mathscr{E}(t)$ for any $t \in \boldsymbol{R}$, and

$$\mathscr{E}_{\rm m} = \frac{1}{T}\int_0^T \mathscr{E}(s)\,ds = 0, \tag{1.28}$$

which is the condition for the AC Stark effect. Let $m_j > 0$, $q_j \in \boldsymbol{R}$ and $r_j \in \boldsymbol{R}^d$, $1 \le j \le N$, denote the mass, charge and position vector of the j-th particle, respectively. We suppose that the particles under consideration interact with one another through the time-independent pair potentials $\bar{V}_{jk}(r_j - r_k)$, $1 \le j < k \le N$. The system under consideration is governed by the total Hamiltonian in the laboratory frame

$$\bar{H}_{\rm LF}(t) = \sum_{j=1}^N \left(-\frac{1}{2m_j}\Delta_j - q_j\langle \mathscr{E}(t), r_j\rangle\right) + \bar{V}(r); \quad \bar{V}(r) = \sum_{1\le j<k\le N} \bar{V}_{jk}(r_j - r_k). \tag{1.29}$$

In the same way as in Møller [**26**] and Adachi [**1, 2**] (as for the detail, see §7), we have only to consider the Hamiltonian in the moving frame

$$\hat{H}_{\rm MF}(t) = \frac{1}{2}p^2 + \bar{V}(x + c(t)) \tag{1.30}$$

acting on $\mathscr{H} = L^2(X)$, where $c(t)$ is defined as follows:

$$\mathscr{E}_{\rm os}(t) = \mathscr{E}(t) - \mathscr{E}_{\rm m}, \quad \bar{\mathscr{E}}(t) = \int_0^t \mathscr{E}_{\rm os}(s)\,ds, \quad \bar{\mathscr{E}}_{\rm m} = \frac{1}{T}\int_0^T \bar{\mathscr{E}}(s)\,ds,$$
$$\bar{\mathscr{E}}_{\rm os}(t) = \bar{\mathscr{E}}(t) - \bar{\mathscr{E}}_{\rm m}, \quad \bar{\bar{\mathscr{E}}}(t) = \int_0^t \bar{\mathscr{E}}_{\rm os}(s)\,ds, \quad c(t) = \pi((q_1/m_1)\bar{\bar{\mathscr{E}}}(t), \ldots, (q_N/m_N)\bar{\bar{\mathscr{E}}}(t)).$$

Now we impose the following condition $(V_{\rm ST})_{\bar\rho}$ on $\bar{V}_{jk}$'s with $\bar\rho > 0$:

$(V_{\rm ST})_{\bar\rho}$ $\bar{V}_{jk}(y)$, $(j,k) \in \mathscr{A}$, belongs to $C^2(\boldsymbol{R}^d; \boldsymbol{R})$, is independent of t, and satisfies the decaying conditions

$$|(\partial_y^\alpha \bar{V}_{jk})(y)| \le C_\alpha \langle y\rangle^{-\bar\rho-|\alpha|}, \quad 0 \le |\alpha| \le 2. \tag{1.31}$$

As is shown in §7, if $\bar\rho > 1$, then one can regard $V_{jk}(t,y) = \bar{V}_{jk}(y + \tilde{e}_{jk}\bar{\bar{\mathscr{E}}}(t))$ with $\tilde{e}_{jk} = q_j/m_j - q_k/m_k$ as a time-periodic potential satisfying the condition $(V_{\rm WR})_{\bar\rho-1}$. Therefore, in the case where $N = 3$, the following theorem is a direct consequence of Theorem 1.1, Corollary 1.2, Theorem 1.3, and Corollary 1.4.

THEOREM 1.5. *Suppose $\mathscr{E}_{\rm m} = 0$ and $N = 3$. Introduce $K = D_t + \hat{H}_{\rm MF}(t)$ acting on $\mathscr{K}$. Assume $\bar{V}_{jk}$'s satisfy $(V_{\rm ST})_{\bar\rho}$ with some $\bar\rho > 1$. Put*

$$\Theta = \bigcup_{a\in\mathscr{A}\setminus\{a_{\max}\}} \sigma_{\rm pp}(K^a), \quad \widehat{\Theta} = \bigcup_{a\in\mathscr{A}} \sigma_{\rm pp}(K^a) = \Theta \cup \sigma_{\rm pp}(K).$$

Then, the statements of Theorem 1.1, Corollary 1.2, Theorem 1.3, and Corollary 1.4 hold.

As for the asymptotic completeness for $\hat{H}_{\mathrm{MF}}(t)$ with $N = 2$, Yajima [**37**] proved it in the short-range case via the Howland-Yajima method, and Kitada-Yajima [**22**] proved it in the long-range case via the Enss method. On the other hand, for $\hat{H}_{\mathrm{MF}}(t)$ with $N = 3$, Korotyaev [**23**] and Nakamura [**28**] gave some partial results on it in the very short-range case via the Howland-Yajima and the Faddeev methods. The study of the problem of the asymptotic completeness for $\hat{H}_{\mathrm{MF}}(t)$ with $N = 3$ should be done by using some useful propagation estimates like (1.22) and (1.24) in future research.

The plan of this paper is as follows: In §2, we will give the definition of the Yafaev partition of unity and its properties which are useful for our analysis. In §3, we collect frequently used propositions which are useful for our analysis. In §4, we will revisit the case where $N = 2$. The construction of A in (1.15) is based on the arguments and results in §4. In §5, we will give the proof of Theorem 1.1, in particular, (1.16). In §6, we will obtain the minimal velocity estimates for $e^{-i\sigma K}$ and $U(t, s)$. In §7, we will deal with the AC Stark effect case.

Acknowledgement

The author is partially supported by the Grant-in-Aid for Scientific Research (C) #25K07031 from JSPS. The author is grateful to the referee for many valuable comments and suggestions.

2. Yafaev partition of unity

In this section, we give the definition of the Yafaev partition of unity $\{\tilde{J}_a\}_{a\in\mathscr{A}}$ of X, and its properties which are useful especially for the sake of dealing with the errors like $i[K_0, \tilde{J}_a(x)](\tilde{A}_0)_a\tilde{J}_a(x)$ and $i[K_0, \tilde{J}_a(x_a)]\bar{A}_a\tilde{J}_a(x_a)$, which come from our construction of A by (1.15).

First of all, for references, we will give an outline of the construction due to Yafaev [**36**] by following the arguments in [**36**], Hunziker-Sigal [**21**] and Gérard-Łaba [**14**] (also refer to Graf [**15**], Dereziński [**11**], Dereziński-Gérard [**12**] and Gérard-Łaba [**14**] on the construction of the Graf partition of unity, for comparison): Let $\sigma = \{\sigma_a\}_{a\in\mathscr{A}}$ be a sequence of numbers greater than 1 indexed with the elements of $\mathscr{A}$ such that $1 < \sigma_a < \sigma_b$ holds if $a \subsetneq b$ with $a, b \in \mathscr{A}_0 = \mathscr{A} \setminus \{a_{\max}\}$. The argument below is based on the fact that there exists an $M > 0$ such that

$$M|x^{a\cup b}| \leq |x^a| + |x^b|$$

holds for any $a, b \in \mathscr{A}$, because $X^{a\cup b} = X^a + X^b$. Now we suppose that for sufficiently small $\epsilon > 0$, $\sigma_a = 1 + O(\epsilon^{d_a})$ holds for $a \in \mathscr{A}_0$, where $d_a = \dim X_a$. For each $x \in X \setminus \{0\}$, we consider the family $\{\sigma_c|x_c|\}_{c\in\mathscr{A}_0}$, and watch its maximum $\max_{c\in\mathscr{A}_0}\{\sigma_c|x_c|\}$. Since $x_{a_{\max}} = 0$, when watching $\max_{c\in\mathscr{A}_0}\{\sigma_c|x_c|\}$, $\sigma_{a_{\max}}$ is insignificant. This $\sigma_{a_{\max}}$ will be used when guaranteeing the smoothness of the functions defined later on a neighborhood of the origin 0 of X. Moreover, in order to control the size of errors, $\sigma_{a_{\max}} > 0$ will be taken as sufficiently large. We put

$$U_a^\sigma = \big\{x \in X \bigm| \sigma_a|x_a| > \sigma_{a_{\min}}|x_{a_{\min}}| = \sigma_{a_{\min}}|x|\big\} \quad (a \in \mathscr{A}). \tag{2.1}$$

We see that $U^\sigma_{a_{\max}} = \emptyset$ since $x_{a_{\max}} = 0$, $U^\sigma_{a_{\min}} = \emptyset$ trivially, and that for $a \in \mathscr{A}^0_0 = \mathscr{A} \setminus \{a_{\max}, a_{\min}\}$,

$$U^\sigma_a = \left\{ x \in X \;\middle|\; |x^a| < (1 - \sigma^2_{a_{\min}}/\sigma^2_a)^{1/2}|x| \right\}$$

is a conical neighborhood of $X_a \setminus \{0\}$. Since $(1-\sigma^2_{a_{\min}}/\sigma^2_a)^{1/2} = O(\epsilon^{d_a/2})$, U^σ_a is considerably sharp. Let $a \in \mathscr{A}^0_0$ be given, $x \in U^\sigma_a$, and $b \in \mathscr{A}$ be such that $b \not\supset a$, that is, $X_b \not\subset X_a$. By virtue of Lemma 3.1 of [**36**], we see that there exist sequences $\sigma = \{\sigma_a\}_{a\in\mathscr{A}}$'s such that

$$\sigma_b|x_b| < \max_{\substack{f\in\mathscr{A}_0 \\ f\supset a}} \{\sigma_f|x_f|\}$$

holds, because $a \cup b \supsetneq a$ by assumption, and $d_a - d_{a\cup b} \geq d = \dim \boldsymbol{R}^d$. Such sequences σ's are called admissible. The above inequality implies that for $x \in U^\sigma_a$,

$$\max_{c\in\mathscr{A}_0}\{\sigma_c|x_c|\} = \max_{\substack{f\in\mathscr{A}_0 \\ f\supset a}}\{\sigma_f|x_f|\}$$

holds. Taking account of these, we put

$$C^\sigma_a = U^\sigma_a \setminus \left(\bigcup_{f\supsetneq a} U^\sigma_f \right) \quad (a \in \mathscr{A}^0_0), \quad C^\sigma_{a_{\min}} = (X \setminus \{0\}) \setminus \left(\bigcup_{a\in\mathscr{A}^0_0} C^\sigma_a \right). \tag{2.2}$$

This definition yields that $\{C^\sigma_a\}_{a\in\mathscr{A}_0}$ is a disjoint covering of $X \setminus \{0\}$, and that for $x \in C^\sigma_a$ with $a \in \mathscr{A}_0$,

$$\max_{c\in\mathscr{A}_0}\{\sigma_c|x_c|\} = \sigma_a|x_a|$$

holds. In particular, for $a \in \mathscr{A}^0_0$, if the pair (j, k) satisfies $(j, k) \subsetneq a$, then $C^\sigma_a \cap C^\sigma_{(j,k)} = \emptyset$. By following the argument of [**21**], we introduce the intercluster distance

$$|x|_a = \min_{\substack{b\in\mathscr{A}_0 \\ b\not\subset a}} |x^b| \tag{2.3}$$

for $a \in \mathscr{A}_0$. By virtue of Lemma 3.1 of [**21**], it is known that if $a \in \mathscr{A}_0$, then

$$|x|_a \geq \left(\min_{\substack{b\in\mathscr{A}_0 \\ b\not\subset a}} (1 - \sigma^2_{a_{\min}}/\sigma^2_b)^{1/2} \right) |x| \tag{2.4}$$

holds for $x \in C^\sigma_a$. Now we will take a family of sets $\{C^\sigma_a\}_{a\in\mathscr{A}_0}$ with an admissible sequence $\sigma = \{\sigma_a\}_{a\in\mathscr{A}}$, and put

$$\widetilde{C}^\sigma_a = \left\{ x \in C^\sigma_a \;\middle|\; \sigma_a|x_a| > \sigma_{a_{\max}} \right\} \quad (a \in \mathscr{A}_0), \quad \widetilde{C}^\sigma_{a_{\max}} = X \setminus \bigcup_{a\in\mathscr{A}_0} \widetilde{C}^\sigma_a. \tag{2.5}$$

This definition yields that $\{\widetilde{C}^\sigma_a\}_{a\in\mathscr{A}}$ is a disjoint covering of X, and that for $x \in \widetilde{C}^\sigma_{a_{\max}}$,

$$\max\left\{\sigma_{a_{\max}}, \max_{c\in\mathscr{A}_0}\{\sigma_c|x_c|\}\right\} = \sigma_{a_{\max}}$$

holds, and that for $x \in \widetilde{C}_a^\sigma$ with $a \in \mathscr{A}_0$,

$$\max\left\{\sigma_{a_{\max}}, \max_{c\in\mathscr{A}_0}\{\sigma_c|x_c|\}\right\} = \sigma_a|x_a|$$

holds. Then we will introduce

$$J_a^\sigma(x) = \chi_{\widetilde{C}_a^\sigma}(x) \quad (a \in \mathscr{A}), \tag{2.6}$$

where $\chi_{\widetilde{C}_a^\sigma}$ is the characteristic function of $\widetilde{C}_a^\sigma$. It is known that

$$\sum_{a\in\mathscr{A}} J_a^\sigma(x) \equiv 1$$

holds. By the arguments of [**36**] and [**21**], those are known that there exist admissible sequences $\sigma^- = \{\sigma_a^-\}_{a\in\mathscr{A}}$ and $\sigma^+ = \{\sigma_a^+\}_{a\in\mathscr{A}}$ such that $\sigma_a^- < \sigma_a^+$ holds for any $a \in \mathscr{A}$, and that if $\sigma = \{\sigma_a\}_{a\in\mathscr{A}}$ satisfies $\sigma_a^- \leq \sigma_a \leq \sigma_a^+$ for any $a \in \mathscr{A}$, then σ is also admissible. We fix such two admissible sequences σ^- and σ^+. For $b \in \mathscr{A}$, we introduce a smoothing function s_b, that is, a function $s_b \in C_0^\infty(\boldsymbol{R};\boldsymbol{R})$ such that

$$\operatorname{supp} s_b \subset [\sigma_b^-, \sigma_b^+], \quad s_b \geq 0, \quad \int_{\boldsymbol{R}} s_b(\sigma_b)\, d\sigma_b = 1.$$

Putting $\tilde{s}_0(\sigma_0) = \prod_{b\in\mathscr{A}_0} s_b(\sigma_b)$, it satisfies

$$\operatorname{supp} \tilde{s}_0 \subset \mathbf{S}_0 = \prod_{b\in\mathscr{A}_0}[\sigma_b^-, \sigma_b^+], \quad \tilde{s}_0 \geq 0, \quad \int_{\mathbf{S}_0} \tilde{s}_0(\sigma_0)\, d\sigma_0 = 1,$$

where $d\sigma_0 = \otimes_{b\in\mathscr{A}_0} d\sigma_b$. Now we define

$$J_a(x) = \int_{\sigma_{a_{\max}}^-}^{\sigma_{a_{\max}}^+} \int_{\mathbf{S}_0} \tilde{s}_0(\sigma_0) s_{a_{\max}}(\sigma_{a_{\max}}) J_a^\sigma(x)\, d\sigma_0 d\sigma_{a_{\max}} \quad (a \in \mathscr{A}). \tag{2.7}$$

$\{J_a\}_{a\in\mathscr{A}}$ is called a Yafaev partition of unity. Then we have the following:

PROPOSITION 2.1. *The family of functions $\{J_a\}_{a\in\mathscr{A}}$ on X satisfies the following: Each $J_a(x)$ is a smooth function on X. If $a \in \mathscr{A}_0$, then $J_a(x)$ is homogeneous of degree 0 outside some compact neighborhood of the origin 0 of X; while, $J_{a_{\max}}(x)$ is supported in some compact neighborhood of the origin 0 of X. Moreover,*

$$\sum_{a\in\mathscr{A}} J_a(x) \equiv 1 \tag{2.8}$$

holds.

PROOF. The proof was already given in [**36**] and [**21**]. As for the detail, see these papers.

Here we will give an outline of the proof of the smoothness of $J_a(x)$ with $a \in \mathscr{A}_0^0 = \mathscr{A} \setminus \{a_{\max}, a_{\min}\}$ only, since we need to know the properties of $(\nabla J_a)(x)$: For the sake of simplicity, we consider the case where $x \in X$ satisfies $\sigma_a^-|x_a| > \sigma_{a_{\max}}^+$ only. If $x \in X$ satisfies $\sigma_a^-|x_a| > \sigma_{a_{\min}}^+|x_{a_{\min}}| = \sigma_{a_{\min}}^+|x|$, then $x \in U_a^\sigma$ for any $\sigma \in \mathbf{S}$, because $\sigma_a \geq \sigma_a^-$ and $\sigma_{a_{\min}}^+ \geq \sigma_{a_{\min}}$. Then $J_a(y) = 1$ holds in a neighborhood of x. If $x \in X$ satisfies $\sigma_a^+|x_a| < \sigma_{a_{\min}}^-|x|$, then $x \notin U_a^\sigma$ for any $\sigma \in \mathbf{S}$, because $\sigma_a \leq \sigma_a^+$ and $\sigma_{a_{\min}}^- \leq \sigma_{a_{\min}}$. Then $J_a(y) = 0$ holds in a neighborhood of x. Thus we have only to watch the case where $x \in X$ satisfies $\sigma_a^-|x_a| \leq \sigma_{a_{\min}}^+|x|$ and $\sigma_a^+|x_a| \geq \sigma_{a_{\min}}^-|x|$. Consider the rectangle $[\sigma_a^-, \sigma_a^+] \times [\sigma_{a_{\min}}^-, \sigma_{a_{\min}}^+]$ in the $\sigma_a\sigma_{a_{\min}}$-plane. Since $\sigma_a^\pm - 1 = O(\epsilon^{d_a})$ and $\sigma_{a_{\min}}^\pm - 1 = O(\epsilon^{d_{a_{\min}}}) < O(\epsilon^{d_a})$ for sufficiently small $\epsilon > 0$, without loss of generality, we may consider the following three cases: (a) x satisfies $\sigma_a^-|x_a| < \sigma_{a_{\min}}^-|x|$ and $\sigma_{a_{\min}}^-|x| \leq \sigma_a^+|x_a| \leq \sigma_{a_{\min}}^+|x|$; (b) x satisfies $\sigma_a^-|x_a| < \sigma_{a_{\min}}^-|x|$ and $\sigma_a^+|x_a| > \sigma_{a_{\min}}^+|x|$; (c) x satisfies $\sigma_{a_{\min}}^-|x| \leq \sigma_a^-|x_a| \leq \sigma_{a_{\min}}^+|x|$ and $\sigma_a^+|x_a| > \sigma_{a_{\min}}^+|x|$. In the case (a), we have

$$\begin{aligned} J_a(x) &= \int_{\sigma_a^-}^{\sigma_{a_{\min}}^-|x|/|x_a|} \int_{\sigma_{a_{\min}}^-}^{\sigma_{a_{\min}}^+} s_a(\sigma_a) s_{a_{\min}}(\sigma_{a_{\min}})\, d\sigma_{a_{\min}} d\sigma_a \\ &\quad + \int_{\sigma_{a_{\min}}^-|x|/|x_a|}^{\sigma_a^+} \int_{\sigma_a|x_a|/|x|}^{\sigma_{a_{\min}}^+} s_a(\sigma_a) s_{a_{\min}}(\sigma_{a_{\min}})\, d\sigma_{a_{\min}} d\sigma_a \\ &= \int_{\sigma_a^-}^{\sigma_{a_{\min}}^-|x|/|x_a|} s_a(\sigma_a)\, d\sigma_a \\ &\quad + \int_{\sigma_{a_{\min}}^-|x|/|x_a|}^{\sigma_a^+} \int_{\sigma_a|x_a|/|x|}^{\sigma_{a_{\min}}^+} s_a(\sigma_a) s_{a_{\min}}(\sigma_{a_{\min}})\, d\sigma_{a_{\min}} d\sigma_a. \end{aligned}$$

In particular, we obtain

$$\begin{aligned} (\nabla J_a)(x) &= \nabla \left(\frac{\sigma_{a_{\min}}^-|x|}{|x_a|} \right) s_a \left(\frac{\sigma_{a_{\min}}^-|x|}{|x_a|} \right) \\ &\quad - \nabla \left(\frac{\sigma_{a_{\min}}^-|x|}{|x_a|} \right) s_a \left(\frac{\sigma_{a_{\min}}^-|x|}{|x_a|} \right) \int_{\sigma_{a_{\min}}^-}^{\sigma_{a_{\min}}^+} s_{a_{\min}}(\sigma_{a_{\min}})\, d\sigma_{a_{\min}} \\ &\quad - \int_{\sigma_{a_{\min}}^-|x|/|x_a|}^{\sigma_a^+} \nabla \left(\frac{\sigma_a|x_a|}{|x|} \right) s_a(\sigma_a) s_{a_{\min}} \left(\frac{\sigma_a|x_a|}{|x|} \right) d\sigma_a \\ &= -\nabla \left(\frac{|x_a|}{|x|} \right) \int_{\sigma_{a_{\min}}^-|x|/|x_a|}^{\sigma_a^+} \sigma_a s_a(\sigma_a) s_{a_{\min}} \left(\frac{\sigma_a|x_a|}{|x|} \right) d\sigma_a. \end{aligned}$$

The cases (b) and (c) can be also treated similarly. These yield the smoothness of $J_a(x)$. □

We put

$$\tilde{J}_a(x) = J_a(x) \left\{ \sum_{b \in \mathscr{A}} J_b(x)^2 \right\}^{-1/2} \qquad (a \in \mathscr{A}). \tag{2.9}$$

We also call $\{\tilde{J}_a\}_{a \in \mathscr{A}}$ a Yafaev partition of unity:

COROLLARY 2.2. *The family of functions $\{\tilde{J}_a\}_{a\in\mathscr{A}}$ on X satisfies the following: Each $\tilde{J}_a(x)$ is a smooth function on X. If $a \in \mathscr{A}_0$, then $\tilde{J}_a(x)$ is homogeneous of degree 0 outside some compact neighborhood of the origin 0 of X; while, $\tilde{J}_{a_{\max}}(x)$ is supported in some compact neighborhood of the origin 0 of X. Moreover,*

$$\sum_{a\in\mathscr{A}} \tilde{J}_a(x)^2 \equiv 1 \tag{2.10}$$

holds.

