DIFFERENTIAL MODULES AND DORMANT OPERS OF HIGHER LEVEL

YASUHIRO WAKABAYASHI

Abstract. The aim of the present paper is to develop the theory of $\mathcal{D}$-modules in positive characteristic. In Sections 2, 3, and 4, we study higher-level generalizations of differential modules in positive characteristic. These objects may be regarded as ring-theoretic counterparts of vector bundles on an algebraic curve equipped with an action of the ring of (logarithmic) differential operators of finite level introduced by P. Berthelot and C. Montagnon. The well-known existence assertion for a cyclic vector of a differential module is generalized to higher level. In Sections 5, 6, and 7, we introduce and discuss (dormant) opers of level $N > 0$ on a pointed smooth curve whose structure group is either GL_n or PGL_n. Some of the results in Sections 3 and 4 are applied to prove a duality theorem between dormant PGL_n-opers of level N and dormant PGL_{p^N-n}-opers of level N. Finally, in the case where the underlying curve is a 3-pointed projective line, we establish a bijective correspondence between dormant PGL_2-opers of level N and certain tamely ramified coverings. These assertions are building blocks to establish the enumerative geometry of higher-level dormant opers.

Contents

1. Introduction

1.1. **Differential modules in positive characteristic.** Let R be a *differential ring*, i.e., a commutative ring equipped with a derivation $\partial : R \to R$. A *differential module* over R is an R-module E equipped with an additive map $\nabla : E \to E$ satisfying the Leibniz rule: $\nabla(a \cdot v) = \partial(a) \cdot v + a \cdot \nabla(v)$ $(a \in R,\ v \in E)$. If R is of characteristic 0, a differential module may be regarded as a ring-theoretic counterpart of a $\mathcal{D}$-module (or equivalently, a sheaf equipped with a flat connection) on an algebraic curve.

Y. Wakabayashi: Graduate School of Information Science and Technology, The University of Osaka, Suita, Osaka 565-0871, Japan;

e-mail: wakabayashi@ist.osaka-u.ac.jp;

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On the other hand, differential modules and $\mathcal{D}$-modules in characteristic $p > 0$ have many different features from those in characteristic 0 (see, e.g., [And], [Hon], [Kat2], [Kat3]). For example, unlike the case of characteristic 0, Picard-Vessiot theory fails for differential modules in characteristic p. Despite this problem, such objects have attracted a lot of attention for various reasons, including in relation to the Grothendieck-Katz p-curvature conjecture.

We should note that there are variations of the sheaf "$\mathcal{D}$" defined on an algebraic variety X in characteristic p. One is the ring of crystalline differential operators (following the wording in [BMR]), which we denote by $\mathcal{D}_X^{(0)}$. Giving a $\mathcal{D}_X^{(0)}$-module is equivalent to giving an $\mathcal{O}_X$-module together with a flat connection. This means that the notion of a $\mathcal{D}_X^{(0)}$-module corresponds exactly to a differential module in the usual sense.

Another variant is the ring of differential operators $\mathcal{D}_X^{(\infty)}$ in the sense of Grothendieck (cf. [Gro, Section 16.8.1]). A $\mathcal{D}_X^{(\infty)}$-module is often called a *stratified sheaf* and interpreted as an $\mathcal{O}_X$-module admitting infinite Frobenius descent. The ring-theoretic counterpart is known as an *iterative differential module* (cf. [Oku, Section 1.2], [vdPS, Section 13.3]). The notion of a stratified sheaf was introduced in D. Gieseker's paper (cf. [Gie, Definition 1.1]) and discussed in, e.g., [dSa], [Esn], [EsMe], [Kin1], and [Kin2]. We also can find descriptions of iterative differential modules in, e.g., [Ern], [MavdP], and [Rös].

Next, let us recall the ring of differential operators $\mathcal{D}_X^{(m)}$ of level $m \in \mathbb{Z}_{\geq 0}$, as introduced by P. Berthelot (cf. [PBer1], [PBer2]); this kind of sheaf is an essential ingredient in defining arithmetic $\mathcal{D}$-modules, and it may be positioned between $\mathcal{D}_X^{(0)}$ and $\mathcal{D}_X^{(\infty)}$. In fact, the ring of crystalline differential operators coincides with Berthelot's $\mathcal{D}_X^{(0)}$ (i.e., $\mathcal{D}_X^{(m)}$ for $m = 0$), and there exists an inductive system

$$\mathcal{D}_X^{(0)} \to \mathcal{D}_X^{(1)} \to \mathcal{D}_X^{(2)} \to \cdots \to \mathcal{D}_X^{(m)} \to \cdots$$

satisfying $\varinjlim_m \mathcal{D}_X^{(m)} = \mathcal{D}_X^{(\infty)}$. Moreover, C. Montagnon generalized $\mathcal{D}_X^{(m)}$ to the case where the underlying scheme is equipped with a log structure (cf. [Mon, Définition 2.3.1]). This generalization allows us to deal with $\mathcal{D}_X^{(m)}$-modules (in the logarithmic sense) for (possibly singular) pointed curves X.

1.2. **Cyclic vectors of higher-level differential modules.** In Sections 2, 3, and 4 of the present paper, we consider ring-theoretic counterparts of (both non-logarithmic and logarithmic versions of) $\mathcal{D}_X^{(m)}$-modules, in other words, higher-level generalizations of differential modules. (Note that some of our discussions are merely paraphrases of previous studies.)

The central character is an *m-differential ring* (resp., an *m-log differential ring*), which is defined as a ring R in characteristic p equipped with a collection of certain additive endomorphisms $\partial_{\langle \bullet \rangle} := \{\partial_{\langle j \rangle}\}_{j \in \mathbb{Z}_{\geq 0}}$ (cf. Definition 2.2.1, (ii)). Each such data $\mathscr{R} := (R, \partial_{\langle \bullet \rangle})$ yields a possibly noncommutative ring $D_{\mathscr{R}}^{(m)}$ generated by the elements of R and the set of abstract symbols $\{\partial_{\langle j \rangle}\}_{j \in \mathbb{Z}_{\geq 0}}$. (The definition of a related ring can be found in [Kin1, Definition 1.1.1].) In particular, we obtain the notion of a $D_{\mathscr{R}}^{(m)}$-module, which corresponds to the sheaf theoretic notion of a $\mathcal{D}_X^{(m)}$-module.

An important ingredient in the theory of differential modules is the concept of a *cyclic vector*. A cyclic vector of a differential module (E, ∇) is an element $v \in E$ such that the elements $\nabla^0(v) \, (= v) \, , \nabla^1(v), \cdots , \nabla^l(v)$ (for some $l \geq 0$) form a basis for E. The choice of such an element enables (E, ∇) to be interpreted as a higher-order linear differential operator on R;

accordingly, each element in E that is horizontal with respect to ∇ can be described as a root function of that operator, i.e., a function annihilated by that operator. A fundamental result is the existence of a cyclic vector in a general situation (cf. [ChKo], [Kat4]). We refer the reader to, e.g., [Adj], [DBer], [Del], and [Kov], for various discussions concerning cyclic vectors.

Given an m-differential ring $\mathscr{R}$ and $D^{(m)}_{\mathscr{R}}$-module (E, ∇), we can describe the notion of an *m-cyclic vector* of (E, ∇) (cf. Definition 2.5.1) as a higher-level generalization of a cyclic vector in the classical sense. An m-cyclic vector of (E, ∇) is, by definition, an element of E such that $\nabla_{\langle 0 \rangle}(v) \, (= v) \, , \nabla_{\langle 1 \rangle}(v), \cdots , \nabla_{\langle l \rangle}(v)$ (for some $l \geq 0$) forms a basis of E.

Also, by a *pinned $D^{(m)}_{\mathscr{R}}$-module*, we mean a $D^{(m)}_{\mathscr{R}}$-module together with an m-cyclic vector. Our study of m-cyclic vectors stems from the fact that a pinned $D^{(0)}_{\mathscr{R}}$-module is regarded as a locally defined GL_n-oper on a curve; as such, various properties of m-cyclic vectors can be used to examine higher-level generalizations of opers. The first main result of the present paper generalizes the classical assertion of the existence of a cyclic vector to higher level.

Theorem A (cf. Theorem 2.5.6)**.** *Let m be a nonnegative integer, n a positive integer, and $(R, \partial_{\langle \bullet \rangle})$ an m-differential field over $\mathbb{F}_p := \mathbb{Z}/p\mathbb{Z}$. Assume that the morphism $D^{(m)}_{\mathscr{R}, <n} \rightarrow \mathrm{End}_{\mathbb{F}_p}(R)$ naturally induced by $\partial_{\langle \bullet \rangle}$ is injective. (This assumption is fulfilled when $n \leq p^{m+1}$ and R is either $k(t)$ or $k(\!(t)\!)$ for a perfect field k over $\mathbb{F}_p$ equipped with the m-derivation $\partial_{\langle \bullet \rangle} := \{\partial_{\langle j \rangle}\}_j$ given by (2.1).) Then, each $D^{(m)}_{\mathscr{R}}$-module (E, ∇) with $\mathrm{rk}(E) = n$ admits an m-cyclic vector.*

After proving the above theorem, we discuss the *p^{m+1}-curvature* of each $D^{(m)}_{\mathscr{R}}$-module in the situation (cf. Section 3.1) that $\mathscr{R} := (R, \partial_{\langle \bullet \rangle})$ (resp., $\mathscr{R} := (R, \breve{\partial}_{\langle \bullet \rangle})$)) is a certain type of m-differential ring (resp., m-log differential ring); we use the notation $D^{(m)}_{R}$ (resp., $\breve{D}^{(m)}_{R}$) to denote the ring $D^{(m)}_{\mathscr{R}}$ for convenience. The p^{m+1}-curvature of a $D^{(m)}_{R}$-module (resp., a $\breve{D}^{(m)}_{R}$-module) (E, ∇) is defined as an invariant measuring the extent to which the element $\partial_{\langle p^{m+1} \rangle}$ (resp., $\breve{\partial}_{\langle p^{m+1} \rangle}$) via ∇ vanishes. We say that (E, ∇) is *dormant* (cf. Definition 3.1.3) if it has vanishing p^{m+1}-curvature.

Here, suppose that (E, ∇) is dormant and the R-module E is free and of rank $n > 0$. In the non-logarithmic case, the structure of (E, ∇) is not difficult because a classical result by Cartier implies that (E, ∇) is isomorphic to the direct sum of finitely many copies of the trivial $D^{(m)}_{R}$-module (cf. Corollary 3.1.7).

On the other hand, in the logarithmic case, the formal completion of (E, ∇) is isomorphic to that of the direct sum $\bigoplus_{j=1}^{n} (R, \nabla_{d_j})$ for various elements d_j $(j = 1, \cdots , n)$ of $\mathbb{Z}/p^{m+1}\mathbb{Z}$, where each ∇_{d_j} denotes a $\breve{D}^{(m)}_{R}$-action on R defined in (4.3). The resulting multiset $[d_1, \cdots , d_n]$ is called the *exponent* of (E, ∇) (cf. Definition 4.3.1). We examine its relationship with the *residue* described in Section 4.1 (cf. Propositions 4.2.1, (i), 4.3.4, and Remark 4.3.3), as well as with the existence of an m-cyclic vector (cf. Proposition 4.4.1). In addition, we establish a duality between dormant pinned $\breve{D}^{(m)}_{R}$-modules of rank n (with $0 < n < p^{m+1}$) and dormant pinned $\breve{D}^{(m)}_{R}$-modules of rank $p^N - n$ (cf. Corollary 3.3.3, Proposition 4.4.3).

1.3. **Duality for dormant opers of higher level.** In Sections 5, 6, and 7 of the present paper, we study (dormant) GL_n-opers and (dormant) PGL_n-opers of level $N > 0$. Here, let $\mathscr{X} := (X, \{\sigma_i\}_{i=1}^{r})$ (cf. Section 5.2), where $r \geq 0$, be an r-pointed smooth curve over an algebraically closed field k of characteristic p. A $\mathrm{GL}_n^{(N)}$-*oper* (or a GL_n-*oper of level* N) on $\mathscr{X}$

is, roughly speaking, a rank n vector bundle on X equipped with both a $\mathcal{D}_X^{(N-1)}$-action and complete flag structure satisfying a strict form of Griffiths transversality. $\mathrm{GL}_n^{(1)}$-opers have been investigated from various points of view (cf. [BeDr1], [BeDr2], [BeBi], [Fre], [Wak5]).

In addition, *dormant* $\mathrm{PGL}_2^{(N)}$-opers (i.e., PGL_2-opers of level N with vanishing p^N-curvature) on an unpointed smooth curve were discussed in [Hos2], [Wak3], and [Wak4] under the identification with F^N*-projective structures*. In the case where the set of marked points $\{\sigma_i\}_i$ of $\mathscr{X}$ is nonempty, we introduce the *radius* of a $\mathrm{PGL}_n^{(N)}$-oper at each marked point σ_i (cf. Definition 5.3.2); this is an element of $\mathfrak{S}_n \backslash (\mathbb{Z}/p^N\mathbb{Z})^n/\Delta$ (where Δ denotes the image of the diagonal embedding $\mathbb{Z}/p^N\mathbb{Z} \hookrightarrow (\mathbb{Z}/p^N\mathbb{Z})^n$ and $\mathfrak{S}_n$ denotes the symmetric group of n letters acting on $(\mathbb{Z}/p^N\mathbb{Z})^n$ by permutation) induced from the exponent of the $\breve{D}_{k[\![t]\!]}$-module obtained by restricting that oper to the formal neighborhood of σ_i.

Given an element $\vec{\rho} \in (\rho_i)_{i=1}^r \in (\mathfrak{S}_n \backslash (\mathbb{Z}/p^N\mathbb{Z})^n/\Delta)^r$, we set

$$\mathrm{Op}_n^{\text{Zzz...}} \quad \left(\text{resp., } \mathrm{Op}_{n,\vec{\rho}}^{\text{Zzz...}}\right)$$

to be the set of dormant $\mathrm{PGL}_n^{(N)}$-opers (resp., dormant $\mathrm{PGL}_n^{(N)}$-opers of radii $\vec{\rho}$) on $\mathscr{X}$. Then, we verify that $\mathrm{Op}_n^{\text{Zzz...}} = \emptyset$ if $n > p^N$ (cf. Corollary 5.2.4) and that $\mathrm{Op}_n^{\text{Zzz...}}$ consists of exactly one element if $n = 1$. Also, by the duality of differential modules established in Sections 3 and 4, we obtain the following assertions, generalizing [Wak1, Theorem A, (i) and (ii)], [Wak1, Corollary 4.3.3], and [Hos1, Theorem A].

Theorem B (cf. Theorem 6.3.1, Corollary 6.3.2)**.** *Suppose that* $0 < n < p^N$. *Then, the following assertions hold:*

(i) *There exists a canonical bijection of sets*

$$\mathcal{C}_n : \mathrm{Op}_n^{\text{Zzz...}} \xrightarrow{\sim} \mathrm{Op}_{p^N-n}^{\text{Zzz...}}$$

satisfying $\mathcal{C}_{p^N-n} \circ \mathcal{C}_n = \mathrm{id}$. *In particular, there exists exactly one isomorphism class of dormant* $\mathrm{PGL}_{p^N-1}^{(N)}$*-oper on* $\mathscr{X}$*; i.e., the following equality holds:*

$$\sharp(\mathrm{Op}_{p^N-1}^{\text{Zzz...}}) = 1.$$

(ii) *Suppose further that* $r > 0$*, and let us take* $\vec{\rho} := (\rho_i)_{i=1}^r \in (\mathfrak{S}_n \backslash (\mathbb{Z}/p^N\mathbb{Z})^n/\Delta)^r$. *Then,* $\mathcal{C}_n$ *restricts to a bijection*

$$\mathcal{C}_{n,\vec{\rho}} : \mathrm{Op}_{n,\vec{\rho}}^{\text{Zzz...}} \xrightarrow{\sim} \mathrm{Op}_{p^N-n,\vec{\rho}^{\blacktriangledown}}^{\text{Zzz...}},$$

where $\vec{\rho}^{\blacktriangledown} := (\rho_i^{\blacktriangledown})_i$ *is the set defined in* (4.8)*, satisfying* $\mathcal{C}_{p^N-n,\vec{\rho}^{\blacktriangledown}} \circ \mathcal{C}_{n,\vec{\rho}} = \mathrm{id}$ *(under the equality* $\vec{\rho}^{\blacktriangledown\blacktriangledown} = \vec{\rho}$*).*

1.4. **Comparison with tamely ramified coverings.** The next topic concerns the classification of tamely ramified coverings of the projective line in characteristic p with specified ramification data and fixed branch points. For related work on this problem, we refer the reader to [BoOs], [BoZa], [Ebe], [Fab], [Oss1], [Oss2], [Oss3], and [Oss4]. We know (cf. [Moc], [Oss2], [Oss4]) that certain tamely ramified coverings with ramification indices $< p$ can be described in terms of dormant PGL_2-opers (i.e., dormant torally indigenous bundles, in the sense of [Moc]). In particular, that description allows us to translate dormant PGL_2-opers on a 3-pointed projective line into simple combinatorial data; this result is the starting point of the enumerative geometry of dormant opers studied in [Wak5]. In the present paper, the situation is generalized to the higher level case in order to deal with tamely ramified coverings having large ramification indices.

Let us consider the 3-pointed projective line $\mathscr{P} := (\mathbb{P}, \{[0], [1], [\infty]\})$ over k, where $[x]$ (for each $x \in \{0, 1, \infty\}$) denotes the point of the projective line $\mathbb{P}$ determined by the value x. (We use the notation "$\mathbb{P}$" as opposed to the usual notation "$\mathbb{P}^1$" because later on we will need to consider $\mathbb{P}^1$ equipped with log structures and $\mathbb{P}^{\log}$ is notationally and typographically simpler than $(\mathbb{P}^1)^{\log}$.) We shall write

$$\mathrm{Cov}$$

for the set of equivalence classes of finite, separable, and tamely ramified coverings $\phi : \mathbb{P} \to \mathbb{P}$ satisfying the following conditions:

- The set of ramification points of ϕ coincides with $\{[0], [1], [\infty]\}$;
- If λ_x $(x = 0, 1, \infty)$ denotes the ramification index of ϕ at $[x]$, then $\lambda_0, \lambda_1, \lambda_\infty$ are all odd and satisfy the inequality $\lambda_0 + \lambda_1 + \lambda_\infty < 2p^N$.

Here, for two such coverings $\phi_1, \phi_2 : \mathbb{P} \to \mathbb{P}$, we say that ϕ_1 and ϕ_2 are equivalent if $\phi_2 = h \circ \phi_1$ for some $h \in \mathrm{PGL}_2(k) = \mathrm{Aut}_k(\mathbb{P})$.

The final result of the present paper establishes, as described below, a bijective correspondence between Cov and $\mathrm{Op}_2^{\text{Zzz...}}$ for $\mathscr{X} = \mathscr{P}$; this generalizes [Moc, Introduction, Theorem 1.3].

Theorem C (cf. Theorem 7.4.3 for the full statement)**.** *Let us consider the set* $\mathrm{Op}_2^{\text{Zzz...}}$ *in the case where the underlying curve "$\mathscr{X}$" is taken to be $\mathscr{P}$. Then, we can construct a canonical bijection of sets*

$$\Upsilon : \mathrm{Cov} \xrightarrow{\sim} \mathrm{Op}_2^{\text{Zzz...}}$$

satisfying the following condition: if ϕ is a tamely ramified covering classified by Cov *whose ramification index at $[x]$ $(x = 0, 1, \infty)$ is λ_x, then the radii of the dormant* PGL_2*-oper on $\mathscr{P}$ determined by $\Upsilon(\phi)$ coincides with the image of $(\frac{1}{2} \cdot \overline{\lambda}_0, \frac{1}{2} \cdot \overline{\lambda}_1, \frac{1}{2} \cdot \overline{\lambda}_\infty)$ via the natural quotient $(\mathbb{Z}/p^N\mathbb{Z})^3 \twoheadrightarrow (\mathfrak{S}_2 \backslash (\mathbb{Z}/p^N\mathbb{Z})^2/\Delta)^3$. In particular, the set* $\mathrm{Op}_2^{\text{Zzz...}}$ *in this case is finite.*

1.5. **Future work.** The results of the present paper are building blocks to establish the enumerative geometry of dormant opers of higher level. (This involves the treatment of dormant opers, as well as linear ODE's, in *prime-power characteristic* by applying what we call *diagonal reduction/lifting*; see [Wak6, Theorem D].) One central theme of the theory is to explicitly figure out how many higher-level dormant opers (and related mathematical objects) exist.

In addressing this problem, the duality assertion in Theorem B allows the study of $\mathrm{PGL}_n^{(N)}$-opers with n large to be translated into the study of them with n small. This approach seems to be valid because it is presumed that higher-level generalizations of the formula obtained in [Wak5, Theorem H] are given only when n is sufficiently small (relative to p).

On the other hand, we proved a certain factorization property of $\mathrm{Op}_n^{\text{Zzz...}}$ using the clutching morphisms between moduli stacks of pointed stable curves (cf. [Wak6, Theorem C]). This factorization property induces a 2d TQFT (= 2-dimensional topological quantum field theory), and thus enables us to reduce the various problems we wish to solve to the simplest case where the underlying curve is the 3-pointed projective line. Therefore, by applying Theorem C together with an argument similar to the proof of [Moc, Introduction, Theorem 3.1], we can classify dormant $\mathrm{PGL}_2^{(N)}$-opers (on the 3-pointed projective line, or more generally, a totally degenerate stable curve) in terms of purely combinatorial data given by their radii (cf. [Wak6, Theorem E]).

2. Differential modules and cyclic vectors of higher level

First, we study differential modules of level $m \geq 0$ and generalize the notion of a cyclic vector to such modules, i.e., m-cyclic vectors. (For convenience, we will occasionally include the case of $m = -1$.) At the end of this section, we prove the existence of an m-cyclic vector under mild conditions (cf. Theorem 2.5.6).

Throughout the present paper, we will fix a prime p. Unless stated otherwise, all the rings appearing in the present paper are assumed to be unital, associative, and commutative.

2.1. **Modified binomial coefficients.** For nonnegative integers m and l, let $(q_l^{(m)}, r_l^{(m)})$ be the pair of nonnegative integers uniquely determined by the condition that $l = p^m \cdot q_l^{(m)} + r_l^{(m)}$ and $0 \leq r_l^{(m)} < p^m$. For each pair of nonnegative integers (j, j') with $j \geq j'$, we set

$$\left\{ \begin{matrix} j \\ j' \end{matrix} \right\}_{(m)} := \frac{q_j^{(m)}!}{q_{j'}^{(m)}! \cdot q_{j-j'}^{(m)}!}, \qquad \left\langle \begin{matrix} j \\ j' \end{matrix} \right\rangle_{(m)} := \binom{j}{j'} \cdot \left\{ \begin{matrix} j \\ j' \end{matrix} \right\}_{(m)}^{-1}$$

(cf. [PBer1, Section 1.1.2]). Moreover, if j'' is an integer with $\max\{j', j - j'\} \leq j'' \leq j$, then we set

$$\left\langle \begin{matrix} j \\ j' \end{matrix} \right\rangle_{(m)}^{[j'']} := \frac{j''!}{(j - j'')! \cdot (j'' - j')! \cdot (j'' + j' - j)!} \cdot \frac{q_{j'}^{(m)}! \cdot q_{j-j'}^{(m)}!}{q_{j''}^{(m)}!}.$$

In particular, we have $\left\langle \begin{matrix} j \\ j' \end{matrix} \right\rangle_{(m)}^{[j]} = \left\langle \begin{matrix} j \\ j' \end{matrix} \right\rangle_{(m)}$. Note that all these values lie in $\mathbb{Z}_{(p)}$ and hence induce elements of $\mathbb{F}_p := \mathbb{Z}/p\mathbb{Z}$ via the natural quotient $\mathbb{Z}_{(p)} \twoheadrightarrow \left(\mathbb{Z}_{(p)}/p\mathbb{Z}_{(p)} =\right) \mathbb{F}_p$ even when the integer j'' is divisible by p. When there is no fear of confusion, we will omit the notation "(m)"; i.e., we will write $q_l := q_l^{(m)}$, $r_l := r_l^{(m)}$, and

$$\left\{ \begin{matrix} j \\ j' \end{matrix} \right\} := \left\{ \begin{matrix} j \\ j' \end{matrix} \right\}_{(m)}, \quad \left\langle \begin{matrix} j \\ j' \end{matrix} \right\rangle := \left\langle \begin{matrix} j \\ j' \end{matrix} \right\rangle_{(m)}, \quad \left\langle \begin{matrix} j \\ j' \end{matrix} \right\rangle^{[j'']} := \left\langle \begin{matrix} j \\ j' \end{matrix} \right\rangle_{(m)}^{[j'']}.$$

2.2. **Differential rings of higher level.** Let us fix an integer $m \geq 0$ and a ring R_0 over $\mathbb{F}_p$. In the following discussion, the non-resp'd portion deals with the non-logarithmic case and the resp'd portion deals with the logarithmic case.

Definition 2.2.1. (i) Let R be a ring over R_0. An m-**derivation** (resp., m-**log derivation**) on R relative to R_0 is a collection

$$\partial_{\langle \bullet \rangle} := \{\partial_{\langle j \rangle}\}_{j \in \mathbb{Z}_{\geq 0}}$$

consisting of R_0-linear endomorphisms $\partial_{\langle j \rangle}$ of R, regarded as an R_0-module, satisfying the following conditions:

(a) If $j = 0$, then $\partial_{\langle j \rangle} = \mathrm{id}_R$;

(b) If $j > 0$, then the following equalities hold:

$*$ $\partial_{\langle j \rangle}(a \cdot b) = \sum_{j'+j''=j} \left\{ \begin{matrix} j \\ j' \end{matrix} \right\} \cdot \partial_{\langle j' \rangle}(a) \cdot \partial_{\langle j'' \rangle}(b)$ for any elements $a, b \in R$;

$*$ $\partial_{\langle j' \rangle} \circ \partial_{\langle j-j' \rangle} = \left\langle \begin{matrix} j \\ j' \end{matrix} \right\rangle \cdot \partial_{\langle j \rangle}$ $\left(\text{resp., } \partial_{\langle j' \rangle} \circ \partial_{\langle j-j' \rangle} = \sum_{j''=\max\{j', j-j'\}}^{j} \left\langle \begin{matrix} j \\ j' \end{matrix} \right\rangle^{[j'']} \cdot \partial_{\langle j'' \rangle}\right)$

for any integer j' with $0 \leq j' \leq j$.

(ii) By an m-**differential ring** (resp., m-**log differential ring**) over R_0, we mean the pair

$$\mathscr{R} := (R, \partial_{\langle \bullet \rangle})$$

consisting of a ring R over R_0 and an m-derivation (resp., m-log derivation) $\partial_{\langle \bullet \rangle}$ on R. For convenience, we refer to each ring R over R_0 as a (-1)-**differential ring** over R_0. Finally, by an m-**differential field**, we mean an m-differential ring $(R, \partial_{\langle \bullet \rangle})$ such that R is a field.

Example 2.2.2. Consider the case where R is taken to be the ring $k[\![t]\!]$ of formal power series with coefficients in a perfect field k of characteristic p. Denote by K the fraction field of R, i.e., $K := k(\!(t)\!)$. Also, for each integer $l \geq 0$, we shall denote by $R^{(l)}$ (resp., $K^{(l)}$) the subring of R (resp., the subfield of K) consisting of elements a^{p^l} for $a \in R$ (resp., $a \in K$). Then, K has basis $1, t, \cdots, t^{p^{m+1}-1}$ over $K^{(m+1)}$, and it admits an m-derivation $\partial_{\langle \bullet \rangle} := \{\partial_{\langle j \rangle}\}_{l \in \mathbb{Z}_{\geq 0}}$ (resp., m-log derivation $\breve{\partial}_{\langle \bullet \rangle} := \{\breve{\partial}_{\langle j \rangle}\}_{j \in \mathbb{Z}_{\geq 0}}$) relative to $K^{(m+1)}$ given by

$$\partial_{\langle j \rangle}(t^n) := q_j! \cdot \binom{n}{j} \cdot t^{n-j} \ \left(\text{resp., } \breve{\partial}_{\langle j \rangle}(t^n) := q_j! \cdot \binom{n}{j} \cdot t^n\right) \tag{2.1}$$

for every $j, n \in \mathbb{Z}_{\geq 0}$. The collection $\partial_{\langle \bullet \rangle} := \{\partial_{\langle j \rangle}\}_j$ (resp., $\breve{\partial}_{\langle \bullet \rangle} := \{\breve{\partial}_{\langle j \rangle}\}_j$) restricts to an m-derivation (resp., m-log derivation) on R relative to $R^{(m+1)}$, which we express in the same notation. For each $l \in \mathbb{Z}_{\geq 0}$, $R^{(l)}$ coincides with R^l $\left(= \bigcap_{j=0}^{l-1} \mathrm{Ker}(\partial_{\langle p^j \rangle}) = \bigcap_{j=0}^{l-1} \mathrm{Ker}(\breve{\partial}_{\langle p^j \rangle})\right)$, and the collection $(R^{(l)}, \{\partial_{jp^l}\}_{j \in \mathbb{Z}_{\geq 0}})$ forms an $(m-l)$-differential ring over $R^{(m+1)}$.

Remark 2.2.3. (i) If we are given a 0-derivation $\{\partial_{\langle j \rangle}\}_{j \in \mathbb{Z}_{\geq 0}}$ on R, then $\partial_{\langle 1 \rangle}$ defines a derivation on R (over R_0) in the usual sense. Under the correspondence $(R, \{\partial_{\langle j \rangle}\}_j) \leftrightarrow (R, \partial_{\langle 1 \rangle})$, the notion of a 0-differential ring coincides with the usual notion a differential ring.

(ii) Let R be a ring over R_0. The second equality in condition (b) above implies that an m-derivation (resp., an m-log derivation) $\{\partial_{\langle j \rangle}\}_{j \in \mathbb{Z}_{\geq 0}}$ on R is uniquely determined by its subset $\{\partial_{\langle p^j \rangle}\}_{0 \leq j \leq m}$.

Let us fix an m-differential ring (resp., m-log differential ring) $\mathscr{R} := (R, \partial_{\langle \bullet \rangle})$ over R_0, where $\partial_{\langle \bullet \rangle} := \{\partial_{\langle j \rangle}\}_{j \in \mathbb{Z}_{\geq 0}}$. Then, we obtain the possibly noncommutative ring

$$D_{\mathscr{R}}^{(m)}$$

over R_0 generated by the collection of symbols $\{\partial_{\langle j \rangle}\}_{j \in \mathbb{Z}_{\geq 0}}$ subject to the following relations:

- $\partial_{\langle 0 \rangle} = 1$;

- $\partial_{\langle j \rangle} \cdot a = \sum_{j'+j''=j} \left\{ \begin{matrix} j \\ j' \end{matrix} \right\} \cdot \partial_{\langle j' \rangle}(a) \cdot \partial_{\langle j'' \rangle}$ for any $j \in \mathbb{Z}_{\geq 0}$ and $a \in R$;
- $\partial_{\langle j' \rangle} \cdot \partial_{\langle j-j' \rangle} = \left\langle \begin{matrix} j \\ j' \end{matrix} \right\rangle \cdot \partial_{\langle j \rangle}$ $\left(\text{resp., } \partial_{\langle j' \rangle} \cdot \partial_{\langle j-j' \rangle} = \sum_{j''=\max\{j', j-j'\}}^{j} \left\langle \begin{matrix} j \\ j' \end{matrix} \right\rangle^{[j'']} \cdot \partial_{\langle j'' \rangle}\right)$ for any integers j, j' with $0 \leq j' \leq j$.

We shall set $D_{\mathscr{R}}^{(-1)} := R$. The ring $D_{\mathscr{R}}^{(m)}$ admits two R-module structures given by left and right multiplications. For each $l \in \mathbb{Z}_{\geq 0}$, we shall denote by $D_{\mathscr{R}, <l}^{(m)}$ the two-sided R-submodule of $D_{\mathscr{R}}^{(m)}$ generated by the products $\partial_{\langle j_1 \rangle}^{a_1} \cdots \partial_{\langle j_s \rangle}^{a_s}$ $(s \geq 1)$ with $j_i \leq p^m$ $(i = 1, \cdots, s)$ and $\sum_{i=1}^{s} a_i j_i < l$. The collection $\{D_{\mathscr{R}, <l}^{(m)}\}_l$ forms an increasing filtration on $D_{\mathscr{R}}^{(m)}$ with $\bigcup_l D_{\mathscr{R}, <l}^{(m)} = D_{\mathscr{R}}^{(m)}$. Also, one may verify that the R_0-algebra $D_{\mathscr{R}}^{(m)}$ is generated by the elements of R and the set $\{\partial_{\langle p^j \rangle}\}_{0 \leq j \leq m}$; and $D_{\mathscr{R}}^{(m)}$ forms a left and right noetherian ring if R is noetherian (cf. [PBer1, Proposition 1.2.4, (i)], [Mon, Proposition 2.3.2, (b)]).

2.3. **Differential modules of higher level.** Let us take an R-module E. By a **(left) $D_{\mathscr{R}}^{(m)}$-module structure** on E, we mean a left $D_{\mathscr{R}}^{(m)}$-action (i.e., an R_0-algebra homomorphism) $\nabla : D_{\mathscr{R}}^{(m)} \to \mathrm{End}_{R_0}(E)$ on E extending its R-module structure. An R-module equipped with a $D_{\mathscr{R}}^{(m)}$-module structure is called a **(left) $D_{\mathscr{R}}^{(m)}$-module**, or a **differential module over $\mathscr{R}$**. Moreover, we can define, in a natural manner, the notion of an isomorphism between $D_{\mathscr{R}}^{(m)}$-modules. Given a $D_{\mathscr{R}}^{(m)}$-module structure ∇ on E and an integer $j \in \mathbb{Z}_{\geq 0}$, we shall write $\nabla_{\langle j \rangle} := \nabla(\partial_{\langle j \rangle})$. If $\mathscr{R}$ is non-logarithmic (resp., logarithmic), then the assignment $\nabla \mapsto \{\nabla_{\langle j \rangle}\}_{j \in \mathbb{Z}_{\geq 0}}$ determines a bijective correspondence between the set of $D_{\mathscr{R}}^{(m)}$-module structures on E and the set of collections of R_0-linear endomorphisms $\{\nabla_{\langle j \rangle}\}_{j \in \mathbb{Z}_{\geq 0}}$ of E satisfying the following conditions:

- $\nabla_{\langle 0 \rangle} = \mathrm{id}_E$;
- $\nabla_{\langle j \rangle}(a \cdot v) = \sum_{j'+j''=j} \left\{ \begin{matrix} j \\ j' \end{matrix} \right\} \cdot \partial_{\langle j' \rangle}(a) \cdot \nabla_{\langle j'' \rangle}(v)$ for any integer $j > 0$ and any elements $a \in R$, $v \in E$;
- $\nabla_{\langle j' \rangle} \circ \nabla_{\langle j-j' \rangle} = \left\langle \begin{matrix} j \\ j' \end{matrix} \right\rangle \cdot \nabla_{\langle j' \rangle}$ $\left(\text{resp., } \nabla_{\langle j' \rangle} \circ \nabla_{\langle j-j' \rangle} = \sum_{j''=\max\{j', j-j'\}}^{j} \left\langle \begin{matrix} j \\ j' \end{matrix} \right\rangle^{[j'']} \cdot \nabla_{\langle j'' \rangle}\right)$ for any integers j, j' with $0 \leq j' \leq j$.

Because of this correspondence, we will not distinguish these two additional structures on E.

Remark 2.3.1. (i) Let us consider the case of $m = 0$. Suppose that (E, ∇) is a $D_{\mathscr{R}}^{(0)}$-module. Then, the R-module E together with the endomorphism $\nabla_{\langle 1 \rangle}$ $\big(:= \nabla(\partial_{\langle 1 \rangle})\big)$ specifies a differential module, in the classical sense, over the differential ring corresponding to $\mathscr{R}$ (cf. Remark 2.2.3, (i)).

(ii) The notion of a $D_{\mathscr{R}}^{(m)}$-module is slightly different from the notion of an *iterative differential module of level* m, discussed in [Kin1]. In fact, the latter one essentially requires the condition of vanishing p^{m+1}-curvature, in the sense of Section 3.1 (cf. [Kin1, Remark 2.3.5]); compare Corollary 3.1.7 and [Kin1, Proposition 1.1.6].

Remark 2.3.2. Let $\{\partial_{\langle j \rangle}\}_{j \in \mathbb{Z}_{\geq 0}}$ be an m-derivation (resp., m-log derivation) on R. To make the integer m explicit, we here write $\partial^{(m)}_{\langle j \rangle} := \partial_{\langle j \rangle}$ ($j \in \mathbb{Z}_{\geq 0}$). For an integer m' with $0 \leq m' \leq m$, the endomorphism $\partial^{(m')}_{\langle j \rangle} := \frac{q_j^{(m')}!}{q_j^{(m)}!} \cdot \partial^{(m)}_{\langle j \rangle}$ of R is well-defined, and the collection $\{\partial^{(m')}_{\langle j \rangle}\}_{j \in \mathbb{Z}_{\geq 0}}$ forms an m'-derivation (resp., m-log derivation) on R. Let us set $D^{(m')}_{\mathscr{R}} := D^{(m')}_{(R, \{\partial^{(m')}_{\langle j \rangle}\}_j)}$. Then, the assignment $\partial^{(m')}_{\langle j \rangle} \mapsto \frac{q_j^{(m')}!}{q_j^{(m)}!} \cdot \partial^{(m)}_{\langle j \rangle}$ ($j \in \mathbb{Z}_{\geq 0}$) determines an R_0-algebra homomorphism $D^{(m')}_{\mathscr{R}} \to D^{(m)}_{\mathscr{R}}$. This homomorphism allows us to construct a $D^{(m')}_{\mathscr{R}}$-module by means of each $D^{(m)}_{\mathscr{R}}$-module.

We shall denote by

$$\mathfrak{Mod}(D^{(m)}_{\mathscr{R}})$$

the category of $D^{(m)}_{\mathscr{R}}$-modules. (In particular, $\mathfrak{Mod}(D^{(-1)}_{\mathscr{R}})$ coincides with the category of R-modules.) This category has the structure of a tensor product: given two $D^{(m)}_{\mathscr{R}}$-modules (E', ∇') and (E'', ∇''), we set

$$(E', \nabla') \otimes (E'', \nabla'') := (E' \otimes_R E'', \nabla' \otimes \nabla''),$$

where $\nabla' \otimes \nabla''$ denotes the $D^{(m)}_{\mathscr{R}}$-module structure on the tensor product $E' \otimes_R E''$ determined by

$$(\nabla' \otimes \nabla'')_{\langle j \rangle}(v' \otimes v'') := \sum_{j' + j'' = j} \begin{Bmatrix} j \\ j' \end{Bmatrix} \cdot \nabla'_{\langle j' \rangle}(v') \otimes \nabla''_{\langle j'' \rangle}(v'')$$

for any $j \in \mathbb{Z}_{\geq 0}$, $v' \in E'$, and $v'' \in E''$. Similarly, we can construct a $D^{(m)}_{\mathscr{R}}$-module structure on $\mathrm{Hom}_R(E', E'')$ arising from ∇' and ∇''. In particular, for a $D^{(m)}_{\mathscr{R}}$-module (E, ∇), we can define the dual $(E^\vee, \nabla^\vee)$ of (E, ∇). In this way, $\mathfrak{Mod}(D^{(m)}_{\mathscr{R}})$ is equipped with a structure of closed monoidal category.

2.4. **Varying levels.** Fix an integer $m \geq 0$ and an integer l with $0 \leq l \leq m+1$. Furthermore, we set

$$R^l := \bigcap_{j=1}^{p^{l-1}} \mathrm{Ker}(\partial_{\langle j \rangle}) \left(= \bigcap_{j=0}^{l-1} \mathrm{Ker}(\partial_{\langle p^j \rangle}) \right),$$

which is an R_0-subalgebra of R. Here, we obtain a sequence of inclusions between R_0-algebras

$$R^{m+1} \subseteq R^m \subseteq \cdots \subseteq R^1 \subseteq R^0 = R.$$

Note that, if $\mathscr{R}$ is an m-differential field, then R^l (for every l) forms a subfield of R. For a nonnegative integer j with $j + l \leq m + 1$ and an R^{j+l}-module E, we shall set $F_j^{(l)*}(E)$ to be the R^j-module defined as

$$F_j^{(l)*}(E) := R^j \otimes_{R^{j+l}} E.$$

For simplicity, we write $F^{(l)*}(E) := F_0^{(l)*}(E)$.

Also, for each $j \in \mathbb{Z}_{\geq 0}$, the endomorphism $\partial_{\langle jp^l \rangle}$ restricts to an R_0-linear endomorphism of R^l; we will abuse the notation by writing $\partial_{\langle jp^l \rangle}$ for this restriction. Then, the collection

$$\mathscr{R}^l := (R^l, \{\partial_{\langle jp^l \rangle}\}_{j \in \mathbb{Z}_{\geq 0}})$$

forms an $(m-l)$-differential ring over R_0. In particular, we obtain the R_0-algebra $D^{(m-l)}_{\mathscr{R}^l}$ and a sequence of inclusions

$$\left(R^{m+1} =\right) D^{(-1)}_{\mathscr{R}^{m+1}} \hookrightarrow D^{(0)}_{\mathscr{R}^{m}} \hookrightarrow D^{(1)}_{\mathscr{R}^{m-1}} \hookrightarrow \cdots \hookrightarrow D^{(m-1)}_{\mathscr{R}^{1}} \hookrightarrow D^{(m)}_{\mathscr{R}}.$$

Next, given a $D^{(m)}_{\mathscr{R}}$-module (E, ∇), we shall write

$$E^l := \bigcap_{j=1}^{p^{l-1}} \mathrm{Ker}(\nabla_{\langle j \rangle}) \left(= \bigcap_{j=0}^{l-1} \mathrm{Ker}(\nabla_{\langle p^j \rangle}) \right),$$

where $E^0 := E$. In particular, we obtain a sequence of inclusions between modules

$$E^{m+1} \subseteq E^m \subseteq \cdots \subseteq E^1 \subseteq E^0 = E.$$

Note that E^l forms an R^l-module via the natural inclusion $R^l \hookrightarrow R$, and $\nabla_{\langle jp^l \rangle}$ (for each j) restricts to an R^l-linear endomorphism of E^l; we will abuse the notation by writing $\nabla_{\langle jp^l \rangle}$ for this restriction. One may verify that the collection

$$\nabla^l := \{\nabla_{\langle jp^l \rangle}\}_j$$

forms a $D^{(m-l)}_{\mathscr{R}^l}$-module structure on E^l. Also, if $f : (E, \nabla) \to (E', \nabla')$ is a morphism of $D^{(m)}_{\mathscr{R}}$-modules, then it restricts to a morphism of $D^{(m-l)}_{\mathscr{R}^l}$-modules $f^l : (E^l, \nabla^l) \to (E'^l, \nabla'^l)$. The resulting assignments $(E, \nabla) \mapsto (E^l, \nabla^l)$ and $f \mapsto f^l$ define a functor

$$\Xi^{\downarrow (l)} : \mathfrak{Mod}(D^{(m)}_{\mathscr{R}}) \to \mathfrak{Mod}(D^{(m-l)}_{\mathscr{R}^l}). \tag{2.2}$$

Conversely, given a $D^{(m-l)}_{\mathscr{R}^l}$-module (E, ∇), we can construct a $D^{(m)}_{\mathscr{R}}$-module structure $F^{(l)*}(\nabla)$ on $F^{(l)*}(E)$ given by

$$F^{(l)*}(\nabla)_{\langle j \rangle}(a \otimes v) := \sum_{j'+j''=j} \left\{ \begin{matrix} j \\ j' \end{matrix} \right\} \cdot \partial_{\langle j' \rangle}(a) \otimes \nabla_{\langle j''/p^j \rangle}(v)$$

for any $a \in R$ and $v \in E$, where $\nabla_{\langle s \rangle} := 0$ if $s \notin \mathbb{Z}_{\geq 0}$ (cf. [PBer2, Proposition 2.2.4, (ii)], [Mon, Proposition 3.4.1, (ii)]). In particular, for an R^{m+1}-module E', we obtain

$$\nabla^{\mathrm{can}}_{\mathscr{R}, E'} := F^{(m+1)*}(\nabla) : D^{(m)}_{\mathscr{R}} \to \mathrm{End}_{R_0}(F^{(m+1)*}(E')). \tag{2.3}$$

Each morphism of $D^{(m-l)}_{\mathscr{R}^l}$-modules $f : (E, \nabla) \to (E', \nabla')$ induces a morphism of $D^{(m)}_{\mathscr{R}}$-modules $F^{(l)*}(f) : (F^{(l)*}(E), F^{(l)*}(\nabla)) \to (F^{(l)*}(E'), F^{(l)*}(\nabla'))$. The resulting assignments $(E, \nabla) \mapsto (F^{(l)*}(E), F^{(l)*}(\nabla))$ and $f \mapsto F^{(l)*}(f)$ define a functor

$$\Xi^{\uparrow (l)} : \mathfrak{Mod}(D^{(m-l)}_{\mathscr{R}^l}) \to \mathfrak{Mod}(D^{(m)}_{\mathscr{R}}). \tag{2.4}$$

This functor is compatible with the formation of the tensor product and is left adjoint to $\Xi^{\downarrow (l)}$.

Here, let us describe the unit and counit morphisms for the adjunction "$\Xi^{\uparrow(l)} \dashv \Xi^{\downarrow(l)}$". If (E, ∇) is a $D^{(m)}_{\mathscr{R}}$-module, then the natural inclusion $E^l \hookrightarrow E$, which is $D^{(m-l)}_{\mathscr{R}^l}$-linear, extends to a morphism of $D^{(m)}_{\mathscr{R}}$-modules

$$\tau^{\downarrow\uparrow(l)}_{(E,\nabla)} : \left((\Xi^{\uparrow(l)} \circ \Xi^{\downarrow(l)})((E,\nabla)) =\right) (F^{(l)*}(E^l), F^{(l)*}(\nabla^l)) \to (E, \nabla). \tag{2.5}$$

On the other hand, if (E, ∇) is a $D^{(m-l)}_{\mathscr{R}^l}$-module, then the morphism $E \to F^{(l)*}(E)$ given by $v \mapsto 1 \otimes v$ restricts to a morphism of $D^{(m-l)}_{\mathscr{R}^l}$-modules

$$\tau^{\uparrow\downarrow(l)}_{(E,\nabla)} : (E, \nabla) \to (F^{(l)*}(E)^l, F^{(l)*}(\nabla)^l) \left(= (\Xi^{\downarrow(l)} \circ \Xi^{\uparrow(l)})((E,\nabla))\right).$$

The formation of $\tau^{\downarrow\uparrow(l)}_{(E,\nabla)}$ (resp., $\tau^{\uparrow\downarrow(l)}_{(E,\nabla)}$) is functorial with respect to (E, ∇).

2.5. **m-cyclic vectors.** Let m be a nonnegative integer. In this subsection, we introduce the notion of an m-cyclic vector, as a higher-level generalization of a cyclic vector.

Definition 2.5.1. Let (E, ∇) be a $D^{(m)}_{\mathscr{R}}$-module. An element v of E is called an **m-cyclic vector** of (E, ∇) if there exists a positive integer n such that the collection

$$\nabla_{\langle 0 \rangle}(v) \, (= v) \, , \nabla_{\langle 1 \rangle}(v), \cdots , \nabla_{\langle n-1 \rangle}(v)$$

forms a basis of the R-module E.

The following assertion follows immediately from the definition of an m-cyclic vector.

Lemma 2.5.2. *Let (E, ∇) be a $D^{(m)}_{\mathscr{R}}$-module, and suppose that there exists an m-cyclic vector of (E, ∇). Then, E is finite and free as an R-module.*

Remark 2.5.3. Suppose that $m = 0$ and $\mathscr{R}$ is non-logarithmic. Then, each $D^{(0)}_{\mathscr{R}}$-module structure ∇ on an R-module satisfies $\nabla_{\langle j \rangle} = \nabla^j_{\langle 1 \rangle}$ for every j. It follows that a 0-cyclic vector is the same as a cyclic vector in the classical sense.

Definition 2.5.4. (i) A **pinned $D^{(m)}_{\mathscr{R}}$-module** is a triple

$$(E, \nabla, v)$$

consisting of a $D^{(m)}_{\mathscr{R}}$-module (E, ∇) and an m-cyclic vector v of it. For a pinned $D^{(m)}_{\mathscr{R}}$-module (E, ∇, v), the **rank** of (E, ∇, v) is defined as the rank of the free R-module E.

(ii) Let (E, ∇, v) and (E', ∇', v') be pinned $D^{(m)}_{\mathscr{R}}$-modules. An **isomorphism of pinned $D^{(m)}_{\mathscr{R}}$-modules** from (E, ∇, v) to (E', ∇', v') is a morphism of $D^{(m)}_{\mathscr{R}}$-modules $f : (E, \nabla) \to (E', \nabla')$ with $f(v) = v'$.

Let us describe several basic properties of pinned $D^{(m)}_{\mathscr{R}}$-modules:

Proposition 2.5.5. (i) *Any morphism of pinned $D^{(m)}_{\mathscr{R}}$-modules is surjective.*

(ii) *Suppose that we are given a pinned $D^{(m)}_{\mathscr{R}}$-module (E, ∇, v) and a $D^{(m)}_{\mathscr{R}}$-module (E', ∇'). Denote by* $\mathrm{Hom}((E, \nabla), (E', \nabla'))$ *the set of morphisms of $D^{(m)}_{\mathscr{R}}$-modules from (E, ∇) to (E', ∇'). Then, the map of sets*

$$\mathrm{Hom}((E, \nabla), (E', \nabla')) \to E'$$

given by $f \mapsto f(v)$ is injective.

(iii) *Let (E,∇) be a $D^{(m)}_{\mathscr{R}}$-module, v an element of E, and m' an integer with $0 \leq m' \leq m$. Denote by $\nabla^{(m')}$ the $D^{(m')}_{\mathscr{R}}$-module structure on E induced from ∇ via the R_0-algebra homomorphism $D^{(m')}_{\mathscr{R}} \to D^{(m)}_{\mathscr{R}}$ (cf. Remark 2.3.2). Then, v forms an m-cyclic vector of (E,∇) if v forms an m'-cyclic vector of the $D^{(m')}_{\mathscr{R}}$-module $(E,\nabla^{(m')})$.*

Proof. To prove assertion (i), let us take a morphism of pinned $D^{(m)}_{\mathscr{R}}$-modules $f : (E,\nabla,v) \to (E',\nabla',v')$. Then, since $f(\nabla_{\langle j\rangle}(v)) = \nabla'_{\langle j\rangle}(f(v)) = \nabla'_{\langle j\rangle}(v')$ $(j = 0,1,2,\cdots)$, the assertion follows from the fact that the set of elements $\{\nabla'_{\langle j\rangle}(v')\}_{j\in\mathbb{Z}_{\geq 0}}$ generates E'.

The remaining assertions, i.e., (ii) and (iii), can be verified from the definition of an m-cyclic vector (we will omit the details). □

Now, let us prove the following theorem, asserting the existence of an m-cyclic vector in a general situation. Our proof is based on the proof of [ChKo, Theorem 3.11].

Theorem 2.5.6 (cf. Theorem A). *Let n be a positive integer and $\mathscr{R} := (R,\partial_{\langle\bullet\rangle})$ be an m-differential field over $\mathbb{F}_p$. Assume that the morphism $D^{(m)}_{\mathscr{R},<n} \to \mathrm{End}_{\mathbb{F}_p}(R)$ naturally induced by $\partial_{\langle\bullet\rangle}$ is injective. (This means that, for each nonzero element $D \in D^{(m)}_{\mathscr{R},<n}$, there exists an element a of R with $D(a) \neq 0$.) Then, each $D^{(m)}_{\mathscr{R}}$-module (E,∇) with $\mathrm{rk}(E) = n$ admits an m-cyclic vector.*

Proof. It suffices to consider the case of $n > 1$. Suppose that a nonzero element v of E is not an m-cyclic vector. Then, there exists an integer l with $1 \leq l < n$ such that

$$v \wedge \nabla_{\langle 1\rangle}(v) \wedge \cdots \wedge \nabla_{\langle l-1\rangle}(v) \neq 0 \quad \text{and} \quad \nabla_{\langle l\rangle}(v) = \sum_{j=0}^{l-1} a_j \cdot \nabla_{\langle j\rangle}(v) \tag{2.6}$$

for some $a_0,\cdots,a_{l-1} \in R$. For simplicity, we set $v_j := \nabla_{\langle j\rangle}(v)$ $(j = 0,1,\cdots,l-1)$. Choose an element u of E not in the span of $\{v_0,v_1,\cdots,v_{l-1}\}$. We extend the R-lineary independent set $\{v_0,v_1,\cdots,v_{l-1}\}$ to a basis of E by first adjoining u, and then, if necessary, some elements $e_1,\cdots,e_{n-l-1}$ of E. For each integer j with $0 \leq j \leq l$, we shall write

$$\nabla_{\langle j\rangle}(u) = \sum_{i=0}^{l-1} \alpha_{ji} \cdot v_i + \beta_j \cdot u + \sum_{i=1}^{n-l-1} \gamma_{ji} \cdot e_i,$$

where $\alpha_{ji},\beta_j,\gamma_{ji} \in R$. In particular, $\alpha_{0i} = \gamma_{0i} = 0$ and $\beta_0 = 1$.

For each integer r with $0 \leq r \leq l$, write L_r for the $\mathbb{F}_p$-linear endomorphism of R given by

$$L_r := \sum_{i=0}^{r} \left\{ \begin{matrix} r \\ i \end{matrix} \right\} \cdot \beta_i \cdot \partial_{\langle r-i\rangle}$$

(hence $L_0 = \mathrm{id}_R$). Next, let us define L to be the $\mathbb{F}_p$-linear endomorphism of R given by

$$L := L_l - \sum_{r=0}^{l-1} a_r \cdot L_r = \partial_{\langle l\rangle} + c_{l-1} \cdot \partial_{\langle l-1\rangle} + \cdots + c_1 \cdot \partial_{\langle 1\rangle} + c_0,$$

where

$$c_i := \left\{ \begin{matrix} l \\ l-i \end{matrix} \right\} \cdot \beta_{l-i} - \sum_{r=i}^{l-1} \left\{ \begin{matrix} r \\ r-i \end{matrix} \right\} \cdot a_r \cdot \beta_{r-i}$$

$(i = 0, \cdots, l-1)$. Since the operator L defines an element of $D^{(m)}_{\mathscr{R},<n}$, the injectivity assumption of the morphism $D^{(m)}_{\mathscr{R},<n} \to \mathrm{End}_{\mathbb{F}_p}(R)$ implies that there exists an element z of R with $L(z) \neq 0$.

Let us choose an indeterminate λ over R. Extend the m-derivation $\partial_{\langle \bullet \rangle}$ on R to the rational function field $R(\lambda)$ by defining $\partial_{\langle j \rangle}(\lambda) = 0$ $(j = 1, 2, \cdots)$. To be precise, this m-derivation can be obtained by first defining $\partial_{\langle \bullet \rangle}$ on $R[\lambda] = R \otimes_{R^{m+1}} R^{m+1}[\lambda]$ by $\partial_{\langle l \rangle}(a \otimes b) = \partial_{\langle l \rangle}(a) \otimes b$ and then extending via the quotient rule. The tensor product $E^\lambda := R(\lambda) \otimes_R E$ has the natural $D^{(m)}_{(R(\lambda), \partial_{\langle \bullet \rangle})}$-module structure obtained by defining

$$\nabla_{\langle j \rangle}(a \otimes w) = \sum_{j'+j''=j} \left\{ \begin{matrix} j \\ j' \end{matrix} \right\} \cdot \partial_{\langle j' \rangle}(a) \otimes \nabla_{\langle j'' \rangle}(w)$$

$(j = 0, 1, 2, \cdots)$ for any $a \in R(\lambda)$ and $w \in E$. Here, we set

$$\hat{v} := v + \lambda \cdot z \cdot u \, (= 1 \otimes v + z \cdot \lambda \otimes u) \in E^\lambda.$$

For each integer r with $0 \leq r \leq l$, we have

$$\begin{aligned} &\nabla_{\langle r \rangle}(\hat{v}) \qquad (2.7) \\ &= \nabla_{\langle r \rangle}(v) + \lambda \cdot \sum_{j=0}^{r} \left\{ \begin{matrix} r \\ j \end{matrix} \right\} \cdot \partial_{\langle r-j \rangle}(z) \cdot \nabla_{\langle j \rangle}(u) \\ &= v_r + \lambda \cdot \sum_{j=0}^{r} \left\{ \begin{matrix} r \\ j \end{matrix} \right\} \cdot \partial_{\langle r-j \rangle}(z) \cdot \left(\sum_{i=0}^{l-1} \alpha_{ji} \cdot v_i + \beta_j \cdot u + \sum_{i=1}^{n-l-1} \gamma_{ji} \cdot e_i \right) \\ &= v_r + \lambda \cdot \sum_{i=0}^{l-1} \sum_{j=0}^{r} \left\{ \begin{matrix} r \\ j \end{matrix} \right\} \cdot \partial_{\langle r-j \rangle}(z) \cdot \alpha_{ji} \cdot v_i \\ &\quad + \lambda \cdot L_r(z) \cdot u + \lambda \cdot \sum_{i=1}^{n-l-1} \sum_{j=0}^{r} \left\{ \begin{matrix} r \\ j \end{matrix} \right\} \cdot \partial_{\langle r-j \rangle}(z) \cdot \gamma_{ji} \cdot e_i \\ &= v_r + \lambda \cdot \sum_{i=0}^{l-1} \theta_{ri} \cdot v_i + \lambda \cdot L_r(z) \cdot u + \lambda \cdot \sum_{i=1}^{n-l-1} \theta'_{ri} \cdot e_i \end{aligned}$$

for some $\theta_{ri}, \theta'_{ri} \in R$. Similarly, it follows from (2.6) that

$$\begin{aligned} &\nabla_{\langle l \rangle}(\hat{v}) \qquad (2.8) \\ &= \nabla_{\langle l \rangle}(v) + \lambda \cdot \sum_{j=0}^{l} \left\{ \begin{matrix} l \\ j \end{matrix} \right\} \cdot \partial_{\langle l-j \rangle}(z) \cdot \nabla_{\langle j \rangle}(u) \\ &= \sum_{i=0}^{l-1} a_i \cdot v_i + \lambda \cdot \sum_{j=0}^{l} \left\{ \begin{matrix} l \\ j \end{matrix} \right\} \cdot \partial_{\langle l-j \rangle}(z) \cdot \left(\sum_{i=0}^{l-1} \alpha_{ji} \cdot v_i + \beta_j \cdot u + \sum_{i=1}^{n-l-1} \gamma_{ji} \cdot e_i \right) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=0}^{l-1} a_i \cdot v_i + \lambda \cdot \sum_{i=0}^{l-1} \sum_{j=0}^{l} \begin{Bmatrix} l \\ j \end{Bmatrix} \cdot \partial_{\langle l-j \rangle}(z) \cdot \alpha_{ji} \cdot v_i \\
&\quad + \lambda \cdot L_l(z) \cdot u + \lambda \cdot \sum_{i=1}^{n-l-1} \sum_{j=0}^{l} \begin{Bmatrix} l \\ j \end{Bmatrix} \cdot \partial_{\langle l-j \rangle}(z) \cdot \gamma_{ji} \cdot e_i \\
&= \sum_{i=0}^{l-1} a_i \cdot v_i + \lambda \cdot \sum_{i=0}^{l-1} \theta_{li} \cdot v_i + \lambda \cdot L_l(z) \cdot u + \lambda \cdot \sum_{i=1}^{n-l-1} \theta'_{li} \cdot e_i
\end{aligned}
$$

for some $\theta_{li}, \theta'_{li} \in R$.