By virtue of (2.4), the following lemma can be obtained immediately:

LEMMA 2.3. *Let $a \in \mathscr{A}_0$. Then*

$$|x|_a \geq c_a|x|; \quad c_a = \left(\min_{\substack{b\in\mathscr{A}_0 \\ b\not\subset a}} (1 - (\sigma^+_{a_{\min}})^2/(\sigma^-_b)^2)^{1/2} \right) > 0, \tag{2.11}$$

holds for $x \in \operatorname{supp} \tilde{J}_a$. In particular, if $(j,k) \in \mathscr{A}$ satisfies $(j,k) \not\subset a$, then

$$|x^{(j,k)}| \geq c_a|x|$$

holds for $x \in \operatorname{supp} \tilde{J}_a$.

COROLLARY 2.4. *Assume V satisfies $(V_{\mathrm{WR}})_\rho$ with some $\rho > 0$. Let $a \in \mathscr{A}_0$, and $m, \ell \in \boldsymbol{N} \cup \{0\}$ such that $0 \leq m + \ell \leq 2$. Then, for any $(j,k) \in \mathscr{A}$ such that $(j,k) \not\subset a$,*

$$(\partial_t^m (\nabla^{(j,k)})^\ell V^{(j,k)})(t, x^{(j,k)})\tilde{J}_a(x) = O(\langle x\rangle^{-\rho-(m+\ell)})$$

holds as $|x| \to \infty$.

Now, for the sake of simplicity, we restrict to the case where $N = 3$. Then the following lemma holds. Roughly speaking, it implies that $\nabla\tilde{J}_a$ with $a \in \mathscr{A}_0^0$ is supported in $\operatorname{supp} \tilde{J}_{a_{\min}}$.

LEMMA 2.5. *Suppose $N = 3$. Let $a \in \mathscr{A}_0^0$, that is, a be some pair (j,k). Then*

$$|x^{(j,k)}| \geq c'_{(j,k)}|x|; \quad c'_{(j,k)} = (1 - (\sigma^+_{a_{\min}})^2/(\sigma^-_{(j,k)})^2)^{1/2} > 0, \tag{2.12}$$

holds for $x \in \operatorname{supp}(\nabla\tilde{J}_{(j,k)})$ with $|x| \geq \sigma^+_{a_{\max}}/\sigma^-_{a_{\min}}$.

PROOF. It follows from the proof of Proposition 2.1 that $(\nabla J_{(j',k')})(x)$ is supported in the complement of

$$\left\{x \in X \mid \sigma^-_{(j',k')}|x_{(j',k')}| > \sigma^+_{a_{\min}}|x|\right\} = \left\{x \in X \mid |x^{(j',k')}| < c'_{(j',k')}|x|\right\}$$

for any pair (j', k'), in which $J_{(j',k')}(x)$ is equal to 1, if $|x| \geq \sigma^+_{a_{\max}}/\sigma^-_{a_{\min}}$. This completes the proof. □

By using the Yafaev partition of unity $\{\tilde{J}_a\}_{a\in\mathscr{A}}$, we will introduce a conjugate operator A

for K by the first representation of (1.15)

$$A = \sum_{a\in\mathscr{A}\setminus\mathscr{A}_0^0} \tilde{J}_a(x)\tilde{A}_0\tilde{J}_a(x) + \sum_{a\in\mathscr{A}_0^0} \tilde{J}_a(x)(\tilde{A}_0)^a\tilde{J}_a(x) + L_0 \sum_{a\in\mathscr{A}_0^0} \tilde{J}_a(x_a)\bar{A}_a\tilde{J}_a(x_a).$$

By using (2.10), we have

$$\begin{aligned}\sum_{a\in\mathscr{A}} \tilde{J}_a(x)\tilde{A}_0\tilde{J}_a(x) &= \sum_{a\in\mathscr{A}} \tilde{J}_a(x)[\tilde{A}_0, \tilde{J}_a(x)] + \sum_{a\in\mathscr{A}} \tilde{J}_a(x)^2\tilde{A}_0 \\ &= -i\langle D_t\rangle_{2\omega}^{-1} \sum_{a\in\mathscr{A}} \tilde{J}_a(x)\{x\cdot(\nabla\tilde{J}_a)(x)\} + \tilde{A}_0 \\ &= -i\langle D_t\rangle_{2\omega}^{-1}\left\{x\cdot\nabla\left(\frac{1}{2}\sum_{a\in\mathscr{A}} \tilde{J}_a(x)^2\right)\right\} + \tilde{A}_0 = \tilde{A}_0.\end{aligned}$$

Since $(\tilde{A}_0)^a - \tilde{A}_0 = -(\tilde{A}_0)_a$, we obtain the second representation of (1.15)

$$A = \tilde{A}_0 + \sum_{a\in\mathscr{A}_0^0} \tilde{J}_a(x)\{-(\tilde{A}_0)_a\}\tilde{J}_a(x) + L_0 \sum_{a\in\mathscr{A}_0^0} \tilde{J}_a(x_a)\bar{A}_a\tilde{J}_a(x_a).$$

By this representation of A, we have

$$i[K_0, A] = i[K_0, \tilde{A}_0] + \sum_{a\in\mathscr{A}_0^0} S_{1,a} + L_0 \sum_{a\in\mathscr{A}_0^0} S_{2,a} \tag{2.13}$$

with $S_{1,a} = i[K_0, \tilde{J}_a(x)\{-(\tilde{A}_0)_a\}\tilde{J}_a(x)]$ and $S_{2,a} = i[K_0, \tilde{J}_a(x_a)\bar{A}_a\tilde{J}_a(x_a)]$. In our analysis, we will regard

$$i[K_0, A]^{\mathrm{m}} = i[K_0, \tilde{A}_0] + \sum_{a\in\mathscr{A}_0^0} S_{1,a}^{\mathrm{m}} + L_0 \sum_{a\in\mathscr{A}_0^0} S_{2,a}^{\mathrm{m}}$$

as the main part of $i[K_0, A]$, where $S_{1,a}^{\mathrm{m}} = \tilde{J}_a(x)i[K_0, \{-(\tilde{A}_0)_a\}]\tilde{J}_a(x)$ and $S_{2,a}^{\mathrm{m}} = \tilde{J}_a(x_a)$ $i[K_0, \bar{A}_a]\tilde{J}_a(x_a)$. Then, $\bar{S}_{1,a} = S_{1,a} - S_{1,a}^{\mathrm{m}}$ is represented by using the factor $(\nabla\tilde{J}_a)(x)$, and $\bar{S}_{2,a} = S_{2,a} - S_{2,a}^{\mathrm{m}}$ is represented by using the factor $(\nabla_a\tilde{J}_a)(x_a)$. Hence, we have to pay attention to the properties of $(\nabla\tilde{J}_a)(x)$ and $(\nabla_a\tilde{J}_a)(x_a)$. As for $(\nabla\tilde{J}_a)(x)$, we refer to the following corollary, which can be obtained immediately from the proof of Proposition 2.1:

COROLLARY 2.6. *Let $a \in \mathscr{A}_0^0$. Then $\max_{x\in X}\{\langle x\rangle|(\nabla\tilde{J}_a)(x)|\}$ can be controlled by $\max_{t\in\boldsymbol{R}} s_{a_{\min}}(t)$.*

Since $\max_{t\in\boldsymbol{R}} s_{a_{\min}}(t) = O(\epsilon^{-d_{a_{\min}}})$, $\langle K_0\rangle^{-1}\bar{S}_{1,a}\langle K_0\rangle^{-1}$ is bounded, but is not small. Roughly speaking, in our strategy, we will show that this is dominated by $L_0\langle K_0\rangle^{-1}S_{2,a}^{\mathrm{m}}$ $\langle K_0\rangle^{-1}/4$, and that $\bar{S}_{2,a}$ is also dominated by $S_{2,a}^{\mathrm{m}}/4$. To this end, we refer to the following lemma and corollary:

LEMMA 2.7. *For $a \in \mathscr{A}_0^0$, $\max_{x_a\in X_a}\{\langle x_a\rangle|(\nabla_a\tilde{J}_a)(x_a)|\}$ can be controlled by $\max_{t\in\boldsymbol{R}}\{ts_{a_{\max}}(t)\}$.*

PROOF. We have only to show the lemma in which $\tilde{J}_a$ is replaced by J_a. The proof is quite similar to the above one of Proposition 2.1: If $x_a \in X_a$ satisfies $\sigma_a^-|x_a| > \sigma_{a_{\max}}^+$, then $x_a \in U_a^\sigma$ for any $\sigma \in \mathbf{S}$, because $\sigma_a \geq \sigma_a^-$ and $\sigma_{a_{\max}}^+ \geq \sigma_{a_{\max}}$. Then $J_a(y_a) = 1$ holds in a neighborhood of x_a. If $x_a \in X_a$ satisfies $\sigma_a^+|x_a| < \sigma_{a_{\max}}^-$, then $x_a \notin U_a^\sigma$ for any $\sigma \in \mathbf{S}$, because $\sigma_a \leq \sigma_a^+$ and $\sigma_{a_{\max}}^- \leq \sigma_{a_{\max}}$. Then $J_a(y_a) = 0$ holds in a neighborhood of x_a. Thus we have only to watch the case where $x_a \in X_a$ satisfies $\sigma_a^-|x_a| \leq \sigma_{a_{\max}}^+$ and $\sigma_a^+|x_a| \geq \sigma_{a_{\max}}^-$. Consider the rectangle $[\sigma_a^-, \sigma_a^+] \times [\sigma_{a_{\max}}^-, \sigma_{a_{\max}}^+]$ in the $\sigma_a\sigma_{a_{\max}}$-plane. Since $\sigma_a^\pm - 1 = O(\epsilon_a^d)$ and $\sigma_{a_{\max}}^\pm - 1 = O(1) > O(\epsilon_a^d)$ for sufficiently small $\epsilon > 0$, without loss of generality, we may consider the following three cases: (a) x_a satisfies $\sigma_a^-|x_a| < \sigma_{a_{\max}}^-$ and $\sigma_{a_{\max}}^- \leq \sigma_a^+|x_a| \leq \sigma_{a_{\max}}^+$; (b) x_a satisfies $\sigma_{a_{\max}}^- \leq \sigma_a^-|x_a| \leq \sigma_{a_{\max}}^+$ and $\sigma_{a_{\max}}^- \leq \sigma_a^+|x_a| \leq \sigma_{a_{\max}}^+$; (c) x_a satisfies $\sigma_{a_{\max}}^- \leq \sigma_a^-|x_a| \leq \sigma_{a_{\max}}^+$ and $\sigma_a^+|x_a| > \sigma_{a_{\max}}^+$. In the case (a), we have

$$J_a(x_a) = \int_{\sigma_a^-}^{\sigma_{a_{\max}}^-/|x_a|} s_a(\sigma_a)\,d\sigma_a + \int_{\sigma_{a_{\max}}^-/|x_a|}^{\sigma_a^+} \int_{\sigma_a|x_a|}^{\sigma_{a_{\max}}^+} s_a(\sigma_a)s_{a_{\max}}(\sigma_{a_{\max}})\,d\sigma_{a_{\max}}d\sigma_a.$$

In particular, we obtain

$$(\nabla_a J_a)(x_a) = -\nabla_a(|x_a|) \int_{\sigma_{a_{\max}}^-/|x_a|}^{\sigma_a^+} \sigma_a s_a(\sigma_a) s_{a_{\max}}(\sigma_a|x_a|)\,d\sigma_a.$$

The cases (b) and (c) can be also treated similarly. These yield the lemma. □

We note that for any $C > 0$, there exist smoothing functions $s_{a_{\max}}$'s satisfying $\max_{t\in\boldsymbol{R}}\{ts_{a_{\max}}(t)\} = C$. In fact,

$$\lim_{\epsilon\to+0} \int_\epsilon^1 Ct^{-1}\,dt = +\infty$$

implies the existence of a smoothing function $\tilde{s}_{a_{\max}}$ satisfying $\max_{t\in\boldsymbol{R}}\{t\tilde{s}_{a_{\max}}(t)\} = C$ with some $0 < \sigma_{a_{\max},1}^- < \sigma_{a_{\max},1}^+ \leq 1$, that is, $\tilde{s}_{a_{\max}} \in C_0^\infty(\boldsymbol{R};\boldsymbol{R})$ such that

$$\mathrm{supp}\,\tilde{s}_{a_{\max}} \subset [\sigma_{a_{\max},1}^-, \sigma_{a_{\max},1}^+], \quad \tilde{s}_{a_{\max}} \geq 0, \quad \int_{\sigma_{a_{\max},1}^-}^{\sigma_{a_{\max},1}^+} \tilde{s}_{a_{\max}}(t)\,dt = 1.$$

Now, for $L_1 > 0$, put $\sigma_{a_{\max}}^\pm = \sigma_{a_{\max},1}^\pm L_1$ and $s_{a_{\max}}(t) = L_1^{-1}\tilde{s}_{a_{\max}}(L_1^{-1}t)$. Then we see that $s_{a_{\max}}$ is also a smoothing function satisfying $\max_{t\in\boldsymbol{R}}\{ts_{a_{\max}}(t)\} = C$ with $\sigma_{a_{\max}}^- < \sigma_{a_{\max}}^+$. In this paper, we will take $\max_{t\in\boldsymbol{R}}\{ts_{a_{\max}}(t)\} > 0$ as sufficiently small, and $L_1 > 0$ as sufficiently large. Then the following corollary can be obtained from the proof of Lemma 2.7.

COROLLARY 2.8. *For $a \in \mathscr{A}_0^0$, $\max_{x_a\in X_a} |(\nabla_a\tilde{J}_a)(x_a)| = O(L_1^{-1})$ holds.*

By virtue of these, one can regard $i[K_0, A]^{\mathrm{m}}$ as the main part of $i[K_0, A]$. As for the detailed argument, see the proof of Lemma 5.1 in §5.

The following lemma will be also used in §5.

LEMMA 2.9. *For $a \in \mathscr{A}_0^0$, $\tilde{J}_a(x_a) \geq \tilde{J}_a(x) \geq 0$ holds.*

PROOF. We have only to show the lemma in which $\tilde{J}_a$ is replaced by J_a. By the definition of $\widetilde{C}_a^\sigma$, if $x \in \widetilde{C}_a^\sigma$, then $x_a \in \widetilde{C}_a^\sigma$ holds trivially, that is, $J_a^\sigma(x) = 1$ implies $J_a^\sigma(x_a) = 1$. And, it is easy to see that there exists an $x \in X$ such that $J_a^\sigma(x) = 0$ and $J_a^\sigma(x_a) = 1$. These and the definition of J_a yield the lemma. □

3. Collection of frequently used propositions

In this section, we collect frequently used propositions which are useful for the proof of the Mourre estimate for K.

First of all, we state the Nelson's commutator theorem, which guarantees the self-adjointness of A in (1.15) (as for the proof, see e.g. Reed-Simon [**30**] and Gérard-Łaba [**14**]).

THEOREM 3.1. *Let $\mathcal{K}$ be a Hilbert space. Suppose that $N_0 \geq c > 0$ is a self-adjoint operator on $\mathcal{K}$ and A is a symmetric operator on $\mathcal{K}$ such that $D(N_0) \subset D(A)$ and there exists a constant $C > 0$ such that*

$$\|Au\| \leq C\|N_0 u\| \quad \textit{for } u \in D(N_0),$$
$$|(Au, N_0 u) - (N_0 u, Au)| \leq C\|{N_0}^{1/2} u\|^2 \quad \textit{for } u \in D(N_0)$$

hold. Then A is essentially self-adjoint on $D(N_0)$. Denoting by $\bar{A}$ the unique self-adjoint extension of A, if $u \in D(\bar{A})$, then $(1 + i\epsilon N_0)^{-1} u$ converges to u in the graph topology of $D(\bar{A})$ as $\epsilon \to 0$.

Applying Theorem 3.1 with $\mathcal{K} = \mathscr{K}$ and $N_0 = \langle D_t \rangle + p^2/2 + x^2/2$, we see that A in (1.15) has its unique self-adjoint extension, which is also denoted by A.

In the usual proof of the Mourre estimate for K, one of the points to be checked is that the condition

$$\sup_{|\kappa| \leq 1} \|K e^{i\kappa A} (K + i)^{-1}\|_{\mathscr{B}(\mathscr{K})} < \infty \tag{3.1}$$

is satisfied by a conjugate operator A (see e.g. Mourre [**25**]). However, it seems not easy to verify directly that A satisfies (3.1). In order to overcome this difficulty, we need the following proposition (see e.g. Lemma 3.2.2 and Proposition 3.2.3 of [**14**]; see also Amrein-Boutet de Monvel-Georgescu [**8**]):

PROPOSITION 3.2. *Let $\mathcal{K}$ be a Hilbert space. Suppose that K, K_0 and N_0 are self-adjoint operators on $\mathcal{K}$ such that $N_0 \geq c > 0$, $D(K) = D(K_0)$ as Banach spaces, and for $z \in \boldsymbol{C} \setminus \sigma(K)$, $(K - z)^{-1}$ preserves $D(N_0)$. Let A be a symmetric operator on $\mathcal{K}$. Suppose that K_0 and A satisfy $D(N_0) \subset D(K_0)$, $D(N_0) \subset D(A)$,*

$$\|K_0 u\| \leq C\|N_0 u\| \quad \textit{for } u \in D(N_0),$$
$$|(K_0 u, N_0 u) - (N_0 u, K_0 u)| \leq C\|{N_0}^{1/2} u\|^2 \quad \textit{for } u \in D(N_0),$$
$$\|Au\| \leq C\|N_0 u\| \quad \textit{for } u \in D(N_0),$$
$$|(Au, N_0 u) - (N_0 u, Au)| \leq C\|{N_0}^{1/2} u\|^2 \quad \textit{for } u \in D(N_0).$$

Denote the unique self-adjoint extension of A also by A. Assume moreover that

$$|(Au, Ku) - (Ku, Au)| \le C(\|Ku\|^2 + \|u\|^2) \quad \textit{for } u \in D(N_0)$$

holds. Then the following hold:
(1) *$D(N_0)$ is dense in $D(K) \cap D(A)$ with the norm $\|Ku\| + \|Au\| + \|u\|$.*
(2) *The commutator $i[K, A]$, defined as a quadratic form on $D(K) \cap D(A)$, is the unique extension of the quadratic form $i[K, A]$ on $D(N_0)$.*
(3) *$K \in C^1(A)$, that is, for some $z \in \boldsymbol{C} \setminus \sigma(K)$, the map*

$$\boldsymbol{R} \ni \kappa \mapsto e^{i\kappa A}(K - z)^{-1} e^{-i\kappa A} \in \mathscr{B}(\mathcal{K})$$

is C^1 in the strong topology of $\mathscr{B}(\mathcal{K})$, which is the algebra of bounded linear operators in $\mathcal{K}$.
(4) *$D(K) \cap D(A)$ is a core for K, and the quadratic form $i[K, A]$ on $D(K) \cap D(A)$ extends uniquely to a bounded operator from $D(K)$ to its dual space $D(K)^*$, which is denoted also by $i[K, A]$.*
(5) *The virial relation holds: For any $\lambda \in \boldsymbol{R}$,*

$$E_K(\{\lambda\}) i[K, A] E_K(\{\lambda\}) = 0$$

holds. Here $E_K(S)$ stands for the spectral projection for K onto $S \subset \boldsymbol{R}$.
(6) *For $z \in \boldsymbol{C} \setminus \sigma(K)$, $i[(K - z)^{-1}, A] = -(K - z)^{-1} i[K, A](K - z)^{-1}$ holds.*
(7) *For $z \in \boldsymbol{C} \setminus \sigma(K)$, $(K - z)^{-1}$ preserves $D(A)$.*

As for the characterization of the operators in $C^1(A)$, we refer to the following proposition (see e.g. Proposition 3.2.1 of [**14**]; see also [**8**]):

PROPOSITION 3.3. *Let $\mathcal{K}$ be a Hilbert space. Suppose that K and A are self-adjoint operators on $\mathcal{K}$. Then the following are equivalent:*
(1) *$K \in C^1(A)$.*
(2) *For some $z \in \boldsymbol{C} \setminus \sigma(K)$,*

$$|(Au, (K - \bar{z})^{-1} u) - ((K - z)^{-1} u, Au)| \le C\|u\|^2 \quad \textit{for } u \in D(A).$$

(3) *For any $z \in \boldsymbol{C} \setminus \sigma(K)$,*

$$|(Au, (K - \bar{z})^{-1} u) - ((K - z)^{-1} u, Au)| \le C\|u\|^2 \quad \textit{for } u \in D(A).$$

(4) *The following two conditions hold:*
 (i) *There exists some $C > 0$ such that*

$$|(Au, Ku) - (Ku, Au)| \le C\|(K + i)u\|^2 \quad \textit{for } u \in D(K) \cap D(A);$$

 (ii) *There exists some $z \in \boldsymbol{C} \setminus \sigma(K)$ such that*

$$\left\{u \in D(A) \;\middle|\; (K - z)^{-1} u \in D(A),\ (K - \bar{z})^{-1} u \in D(A)\right\}$$

 is a core for A.