Now, let us consider the vector

$$
\hat{w} := \hat{v} \wedge \nabla_{\langle 1 \rangle}(\hat{v}) \wedge \cdots \wedge \nabla_{\langle l \rangle}(\hat{v}) \in \bigwedge\nolimits_{R(\lambda)}^{l+1} E^{\lambda}.
$$

Under the natural identification $\bigwedge_{R(\lambda)}^{l+1} E^{\lambda} = \left(\bigwedge_{R}^{l+1} E\right) \otimes_R R(\lambda)$, we can write

$$
\hat{w} = w_0 + w_1 \cdot \lambda + w_2 \cdot \lambda^2 + \cdots + w_{l+1} \cdot \lambda^{l+1}
$$

for some $w_0, \cdots, w_{l+1} \in \bigwedge_R^{l+1} E$. Since $\hat{w}|_{\lambda=0} = 0$, we have $w_0 = 0$. By (2.7) and (2.8), the coefficient of $v_0 \wedge v_1 \wedge \cdots \wedge v_{l-1} \wedge u$ for w_1 is given by

$$
\begin{aligned}
&\left(\sum_{r=0}^{l-1} v_0 \wedge \cdots \wedge v_{r-1} \wedge (L_r(z) \cdot u) \wedge v_{r+1} \wedge \cdots \wedge v_{l-1} \wedge \sum_{j=0}^{l-1} a_j \cdot v_j \right) \\
&\quad + v_0 \wedge \cdots \wedge v_{l-1} \wedge (L_l(z) \cdot u) \\
&= \left(\sum_{r=0}^{l-1} a_r \cdot L_r(z) \cdot v_0 \wedge \cdots \wedge v_{r-1} \wedge u \wedge v_{r+1} \wedge \cdots \wedge v_{l-1} \wedge v_r \right) \\
&\quad + L_l(z) \cdot v_0 \wedge \cdots \wedge v_{l-1} \wedge u \\
&= \left(L_l(z) - \sum_{r=0}^{l-1} a_r \cdot L_r(z) \right) \cdot v_0 \wedge \cdots \wedge v_{l-1} \wedge u \\
&= L(z) \cdot v_0 \wedge \cdots \wedge v_{l-1} \wedge u \, (\neq 0) \, .
\end{aligned}
$$

It follows that the coefficient of $v_0 \wedge v_1 \wedge \cdots \wedge v_{l-1} \wedge u$ for $\hat{w}$ is a nonzero polynomial in $R[\lambda]$ whose degree is at most $l+1$ and whose constant term is 0. Since the injectivity of $D^{(m)}_{\mathscr{R},<n} \to \mathrm{End}_{\mathbb{F}_p}(R)$ implies $\partial_{\langle 1 \rangle} \neq 0$, we see that R is not a finite field, and that R^{m+1} has at least $n \, (> l)$ nonzero elements. Hence, there exists an element $\lambda_0 \in R^{m+1}$ which is not a zero of that polynomial. Then, the element $\overline{v} := v + \lambda_0 \cdot z \cdot u \in E$ satisfies $\overline{v} \wedge \nabla_{\langle 1 \rangle}(\overline{v}) \wedge \cdots \wedge \nabla_{\langle l \rangle}(\overline{v}) \neq 0$. By repeating the procedure for constructing $\overline{v}$ using v, we obtain an m-cyclic vector of (E, ∇). This completes the proof of this assertion. □

3. Dormant differential modules

This section deals with the p^{m+1}-curvature of a differential module of level $m \geq 0$. In particular, we focus on differential modules with vanishing p^{m+1}-curvature, which will be called *dormant* differential modules. At the end of this section, we provide a functorial construction of duality between dormant pinned differential modules of rank n (with $0 < n < p^{m+1}$) and those of rank $p^{m+1} - n$ (cf. Theorem 3.3.2, Corollary 3.3.3).

3.1. **p^{m+1}-curvature and dormant differential modules.** Let us fix an integer $m \geq 0$ and a field K of characteristic p. Since a perfect field of characteristic p has only the zero derivation, we should impose the condition that $K \neq K^{(1)} := \{a^p \mid a \in K\}$. In particular, suppose here that $[K : K^{(1)}] = p$ and there exists a discrete valuation ring R whose fraction field coincides with K. Examples of fields K satisfying this condition are $k(t)$ and $k(\!(t)\!)$ with k a perfect field of characteristic p (cf. Example 2.2.2).

For simplicity, we write

$$D_S^{(m)} := D^{(m)}_{(S, \partial_{\langle \bullet \rangle})} \quad \text{and} \quad \breve{D}_S^{(m)} := D^{(m)}_{(S, \breve{\partial}_{\langle \bullet \rangle})}, \tag{3.1}$$

where $S \in \{R, K\}$. Let "$\dot{(-)}$" denote either the absence or presence of "$\breve{(-)}$". Each $\dot{D}_R^{(m)}$-module structure ∇ on an R-module E naturally extends to a $\dot{D}_K^{(m)}$-module structure $\nabla_{\otimes K}$ on $K \otimes_R E$. The assignment $(E, \nabla) \mapsto (K \otimes_R E, \nabla_{\otimes K})$ defines a functor

$$\dot{\kappa} : \mathfrak{Mod}(\dot{D}_R^{(m)}) \rightarrow \mathfrak{Mod}(\dot{D}_K^{(m)}).$$

Next, let $S \in \{R, K\}$. For a $D_S^{(m)}$-module (E, ∇), the collection $\{t^j \cdot \nabla_{\langle j \rangle}\}_j$ determines a structure of $\breve{D}_S^{(m)}$-module on E. Conversely, suppose that we are given a $\breve{D}_S^{(m)}$-bundle $(E, \breve{\nabla})$ such that, for every $j \in \mathbb{Z}_{\geq 0}$, $\breve{\nabla}_{\langle j \rangle}$ may be expressed as $\breve{\nabla}_{\langle j \rangle} = t^j \cdot \nabla_{\langle j \rangle}$ for some $\nabla_{\langle j \rangle} \in \mathrm{End}_{S^{(m+1)}}(E)$. Then, the collection $\{\nabla_{\langle j \rangle}\}_j$ determines a structure of $D_S^{(m)}$-module on E. The assignment $(E, \nabla) \mapsto (E, \{t^j \cdot \nabla_{\langle j \rangle}\}_j)$ defines a functor

$$\eta_S : \mathfrak{Mod}(D_S^{(m)}) \rightarrow \mathfrak{Mod}(\breve{D}_S^{(m)}), \tag{3.2}$$

and it becomes an equivalence of categories for $S = K$. Moreover, the following square diagram of categories is 1-commutative:

$$\begin{array}{ccc} \mathfrak{Mod}(D_R^{(m)}) & \xrightarrow{\kappa} & \mathfrak{Mod}(D_K^{(m)}) \\ {\scriptstyle \eta_R} \downarrow & & \wr \downarrow {\scriptstyle \eta_K} \\ \mathfrak{Mod}(\breve{D}_R^{(m)}) & \xrightarrow[\breve{\kappa}]{} & \mathfrak{Mod}(\breve{D}_K^{(m)}). \end{array}$$

Remark 3.1.1. Let a be an integer and (E, ∇) a $\breve{D}_R^{(m)}$-module such that the R-module E is free. Then, ∇ naturally induces a $\breve{D}_R^{(m)}$-module structure on the R-module $t^a \cdot E \subseteq K \otimes_R E$, which will be denoted by $\nabla|_{t^a \cdot E}$.

Remark 3.1.2. Let E be a $S^{(m+1)}$-module. Then, since $\Xi^{\uparrow (m+1)}$ (cf. (2.4)) is compatible with η_S (cf. (3.2)), $\nabla^{\mathrm{can}}_{(S, \breve{\partial}_{\langle \bullet \rangle}), E}$ comes from the $D_S^{(m+1)}$-module structure $\nabla^{\mathrm{can}}_{(S, \partial_{\langle \bullet \rangle}), E}$ via η_S. This means that the following equality holds:

$$\nabla^{\mathrm{can}}_{(S,\breve{\partial}_{\langle\bullet\rangle}),E} = \{t^j(\nabla^{\mathrm{can}}_{(S,\partial_{\langle\bullet\rangle}),E})_{\langle j\rangle}\}_j.$$

Let (E,∇) be a $D_S^{(m)}$-module (resp., a $\breve{D}_S^{(m)}$-module), where $S\in\{R,K\}$. The p^{m+1}**-curvature** of (E,∇) is defined as

$${}^p\psi_{(E,\nabla)} := \nabla_{\langle p^{m+1}\rangle} \in \mathrm{End}_{S^{(m+1)}}(E).$$

It can immediately be seen that ${}^p\psi_{(E,\nabla)}$ belongs to $\mathrm{End}_R(E)$.

Definition 3.1.3. With the above notation, we shall say that (E,∇) is **dormant** if the equality

$${}^p\psi_{(E,\nabla)} = 0$$

holds. Also, a pinned $\dot{D}_S^{(m)}$-module (E,∇,v) is called **dormant** if (E,∇) is dormant.

Remark 3.1.4. Let us consider the case of $m=0$. Then, the p^1-curvature ${}^p\psi_{(E,\nabla)}$ of a $D_S^{(0)}$-module (E,∇) coincides with the p-curvature of a differential module over the differential ring $(S,\partial_{\langle 1\rangle})$ (cf. Remark 2.3.1 and [vdPS, Section 13.1]). It is well-known from [Kat1, Theorem (5.1)] that the functors $\Xi^{\downarrow(1)}$ and $\Xi^{\uparrow(1)}$ define an equivalence of categories

$$\begin{pmatrix}\text{the category of}\\ \text{dormant } D_S^{(0)}\text{-modules}\end{pmatrix} \xrightarrow{\sim} \begin{pmatrix}\text{the category of}\\ S^{(1)}\text{-modules}\end{pmatrix}. \tag{3.3}$$

This equivalence for $S=K$ can also be found in [vdPS, Lemma 13.2].

The following assertion is essentially not a new result because it may be regarded as a version of [PBer2, Théorèm 2.3.6] for differential modules.

Proposition 3.1.5. *Let l be an integer with $0\leq l\leq m$ and let $S\in\{R,K\}$. Then, the following assertions hold:*

(i) *The functors $\Xi^{\downarrow(l)}$ (cf. (2.2)) and $\Xi^{\uparrow(l)}$ (cf. (2.4)) define an equivalence of categories*

$$\mathfrak{Mod}(D_S^{(m)}) \xrightarrow{\sim} \mathfrak{Mod}(D_{S^{(l)}}^{(m-l)}). \tag{3.4}$$

(ii) *Let (E,∇) be a $D_{S^{(l)}}^{(m-l)}$-module. Then, the p^{m-l+1}-curvature ${}^p\psi_{(E,\nabla)}$ of (E,∇) and the p^{m+1}-curvature ${}^p\psi_{(F^{(l)*}(E),F^{(l)*}(\nabla))}$ of $(F^{(l)*}(E),F^{(l)*}(\nabla))$ $\big(=\Xi^{\uparrow(l)}((E,\nabla))\big)$ satisfy the equality*

$${}^p\psi_{(F^{(l)*}(E),F^{(l)*}(\nabla))} = \mathrm{id}_R\otimes{}^p\psi_{(E,\nabla)} \in \mathrm{End}_R(F^{(l)*}(E)).$$

In particular, (E,∇) is dormant if and only if $(F^{(l)*}(E),F^{(l)*}(\nabla))$ is dormant.

Proof. First, we shall prove assertion (i) by induction on l. The base step, i.e., $l=0$, is trivial. For the induction step, let us take a $D_S^{(m)}$-module (E,∇). This $D_S^{(m)}$-module induces a $D_{S^{(1)}}^{(m-1)}$-module of the form (E^1,∇^1) $\big(=\Xi^{\downarrow(1)}((E,\nabla))\big)$. Since $(E^1)^{l-1}=E^l$, the induction hypothesis implies that the morphism $\tau^{\downarrow\uparrow(l-1)}_{(E^1,\nabla^1)} : F_1^{(l-1)*}(E^l)\to E^1$ (cf. (2.5)) is an isomorphism. On the other hand, the p-curvature of the differential module $(E,\nabla_{\langle 1\rangle})$ vanishes, so it follows from the equivalence of categories (3.3) that the morphism $\tau_1 : F^{(1)*}(E^1)\xrightarrow{\sim} E$ extending the inclusion

$E^1 \hookrightarrow E$ is an isomorphism. Hence, $\tau^{\downarrow\uparrow(l)}_{(E,\nabla)}$ turns out to be an isomorphism because it coincides with the composite isomorphism

$$F^{(l)*}(E^l)\left(= F^{(1)*}(F_1^{(l-1)*}(E^l))\right) \xrightarrow{\Xi^{\uparrow(1)*}(\tau^{\downarrow\uparrow(l-1)}_{(E^1,\nabla^1)})} F^{(1)*}(E^1) \xrightarrow{\tau_1} E.$$

Since $\tau^{\downarrow\uparrow(l)}_{(E,\nabla)}$ is functorial with respect to (E,∇), we see that the composite functor $\Xi^{\uparrow(l)} \circ \Xi^{\downarrow(l)}$ is isomorphic to the identity functor of $\mathfrak{Mod}(D_S^{(m)})$.

Next, let (E,∇) be a $D^{(m-l)}_{S^l}$-module. By applying $\Xi^{\uparrow(l-1)}$ to (E,∇), we obtain a $D^{(m-1)}_{S^{(1)}}$-module of the form $(F_1^{(l-1)*}(E), F_1^{(l-1)*}(\nabla))\left(= \Xi^{\uparrow(l-1)}((E,\nabla))\right)$. It follows from (3.3) again that the natural morphism $\tau_2 : F_1^{(l-1)*}(E) \to F^{(1)*}(F_1^{(l-1)*}(E))^1$ is an isomorphism. Also, the induction hypothesis implies that the morphism $\tau^{\uparrow\downarrow(l-1)}_{(E,\nabla)} : E \to F_1^{(l-1)*}(E)^{l-1}$ is an isomorphism. Hence, $\tau^{\uparrow\downarrow(l)}_{(E,\nabla)}$ turns out to be an isomorphism because it coincides with the composite isomorphism

$$E \xrightarrow{\tau^{\uparrow\downarrow(l-1)}_{(E,\nabla)}} F_1^{(l-1)*}(E)^{l-1} \xrightarrow{\Xi^{\downarrow(l-1)}(\tau_2)} \left((F^{(1)*}(F_1^{(l-1)*}(E))^1)^{l-1} =\right) F^{(l)*}(E)^l.$$

Since $\tau^{\uparrow\downarrow(l)}_{(E,\nabla)}$ is functorial with respect to (E,∇), the composite functor $\Xi^{\downarrow(l)} \circ \Xi^{\uparrow(l)}$ is isomorphic to the identity functor of $\mathfrak{Mod}(D^{(m-l)}_{S^{(l)}})$. This completes the proof of assertion (i).

Assertion (ii) can be verified immediately from the definitions of $p^{(-)}$-curvature and the functor $\Xi^{\uparrow(l)}$. □

Remark 3.1.6. By the equivalence (3.4) resulting from the above proposition, $D_R^{(m)}$-modules are equivalent to $D^{(0)}_{R^{(m)}}$-modules, i.e., differential modules in the classical sense. (But, as we will see in the next section, this is not true for the logarithmic case, i.e., $\breve{D}_R$-modules.)

Moreover, the above theorem for $l = m$ and the equivalence of categories (3.3) (in the case where S is replaced by $S^{(m)}$) together imply the following assertion, which is already obtained in [LeQu, Corollary 3.2.4] and [Kin1, Proposition 1.1.6].

Corollary 3.1.7. *Let $S \in \{R, K\}$. Then, the functors $\Xi^{\downarrow(m+1)}$ and $\Xi^{\uparrow(m+1)}$ (i.e., the assignments $(E,\nabla) \mapsto E^{m+1}$ and $E' \mapsto (F^{(m+1)*}(E'), \nabla^{\mathrm{can}}_{(S,\partial_{\langle\bullet\rangle}),E'})$) induce an equivalence of categories*

$$\begin{pmatrix} \textit{the category of} \\ \textit{dormant } D_S^{(m)}\textit{-modules} \end{pmatrix} \xrightarrow{\sim} \begin{pmatrix} \textit{the category of} \\ S^{(m+1)}\textit{-modules} \end{pmatrix}. \tag{3.5}$$

3.2. **Dormant pinned $D_R^{(m)}$-module of rank p^{m+1}.** Let $S \in \{R, K\}$, and let "$\dot{(-)}$" denote either the absence or presence of "$\breve{(-)}$". Here, let us construct an example of a dormant pinned $D_S^{(m)}$-module of rank p^{m+1}. The S-module $\dot{P}_S := \dot{D}_S^{(m)}/\dot{D}_S^{(m)} \cdot \dot{\partial}_{\langle p^{m+1}\rangle}$ has the a $\dot{D}_S^{(m)}$-module structure $\nabla_{\dot{P}_S}$ induced from the left $\dot{D}_S^{(m)}$-module structure of $\dot{D}_S^{(m)}$ itself. One can verify that $(\dot{P}_S, \nabla_{\dot{P}_S})$ is dormant. If $\dot{\delta}_{\langle l\rangle}$ $(l = 0, \cdots, p^{m+1}-1)$ is the image of $\dot{\partial}_{\langle l\rangle}$ via the quotient $\dot{D}_S^{(m)} \twoheadrightarrow \dot{P}_S$, then we have $\dot{P}_S = \bigoplus_{l=0}^{p^{m+1}-1} S \cdot \dot{\delta}_{\langle l\rangle}$. Hence, the triple

$$(\dot{P}_S, \nabla_{\dot{P}_S}, v_{\dot{P}_S}), \tag{3.6}$$

where $v_{\dot{P}_S} := \dot{\delta}_{\langle 0 \rangle}$, forms a dormant pinned $\dot{D}_S^{(m)}$-module of rank p^{m+1}.

Next, let (E, ∇, v) be a dormant pinned $\breve{D}_S^{(m)}$-module. The S-linear injection $S \hookrightarrow E$ given by $a \mapsto a \cdot v$ (for any $a \in S$) extends to a $\dot{D}_S^{(m)}$-linear morphism $\widetilde{\nu}_{(E,\nabla,v)} : \left(\dot{D}_S^{(m)} \otimes_S S =\right) \dot{D}_S^{(m)} \to E$. This morphism preserves the $\dot{D}_S^{(m)}$-action. Since (E, ∇) has vanishing p^{m+1}-curvature, $\widetilde{\nu}_{(E,\nabla,v)}$ factors through the quotient $\dot{D}_S^{(m)} \twoheadrightarrow \dot{P}_S$. Thus, $\widetilde{\nu}_{(E,\nabla,v)}$ induces a morphism

$$\nu_{(E,\nabla,v)} : \dot{P}_S \twoheadrightarrow E, \tag{3.7}$$

which forms a morphism of pinned $\dot{D}_S^{(m)}$-modules $(\dot{P}_S, \nabla_{\dot{P}_S}, v_{\dot{P}_S}) \to (E, \nabla, v)$. It follows from Proposition 2.5.5, (i), that $\nu_{(\mathcal{E},\nabla,v)}$ is surjective. By combining Theorem 2.5.6 with the following proposition (in the case where $S = K$ and "$\dot{(-)}$" denotes the absence of $\breve{(-)}$), we see that the existence of an m-cyclic vector for a dormant $D_K^{(m)}$-module depends only on the rank of the underlying K-vector space.

Proposition 3.2.1. *Let n be a positive integer and (E, ∇) a dormant $\dot{D}_S^{(m)}$-module such that the S-module E is free and of rank n. Then, the following assertions hold:*

(i) *If the inequality $n > p^{m+1}$ holds, then there are no m-cyclic vectors of (E, ∇).*
(ii) *If the equality $n = p^{m+1}$ holds and there exists an m-cyclic vector of (E, ∇), then $\nu_{(E,\nabla,v)}$ is an isomorphism. In particular, the isomorphism class of a dormant $\dot{D}_S^{(m)}$-module whose underlying S-module is free and of rank p^{m+1} is uniquely determined, i.e., the class represented by $(\dot{P}_S, \nabla_{\dot{P}_S}, v_{\dot{P}_S})$.*

Proof. The assertions follow immediately from the surjectivity of the morphism $\nu_{(\mathcal{E},\nabla,v)}$ for each m-cyclic vector v of (E, ∇). □

Example 3.2.2. Let us consider the dual of $\dot{P}_S \left(= \bigoplus_{l=0}^{p^{m+1}-1} S \cdot \dot{\delta}_{\langle l \rangle}\right)$. Denote the dual basis of $\dot{\delta}_{\langle 0 \rangle}, \cdots, \dot{\delta}_{\langle p^{m+1}-1 \rangle}$ by $\dot{\delta}^\vee_{\langle 0 \rangle}, \cdots, \dot{\delta}^\vee_{\langle p^{m+1}-1 \rangle}$. From the definition of $\nabla_{\dot{P}_S}$, the element $v_{\dot{P}_S^\vee} := \dot{\delta}^\vee_{\langle p^{m+1}-1 \rangle}$ defines an m-cyclic vector of the dual $\dot{D}_S^{(m)}$-module $(\dot{P}_S^\vee, \nabla^\vee_{\dot{P}_S})$ of $(\dot{P}_S, \nabla_{\dot{P}_S})$. Hence, we obtain the dormant pinned $\dot{D}_S^{(m)}$-module

$$(\dot{P}_S^\vee, \nabla^\vee_{\dot{P}_S}, v_{\dot{P}_S^\vee}). \tag{3.8}$$

Since the free S-module $\dot{P}_S^\vee$ is of rank p^{m+1}, it follows from Proposition 3.2.1, (ii), that the induced morphism

$$\nu_{(\dot{P}_S^\vee, \nabla^\vee_{\dot{P}_S}, v_{\dot{P}_S^\vee})} : (\dot{P}_S, \nabla_{\dot{P}_S}, v_{\dot{P}_S}) \to (\dot{P}_S^\vee, \nabla^\vee_{\dot{P}_S}, v_{\dot{P}_S^\vee}) \tag{3.9}$$

defines an isomorphism of pinned $\dot{D}_S^{(m)}$-modules.

3.3. **Duality of dormant pinned $D_R^{(m)}$-modules.** Let us keep the above notation. Also, let us denote by ∇_{Ker} the $\dot{D}_S^{(m)}$-module structure on $\mathrm{Ker}(\nu_{(E,\nabla,v)})$ obtained by restricting $\nabla_{\dot{P}_S}$. The $\dot{D}_S^{(m)}$-module $(\mathrm{Ker}(\nu_{(E,\nabla,v)})), \nabla_{\mathrm{Ker}})$ has vanishing p^{m+1}-curvature, so do its dual $(E^\blacktriangledown, \nabla^\blacktriangledown) := (\mathrm{Ker}(\nu_{(E,\nabla,v)}))^\vee, \nabla^\vee_{\mathrm{Ker}})$. We shall write $v^\blacktriangledown$ for the element of $E^\blacktriangledown$ defined to be the image of 1 via the dual of the composite

$$\mathrm{Ker}(\nu_{(E,\nabla,v)}) \xrightarrow{\text{inclusion}} \dot{P}_S \left(= \bigoplus_{l=0}^{p^{m+1}-1} S \cdot \dot{\delta}_{\langle l \rangle} \right) \to \left(S \cdot \dot{\delta}_{\langle p^{m+1}-1 \rangle} = \right) S,$$

where the second arrow denotes the projection to the last factor.

Lemma 3.3.1. *The element $v^\blacktriangledown$ forms an m-cyclic vector of $(E^\blacktriangledown, \nabla^\blacktriangledown)$.*

Proof. For each $j \in \mathbb{Z}_{\geq 0}$, we write

$$\dot{P}_{S,j} := \mathrm{Im}\left(\dot{D}^{(m)}_{S,<j} \hookrightarrow \dot{D}^{(m)}_S \twoheadrightarrow \dot{P}_S \right) \left(= \bigoplus_{l=0}^{j-1} S \cdot \dot{\delta}_{\langle l \rangle} \right).$$

Let us set h to be the composite

$$h : \dot{P}_{S,n} \xrightarrow{\text{inclusion}} \dot{P}_S \xrightarrow{\nu_{(E,\nabla,v)}} E.$$

By the definition of $\nu_{(E,\nabla,v)}$, we have

$$h(\dot{\delta}_{\langle j \rangle}) = h((\nabla_{\dot{P}_S})_{\langle j \rangle}(\dot{\delta}_{\langle 0 \rangle})) = \nabla_{\langle j \rangle}(h(\dot{\delta}_{\langle 0 \rangle})) = \nabla_{\langle j \rangle}(v)$$

for every $j = 0, \cdots, n-1$. On the other hand, since $\{\dot{\delta}_{\langle j \rangle}\}_{j=0}^{n-1}$ and $\{\nabla_{\langle j \rangle}(v)\}_{j=0}^{n-1}$ form bases for $\dot{P}_{S,n}$ and E, respectively, h is an isomorphism. This implies that the composite

$$\lambda : \mathrm{Ker}(\nu_{(E,\nabla,v)}) \xrightarrow{\text{inclusion}} \dot{P}_S \twoheadrightarrow \dot{P}_S/\dot{P}_{S,n}$$

is an isomorphism. Since the element $v_{\dot{P}_S^\vee}$ of $\dot{P}_S^\vee$ (cf. Example 3.2.2) forms an m-cyclic vector, the elements $\left(v_{\dot{P}_S^\vee} =\right) \dot{\partial}^\vee_{\langle p^{m+1}-1 \rangle}, \cdots, \dot{\partial}^\vee_{\langle n \rangle}$ generate an S-submodule $(\dot{P}_S/\dot{P}_{S,n})^\vee \subseteq \dot{P}_S^\vee$. In particular, the elements $\lambda^\vee(\dot{\partial}_{\langle p^{m+1}-1 \rangle}), \cdots, \lambda^\vee(\dot{\partial}_{\langle n \rangle})$ generate $E^\blacktriangledown$, where $\lambda^\vee$ denotes the dual $(\dot{P}_S/\dot{P}_{S,n})^\vee \xrightarrow{\sim} E^\blacktriangledown \left(= \mathrm{Ker}(\nu_{(E,\nabla,v)})^\vee\right)$ of λ. On the other hand, it follows from the various definitions involved that the equality $\nabla^\blacktriangledown_{\langle j \rangle}(v^\blacktriangledown) = \lambda^\vee(\dot{\partial}^\vee_{\langle p^{m+1}-1-j \rangle})$ holds for every $j = 0, \cdots, p^{m+1} - n - 1$. Thus, $v^\blacktriangledown$ turns out to form an m-cyclic vector of $(E^\blacktriangledown, \nabla^\blacktriangledown)$. This completes the proof of the assertion. ☐

By the above lemma, we obtain a dormant pinned $\dot{D}^{(m)}_S$-module

$$(E^\blacktriangledown, \nabla^\blacktriangledown, v^\blacktriangledown) \tag{3.10}$$

of rank $p^m - n$, which we call the **dual** of (E, ∇, v).

Next, let us take a morphism $f : (E, \nabla, v) \to (E', \nabla', v')$ between dormant pinned $\dot{D}^{(m)}_S$-modules. The construction of $\nu_{(-)}$ yields the equality $f \circ \nu_{(E,\nabla,v)} = \nu_{(E',\nabla',v')}$. Hence, f restricts to the inclusion $f_{\mathrm{Ker}} : \mathrm{Ker}(\nu_{(E,\nabla,v)}) \hookrightarrow \mathrm{Ker}(\nu_{(E',\nabla',v')})$. Taking its dual gives a morphism of pinned $\dot{D}^{(m)}_S$-modules

$$f^\blacktriangledown : (E'^\blacktriangledown, \nabla'^\blacktriangledown, v'^\blacktriangledown) \to (E^\blacktriangledown, \nabla^\blacktriangledown, v^\blacktriangledown).$$

Here, we shall write

$$\mathfrak{Mod}(\dot{D}^{(m)}_S)^\circledast$$

for the category of dormant pinned $\dot{D}^{(m)}_S$-modules.

Theorem 3.3.2. (i) *The assignments $(E, \nabla, v) \mapsto (E^\blacktriangledown, \nabla^\blacktriangledown, v^\blacktriangledown)$ and $f \mapsto f^\blacktriangledown$ constructed above define a self-equivalence*

$$\dot{\mathcal{C}} : \mathfrak{Mod}(\dot{D}_S^{(m)})^\circledast \xrightarrow{\sim} \mathfrak{Mod}(\dot{D}_S^{(m)})^\circledast$$

of the category $\mathfrak{Mod}(\dot{D}_S^{(m)})^\circledast$ with $\dot{\mathcal{C}} \circ \dot{\mathcal{C}} = \mathrm{id}$. In particular, for each pinned $\dot{D}_S^{(m)}$-module (E, ∇, v), there exists an isomorphism of pinned $\dot{D}_S^{(m)}$-modules

$$(E, \nabla, v) \xrightarrow{\sim} (E^{\blacktriangledown\blacktriangledown}, \nabla^{\blacktriangledown\blacktriangledown}, v^{\blacktriangledown\blacktriangledown}).$$

(ii) *The following diagram of functors is 1-commutative:*

$$\begin{array}{ccccc}
\mathfrak{Mod}(D_R^{(m)})^\circledast & \xrightarrow{\eta_R^\circledcirc} & \mathfrak{Mod}(\breve{D}_R^{(m)})^\circledast & \xrightarrow{(\eta_K^{-1} \circ \breve{\kappa})^\circledcirc} & \mathfrak{Mod}(D_K^{(m)})^\circledast \\
\wr \downarrow \mathcal{C} & & \wr \downarrow \breve{\mathcal{C}} & & \wr \downarrow \mathcal{C} \\
\mathfrak{Mod}(D_R^{(m)})^\circledast & \xrightarrow[\eta_R^\circledcirc]{} & \mathfrak{Mod}(\breve{D}_R^{(m)})^\circledast & \xrightarrow[(\eta_K^{-1} \circ \breve{\kappa})^\circledcirc]{} & \mathfrak{Mod}(D_K^{(m)})^\circledast,
\end{array}$$

where $\eta_R^\circledcirc$ and $(\eta_K^{-1} \circ \breve{\kappa})^\circledcirc$ denote the functors induced, via restriction, from η_R and $\eta_K^{-1} \circ \breve{\kappa}$, respectively.

Proof. Let us prove assertion (i). Let (E, ∇, v) be a dormant pinned $D_S^{(m)}$-module. Then, it induces the following short exact sequence:

$$0 \longrightarrow \mathrm{Ker}(\nu_{(E,\nabla,v)}) \xrightarrow{\iota} \dot{P}_S \xrightarrow{\nu_{(E,\nabla,v)}} E \longrightarrow 0,$$

where we write ι for the natural inclusion. The dual of this sequence fits into the following diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Ker}(\nu_{(E^\blacktriangledown,\nabla^\blacktriangledown,v^\blacktriangledown)}) & \longrightarrow & \dot{P}_S & \xrightarrow{\nu_{(E^\blacktriangledown,\nabla^\blacktriangledown,v^\blacktriangledown)}} & E^\blacktriangledown & \longrightarrow & 0 \\
 & & & & \downarrow \nu_{(\dot{P}_S, \nabla_{\dot{P}_S}, v_{\dot{P}_S})} & & \wr \downarrow \mathrm{id} & & \\
0 & \longrightarrow & E^\vee & \xrightarrow[\nu^\vee_{(E,\nabla,v)}]{} & \dot{P}_S^\vee & \xrightarrow[\iota^\vee]{} & E^\blacktriangledown & \longrightarrow & 0,
\end{array}$$

where the right-hand square is commutative because of Proposition 2.5.5, (ii), together with the equality $(\iota^\vee \circ \nu_{(\dot{P}_S, \nabla_{\dot{P}_S}, v_{\dot{P}_S})})(v_{\dot{P}_S}) = (\mathrm{id} \circ \nu_{(E^\blacktriangledown,\nabla^\blacktriangledown,v^\blacktriangledown)})(v_{\dot{P}_S})$. Hence, this diagram induces an isomorphism $\mathrm{Ker}(\nu_{(E^\blacktriangledown,\nabla^\blacktriangledown,v^\blacktriangledown)}) \xrightarrow{\sim} E^\vee$. By taking the dual of this isomorphism, we obtain an isomorphism $\xi : E \xrightarrow{\sim} E^{\blacktriangledown\blacktriangledown}$. Note that $\nu_{(\dot{P}_S, \nabla_{\dot{P}_S}, v_{\dot{P}_S})}$ preserves the $\dot{D}_S^{(m)}$-action and it is compatible with the respective projections onto S, i.e., $v_{\dot{P}_S^\vee}$ and $v^\vee_{\dot{P}_S}$. This implies that ξ defines an isomorphism of pinned $\dot{D}_S^{(m)}$-modules $(E, \nabla, v) \xrightarrow{\sim} (E^\blacktriangledown, \nabla^{\blacktriangledown\blacktriangledown}, v^{\blacktriangledown\blacktriangledown})$. Moreover, the formation of this isomorphism is functorial with respect to (E, ∇, v), so it induces an isomorphism $\mathrm{id}_{\mathfrak{Mod}(\dot{D}_S^{(m)})^\circledast} \xrightarrow{\sim} \dot{\mathcal{C}} \circ \dot{\mathcal{C}}$ of functors. This completes the proof of assertion (i).

Assertion (ii) follows immediately from the construction of $\dot{\mathcal{C}}$. □

For each integer n with $1 \leq n \leq p^m - 1$, we shall write

$$\mathrm{Mod}(\dot{D}_S^{(m)})^\circledast_n$$

for the set of isomorphism classes of dormant pinned $\dot{D}_S^{(m)}$-modules of rank n. In the case of $n = 1$, the set $\mathrm{Mod}(\dot{D}_S^{(m)})_1^{\circledast}$ forms a group with a binary operation given by $((E, \nabla, v), (E', \nabla', v')) \mapsto (E \otimes E, \nabla \otimes \nabla', v \otimes v')$ (cf. Example 3.3.4 below).

The following assertion is a direct consequence of the above theorem.

Corollary 3.3.3. *The assignment* $(E, \nabla, v) \mapsto (E^{\blacktriangledown}, \nabla^{\blacktriangledown}, v^{\blacktriangledown})$ *defines a bijection of sets*

$$[\dot{\mathcal{U}}]_n : \mathrm{Mod}(\dot{D}_S^{(m)})_n^{\circledast} \xrightarrow{\sim} \mathrm{Mod}(\dot{D}_S^{(m)})_{p^{m+1}-n}^{\circledast} \tag{3.11}$$

satisfying $[\dot{\mathcal{U}}]_{p^{m+1}-n} \circ [\dot{\mathcal{U}}]_n = \mathrm{id}$.

Example 3.3.4. Let us examine the group $\mathrm{Mod}(D_S^{(m)})_1^{\circledast}$. For each element $a \in S^{\times}$, denote by μ_a the automorphism of the trivial S-module S given by multiplication by a. There exists a unique $D_S^{(m)}$-module structure $\mu_{a*}(\partial_{\langle \bullet \rangle})$ on S such that μ_a becomes an isomorphism of $D_S^{(m)}$-modules $(S, \partial_{\langle \bullet \rangle}) \xrightarrow{\sim} (S, \mu_{a*}(\partial_{\langle \bullet \rangle}))$. Then, the triple

$$(S, \mu_{a*}(\partial_{\langle \bullet \rangle}), 1)$$

forms a dormant pinned $D_S^{(m)}$-module of rank 1. Since $\Xi^{\uparrow(m+1)}(S^{(m+1)}) = (S, \partial_{\langle \bullet \rangle})$, the equivalence of categories (3.5) shows that any dormant pinned $D_S^{(m)}$-module of rank 1 is isomorphic to $(S, \mu_{a*}(\partial_{\langle \bullet \rangle}), 1)$ for some $a \in S^{\times}$. Also, for $a, b \in S^{\times}$, $(S, \mu_{a*}(\partial_{\langle \bullet \rangle}), 1) \cong (S, \mu_{b*}(\partial_{\langle \bullet \rangle}), 1)$ $(a, b \in S^{\times})$ if and only if there exists $c \in (S^{(m+1)})^{\times}$ satisfying $b = a \cdot c$. Hence, the assignment $a \mapsto (S, \mu_{a*}(\partial_{\langle \bullet \rangle}), 1)$ gives a well-defined bijection of sets

$$S^{\times}/(S^{(m+1)})^{\times} \xrightarrow{\sim} \mathrm{Mod}(D_S^{(m)})_1^{\circledast}. \tag{3.12}$$

Since $\mu_a \circ \mu_b = \mu_{a \cdot b}$, we have $(S, \mu_{a*}(\partial_{\langle \bullet \rangle}), 1) \otimes (S, \mu_{b*}(\partial_{\langle \bullet \rangle}), 1) \cong (S, \mu_{a \cdot b*}(\partial_{\langle \bullet \rangle}), 1)$. This implies that (3.12) becomes an isomorphism of groups.

Moreover, by composing with $[\mathcal{U}]_1$, we obtain a bijection

$$S^{\times}/(S^{(m+1)})^{\times} \xrightarrow{\sim} \mathrm{Mod}(D_S^{(m)})_{p^{m+1}-1}^{\circledast}.$$

4. Residues and exponents of log differential modules

Let m, K, R, t, and $\partial_{\langle \bullet \rangle}$ be as in the previous section. This section discusses the residue and the exponent of a dormant $\breve{D}_R^{(m)}$-module. (For the previous work dealing with related concepts, we refer to [Kin1].) In particular, we examine the exponent of a dormant $\breve{D}_R^{(m)}$-module admitting an m-cyclic vector (cf. Proposition 4.4.1).

4.1. **Residue of a $\breve{D}_R^{(m)}$-module.** Let (E, ∇) be a dormant $\breve{D}_R^{(m)}$-bundle such that the R-module E is free and of rank $n > 0$. For each $l = 0, \cdots, m$, the pair

$$(E^l, \overline{\nabla}^l), \tag{4.1}$$

where $\overline{\nabla}^l := \nabla_{\langle p^l \rangle}|_{E^l}$, determines a dormant $\breve{D}_{R^{(l)}}^{(0)}$-module (cf. Remark 2.3.1).

From the equivalence of categories (3.5), the injective morphism $\tau^{\downarrow\uparrow(m+1)}_{(E,\nabla)} : F^{(m+1)*}(E^{m+1}) \to E$ (cf (2.5)) becomes bijective after tensoring with K. Hence, the cokernel

$$\mathrm{Res}(\nabla) := \mathrm{Coker}(\tau^{\downarrow\uparrow(m+1)}_{(E,\nabla)})$$

of $\tau^{\downarrow\uparrow(m+1)}_{(E,\nabla)}$ is of finite length. We will refer to $\mathrm{Res}(\nabla)$ as the **residue** of ∇.

Note that there exists a natural sequence of inclusions

$$F^{(m+1)*}(E^{m+1}) \subseteq F^{(m)*}(E^{m}) \subseteq \cdots \subseteq F^{(1)*}(E^{1}) \subseteq F^{(0)*}(E^{0}) = E.$$

For each integer l with $0 \leq l \leq m+1$, we shall write

$$\mathrm{Res}(\nabla)^l := \mathrm{Im}\left(F^{(l)*}(E^l) \xrightarrow{\tau^{\downarrow\uparrow(l)}_{(E,\nabla)}} E \xrightarrow{\text{quotient}} \mathrm{Res}(\nabla)\right) (\subseteq \mathrm{Res}(\nabla)).$$

The natural surjection $F^{(l)*}(E^l) \twoheadrightarrow \mathrm{Res}(\nabla)^l$ induces an isomorphism of R-modules

$$F^{(l)*}(E^l)/F^{(m+1)*}(E^{m+1}) \xrightarrow{\sim} \mathrm{Res}(\nabla)^l. \tag{4.2}$$

Proposition 4.1.1. *The collection* $\{\mathrm{Res}(\nabla)^l\}_{0\leq l\leq m+1}$ *forms a decreasing filtration of* $\mathrm{Res}(\nabla)$ *satisfying that* $\mathrm{Res}(\nabla)^0 = \mathrm{Res}(\nabla)$, $\mathrm{Res}(\nabla)^{m+1} = 0$, *and*

$$\mathrm{Res}(\nabla)^l/\mathrm{Res}(\nabla)^{l+1} \cong F^{(l)*}(\mathrm{Res}(\overline{\nabla}^l))$$

for every $l = 0, \cdots, m$. *In particular, the following equality holds:*

$$\mathrm{length}_R(\mathrm{Res}(\nabla)) = \sum_{l=0}^{m} p^l \cdot \mathrm{length}_R(\mathrm{Res}(\overline{\nabla}^l)).$$

Proof. Let us prove the former assertion. It is clear that $\mathrm{Res}(\nabla)^0 = \mathrm{Res}(\nabla)$ and $\mathrm{Res}(\nabla)^{m+1} = 0$. Next, let us take $l \in \{0, \cdots, m\}$. The short exact sequence of $R^{(l)}$-modules

$$0 \longrightarrow F_l^{(1)*}(E^{l+1}) \longrightarrow E^l \longrightarrow \mathrm{Res}(\overline{\nabla}^l) \longrightarrow 0$$

induces, via application of the functor $F^{(l)*}(-)$, a short exact sequence

$$0 \longrightarrow F^{(l+1)*}(E^{l+1}) \left(= F^{(l)*}(F_l^{(1)*}(E^{l+1}))\right) \longrightarrow F^{(l)*}(E_l) \longrightarrow F^{(l)*}(\mathrm{Res}(\overline{\nabla}^l)) \longrightarrow 0.$$

Hence, the assertion follows from this sequence together with (4.2).

The latter assertion follows from the former one because

$$\begin{aligned}\mathrm{length}_R(\mathrm{Res}(\nabla)) &= \sum_{l=0}^{m} \mathrm{length}_R(\mathrm{Res}(\nabla)^l/\mathrm{Res}(\nabla)^{l+1}) \\ &= \sum_{l=0}^{m} \mathrm{length}_R(F^{(l)*}(\mathrm{Res}(\overline{\nabla}^l))) \\ &= \sum_{l=0}^{m} p^l \cdot \mathrm{length}_R(\mathrm{Res}(\overline{\nabla}^l)).\end{aligned}$$

This completes the proof of this proposition. ∎

Example 4.1.2. Let E be a free $R^{(m+1)}$-module of finite rank. Then, the natural morphism $E \to (F^{(m+1)*}(E))^{m+1}$ with respect to the $\breve{D}_R^{(m+1)}$-module structure $\nabla^{\mathrm{can}}_{(R,\breve{\partial}_{\langle\bullet\rangle}),E}$ (cf. (2.3)) is an isomorphism. This implies that the residue $\mathrm{Res}(\nabla^{\mathrm{can}}_{(R,\breve{\partial}_{\langle\bullet\rangle}),E})$ of $\nabla^{\mathrm{can}}_{(R,\breve{\partial}_{\langle\bullet\rangle}),E}$ vanishes.

4.2. **Dormant $\breve{D}_R^{(m)}$-modules of rank** 1. Let us take an element a of $\mathbb{Z}/p^{m+1}\mathbb{Z}$. Denote by $\widetilde{a}$ the integer defined as the unique lifting of a via the natural surjection $\mathbb{Z} \twoheadrightarrow \mathbb{Z}/p^{m+1}\mathbb{Z}$ satisfying $0 \leq \widetilde{a} < p^{m+1}$. Then, there exists a unique $\breve{D}_R^{(m)}$-module structure

$$\nabla_a := \{\nabla_{a,\langle j\rangle}\}_{j\in\mathbb{Z}_{\geq 0}} \tag{4.3}$$

on R satisfying $\nabla_{a,\langle j\rangle}(t^n) = q_j! \cdot \binom{n-\widetilde{a}}{j} \cdot t^n$ (where $\binom{n-\widetilde{a}}{j} := \frac{(n-\widetilde{a})\cdots(n-\widetilde{a}-j+1)}{j!}$) for every $j, n \in \mathbb{Z}_{\geq 0}$. In particular, we have $\nabla_0 = \breve{\partial}_{\langle\bullet\rangle}$. The $\breve{D}_R^{(m)}$-module (R, ∇_a) is isomorphic to the unique extension of the trivial $\breve{D}_R^{(m)}$-module $(R, \breve{\partial}_{\langle\bullet\rangle})$ to $t^{-\widetilde{a}} \cdot R\,(\subseteq K)$ (cf. Remark 3.1.1). It follows immediately that ∇_a has vanishing p^{m+1}-curvature.

Proposition 4.2.1. (i) *Let us express the integer $\widetilde{a}$ as $\widetilde{a} = \sum_{l=0}^{m} p^l \cdot \widetilde{a}_l$, where $0 \leq \widetilde{a}_l < p$ $(l = 0, \cdots, m)$. Then, the following equalities hold:*

$$\mathrm{length}_R(\mathrm{Res}(\nabla_a)) = \widetilde{a}, \quad \mathrm{length}_R(\mathrm{Res}(\nabla_a)^l/\mathrm{Res}(\nabla_a)^{l+1}) = p^l \cdot \widetilde{a}_l$$

$(l = 0, \cdots, m)$.

(ii) *Let $a, b \in \mathbb{Z}/p^{m+1}\mathbb{Z}$, and write $c := b - a$. Then, we have*

$$\mathrm{Hom}((R, \nabla_a), (R, \nabla_b)) = \{\mu_{s\cdot t^{\widetilde{c}}} \mid s \in (R^{(m+1)})^\times\},$$

where $\mu_{s'}$ (for each $s' \in R^{(m+1)}$) denotes the endomorphism of R given by multiplication by s'.

(iii) *For each $a, b \in \mathbb{Z}/p^{m+1}\mathbb{Z}$, we have*

$$(R, \nabla_a) \otimes (R, \nabla_b) \cong (R, \nabla_{a+b}), \quad (R^\vee, \nabla_a^\vee) \cong (R, \nabla_{-a}).$$

(iv) *Let (E, ∇) be a dormant $\breve{D}_R^{(m)}$-module with $E = R$. Then, it is isomorphic to (R, ∇_a) for a unique element a of $\mathbb{Z}/p^{m+1}\mathbb{Z}$.*

Proof. Assertion (i) follows from (4.2) together with the equality $\bigcap_{j=0}^{p^l} \mathrm{Ker}(\nabla_{a,\langle j\rangle}) = t^{\sum_{j=0}^{l} p^j \cdot \widetilde{a}_j} \cdot R^{l+1}\,(\subseteq R)$, which can be verified by induction on l. Assertions (ii) and (iii) follow immediately from the definition of ∇_a.

Here, we shall prove assertion (iv). Let us write $\widetilde{a} := \mathrm{length}_R(\mathrm{Res}(\nabla))$ and write a for the image of $\widetilde{a}$ via the quotient $\mathbb{Z} \twoheadrightarrow \mathbb{Z}/p^{m+1}\mathbb{Z}$. From Proposition 4.1.1 and [Oss4, Proposition 2.8], we have

$$\widetilde{a} = \sum_{l=0}^{m} p^l \cdot \mathrm{length}_R(\mathrm{Res}(\overline{\nabla}^l)) < \sum_{l=0}^{m} p^l \cdot (p-1) \leq p^{m+1}. \tag{4.4}$$

Since $\tau^{\downarrow\uparrow(m+1)}_{(E,\nabla)} : F^{(m+1)*}(E^{m+1}) \to E$ is injective and E is a free R-module of rank 1, the $R^{(m+1)}$-module E^{m+1} may be identified with $R^{(m+1)}$. This identification gives an identification $F^{(m+1)*}(E^{m+1}) = \big(F^{(m+1)*}(R^{(m+1)}) =\big)\, R$, by which the $\breve{D}_R^{(m+1)}$-module structure $\nabla^{\mathrm{can}}_{(R,\breve{\partial}_{\langle\bullet\rangle}),E^{m+1}}$ (cf. (2.3)) corresponds to the trivial one $\breve{\partial}_{\langle\bullet\rangle}$. Hence, since $\tau^{\downarrow\uparrow(m+1)}_{(E,\nabla)}$ is $\breve{D}_R^{(m+1)}$-linear, ∇ may be identified, via $\tau^{\downarrow\uparrow(m+1)}_{(E,\nabla)}$, with a unique $\breve{D}^{(m+1)}_R$-module structure on $t^{-\widetilde{a}} \cdot R\,(\subseteq K)$ extending $\breve{\partial}_{\langle \bullet \rangle}$. This implies $\nabla = \nabla_a$ by (4.4), which completes the proof. □

4.3. **Exponent of a dormant $\breve{D}^{(m)}_R$-module.** Denote by k the residue field of the discrete valuation ring R. Since k is perfect, $R/(t) = k$, and $R \supseteq k$, the t-adic completion $\widehat{R}$ of R is naturally isomorphic to $k[\![t]\!]$, i.e., the ring of formal power series with coefficients in k (cf. [Ser, Chapter I, Section 4, Theorem 2]). Now, let (E, ∇) be a dormant $\breve{D}^{(m)}_R$-module such that the R-module E is free. The t-adic completion of (E, ∇) defines a $\breve{D}^{(m)}_{k[\![t]\!]}$-module $(\widehat{E}, \widehat{\nabla})$. According to [Kin1, Proposition 1.1.12], there exists an isomorphism of $\breve{D}^{(m)}_{k[\![t]\!]}$-modules

$$\xi : (\widehat{E}, \widehat{\nabla}) \xrightarrow{\sim} \bigoplus_{i=1}^{n} (k[\![t]\!], \widehat{\nabla}_{d_i}), \tag{4.5}$$

where $d_1, \cdots, d_n \in \mathbb{Z}/p^{m+1}\mathbb{Z}$ and each $\widehat{\nabla}_{d_i}$ $(i = 1, \cdots, n)$ denotes the $\breve{D}^{(m)}_{k[\![t]\!]}$-module structure on $k[\![t]\!]$ defined as the t-adic completion of ∇_{d_i}. It follows from Proposition 4.2.1, (ii), that the multiset $[d_1, \cdots, d_n]$ depends only on the isomorphism class of (E, ∇). (For the definition and various descriptions concerning a *multiset*, we refer the reader to [SIYS].)

Definition 4.3.1. In the above situation, the multiset $[d_1, \cdots, d_n]$ is called the **exponent** of (E, ∇).

Example 4.3.2. Let (E, ∇) be as above and (L, ∇_L) be a dormant $\breve{D}^{(m)}_R$-module with $L \cong R$. According to Proposition 4.2.1, (iv), (L, ∇_L) is isomorphic to (R, ∇_a) for some $a \in \mathbb{Z}/p^{m+1}\mathbb{Z}$. Then, the tensor product $(L \otimes E, \nabla_L \otimes \nabla)$ forms a dormant $\breve{D}^{(m)}_R$-module whose exponent is $[d_1 + a, d_2 + a, \cdots, d_n + a]$.

Remark 4.3.3. For each $\breve{D}^{(0)}_R$-module (E', ∇'), the **monodromy (operator)** of (E', ∇') is the element $\mu_{(E',\nabla')}$ of $\mathrm{End}_k(k \otimes_R E')$ naturally induced by $\nabla'_{\langle 1 \rangle}$ via reduction modulo (t) (cf. [Wak5, Definition 1.46]). If $E' = R$, we have

$$\mu_{(E',\nabla')} \equiv -\mathrm{length}_R(\mathrm{Res}(\nabla')) \bmod p. \tag{4.6}$$

Now, suppose that the exponent of a dormant $\breve{D}^{(m)}_R$-module (E, ∇) as above is $[d_1, \cdots, d_n]$. For each $i = 1, \cdots, n$, we express the integer $\widetilde{d}_i$ (cf. Section 4.2) as $\widetilde{d}_i = \sum_{l=0}^{m} p^l \cdot \widetilde{d}_{il}$, where $0 \leq \widetilde{d}_{il} < p$. Then, it follows from (4.6) together with Propositions 4.1.1 and 4.2.1, (i), that, for each $l = 0, \cdots, m$, the monodromy $\mu_{(E^l, \overline{\nabla}^l)}$ of the $\breve{D}^{(0)}_{R^{(l)}}$-module $(E^l, \overline{\nabla}^l)$ (cf. (4.1)) is diagonalized and conjugate to the diagonal matrix with diagonal entries $-\widetilde{d}_{1l}, \cdots, -\widetilde{d}_{nl}$ mod p.

Proposition 4.3.4. *Let (E, ∇) be a dormant $\breve{D}^{(m)}_R$-module such that the R-module E is free. Then, the following three conditions (a)-(c) are equivalent to each other:*

- (a) *The residue $\mathrm{Res}(\nabla)$ of ∇ vanishes;*
- (b) *The exponent of (E, ∇) coincides with $[0, 0, \cdots, 0]$;*
- (c) *(E, ∇) comes from a $D^{(N-1)}_R$-module via η_R (cf. (3.2)), meaning that there exists a $D^{(N-1)}_R$-module structure $\nabla' \left(= \{\nabla'_{\langle j \rangle}\}_j\right)$ with $\nabla = \{t^j \cdot \nabla'_{\langle j \rangle}\}_j$.*

Proof. The equivalence (a) $\Longleftrightarrow$ (b) follows from Proposition 4.2.1, (i), and the existence of the decomposition (4.5). The equivalence (a) $\Longleftrightarrow$ (c) follows from the equivalence of categories asserted in Corollary 3.1.7 and the comments in Remark 3.1.2 and Example 4.1.2 (together with the fact that $\tau^{\downarrow\uparrow(m+1)}_{(E,\nabla)}$ is compatible with the respective $D^{(m+1)}_R$-module structures, i.e, $\nabla^{\mathrm{can}}_{(R,\breve{\partial}_{\langle\bullet\rangle})}$ and ∇). $\square$

4.4. **Relationship with the existence of an m-cyclic vector.** The following assertion helps us to understand the exponents of dormant pinned $\breve{D}^{(m)}_R$-modules.

Proposition 4.4.1. *Let (E,∇) be a dormant $\breve{D}^{(m)}_R$-module such that the R-module E is free and of rank $n > 0$. Let $[d_1,\cdots,d_n]$ be the exponent of (E,∇). Then, the following two conditions (a), (b) are equivalent to each other:*

(a) *(E,∇) admits an m-cyclic vector;*
(b) *The inequality $n \leq p^{m+1}$ and the equality*

$$\sum_{1\leq i<i'\leq n} \nu_p\left(\widetilde{d}_{i'} - \widetilde{d}_i\right) = \sum_{s\in\mathbb{Z}_{>0}} \sum_{j=0}^{n-1} \left\lfloor \frac{j}{p^s} \right\rfloor \tag{4.7}$$

hold, where for each integer a we denote by $\nu_p(a)\,(\in \mathbb{Z}_{\geq 0}\sqcup\{\infty\})$ the p-adic order of a.

In particular, under the assumption that $n \leq p$, (E,∇) admits an m-cyclic vector if and only if the mod p reductions of $d_1,\cdots,d_n$ are mutually distinct.

Proof. Denote by $(\widehat{E},\widehat{\nabla})$ the t-adic completion of (E,∇), and choose an isomorphism $\xi : (\widehat{E},\widehat{\nabla}) \overset{\sim}{\to} \bigoplus_{i=1}^n (k[\![t]\!],\widehat{\nabla}_{d_i})$ as in (4.5). Let us take an element v of E, which determines an element $\widehat{v}$ of $\widehat{E}$ via the natural morphism $E \to \widehat{E}$. Write $(u_i)_{i=1}^n := \xi(\widehat{v}) \in k[\![t]\!]^{\oplus n}$. Also, for each $i = 1,\cdots,n$, we set $u_i = \sum_{s=0}^\infty u_{i,s}\cdot t^s$ (where $u_{i,s}\in k$). Then, $\widehat{\nabla}_{d_i,\langle j\rangle}(u_i)$ is expressed as

$$\widehat{\nabla}_{d_i,\langle j\rangle}(u_i) = \sum_{s=0}^\infty q_j!\cdot \binom{s-\widetilde{d}_i}{j}\cdot u_{i,s}\cdot t^s.$$

The isomorphism ξ preserves the $\breve{D}^{(m)}_{k[\![t]\!]}$-action, so the equality $\xi(\widehat{\nabla}_{\langle j\rangle}(\widehat{v})) = (\widehat{\nabla}_{d_i,\langle j\rangle}(u_i))_{i=1}^n$ holds for every $j = 0,\cdots,n-1$. Hence, $\{\widehat{\nabla}_{\langle j\rangle}(\widehat{v})\}_{0\leq j\leq n-1}$ forms a basis of $\widehat{E}$ if and only if the collection $\{(q_j!\cdot\binom{-\widetilde{d}_i}{j}\cdot u_{i,0})_{i=1}^n\}_{0\leq j\leq n-1}$ $\left(= \{(\nabla_{d_i,\langle j\rangle}(u_i)|_{t=0})_{i=1}^n\}_{0\leq j\leq n-1}\right)$ forms a basis of $k^{\oplus n}$, i.e., the following three conditions (1)-(3) are fulfilled:

(1) For every $i = 1,\cdots,n$, the element $u_{i,0}$ belongs to $k^\times$ (or equivalently, $u_i \in k[\![t]\!]^\times$);
(2) For every $j = 0,\cdots,n-1$, the integer $q_j!$ is invertible in k;
(3) The $n\times n$ matrix $\left(\binom{-\widetilde{d}_i}{j}\right)_{1\leq i\leq n, 0\leq j\leq n-1}$ is invertible.