From now on, we will give several propositions about the Floquet Hamiltonians under consideration, which are frequently used in this paper:

PROPOSITION 3.4. *Assume V satisfies $(V_{\rm WR})_\rho$ with some $\rho > 0$. Then the spectrum $\sigma(K)$ of the Floquet Hamiltonian K is ω-periodic, that is, $\sigma(K) = \sigma(K) + \omega \boldsymbol{Z}$.*

As is well-known, the periodic structure of $\sigma(K)$ stated in Proposition 3.4 has been used frequently in the previous works. One has only to take account of the formula $e^{-i\omega t} K e^{i\omega t} = K + \omega$.

PROPOSITION 3.5. (1) *$\langle p\rangle^2 \langle D_t\rangle^{-1} \langle K_0\rangle^{-1}$ and $\langle p\rangle \langle D_t\rangle^{-1/2} \langle K_0\rangle^{-1/2}$ are bounded. Here $\langle p\rangle = (1+p^2)^{1/2}$, $\langle D_t\rangle = (1+(D_t)^2)^{1/2}$ and $\langle K_0\rangle = (1+(K_0)^2)^{1/2}$.*
(2) *Suppose that $g \in C^2(\boldsymbol{R})$ satisfies* $\operatorname{supp} g \subset (0,\infty)$, *and $g^{(k)}$, $k = 0$, 1, 2, are all bounded. Then $\langle p\rangle g(D_t) \langle K_0\rangle^{-1}$ and $\langle D_t\rangle^{1/2} g(D_t) \langle K_0\rangle^{-1}$ are bounded.*

Proposition 3.5 can be proved by the same argument as in the case of the free Stark Hamiltonian $p^2/2 - E\cdot x$ with $E \neq 0$ (see e.g. [**9**], [**17**] and Simon [**33**]). So we omit the proof.

The following proposition says the so-called local compactness property of K_0.

PROPOSITION 3.6. *Let $R > 0$ and $z \in \boldsymbol{C} \setminus \boldsymbol{R}$. Then $F(|x| \le R)(K_0 - z)^{-1}$ is compact on $\mathscr{K}$. Here $F(|x| \le R)$ stands for a smoothed one of the characteristic function of $\{x \in X \mid |x| \le R\}$, whose definition is given in §1.*

In the same way as in the proof of Lemma 3.1 of Yajima [**37**] and Lemma 4.6 of Møller [**26**], the compactness of

$$
\begin{aligned}
&(F(|x|\le R)(K_0-z)^{-1})(F(|x|\le R)(K_0-z)^{-1})^* \\
&= \frac{1}{z-\bar z} F(|x|\le R)\{(K_0-z)^{-1} - (K_0-\bar z)^{-1}\} F(|x|\le R)
\end{aligned}
$$

can be shown, which yields Proposition 3.6 immediately. So we omit the proof.

COROLLARY 3.7. *Assume V satisfies $(V_{\rm WR})_\rho$ with some $\rho > 0$. Then the following hold:*
(1) *Let $R > 0$ and $z \in \boldsymbol{C}\setminus\boldsymbol{R}$. Then $F(|x|\le R)(K-z)^{-1}$ is also compact on $\mathscr{K}$.*
(2) *Let $\mu > 0$ and $z \in \boldsymbol{C}\setminus\boldsymbol{R}$. Then $\langle x\rangle^{-\mu}(K-z)^{-1}$ as well as $\langle x\rangle^{-\mu}(K_0-z)^{-1}$ is compact on $\mathscr{K}$.*

By using the second resolvent identity

$$
(K-z)^{-1} - (K_0-z)^{-1} = -(K_0-z)^{-1} V (K-z)^{-1}, \tag{3.2}
$$

Corollary 3.7 follows from Proposition 3.6 immediately.

As is well-known, in the proof due to Froese-Herbst [**13**] of the Mourre estimate for time-independent N-body Schrödinger operators, the compactness of $f(H)\tilde J_a(x) - \tilde J_a(x) f(H_a)$ is one of the keys in the induction process with respect to N, where H is the full Hamiltonian, H_a is the cluster Hamiltonian with $a \in \mathscr{A}_0 = \mathscr{A}\setminus\{a_{\max}\}$, and f is compactly supported.

In the case of the Floquet Hamiltonians, $f(K)\tilde{J}_a(x) - \tilde{J}_a(x)f(K_a)$ is not compact, in general. However, by modifying $f(K)\tilde{J}_a(x) - \tilde{J}_a(x)f(K_a)$ by some appropriate cut-off with respect to D_t, one can acquire the compactness of the modified one. The following proposition is one of the keys in the proof of the Mourre estimate for K stated in §5: Before stating the proposition, we introduce two functions η_1, $\bar{\eta}_1 \in C^\infty(\boldsymbol{R};\boldsymbol{R})$ such that η_1 is supported in $(-2,\infty)$, $\bar{\eta}_1$ is supported in $(-\infty,-1)$, $0 \le \eta_1(s) \le 1$, $0 \le \bar{\eta}_1(s) \le 1$, $\eta_1(s) = 1$ on $[-1,\infty)$, $\bar{\eta}_1(s) = 1$ on $(-\infty,-2]$, and $\eta_1(s)^2 + \bar{\eta}_1(s)^2 \equiv 1$. Moreover, for $S > 0$, we put $\eta_S(s) = \eta_1(s/S)$ and $\bar{\eta}_S(s) = \bar{\eta}_1(s/S)$.

PROPOSITION 3.8. *Assume $N \ge 3$, and V satisfies $(V_{\mathrm{WR}})_\rho$ with some $\rho > 0$. By using η_S and $\bar{\eta}_S$ with $S > 0$, let us introduce the partition of unity $\eta_S(D_t)$ and $\bar{\eta}_S(D_t)$ such that $\eta_S(D_t)^2 + \bar{\eta}_S(D_t)^2 = 1$. Let $a \in \mathscr{A}_0$ and $f \in C_0^\infty(\boldsymbol{R};\boldsymbol{R})$. Then $\{f(K)\tilde{J}_a(x) - \tilde{J}_a(x)f(K_a)\}\eta_S(D_t)$ is compact on $\mathscr{K}$.*

PROOF. We will prove the proposition by the almost analytic extension method: Take an almost analytic extension $\tilde{f} \in C_0^\infty(\boldsymbol{C})$ of $f \in C_0^\infty(\boldsymbol{R};\boldsymbol{R})$. Then we see that

$$\begin{aligned} & f(K)\tilde{J}_a(x) - \tilde{J}_a(x)f(K_a) \\ = {} & \frac{1}{2\pi i}\int_{\boldsymbol{C}} \overline{\partial}_\zeta \tilde{f}(\zeta)\{(\zeta-K)^{-1}\tilde{J}_a(x) - \tilde{J}_a(x)(\zeta-K_a)^{-1}\}\, d\zeta\wedge d\overline{\zeta} \\ = {} & \frac{1}{2\pi i}\int_{\boldsymbol{C}} \overline{\partial}_\zeta \tilde{f}(\zeta)(\zeta-K)^{-1} \\ & \times \{-i(\nabla\tilde{J}_a)(x)\cdot p - (\Delta\tilde{J}_a)(x)/2 + \tilde{J}_a(x)I_a(t)\}(\zeta-K_a)^{-1}\, d\zeta\wedge d\overline{\zeta} \end{aligned}$$

holds. Since $(\Delta\tilde{J}_a)(x) = O(\langle x\rangle^{-2})$ and $\tilde{J}_a(x)I_a(t) = O(\langle x\rangle^{-\rho})$, by Proposition 2.1 and Corollary 2.4, the last two operators of the right-hand side are compact, by virtue of Corollary 3.7 (2). In order to watch the first operator of the right-hand side, we introduce $\eta_0 \in C^\infty(\boldsymbol{R};\boldsymbol{R})$ such that η_0 is supported in $(0,\infty)$, $0 \le \eta_0(s) \le 1$, and $\eta_0(s) = 1$ on $[1,\infty)$, and use the cut-off $\eta_0(D_t)$ with respect to D_t. By virtue of Proposition 3.5 (2), we see that $p(\zeta-K_a)^{-1}\eta_0(D_t)$ is bounded, which is the key in this proof. Then it follows from Corollary 3.7 (2) that

$$\left(\int_{\boldsymbol{C}} \overline{\partial}_\zeta \tilde{f}(\zeta)(\zeta-K)^{-1}\{(\nabla\tilde{J}_a)(x)\cdot p\}(\zeta-K_a)^{-1}\, d\zeta\wedge d\overline{\zeta}\right)\eta_0(D_t)$$

is also compact, since $(\nabla\tilde{J}_a)(x) = O(\langle x\rangle^{-1})$ by Proposition 2.1. Since $\eta_S(s) - \eta_0(s)$ is compactly supported, we see that $p(\zeta-K_a)^{-1}(\eta_S(D_t) - \eta_0(D_t))$ is also bounded. In fact, by Proposition 3.5 (1), $p(\zeta-K_a)^{-1}\langle D_t\rangle^{-1/2}$ is bounded, and $\langle D_t\rangle^{1/2}(\eta_S(D_t) - \eta_0(D_t))$ is also bounded, by the compactness of the support of $\eta_S(s) - \eta_0(s)$. Thus one can show that

$$\left(\int_{\boldsymbol{C}} \overline{\partial}_\zeta \tilde{f}(\zeta)(\zeta-K)^{-1}\{(\nabla\tilde{J}_a)(x)\cdot p\}(\zeta-K_a)^{-1}\, d\zeta\wedge d\overline{\zeta}\right)(\eta_S(D_t) - \eta_0(D_t))$$

is also compact in the same way as above. These show the proposition. □

REMARK 3.1. We do not know whether $f(K)\tilde{J}_a(x) - \tilde{J}_a(x)f(K_a)$ is compact or not, as mentioned also in [**27**]. As will be seen in §5, the cut-off $\eta_S(D_t)$ does work effectively in the proof of the Mourre estimate for K. Thus this cut-off seems have no disadvantage.

On the other hand, in the case where $N = 2$, by virtue of (3.2), the compactness of $\langle K_0\rangle\{f(K) - f(K_0)\}$ (cf. Lemma 2.2 of [**13**]), which has no cut-off like $\eta_S(D_t)$, can be proved in the same way as above. So we omit the proof.

PROPOSITION 3.9. *Assume $N = 2$, and V satisfies $(V_{\mathrm{WR}})_\rho$ with some $\rho > 0$. Let $f \in C_0^\infty(\boldsymbol{R}; \boldsymbol{R})$. Then $\langle K_0\rangle\{f(K) - f(K_0)\}$ is compact on $\mathscr{K}$.*

4. Two-body case revisited

In this section, we revisit the proof of the Mourre estimate for K with $N = 2$ throughout this section. So we suppose $N = 2$. For the sake of simplicity, we will consider a regular potential $V^{(1,2)}$. So we impose $(V_{\mathrm{WR}})_\rho$ on $V^{(1,2)}$ with $\rho > 0$.

First we state some properties of $A = \tilde{A}_0$ given by (1.13). Unlike in Adachi-Kiyose [**5**], the weight in (1.13) is not the signed one $(\lambda_0 - \delta - D_t)^{-1}$ but the positive one $\langle D_t\rangle_{2\omega}^{-1} = ((2\omega)^2 + (D_t)^2)^{-1/2}$.

LEMMA 4.1. *As for $i[K_0, A]$ and $i[i[K_0, A], A]$,*

$$\begin{aligned} i[K_0, A] &= \langle D_t\rangle_{2\omega}^{-1} p^2 = 2\langle D_t\rangle_{2\omega}^{-1}(K_0 - D_t), \\ i[i[K_0, A], A] &= 2\langle D_t\rangle_{2\omega}^{-2} p^2 = 4\langle D_t\rangle_{2\omega}^{-2}(K_0 - D_t) \end{aligned} \tag{4.1}$$

hold. Hence, $i[K_0, A]\langle K_0\rangle^{-1}$ and $i[i[K_0, A], A]\langle K_0\rangle^{-1}$ are bounded.

LEMMA 4.2. *Assume $N = 2$, and $V = V^{(1,2)}$ satisfies $(V_{\mathrm{WR}})_\rho$ with some $\rho > 0$. As for $i[V, A]$ and $i[i[V, A], A]$,*

$$\begin{aligned} i[V, A] &= i[V, \langle D_t\rangle_{2\omega}^{-1}]\hat{A}_0 + \langle D_t\rangle_{2\omega}^{-1} i[V, \hat{A}_0], \\ i[i[V, A], A] &= i[i[V, \langle D_t\rangle_{2\omega}^{-1}], \langle D_t\rangle_{2\omega}^{-1}]\hat{A}_0^2 + \langle D_t\rangle_{2\omega}^{-1} i[i[V, \langle D_t\rangle_{2\omega}^{-1}], \hat{A}_0]\hat{A}_0 \\ &\quad + \langle D_t\rangle_{2\omega}^{-1} i[i[V, \hat{A}_0], \langle D_t\rangle_{2\omega}^{-1}]\hat{A}_0 + \langle D_t\rangle_{2\omega}^{-2} i[i[V, \hat{A}_0], \hat{A}_0] \end{aligned} \tag{4.2}$$

hold. Hence, $i[V, A]\langle K_0\rangle^{-1}$ and $i[i[V, A], A]\langle K_0\rangle^{-1}$ are bounded. Moreover, $\langle K_0\rangle^{-1} i[V, A] \langle K_0\rangle^{-1}$ is compact.

By simple computation, (4.1) and (4.2) can be obtained. Therefore Lemma 4.1 follows trivially. Lemma 4.2 can be shown in the same way as in [**5**]: For instance, as for $i[V, A]$, by $(\partial_t V)(t, x) = O(\langle x\rangle^{-\rho-1})$ and $(\nabla V)(t, x) = O(\langle x\rangle^{-\rho-1})$, we see that $i[V, A]\langle K_0\rangle^{-1}$ is bounded, by virtue of Proposition 3.5 (1). Here we used

$$i[V, \langle D_t\rangle_{2\omega}^{-1}] = \frac{1}{2\pi i}\int_{\boldsymbol{C}} \overline{\partial}_\zeta \tilde{g}(\zeta)(\zeta - D_t)^{-1} i[V, D_t](\zeta - D_t)^{-1}\, d\zeta \wedge d\overline{\zeta}$$

with $i[V, D_t] = -\partial_t V$, where $\tilde{g} \in C^\infty(\boldsymbol{C})$ is an almost analytic extension of $g(\nu) = \langle \nu\rangle_{2\omega}^{-1}$, which satisfies $|\overline{\partial}_z \tilde{g}(z)| \le C_L\langle z\rangle^{-1-1-L}|\mathrm{Im}\, z|^L$ with $L \in \boldsymbol{N} \cup \{0\}$. Moreover, by virtue of Corollary 3.7 (2), we also see that $\langle K_0\rangle^{-1} i[V, A]\langle K_0\rangle^{-1}$ is compact.

One can show the following theorem and corollary:

THEOREM 4.3. *Suppose $N = 2$. Assume V satisfies $(V_{\rm WR})_\rho$ with some $\rho > 0$. Put*

$$\Theta = \sigma_{\rm pp}(K^{a_{\min}}) = \sigma_{\rm pp}(D_t) = \omega \boldsymbol{Z}, \quad \widehat{\Theta} = \bigcup_{a \in \mathscr{A}} \sigma_{\rm pp}(K^a) = \Theta \cup \sigma_{\rm pp}(K).$$

Let $\lambda_0 \in [0, \omega)$. Then the following hold:
(1) *Put*

$$d_0(\lambda) = {\rm dist}(\lambda, \Theta), \quad d_1(\lambda) = {\rm dist}(\lambda, \Theta \cap (-\infty, \lambda])$$

for $\lambda \in \boldsymbol{R}$. Note $d_0(\lambda) \le d_1(\lambda)$. Suppose $\lambda_0 \in [0, \omega) \setminus \Theta$, that is, $\lambda_0 \in (0, \omega)$. Take δ as $0 < \delta \le d_0(\lambda_0)/2$. Then, for any $f_{\lambda_0,\delta} \in C_0^\infty(\boldsymbol{R}; \boldsymbol{R})$ supported in $[\lambda_0 - \delta, \lambda_0 + \delta]$,

$$f_{\lambda_0,\delta}(K) i[K, A] f_{\lambda_0,\delta}(K) \ge \frac{d_1(\lambda_0) - \delta}{\omega} f_{\lambda_0,\delta}(K)^2 + C_0 \tag{4.3}$$

holds with some compact operator C_0 on $\mathscr{K}$. For $\lambda_0 \in [0, \omega)$, $d_1(\lambda_0) = \lambda_0$ holds.

Suppose $\lambda_0 \in [0, \omega) \cap \Theta$, that is, $\lambda_0 = 0$. Take δ as $0 < \delta \le \omega/2$. Then, for any $f_{\lambda_0,\delta} \in C_0^\infty(\boldsymbol{R}; \boldsymbol{R})$ supported in $[\lambda_0 - \delta, \lambda_0 + \delta]$,

$$f_{\lambda_0,\delta}(K) i[K, A] f_{\lambda_0,\delta}(K) \ge C_0 \tag{4.4}$$

holds with some compact operator C_0 on $\mathscr{K}$.

Hence, for any $\hat{\delta}$ such that $0 < \hat{\delta} < \delta$, $\sigma_{\rm pp}(K) \cap (\lambda_0 - \hat{\delta}, \lambda_0 + \hat{\delta})$ is finite, and that the eigenvalues of K in $(\lambda_0 - \hat{\delta}, \lambda_0 + \hat{\delta})$ are of finite multiplicity.
(2) *Suppose $\lambda_0 \in [0, \omega) \setminus \widehat{\Theta}$. Take ϵ as $0 < \epsilon < d_1(\lambda_0) - d_0(\lambda_0)/2$. Then there exists a small $\delta_{\epsilon,1} > 0$ such that $\delta_{\epsilon,1} \le d_0(\lambda_0)/2$ and for any $f_{\lambda_0,\delta} \in C_0^\infty(\boldsymbol{R}; \boldsymbol{R})$ supported in $[\lambda_0 - \delta, \lambda_0 + \delta]$ with $0 < \delta \le \delta_{\epsilon,1}$,*

$$f_{\lambda_0,\delta}(K) i[K, A] f_{\lambda_0,\delta}(K) \ge \frac{d_1(\lambda_0) - \delta - \epsilon}{\omega} f_{\lambda_0,\delta_{\epsilon,1}}(K)^2 \tag{4.5}$$

holds. Suppose $s > 1/2$ and $0 < \hat{\delta} < \delta_{\epsilon,1}$. Then

$$\sup_{\substack{{\rm Re}\, z \in [\lambda_0 - \hat{\delta}, \lambda_0 + \hat{\delta}] \\ {\rm Im}\, z \ne 0}} \| \langle A \rangle^{-s} (K - z)^{-1} \langle A \rangle^{-s} \|_{\mathscr{B}(\mathscr{K})} < \infty \tag{4.6}$$

holds. Moreover, $\langle A \rangle^{-s} (K - z)^{-1} \langle A \rangle^{-s}$ is a $\mathscr{B}(\mathscr{K})$-valued $\theta(s)$-Hölder continuous function on $z \in S_{\lambda_0,\hat{\delta},\pm}$ with some $0 < \theta(s) < 1$. And, there exist the norm limits

$$\langle A \rangle^{-s} (K - (\lambda \pm i0))^{-1} \langle A \rangle^{-s} = \lim_{\varepsilon \to +0} \langle A \rangle^{-s} (K - (\lambda \pm i\varepsilon))^{-1} \langle A \rangle^{-s}$$

in $\mathscr{B}(\mathscr{K})$ for any $\lambda \in [\lambda_0 - \hat{\delta}, \lambda_0 + \hat{\delta}]$. $\langle A \rangle^{-s} (K - (\lambda \pm i0))^{-1} \langle A \rangle^{-s}$ are also $\theta(s)$-Hölder continuous in λ.