Condition (2) is equivalent to the inequality $n \leq p^{m+1}$. Regarding Condition (3), it follows from Vandermonde's determinant that

$$\det\left(\left(\binom{-\widetilde{d}_i}{j}\right)_{ij}\right) = \left(\prod_{j=0}^{n-1}\frac{(-1)^j}{j!}\right)\cdot\det\left((\widetilde{d}_i^j)_{ij}\right) = (-1)^{\frac{n(n-1)}{2}}\cdot\frac{\prod_{1\leq i<i'\leq n}(\widetilde{d}_{i'}-\widetilde{d}_i)}{\prod_{j=0}^{n-1} j!}.$$

The p-adic order of this value (which is nonnegative because it is well-defined as an element of $\mathbb{F}_p \subseteq k$) can be computed as follows:

$$\begin{aligned}\nu_p\left(\frac{\prod_{1\leq i<i'\leq n}(\widetilde{d}_{i'}-\widetilde{d}_i)}{\prod_{j=0}^{n-1} j!}\right) &= \nu_p\left(\prod_{1\leq i<i'\leq n}(\widetilde{d}_{i'}-\widetilde{d}_i)\right) - \nu_p\left(\prod_{j=0}^{n-1} j!\right) \\ &= \sum_{1\leq i<i'\leq n} \nu_p\left(\widetilde{d}_{i'}-\widetilde{d}_i\right) - \sum_{j=0}^{n-1}\nu_p\left(j!\right) \\ &= \sum_{1\leq i<i'\leq n} \nu_p\left(\widetilde{d}_{i'}-\widetilde{d}_i\right) - \sum_{s\in\mathbb{Z}_{>0}}\sum_{j=0}^{n-1}\left\lfloor\frac{j}{p^s}\right\rfloor.\end{aligned}$$

Hence, Condition (3) is fulfilled if and only if the equality (4.7) holds. It follows that we obtain the implication (a) $\Rightarrow$ (b).

Conversely, suppose that Condition (b) is satisfied. Then, since $k\otimes_R E = k\otimes \widehat{E} = k^{\oplus n}$, we can take an element v of E such that the induced element $\widehat{v}$ of $\widehat{E}$ satisfies $\xi(\widehat{v}) \in (k[\![t]\!]^\times)^{\oplus n}$. According to the above discussion, $\widehat{v}$ defines an m-cyclic vector of $(\widehat{E},\widehat{\nabla})$. By the faithful flatness of the natural homomorphism $R \to \left(\widehat{R}=\right) k[\![t]\!]$, we see that v forms an m-cyclic vector of (E,∇). This implies (b) $\Rightarrow$ (a), which completes the proof of this proposition. $\square$

Corollary 4.4.2. *The exponent of the dormant $\breve{D}_R^{(m)}$-module $(\breve{P}_R, \nabla_{\breve{P}_R})$ (cf.* (3.6)*) coincides with* $[0,1,\cdots,p^{m+1}-1]$.

Proof. Recall that $(\breve{P}_R, \nabla_{\breve{P}_R})$ has an m-cyclic vector (by the discussion in Section 3.2) and the free R-module $\breve{P}_R$ is of rank p^{m+1}. Hence, the assertion follows from the above proposition. $\square$

Let $\delta := \{d_1,\cdots,d_n\}$ (where $d_i\neq d_{i'}$ if $i\neq i'$) be a subset of $\mathbb{Z}/p^{m+1}\mathbb{Z}$ whose cardinality equals n. We shall set

$$\delta^{\blacktriangledown} := \{d_1^{\blacktriangledown},\cdots,d^{\blacktriangledown}_{p^{m+1}-n}\} \tag{4.8}$$

to be the subset of $\mathbb{Z}/p^{m+1}\mathbb{Z}$ with $\delta\sqcup\{-d_1^{\blacktriangledown},\cdots,-d^{\blacktriangledown}_{p^{m+1}-n}\} = \mathbb{Z}/p^{m+1}\mathbb{Z}$. It can immediately be seen that $\delta^{\blacktriangledown\blacktriangledown}=\delta$.

If a dormant $\breve{D}_R^{(m)}$-module admits an m-cyclic vector, then it follows from Proposition 4.4.1 that the elements in its exponent are mutually distinct, and hence form a subset of $\mathbb{Z}/p^{m+1}\mathbb{Z}$. In particular, for each subset δ of $\mathbb{Z}/p^{m+1}\mathbb{Z}$, it makes sense to speak of a dormant pinned $\breve{D}_R^{(m)}$-module *of exponent* δ. For each subset $\delta := \{d_1,\cdots,d_n\}$ of $\mathbb{Z}/p^{m+1}\mathbb{Z}$ whose cardinality equals n, we shall denote by

$$\mathrm{Mod}(\breve{D}_R^{(m)})^{\circledast}_{n,\delta}$$

the subset of $\mathrm{Mod}(\breve{D}_R^{(m)})^{\circledast}_n$ consisting of dormant pinned $\breve{D}_R^{(m)}$-modules of rank n and exponent δ.

Proposition 4.4.3. *The bijection $[\breve{\mathcal{O}}]_n : \mathrm{Mod}(\breve{D}_R^{(m)})^{\circledast}_n \xrightarrow{\sim} \mathrm{Mod}(\breve{D}_R^{(m)})^{\circledast}_{p^{m+1}-n}$ (cf.* (3.11)*) restricts to a bijection of sets*

$$[\breve{\mathcal{O}}]_{n,\delta} : \mathrm{Mod}(\breve{D}_R^{(m)})^{\circledast}_{n,\delta} \xrightarrow{\sim} \mathrm{Mod}(\breve{D}_R^{(m)})^{\circledast}_{p^{m+1}-n,\delta^{\blacktriangledown}}.$$

Proof. Let us take a dormant pinned $\breve{D}_R^{(m)}$-module (E, ∇, v) classified by $\mathrm{Mod}(\breve{D}_R^{(m)})^{\circledast}_{n,\delta}$. Denote by $(\widehat{E}, \widehat{\nabla})$ the t-adic completion of (E, ∇). Note that the t-adic completion of $(\breve{P}_R, \nabla_{\breve{P}_R})$ may be identified with $(\breve{P}_{k[\![t]\!]}, \nabla_{\breve{P}_{k[\![t]\!]}})$. Hence, the t-adic completion $\widehat{\nu}_{(E,\nabla,v)}$ of $\nu_{(E,\nabla,v)}$ (cf. (3.7)) specifies a morphism $(\breve{P}_{k[\![t]\!]}, \nabla_{\breve{P}_{k[\![t]\!]}}) \to (\widehat{E}, \widehat{\nabla})$. We shall fix an isomorphism of $\breve{D}_{k[\![t]\!]}^{(m)}$-modules $\xi : (\widehat{E}, \widehat{\nabla}) \xrightarrow{\sim} \bigoplus_{i=1}^n (k[\![t]\!], \widehat{\nabla}_{d_i})$ as in (4.5). On the other hand, according to Corollary 4.4.2, there exists an isomorphism $\xi_P : (\breve{P}_{k[\![t]\!]}, \nabla_{\breve{P}_{k[\![t]\!]}}) \xrightarrow{\sim} \bigoplus_{d \in \mathbb{Z}/p^{m+1}\mathbb{Z}} (k[\![t]\!], \widehat{\nabla}_d)$. In particular, we obtain a surjective morphism

$$\xi \circ \widehat{\nu}_{(E,\nabla,v)} \circ \xi_P^{-1} : \bigoplus_{d \in \mathbb{Z}/p^{m+1}\mathbb{Z}} (k[\![t]\!], \widehat{\nabla}_d) \to \bigoplus_{i=1}^n (k[\![t]\!], \widehat{\nabla}_{d_i}).$$

From Proposition 4.2.1, (ii), we see that the kernel of this morphism is isomorphic to $\bigoplus_{d \notin \delta} (k[\![t]\!], \widehat{\nabla}_d)$. Hence, the dual of this dormant $\breve{D}_{k[\![t]\!]}^{(m)}$-module, i.e., the t-adic completion of $(E^\blacktriangledown, \nabla^\blacktriangledown)$, is isomorphic to $\bigoplus_{d \in \delta^\blacktriangledown} (k[\![t]\!], \widehat{\nabla}_d)$ (cf. Proposition 4.2.1, (iii)). This means that the exponent of $(E^\blacktriangledown, \nabla^\blacktriangledown)$ coincides with $\delta^\blacktriangledown$, which completes the proof. □

5. Dormant PGL_n-opers of higher level

In this and the remaining sections, we apply the results on higher-level differential modules proved so far to discuss the corresponding objects defined on an algebraic curve in characteristic $p > 0$. The notion of a $\mathrm{PGL}_n^{(N)}$-oper (i.e., a PGL_n-oper of level N) for $N > 0$ will be defined in terms of the ring of differential operators of level $N-1$ (cf. Definition 5.2.5). Also, we introduce the radius of a $\mathrm{PGL}_n^{(N)}$-oper by using the local description at each marked point of the underlying pointed curve (cf. Definition 5.3.2).

5.1. **Logarithmic differential operators.** In the rest of the present paper, let us fix a positive integer N, a nonnegative integer r, and an algebraically closed field k of characteristic p. Also, let us fix an r-pointed (possibly nonproper) smooth curve

$$\mathscr{X} := (f : X \to \mathrm{Spec}(k), \{\sigma_i\}_{1 \leq i \leq r}) \tag{5.1}$$

over k, i.e., a smooth curve X over k together with r marked points $\{\sigma_i\}_{1 \leq i \leq r}$ $(\subseteq X(k))$. The divisor on X defined as the union of the marked points σ_i determines a log structure on X; we shall denote the resulting log scheme by $X^{\log}$. Since $X^{\log}$ is log smooth over k, the sheaf of logarithmic 1-forms $\Omega_{X^{\log}}$ of $X^{\log}/k$, as well as its dual $\mathcal{T}_{X^{\log}} := \Omega_{X^{\log}}^\vee$, is a line bundle. When there is no fear of confusion, we write Ω and $\mathcal{T}$ instead of $\Omega_{X^{\log}}$ and $\mathcal{T}_{X^{\log}}$, respectively.

Next, write F_k (resp., F_X) for the absolute Frobenius endomorphism of $\mathrm{Spec}(k)$ (resp., X). We shall denote by $X^{(N)}$ the base-change $k \times_{F_k^N, k} X$ of X by the N-th iterate F_k^N of F_k; we will refer to it as the **N-th Frobenius twist** of X over k. Also, the morphism $F_{X/k}^{(N)} \left(:= (f, F_X^N)\right) : X \to X^{(N)}$ is called the **N-th relative Frobenius morphism** of X over k.

Denote by $\mathcal{D}_{X^{\log}}^{(N-1)}$ (cf. [Mon, Définition 2.3.1]) the ring of logarithmic differential operators on $X^{\log}/k$ (equipped with the trivial $(N-1)$-PD structure) of level $N-1$. For each integer j, we shall write $\mathcal{D}^{(N-1)}_{X^{\mathrm{log}}, <j}$ for the $\mathcal{O}_X$-submodule of $\mathcal{D}^{(N-1)}_{X^{\mathrm{log}}}$ consisting of logarithmic differential operators of order $< j$. When there is no fear of confusion, we will write $\mathcal{D}^{(N-1)}$ (resp., $\mathcal{D}^{(N-1)}_{<j}$) instead of $\mathcal{D}^{(N-1)}_{X^{\mathrm{log}}}$ (resp., $\mathcal{D}^{(N-1)}_{X^{\mathrm{log}}, <j}$). If N' is a positive integer with $N' \geq N$, then there exists a canonical morphism $\mathcal{D}^{(N-1)} \rightarrow \mathcal{D}^{(N'-1)}$ (cf. [Mon, Section 2.5.1]).

A **(left)** $\mathcal{D}^{(N-1)}$**-module structure** on an $\mathcal{O}_X$-module $\mathcal{E}$ is a left $\mathcal{D}^{(N-1)}$-action $\nabla : \mathcal{D}^{(N-1)} \rightarrow \mathcal{E}nd_k(\mathcal{E})$ on $\mathcal{E}$ extending its $\mathcal{O}_X$-module structure. An $\mathcal{O}_X$-module equipped with a $\mathcal{D}^{(N-1)}$-module structure is called a **(left)** $\mathcal{D}^{(N-1)}$**-module**. Given a $\mathcal{D}^{(N-1)}$-module $(\mathcal{E}, \nabla)$, we shall write $\mathcal{E}^{\nabla}$ for the subsheaf of $\mathcal{E}$ on which $\mathcal{D}^{(N-1)}_{+}$ acts as zero, where $\mathcal{D}^{(N-1)}_{+}$ denotes the kernel of the canonical projection $\mathcal{D}^{(N-1)} \twoheadrightarrow \mathcal{O}_X$. The sheaf $\mathcal{E}^{\nabla}$ may be regarded as an $\mathcal{O}_{X^{(N)}}$-module via the underlying homeomorphism of $F^{(N)}_{X/k}$.

Recall that giving a $\mathcal{D}^{(0)}$-module structure on an $\mathcal{O}_X$-module $\mathcal{E}$ is equivalent to giving a logarithmic connection on $\mathcal{E}$, i.e., a k-linear morphism $\mathcal{E} \rightarrow \Omega \otimes \mathcal{E}$ satisfying the Leibniz rule. For each $\mathcal{D}^{(N-1)}$-module structure ∇ on an $\mathcal{O}_X$-module $\mathcal{E}$, we shall write

$$\overline{\nabla} : \mathcal{E} \rightarrow \Omega \otimes \mathcal{E}$$

for the logarithmic connection on $\mathcal{E}$ corresponding to the $\mathcal{D}^{(0)}$-module structure induced from ∇ via the canonical morphism $\mathcal{D}^{(0)} \rightarrow \mathcal{D}^{(N-1)}$.

Denote by ${}^p\psi_{X^{\mathrm{log}}}$ the p^N-curvature map $\mathcal{T}^{\otimes p^N} \rightarrow \mathcal{D}^{(N-1)}$ defined in [Ohk, Definition 3.10]. Given a $\mathcal{D}^{(N-1)}$-module $(\mathcal{E}, \nabla)$, we shall set

$${}^p\psi_{(\mathcal{E}, \nabla)} := \nabla \circ {}^p\psi_{X^{\mathrm{log}}} : \mathcal{T}^{\otimes p^N} \rightarrow \mathcal{E}nd_k(\mathcal{E}),$$

which will be called the p^N**-curvature** of $(\mathcal{E}, \nabla)$. The p^1-curvature of a $\mathcal{D}^{(0)}$-module is essentially the same as the p-curvature of the corresponding logarithmic connection (cf. [Ogu, Section 1.2]).

Definition 5.1.1. Let $(\mathcal{E}, \nabla)$ be a $\mathcal{D}^{(N-1)}$-module. Then, we shall say that $(\mathcal{E}, \nabla)$ is **dormant** if ${}^p\psi_{(\mathcal{E}, \nabla)} = 0$.

Remark 5.1.2. Let us review a result in the case where $r = 0$, or equivalently, the log structure of X^{log} is trivial. Then, our definition of p^N-curvature coincides with the p-$(N-1)$-curvature in the sense of [LeQu, Definition 3.1.1]. For each $\mathcal{O}_{X^{(N)}}$-module $\mathcal{E}$, there exists a canonical (non-logarithmic) $\mathcal{D}^{(N-1)}_X$-module structure $\nabla^{\mathrm{can}}_{\mathcal{E}}$ on $F^{(N)*}_{X/k}(\mathcal{E})$ with vanishing p^N-curvature. According to [LeQu, Corollary 3.2.4], the assignments $(\mathcal{E}, \nabla) \mapsto \mathcal{E}^{\nabla}$ and $\mathcal{E} \mapsto (F^{(N)*}_{X/k}(\mathcal{E}, \nabla^{\mathrm{can}}_{\mathcal{E}}))$ determine an equivalence of categories

$$\begin{pmatrix} \text{the category of} \\ \text{dormant } \mathcal{D}^{(N-1)}_X\text{-modules} \end{pmatrix} \xrightarrow{\sim} \begin{pmatrix} \text{the category of} \\ \mathcal{O}_{X^{(N)}}\text{-modules} \end{pmatrix}. \tag{5.2}$$

The ring-theoretic counterpart of this equivalence was already mentioned in Corollary 3.1.7.

Here, we shall consider the local description of $\mathcal{D}^{(N-1)}$-modules. Let x be a k-rational point of X and fix a local function t on X defining x. Suppose that x lies in $X \setminus \bigcup_{i=1}^r \{\sigma_i\}$ (resp., $\bigcup_{i=1}^r \{\sigma_i\}$). Then, it follows from [PBer1, Proposition 2.2.4] (resp., [Mon, Lemme 2.3.3]) that (the sections of) the restriction of $\mathcal{D}^{(N-1)}$ to $D_x := \mathrm{Spec}(\mathcal{O}_{X,x})$ may be identified with $D^{(N-1)}_{\mathcal{O}_{X,x}}$ (resp., $\breve{D}^{(N-1)}_{\mathcal{O}_{X,x}}$) defined in (3.1) for $R = \mathcal{O}_{X,x}$. Each $\mathcal{D}^{(N-1)}$-module $(\mathcal{E}, \nabla)$ induces, via restriction to D_x, a $D^{(N-1)}_{\mathcal{O}_{X,x}}$-module (resp., $\check{D}^{(N-1)}_{\mathcal{O}_{X,x}}$-module) $(\mathcal{E}, \nabla)|_{D_x}$. According to [Ohk, Proposition 3.11], the p^N-curvature of $(\mathcal{E}, \nabla)|_{D_x}$ may be regarded as the restriction to D_x of the p^N-curvature of $(\mathcal{E}, \nabla)$. In particular, if $(\mathcal{E}, \nabla)$ is dormant, then the restriction $(\mathcal{E}, \nabla)|_{D_x}$ is dormant.

Definition 5.1.3. Let $(\mathcal{E}, \nabla)$ be a dormant $\mathcal{D}^{(N-1)}$-module such that $\mathcal{E}$ is a vector bundle on X of rank $n > 0$. Suppose that $r > 0$. Then, for each $i \in \{1, \cdots, r\}$, the **(local) exponent** of $(\mathcal{E}, \nabla)$ (or, of ∇) at the marked point σ_i is defined as the exponent of $(\mathcal{E}, \nabla)|_{D_{\sigma_i}}$ (cf. Definition 4.3.1).

The following assertion will be applied in the proof of Proposition 7.3.1.

Proposition 5.1.4. *Let l be a positive integer with $p \nmid l$, $\mathcal{N}$ a line bundle on X, and $\nabla_{\mathcal{N}^{\otimes l}}$ a $\mathcal{D}^{(N-1)}$-module structure on the l-th tensor product $\mathcal{N}^{\otimes l}$ of $\mathcal{N}$ with vanishing p^N-curvature. Then, there exists a unique $\mathcal{D}^{(N-1)}$-module structure $\nabla_{\mathcal{N}}$ on $\mathcal{N}$ with vanishing p^N-curvature whose l-th tensor product $\nabla_{\mathcal{N}}^{\otimes l}$ coincides with $\nabla_{\mathcal{N}^{\otimes l}}$.*

Proof. For each $i = 1, \cdots, r$, denote by e_i the exponent of $\nabla_{\mathcal{N}^{\otimes l}}$ at σ_i. Write $\widetilde{e}'_i$ for the unique integer with $0 \leq \widetilde{e}'_i < p^N$ and $\widetilde{e}'_i \equiv e_i/l \mod p^N$. According to the discussion in Remark 3.1.1, the trivial $\mathcal{D}^{(N-1)}$-module structure on $\mathcal{O}_X$ extends uniquely to a $\mathcal{D}^{(N-1)}$-module structure ∇_+ on $\mathcal{O}_X(\sum_{i=1}^r \widetilde{e}'_i \cdot \sigma_i) \, (\supseteq \mathcal{O}_X)$ with vanishing p^N-curvature. The exponent of $\nabla_+^{\otimes l} \otimes \nabla_{\mathcal{N}^{\otimes l}}$ at σ_i is $\overline{l \cdot \widetilde{e}'_i} - e_i = l \cdot (e_i/l) - e_i = 0$. Hence, by Proposition 4.3.4 together with the equivalence of categories (5.2), there exists a line bundle $\mathcal{M}$ on $X^{(N)}$ with $(F^{(N)*}(\mathcal{M}), \nabla^{\mathrm{can}}_{\mathcal{M}}) \cong (\mathcal{N}'^{\otimes l}, \nabla_+^{\otimes l} \otimes \nabla_{\mathcal{N}^{\otimes l}})$, where $\mathcal{N}' := \mathcal{O}_X(\sum_{i=1}^r \widetilde{e}'_i \cdot \sigma_i) \otimes \mathcal{N}$. If $\mathcal{M}'$ denotes the line bundle on X corresponding to $\mathcal{M}$ via base-change $X^{(N)} \xrightarrow{\sim} X$ by F_k^N, then we have $\mathcal{M}'^{\otimes p^N} \cong \mathcal{N}'^{\otimes l}$. Here, let us take a pair of integers (a, b) with $a \cdot p^N + b \cdot l = 1$. Then,

$$\mathcal{N}' = \mathcal{N}'^{\otimes (a \cdot p^N + b \cdot l)} \cong \mathcal{N}'^{\otimes a p^N} \otimes \mathcal{M}'^{\otimes b p^N} = (\mathcal{N}'^{\otimes a} \otimes \mathcal{M}'^{\otimes b})^{\otimes p^N}. \tag{5.3}$$

Let us define $\mathcal{L}$ to be the line bundle on $X^{(N)}$ corresponding to $\mathcal{N}'^{\otimes a} \otimes \mathcal{M}'^{\otimes b}$ via base-change by F_k^N. By (5.3), we see that $F^{(N)*}(\mathcal{L}) \cong \mathcal{N}'$, i.e., there exists an isomorphism $F^{(N)*}(\mathcal{L}) \otimes \mathcal{O}_X(-\sum_i \widetilde{e}'_i \cdot \sigma_i) \xrightarrow{\sim} \mathcal{N}$. The line bundle $\mathcal{N}$ is equipped with the $\mathcal{D}^{(N-1)}$-module structure $\nabla_{\mathcal{N}}$ corresponding to $\nabla^{\mathrm{can}}_{\mathcal{L}} \otimes \nabla_+^{\vee}$ via this isomorphism. The l-th tensor product $\nabla_{\mathcal{N}}^{\otimes l}$ of $\nabla_{\mathcal{N}}$ has vanishing p^N-curvature and coincides with $\nabla_{\mathcal{N}^{\otimes l}}$ by its construction. This completes the proof of this proposition. ☐

5.2. GL_n**-opers and** PGL_n**-opers of level** N**.** Let us fix a positive integer n. We shall define the notion of a $\mathrm{GL}_n^{(N)}$-oper, as follows. (Note that a $\mathrm{GL}_n^{(1)}$-oper is the same as a GL_n-oper in the classical sense.)

Definition 5.2.1. (i) Let us consider a collection of data

$$\mathscr{F}^{\heartsuit} := (\mathcal{F}, \nabla, \{\mathcal{F}^j\}_{0 \leq j \leq n}),$$

where
- $\mathcal{F}$ is a vector bundle on X of rank n;
- ∇ is a $\mathcal{D}^{(N-1)}$-module structure on $\mathcal{F}$;

– $\{\mathcal{F}^j\}_{0\leq j\leq n}$ is an n-step decreasing filtration

$$0 = \mathcal{F}^n \subseteq \mathcal{F}^{n-1} \subseteq \cdots \subseteq \mathcal{F}^0 = \mathcal{F}$$

on $\mathcal{F}$ consisting of subbundles such that the subquotients $\mathcal{F}^j/\mathcal{F}^{j+1}$ are line bundles.

Then, we say that $\mathscr{F}^\heartsuit$ is a $\mathrm{GL}_n^{(N)}$**-oper** (or a GL_n**-oper of level** N) on $\mathscr{X}$ if, for every $j = 0, \cdots, n-1$, the $\mathcal{O}_X$-linear morphism $\mathcal{D}^{(N-1)} \otimes \mathcal{F} \to \mathcal{F}$ induced by ∇ restrict to an isomorphism

$$\mathcal{D}^{(N-1)}_{<n-j} \otimes \mathcal{F}^{n-1} \xrightarrow{\sim} \mathcal{F}^j. \tag{5.4}$$

The notion of an isomorphism between two $\mathrm{GL}_n^{(N)}$-opers can be defined in a natural manner (so we omit the details).

(ii) Let $\mathscr{F}^\heartsuit := (\mathcal{F}, \nabla, \{\mathcal{F}^j\}_j)$ be a $\mathrm{GL}_n^{(N)}$-oper. Then, we shall say that $\mathcal{F}^\heartsuit$ is **dormant** if ${}^p\psi_{(\mathcal{F},\nabla)} = 0$.

Remark 5.2.2. (i) A (dormant) $\mathrm{GL}_1^{(N)}$-oper is the same as a (dormant) $\mathcal{D}^{(N-1)}$-module $(\mathcal{F}, \nabla)$ such that $\mathcal{F}$ is a line bundle.

(ii) In the case of $n = 2$, a $\mathrm{GL}_2^{(N)}$-oper on $\mathscr{X}$ is given as a triple $(\mathcal{F}, \nabla, \mathcal{L})$ consisting of
  - a $\mathcal{D}^{(N-1)}$-module $(\mathcal{F}, \nabla)$ such that $\mathcal{F}$ is a rank 2 vector bundle, and
  - a line subbundle $\mathcal{L}$ of $\mathcal{F}$ such that the $\mathcal{O}_X$-linear composite

$$\mathcal{L} \xrightarrow{\text{inclusion}} \mathcal{F} \xrightarrow{\overline{\nabla}} \Omega \otimes \mathcal{F} \xrightarrow{\text{quotient}} \Omega \otimes (\mathcal{F}/\mathcal{L}) \tag{5.5}$$

defines an isomorphism between line bundles.

The following assertion implies that higher-level differential modules with a cyclic vector may be regarded as ring-theoretic counterparts of GL_n-opers of higher level.

Proposition 5.2.3. *Let $(\mathcal{F}, \nabla)$ be a $\mathcal{D}^{(N-1)}$-module such that $\mathcal{F}$ is a vector bundle of rank n. Also, let x be a (possibly generic) point x of $X \setminus \bigcup_{i=1}^r \{\sigma_i\}$ (resp., $\bigcup_{i=1}^r \{\sigma_i\}$). Write $(\mathcal{F}_x, \nabla_x) := (\mathcal{F}, \nabla)|_{D_x}$ for the $D^{(N-1)}_{\mathcal{O}_{X,x}}$-module (resp., $\breve{D}^{(N-1)}_{\mathcal{O}_{X,x}}$-module) obtained as the restriction of $(\mathcal{F}, \nabla)$ to D_x $(:= \mathrm{Spec}(\mathcal{O}_{X,x}))$.*

(i) *Suppose that there exists an n-step decreasing filtration $\{\mathcal{F}^j\}_{0\leq j\leq n}$ on $\mathcal{F}$ for which the collection $(\mathcal{F}, \nabla, \{\mathcal{F}^j\}_{0\leq j\leq n})$ forms a $\mathrm{GL}_n^{(N)}$-oper. Then, each generator of the restriction of $\mathcal{F}^{n-1}$ to D_x defines an $(N-1)$-cyclic vector of $(\mathcal{F}_x, \nabla_x)$.*

(ii) *Suppose that there exists an $(N-1)$-cyclic vector of $(\mathcal{F}_x, \nabla_x)$. For each $j = 0, \cdots, n$, we shall write $\mathcal{F}_x^j$ for the $\mathcal{O}_{X,x}$-submodule of $\mathcal{F}_x$ generated by the elements $\nabla_{x,\langle l\rangle}(v)$ for $l \leq n-j-1$. Then, there exists an open neighborhood U of x satisfying the following condition: the filtration $\{\mathcal{F}_x^j\}_j$ extends to a decreasing filtration $\{(\mathcal{F}|_U)^j\}_{0\leq j\leq n}$ of $\mathcal{F}|_U$ (i.e., $(\mathcal{F}|_U)^j|_{D_x} = \mathcal{F}_x^j$ for every j) for which the collection $(\mathcal{F}|_U, \nabla|_U, \{(\mathcal{F}|_U)^j\}_{0\leq j\leq n})$ forms a $\mathrm{GL}_n^{(N)}$-oper on the pointed curve $\mathscr{X}$ restricted to U.*

Proof. The assertions follow immediately from the definitions of a $\mathrm{GL}_n^{(N)}$-oper and an $(N-1)$-cyclic vector. □

By applying results on differential modules proved in Sections 2 and 3, we can obtain the following assertion.

Corollary 5.2.4. (i) *Let $(\mathcal{F}, \nabla)$ be a $\mathcal{D}^{(N-1)}$-module such that $\mathcal{F}$ is a vector bundle of rank $n \leq p^N$. Also, let x be a k-rational point of X. Then, there exists an open neighborhood U of x in X and an n-step decreasing filtration $\{(\mathcal{F}|_U)^j\}_{0 \leq j \leq n}$ on $\mathcal{F}|_U$ such that the collection $(\mathcal{F}|_U, \nabla|_U, \{(\mathcal{F}|_U)^j\}_{0\leq j \leq n})$ forms a $\mathrm{GL}_n^{(N)}$-oper on the pointed curve $\mathscr{X}$ restricted to U.*

(ii) *If $n > p^N$, then there are no dormant $\mathrm{GL}_n^{(N)}$-opers on $\mathscr{X}$.*

Proof. Assertion (i) follows from Theorem 2.5.6 and Proposition 5.2.3, (ii). Assertion (ii) follows from Propositions 3.2.1, (i), and 5.2.3, (i). □

Next, we shall define an equivalence relation in the set of dormant $\mathrm{GL}_n^{(N)}$-opers. Let $\mathscr{F}^\heartsuit := (\mathcal{F}, \nabla, \{\mathcal{F}^j\}_j)$ be a $\mathrm{GL}_n^{(N)}$-oper on $\mathscr{X}$ and $(\mathcal{N}, \nabla_\mathcal{N})$ a line bundle on X equipped with a $\mathcal{D}^{(N-1)}$-module structure. According to [Mon, Corollaire 2.6.1], there exists a canonical $\mathcal{D}^{(N-1)}$-module structure $\nabla \otimes \nabla_\mathcal{N}$ on the tensor product $\mathcal{F} \otimes \mathcal{N}$ naturally arising from ∇ and $\nabla_\mathcal{N}$. One may verify that the collection

$$\mathscr{F}^\heartsuit_{\otimes(\mathcal{N}, \nabla_\mathcal{N})} := (\mathcal{N} \otimes \mathcal{F}, \nabla_\mathcal{N} \otimes \nabla, \{\mathcal{N} \otimes \mathcal{F}^j\}_{0 \leq j \leq n})$$

forms a $\mathrm{GL}_n^{(N)}$-oper. If both $\mathscr{F}^\heartsuit$ and $(\mathcal{N}, \nabla_\mathcal{N})$ are dormant, then $\mathscr{F}^\heartsuit_{\otimes \mathcal{N}}$ is dormant. Now, let us consider the binary relation "$\sim$" in the set of dormant $\mathrm{GL}_n^{(N)}$-opers on $\mathscr{X}$ defined by $\mathscr{F}^\heartsuit \sim \mathscr{F}'^\heartsuit$ if and only if $\mathscr{F}^\heartsuit_{\otimes(\mathcal{N}, \nabla_\mathcal{N})} \cong \mathscr{F}'^\heartsuit$ for some $(\mathcal{N}, \nabla_\mathcal{N})$ as above; this relation in fact defines an equivalence relation. For each $\mathscr{F}^\heartsuit$ as above, we shall write

$$\mathscr{F}^{\heartsuit \Rightarrow \spadesuit}$$

for the equivalence class represented by $\mathscr{F}^\heartsuit$.

Definition 5.2.5. A $\mathrm{PGL}_n^{(N)}$**-oper** (or a PGL_n**-oper of level** N) on $\mathscr{X}$ is the equivalence class $\mathscr{F}^\spadesuit \left(= \mathscr{F}^{\heartsuit \Rightarrow \spadesuit}\right)$ of a $\mathrm{GL}_n^{(N)}$-oper $\mathcal{F}^\heartsuit$ on $\mathscr{X}$. A $\mathrm{PGL}_n^{(N)}$-oper is called **dormant** if it may be represented by a dormant $\mathrm{GL}_n^{(N)}$-oper.

We shall denote by

$$\mathrm{Op}^{\text{Zzz...}}_{N,n,\mathscr{X}}, \text{ or simply } \mathrm{Op}^{\text{Zzz...}}_n$$

the set of dormant $\mathrm{PGL}_n^{(N)}$-opers on $\mathscr{X}$.

Remark 5.2.6. It can immediately be seen that $\sharp(\mathrm{Op}^{\text{Zzz...}}_1) = 1$ (cf. Remark 5.2.2, (i)). Also, according to [Moc, Chapter II, Theorem 2.8], $\mathrm{Op}^{\text{Zzz...}}_2$ is nonempty if $N = 1$. For a general n, we know that the set $\mathrm{Op}^{\text{Zzz...}}_n$ for $N = 1$ is nonempty when n is sufficiently small relative to p (cf. [Wak5, Theorem 3.38]). On the other hand, it follows from Corollary 5.2.4 that $\mathrm{Op}^{\text{Zzz...}}_n = \emptyset$ if $n > p^N$.

5.3. **Radius of a dormant** $\mathrm{PGL}_n^{(N)}$**-oper.** Denote by Δ the image of the diagonal embedding $\mathbb{Z}/p^N\mathbb{Z} \hookrightarrow (\mathbb{Z}/p^N\mathbb{Z})^n$. In particular, by regarding it as a group homomorphism, we obtain the quotient $(\mathbb{Z}/p^N\mathbb{Z})^n/\Delta$. Note that the set $(\mathbb{Z}/p^N\mathbb{Z})^n$ is equipped with the action of the symmetric group $\mathfrak{S}_n$ of n letters by permutation; this action induces a well-defined $\mathfrak{S}_n$-action on $(\mathbb{Z}/p^N\mathbb{Z})^n/\Delta$. Hence, we obtain the sets $\mathfrak{S}_n \backslash (\mathbb{Z}/p^N\mathbb{Z})^n$, $\mathfrak{S}_n \backslash (\mathbb{Z}/p^N\mathbb{Z})^n/\Delta$, and moreover, obtain the natural projection

$$\pi : \mathfrak{S}_n \backslash (\mathbb{Z}/p^N\mathbb{Z})^n \twoheadrightarrow \mathfrak{S}_n \backslash (\mathbb{Z}/p^N\mathbb{Z})^n/\Delta. \tag{5.6}$$

Each element of $\mathfrak{S}_n \backslash (\mathbb{Z}/p^N\mathbb{Z})^n$ may be regarded as a multiset of $\mathbb{Z}/p^N\mathbb{Z}$ whose cardinality equals n.

Remark 5.3.1. Let us consider the case of $n = 2$. Denote by $(\mathbb{Z}/p^N\mathbb{Z})/\{\pm 1\}$ the set of equivalence classes of elements $a \in \mathbb{Z}/p^N\mathbb{Z}$, in which a and $-a$ are identified. Then, the assignment $a \mapsto [a, -a]$ determines a well-defined bijection

$$(\mathbb{Z}/p^N\mathbb{Z})/\{\pm 1\} \xrightarrow{\sim} \mathfrak{S}_2 \backslash (\mathbb{Z}/p^N\mathbb{Z})^2/\Delta. \tag{5.7}$$

By using this bijection, we will identify $\mathfrak{S}_2 \backslash (\mathbb{Z}/p^N\mathbb{Z})^2/\Delta$ with $(\mathbb{Z}/p^N\mathbb{Z})/\{\pm 1\}$ (cf. the discussion in Section 7).

Let $\mathscr{F}^\spadesuit$ be a dormant $\mathrm{PGL}_n^{(N)}$-oper on $\mathscr{X}$, and choose a dormant $\mathrm{GL}_n^{(N)}$-oper $\mathscr{F}^\heartsuit := (\mathcal{F}, \nabla, \{\mathcal{F}^j\}_j)$ representing $\mathscr{F}^\spadesuit$. Suppose that $r > 0$. For each $i = 1, \cdots, r$, denote by δ_i the exponent of $(\mathcal{F}, \nabla)$ at σ_i. Let us write $\rho_{\mathscr{F}^\spadesuit, i} := \pi(\delta_i) \in \mathfrak{S}_n \backslash (\mathbb{Z}/p^N\mathbb{Z})^n/\Delta$. It follows from the fact mentioned in Example 4.3.2 that the element $\rho_{\mathscr{F}^\spadesuit, i}$ does not depend on the choice of the representative $\mathscr{F}^\heartsuit$ of $\mathscr{F}^\spadesuit$.

Definition 5.3.2. (i) We shall refer to $\rho_{\mathscr{F}^\spadesuit, i}$ as the **radius** of $\mathscr{F}^\spadesuit$ at σ_i.
(ii) Let $\vec{\rho} := (\rho_i)_{i=1}^r$ be an element of $(\mathfrak{S}_n \backslash (\mathbb{Z}/p^N\mathbb{Z})^n/\Delta)^r$. We shall say that $\mathscr{F}^\spadesuit$ is **of radii** $\vec{\rho}$ if $\rho_i = \rho_{\mathscr{F}^\spadesuit, i}$ for every $i = 1, \cdots, r$.

For each $\vec{\rho} \in (\mathfrak{S}_n \backslash (\mathbb{Z}/p^N\mathbb{Z})^n/\Delta)^r$, we shall denote by

$$\mathrm{Op}^{\text{Zzz...}}_{N,n,\mathscr{X},\vec{\rho}}, \text{ or simply } \mathrm{Op}^{\text{Zzz...}}_{n,\vec{\rho}}$$

the subset of $\mathrm{Op}^{\text{Zzz...}}_n$ consisting of dormant $\mathrm{PGL}_n^{(N)}$-opers of radii $\vec{\rho}$.

Remark 5.3.3. Let us recall the previous study for $N = 1$. A $\mathrm{PGL}_2^{(1)}$-oper is essentially the same as a *torally indigenous bundle* in the sense of [Moc, Chapter I, Definition 4.1]. Also, the radii of a dormant torally indigenous bundle on $\mathscr{X}$ (which belong to the set $\mathbb{F}_p$, as proved in [Moc, Chapter II, Proposition 1.5]) is consistent with the radii of the corresponding PGL_2-oper via the quotient $\mathbb{F}_p \twoheadrightarrow \mathbb{F}_p/\{\pm 1\}$ $(= (\mathbb{Z}/p\mathbb{Z})/\{\pm 1\})$. According to [Moc, Chapter II, Proposition 1.4], the set $\mathrm{Op}^{\text{Zzz...}}_{2,(\rho)_{i=1}^r}$ in the case of $N = 1$ is empty unless $\rho_i \in \mathbb{F}_p^\times/\{\pm 1\}$ for every $i = 1, \cdots, r$.

Moreover, for a general n, the radii of a dormant $\mathrm{PGL}_n^{(1)}$-oper introduced above coincides with the one in the sense of [Wak5, Definition 2.32] under the identification of each element in $\mathfrak{S}_n \backslash \mathbb{F}_p^n/\Delta$ with an $\mathbb{F}_p$-rational point in the adjoint quotient of the Lie algebra $\mathfrak{pgl}_n$.

6. Duality of dormant PGL_n-opers

In this section, we establish a duality between dormant $\mathrm{PGL}_n^{(N)}$-opers and dormant $\mathrm{PGL}_{p^N-n}^{(N)}$-opers (cf. Theorem 6.3.1). As a corollary, we will see that there is exactly one isomorphism class of dormant $\mathrm{PGL}_{p^N-1}^{(N)}$-oper (cf. Corollary 6.3.2, (ii)).

We keep the notation in the previous section.

6.1. **Dormant $\mathrm{GL}_{p^N}^{(N)}$-opers.** Let $\mathcal{L}$ be a line bundle on X. We equip $\mathcal{D}^{(N-1)} \otimes \mathcal{L}$ with the $\mathcal{D}^{(N-1)}$-module structure given by left multiplication. We shall write $\mathcal{P}_{\mathcal{L}}$ for the quotient of the left $\mathcal{D}^{(N-1)}$-module $\mathcal{D}^{(N-1)} \otimes \mathcal{L}$ by the $\mathcal{D}^{(N-1)}$-submodule generated by the image of ${}^p\psi_{X^{\log}} \otimes \mathrm{id}_{\mathcal{L}} : \mathcal{T}^{\otimes p^N} \otimes \mathcal{L} \to \mathcal{D}^{(N-1)} \otimes \mathcal{L}$. Denote by $\nabla_{\mathcal{P}_{\mathcal{L}}}$ the resulting $\mathcal{D}^{(N-1)}$-module structure of $\mathcal{P}_{\mathcal{L}}$; by construction, $(\mathcal{P}_{\mathcal{L}}, \nabla_{\mathcal{P}_{\mathcal{L}}})$ has vanishing p^N-curvature. Also, for each $j = 0, \cdots, p^N$, we shall set $\mathcal{P}_{\mathcal{L}}^j$ to be the subbundle of $\mathcal{P}_{\mathcal{L}}$ defines as

$$\mathcal{P}_{\mathcal{L}}^j := \mathrm{Im}\left(\mathcal{D}_{<p^N-j}^{(N-1)} \otimes \mathcal{L} \xrightarrow{\text{inclusion}} \mathcal{D}^{(N-1)} \otimes \mathcal{L} \xrightarrow{\text{quotient}} \mathcal{P}_{\mathcal{L}}\right).$$

The collection of data

$$\mathscr{P}_{\mathcal{L}}^{\heartsuit} := (\mathcal{P}_{\mathcal{L}}, \nabla_{\mathcal{P}_{\mathcal{L}}}, \{\mathcal{P}_{\mathcal{L}}^j\}_{0 \leq j \leq p^N})$$

forms a dormant $\mathrm{GL}_{p^N}^{(N)}$-oper on $\mathscr{X}$. Indeed, as discussed in Section 3.2, the restriction of this data to D_x $(= \mathrm{Spec}(\mathcal{O}_{X,x}))$ for each k-rational point x of $X \setminus \bigcup_{i=1}^r \{\sigma_i\}$ (resp., $\bigcup_{i=1}^r \{\sigma_i\}$) defines a dormant pinned $D_{\mathcal{O}_{X,x}}^{(N-1)}$-module (resp., a dormant pinned $\breve{D}_{\mathcal{O}_{X,x}}^{(N-1)}$-module). In particular, we obtain a dormant $\mathrm{PGL}_{p^N}^{(N)}$-oper $\mathscr{P}_{\mathcal{L}}^{\heartsuit \Rightarrow \spadesuit}$ on $\mathscr{X}$.

Next, let $(\mathcal{N}, \nabla_{\mathcal{N}})$ be a dormant $\mathcal{D}^{(N-1)}$-module such that $\mathcal{N}$ is a line bundle. Since the tensor product $\nabla_{\mathcal{N}} \otimes \nabla$ has vanishing p^N-curvature, the composite

$$\begin{aligned}\mathcal{D}^{(N-1)} \otimes (\mathcal{N} \otimes \mathcal{L}) \left(= \mathcal{D}^{(N-1)} \otimes (\mathcal{N} \otimes \mathcal{P}_{\mathcal{L}}^{p^N-1})\right) &\xrightarrow{\text{inclusion}} \mathcal{D}^{(N-1)} \otimes (\mathcal{N} \otimes \mathcal{P}_{\mathcal{L}}) \\ &\xrightarrow{\nabla_{\mathcal{N}} \otimes \nabla} \mathcal{N} \otimes \mathcal{P}_{\mathcal{L}}\end{aligned}$$

factors through the quotient $\mathcal{D}^{(N-1)} \otimes (\mathcal{N} \otimes \mathcal{L}) \twoheadrightarrow \mathcal{P}_{\mathcal{N} \otimes \mathcal{L}}$. By considering the local description, we can see that the resulting morphism $\mathcal{P}_{\mathcal{N} \otimes \mathcal{L}} \to \mathcal{N} \otimes \mathcal{P}_{\mathcal{L}}$ defines an isomorphism between dormant $\mathrm{GL}_n^{(N)}$-opers

$$\mathscr{P}_{\mathcal{N} \otimes \mathcal{L}}^{\heartsuit} \xrightarrow{\sim} (\mathscr{P}_{\mathcal{L}}^{\heartsuit})_{\otimes (\mathcal{N}, \nabla_{\mathcal{N}})}. \tag{6.1}$$

6.2. **Duality for dormant $\mathrm{GL}_n^{(N)}$ opers.** Next, let $\mathscr{F}^{\heartsuit} := (\mathcal{F}, \nabla, \{\mathcal{F}^j\}_j)$ be a dormant $\mathrm{GL}_n^{(N)}$-oper on $\mathscr{X}$ with $\mathcal{F}^{n-1} = \mathcal{L}$. The inclusion $\mathcal{L} \left(= \mathcal{D}_{<1}^{(N-1)} \otimes \mathcal{L}\right) \hookrightarrow \mathcal{F}$ extends uniquely to a $\mathcal{D}^{(N-1)}$-linear morphism $\mathcal{D}^{(N-1)} \otimes \mathcal{L} \to \mathcal{F}$. Since $(\mathcal{F}, \nabla)$ has vanishing p^N-curvature, this morphism factors through the quotient $\mathcal{D}^{(N-1)} \otimes \mathcal{L} \twoheadrightarrow \mathcal{P}_{\mathcal{L}}$. Thus, we obtain a morphism of $\mathcal{D}^{(N-1)}$-modules

$$\nu_{\mathscr{F}^{\heartsuit}} : (\mathcal{P}_{\mathcal{L}}, \nabla_{\mathcal{P}_{\mathcal{L}}}) \to (\mathcal{F}, \nabla).$$

By Proposition 5.2.3, (i), the restriction of this morphism to D_x for each point x of $X \setminus \bigcup_{i=1}^r \{\sigma_i\}$ (resp., $\bigcup_{i=1}^r \{\sigma_i\}$) may be regarded as a morphism of pinned $D_{\mathcal{O}_{X,x}}^{(N-1)}$-modules (resp., pinned $\breve{D}_{\mathcal{O}_{X,x}}^{(N-1)}$-modules). Hence, it follows from Proposition 2.5.5, (i), that $\nu_{\mathscr{F}^{\heartsuit}}$ is verified to be surjective.

Example 6.2.1. Let us consider the dual $(\mathcal{P}_{\mathcal{L}}^{\vee}, \nabla_{\mathcal{P}_{\mathcal{L}}}^{\vee})$ of $(\mathcal{P}_{\mathcal{L}}, \nabla_{\mathcal{P}_{\mathcal{L}}})$. For each $j = 0, \cdots, p^N$, we shall write $\mathcal{P}_{\mathcal{L}}^{\vee j}$ for the $\mathcal{O}_X$-submodule of $\mathcal{P}_{\mathcal{L}}^{\vee}$ defined as the image of the natural injection

$(\mathcal{P}_{\mathcal{L}}/\mathcal{P}_{\mathcal{L}}^{p^N-j})^\vee \hookrightarrow \mathcal{P}_{\mathcal{L}}^\vee$. In particular, the line subbundle $\mathcal{P}_{\mathcal{L}}^{\vee p^N-1}$ can be identified with $\Omega^{\otimes(p^N-1)} \otimes \mathcal{L}^\vee$, and we obtain a collection of data

$$\mathscr{P}_{\mathcal{L}}^{\heartsuit\vee} := (\mathcal{P}_{\mathcal{L}}^\vee, \nabla_{\mathcal{P}_{\mathcal{L}}}^\vee, \{\mathcal{P}_{\mathcal{L}}^{\vee j}\}_{0 \leq j \leq p^N}).$$

For each k-rational point x of X, the restriction of $(\mathcal{P}_{\mathcal{L}}^\vee, \nabla_{\mathcal{P}_{\mathcal{L}}}^\vee)$ to D_x together with a generator of $\mathcal{P}_{\mathcal{L}}^{\vee p^N-1}|_{D_x}$ may be regarded as the data (3.8). It follows that $\mathscr{P}_{\mathcal{L}}^{\heartsuit\vee}$ forms a dormant $\mathrm{GL}_{p^N}^{(N)}$-oper. Also, the morphism

$$\nu_{\mathscr{P}_{\mathcal{L}}^{\heartsuit\vee}} : (\mathcal{P}_{\Omega^{\otimes(p^N-1)}\otimes\mathcal{L}^\vee}, \nabla_{\mathcal{P}_{\Omega^{\otimes(p^N-1)}\otimes\mathcal{L}^\vee}}) \to (\mathcal{P}_{\mathcal{L}}^\vee, \nabla_{\mathcal{P}_{\mathcal{L}}}^\vee)$$

is compatible, via restriction to D_x, with the morphism (3.9). It follows that $\nu_{\mathscr{F}_{\mathcal{L}}^{\heartsuit\vee}}$ defines an isomorphism $\mathscr{P}^{\heartsuit}_{\Omega^{\otimes(p^N-1)}\otimes\mathcal{L}^\vee} \xrightarrow{\sim} \mathscr{P}_{\mathcal{L}}^{\heartsuit\vee}$ of $\mathrm{GL}_{p^N}^{(N)}$-opers.

Let us write $\mathcal{F}^\blacktriangledown := \mathrm{Ker}(\nu_{\mathscr{F}^\heartsuit})^\vee$. Since $\nu_{\mathscr{F}^\heartsuit}$ preserves the $\mathcal{D}^{(N-1)}$-action, $\nabla_{\mathcal{P}_{\mathcal{L}}}$ restricts to a $\mathcal{D}^{(N-1)}$-module structure $\nabla_{\mathrm{Ker}(\nu_{\mathscr{F}^\heartsuit})}$ on the kernel $\mathrm{Ker}(\nu_{\mathscr{F}^\heartsuit})$; it induces a $\mathcal{D}^{(N-1)}$-module structure on $\mathcal{F}^\blacktriangledown$, which we denote by $\nabla^\blacktriangledown$. For each $j = 0, \cdots, p^N - n$, let us define γ_j to be the composite

$$\gamma_j : \mathrm{Ker}(\nu_{\mathscr{F}^\heartsuit}) \xrightarrow{\text{inclusion}} \mathcal{P}_{\mathcal{L}} \twoheadrightarrow \mathcal{P}_{\mathcal{L}}/\mathcal{P}_{\mathcal{L}}^{p^N-n-j}.$$

By letting $\mathcal{F}^{\blacktriangledown j} := \mathrm{Im}(\gamma_j^\vee)\ (\subseteq \mathcal{F}^\blacktriangledown)$, we obtain a collection of data

$$\mathscr{F}^{\heartsuit\blacktriangledown} := (\mathcal{F}^\blacktriangledown, \nabla^\blacktriangledown, \{\mathcal{F}^{\blacktriangledown j}\}_{0 \leq j \leq p^N-n}).$$

Proposition 6.2.2. *Let us keep the above notation.*

(i) *$\mathscr{F}^{\heartsuit\blacktriangledown}$ forms a dormant $\mathrm{GL}_{p^N-n}^{(N)}$-oper on $\mathscr{X}$. Moreover, there exists a canonical isomorphism $\mathscr{F}^\heartsuit \xrightarrow{\sim} \mathscr{F}^{\heartsuit\blacktriangledown\blacktriangledown}$ of $\mathrm{GL}_n^{(N)}$-opers.*

(ii) *Let $(\mathcal{N}, \nabla_{\mathcal{N}})$ be a dormant $\mathcal{D}^{(N-1)}$-module such that $\mathcal{N}$ is a line bundle. Then, there exists a canonical isomorphism of $\mathrm{GL}_{p^N-n}^{(N)}$-opers*

$$(\mathscr{F}^{\heartsuit\blacktriangledown})_{\otimes(\mathcal{N}^\vee, \nabla_{\mathcal{N}}^\vee)} \cong (\mathscr{F}^\heartsuit_{\otimes(\mathcal{N}, \nabla_{\mathcal{N}})})^\blacktriangledown.$$

Proof. First, let us consider assertion (i). The formation of $\mathscr{F}^{\heartsuit\blacktriangledown}$ is compatible with that of $(E^\blacktriangledown, \nabla^\blacktriangledown, v^\blacktriangledown)$ (cf. (3.10)) via restriction to D_x for every point x of X. This implies that $\mathscr{F}^{\heartsuit\blacktriangledown}$ forms a dormant $\mathrm{GL}_{p^N-n}^{(N)}$-oper (cf. Proposition 5.2.3). Also, let us consider the following diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Ker}(\nu_{\mathscr{F}^{\heartsuit\blacktriangledown}}) & \xrightarrow{\text{inclusion}} & \mathcal{P}_{\Omega^{\otimes(p^N-1)}\otimes\mathcal{L}^\vee} & \xrightarrow{\nu_{\mathscr{F}^{\heartsuit\blacktriangledown}}} & \mathcal{F}^\blacktriangledown & \longrightarrow & 0 \\
 & & & & \Big\downarrow {\scriptstyle \nu_{\mathscr{P}_{\mathcal{L}}^{\heartsuit\blacktriangledown}}} & & \Big\downarrow {\scriptstyle \wr\ \mathrm{id}} & & \\
0 & \longrightarrow & \mathcal{F}^\vee & \xrightarrow[\nu_{\mathscr{F}^\heartsuit}^\vee]{} & \mathcal{P}_{\mathcal{L}}^\vee & \xrightarrow[\text{quotient}]{} & \mathcal{F}^\blacktriangledown & \longrightarrow & 0.
\end{array}$$

The right-hand square is commutative because its restriction to D_x for every point x of X is commutative, as observed in the proof of Theorem 3.3.2, (i). Hence, it induces a morphism $\mathrm{Ker}(\nu_{\mathscr{F}^{\heartsuit\blacktriangledown}}) \xrightarrow{\sim} \mathcal{F}^\vee$. By considering the local description again, one verifies that the dual of this isomorphism specifies an isomorphism $\mathscr{F}^\heartsuit \xrightarrow{\sim} \mathscr{F}^{\heartsuit\blacktriangledown\blacktriangledown}$ of $\mathrm{GL}_n^{(N)}$-opers. This completes the proof of assertion (i).

Next, we shall prove assertion (ii). Consider the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{N}^\vee \otimes \mathcal{F}^\vee & \xrightarrow{\mathrm{id}_{\mathcal{N}^\vee}\otimes \nu^\vee_{\mathscr{F}^\heartsuit}} & \mathcal{N}^\vee \otimes \mathcal{P}^\vee_{\mathcal{L}} & \xrightarrow{\text{quotient}} & \mathcal{N}^\vee \otimes \mathrm{Ker}(\nu_{\mathscr{F}^\heartsuit})^\vee & \longrightarrow & 0 \\ & & \downarrow \wr & & \downarrow \wr & & & & \\ 0 & \longrightarrow & (\mathcal{N}\otimes \mathcal{F})^\vee & \xrightarrow[\nu^\vee_{\mathscr{F}^\heartsuit}]{} & \mathcal{P}^\vee_{\mathcal{N}\otimes\mathcal{L}} & \xrightarrow[\text{quotient}]{} & \mathrm{Ker}(\nu_{\mathscr{F}^\heartsuit_{\otimes(\mathcal{N},\nabla_{\mathcal{N}})}})^\vee & \longrightarrow & 0, \end{array}$$

where the middle vertical arrow denotes the dual of (6.1) and the left-hand vertical arrow denotes the canonical isomorphism. Since the left-hand square diagram is commutative, this diagram induces an isomorphism $\mathcal{N}^\vee \otimes \mathrm{Ker}(\nu_{\mathscr{F}^\heartsuit})^\vee \xrightarrow{\sim} \mathrm{Ker}(\nu_{\mathscr{F}^\heartsuit_{\otimes(\mathcal{N},\nabla_{\mathcal{N}})}})^\vee$. This isomorphism specifies an isomorphism $(\mathscr{F}^{\heartsuit\blacktriangledown})_{\otimes(\mathcal{N}^\vee,\nabla^\vee_{\mathcal{N}})} \xrightarrow{\sim} (\mathscr{F}^\heartsuit_{\otimes(\mathcal{N},\nabla_{\mathcal{N}})})^\blacktriangledown$ of $\mathrm{GL}^{(N)}_{p^N-n}$-opers. This completes the proof of assertion (ii). □

Remark 6.2.3. In this remark, we shall examine the determinant of a GL_n-oper. Let $\mathscr{F}^\heartsuit := (\mathcal{F}, \nabla, \{\mathcal{F}^j\}_j)$ be as above. Since $\mathcal{D}^{(N-1)}_{<j+1}/\mathcal{D}^{(N-1)}_{<j} \cong \mathcal{T}^{\otimes j}$ $(j = 0, 1, 2, \cdots)$, we obtain the composite of canonical isomorphisms

$$\begin{aligned} \det(\mathcal{P}_{\mathcal{L}}) &\xrightarrow{\sim} \bigotimes_{j=0}^{p^N-1} \mathcal{P}^j_{\mathcal{L}}/\mathcal{P}^{j+1}_{\mathcal{L}} \\ &\xrightarrow{\sim} \bigotimes_{j=0}^{p^N-1} \mathcal{T}^{\otimes(p^N-j-1)} \otimes \mathcal{L} \\ &\xrightarrow{\sim} \mathcal{T}^{\otimes p^N(p^N-1)/2} \otimes \mathcal{L}^{\otimes p^N} \\ &\xrightarrow{\sim} F^{(N)*}_{X/k}(\mathcal{N}_{\mathcal{L}}), \end{aligned} \tag{6.2}$$

where $\mathcal{N}_{\mathcal{L}}$ denotes the line bundle on $X^{(N)}$ corresponding to $\mathcal{T}^{\otimes(p^N-1)/2} \otimes \mathcal{L}$ via base-change $X^{(N)} \xrightarrow{\sim} X$ by F^N_k. Similarly, there exists an isomorphism

$$\det(\mathcal{F}) \xrightarrow{\sim} \mathcal{T}^{\otimes n(n-1)/2} \otimes \mathcal{L}^{\otimes n}. \tag{6.3}$$

The $\mathcal{D}^{(N-1)}$-module structure on the determinant bundle $\det(\mathcal{P}_{\mathcal{L}})$ induced by $\nabla_{\mathcal{P}_{\mathcal{L}}}$ corresponds to $\nabla^{\mathrm{can}}_{\mathcal{N}_{\mathcal{L}}}$ via (6.2). Hence, the determinant of $\nabla^\blacktriangledown$ corresponds to $(\nabla^{\mathrm{can}}_{\mathcal{N}_{\mathcal{L}}})^\vee \otimes \det(\nabla)$ via the following composite of natural isomorphisms:

$$\det(\mathcal{F}^\blacktriangledown) \xrightarrow{\sim} \det(\mathrm{Ker}(\nu_{\mathscr{F}^\heartsuit}))^\vee \xrightarrow{\sim} \det(\mathcal{P}_{\mathcal{L}})^\vee \otimes \det(\mathcal{F}) \xrightarrow{\sim} F^{(N)*}_{X/k}(\mathcal{N}_{\mathcal{L}})^\vee \otimes \det(\mathcal{F}).$$

6.3. **Duality for dormant** $\mathrm{PGL}^{(N)}_n$ **opers.** By applying Proposition 6.2.2, (i), we obtain a bijective correspondence

$$\begin{pmatrix} \text{the set of isomorphism classes} \\ \text{of dormant } \mathrm{GL}^{(N)}_n\text{-opers on } \mathscr{X} \end{pmatrix} \xrightarrow{\sim} \begin{pmatrix} \text{the set of isomorphism classes} \\ \text{of dormant } \mathrm{GL}^{(N)}_{p^N-n}\text{-opers on } \mathscr{X} \end{pmatrix}. \tag{6.4}$$

Moreover, this correspondence and Proposition 6.2.2, (ii), together imply the following assertion, which is a part of Theorem B.