COROLLARY 4.4. *Assume V satisfies $(V_{\rm WR})_\rho$ with some $\rho > 0$. Then the following hold:*

(1) *The eigenvalues of K in $\boldsymbol{R}\setminus\Theta$ can accumulate only at Θ. Moreover, $\widehat{\Theta}$ is a countable closed set.*
(2) *Let I be a compact interval in $\boldsymbol{R}\setminus\widehat{\Theta}$. Suppose $1/2 < s \leq 1$. Then*

$$\sup_{\substack{\mathrm{Re}\, z\in I\\ \mathrm{Im}\, z\neq 0}} \|\langle x\rangle^{-s}(K-z)^{-1}\langle x\rangle^{-s}\|_{\mathscr{B}(\mathscr{K})} < \infty \tag{4.7}$$

holds. Moreover, $\langle x\rangle^{-s}(K-z)^{-1}\langle x\rangle^{-s}$ is a $\mathscr{B}(\mathscr{K})$-valued $\theta(s)$-Hölder continuous function on $z\in S_{I,\pm}$. And, there exist the norm limits

$$\langle x\rangle^{-s}(K-(\lambda\pm i0))^{-1}\langle x\rangle^{-s} = \lim_{\varepsilon\to+0}\langle x\rangle^{-s}(K-(\lambda\pm i\varepsilon))^{-1}\langle x\rangle^{-s}$$

in $\mathscr{B}(\mathscr{K})$ for $\lambda\in I$. $\langle x\rangle^{-s}(K-(\lambda\pm i0))^{-1}\langle x\rangle^{-s}$ are also $\theta(s)$-Hölder continuous in λ.

We will sketch the proof of the estimates (4.3) and (4.4) only. Thus Theorem 4.3 and Corollary 4.4 can be shown by the standard argument in the Mourre theory. In particular, for the proof of Corollary 4.4, we use the argument due to Perry-Sigal-Simon [**29**], and the boundedness of $A(K-\lambda_0-i)^{-1}\langle x\rangle^{-1}$, which follows from Proposition 3.5 (1).

PROOF OF (4.3) AND (4.4). Let $\lambda_0 \in [0,\omega)$. For a while, take δ as $0<\delta\leq\omega/2$. Denote by $f_{\lambda_0,\delta}$ a function in $C_0^\infty(\boldsymbol{R};\boldsymbol{R})$ such that $\mathrm{supp}\, f_{\lambda_0,\delta}\subset[\lambda_0-\delta,\lambda_0+\delta]$. Since $\langle K_0\rangle^{-1} i[V,A]\langle K_0\rangle^{-1}$ is compact by Lemma 4.2, we see that

$$f_{\lambda_0,\delta}(K)i[K,A]f_{\lambda_0,\delta}(K) = f_{\lambda_0,\delta}(K)i[K_0,A]f_{\lambda_0,\delta}(K) + C_1 \tag{4.8}$$

holds with some compact operator C_1 on $\mathscr{K}$. Since $i[K_0,A]\langle K_0\rangle^{-1}$ is bounded by Lemma 4.1, and $\langle K_0\rangle\{f_{\lambda_0,\delta}(K)-f_{\lambda_0,\delta}(K_0)\}$ is compact by Proposition 3.9, we obtain

$$f_{\lambda_0,\delta}(K)i[K,A]f_{\lambda_0,\delta}(K) = f_{\lambda_0,\delta}(K_0)i[K_0,A]f_{\lambda_0,\delta}(K_0) + C_2 \tag{4.9}$$

with some compact operator C_2 on $\mathscr{K}$. Let $\hat{\mathscr{F}} : L^2(\boldsymbol{T}) \to \ell^2(\boldsymbol{Z})$ be the Fourier transform which is defined by

$$\hat{\mathscr{F}}[g](n) = T^{-1/2}\int_0^T g(t)e^{-in\omega t}\,dt, \qquad n\in\boldsymbol{Z};\quad g\in L^2(\boldsymbol{T}).$$

Then D_t becomes the multiplication by $n\omega$ on $\ell^2(\boldsymbol{Z})$ for $n\in\boldsymbol{Z}$, that is, D_t can be decomposed into the direct sum $\bigoplus_{n\in\boldsymbol{Z}} n\omega$, via the Fourier transform $\hat{\mathscr{F}}$. By using $\hat{\mathscr{F}}$, $f_{\lambda_0,\delta}(K_0)i[K_0,A]$ $f_{\lambda_0,\delta}(K_0)$ can be decomposed into the direct sum

$$\bigoplus_{n\in\boldsymbol{Z}}\frac{p^2 f_{\lambda_0,\delta}(n\omega+p^2/2)^2}{((2\omega)^2+(n\omega)^2)^{1/2}} = \bigoplus_{n\in\boldsymbol{Z}}\frac{p^2 f_{\lambda_0-n\omega,\delta}(p^2/2)^2}{((2\omega)^2+(n\omega)^2)^{1/2}} = \bigoplus_{n\in\boldsymbol{Z}} I_0(n\omega).$$

We first consider the case where $\lambda_0\in[0,\omega)\setminus\Theta$. Take δ as $0<\delta\leq d_0(\lambda_0)/2$. If $n\leq 0$, then the fibered operator $I_0(n\omega)$ can be estimated as

$$I_0(n\omega) \geq \frac{2\{(\lambda_0-n\omega)-\delta\}}{((2\omega)^2+(n\omega)^2)^{1/2}} f_{\lambda_0-n\omega,\delta}(p^2/2)^2$$

$$\geq \frac{2\{(\lambda_0+|n|\omega)-\delta\}}{2\omega+|n|\omega} f_{\lambda_0-n\omega,\delta}(p^2/2)^2 \geq \frac{\lambda_0-\delta}{\omega} f_{\lambda_0-n\omega,\delta}(p^2/2)^2;$$

while, if $n \geq 1$, then $I_0(n\omega) = 0$. Here we used that that if $n \geq 1$, then $\lambda_0 - n\omega + \delta \leq \lambda_0 - \omega + \delta \leq \lambda_0 - \omega + d_0(\lambda_0)/2 < 0$ holds. Then we obtain

$$f_{\lambda_0,\delta}(K_0) i[K_0, A] f_{\lambda_0,\delta}(K_0) \geq \frac{\lambda_0-\delta}{\omega} f_{\lambda_0,\delta}(K_0)^2.$$

It follows from this, (4.9) and the compactness of $\langle K_0\rangle\{f_{\lambda_0,\delta}(K) - f_{\lambda_0,\delta}(K_0)\}$ that (4.3) holds. Here we used $d_1(\lambda_0) = \lambda_0$.

We next consider the case where $\lambda_0 \in [0, \omega) \cap \Theta$. Take δ as $0 < \delta \leq \omega/2$. Then

$$f_{\lambda_0,\delta}(K_0) i[K_0, A] f_{\lambda_0,\delta}(K_0) \geq 0$$

holds. In the same way as above, it can be shown easily that (4.4) holds. □

5. Proof of Theorem 1.1

In this section, we prove Theorem 1.1. As in §2, we will show the estimate (1.16) only. (1.16) yields the Mourre estimate (1.17). Thus Theorem 1.1 and Corollary 1.2 can be shown by the standard argument in the Mourre theory. Throughout this section, we assume V satisfies $(V_{\mathrm{WR}})_\rho$ with some $\rho > 0$.

Take the Yafaev partition of unity $\{\tilde{J}_a\}_{a\in\mathscr{A}}$, and introduce the operator A by (1.15). By the second representation of (1.15), we have (2.13):

$$i[K_0, A] = i[K_0, \tilde{A}_0] + \sum_{a\in\mathscr{A}_0^0} S_{1,a} + L_0 \sum_{a\in\mathscr{A}_0^0} S_{2,a}$$

with $S_{1,a} = i[K_0, \tilde{J}_a(x)\{-(\tilde{A}_0)_a\}\tilde{J}_a(x)]$ and $S_{2,a} = i[K_0, \tilde{J}_a(x_a)\bar{A}_a\tilde{J}_a(x_a)]$. Then we obtain the following estimate for $f_{\lambda_0,\delta}(K) i[K_0, A] f_{\lambda_0,\delta}(K)$:

LEMMA 5.1. *Let $\lambda_0 \in [0, \omega)$ and $\delta > 0$. Then*

$$\begin{aligned}
&f_{\lambda_0,\delta}(K) i[K_0, A] f_{\lambda_0,\delta}(K)\\
&\geq f_{\lambda_0,\delta}(K)(\langle D_t\rangle_{2\omega}^{-1} p^2) f_{\lambda_0,\delta}(K)\\
&\quad + \sum_{a\in\mathscr{A}_0^0} f_{\lambda_0,\delta}(K)\tilde{J}_a(x)(-\langle D_t\rangle_{2\omega}^{-1}(p_a)^2)\tilde{J}_a(x) f_{\lambda_0,\delta}(K)\\
&\quad + \frac{L_0}{2}\sum_{a\in\mathscr{A}_0^0} f_{\lambda_0,\delta}(K)\tilde{J}_a(x_a)(\langle D_t\rangle_{2\omega}^{-1}(p_a)^2)\tilde{J}_a(x_a) f_{\lambda_0,\delta}(K)\\
&\quad + f_{\lambda_0,\delta}(K)\{O(L_0^{-1}) + L_0 O(L_1^{-1})\} f_{\lambda_0,\delta}(K) + C
\end{aligned} \tag{5.1}$$

holds with some compact operator C on $\mathscr{K}$, where $L_1 > 0$ is the parameter which comes from Corollary 2.8.

PROOF. Lemma 4.1 yields $i[K_0, \tilde{A}_0] = \langle D_t\rangle_{2\omega}^{-1} p^2$ and $i[K_0, -(\tilde{A}_0)_a] = -\langle D_t\rangle_{2\omega}^{-1}(p_a)^2$.

We first watch $S_{2,a} = i[K_0, \tilde{J}_a(x_a)\bar{A}_a\tilde{J}_a(x_a)]$ with $a \in \mathscr{A}_0^0$. One can show easily that

$$\begin{aligned} S_{2,a} &= 2\mathrm{Re}\,[\{\mathrm{Re}\,(p_a \cdot (\nabla_a \tilde{J}_a)(x_a))\}\bar{A}_a\tilde{J}_a(x_a)] + \tilde{J}_a(x_a)(\langle p_a\rangle^{-2}(p_a)^2)\tilde{J}_a(x_a) \\ &= 2\mathrm{Re}\,[\{\mathrm{Re}\,(p_a \cdot (\nabla_a \tilde{J}_a)(x_a))\}\bar{A}_a\tilde{J}_a(x_a)] \\ &\quad + (\tilde{J}_a(x_a) + L_0^{-2})(\langle p_a\rangle^{-2}(p_a)^2)(\tilde{J}_a(x_a) + L_0^{-2}) + O(L_0^{-2}) \end{aligned}$$

is bounded. By simple calculation, $\bar{A}_a$ can be recognized as $x_a \cdot p_a\langle p_a\rangle^{-2}$ up to the bounded error. Then the main part of $S_{2,a}$ can be recognized as

$$\begin{aligned} &(p_a\langle p_a\rangle^{-1})(\tilde{J}_a(x_a) + L_0^{-2})^2(p_a\langle p_a\rangle^{-1})^* \\ &+ 2\mathrm{Re}\,[(p_a\langle p_a\rangle^{-1})(\nabla_a\tilde{J}_a)(x_a)^*(x_a\tilde{J}_a(x_a))(p_a\langle p_a\rangle^{-1})^*] \end{aligned} \tag{5.2}$$

by neglecting errors of the orders $O(L_0^{-2})$ and $O(L_1^{-1})$, where we used the notation $(p_a)^2 = p_a \cdot p_a = p_a(p_a)^*$ and $x_a \cdot p_a = x_a(p_a)^*$. We also used Corollary 2.8. Introducing a bounded operator $B_{a,0} = (p_a\langle p_a\rangle^{-1})(\tilde{J}_a(x_a) + L_0^{-2})$, the first term of (5.2) is written as $B_{a,0}B_{a,0}^*$. By virtue of Lemma 2.7 and the remark directly below it, (5.2) can be estimated from below by $(3/4)B_{a,0}B_{a,0}^*$.

We next watch $S_{1,a} = i[K_0, \tilde{J}_a(x)\{-(\tilde{A}_0)_a\}\tilde{J}_a(x)]$ with $a \in \mathscr{A}_0^0$. By virtue of Proposition 3.5 (1), one can show easily that

$$S_{1,a} = 2\mathrm{Re}\,[\{\mathrm{Re}\,(p \cdot (\nabla\tilde{J}_a)(x))\}\{-(\tilde{A}_0)_a\}\tilde{J}_a(x)] + \tilde{J}_a(x)(-\langle D_t\rangle_{2\omega}^{-1}(p_a)^2)\tilde{J}_a(x)$$

is K_0-bounded. By simple calculation, $(\tilde{A}_0)_a$ can be recognized as $x_a \cdot p_a\langle D_t\rangle_{2\omega}^{-1}$ up to the bounded error. Then the main part of $S_{1,a}$ can be recognized as

$$\tilde{J}_a(x)(-\langle D_t\rangle_{2\omega}^{-1}(p_a)^2)\tilde{J}_a(x) - 2\mathrm{Re}\,[p\langle D_t\rangle_{2\omega}^{-1}(\nabla\tilde{J}_a)(x)^*(x_a\tilde{J}_a(x))(p_a)^*] \tag{5.3}$$

by neglecting K_0-compact errors. Introducing a K_0-bounded operator $B_{a,1} = p\langle D_t\rangle_{2\omega}^{-1}(\nabla\tilde{J}_a)(x)^*(x_a\tilde{J}_a(x))\langle p_a\rangle(\tilde{J}_a(x_a) + L_0^{-2})^{-1}$, we see that

$$\begin{aligned} &-2\mathrm{Re}\,[p\langle D_t\rangle_{2\omega}^{-1}(\nabla\tilde{J}_a)(x)^*(x_a\tilde{J}_a(x))(p_a)^*] + \frac{L_0}{4}B_{a,0}B_{a,0}^* \\ &= \left(\frac{\sqrt{L_0}}{2}B_{a,0} - \frac{2}{\sqrt{L_0}}B_{a,1}\right)\left(\frac{\sqrt{L_0}}{2}B_{a,0} - \frac{2}{\sqrt{L_0}}B_{a,1}\right)^* - \frac{4}{L_0}B_{a,1}B_{a,1}^* \\ &\geq -\frac{4}{L_0}B_{a,1}B_{a,1}^*. \end{aligned}$$

Here we note that $\langle K_0\rangle^{-1}B_{a,1}$ is bounded in L_0, because $0 \leq \tilde{J}_a(x)(\tilde{J}_a(x_a) + L_0^{-2})^{-1} \leq 1$ by Lemma 2.9. Since $(3/4)B_{a,0}B_{a,0}^* - (1/4)B_{a,0}B_{a,0}^* = (1/2)B_{a,0}B_{a,0}^*$ and $B_{a,0}B_{a,0}^* = \tilde{J}_a(x_a)(\langle p_a\rangle^{-2}(p_a)^2)\tilde{J}_a(x_a) + O(L_0^{-2}) + O(L_1^{-1})$ by Corollary 2.8, we obtain (5.1). □

This estimate (5.1) is important for the sake of obtaining the Mourre estimate for K. Since $\langle D_t\rangle_{2\omega}^{-1}\langle p_a\rangle^2$ is K_0-bounded by Proposition 3.5 (1), for relatively large $L_0 > 0$,

$$
\begin{aligned}
&f_{\lambda_0,\delta}(K)\tilde{J}_a(x)(-\langle D_t\rangle_{2\omega}^{-1}(p_a)^2)\tilde{J}_a(x)f_{\lambda_0,\delta}(K)\\
&+\frac{L_0}{4}f_{\lambda_0,\delta}(K)\tilde{J}_a(x_a)(\langle p_a\rangle^{-2}(p_a)^2)\tilde{J}_a(x_a)f_{\lambda_0,\delta}(K)\geq C'_a
\end{aligned}
\tag{5.4}
$$

holds with some compact operator C'_a on $\mathscr{K}$, for each $a\in\mathscr{A}_0^0$. Here we used $\tilde{J}_a(x_a)\geq\tilde{J}_a(x)\geq 0$ by Lemma 2.9, and $(\nabla_a\tilde{J}_a)(x)=O(\langle x\rangle^{-1})$ by Proposition 2.1, which is K_0-compact by Corollary 3.7 (2). This estimate (5.4) will be used later.

On the other hand, by the first representation of (1.15), $i[V,A]$ can be written as

$$
\begin{aligned}
i[V,A]=&\sum_{a\in\mathscr{A}\setminus\mathscr{A}_0^0}\tilde{J}_a(x)i[V,\tilde{A}_0]\tilde{J}_a(x)\\
&+\sum_{a\in\mathscr{A}_0^0}\tilde{J}_a(x)i[V,(\tilde{A}_0)^a]\tilde{J}_a(x)+L_0\sum_{a\in\mathscr{A}_0^0}\tilde{J}_a(x_a)i[I_a,\bar{A}_a]\tilde{J}_a(x_a)
\end{aligned}
$$

by $i[V^a,\bar{A}_a]=0$ for $a\in\mathscr{A}_0^0$. Then we obtain the following estimate for $f_{\lambda_0,\delta}(K)i[V,A]$ $f_{\lambda_0,\delta}(K)$:

LEMMA 5.2. *Let $\lambda_0\in[0,\omega)$ and $\delta>0$. Then*

$$
\begin{aligned}
f_{\lambda_0,\delta}(K)i[V,A]f_{\lambda_0,\delta}(K)=&\sum_{a\in\mathscr{A}_0^0}f_{\lambda_0,\delta}(K)\tilde{J}_a(x)i[V^a,(\tilde{A}_0)^a]\tilde{J}_a(x)f_{\lambda_0,\delta}(K)\\
&+f_{\lambda_0,\delta}(K)\{L_0O(L_1^{-\rho})\}f_{\lambda_0,\delta}(K)+C'
\end{aligned}
\tag{5.5}
$$

holds with some compact operator C' on $\mathscr{K}$, where $L_1>0$ is the parameter which comes from Corollary 2.8.

PROOF. By virtue of Proposition 3.6 and Corollary 3.7, one can show easily that

$$
\langle K_0\rangle^{-1}\left(\sum_{a\in\mathscr{A}\setminus\mathscr{A}_0^0}\tilde{J}_a(x)i[V,\tilde{A}_0]\tilde{J}_a(x)+\sum_{a\in\mathscr{A}_0^0}\tilde{J}_a(x)i[I_a,(\tilde{A}_0)^a]\tilde{J}_a(x)\right)\langle K_0\rangle^{-1}
\tag{5.6}
$$

is compact. Here we used the compactness of supp $\tilde{J}_{a_{\max}}$, and Corollary 2.4. Moreover, we see that for $a\in\mathscr{A}_0^0$,

$$
\tilde{J}_a(x_a)i[I_a,\bar{A}_a]\tilde{J}_a(x_a)=O(L_1^{-\rho})
\tag{5.7}
$$

holds, by virtue of Lemma 2.3. Here we used that $\tilde{J}_a(x_a)$ is supported in $\{x_a\in X_a\mid|x_a|\geq(\sigma_{a_{\max},1}^-/2)L_1\}$. Therefore we obtain (5.5). □

By the above argument, the following lemma can be obtained as Lemmas 4.1 and 4.2:

LEMMA 5.3. *Assume V satisfies $(V_{\mathrm{WR}})_\rho$ with some $\rho>0$. Then $i[K_0,A]\langle K_0\rangle^{-1}$ and $i[V,A]\langle K_0\rangle^{-1}$ are bounded. This yields that $i[K,A]\langle K_0\rangle^{-1}$ is bounded. Moreover, $\langle K_0\rangle^{-1}i[i[K,A],A]\langle K_0\rangle^{-1}$ is also bounded.*

Based on these, one can show (1.16) as follows:

PROOF OF (1.16). Suppose $\lambda_0 \in [0,\omega) \setminus \Theta$ and $\epsilon > 0$. Put

$$\Theta = \bigcup_{a \in \mathscr{A}_0} \sigma_{\mathrm{pp}}(K^a), \quad d_0(\lambda) = \mathrm{dist}(\lambda, \Theta), \quad d_1(\lambda) = \mathrm{dist}(\lambda, \Theta \cap (-\infty, \lambda])$$

for $\lambda \in \boldsymbol{R}$. For a while, take δ as $0 < \delta \le d_0(\lambda_0)/2$. Then both $\lambda_0 - \delta \ge \lambda_0 - d_0(\lambda_0)/2 > 0$ and $\lambda_0 + \delta \le \lambda_0 + d_0(\lambda_0)/2 < \omega$ hold. By Lemmas 5.1 and 5.2, we have

$$\begin{aligned} &f_{\lambda_0,\delta}(K) i[K,A] f_{\lambda_0,\delta}(K) \\ &\ge f_{\lambda_0,\delta}(K) T_0 f_{\lambda_0,\delta}(K) + f_{\lambda_0,\delta}(K)\{O(L_0^{-1}) + L_0 O(L_1^{\max\{-1,-\rho\}})\} f_{\lambda_0,\delta}(K) + C_1 \end{aligned} \tag{5.8}$$

with some compact operator C_1 on $\mathscr{K}$, where

$$\begin{aligned} &T_0 = T_1 + \sum_{a \in \mathscr{A}_0^0} \left(T_{2,a} + \frac{L_0}{2} T_{3,a} + T_{4,a} \right); \\ &T_1 = \langle D_t \rangle_{2\omega}^{-1} p^2, \quad T_{2,a} = \tilde{J}_a(x)(-\langle D_t \rangle_{2\omega}^{-1}(p_a)^2)\tilde{J}_a(x), \\ &T_{3,a} = \tilde{J}_a(x_a)(\langle p_a \rangle^{-2}(p_a)^2)\tilde{J}_a(x_a), \quad T_{4,a} = \tilde{J}_a(x) i[V^a, (\tilde{A}_0)^a]\tilde{J}_a(x). \end{aligned}$$

By using the partition of unity $\eta_S(D_t)$ and $\bar{\eta}_S(D_t)$ with sufficiently large $S > 0$ such that $\eta_S(D_t)^2 + \bar{\eta}_S(D_t)^2 = 1$ introduced in Proposition 3.8, we will decompose the term $f_{\lambda_0,\delta}(K) T_0 f_{\lambda_0,\delta}(K)$ into the sum

$$f_{\lambda_0,\delta}(K)\eta_S(D_t) T_0 \eta_S(D_t) f_{\lambda_0,\delta}(K) + f_{\lambda_0,\delta}(K)\bar{\eta}_S(D_t) T_0 \bar{\eta}_S(D_t) f_{\lambda_0,\delta}(K)$$

up to the errors of order $O(S^{-1})$, which come from $[T_{4,a}, \eta_S(D_t)]$, $[T_{4,a}, \bar{\eta}_S(D_t)]$'s.