Theorem 6.3.1. (i) *The assignment* $\mathscr{F}^{\heartsuit \Rightarrow \spadesuit} \mapsto \mathscr{F}^{\heartsuit \blacktriangledown \Rightarrow \spadesuit}$ *defines a well-defined bijection of sets*

$$\mathcal{C}_n : \mathrm{Op}_n^{\mathrm{Zzz...}} \xrightarrow{\sim} \mathrm{Op}_{p^N-n}^{\mathrm{Zzz...}}$$

satisfying $\mathcal{C}_{p^N-n} \circ \mathcal{C}_n = \mathrm{id}$.

(ii) *Suppose further that* $r > 0$. *Let* $\vec{\rho} := (\rho_i)_{i=1}^r$ *be an element of* $(\mathfrak{S}_n \backslash (\mathbb{Z}/p^N\mathbb{Z})^n/\Delta)^r$. *Then,* $\mathcal{C}_n$ *restricts to a bijection*

$$\mathcal{C}_{n,\vec{\rho}} : \mathrm{Op}_{n,\vec{\rho}}^{\mathrm{Zzz...}} \xrightarrow{\sim} \mathrm{Op}_{p^N-n,\vec{\rho}^{\blacktriangledown}}^{\mathrm{Zzz...}},$$

where $\vec{\rho}^{\blacktriangledown} := (\rho_i^{\blacktriangledown})_{i=1}^r$ *(cf. (4.8)).*

Proof. Assertions follow from Propositions 4.4.3 and 6.2.2. □

Moreover, the duality theorem established above implies the following assertion, which is the remaining portion of Theorem B; note that this assertion generalizes results proved by Y. Hoshi (cf. [Hos1, Theorem A]) and the author (cf. [Wak1, Corollary 4.3.3]).

Corollary 6.3.2. (i) *Let* $\mathcal{L}$ *be a line bundle on* X *and* $\nabla_{\circledcirc}$ *a dormant* $\mathcal{D}^{(N-1)}$*-module structure on* $\mathcal{T}^{\otimes n(n-1)/2} \otimes \mathcal{L}^{\otimes n}$. *Then, there exists exactly one isomorphism class of dormant* $\mathrm{GL}_{p^N-1}^{(N)}$*-oper* $\mathscr{F}^{\heartsuit} := (\mathcal{F}, \nabla, \{\mathcal{F}^j\}_j)$ *on* $\mathscr{X}$ *such that* $\mathcal{F}^{n-1} = \mathcal{L}$ *and* $\det(\nabla) = \nabla_{\circledcirc}$ *under the identification* $\det(\mathcal{F}) = \mathcal{T}^{\otimes n(n-1)/2} \otimes \mathcal{L}^{\otimes n}$ *given by* (6.3)*.*

(ii) *There exists exactly one isomorphism class of dormant* $\mathrm{PGL}_{p^N-1}^{(N)}$*-oper, i.e., the following equality holds:*

$$\sharp(\mathrm{Op}_{p^N-1}^{\mathrm{Zzz...}}) = 1.$$

Moreover, if $r > 0$*, then the radius of the unique dormant* $\mathrm{PGL}_{p^N-1}^{(N)}$*-oper at any marked point coincides with* $\pi([1, 2, \cdots, p^N - 1])$ *(cf. (5.6)).*

Proof. Assertion (i) follows from the observation that the desired $\mathrm{GL}_{p^N-1}^{(N)}$-oper is the unique one corresponding, via (6.4), to the dormant $\mathrm{GL}_1^{(N)}$-oper

$$(F_{X/k}^{(N)*}(\mathcal{N}_{\mathcal{L}})^{\vee} \otimes \mathcal{T}^{\otimes n(n-1)/2} \otimes \mathcal{L}^{\otimes n}, (\nabla_{\mathcal{N}_{\mathcal{L}}}^{\mathrm{can}})^{\vee} \otimes \nabla_{\circledcirc})$$

(cf. Remarks 5.2.2, (i), and 6.2.3). Assertion (ii) follows from the bijection $\mathcal{C}_1$ asserted in Theorem 6.3.1, (i), and the equalities $\sharp(\mathrm{Op}_1^{\mathrm{Zzz...}}) = \sharp(\mathrm{Op}_{1,(\pi([0]),\cdots,\pi([0]))}^{\mathrm{Zzz...}}) = 1$. □

7. Tamely ramified coverings and dormant PGL_2-opers

Recall (cf. [Moc], [Oss2], [Oss4]) that certain tamely ramified coverings with ramification indices $< p$ between two copies of the projective line can be described in terms of dormant $\mathrm{PGL}_2^{(1)}$-opers. That description is the starting point of the enumerative geometry of dormant opers because it allows us to translate dormant PGL_2-opers on a 3-pointed projective line into simple combinatorial data. In this section, the situation is generalized to the case of higher level in order to deal with tamely ramified covering having large ramification indices. Theorem C will be proved at the end of this section.

7.1. **Dormant PGL_2-opers arising from tamely ramified coverings.** Denote by $\mathbb{P}$ the projective line over k, i.e., $\mathbb{P} := \mathrm{Proj}(k[x_1, x_2])$. Let $\mathscr{X} := (X, \{\sigma_i\}_{i=1}^r)$ be as before, and take an r-tuple of integers $\vec{\lambda} := (\lambda_1, \cdots, \lambda_r)$ with $0 < \lambda_i < p^N$ $(i = 1, \cdots, r)$. We shall write $\vec{\rho} := (\rho_1, \cdots, \rho_r)$, where $\rho_i := \frac{1}{2} \cdot \overline{\lambda}_i \in (\mathbb{Z}/p^N\mathbb{Z})/\{\pm 1\}$. Under the identification $(\mathbb{Z}/p^N\mathbb{Z})/\{\pm 1\} = \mathfrak{S}_2 \backslash (\mathbb{Z}/p^N\mathbb{Z})^2/\Delta$ defined in (5.7), $\vec{\rho}$ may be regarded as an element of $(\mathfrak{S}_2 \backslash (\mathbb{Z}/p^N\mathbb{Z})^2/\Delta)^r$. We shall denote by

$$\mathrm{Cov}_{\vec{\lambda}}$$

the set of equivalence classes of finite, separable, and tamely ramified coverings $\phi : X \to \mathbb{P}$ that are ramified at σ_i with index λ_i and étale elsewhere. Here, the equivalence relation is defined in such a way that two coverings $\phi_1, \phi_2 : X \to \mathbb{P}^1$ are equivalent if there exists an element $h \in \mathrm{PGL}_2(k)\,(= \mathrm{Aut}_k(\mathbb{P}))$ with $\phi_2 = h \circ \phi_1$. For each ϕ as above, we shall denote by $[\phi]$ the element of $\mathrm{Cov}_{\vec{\lambda}}$ (i.e., the equivalence class) represented by ϕ.

In what follows, let us construct a map of sets $\mathrm{Cov}_{\vec{\lambda}} \to \mathrm{Op}^{\mathrm{Zzz...}}_{2,\vec{\rho}}$. Let us take an element $[\phi]$ of $\mathrm{Cov}_{\vec{\lambda}}$, and choose a tamely ramified covering ϕ representing $[\phi]$. Let $\sigma'_1, \cdots, \sigma'_{r'}$ $(0 < r' \leq r)$ be the mutually distinct points of $\mathbb{P}$ such that $\bigcup_{i=1}^r \{\phi(\sigma_i)\} = \{\sigma'_j\}_{j=1}^{r'}$. In particular, $\mathscr{P}' := (\mathbb{P}, \{\sigma'_j\}_{j=1}^{r'})$ defines an r'-pointed genus-0 curve; we denote the induced log curve by $\mathbb{P}^{\mathrm{log}'}$. Since ϕ is tamely ramified, the morphism ϕ extends to a *log étale* morphism $\phi^{\mathrm{log}} : X^{\mathrm{log}} \to \mathbb{P}^{\mathrm{log}'}$. Write $\mathcal{L} := \mathcal{O}_{\mathbb{P}}(-1) \otimes \mathcal{O}_{\mathbb{P}}(\sum_{j=1}^{r'} \sigma'_j)$, and write τ_0 for the $\mathcal{O}_{\mathbb{P}}$-linear injection $\mathcal{O}_{\mathbb{P}}(-1) \hookrightarrow \mathcal{O}_{\mathbb{P}}^{\oplus 2}$ given by $w \mapsto (wx_1, wx_2)$ for each local section $w \in \mathcal{O}_{\mathbb{P}}(-1)$. Also, let $\mathcal{F}$ be a rank 2 vector bundle on $\mathbb{P}$ which makes the following square diagram cocartesian:

$$\begin{array}{ccc} \mathcal{O}_{\mathbb{P}}(-1) & \xrightarrow{\tau_0} & \mathcal{O}_{\mathbb{P}}^{\oplus 2} \\ {\scriptstyle \mathrm{inclusion}} \downarrow & & \downarrow \\ \mathcal{L} & \longrightarrow & \mathcal{F}. \end{array}$$

The trivial $\mathcal{D}^{(N-1)}_{\mathbb{P}^{\mathrm{log}'}}$-module structure on $\mathcal{O}_{\mathbb{P}}^{\oplus 2}$ extends uniquely to a $\mathcal{D}^{(N-1)}_{\mathbb{P}^{\mathrm{log}'}}$-module structure $\nabla_{\mathcal{F}}$ on $\mathcal{F}$. It follows from the various definitions involved that the composite

$$\mathcal{L} \xrightarrow{\mathrm{inclusion}} \mathcal{F} \xrightarrow{\nabla_{\mathcal{F}}} \Omega_{\mathbb{P}^{\mathrm{log}'}/k} \otimes \mathcal{F} \twoheadrightarrow \Omega_{\mathbb{P}^{\mathrm{log}'}/k} \otimes (\mathcal{F}/\mathcal{L})$$

is $\mathcal{O}_{\mathbb{P}}$-linear and injective. Moreover, since $\deg(\mathcal{L}) = \det(\Omega_{\mathbb{P}^{\mathrm{log}'}/k} \otimes (\mathcal{F}/\mathcal{L}))\,(= r' - 1)$, this morphism is an isomorphism. This means that the triple $(\mathcal{F}, \nabla_{\mathcal{F}}, \mathcal{L})$ forms a dormant $\mathrm{GL}_2^{(N)}$-oper on $\mathscr{P}'$ (cf. Remark 5.2.2, (ii)). Hence, the pull-back of this data via the log étale morphism ϕ^{log} defines a dormant $\mathrm{GL}_2^{(N)}$-oper

$$\mathscr{F}_{\phi}^{\heartsuit} := \phi^{\mathrm{log}*}(\mathcal{F}, \nabla_{\mathcal{F}}, \mathcal{L})$$

on $\mathscr{X}$.

Proposition 7.1.1. *The dormant $\mathrm{PGL}_2^{(N)}$-oper $\mathscr{F}_{\phi}^{\heartsuit\Rightarrow\spadesuit}$ on $\mathscr{X}$ represented by $\mathscr{F}_{\phi}^{\heartsuit}$ is of radii $\vec{\rho}$.*

Proof. The problem is the computation of the radii of $\mathscr{F}_{\phi}^{\spadesuit}$. Let us take $i \in \{1, \cdots, r\}$, and choose $j \in \{1, \cdots, r'\}$ such that $\sigma'_j = \phi(\sigma_i)$. Also, choose a local function t on $\mathbb{P}$ defining σ'_j. This local function allows us to identify the formal neighborhood $\widehat{D}_{\sigma'_j}$ of σ'_j in $\mathbb{P}$ with

$\mathrm{Spec}(k[\![t]\!])$. Since the ramification index of ϕ at σ_i is λ_i, the formal neighborhood $\widehat{D}_{\sigma_i}$ of σ_i in X may be identified with $\mathrm{Spec}(k[\![t^{1/\lambda_i}]\!])$ and the restriction of ϕ to $\widehat{D}_{\sigma_i}$ may be identified with the morphism $\mathrm{Spec}(k[\![t^{1/\lambda_i}]\!]) \to \mathrm{Spec}(k[\![t]\!])$ induced by the natural inclusion $k[\![t]\!] \hookrightarrow k[\![t^{1/\lambda_i}]\!]$. The $\breve{D}^{(N-1)}_{k[\![t]\!]}$-module corresponding to the restriction of $(\mathcal{F}, \nabla)$ to $\widehat{D}_{\sigma'_j}$ is isomorphic to $(k[\![t]\!], \widehat{\nabla}_0) \oplus (t^{-1} \cdot k[\![t]\!], \widehat{\nabla}_0)$. It follows that the pull-back of $(\mathcal{F}, \nabla)$ to $\phi^{\log}$ restricted to $\widehat{D}_{\sigma_i}$ is isomorphic to $(k[\![t^{1/\lambda_i}]\!], \widehat{\nabla}_0) \oplus ((t^{1/\lambda_i})^{-\lambda_i} \cdot k[\![t^{1/\lambda_i}]\!], \widehat{\nabla}_0)$ (which is also isomorphic to $(k[\![s]\!], \widehat{\nabla}_0) \oplus (k[\![s]\!], \widehat{\nabla}_{\overline{\lambda}_i})$ by putting $s := t^{1/\lambda_i}$). Hence, the exponent of $\mathscr{F}^{\heartsuit}_{\phi}$ at σ_i coincides with $[0, \overline{\lambda}_i]$, which implies $\rho_{\mathscr{F}^{\heartsuit \Rightarrow \spadesuit}_{\phi}, i} = \rho_i$. This completes the proof of this proposition. $\square$

Since τ_0 is invariant under pull-back by automorphisms of $\mathbb{P}$, the isomorphism class of $\mathscr{F}^{\heartsuit}_{\phi}$ does not depend on the choice of the representative ϕ of $[\phi]$. Hence, the above proposition implies that the assignment $[\phi] \mapsto \mathscr{F}^{\spadesuit}_{\phi}$ gives a well-defined map of sets

$$\Upsilon_{\vec{\lambda}} : \mathrm{Cov}_{\vec{\lambda}} \to \mathrm{Op}^{\mathrm{Zzz...}}_{2, \vec{\rho}} . \tag{7.1}$$

7.2. **Tamely ramified endomorphisms of a 3-pointed projective line.** Denote by $[0]$, $[1]$, and $[\infty]$ the k-rational points of $\mathbb{P}$ determined by the values 0, 1, and ∞ respectively. After ordering the three points $[0], [1], [\infty]$, we obtain a unique (up to isomorphism) 3-pointed proper smooth curve

$$\mathscr{P} := (\mathbb{P}/k, \{[0], [1], [\infty]\})$$

of genus 0 over k. In particular, we obtain a log curve $\mathbb{P}^{\log}$ over k.

Next, let us take a triple $(\rho_0, \rho_1, \rho_\infty)$ of elements of $(\mathbb{Z}/p^N\mathbb{Z})^\times/\{\pm 1\}$. There exists the triple of integers $(\lambda_0, \lambda_1, \lambda_\infty)$ satisfying the following conditions:

(a) $2 \cdot \rho_x = \lambda_x$ as elements of $(\mathbb{Z}/p^N\mathbb{Z})/\{\pm 1\}$ and $0 < \lambda_x < p^N$ for every $x = 0, 1, \infty$;
(b) The sum $\lambda_0 + \lambda_1 + \lambda_\infty$ is odd $< 2 \cdot p^N$.

Let us write $\mathcal{O}^+_{\mathbb{P}} := \mathcal{O}_{\mathbb{P}}(\lambda_0 \cdot [0] + \lambda_1 \cdot [1] + \lambda_\infty \cdot [\infty])$. Note that there is a unique $\mathcal{D}^{(N-1)}_{\mathbb{P}^{\log}}$-module structure ∇^+ on $\mathcal{O}^+_{\mathbb{P}}$ whose restriction to $U := \mathbb{P} \setminus \{[0], [1], [\infty]\}$ coincides with the trivial $\mathcal{D}^{(N-1)}_U$-module structure on $\mathcal{O}^+_{\mathbb{P}}|_U = \mathcal{O}_U$ (cf. Remark 3.1.1).

The following assertion is a special case of [Oss1, Theorem 3.3, (ii)]; we shall prove it by a relatively elementary argument.

Proposition 7.2.1. *Let $\phi : \mathbb{P} \to \mathbb{P}$ be a tamely ramified covering classified by $\mathrm{Cov}_{(\lambda_0, \lambda_1, \lambda_\infty)}$ in the case where "$\mathscr{X}$" is taken to be $\mathscr{P}$. Then, the points $\phi([0]), \phi([1]), \phi([\infty])$ are mutually distinct.*

Proof. First, we shall suppose that the set $\phi(\{[0], [1], [\infty]\})$ consists of one point. By considering the fiber of ϕ over this point, we see that $\deg(\phi) \geq \lambda_0 + \lambda_1 + \lambda_\infty$. It follows from the Riemann-Hurwitz formula that

$$-2 \, (= 2 \cdot (\text{genus of } \mathbb{P}) - 2) = -2 \cdot \deg(\phi) + \sum_{x = 0, 1, \infty} (\lambda_x - 1) \leq -(\lambda_0 + \lambda_1 + \lambda_\infty) - 3.$$

Thus, we obtain a contradiction. Next, suppose that $\phi(\{[0], [1], [\infty]\})$ consists of two points. After possibly applying a linear transformation, we may assume that these two elements coincide with $\{[0], [\infty]\}$, and that $\{[0]\} \subseteq \phi^{-1}([0])$ and $\{[1], [\infty]\} \subseteq \phi^{-1}([\infty])$. Hence, ϕ defines a tamely ramified covering of $\mathbb{G}_m \, (= \mathbb{P} \setminus \{[0], [\infty]\})$. Recall that the tame fundamental group

$\pi_1^{\mathrm{tame}}(\mathbb{G}_m)$ of $\mathbb{G}_m$ is isomorphic to $\widehat{\mathbb{Z}}^{p'}$, the maximal prime-to-$p$ quotient of $\widehat{\mathbb{Z}} := \varprojlim_{n \in \mathbb{Z}_{\geq 0}} \mathbb{Z}/n\mathbb{Z}$. A topological generator σ of $\pi_1^{\mathrm{tame}}(\mathbb{G}_m)$ acts on the fiber over the point near $[0]$ as a cyclic permutation. On the other hand, σ^{-1} acts on the fiber over the point near $[\infty]$ as a product of two disjoint cyclic permutations. This is a contradiction. Hence, the the image $\phi(\{[0], [1], [\infty]\})$ consists of three points. This completes the proof of this assertion. □

Remark 7.2.2. Because of Proposition 7.2.1, each element of $\mathrm{Cov}_{(\lambda_0, \lambda_1, \lambda_\infty)}$ has a unique representative $\phi : \mathbb{P} \to \mathbb{P}$ satisfying $\phi([x]) = [x]$ for every $x = 0, 1, \infty$. Following [ABEGKM] (or [BEK]), we call such a covering a *dynamical Belyi map*.

Now, let us take a dynamical Belyi map $\phi : \mathbb{P} \to \mathbb{P}$ classified by $\mathrm{Cov}_{(\lambda_0, \lambda_1, \lambda_\infty)}$; this covering corresponds to a representation of the tame fundamental group $\pi_1^{\mathrm{tame}}(\mathbb{P} \setminus \{[0], [1], [\infty]\})$ (which is obtained as a quotient of the profinite completion $\widehat{\Pi}_{0,3}$ of the group $\Pi_{0,3} := \langle \gamma_0, \gamma_1, \gamma_\infty \mid \gamma_0 \gamma_1 \gamma_\infty = 1 \rangle$). Hence, by the above proposition, ϕ determines three cyclic permutations σ_0, σ_1, and σ_∞ (in the symmetric group $\mathfrak{S}_d$ of d letters for some $d \geq 1$) of orders λ_0, λ_1, and λ_∞, respectively, satisfying $\sigma_0 \circ \sigma_1 = \sigma_\infty$. A trivial elementary argument shows that this condition implies the following inequalities:

$$|\lambda_0 - \lambda_1| < \lambda_\infty < \lambda_0 + \lambda_1. \tag{7.2}$$

These inequalities also can be obtained by the inequality $\deg(\phi) \left(= \frac{\lambda_0 + \lambda_1 + \lambda_\infty - 1}{2}\right) \geq \lambda_0, \lambda_1, \lambda_\infty$.

Conversely, suppose that a subgroup of $\mathfrak{S}_d$ generated by three cyclic permutations σ_0, σ_1, σ_∞ with $\sigma_0 \circ \sigma_1 = \sigma_\infty$ has order prime to p. Then, the assignment $\gamma_x \mapsto \sigma_x$ ($x = 0, 1, \infty$) induces a representation $\pi_1^{\mathrm{tame}}(\mathbb{P} \setminus \{[0], [1], [\infty]\}) \to \mathfrak{S}_d$ because the surjection $\widehat{\Pi}_{0,3} \twoheadrightarrow \pi_1^{\mathrm{tame}}(\mathbb{P} \setminus \{[0], [1], [\infty]\})$ becomes bijective after taking their maximal prime-to-p quotients. In particular, the corresponding tamely ramified covering is classified by $\mathrm{Cov}_{(\lambda_0, \lambda_1, \lambda_\infty)}$. See [BEK] for the study concerning the relationship between such cyclic permutations and dynamical Belyi maps in characteristic p.

7.3. **Dormant PGL_2-opers on a 3-pointed projective line.** In what follows, we shall prove that the map $\Upsilon_{\vec{\lambda}}$ defined in (7.1) becomes a bijection when $\mathscr{X} = \mathscr{P}$. To do this, we construct its inverse map. We first prove the following proposition.

Proposition 7.3.1. *Let $\mathscr{F}^\spadesuit$ be a dormant $\mathrm{PGL}_2^{(N)}$-oper on $\mathscr{P}$. Then, there exists a dormant $\mathrm{GL}_2^{(N)}$-oper $\mathscr{F}^\heartsuit := (\mathcal{F}, \nabla, \mathcal{L})$ on $\mathscr{P}$ satisfying $\mathscr{F}^{\heartsuit \Rightarrow \spadesuit} \cong \mathscr{F}^\spadesuit$ and $\det(\mathcal{F}, \nabla) \cong (\mathcal{O}_{\mathbb{P}}^+, \nabla^+)$, Moreover, such a $\mathrm{GL}_2^{(N)}$-oper is uniquely determined up isomorphism.*

Proof. First, we prove the existence portion. Let us take a dormant $\mathrm{GL}_2^{(N)}$-oper $\mathscr{F}^\heartsuit := (\mathcal{F}, \nabla, \mathcal{L})$ on $\mathscr{P}$ with $\mathscr{F}^{\heartsuit \Rightarrow \spadesuit} \cong \mathscr{F}^\spadesuit$. Since $\mathcal{L} \cong \Omega_{\mathbb{P}^{\mathrm{log}}} \otimes (\mathcal{F}/\mathcal{L})$ (cf. (5.5)), the following equalities hold:

$$\begin{aligned} \deg(\mathcal{O}_{\mathbb{P}}^+ \otimes \det(\mathcal{F})^\vee) &= \lambda_0 + \lambda_1 + \lambda_\infty - \deg(\mathcal{L}) - \deg(\mathcal{F}/\mathcal{L}) \\ &= \lambda_0 + \lambda_1 + \lambda_\infty + 1 - 2 \cdot \deg(\mathcal{L}). \end{aligned}$$

In particular, the degree of $\mathcal{O}_{\mathbb{P}}^+ \otimes \det(\mathcal{F})^\vee$ is even. Hence, it follows from Lemma 5.1.4 that there exists a dormant $\mathcal{D}_{\mathbb{P}^{\mathrm{log}}}^{(N-1)}$-bundle $(\mathcal{N}, \nabla_{\mathcal{N}})$ such that $\mathcal{N}$ is a line bundle with $\mathcal{N}^{\otimes 2} \cong (\mathcal{O}_{\mathbb{P}}^+, \nabla^+) \otimes \det(\mathcal{F}, \nabla_{\mathcal{F}})^\vee$. By putting $(\mathcal{F}', \nabla', \mathcal{L}') := \mathscr{F}^\heartsuit_{\otimes (\mathcal{N}, \nabla_{\mathcal{N}})}$, we have

$$\det(\mathcal{F}', \nabla') \cong (\mathcal{N}, \nabla_{\mathcal{N}})^{\otimes 2} \otimes \det(\mathcal{F}, \nabla) \cong (\mathcal{O}_{\mathbb{P}}^+, \nabla^+).$$

Thus, $(\mathcal{F}', \nabla', \mathcal{L}')$ specifies the required $\mathrm{GL}_2^{(N)}$-oper.

Next, we shall prove the uniqueness portion. Let $\mathscr{F}_i^\heartsuit := (\mathcal{F}_i, \nabla_i, \mathcal{L}_i)$ $(i = 1, 2)$ be dormant $\mathrm{GL}_2^{(N)}$-opers on $\mathscr{P}$ satisfying the required conditions. Since $\mathscr{F}_1^{\heartsuit \Rightarrow \spadesuit} = \mathscr{F}_2^{\heartsuit \Rightarrow \spadesuit}$, there exists a dormant $\mathcal{D}_{\mathbb{P}^{\log}}^{(N-1)}$-module $(\mathcal{N}, \nabla_{\mathcal{N}})$ such that $\mathcal{N}$ is a line bundle and $\mathscr{F}_2^\heartsuit \cong (\mathscr{F}_1^\heartsuit)_{\otimes(\mathcal{N}, \nabla_{\mathcal{N}})}$. If ∇^{triv} denotes the trivial $\mathcal{D}_{\mathbb{P}^{\log}}^{(N-1)}$-module structure on $\mathcal{O}_{\mathbb{P}}$, then we have

$$\begin{aligned}(\mathcal{O}_{\mathbb{P}}, \nabla^{\mathrm{triv}}) &\cong \det(\mathcal{F}_2, \nabla_2) \otimes (\mathcal{O}_{\mathbb{P}}^+, \nabla^+)^\vee \\ &\cong \det((\mathcal{N}, \nabla_{\mathcal{N}}) \otimes (\mathcal{F}_1, \nabla_1)) \otimes (\mathcal{O}_{\mathbb{P}}^+, \nabla^+)^\vee \\ &\cong (\mathcal{N}, \nabla_{\mathcal{N}})^{\otimes 2} \otimes \det(\mathcal{F}_1, \nabla_1) \otimes (\mathcal{O}_{\mathbb{P}}^+, \nabla^+)^\vee \\ &\cong (\mathcal{N}, \nabla_{\mathcal{N}})^{\otimes 2}.\end{aligned}$$

Since $\mathrm{Pic}(\mathbb{P}) \cong [\mathcal{O}_{\mathbb{P}}(1)] \cdot \mathbb{Z}$, the line bundle $\mathcal{N}$ may be identified with $\mathcal{O}_{\mathbb{P}}$. Moreover, by the uniqueness portion of Proposition 5.1.4, $\nabla_{\mathcal{N}}$ coincides with ∇^{triv} via a fixed identification $\mathcal{N} = \mathcal{O}_{\mathbb{P}}$. Thus, we have $(\mathscr{F}_1^\heartsuit)_{\otimes(\mathcal{N}, \nabla_{\mathcal{N}})} \cong \mathscr{F}_1^\heartsuit$, which implies that $\mathscr{F}_2^\heartsuit$ is isomorphic to $\mathscr{F}_1^\heartsuit$. This completes the proof of the uniqueness portion. □

Now, let us take a dormant $\mathrm{PGL}_2^{(N)}$-oper $\mathscr{F}^\spadesuit$ on $\mathscr{P}$ of radii $(\rho_0, \rho_1, \rho_\infty)$. Also, let $\mathscr{F}^\heartsuit := (\mathcal{F}, \nabla, \mathcal{L})$ be the dormant $\mathrm{GL}_2^{(N)}$-oper resulting from Proposition 7.3.1 applied to $\mathscr{F}^\spadesuit$. In particular, we have $\deg(\mathcal{L}) = \frac{\lambda_0 + \lambda_1 + \lambda_\infty + 1}{2}$ and $\deg(\mathcal{F}/\mathcal{L}) = \frac{\lambda_0 + \lambda_1 + \lambda_\infty - 1}{2}$. Denote by τ the $\mathcal{O}_{\mathbb{P}}$-linear morphism $F_{\mathbb{P}/k}^{(N)*}(\mathcal{F}^\nabla) \to \mathcal{F}$ extending the ($\mathcal{O}_{\mathbb{P}^{(N)}}$-linear) inclusion $\mathcal{F}^\nabla \hookrightarrow \mathcal{F}$; the morphism τ is compatible with $\nabla_{\mathcal{F}^\nabla}^{\mathrm{can}}$ (cf. Remark 5.1.2) and ∇. We shall write $\mathcal{L}^\natural := \mathcal{L} \cap \mathrm{Im}(\tau)$. Since the restriction of τ to $U := \mathbb{P} \setminus \{[0], [1], [\infty]\}$ is an isomorphism, the quotient sheaf $\mathcal{L}/\mathcal{L}^\natural$ is a torsion sheaf supported on $\{[0], [1], [\infty]\}$.