We first watch $f_{\lambda_0,\delta}(K)\bar{\eta}_S(D_t) T_0 \bar{\eta}_S(D_t) f_{\lambda_0,\delta}(K)$. Taking account of $T_1 = 2\langle D_t \rangle_{2\omega}^{-1}$ $(K_0 - D_t)$ by Lemma 4.1, we will focus on $\langle D_t \rangle_{2\omega}^{-1}(-D_t)$. Via the Fourier transform $\hat{\mathscr{F}} : L^2(\boldsymbol{T})$ $\to \ell^2(\boldsymbol{Z})$ introduced in §4, $\langle D_t \rangle_{2\omega}^{-1}(-D_t)$ can be decomposed into the direct sum

$$\bigoplus_{n \in \boldsymbol{Z}} \frac{-n\omega}{((2\omega)^2 + (n\omega)^2)^{1/2}}.$$

Suppose $n \in \boldsymbol{Z}$ such that $n < -S < 0$. Then we note

$$\frac{-n\omega}{((2\omega)^2 + (n\omega)^2)^{1/2}} \ge \frac{|n|\omega}{2\omega + |n|\omega} = 1 - \frac{2\omega}{2\omega + |n|\omega}.$$

By using this and that K_0 is K-bounded, one can obtain easily

$$f_{\lambda_0,\delta}(K)\bar{\eta}_S(D_t) T_1 \bar{\eta}_S(D_t) f_{\lambda_0,\delta}(K) \ge (2 + O(S^{-1})) f_{\lambda_0,\delta}(K)\bar{\eta}_S(D_t)^2 f_{\lambda_0,\delta}(K).$$

By using (5.4), $\{(L_0/2) - (L_0/4)\} T_{3,a} = (L_0/4) T_{3,a} \ge 0$ and $\bar{\eta}_S(D_t) T_{4,a} \bar{\eta}_S(D_t) = O(S^{-1})$ for $a \in \mathscr{A}_0^0$, one can obtain

$$\begin{aligned} &f_{\lambda_0,\delta}(K)\bar{\eta}_S(D_t) T_0 \bar{\eta}_S(D_t) f_{\lambda_0,\delta}(K) \\ &\ge (2 + O(S^{-1})) f_{\lambda_0,\delta}(K)\bar{\eta}_S(D_t)^2 f_{\lambda_0,\delta}(K) + C_2 \end{aligned} \tag{5.9}$$

with some compact operator C_2 on $\mathscr{K}$.

We next watch $f_{\lambda_0,\delta}(K)\eta_S(D_t)T_0\eta_S(D_t)f_{\lambda_0,\delta}(K)$. By the IMS localization formula for $\{\tilde{J}_a\}_{a\in\mathscr{A}}$

$$p^2 = \sum_{a\in\mathscr{A}} \tilde{J}_a(x)p^2\tilde{J}_a(x) + \sum_{a\in\mathscr{A}} |(\nabla\tilde{J}_a)(x)|^2$$

and $\sum_{a\in\mathscr{A}} |(\nabla\tilde{J}_a)(x)|^2 = O(\langle x\rangle^{-2})$ obtained by Proposition 2.1, we have

$$\begin{aligned}
&f_{\lambda_0,\delta}(K)\eta_S(D_t)T_1\eta_S(D_t)f_{\lambda_0,\delta}(K)\\
&= \sum_{a\in\mathscr{A}} f_{\lambda_0,\delta}(K)\eta_S(D_t)\tilde{J}_a(x)T_1\tilde{J}_a(x)\eta_S(D_t)f_{\lambda_0,\delta}(K) + C_3\\
&= f_{\lambda_0,\delta}(K)\eta_S(D_t)\tilde{J}_{a_{\min}}(x)T_1\tilde{J}_{a_{\min}}(x)\eta_S(D_t)f_{\lambda_0,\delta}(K)\\
&\quad + \sum_{a\in\mathscr{A}_0^0} f_{\lambda_0,\delta}(K)\eta_S(D_t)\tilde{J}_a(x)T_1\tilde{J}_a(x)\eta_S(D_t)f_{\lambda_0,\delta}(K) + C_3'
\end{aligned}$$

with some compact operators C_3 and C_3' on $\mathscr{K}$. Here we also used the compactness of $\operatorname{supp}\tilde{J}_{a_{\max}}$ and Corollary 3.7. Then we obtain

$$\begin{aligned}
&f_{\lambda_0,\delta}(K)\eta_S(D_t)T_0\eta_S(D_t)f_{\lambda_0,\delta}(K)\\
&= f_{\lambda_0,\delta}(K)\eta_S(D_t)T_{1,a_{\min}}\eta_S(D_t)f_{\lambda_0,\delta}(K)\\
&\quad + \sum_{a\in\mathscr{A}_0^0} f_{\lambda_0,\delta}(K)\eta_S(D_t)\left(T_{1,a}' + \frac{L_0}{2}T_{3,a} + T_{4,a}\right)\eta_S(D_t)f_{\lambda_0,\delta}(K) + C_3',
\end{aligned} \tag{5.10}$$

where $T_{1,a_{\min}} = \tilde{J}_{a_{\min}}(x)(\langle D_t\rangle_{2\omega}^{-1}p^2)\tilde{J}_{a_{\min}}(x)$, and $T_{1,a}' = \tilde{J}_a(x)(\langle D_t\rangle_{2\omega}^{-1}(p^a)^2)\tilde{J}_a(x)$ with $a\in\mathscr{A}_0^0$. Here we used $p^2 + \{-(p_a)^2\} = (p^a)^2$.

As for the term $f_{\lambda_0,\delta}(K)\eta_S(D_t)T_{1,a_{\min}}\eta_S(D_t)f_{\lambda_0,\delta}(K)$, the estimate

$$\begin{aligned}
&f_{\lambda_0,\delta}(K)\eta_S(D_t)T_{1,a_{\min}}\eta_S(D_t)f_{\lambda_0,\delta}(K)\\
&\geq \frac{\widehat{d}_{1,a_{\min}}(\lambda_0)-\delta}{\omega} f_{\lambda_0,\delta}(K)\tilde{J}_{a_{\min}}(x)\eta_S(D_t)^2\tilde{J}_{a_{\min}}(x)f_{\lambda_0,\delta}(K) + C_{4,a_{\min}}
\end{aligned} \tag{5.11}$$

holds with some compact operator $C_{4,a_{\min}}$ on $\mathscr{K}$, where

$$\widehat{d}_{1,a_{\min}}(\lambda) = \operatorname{dist}(\lambda, \widehat{\Theta}_{a_{\min}}\cap(-\infty,\lambda]);\quad \widehat{\Theta}_{a_{\min}} = \sigma_{\mathrm{pp}}(D_t) = \omega\boldsymbol{Z},$$

for $\lambda\in\boldsymbol{R}$; as for the term $f_{\lambda_0,\delta}(K)\eta_S(D_t)(T_{1,a}' + (L_0/2)T_{3,a} + T_{4,a})\eta_S(D_t)f_{\lambda_0,\delta}(K)$ with $a\in\mathscr{A}_0^0$, the estimate

$$\begin{aligned}
&f_{\lambda_0,\delta}(K)\eta_S(D_t)\left(T_{1,a}' + \frac{L_0}{2}T_{3,a} + T_{4,a}\right)\eta_S(D_t)f_{\lambda_0,\delta}(K)\\
&\geq \frac{\widehat{d}_{1,a}(\lambda_0)-\delta-\epsilon/2}{\omega} f_{\lambda_0,\delta}(K)\tilde{J}_a(x)\eta_S(D_t)^2\tilde{J}_a(x)f_{\lambda_0,\delta}(K) + O(S^{-1}) + C_{4,a}
\end{aligned} \tag{5.12}$$

holds with some compact operator $C_{4,a}$ on $\mathscr{K}$, where

$$\widehat{d}_{1,a}(\lambda) = \mathrm{dist}(\lambda, \widehat{\Theta}_a \cap (-\infty, \lambda]); \quad \widehat{\Theta}_a = \omega \boldsymbol{Z} \cup \sigma_{\mathrm{pp}}(K^a),$$

for $\lambda \in \boldsymbol{R}$. Here we used that $a \in \mathscr{A}_0^0$ is a pair, since $N = 3$. The estimates (5.11) and (5.12) will be shown later as Lemmas 5.4 and 5.5. Let us continue the proof of (1.16), on the assumption that (5.11) and (5.12) has been obtained. By using (5.10), (5.11) and (5.12), one can obtain

$$\begin{aligned}
& f_{\lambda_0,\delta}(K)\eta_S(D_t)T_0\eta_S(D_t)f_{\lambda_0,\delta}(K) \\
\geq\ & \frac{\widehat{d}_{1,a_{\min}}(\lambda_0) - \delta}{\omega} f_{\lambda_0,\delta}(K)\tilde{J}_{a_{\min}}(x)\eta_S(D_t)^2\tilde{J}_{a_{\min}}(x)f_{\lambda_0,\delta}(K) \\
& + \sum_{a \in \mathscr{A}_0^0} \frac{\widehat{d}_{1,a}(\lambda_0) - \delta - \epsilon/2}{\omega} f_{\lambda_0,\delta}(K)\tilde{J}_a(x)\eta_S(D_t)^2\tilde{J}_a(x)f_{\lambda_0,\delta}(K) + O(S^{-1}) + C_5 \\
\geq\ & \sum_{a \in \mathscr{A}_0} \frac{d_1(\lambda_0) - \delta - \epsilon/2}{\omega} f_{\lambda_0,\delta}(K)\tilde{J}_a(x)\eta_S(D_t)^2\tilde{J}_a(x)f_{\lambda_0,\delta}(K) + O(S^{-1}) + C_5
\end{aligned}$$

with some compact operator C_5 on $\mathscr{A}$. Here we used $d_1(\lambda_0) = \min\{\widehat{d}_{1,a}(\lambda_0) \mid a \in \mathscr{A}_0\}$. Moreover, by using (2.10), the compactness of $\mathrm{supp}\, \tilde{J}_{a_{\max}}$, and Corollary 3.7, we have

$$\begin{aligned}
& f_{\lambda_0,\delta}(K)\eta_S(D_t)T_0\eta_S(D_t)f_{\lambda_0,\delta}(K) \\
\geq\ & \sum_{a \in \mathscr{A}} \frac{d_1(\lambda_0) - \delta - \epsilon/2}{\omega} f_{\lambda_0,\delta}(K)\tilde{J}_a(x)\eta_S(D_t)^2\tilde{J}_a(x)f_{\lambda_0,\delta}(K) + O(S^{-1}) + C_5' \\
=\ & \frac{d_1(\lambda_0) - \delta - \epsilon/2}{\omega} f_{\lambda_0,\delta}(K)\eta_S(D_t)^2 f_{\lambda_0,\delta}(K) + O(S^{-1}) + C_5'
\end{aligned} \tag{5.13}$$

with some compact operator C_5' on $\mathscr{A}$.

By virtue of (5.8), (5.9) and (5.13), we obtain the estimate

$$\begin{aligned}
& f_{\lambda_0,\delta}(K)i[K, A]f_{\lambda_0,\delta}(K) \\
\geq\ & (2 + O(S^{-1}))f_{\lambda_0,\delta}(K)\bar{\eta}_S(D_t)^2 f_{\lambda_0,\delta}(K) \\
& + \frac{d_1(\lambda_0) - \delta - \epsilon/2}{\omega} f_{\lambda_0,\delta}(K)\eta_S(D_t)^2 f_{\lambda_0,\delta}(K) \\
& + f_{\lambda_0,\delta}(K)\{O(L_0^{-1}) + L_0 O(L_1^{\max\{-1,-\rho\}}) + O(S^{-1})\}f_{\lambda_0,\delta}(K) + C_6
\end{aligned}$$

with some compact operator C_6 on $\mathscr{A}$, by sandwiching this estimate by two $f_{\lambda_0,\delta}(K)$'s with smaller $\delta > 0$ if necessary. This estimate yields

$$\begin{aligned}
& f_{\lambda_0,\delta}(K)i[K, A]f_{\lambda_0,\delta}(K) \\
\geq\ & \frac{d_1(\lambda_0) - \delta - \epsilon/2}{\omega} f_{\lambda_0,\delta}(K)f_{\lambda_0,\delta}(K) \\
& + f_{\lambda_0,\delta}(K)\{O(L_0^{-1}) + L_0 O(L_1^{\max\{-1,-\rho\}}) + O(S^{-1})\}f_{\lambda_0,\delta}(K) + C_6.
\end{aligned} \tag{5.14}$$

Here we used that $(d_1(\lambda_0) - \delta - \epsilon/2)/\omega < 1 < 2 + O(S^{-1})$ holds for relatively large $S > 0$. After taking $L_0 > 0$ and $S > 0$ as sufficiently large, we have only to take $L_1 > 0$ as sufficiently large for the sake of obtaining the estimate

$$f_{\lambda_0,\delta}(K)\{O(L_0^{-1})+L_0O(L_1^{\max\{-1,-\rho\}})+O(S^{-1})\}f_{\lambda_0,\delta}(K)\geq\frac{-\epsilon/2}{\omega}f_{\lambda_0,\delta}(K)^2.$$

This and (5.14) yield (1.16). □

Now we will show the estimates (5.11) and (5.12).

LEMMA 5.4. *The estimate* (5.11) *holds with some compact operator* $C_{4,a_{\min}}$ *on* $\mathscr{K}$.

PROOF. The term $f_{\lambda_0,\delta}(K)\eta_S(D_t)T_{1,a_{\min}}\eta_S(D_t)f_{\lambda_0,\delta}(K)$ can be written as

$$\begin{aligned}&f_{\lambda_0,\delta}(K)\eta_S(D_t)T_{1,a_{\min}}\eta_S(D_t)f_{\lambda_0,\delta}(K)\\&=\tilde{J}_{a_{\min}}(x)f_{\lambda_0,\delta}(K_0)\eta_S(D_t)(\langle D_t\rangle_{2\omega}^{-1}p^2)\eta_S(D_t)f_{\lambda_0,\delta}(K_0)\tilde{J}_{a_{\min}}(x)+C'_{4,a_{\min}}\end{aligned}$$

with some compact operator $C'_{4,a_{\min}}$ on $\mathscr{K}$. Here we used Proposition 3.8 and $K_{a_{\min}}=K_0$. As for the factor

$$\begin{aligned}&f_{\lambda_0,\delta}(K_0)\eta_S(D_t)(\langle D_t\rangle_{2\omega}^{-1}p^2)\eta_S(D_t)f_{\lambda_0,\delta}(K_0)\\&=\eta_S(D_t)f_{\lambda_0,\delta}(K_0)(\langle D_t\rangle_{2\omega}^{-1}p^2)f_{\lambda_0,\delta}(K_0)\eta_S(D_t),\end{aligned}$$

it follows from the estimate

$$f_{\lambda_0,\delta}(K_0)(\langle D_t\rangle_{2\omega}^{-1}p^2)f_{\lambda_0,\delta}(K_0)\geq\frac{\lambda_0-\delta}{\omega}f_{\lambda_0,\delta}(K_0)^2\tag{5.15}$$

which can be obtained in the same way as in §4 that

$$\begin{aligned}&f_{\lambda_0,\delta}(K_0)\eta_S(D_t)(\langle D_t\rangle_{2\omega}^{-1}p^2)\eta_S(D_t)f_{\lambda_0,\delta}(K_0)\\&\geq\frac{\widehat{d}_{1,a_{\min}}(\lambda_0)-\delta}{\omega}f_{\lambda_0,\delta}(K_0)\eta_S(D_t)^2f_{\lambda_0,\delta}(K_0)\end{aligned}$$

holds, since $\widehat{d}_{1,a_{\min}}(\lambda_0)=\lambda_0$. By sandwiching this estimate by two $\tilde{J}_{a_{\min}}(x)$'s, and using Proposition 3.8, we obtain (5.11). □

LEMMA 5.5. *Let* $a\in\mathscr{A}_0^0$. *The estimate* (5.12) *holds with some compact operator* $C_{4,a}$ *on* $\mathscr{K}$.

PROOF. The term $f_{\lambda_0,\delta}(K)\eta_S(D_t)(T'_{1,a}+(L_0/2)T_{3,a}+T_{4,a})\eta_S(D_t)f_{\lambda_0,\delta}(K)$ can be estimated as

$$\begin{aligned}&f_{\lambda_0,\delta}(K)\eta_S(D_t)\left(T'_{1,a}+\frac{L_0}{2}T_{3,a}+T_{4,a}\right)\eta_S(D_t)f_{\lambda_0,\delta}(K)\\&\geq f_{\lambda_0,\delta}(K)\eta_S(D_t)\left(T'_{1,a}+\frac{L_0}{2}T'_{3,a}+T_{4,a}\right)\eta_S(D_t)f_{\lambda_0,\delta}(K),\end{aligned}\tag{5.16}$$

where

$$T'_{3,a}=\tilde{J}_a(x)(\langle p_a\rangle^{-2}(p_a)^2)\tilde{J}_a(x).$$

Here we used $\tilde{J}_a(x_a) \geq \tilde{J}_a(x) \geq 0$ by Lemma 2.9.

As for the term $f_{\lambda_0,\delta}(K)\eta_S(D_t)(T'_{1,a} + (L_0/2)T'_{3,a} + T_{4,a})\eta_S(D_t)f_{\lambda_0,\delta}(K)$, we have

$$\begin{aligned} & f_{\lambda_0,\delta}(K)\eta_S(D_t)\left(T'_{1,a} + \frac{L_0}{2}T'_{3,a} + T_{4,a}\right)\eta_S(D_t)f_{\lambda_0,\delta}(K) \\ &= \tilde{J}_a(x)f_{\lambda_0,\delta}(K_a)\eta_S(D_t)i[K^a, (\tilde{A}_0)^a]\eta_S(D_t)f_{\lambda_0,\delta}(K_a)\tilde{J}_a(x) \\ &\quad + \frac{L_0}{2}\tilde{J}_a(x)f_{\lambda_0,\delta}(K_a)\eta_S(D_t)(\langle p_a\rangle^{-2}(p_a)^2)\eta_S(D_t)f_{\lambda_0,\delta}(K_a)\tilde{J}_a(x) + C'_{4,a} \\ &= \tilde{J}_a(x)\eta_S(D_t)f_{\lambda_0,\delta}(K_a)i[K^a, (\tilde{A}_0)^a]f_{\lambda_0,\delta}(K_a)\eta_S(D_t)\tilde{J}_a(x) \\ &\quad + \frac{L_0}{2}\tilde{J}_a(x)\eta_S(D_t)f_{\lambda_0,\delta}(K_a)(\langle p_a\rangle^{-2}(p_a)^2)f_{\lambda_0,\delta}(K_a)\eta_S(D_t)\tilde{J}_a(x) \\ &\quad + O(S^{-1}) + C'_{4,a} \end{aligned} \tag{5.17}$$

with some compact operator $C'_{4,a}$ on $\mathscr{K}$. Here we used

$$T'_{1,a} + T_{4,a} = \tilde{J}_a(x)i[K^a, (\tilde{A}_0)^a]\tilde{J}_a(x),$$

and Proposition 3.8.

As for the factor $(L_0/2)f_{\lambda_0,\delta}(K_a)(\langle p_a\rangle^{-2}(p_a)^2)f_{\lambda_0,\delta}(K_a)$, via the Fourier transform $\mathscr{F}_a : \mathscr{H}_a \to \mathscr{H}_a$, this can be decomposed into the direct integral

$$\int_{[0,\infty)}^{\oplus} F_{a,1}(\lambda_a)\,d\lambda_a;\;\; F_{a,1}(\lambda_a) = \frac{L_0}{2}\tilde{F}_{a,1}(\lambda_a)f_{\lambda_0-\lambda_a,\delta}(K^a)^2,\;\; \tilde{F}_{a,1}(\lambda_a) = \frac{2\lambda_a}{1+2\lambda_a}.$$

On the other hand, as for the factor $f_{\lambda_0,\delta}(K_a)i[K^a, (\tilde{A}_0)^a]f_{\lambda_0,\delta}(K_a)$, via the Fourier transform $\mathscr{F}_a : \mathscr{H}_a \to \mathscr{H}_a$, this can be decomposed into the direct integral

$$\int_{[0,\infty)}^{\oplus} F_{a,0}(\lambda_a)\,d\lambda_a;\quad F_{a,0}(\lambda_a) = f_{\lambda_0-\lambda_a,\delta}(K^a)i[K^a, (\tilde{A}_0)^a]f_{\lambda_0-\lambda_a,\delta}(K^a).$$

Then we study the sum of the fibered operators $F_a(\lambda_a) = F_{a,0}(\lambda_a) + F_{a,1}(\lambda_a)$. Here we note that there exists a unique $\kappa_a \in \widehat{\Theta}_a \cap [0, \lambda_0]$ such that $\widehat{d}_{1,a}(\lambda_0) = \lambda_0 - \kappa_a > 0$, since $\lambda_0 \notin \Theta$. We first consider the case where $\lambda_a > \omega$. Since $i[K^a_0, (\tilde{A}_0)^a] = (\langle D_t\rangle_{2\omega}^{-1/2}p^a)(\langle D_t\rangle_{2\omega}^{-1/2}p^a)^*$ with $K^a_0 = D_t + (p_a)^2/2$, and

$$i[V^a, (\tilde{A}_0)^a] = \mathrm{Re}\,\{(\langle D_t\rangle_{2\omega}^{-1/2}p^a)(B^{a,1})^* - B^{a,2}\}$$

with $B^{a,1} = \langle D_t\rangle_{2\omega}^{-1}i[\langle D_t\rangle_{2\omega}, V^a]\langle D_t\rangle_{2\omega}^{-1/2}x^a$ and $B^{a,2} = (x^a\cdot(\nabla^a V^a))\langle D_t\rangle_{2\omega}^{-1}$ by (4.1) and (4.2), we have

$$i[K^a, (\tilde{A}_0)^a] \geq -\frac{1}{4}(B^{a,1})(B^{a,1})^* - \mathrm{Re}\,B^{a,2}.$$

Here we used $(\langle D_t\rangle_{2\omega}^{-1/2}p^a + B^{a,1}/2)(\langle D_t\rangle_{2\omega}^{-1/2}p^a + B^{a,1}/2)^* \geq 0$. Since both $B^{a,1}$ and $B^{a,2}$ are bounded by assumption, there exists some $M_a \in \boldsymbol{R}$ such that

$$F_{a,0}(\lambda_a) \geq M_a f_{\lambda_0-\lambda_a,\delta}(K^a)^2$$

holds. On the other hand, by the monotone increasing of $\tilde{F}_{a,1}(\lambda_a)$ in λ_a, $\tilde{F}_{a,1}(\lambda_a) > \tilde{F}_{a,1}(\omega) = 2\omega/(1+2\omega)$ holds, which implies

$$F_{a,1}(\lambda_a) \geq \frac{L_0}{2}\tilde{F}_{a,1}(\omega) f_{\lambda_0-\lambda_a,\delta}(K^a)^2$$

holds. By taking $L_0 > 0$ so large that $M_a + (L_0/2)\{2\omega/(1+2\omega)\} \geq 1$, that is, $L_0 \geq (1 - M_a)(1+2\omega)/\omega$,

$$F_a(\lambda_a) = F_{a,0}(\lambda_a) + F_{a,1}(\lambda_a) \geq f_{\lambda_0-\lambda_a,\delta}(K^a)^2$$

holds. We next consider the case where $0 \leq \lambda_a \leq \omega$. By virtue of the arguments of [**13**] and the results of §4, we see that there exists a small $\delta_{\epsilon,a} > 0$ such that $\delta_{\epsilon,a} \leq d_0(\lambda_0)/2$, and if $0 < \delta \leq \delta_{\epsilon,a}$, then

$$F_{a,0}(\lambda_a) \geq \frac{\widehat{d}_{1,a}(\lambda_0-\lambda_a) - \delta - \epsilon/2}{\omega} f_{\lambda_0-\lambda_a,\delta}(K^a)^2$$

holds for $\lambda_0 - \lambda_a \notin \widehat{\Theta}_a$; while,

$$F_{a,0}(\lambda_a) \geq \frac{-\epsilon/2}{\omega} f_{\lambda_0-\lambda_a,\delta}(K^a)^2$$

holds for $\lambda_0 - \lambda_a \in \widehat{\Theta}_a$. Here we used that $\lambda_0 - \lambda_a$ belongs to the compact interval $[\lambda_0 - \omega, \lambda_0]$, which makes the key argument of [**13**] available. If $\lambda_0 - \lambda_a > \kappa_a$, that is, $0 \leq \lambda_a < \lambda_0 - \kappa_a \leq \lambda_0 < \omega$, then

$$F_{a,0}(\lambda_a) \geq \frac{(\lambda_0-\lambda_a) - \kappa_a - \delta - \epsilon/2}{\omega} f_{\lambda_0-\lambda_a,\delta}(K^a)^2$$

holds because $\lambda_0 - \lambda_a \notin \widehat{\Theta}_a$. Since $\tilde{F}_{a,0}(\lambda_a) = 2\lambda_a/(1+2\lambda_a) \geq 2\lambda_a/(1+2\omega)$, by taking L_0 so large that $(L_0/2)\{2\lambda_a/(1+2\omega)\} - \lambda_a/\omega \geq 0$, that is, $L_0 \geq (1+2\omega)/\omega$,

$$F_a(\lambda_a) \geq \frac{\lambda_0 - \kappa_a - \delta - \epsilon/2}{\omega} f_{\lambda_0-\lambda_a,\delta}(K^a)^2$$

holds. Now we focus on the case where $\lambda_0 - \lambda_a \leq \kappa_a$, that is, $\omega \geq \lambda_a \geq \lambda_0 - \kappa_a$. As for $F_{a,0}(\lambda_a)$, even if $\lambda_0 - \lambda_a \notin \widehat{\Theta}_a$, we will utilize the estimate

$$F_{a,0}(\lambda_a) \geq \frac{-\delta - \epsilon/2}{\omega} f_{\lambda_0-\lambda_a,\delta}(K^a)^2$$

consistently. By the monotone increasing of $\tilde{F}_{a,1}(\lambda_a)$ in λ_a, $\tilde{F}_{a,1}(\lambda_a) \geq \tilde{F}_{a,1}(\lambda_0 - \kappa_a) = 2(\lambda_0-\kappa_a)/\{1+2(\lambda_0-\kappa_a)\} \geq 2(\lambda_0-\kappa_a)/(1+2\omega)$ holds, which implies

$$F_{a,1}(\lambda_a) \geq \frac{L_0}{2}\frac{2(\lambda_0-\kappa_a)}{1+2\omega} f_{\lambda_0-\lambda_a,\delta}(K^a)^2$$

holds. By taking L_0 so large that $L_0/(1+2\omega) \geq 1/\omega$, that is, $L_0 \geq (1+2\omega)/\omega$,

$$F_a(\lambda_a) \geq \frac{\lambda_0 - \kappa_a - \delta - \epsilon/2}{\omega} f_{\lambda_0-\lambda_a,\delta}(K^a)^2$$

holds. By combining these estimates, we obtain

$$\begin{aligned} &f_{\lambda_0,\delta}(K_a)\left(i[K^a, (\tilde{A}_0)^a] + \frac{L_0}{2}(\langle p_a\rangle^{-2}(p_a)^2)\right) f_{\lambda_0,\delta}(K_a) \\ &\geq \frac{\widehat{d}_{1,a}(\lambda_0) - \delta - \epsilon/2}{\omega} f_{\lambda_0,\delta}(K_a)^2 \end{aligned} \tag{5.18}$$

since $\widehat{d}_{1,a}(\lambda_0) = \lambda_0 - \kappa_a < \omega$. By sandwiching (5.18) by two $\eta_S(D_t)$'s and $\tilde{J}_a(x)$'s, and using $[\eta_S(D_t), f_{\lambda_0,\delta}(K_a)] = O(S^{-1})$ and Proposition 3.8, it follows from (5.17) that

$$\begin{aligned} &f_{\lambda_0,\delta}(K)\eta_S(D_t)\left(T'_{1,a} + \frac{L_0}{2}T'_{3,a} + T_{4,a}\right)\eta_S(D_t)f_{\lambda_0,\delta}(K) \\ &\geq \frac{\widehat{d}_{1,a}(\lambda_0) - \delta - \epsilon/2}{\omega}\tilde{J}_a(x)\eta_S(D_t)f_{\lambda_0,\delta}(K_a)^2\eta_S(D_t)\tilde{J}_a(x) + O(S^{-1}) + C'_{4,a} \\ &= \frac{\widehat{d}_{1,a}(\lambda_0) - \delta - \epsilon/2}{\omega}\tilde{J}_a(x)f_{\lambda_0,\delta}(K_a)\eta_S(D_t)^2 f_{\lambda_0,\delta}(K_a)\tilde{J}_a(x) + O(S^{-1}) + C'_{4,a} \\ &= \frac{\widehat{d}_{1,a}(\lambda_0) - \delta - \epsilon/2}{\omega} f_{\lambda_0,\delta}(K)\tilde{J}_a(x)\eta_S(D_t)^2\tilde{J}_a(x)f_{\lambda_0,\delta}(K) + O(S^{-1}) + C''_{4,a} \end{aligned}$$

holds with some compact operators $C'_{4,a}$ and $C''_{4,a}$ on $\mathscr{K}$. This estimate and (5.16) yield (5.12). □

6. Minimal velocity estimates

As was shown by some works of Sigal-Soffer (see e.g. [**31, 32**]), by virtue of the Mourre estimate, one can obtain the so-called minimal velocity estimate, which is one of the useful propagation estimates for the time evolution of scattering states. As for the Floquet Hamiltonian K under consideration, we have obtained the Mourre estimate (1.17). The aim of this section is to show Theorem 1.3, by utilization of (1.17). As mentioned in §1, Corollary 1.4 is a direct consequence of Theorem 1.3, by virtue of the arguments of Yajima-Kitada [**40**] and Møller-Skibsted [**27**].

We first state the abstract theory for getting the minimal velocity estimate of the integral type, by following Gérard-Łaba [**14**]. The theory was initiated by Sigal-Soffer. The following proposition is Proposition 4.4.1 of [**14**] with $P = 1$ and $I(\sigma) \equiv 0$:

PROPOSITION 6.1. *Let $\mathcal{K}$ be a Hilbert space. Let K and A be self-adjoint operators on $\mathcal{K}$. Assume that $K \in C^\mu(A)$ for some $\mu > 1$. Let $\Delta \subset \boldsymbol{R}$ be an open interval such that*

$$E_K(\Delta)i[K, A]E_K(\Delta) \geq c_0 E_K(\Delta)$$

holds for some $c_0 > 0$. Then for any $f \in C_0^\infty(\boldsymbol{R}; \boldsymbol{R})$ supported in Δ

$$\int_1^\infty \left\| F\left(\frac{A}{\sigma} \le c_0 - \varepsilon\right) e^{-i\sigma K} f(K)u \right\|_{\mathcal{K}}^2 \frac{d\sigma}{\sigma} \le C\|u\|_{\mathcal{K}}^2,$$
$$\operatorname*{s-lim}_{\sigma\to\infty} F\left(\frac{A}{\sigma} \le c_0 - \varepsilon\right) e^{-i\sigma K} f(K) = 0$$

hold for any $\varepsilon > 0$*.*

By virtue of Proposition 6.1 and the Mourre estimate (1.17), one can obtain the following:

PROPOSITION 6.2. *Suppose that the hypotheses of Theorem 1.3 are satisfied. Then*

$$\int_1^\infty \left\| F\left(\frac{A}{\sigma} \le c_0'(\lambda_0)\right) f_{\lambda_0,\delta}(K) e^{-i\sigma K}\Phi \right\|_{\mathscr{K}}^2 \frac{d\sigma}{\sigma} \le C\|\Phi\|_{\mathscr{K}}^2, \tag{6.1}$$
$$\operatorname*{s-lim}_{\sigma\to\infty} F\left(\frac{A}{\sigma} \le c_0'(\lambda_0)\right) f_{\lambda_0,\delta}(K) e^{-i\sigma K} = 0 \tag{6.2}$$

hold, where

$$c_0(\lambda_0) = c_0(\lambda_0; \epsilon, \delta) = \frac{d_1(\lambda_0) - \delta - 2\epsilon}{\omega},$$
$$0 < c_0'(\lambda_0) = c_0'(\lambda_0; \epsilon, \delta) = \frac{d_1(\lambda_0) - \delta - 3\epsilon}{\omega} < c_0(\lambda_0)$$

for sufficiently small $\epsilon > 0$*.*

However, in the study of the problem of the asymptotic completeness, we have to translate (6.1) into a certain minimal velocity estimate with the localization of the propagation with respect to $x \in X$. To this end, we need the following lemma and proposition (see Lemma 4.4.8 and Proposition 4.4.9 of [**14**]). Lemma 6.3 is used for proving Proposition 6.4:

LEMMA 6.3. *Let* $\mathcal{K}$ *be a Hilbert space. Let* K*,* A *and* B *be self-adjoint operators on* $\mathcal{K}$*. Let* $\Delta \subset \boldsymbol{R}$ *be an open interval. Assume that*
(1) $K \in C^1(A) \cap C^1(B)$ *and* $B \in C^1(A)$*;*
(2) $[B, A](B+i)^{-1}$ *and* $(B+i)^{-1}A(K+i)^{-1}$ *are bounded;*
(3) $B \ge 1/2$*;*
(4) $-cB \le f_1(K)Af_1(K) \le cB$ *holds for some* $c > 0$*, and* $f_1 \in C_0^\infty(\boldsymbol{R}; \boldsymbol{R})$ *such that* $f_1 = 1$ *on* Δ*.*

Then, for any $c_0 > 0$*, there is an* $\varepsilon_0 > 0$ *such that for any* $f \in C_0^\infty(\boldsymbol{R}; \boldsymbol{R})$ *supported in* Δ

$$F\left(\frac{B}{\sigma} \le \varepsilon_0\right) f(K) F\left(\frac{|A|}{\sigma} \ge c_0\right) = O(\sigma^{-1})$$

holds.

PROPOSITION 6.4. *Let* $\mathcal{K}$ *be a Hilbert space. Let* K*,* A *and* B *be self-adjoint operators on* $\mathcal{K}$*. Let* $\Delta \subset \boldsymbol{R}$ *be an open interval. Suppose that the hypotheses of Proposition 6.1 and Lemma 6.3 are satisfied. Then, there exists an* $\varepsilon_0 > 0$ *such that for any* $f \in C_0^\infty(\boldsymbol{R}; \boldsymbol{R})$ *supported in* Δ

$$\int_1^\infty \left\| F\left(\frac{B}{\sigma}\leq \varepsilon_0\right) e^{-i\sigma K} f(K)u \right\|_{\mathcal{K}}^2 \frac{d\sigma}{\sigma} \leq C\|u\|_{\mathcal{K}}^2,$$
$$\operatorname*{s-lim}_{\sigma\to\infty} F\left(\frac{B}{\sigma}\leq \varepsilon_0\right) e^{-i\sigma K} f(K) = 0$$

hold.

In the case where K is a time-independent Schrödinger operator $p^2/2+V$, one can take B as $\langle x\rangle$. In fact, K is in $C^1(\langle x\rangle)$, because $i[K,\langle x\rangle] = \mathrm{Re}\,\{(x/\langle x\rangle)\cdot p\}$ is K-bounded. However, in the case where K is a Floquet Hamiltonian $D_t + p^2/2 + V(t)$, K is not in $C^1(\langle x\rangle)$, even if $V(t)$ is independent of t. In fact, $i[K,\langle x\rangle] = \mathrm{Re}\,\{(x/\langle x\rangle)\cdot p\}$ is not K-bounded. In our analysis, by modifying $(1+\langle D_t\rangle^{-1/2}x^2\langle D_t\rangle^{-1/2})^{1/2}$, we will take B as (1.21)

$$B = (1+\tilde{B}_0+\tilde{B}_1)^{1/2}; \quad \tilde{B}_0 = \langle D_t\rangle^{-1/2} Q_0(x) \langle D_t\rangle^{-1/2},$$
$$\tilde{B}_1 = \sum_{a\in\mathscr{A}_0^0} \tilde{B}_{1,a}, \quad \tilde{B}_{1,a} = \langle p_a\rangle^{-1} Q_{1,a}(x_a)\langle p_a\rangle^{-1},$$

where

$$Q_0(x) = \sum_{a\in\mathscr{A}\setminus\mathscr{A}_0^0} x^2 \tilde{J}_a(x)^2 + \sum_{a\in\mathscr{A}_0^0} (x^a)^2\tilde{J}_a(x)^2 = x^2 - \sum_{a\in\mathscr{A}_0^0}(x_a)^2\tilde{J}_a(x)^2,$$
$$Q_{1,a}(x_a) = (x_a)^2\tilde{J}_a(x_a)^2,$$

with $a\in\mathscr{A}_0^0$. Then we will show Theorem 1.3 as follows:

PROOF OF THEOREM 1.3. We first show that K is in $C^1(B)$, that is, $(K-i)^{-1}i[K,B]$ $(K+i)^{-1}$ is bounded, by virtue of Proposition 3.3: Let $g_{1/2}\in C^\infty(\boldsymbol{R};\boldsymbol{R})$ be such that $g_{1/2}\geq 0$, $g_{1/2}(\nu) = 0$ for $\nu\leq 0$, and $g_{1/2}(\nu) = \nu^{1/2}$ for $\nu\geq 1$. Take an almost analytic extension $\tilde{g}_{1/2}\in C^\infty(\boldsymbol{C})$ of $g_{1/2}$, which satisfies $|\overline{\partial}_z \tilde{g}_{1/2}(z)| \leq C_L\langle z\rangle^{1/2-1-L}|\mathrm{Im}\, z|^L$ with $L\in\boldsymbol{N}\cup\{0\}$. Then, by virtue of the commutator expansion formula (see e.g. [**12**]), $[K,B] = [K, g_{1/2}(B^2)]$ is represented as

$$[K,B] = g_{1/2}^{(1)}(B^2)[K,B^2] + R_2' = \frac{1}{2}B^{-1}[K,\tilde{B}_0+\tilde{B}_1] + R_2';$$
$$R_2' = \frac{1}{2\pi i}\int_{\boldsymbol{C}} \overline{\partial}_\zeta \tilde{g}_{1/2}(\zeta)(\zeta-B^2)^{-2}[[K,B^2],B^2](\zeta-B^2)^{-1}\, d\zeta\wedge d\overline{\zeta}.$$

We will show that $B^{-1}i[K,\tilde{B}_0](K+i)^{-1}$, $B^{-1}i[K,\tilde{B}_{1,a}]$ and R_2' are bounded.

We first consider $i[K,\tilde{B}_0] = S_{0,0}+S_{0,1}+S_{0,2}$ with $S_{0,0} = \langle D_t\rangle^{-1}(\nabla Q_0)(x)\cdot p$, $S_{0,1} = -i\langle D_t\rangle^{-1}(\Delta Q_0)(x)/2$ and $S_{0,2} = i[V,\langle D_t\rangle^{-1}]Q_0(x)$. Trivially, $(1+\tilde{B}_0)^{-1/2}\langle D_t\rangle^{-1/2}O(\langle x\rangle)$ is bounded. By Proposition 3.5 (1), $\langle D_t\rangle^{-1/2}p(K+i)^{-1}$ is bounded. Thus $(1+\tilde{B}_0)^{-1/2}S_{0,0}$ $(K+i)^{-1}$ is bounded. Since $(\Delta Q_0)(x)$ is bounded, $S_{0,1}$ is bounded trivially. By $(V_{\mathrm{WR}})_\rho$ with $\rho>0$, for any $b\in\mathscr{A}_0^0$,

$$(\partial_t V^b)(t,x^b)Q_0(x) = \sum_{a\in\mathscr{A}\setminus\mathscr{A}_0^0}(\partial_t V^b)(t,x^b)x^2\tilde{J}_a(x)^2 + \sum_{\substack{a\in\mathscr{A}_0^0\\ a\neq b}}(\partial_t V^b)(t,x^b)(x^a)^2\tilde{J}_a(x)^2$$

$$
\begin{aligned}
&+ (\partial_t V^b)(t, x^b)(x^b)^2 \tilde{J}_b(x)^2 \\
&= O(1)\tilde{J}_{a_{\max}}(x)^2 + \sum_{\substack{a \in \mathscr{A}_0 \\ a \neq b}} O(\langle x \rangle^{1-\rho})\tilde{J}_a(x)^2 + O(\langle x^b \rangle^{1-\rho})\tilde{J}_b(x)^2
\end{aligned}
$$

holds, which implies $(\partial_t V^b)(t, x^b)Q_0(x) = O(\langle x \rangle^{\max\{1-\rho,0\}})$. Here we used the compactness of $\operatorname{supp} \tilde{J}_{a_{\max}}$, and Corollary 2.4. We note that $(1 + \tilde{B}_0)^{-1/2}\langle D_t \rangle^{-1/2} O(\langle x \rangle^{\max\{1-\rho,0\}})$ is bounded; more precisely, by complex interpolation, $(1+\tilde{B}_0)^{-\max\{1-\rho,0\}/2} \langle D_t \rangle^{-\max\{1-\rho,0\}/2} O(\langle x \rangle^{\max\{1-\rho,0\}})$ is bounded. In particular, $B^{-\max\{1-\rho,0\}} S_{0,2}$ is bounded. These can show easily that $B^{-1} i[K, \tilde{B}_0](K + i)^{-1}$ is bounded.

In the same way as above, we consider $i[K, \tilde{B}_{1,a}] = S_{1,a,0} + S_{1,a,1} + S_{1,a,2}$ with $S_{1,a,0} = \langle p_a \rangle^{-1}(\nabla_a Q_{1,a})(x_a) \cdot p_a \langle p_a \rangle^{-1}$, $S_{1,a,1} = -i\langle p_a \rangle^{-1}(\Delta_a Q_{1,a})(x_a)\langle p_a \rangle^{-1}/2$ and $S_{1,a,2} = i[I_a, \tilde{B}_{1,a}]$. Here we used $i[V^a, \tilde{B}_{1,a}] = 0$. Since $(1 + \tilde{B}_{1,a})^{-1/2}\langle p_a \rangle^{-1} O(\langle x_a \rangle)$ is bounded, $(1+\tilde{B}_{1,a})^{-1/2} S_{1,a,0}$ is bounded. Since $(\Delta_a Q_{1,a})(x_a) = O(1)$, $S_{1,a,1}$ is also bounded. Now we watch $S_{1,a,2}$. As for the commutator $[h(x), \langle p_a \rangle^{-1}]$, where $h(x)$ is the multiplication by $h(x)$, we use the following commutator expansion formula: Let $g_{-1/2} \in C^\infty(\boldsymbol{R}; \boldsymbol{R})$ be such that $g_{-1/2} \geq 0$, $g_{-1/2}(\nu) = 0$ for $\nu \leq 0$, and $g_{-1/2}(\nu) = \nu^{-1/2}$ for $\nu \geq 1$. Take an almost analytic extension $\tilde{g}_{-1/2} \in C^\infty(\boldsymbol{C})$ of $g_{-1/2}$, which satisfies $|\overline{\partial}_z \tilde{g}_{-1/2}(z)| \leq C_L \langle z \rangle^{-1/2-1-L} |\operatorname{Im} z|^L$ with $L \in \boldsymbol{N} \cup \{0\}$. Then $[h(x), \langle p_a \rangle^{-1}] = [h(x), g_{-1/2}(\langle p_a \rangle^2)]$ is represented as

$$
\begin{aligned}
&[h(x), \langle p_a \rangle^{-1}] = -\frac{1}{2}\langle p_a \rangle^{-3}[h(x), \langle p_a \rangle^2] + R_2''; \\
&R_2'' = \frac{1}{2\pi i}\int_{\boldsymbol{C}} \overline{\partial}_\zeta \tilde{g}_{-1/2}(\zeta)(\zeta - \langle p_a \rangle^2)^{-2} \\
&\qquad\qquad \times [[h(x), \langle p_a \rangle^2], \langle p_a \rangle^2](\zeta - \langle p_a \rangle^2)^{-1}\, d\zeta \wedge d\overline{\zeta}.
\end{aligned}
$$

By virtue of this formula, we have $\tilde{B}_{1,a} = \tilde{B}_{1,a,0} + \tilde{B}_{1,a,1} + \tilde{B}_{1,a,2}$ with $\tilde{B}_{1,a,0} = \langle p_a \rangle^{-2} Q_{1,a}(x_a)$, $\tilde{B}_{1,a,1} = i\langle p_a \rangle^{-4} p_a \cdot (\nabla_a Q_{1,a})(x_a)$ and $\tilde{B}_{1,a,2} = \langle p_a \rangle^{-4} R''$, where R'' is bounded. Trivially, $i[I_a, \tilde{B}_{1,a,2}] = i(I_a \tilde{B}_{1,a,2} - \tilde{B}_{1,a,2} I_a)$ is bounded. By using $\langle p_a \rangle^{-2} = (1 + (p_a)^2)^{-1}$, we consider

$$
i[I_a, \tilde{B}_{1,a,0}] = \langle p_a \rangle^{-2}\{2p_a \cdot (\nabla_a I_a) + i(\Delta_a I_a)\}\langle p_a \rangle^{-2} Q_{1,a}(x_a).
$$

Since $[\langle p_a \rangle^{-2}, Q_{1,a}(x_a)]\langle p_a \rangle \langle x_a \rangle^{-1} = \langle p_a \rangle^{-2}\{p_a \cdot O(\langle x_a \rangle) + O(1)\}\langle p_a \rangle^{-1}\langle x_a \rangle^{-1}$ is bounded, $(\nabla_a I_a) Q_{1,a}(x_a) = O(\langle x_a \rangle^{1-\rho})$ and $(\Delta_a I_a) Q_{1,a}(x_a) = O(\langle x_a \rangle^{-\rho})$ by Lemma 2.3, it follows from the boundedness of $\langle x_a \rangle \langle p_a \rangle^{-1}(1+\tilde{B}_{1,a})^{-1/2}$ that $i[I_a, \tilde{B}_{1,a,0}](1+\tilde{B}_{1,a})^{-1/2}$ is bounded. One can show similarly that $i[I_a, \tilde{B}_{1,a,1}]$ is bounded, because $(\nabla_a Q_{1,a})(x_a) = O(\langle x_a \rangle)$. These imply that $B^{-1} i[K, \tilde{B}_{1,a}]$ is bounded.

Now we show that $B^{-2} i[i[K, B^2], B^2]$ is bounded, which yields the boundedness of R_2', by $1/2 - 1 - 2 + 1 = -3/2 < -1$. We first consider $i[i[K, \tilde{B}_0], \tilde{B}_0] = i[S_{0,0} + S_{0,1} + S_{0,2}, \tilde{B}_0]$. Since $i[S_{0,0}, \tilde{B}_0] = \langle D_t \rangle^{-2} |(\nabla Q_0)(x)|^2$, we see that $(1 + \tilde{B}_0)^{-1} i[S_{0,0}, \tilde{B}_0]$ is bounded, by $|(\nabla Q_0)(x)|^2 = O(\langle x \rangle^2)$. We note $i[S_{0,1}, \tilde{B}_0] = 0$. Since $i[S_{0,2}, \tilde{B}_0] = i[i[V, \langle D_t \rangle^{-1}], \langle D_t \rangle^{-1}](Q_0(x))^2$, one can show that $(1 + \tilde{B}_0)^{-\max\{2-\rho,0\}/2} i[S_{0,2}, \tilde{B}_0]$ is bounded, by $(\partial_t^2 V)(Q_0(x))^2 = O(\langle x \rangle^{\max\{2-\rho,0\}})$. Hence, $B^{-2} i[i[K, \tilde{B}_0], \tilde{B}_0]$ is bounded. One can show similarly the boundedness of $B^{-2} i[i[K, \tilde{B}_{1,a}], \tilde{B}_0]$, $B^{-2} i[i[K, \tilde{B}_0], \tilde{B}_{1,a}]$ and

$B^{-2}i[i[K,\tilde{B}_{1,a}],\tilde{B}_{1,a}]$.

Summing up these, we see that $i[K,B](K+i)^{-1}$ is bounded. In particular, K is in $C^1(B)$. On the other hand, one can prove easily that B is in $C^1(A)$, and that $[B,A](B+i)^{-1}$ and $(B+i)^{-1}A(K+i)^{-1}$ are bounded. Now we will take $f_{\lambda_0,\delta_{\epsilon,1},1} \in C_0^\infty(\boldsymbol{R};\boldsymbol{R})$ supported in $[\lambda_0-2\delta_{\epsilon,1},\lambda_0+2\delta_{\epsilon,1}]$ such that $f_{\lambda_0,\delta_{\epsilon,1},1}=1$ on $[\lambda_0-\delta_{\epsilon,1},\lambda_0+\delta_{\epsilon,1}]$, and put

$$c=\|B^{-1/2}f_{\lambda_0,\delta_{\epsilon,1},1}(K)Af_{\lambda_0,\delta_{\epsilon,1},1}(K)B^{-1/2}\|_{\mathscr{B}(\mathscr{K})}$$

Then

$$-cB\le f_{\lambda_0,\delta_{\epsilon,1},1}(K)Af_{\lambda_0,\delta_{\epsilon,1},1}(K)\le cB$$

holds. Then, Theorem 1.3 follows from Proposition 6.2 immediately, by virtue of Lemma 6.3 and Proposition 6.4. □

For the sake of comparison, also in the case where $N=2$, we will give the results corresponding to Theorem 1.3 and Corollary 1.4, without proof.

THEOREM 6.5. *Suppose $N=2$. Assume V satisfies $(V_{\mathrm{WR}})_\rho$ with some $\rho>0$. Let $\lambda_0\in[0,\omega)\setminus\widehat{\Theta}$. Put*

$$B=\left(1+\langle D_t\rangle^{-1/2}x^2\langle D_t\rangle^{-1/2}\right)^{1/2}. \tag{6.3}$$

Then there exists an $\varepsilon_0(\lambda_0)=\varepsilon_0(\lambda_0;\epsilon,\delta)>0$, which is determined by the positive constant $(d_1(\lambda_0)-\delta-\epsilon)/\omega$ in (4.5)*,*

$$\int_1^\infty\left\|F\left(\frac{B}{\sigma}\le\varepsilon_0(\lambda_0)\right)e^{-i\sigma K}f_{\lambda_0,\delta}(K)\Phi\right\|_{\mathscr{K}}^2\frac{d\sigma}{\sigma}\le C\|\Phi\|_{\mathscr{K}}^2, \tag{6.4}$$

$$\operatorname*{s-lim}_{\sigma\to\infty}F\left(\frac{B}{\sigma}\le\varepsilon_0(\lambda_0)\right)e^{-i\sigma K}f_{\lambda_0,\delta}(K)=0 \tag{6.5}$$

hold. In particular, these yield

$$\int_1^\infty\left\|F\left(\frac{|x|}{\sigma}\le\frac{\varepsilon_0(\lambda_0)}{2}\right)e^{-i\sigma K}f_{\lambda_0,\delta}(K)\Phi\right\|_{\mathscr{K}}^2\frac{d\sigma}{\sigma}\le C\|\Phi\|_{\mathscr{K}}^2, \tag{6.6}$$

$$\operatorname*{s-lim}_{\sigma\to\infty}F\left(\frac{|x|}{\sigma}\le\frac{\varepsilon_0(\lambda_0)}{2}\right)e^{-i\sigma K}f_{\lambda_0,\delta}(K)=0. \tag{6.7}$$

COROLLARY 6.6. *Suppose that the hypotheses of Theorem 6.5 are satisfied. Let $g_{\lambda_0,\delta}$ be the function on $\{z\in\boldsymbol{C}\mid |z|=1\}$ such that $g_{\lambda_0,\delta}(e^{-i2\pi\lambda/\omega})=f_{\lambda_0,\delta}(\lambda)$ for $\lambda\in[0,\omega)$. Then*

$$\int_1^\infty\left\|F\left(\frac{|x|}{t}\le\frac{\varepsilon_0(\lambda_0)}{2}\right)U(t,0)g_{\lambda_0,\delta}(U(T,0))\phi\right\|_{\mathscr{H}}^2\frac{dt}{t}\le C\|\phi\|_{\mathscr{H}}^2, \tag{6.8}$$

$$\operatorname*{s-lim}_{t\to\infty}F\left(\frac{|x|}{t}\le\frac{\varepsilon_0(\lambda_0)}{2}\right)U(t,0)g_{\lambda_0,\delta}(U(T,0))=0 \tag{6.9}$$

hold.

REMARK 6.1. In the case where $N = 2$, as for the minimal velocity estimate of the time-wise type, by virtue of the abstract theory of Skibsted [**34**], which was also initiated by Sigal-Soffer, Yokoyama [**41**] gave the result that for $0 < s' < s$,

$$\left\| F\left(\frac{|x|}{t} \le c_1(\lambda_0;\epsilon,\delta)\right) e^{-i\sigma K} f_{\lambda_0,\delta}(K)\langle x\rangle^{-s} \right\|_{\mathscr{B}(\mathscr{K})} = O(\sigma^{-s'}) \tag{6.10}$$

holds with

$$c_1(\lambda_0;\epsilon,\delta) = \left(\frac{2(d_1(\lambda_0)-\delta-\epsilon)}{1+2(d_1(\lambda_0)-\delta)}\right)^{1/2},$$

under the assumption that $V_{12}(t,y) \in C^0(\boldsymbol{R}; C^\infty(\boldsymbol{R}^d;\boldsymbol{R}))$ is T-periodic in t, and satisfies the decaying conditions

$$\sup_{t\in\boldsymbol{R}} |(\partial_y^\alpha V_{12})(t,y)| \le C_\alpha \langle y\rangle^{-\rho-|\alpha|}, \tag{6.11}$$

by using the minimal velocity estimate like

$$\left\| F\left(\frac{\bar{A}_{0,1}}{t} \le c_1(\lambda_0;\epsilon,\delta)\right) e^{-i\sigma K} f_{\lambda_0,\delta}(K)\langle \bar{A}_{0,1}\rangle^{-s} \right\|_{\mathscr{B}(\mathscr{K})} = O(\sigma^{-s}) \tag{6.12}$$

in terms of the conjugate operator $\bar{A}_{0,1}$ in (1.7). The advantage of (6.10) is that $c_1(\lambda_0;\epsilon,\delta)$ can be taken as the one nearly equal to the square root of the positive constant of the Mourre estimate (1.9).

Getting the minimal velocity estimate like (6.10) also in the case where $N = 3$ is one of the future tasks. We think that to this end, we need a rather strong assumption on V_{jk}'s like that each $V_{jk} \in C^\infty(\boldsymbol{R}; C^\infty(\boldsymbol{R}^d;\boldsymbol{R}))$ is T-periodic in t, and satisfies the decaying conditions

$$\sup_{t\in\boldsymbol{R}} |(\partial_t^m \partial_y^\alpha V_{jk})(t,y)| \le C_{m,\alpha} \langle y\rangle^{-\rho-(m+|\alpha|)}. \tag{6.13}$$

7. AC Stark effect case

In this section, we consider a quantum system of N particles moving in a given time-periodic electric field $\mathscr{E}(t) \in \boldsymbol{R}^d$. We suppose that $\mathscr{E}(t) \in C^0(\boldsymbol{R};\boldsymbol{R}^d)$, and that $\mathscr{E}(t)$ has a period $T > 0$, that is, $\mathscr{E}(t+T) = \mathscr{E}(t)$ for any $t \in \boldsymbol{R}$.

Let $m_j > 0$, $q_j \in \boldsymbol{R}$ and $r_j \in \boldsymbol{R}^d$, $1 \le j \le N$, denote the mass, charge and position vector of the j-th particle, respectively. We suppose that the particles under consideration interact with one another through the time-independent pair potentials $\bar{V}_{jk}(r_j - r_k)$, $1 \le j < k \le N$. The system under consideration is governed by the total Hamiltonian in the laboratory frame

$$\begin{aligned} \bar{H}_{\mathrm{LF}}(t) &= \sum_{j=1}^N \left(-\frac{1}{2m_j}\Delta_j - q_j\langle \mathscr{E}(t), r_j\rangle\right) + \bar{V}(r); \\ \bar{V}(r) &= \sum_{1\le j<k\le N} \bar{V}_{jk}(r_j - r_k) \end{aligned} \tag{7.1}$$

acting on $L^2(\boldsymbol{R}^{d\times N})$. $\bar{H}_{\rm LF}(t)$ can be written as

$$\bar{H}_{\rm LF}(t) = \bar{H}_{\rm LF,0}(t) + \bar{V}(r); \quad \bar{H}_{\rm LF,0}(t) = \frac{1}{2}(p_{\bar{X}})^2 - E_{\bar{X}}(t)\cdot r \tag{7.2}$$

acting on $L^2(\bar{X})$, where

$$E_{\bar{X}}(t) = ((q_1/m_1)\mathscr{E}(t), \ldots, (q_N/m_N)\mathscr{E}(t))$$

is a $\bar{X}$-valued T-periodic function. q_j/m_j is called the specific charge of the j-th particle.

Now we would like to state the so-called Avron-Herbst formula for the propagator $\bar{U}_{\rm LF}(t,s)$ generated by $\bar{H}_{\rm LF}(t)$: In the same way as in Møller [**26**] and Adachi [**2, 3**], introduce $\bar{X}$-valued T-periodic functions $E_{\bar{X},\rm os}(t)$, $b_{\bar{X}}(t)$, $b_{\bar{X},\rm os}(t)$ and $c_{\bar{X}}(t)$ as

$$\begin{aligned}
&E_{\bar{X},\rm m} = \frac{1}{T}\int_0^T E_{\bar{X}}(s)\,ds, \quad E_{\bar{X},\rm os}(t) = E_{\bar{X}}(t) - E_{\bar{X},\rm m},\\
&b_{\bar{X}}(t) = \int_0^t E_{\bar{X},\rm os}(s)\,ds, \quad b_{\bar{X},\rm m} = \frac{1}{T}\int_0^T b_{\bar{X}}(s)\,ds,\\
&b_{\bar{X},\rm os}(t) = b_{\bar{X}}(t) - b_{\bar{X},\rm m}, \quad c_{\bar{X}}(t) = \int_0^t b_{\bar{X},\rm os}(s)\,ds.
\end{aligned}$$

By introducing $\boldsymbol{R}^d$-valued T-periodic functions $\mathscr{E}_{\rm os}(t)$, $\bar{\mathscr{E}}(t)$, $\bar{\mathscr{E}}_{\rm os}(t)$ and $\bar{\bar{\mathscr{E}}}(t)$ as

$$\begin{aligned}
&\mathscr{E}_{\rm m} = \frac{1}{T}\int_0^T \mathscr{E}(s)\,ds, \quad \mathscr{E}_{\rm os}(t) = \mathscr{E}(t) - \mathscr{E}_{\rm m},\\
&\bar{\mathscr{E}}(t) = \int_0^t \mathscr{E}_{\rm os}(s)\,ds, \quad \bar{\mathscr{E}}_{\rm m} = \frac{1}{T}\int_0^T \bar{\mathscr{E}}(s)\,ds,\\
&\bar{\mathscr{E}}_{\rm os}(t) = \bar{\mathscr{E}}(t) - \bar{\mathscr{E}}_{\rm m}, \quad \bar{\bar{\mathscr{E}}}(t) = \int_0^t \bar{\mathscr{E}}_{\rm os}(s)\,ds,
\end{aligned}$$

we have

$$\begin{aligned}
&E_{\bar{X},\rm m} = ((q_1/m_1)\mathscr{E}_{\rm m}, \ldots, (q_N/m_N)\mathscr{E}_{\rm m}),\\
&E_{\bar{X},\rm os}(t) = ((q_1/m_1)\mathscr{E}_{\rm os}(t), \ldots, (q_N/m_N)\mathscr{E}_{\rm os}(t)),\\
&b_{\bar{X}}(t) = ((q_1/m_1)\bar{\mathscr{E}}(t), \ldots, (q_N/m_N)\bar{\mathscr{E}}(t)),\\
&b_{\bar{X},\rm os}(t) = ((q_1/m_1)\bar{\mathscr{E}}_{\rm os}(t), \ldots, (q_N/m_N)\bar{\mathscr{E}}_{\rm os}(t)),\\
&c_{\bar{X}}(t) = ((q_1/m_1)\bar{\bar{\mathscr{E}}}(t), \ldots, (q_N/m_N)\bar{\bar{\mathscr{E}}}(t)).
\end{aligned}$$

$\mathscr{E}_{\rm m}$ is the time-mean of $\mathscr{E}(t)$. Also introduce the time-dependent Hamiltonian

$$\begin{aligned}
&\bar{H}_{\rm MF}(t) = \bar{H}_{\rm MF,0} + \bar{V}(r + c_{\bar{X}}(t)); \quad \bar{H}_{\rm MF,0} = \frac{1}{2}(p_{\bar{X}})^2 - E_{\bar{X},\rm m}\cdot r,\\
&\bar{V}(r + c_{\bar{X}}(t)) = \sum_{1\le j<k\le N} \bar{V}_{jk}((r_j + c_{\bar{X},j}(t)) - (r_k + c_{\bar{X},k}(t)))
\end{aligned} \tag{7.3}$$

acting on $L^2(\bar{X})$, which governs the system in the moving frame accelerated by $E_{\bar{X},\rm os}(t) =$

$E_{\bar X}(t) - E_{\bar X,\mathrm{m}}$. $c_{\bar X,j}(t) = (q_j/m_j)\bar{\bar{\mathscr{E}}}(t)$ stands for the j-th component of $c_{\bar X}(t)$. Here we will emphasize that the free Hamiltonian $\bar H_{\mathrm{MF},0}$ is time-independent even if $\bar H_{\mathrm{LF},0}(t)$ is time-dependent. The time-independence of $\bar H_{\mathrm{MF},0}$ makes the dynamics of the system governed by $\bar H_{\mathrm{MF}}(t)$ easy to handle.

If $E_{\bar X,\mathrm{m}} = 0$, then $\bar H_{\mathrm{MF},0}$ is called the free Schrödinger operator; while, if $E_{\bar X,\mathrm{m}} \neq 0$, then $\bar H_{\mathrm{MF},0}$ is called the free DC Stark Hamiltonian. Also, if $q_j/m_j = q_k/m_k$, then

$$\begin{aligned} &\bar V_{jk}((r_j + c_{\bar X,j}(t)) - (r_k + c_{\bar X,k}(t))) \\ &= \bar V_{jk}(r_j - r_k + ((q_j/m_j) - (q_k/m_k))\bar{\bar{\mathscr{E}}}(t)) \equiv \bar V_{jk}(r_j - r_k) \end{aligned}$$

holds, that is, $\bar V_{jk}((r_j+c_{\bar X,j}(t))-(r_k+c_{\bar X,k}(t)))$ is time-independent; while, if $q_j/m_j \neq q_k/m_k$, then the periodicity of $\bar{\bar{\mathscr{E}}}(t)$ in t yields that of $\bar V_{jk}((r_j+c_{\bar X,j}(t))-(r_k+c_{\bar X,k}(t)))$. Hence, if there is no pair (j,k) such that $q_j/m_j \neq q_k/m_k$, then $\bar H_{\mathrm{MF}}(t) \equiv \bar H_{\mathrm{MF},0}+\bar V(r)$ holds, that is, $\bar H_{\mathrm{MF}}(t)$ is also time-independent. So, from now on, we suppose there exists at least one pair (j,k) such that $q_j/m_j \neq q_k/m_k$. But, even under this assumption, $\bar H_{\mathrm{MF}}(t)$ is still time-independent if $\mathscr{E}(t) \equiv \mathscr{E}_\mathrm{m}$, that is, $\mathscr{E}(t)$ is constant in t, because $\bar{\bar{\mathscr{E}}}(t) \equiv 0$.

As for the case where $\bar H_{\mathrm{MF}}(t)$ is time-independent, the problem of the asymptotic completeness of the systems governed by such Hamiltonians was studied intently in the 1980's and 1990's; as for the case where $\mathscr{E}(t) \equiv \mathscr{E}_\mathrm{m} = 0$, see e.g. Sigal-Soffer [**31**], Graf [**15**], Yafaev [**36**], Dereziński [**11**], and so on; while, as for the case where $\mathscr{E}(t) \equiv \mathscr{E}_\mathrm{m} \neq 0$, see e.g. Adachi-Tamura [**6, 7**] and Herbst-Møller-Skibsted [**17, 18**]. So, from now on, we assume that $\mathscr{E}(t)$ is not constant but periodic in t. Let $\bar U_{\mathrm{MF}}(t,s)$ denote the propagator generated by $\bar H_{\mathrm{MF}}(t)$. Then the Avron-Herbst formula for $\bar U_{\mathrm{LF}}(t,s)$

$$\begin{aligned} &\bar U_{\mathrm{LF}}(t,s) = \mathscr{T}_{\bar X}(t)\bar U_{\mathrm{MF}}(t,s)\mathscr{T}_{\bar X}(s)^*; \\ &\mathscr{T}_{\bar X}(t) = e^{-ia_{\bar X}(t)}e^{ib_{\bar X,\mathrm{os}}(t)\cdot r}e^{-ic_{\bar X}(t)\cdot p_{\bar X}} \end{aligned} \tag{7.4}$$

holds, where

$$a_{\bar X}(t) = \int_0^t \left(\frac{1}{2}b_{\bar X,\mathrm{os}}(s)^2 - E_{\bar X,\mathrm{m}}\cdot c_{\bar X}(s)\right) ds.$$

By virtue of the Avron-Herbst formula (7.4), the understanding of $\bar U_{\mathrm{MF}}(t,s)$ yields that of $\bar U_{\mathrm{LF}}(t,s)$ immediately.

Next we would like to watch the motion of the systems in the center-of-mass frame. Here we note that $\bar V(r)$ is independent of x_{cm}, and that $\bar V(r+c_{\bar X}(t))$ is independent of $\pi_{\mathrm{cm}}(r+c_{\bar X}(t)) = x_{\mathrm{cm}}+c_{\mathrm{cm}}(t)$. Hence we will write $\bar V(r)$ and $\bar V(r+c_{\bar X}(t))$ as $\bar V(x)$ and $\bar V(x+c(t))$, respectively. Here we put $c(t) = \pi c_{\bar X}(t)$ and $c_{\mathrm{cm}}(t) = \pi_{\mathrm{cm}}c_{\bar X}(t)$. Now we introduce the Hamiltonians

$$\begin{aligned} &\hat H_{\mathrm{LF}}(t) = \hat H_{\mathrm{LF},0}(t) + \bar V(x); \quad \hat H_{\mathrm{LF},0}(t) = \frac{1}{2}p^2 - E(t)\cdot x, \\ &\hat H_{\mathrm{MF}}(t) = \hat H_{\mathrm{MF},0} + \bar V(x+c(t)); \quad \hat H_{\mathrm{MF},0} = \frac{1}{2}p^2 - E_\mathrm{m}\cdot x \end{aligned} \tag{7.5}$$

acting on $\mathscr{H} = L^2(X)$, and

$$\bar{T}_{\mathrm{LF,cm}}(t) = \frac{1}{2}(p_{\mathrm{cm}})^2 - E_{\mathrm{cm}}(t)\cdot x_{\mathrm{cm}}, \quad \bar{T}_{\mathrm{MF,cm}} = \frac{1}{2}(p_{\mathrm{cm}})^2 - E_{\mathrm{cm,m}}\cdot x_{\mathrm{cm}} \tag{7.6}$$

acting on $L^2(X_{\mathrm{cm}})$, where

$$E(t) = \pi E_{\bar{X}}(t), \quad E_{\mathrm{m}} = \pi E_{\bar{X},\mathrm{m}}, \quad E_{\mathrm{cm}}(t) = \pi_{\mathrm{cm}} E_{\bar{X}}(t), \quad E_{\mathrm{cm,m}} = \pi_{\mathrm{cm}} E_{\bar{X},\mathrm{m}}.$$

Then $\bar{H}_{\mathrm{LF}}(t)$ and $\bar{H}_{\mathrm{MF}}(t)$ are represented as

$$\begin{aligned} \bar{H}_{\mathrm{LF}}(t) &= \hat{H}_{\mathrm{LF}}(t)\otimes \mathrm{Id} + \mathrm{Id}\otimes \bar{T}_{\mathrm{LF,cm}}(t), \\ \bar{H}_{\mathrm{MF}}(t) &= \hat{H}_{\mathrm{MF}}(t)\otimes \mathrm{Id} + \mathrm{Id}\otimes \bar{T}_{\mathrm{MF,cm}} \end{aligned} \tag{7.7}$$

on $L^2(\bar{X}) = \mathscr{H}\otimes L^2(X_{\mathrm{cm}})$. Since $\bar{T}_{\mathrm{LF,cm}}(t)$ (resp. $\bar{T}_{\mathrm{MF,cm}}$) does not depend on pair interactions, the understanding of the dynamics of the system governed by $\hat{H}_{\mathrm{LF}}(t)$ (resp. $\hat{H}_{\mathrm{MF}}(t)$) yields that of the system governed by $\bar{H}_{\mathrm{LF}}(t)$ (resp. $\bar{H}_{\mathrm{MF}}(t)$) immediately.

Now we will focus on $\hat{H}_{\mathrm{LF}}(t)$ and $\hat{H}_{\mathrm{MF}}(t)$. Let $\hat{U}_{\mathrm{LF}}(t,s)$ and $\hat{U}_{\mathrm{MF}}(t,s)$ denote the propagators generated by $\hat{H}_{\mathrm{LF}}(t)$ and $\hat{H}_{\mathrm{MF}}(t)$, respectively. Then, in the same way as in (7.4), the Avron-Herbst formula for $\hat{U}_{\mathrm{LF}}(t,s)$

$$\hat{U}_{\mathrm{LF}}(t,s) = \hat{\mathscr{T}}(t)\hat{U}_{\mathrm{MF}}(t,s)\hat{\mathscr{T}}(s)^*; \quad \hat{\mathscr{T}}(t) = e^{-ia(t)}e^{ib(t)\cdot x}e^{-ic(t)\cdot p} \tag{7.8}$$

holds, where

$$b(t) = \pi b_{\bar{X},\mathrm{os}}(t), \quad a(t) = \int_0^t \left(\frac{1}{2}b(s)^2 - E_{\mathrm{m}}\cdot c(s)\right) ds.$$

By virtue of the Avron-Herbst formula (7.8), the understanding of $\hat{U}_{\mathrm{MF}}(t,s)$ yields that of $\hat{U}_{\mathrm{LF}}(t,s)$ immediately. Hence, we would like to focus on the dynamics of the system governed by $\hat{H}_{\mathrm{MF}}(t)$ in the center-of-mass frame.

The case where $\mathscr{E}_{\mathrm{m}} = 0$ is that of the so-called AC Stark effect. In the case where $\mathscr{E}_{\mathrm{m}} \neq 0$, there are some desirable results on the asymptotic completeness of many body systems governed by such Hamiltonians. In fact, in [**1, 2**], the author obtained the result of the asymptotic completeness for the system under consideration, both in the short-range and the long-range cases, by introducing the Floquet Hamiltonian

$$K = D_t + \hat{H}_{\mathrm{MF}}(t) = D_t + \frac{1}{2}p^2 - E_{\mathrm{m}}\cdot x + \bar{V}(x + c(t)) \tag{7.9}$$

associated with $\hat{H}_{\mathrm{MF}}(t)$. Because of $E_{\mathrm{m}} \neq 0$,

$$A = \frac{E_{\mathrm{m}}}{|E_{\mathrm{m}}|}\cdot p \tag{7.10}$$

can be taken as a conjugate operator for K in the standard Mourre theory. Here we emphasize that in the two-body case, Møller [**26**] first proposed this operator as a conjugate operator for K.

Now we will focus on the case where $\mathscr{E}_{\mathrm{m}} = 0$. Because of $E_{\mathrm{m}} = 0$, we will deal with an N-body Schrödinger operator with time-periodic potentials

$$\hat{H}_{\rm MF}(t) = \frac{1}{2}p^2 + V(t); \quad V(t) = \bar{V}(x + c(t)) = \sum_{1\le j<k\le N} V_{jk}(t, r_j - r_k), \tag{7.11}$$

where

$$V_{jk}(t,y) = \bar{V}_{jk}(y + \tilde{e}_{jk}\bar{\bar{\mathscr{E}}}(t)); \quad \tilde{e}_{jk} = (q_j/m_j) - (q_k/m_k).$$

Here we note that $V_{jk}(t,y)$'s are T-periodic in t. Now we impose the following condition $(V_{\rm ST})_{\bar\rho}$ on $\bar{V}_{jk}$'s with $\bar\rho > 0$:

$(V_{\rm ST})_{\bar\rho}$ $\bar{V}_{jk}(y)$, $(j,k) \in \mathscr{A}$, belongs to $C^2(\boldsymbol{R}^d; \boldsymbol{R})$, is independent of t, and satisfies the decaying conditions

$$|(\partial_y^\alpha \bar{V}_{jk})(y)| \le C_\alpha \langle y\rangle^{-\bar\rho-|\alpha|}, \quad 0 \le |\alpha| \le 2. \tag{7.12}$$

Since

$$\begin{aligned}
(\partial_t V_{jk})(t,y) = &\sum_{\ell=1}^d \tilde{e}_{jk}\bar{\mathscr{E}}_{{\rm os},\ell}(t)(\partial_{y_\ell}\bar{V}_{jk})(y + \tilde{e}_{jk}\bar{\bar{\mathscr{E}}}(t)),\\
(\partial_t^2 V_{jk})(t,y) = &\sum_{\ell=1}^d \tilde{e}_{jk}\mathscr{E}_{{\rm os},\ell}(t)(\partial_{y_\ell}\bar{V}_{jk})(y + \tilde{e}_{jk}\bar{\bar{\mathscr{E}}}(t))\\
&+ \sum_{\ell_2=1}^d\sum_{\ell_1=1}^d \tilde{e}_{jk}\bar{\mathscr{E}}_{{\rm os},\ell_1}(t)\tilde{e}_{jk}\bar{\mathscr{E}}_{{\rm os},\ell_2}(t)(\partial_{y_{\ell_1}}\partial_{y_{\ell_2}}\bar{V}_{jk})(y + \tilde{e}_{jk}\bar{\bar{\mathscr{E}}}(t)),
\end{aligned}$$

we see that $V_{jk}(t,y)$'s satisfy

$$\begin{aligned}
&|(\partial_y^\alpha V_{jk})(t,y)| \le C_{0,\alpha}\langle y\rangle^{-\bar\rho-|\alpha|}, \quad 0 \le |\alpha| \le 2,\\
&|(\partial_t\partial_y^\alpha V_{jk})(t,y)| \le C_{1,\alpha}\langle y\rangle^{-\bar\rho-1-|\alpha|}, \quad 0 \le |\alpha| \le 1,\\
&|(\partial_t^2 V_{jk})(t,y)| \le C_{2,\alpha}\langle y\rangle^{-\bar\rho-1}.
\end{aligned}$$

What we emphasize here is that the decaying rate $O(\langle y\rangle^{-\bar\rho-1})$ of $(\partial_t^2 V_{jk})(t,y)$ is more moderate than $O(\langle y\rangle^{-\bar\rho-2})$. If $V_{jk}(t,y)$ satisfies $(V_{\rm WR})_\rho$ with $\rho > 0$, then the decaying rate of $(\partial_t^2 V_{jk})(t,y)$ is $O(\langle y\rangle^{-\rho-2})$. But, if $\bar\rho > 1$, then one can regard $V_{jk}(t,y) = \bar{V}_{jk}(y + \tilde{e}_{jk}\bar{\bar{\mathscr{E}}}(t))$ as a time-periodic potential satisfying the condition $(V_{\rm WR})_{\bar\rho-1}$. Therefore, in the case where $N = 3$, Theorem 1.5 is a direct consequence of Theorem 1.1, Corollary 1.2, Theorem 1.3, and Corollary 1.4.

References

[1] T. Adachi, Scattering theory for N-body quantum systems in a time-periodic electric field, Funkcial. Ekvac. **44** (2001), 335–376.

[2] T. Adachi, Asymptotic completeness for N-body quantum systems with long-range interactions in a time-periodic electric field, Comm. Math. Phys. **275** (2007), 443–477.

[3] T. Adachi, Asymptotic completeness for N-body quantum systems with long-range interactions in a time-periodic electric field, Comm. Math. Phys. **275** (2007), 443–477.

[4] T. Adachi, T. Kimura and Y. Shimizu, Scattering theory for two-body quantum systems with singular potentials in a time-periodic electric field, J. Math. Phys. **51** (2010), 032103, 23 pp.

[5] T. Adachi and A. Kiyose, On the Mourre estimates for Floquet Hamiltonians, Lett. Math. Phys. **109** (2019), 2513–2529.

[6] T. Adachi and H. Tamura, Asymptotic completeness for long-range many-particle systems with Stark effect, J. Math. Sci. Univ. Tokyo **2** (1995), 76–116.

[7] T. Adachi and H. Tamura, Asymptotic completeness for long-range many-particle systems with Stark effect, II, Comm. Math. Phys. **174** (1996), 537–559.

[8] W. O. Amrein, A. Boutet de Monvel and V. Georgescu, C_0-groups, commutator methods and spectral theory of N-body Hamiltonians, Progress in Mathematics 135, Birkhäuser Verlag, Basel, 1996.

[9] J. E. Avron and I. W. Herbst, Spectral and scattering theory of Schrödinger operators related to the Stark effect, Comm. Math. Phys. **52** (1977), 239–254.

[10] H. L. Cycon, R. G. Froese, W. Kirsch and B. Simon, Schrödinger operators with application to quantum mechanics and global geometry, Texts and Monographs in Physics, Springer-Verlag, Berlin, 1987.

[11] J. Dereziński, Asymptotic completeness of long-range N-body quantum systems, Ann. of Math. (2) **138** (1993), 427–476.

[12] J. Dereziński and C. Gérard, *Scattering theory of classical and quantum mechanical N-particle systems*, Springer-Verlag, 1997.

[13] R. Froese and I. W. Herbst, A new proof of the Mourre estimate, Duke Math. J. **49** (1982), 1075–1085.

[14] C. Gérard and I. Łaba, *Multiparticle quantum scattering in constant magnetic fields*, Mathematical Surveys and Monographs, 90, American Mathematical Society, Providence, RI, 2002.

[15] G. M. Graf, Asymptotic completeness for N-body short-range quantum systems: a new proof, Comm. Math. Phys. **132** (1990), 73–101.

[16] B. Helffer and J. Sjöstrand, Equation de Schrödinger avec champ magnétique et équation de Harper, Lecture Notes in Physics 345 (1989), Springer-Verlag, 118–197.

[17] I. Herbst, J. S. Møller and E. Skibsted, Spectral analysis of N-body Stark Hamiltonians, Comm. Math. Phys. **174** (1995), 261–294.

[18] I. Herbst, J. S. Møller and E. Skibsted, Asymptotic completeness for N-body Stark Hamiltonians, Comm. Math. Phys. **174** (1996), 509–535.

[19] J. S. Howland, Stationary scattering theory for time-dependent Hamiltonians, Math. Ann. **207** (1974), 315–335.

[20] J. S. Howland, Scattering theory for Hamiltonians periodic in time, Indiana Univ. Math. J. **28** (1979), 471–494.

[21] W. Hunziker and I. M. Sigal, Time-dependent scattering theory of N-body quantum systems, Rev. Math. Phys. **12** (2000), 1033–1084.

[22] H. Kitada and K. Yajima, A scattering theory for time-dependent long-range potentials, Duke Math. J. **49** (1982), 341–376.

[23] E. L. Korotyaev, Scattering theory for a three-particle system with two-body interactions periodic in time, Theoret. and Math. Phys. **62** (1985), 163–170.

[24] Y. Kuwabara and K. Yajima, The limiting absorption principle for Schrödinger operators with long-range time-periodic potentials, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **34** (1987), 833–851.

[25] E. Mourre, Absence of singular continuous spectrum for certain selfadjoint operators, Comm. Math. Phys. **78** (1981), 391–408.

[26] J. S. Møller, Two-body short-range systems in a time-periodic electric field, Duke Math. J. **105** (2000), 135–166.

[27] J. S. Møller and E. Skibsted, Spectral theory of time-periodic many-body systems, Adv. Math. **188** (2004), 137–221.

[28] S. Nakamura, Asymptotic completeness for three-body Schrödinger equations with time-periodic potentials, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **33** (1986), 379–402.

[29] P. Perry, I. M. Sigal and B. Simon, Spectral analysis of N-body Schrödinger operators, Ann. of Math. (2) **114** (1981), 519–567.

[30] M. Reed and B. Simon, Methods of Modern Mathematical Physics I–IV, Academic Press.

[31] I. M. Sigal and A. Soffer, The N-particle scattering problem: asymptotic completeness for short-range systems, Ann. of Math. (2) **126** (1987), 35–108.

[32] I. M. Sigal and A. Soffer, Long-range many-body scattering. Asymptotic clustering for Coulomb-type potentials, Invent. Math. **99** (1990), 115–143.

[33] B. Simon, Phase space analysis of simple scattering systems: extensions of some work of Enss, Duke Math. J. **46**

(1979), 119–168.
[34] E. Skibsted, Spectral analysis of N-body systems coupled to a bosonic field, Rev. Math. Phys. **10** (1998), 989–1026.
[35] H. Tamura, Principle of limiting absorption for N-body Schrödinger operators — a remark on the commutator method, Lett. Math. Phys. **17** (1989), 31–36.
[36] D. Yafaev, Radiation conditions and scattering theory for N-particle Hamiltonians, Comm. Math. Phys. **154** (1993), 523–554.
[37] K. Yajima, Scattering theory for Schrödinger equations with potentials periodic in time, J. Math. Soc. Japan **29** (1977), 729–743.
[38] K. Yajima, Existence of solutions for Schrödinger evolution equations, Comm. Math. Phys. **110** (1987), 415–426.
[39] K. Yajima, Existence and regularity of propagators for multi-particle Schrödinger equations in external fields, Comm. Math. Phys. **347** (2016), 103–126.
[40] K. Yajima and H. Kitada, Bound states and scattering states for time periodic Hamiltonians, Ann. Inst. H. Poincaré Sect. A (N.S.) **39** (1983), 145–157.
[41] K. Yokoyama, Mourre theory for time-periodic systems, Nagoya Math. J. **149** (1998), 193–210.

Tadayoshi ADACHI
Division of Mathematical and Information Sciences
Graduate School of Human and Environmental Studies
Kyoto University
Yoshida-Nihonmatsu-cho, Sakyo-ku, Kyoto-shi, Kyoto 606-8501, Japan
E-mail: adachi@math.h.kyoto-u.ac.jp