Lemma 7.3.2. *The length of $\mathcal{L}/\mathcal{L}^\natural$ at the marked point $x \in \{[0], [1], [\infty]\}$ is λ_x. Moreover, the natural inclusion $\mathcal{L}/\mathcal{L}^\natural \hookrightarrow \mathrm{Coker}(\tau)$ is an isomorphism.*

Proof. Let us fix $x \in \{[0], [1], [\infty]\}$, and choose a local function t defining x. The restriction of $(\mathcal{F}, \nabla)$ to the formal neighborhood $\widehat{D}_x$ of x may be expressed as $(k[\![t]\!], \widehat{\nabla}_{\overline{a}}) \oplus (k[\![t]\!], \widehat{\nabla}_{\overline{b}})$ for some integers a, b with $0 \leq b \leq a \leq p^N - 1$. The radius of ∇ at x coincides with ρ_x by assumption, so the equality $2 \cdot \rho_x = \overline{a - b}$ of elements in $(\mathbb{Z}/p^N\mathbb{Z})/\{\pm 1\}$ holds. Since $\det(\mathcal{F}, \nabla) \cong (\mathcal{O}_{\mathbb{P}}^+, \nabla^+)$, a computation using Proposition 4.2.1, (i), of the lengths at x of $\det(\mathcal{F}, \nabla)$ and $(\mathcal{O}_{\mathbb{P}}^+, \nabla^+)$ implies $a + b \equiv \lambda_x \pmod{p^N}$. Hence, since $a + b \leq 2 \cdot p^N$, it follows that $a + b$ is either λ_x or $\lambda_x + p^N$. We shall prove the claim that $a + b = \lambda_x$. Suppose, on the contrary, that $a + b = p^N + \lambda_x$. The condition that $0 \leq a - b \leq p^N - 1$ and $2 \cdot \rho_x = \overline{a - b}$ in $(\mathbb{Z}/p^N\mathbb{Z})/\{\pm 1\}$ implies that $a - b$ is either λ_x or $p^N - \lambda_x$. Since $a + b$ and $a - b$ have the same parity, the equality $a - b = p^N - \lambda_x$ holds. Hence, we have

$$2 \cdot a = (a + b) + (a - b) = (p^N + \lambda_x) + (p^N - \lambda_x) = 2 \cdot p^N.$$

This implies the equality $a = p^N$, which is a contradiction. This proves the claim.

By the claim just established, we have $a + b = \lambda_x$ and $a - b = \lambda_x$. In particular, $a = \lambda_x$ and $b = 0$, which means that the restriction of $(\mathcal{F}, \nabla)$ to $\widehat{D}_x$ is isomorphic to $(k[\![t]\!], \widehat{\nabla}_{\overline{\lambda_x}}) \oplus (k[\![t]\!], \widehat{\nabla}_0)$. It follows that the restriction of $\mathrm{Im}(\tau)$ to $\widehat{D}_x$ coincides with $t^{\lambda_x} \cdot k[\![t]\!] \oplus k[\![t]\!]$ $(\subseteq k[\![t]\!]^{\oplus 2})$. On the other hand, according to the proof in Proposition 4.4.1, the inclusion $\mathcal{L} \hookrightarrow \mathcal{F}$ corresponds, after choosing a suitable trivialization $\Gamma(\widehat{D}_x, \mathcal{L}|_{\widehat{D}_x}) \xrightarrow{\sim} k[\![t]\!]$, to the $k[\![t]\!]$-linear morphism $k[\![t]\!] \to$

$k[\![t]\!]^{\oplus 2}$ given by $1 \mapsto (u_1, u_2)$ for some $u_1, u_2 \in k[\![t]\!]^\times$. By taking account of this observation, we see that the length of $\mathcal{L}/\mathcal{L}^\natural$ at x coincides with λ_x. Moreover, since the length of $\mathrm{Coker}(\tau)$ at x is $a + b = \lambda_x$, the inclusion $\mathcal{L}/\mathcal{L}^\natural \hookrightarrow \mathrm{Coker}(\tau)$ turns out to be an isomorphism. □

Lemma 7.3.3. *There exists an $\mathcal{O}_{\mathbb{P}^{(N)}}$-linear isomorphism $\gamma : \mathcal{O}_{\mathbb{P}^{(N)}}^{\oplus 2} \xrightarrow{\sim} \mathcal{F}^\nabla$.*

Proof. Since $\mathcal{F}^\nabla$ is a rank 2 vector bundle on the genus-0 curve $\mathbb{P}^{(N)}$, it is isomorphic to the direct sum of two line bundles. Let us fix an isomorphism $\gamma : \mathcal{O}_{\mathbb{P}^{(N)}}(a) \oplus \mathcal{O}_{\mathbb{P}^{(N)}}(b) \xrightarrow{\sim} \mathcal{F}^\nabla$, where a and b are some integers with $a \geq b$. The pull-back of γ via $F^{(N)}_{\mathbb{P}/k}$ defines an isomorphism $\gamma^F : \mathcal{O}_{\mathbb{P}}(p^N \cdot a) \oplus \mathcal{O}_{\mathbb{P}}(p^N \cdot b) \xrightarrow{\sim} F^{(N)*}_{\mathbb{P}/k}(\mathcal{F}^\nabla)$. It follows from Lemma 7.3.2 that

$$\begin{aligned}
\lambda_0 + \lambda_1 + \lambda_\infty &= \mathrm{length}(\mathcal{L}/\mathcal{L}^\natural) \\
&= \mathrm{length}(\mathrm{Coker}(\tau)) \\
&= \deg(\mathcal{F}) - \deg(F^{N*}(\mathcal{F}^\nabla)) \\
&= \lambda_0 + \lambda_1 + \lambda_\infty + p^N(a+b).
\end{aligned}$$

This implies $b = -a$. Next, let us consider the composite

$$\mathcal{O}_{\mathbb{P}}(p^N \cdot a) \xrightarrow{v \mapsto (v,0)} \mathcal{O}_{\mathbb{P}}(p^N \cdot a) \oplus \mathcal{O}_{\mathbb{P}}(-p^N \cdot a) \xrightarrow{\gamma^F} F^{(N)*}_{\mathbb{P}/k}(\mathcal{F}^\nabla) \xrightarrow{\tau} \mathcal{F} \twoheadrightarrow \mathcal{F}/\mathcal{L}. \tag{7.3}$$

We shall suppose that $a > 0$. The assumption $\lambda_0 + \lambda_1 + \lambda_\infty < 2 \cdot p^N$ implies

$$\deg(\mathcal{O}_{\mathbb{P}}(p^N \cdot a)) \geq p^N > \frac{\lambda_0 + \lambda_1 + \lambda_\infty - 1}{2} = \deg(\mathcal{F}/\mathcal{L}).$$

Hence, the composite (7.3) must be the zero map. It follows that the image $\mathcal{I}$ of the inclusion into the first factor $\mathcal{O}_{\mathbb{P}}(p^N \cdot a) \hookrightarrow \mathcal{O}_{\mathbb{P}}(p^N \cdot a) \oplus \mathcal{O}_{\mathbb{P}}(-p^N \cdot a)$ is contained in $\mathcal{L}\ (\subseteq \mathcal{F})$ via $\tau \circ \gamma^F$. But, since $\mathcal{I}$ is closed under $\nabla^{\mathrm{can}}_{\mathcal{O}_{\mathbb{P}^{(N)}}(a)} \oplus \nabla^{\mathrm{can}}_{\mathcal{O}_{\mathbb{P}^{(N)}}(b)}$, the line subbundle $\mathcal{L}$ must be closed under ∇. This contradicts the fact that the morphism (5.4) for $j = 1$ is an isomorphism. Thus, we conclude that $a = 0$, i.e., γ defines an isomorphism $\mathcal{O}_{\mathbb{P}}^{\oplus 2} \xrightarrow{\sim} \mathcal{F}^\nabla$. □

Let γ be as asserted in the above lemma. For convenience, we occasionally use the notation X to denote the underlying projective line $\mathbb{P}$. The pull-back $\gamma^F : \mathcal{O}_X^{\oplus 2} \xrightarrow{\sim} F^{(N)*}_{X/k}(\mathcal{F}^\nabla)$ of γ by $F^{(N)}_{X/k}$ induces a trivialization $\mathbb{P}(\gamma^F) : \mathbb{P} \times_k X \xrightarrow{\sim} \mathbb{P}(F^{(N)*}_{X/k}(\mathcal{F}^\nabla))$ of the $\mathbb{P}$-bundle $\mathbb{P}(F^{(N)*}_{X/k}(\mathcal{F}^\nabla))$ associated to $F^{(N)*}_{X/k}(\mathcal{F}^\nabla)$. The sheaf $\mathcal{L}^\natural$, regarded as a line bundle of $F^{(N)*}_{X/k}(\mathcal{F}^\nabla)$ via τ, defines a global section $\sigma : X \to \mathbb{P}(F^{(N)*}_{X/k}(\mathcal{F}^\nabla))$. Thus, we obtain the composite

$$\phi_{\mathscr{F}^\spadesuit} : X \xrightarrow{\sigma} \mathbb{P}(F^{(N)*}_{X/k}(\mathcal{F}^\nabla)) \xrightarrow{\mathbb{P}(\gamma^F)^{-1}} \mathbb{P} \times_k X \xrightarrow{\mathrm{pr}_1} \mathbb{P}.$$

Lemma 7.3.4. *The morphism $\phi_{\mathscr{F}^\spadesuit} : X \to \mathbb{P}^1$ defines a tamely ramified covering classified by the set $\mathrm{Cov}_{(\lambda_0, \lambda_1, \lambda_\infty)}$ (for $\mathscr{X} = \mathscr{P}$).*

Proof. Let x be a k-rational point of X. To complete the proof, we shall describe the morphism $\phi_{\mathscr{F}^\spadesuit}$ restricted to the formal neighborhood $\widehat{D}_x$ of x.

First, suppose that $x \in \{[0], [1], [\infty]\}$. As observed in the proof of Lemma 7.3.2, the $\breve{D}^{(N-1)}_{k[[t]]}$-module corresponding to the restriction of $(\mathcal{F}, \nabla)$ to $\widehat{D}_x$ is isomorphic to $(t^{-\lambda_x} \cdot k[\![t]\!], \widehat{\nabla}_0) \oplus$

$(k[\![t]\!], \widehat{\nabla}_0) \left(\cong (k[\![t]\!], \widehat{\nabla}_{\overline{\lambda}_x}) \oplus (k[\![t]\!], \widehat{\nabla}_0)\right)$. According to the proof of Proposition 4.4.1, the inclusion $\mathcal{L} \hookrightarrow \mathcal{F}$ restricted to $\widehat{D}_x$ may be identified, after choosing a suitable trivialization $\Gamma(\widehat{D}_x, \mathcal{L}|_{\widehat{D}_x}) \xrightarrow{\sim} k[\![t]\!]$, with the $k[\![t]\!]$-linear morphism $k[\![t]\!] \to t^{-\lambda_x} \cdot k[\![t]\!] \oplus k[\![t]\!]$ determined by $1 \mapsto (t^{-\lambda_x} \cdot 1, u)$ for some $u \in k[\![t]\!]^\times$. Under this identification, the inclusion $\mathcal{L}^\natural \hookrightarrow F^{(N)*}_{X/k}(\mathcal{F}^\nabla)$ restricted to $\widehat{D}_x$ corresponds to the inclusion $t^{\lambda_x} \cdot k[\![t]\!] \to k[\![t]\!]^{\oplus 2}$ determined by $t^{\lambda_x} \cdot 1 \mapsto (1, t^{\lambda_x} \cdot u)$. This implies that the restriction of $\phi_{\mathscr{F}^\spadesuit}$ to $\widehat{D}_x$ arises from the k-algebra endomorphism of $k[\![t]\!]$ given by $t \mapsto t^{\lambda_x} \cdot u$. Hence, the ramification index of $\phi_{\mathscr{F}^\spadesuit}$ at x coincides with λ_x.

Next, suppose that $x \in X \setminus \{[0], [1], [\infty]\}$. The inclusion $\mathcal{L} \hookrightarrow \mathcal{F}$ restricted to $\widehat{D}_x$ may be described, after choosing suitable trivializations of $\mathcal{L}|_{\widehat{D}_x}$ and $\mathcal{F}|_{\widehat{D}_x}$, as the $k[\![t]\!]$-linear morphism $k[\![t]\!] \to k[\![t]\!]^{\oplus 2}$ determined by $1 \mapsto (1, t \cdot v)$ for some $v \in k[\![t]\!]^\times$. Hence, by the same reason as above, the ramification index of $\phi_{\mathscr{F}^\spadesuit}$ at x turns out to be 1, which means that $\phi_{\mathscr{F}^\spadesuit}$ is étale at x. This completes the proof of this lemma. □

Note that the resulting element $[\phi_{\mathscr{F}^\spadesuit}] \in \mathrm{Cov}_{(\lambda_0, \lambda_1, \lambda_\infty)}$ does not depend on the choice of the trivialization $\gamma : \mathcal{O}^{\oplus 2}_{\mathbb{P}^{(N)}} \xrightarrow{\sim} \mathcal{F}^\nabla$ because γ is uniquely determined up to forward composition with an element of $\mathrm{Aut}_{\mathcal{O}_{\mathbb{P}^{(N)}}}(\mathcal{O}^{\oplus 2}_{\mathbb{P}^{(N)}}) (= \mathrm{PGL}_2(k))$. Thus, the assignment $\mathscr{F}^\spadesuit \mapsto [\phi_{\mathscr{F}^\spadesuit}]$ determines a well-defined map of sets $\mathrm{Op}^{\mathrm{Zzz...}}_{2,(\rho_0,\rho_1,\rho_\infty)} \to \mathrm{Cov}_{(\lambda_0, \lambda_1, \lambda_\infty)}$. One may verify that this map specifies, by construction, the inverse to the map $\Upsilon_{(\lambda_0, \lambda_1, \lambda_\infty)}$. Thus, we have obtained the following assertion.

Proposition 7.3.5. *(Recall that the underlying curve "$\mathscr{X}$" has been taken to be $\mathscr{P}$.) The assignments $[\phi] \mapsto \mathscr{F}^\spadesuit_\phi$ (i.e, $\Upsilon_{(\lambda_0, \lambda_1, \lambda_\infty)}$) and $\mathscr{F}^\spadesuit \mapsto [\phi_{\mathscr{F}^\spadesuit}]$ constructed above give a bijective correspondence*

$$\mathrm{Op}^{\mathrm{Zzz...}}_{2,(\rho_0,\rho_1,\rho_\infty)} \cong \mathrm{Cov}_{(\lambda_0, \lambda_1, \lambda_\infty)}$$

7.4. **Correspondence between tamely ramified coverings and dormant $\mathrm{PGL}_2^{(N)}$-opers.** The following proposition (together with Proposition 7.3.5) may be regarded as a variant of the rigidity assertion for dynamical Belyi maps proved in [LiOs2, Lemma 2.1].

Proposition 7.4.1. *Let $(\rho_0, \rho_1, \rho_\infty)$ be an element of $((\mathbb{Z}/p^N\mathbb{Z})/\{\pm 1\})^3$. Then, a dormant $\mathrm{PGL}_2^{(N)}$-oper on $\mathscr{P}$ of radii $(\rho_0, \rho_1, \rho_\infty)$ is, if it exists, uniquely determined. That is to say, the following inequality holds:*

$$\sharp(\mathrm{Op}^{\mathrm{Zzz...}}_{2,(\rho_0,\rho_1,\rho_\infty)}) \leq 1.$$

Proof. Suppose that $\mathrm{Op}^{\mathrm{Zzz...}}_{2,(\rho_0,\rho_1,\rho_\infty)} \neq \emptyset$. By the canonical morphism $\mathcal{D}^{(0)}_{\mathbb{P}^{\mathrm{log}}} \to \mathcal{D}^{(N-1)}_{\mathbb{P}^{\mathrm{log}}}$, each element of $\mathrm{Op}^{\mathrm{Zzz...}}_{2,(\rho_0,\rho_1,\rho_\infty)}$ induces a dormant $\mathrm{PGL}_2^{(1)}$-oper of radii $(\overline{\rho}_0, \overline{\rho}_1, \overline{\rho}_\infty)$, where $\overline{\rho}_x$ (for $x = 0, 1, \infty$) denotes the image of ρ_x via the natural surjection $(\mathbb{Z}/p^N\mathbb{Z})/\{\pm 1\} \twoheadrightarrow \mathbb{F}_p/\{\pm 1\}$. Hence, $\mathrm{Op}^{\mathrm{Zzz...}}_{1,2,\mathscr{P},(\overline{\rho}_0,\overline{\rho}_1,\overline{\rho}_\infty)}$ must be nonempty. This fact together with a comment in Remark 5.3.3 implies that $(\overline{\rho}_0, \overline{\rho}_1, \overline{\rho}_\infty) \in (\mathbb{F}_p^\times/\{\pm 1\})^3$, or equivalently, $(\rho_0, \rho_1, \rho_\infty) \in ((\mathbb{Z}/p^N\mathbb{Z})^\times/\{\pm 1\})^3$. In particular, there exists a triple of integers $(\lambda_0, \lambda_1, \lambda_\infty)$ associated to $(\rho_0, \rho_1, \rho_\infty)$ satisfying the conditions (a), (b) described in Section 7.2.

Now, suppose that there exist two dormant $\mathrm{PGL}_2^{(N)}$-opers $\mathscr{F}_1^\spadesuit$, $\mathscr{F}_2^\spadesuit$ on $\mathscr{P}$ of radii $(\rho_0, \rho_1, \rho_\infty)$. For each $j = 1, 2$, denote by $\mathscr{F}_j^\heartsuit := (\mathcal{F}_j, \nabla_j, \mathcal{L}_j)$ the dormant $\mathrm{GL}_n^{(N)}$-oper resulting from

Proposition 7.3.1 applied to $\mathscr{F}_j^{\spadesuit}$. The inclusion $\mathcal{F}_j^{\nabla} \hookrightarrow \mathcal{F}_j$ extends to an $\mathcal{O}_{\mathbb{P}}$-linear morphism $\tau_j : F_{\mathbb{P}/k}^{(N)*}(\mathcal{F}_j^{\nabla}) \to \mathcal{F}_j$. Let us write $\mathcal{L}_j^{\natural} := \mathcal{L}_j \cap \mathrm{Im}(\tau_j)$ and write ι_j for the natural isomorphism $\mathcal{L}_j/\mathcal{L}_j^{\natural} \xrightarrow{\sim} \mathrm{Coker}(\tau_j)$ (cf. Lemma 7.3.2). Also, denote by $\overline{\nabla}_j$ the $\mathcal{D}_{\mathbb{P}^{\log}}^{(0)}$-module structure on $\mathcal{F}_j$ induced from ∇_j. The triple $\overline{\mathscr{F}}_j^{\heartsuit} := (\mathcal{F}_j, \overline{\nabla}_j, \mathcal{L}_j)$ defines a dormant $\mathrm{GL}_2^{(1)}$-oper, in particular, induces a dormant $\mathrm{PGL}_2^{(1)}$-oper $\overline{\mathscr{F}}_i^{\heartsuit\Rightarrow\spadesuit}$ on $\mathscr{P}$ of radii $(\overline{\rho}_0, \overline{\rho}_1, \overline{\rho}_\infty)$. Recall from [Moc, Chapter I, Theorem 4.4] (cf. Remark 5.3.3) that dormant $\mathrm{PGL}_2^{(1)}$-opers on $\mathscr{P}$ are completely determined by their radii. It follows that $\overline{\mathscr{F}}_1^{\heartsuit\Rightarrow\spadesuit} = \overline{\mathscr{F}}_2^{\heartsuit\Rightarrow\spadesuit}$. By the uniqueness assertion in Proposition 7.3.1, there exists an isomorphism of GL_2-opers $\alpha : \overline{\mathscr{F}}_1^{\heartsuit} \xrightarrow{\sim} \overline{\mathscr{F}}_2^{\heartsuit}$. This isomorphism restricts to an isomorphism $\alpha|_{\mathcal{L}_1} : \mathcal{L}_1 \xrightarrow{\sim} \mathcal{L}_2$, which induces, via taking the respective quotients, an isomorphism $\alpha|_{\mathcal{L}_1/\mathcal{L}_1^{\natural}} : \mathcal{L}_1/\mathcal{L}_1^{\natural} \xrightarrow{\sim} \mathcal{L}_2/\mathcal{L}_2^{\natural}$. The composite $\alpha|_{\mathrm{Coker}(\tau_1)} := \iota_2 \circ \alpha|_{\mathcal{L}_1/\mathcal{L}_1^{\natural}} \circ \iota_1^{-1}$ specifies an isomorphism $\mathrm{Coker}(\tau_1) \xrightarrow{\sim} \mathrm{Coker}(\tau_2)$.

In what follows, we shall prove the commutativity of the following square diagram:

$$\begin{array}{ccc} \mathcal{F}_1 & \xrightarrow{\quad \alpha \quad} & \mathcal{F}_2 \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi_2} \\ \mathrm{Coker}(\tau_1) & \xrightarrow[\alpha|_{\mathrm{Coker}(\tau_1)}]{} & \mathrm{Coker}(\tau_2), \end{array} \tag{7.4}$$

where π_j ($j = 1, 2$) denotes the natural projection $\mathcal{F}_j \twoheadrightarrow \mathrm{Coker}(\tau_j)$. Let us take $j \in \{1, 2\}$, $x \in \{[0], [1], [\infty]\}$. Also, choose a local function t on $\mathbb{P}$ defining x. Denote by $\widehat{D}_x$ the formal neighborhood of x in $\mathbb{P}$, which may be identified with $\mathrm{Spec}(k[\![t]\!])$. Fix an identification of the restriction of $(\mathcal{F}_j, \nabla_j)$ to $\widehat{D}_x$ with $(k[\![t]\!], \widehat{\nabla}_{\overline{\lambda}_x}) \oplus (k[\![t]\!], \widehat{\nabla}_0)$ (cf. the proof of Lemma 7.3.2). The isomorphism α restricted to $\widehat{D}_x$ defines an automorphism $\alpha|_{\widehat{D}_x}$ of $(k[\![t]\!], \widehat{\nabla}_{\overline{\lambda}_x}) \oplus (k[\![t]\!], \widehat{\nabla}_0)$. Here, we shall use the notation $\mu_{(-)}$ to denote the endomorphism of $k[\![t]\!]$ given by multiplication by $(-)$. Since $\rho_x \neq 0$ in $\mathbb{F}_p/\{\pm 1\}$ (or equivalently, $\overline{\lambda}_x \neq 0$), the automorphism $\alpha|_{\widehat{D}_x}$ may be expressed as $\mu_v \oplus \mu_w$ for some $v, w \in k[\![t]\!]^{\times}$ after possibly replacing the fixed identification $(\mathcal{F}_j, \nabla_j)|_{\widehat{D}_x} = (k[\![t]\!], \widehat{\nabla}_{\overline{\lambda}_x}) \oplus (k[\![t]\!], \widehat{\nabla}_0)$ with another (cf. Proposition 4.2.1, (ii)). As observed in the proof of Proposition 4.4.1, the inclusion $\mathcal{L}_j \hookrightarrow \mathcal{F}_j$ corresponds, after choosing a suitable trivialization $\Gamma(\widehat{D}_x, \mathcal{L}_j|_{\widehat{D}_x}) \xrightarrow{\sim} k[\![t]\!]$, to the $k[\![t]\!]$-linear morphism $k[\![t]\!] \to k[\![t]\!]^{\oplus 2}$ given by $1 \mapsto (u_j, 1)$ for some $u_j \in k[\![t]\!]^{\times}$. Then, the restriction of $\alpha|_{\mathcal{L}_1}$ to $\widehat{D}_x$ may be expressed as μ_w, and the equality $vu_1 = wu_2$ holds. Hence, for each $(g, h) \in k[\![t]\!]^{\oplus 2}$ $\left(= \mathcal{F}_1|_{\widehat{D}_x}\right)$, we have

$$\begin{aligned} (\pi_2 \circ \alpha)((g, h)) &= \pi_2((vg, wh)) \\ &= \left(\frac{vg}{u_2} \cdot u_2, \frac{vg}{u_2} \cdot 1\right) \bmod \mathrm{Im}(\tau_2) \\ &= \iota_2 \left(\frac{vg}{u_2} \bmod \mathcal{L}_2^{\natural}\right) \end{aligned}$$

$$
\begin{aligned}
&= (\iota_2 \circ \alpha|_{\mathcal{L}_1/\mathcal{L}_1^\natural}) \left(\frac{vg}{wu_2} \text{ mod } \mathcal{L}_1^\natural \right) \\
&= (\alpha|_{\mathrm{Coker}(\tau_1)} \circ \iota_1) \left(\frac{g}{u_1} \text{ mod } \mathcal{L}_1^\natural \right) \\
&= \alpha|_{\mathrm{Coker}(\tau_1)} \left(\left(\frac{g}{u_1} \cdot u_1, \frac{g}{u_1} \cdot 1 \right) \text{ mod } \mathrm{Im}(\tau_1) \right) \\
&= (\alpha|_{\mathrm{Coker}(\tau_1)} \circ \pi_1)((g, h)).
\end{aligned}
$$

This shows the desired commutativity of (7.4).

Moreover, the commutativity of (7.4) just proved implies that α restricts, via ι_1 and ι_2, to an isomorphism $\alpha' : F^{(N)*}_{\mathbb{P}/k}(\mathcal{F}_1^\nabla) \xrightarrow{\sim} F^{(N)*}_{\mathbb{P}/k}(\mathcal{F}_2^\nabla)$. Since $\mathcal{F}_1^\nabla \cong \mathcal{F}_2^\nabla \cong \mathcal{O}_{\mathbb{P}^{(N)}}^{\oplus 2}$ (cf. Lemma 7.3.3), the morphism

$$
\mathrm{Hom}_{\mathcal{O}_{\mathbb{P}^{(N)}}}(\mathcal{F}_1^\nabla, \mathcal{F}_2^\nabla) \to \mathrm{End}_{\mathcal{O}_{\mathbb{P}}}(F^{(N)*}_{\mathbb{P}/k}(\mathcal{F}_1^\nabla), F^{(N)*}_{\mathbb{P}/k}(\mathcal{F}_2^\nabla))
$$

arising from pull-back by $F^{(N)}_{\mathbb{P}/k}$ is bijective. In particular, α' comes from an isomorphism $\mathcal{F}_1^\nabla \xrightarrow{\sim} \mathcal{F}_2^\nabla$, and hence α' is compatible with the respective $\mathcal{D}^{(N-1)}_{\mathbb{P}^{\log}}$-actions $\nabla^{\mathrm{can}}_{\mathcal{F}_1^\nabla}$, $\nabla^{\mathrm{can}}_{\mathcal{F}_2^\nabla}$ (cf. Remark 5.1.2). Since ∇_j is the unique $\mathcal{D}^{(N-1)}_{\mathbb{P}^{\log}}$-module structure on $\mathcal{F}_j$ extending $\nabla^{\mathrm{can}}_{\mathcal{F}_j^\nabla}$ via τ_j, the isomorphism α, being an extension of α', preserves the $\mathcal{D}^{(N-1)}_{\mathbb{P}^{\log}}$-action. It follows that α defines an isomorphism of $\mathrm{GL}_2^{(N)}$-opers $\mathscr{F}_1^\heartsuit \xrightarrow{\sim} \mathscr{F}_2^\heartsuit$, which induces the equality $\mathscr{F}_1^\spadesuit = \mathscr{F}_2^\spadesuit$. This completes the proof of this proposition. □

Remark 7.4.2. The proof of the above proposition shows that $\overset{\text{Zzz...}}{\mathrm{Op}}_{2,(\rho_0,\rho_1,\rho_\infty)} = \emptyset$ unless $(\rho_0, \rho_1, \rho_\infty) \in ((\mathbb{Z}/p^N\mathbb{Z})^\times/\{\pm 1\})^3$. As mentioned in Remark 5.3.3, this fact for $N = 1$ was already verified in [Moc, Chapter II, Proposition 1.4].

We shall write

$$
\mathrm{Cov}_+ \ \ (\text{resp., } \mathrm{Cov})
$$

for the set of equivalence classes of finite, separable, and tamely ramified coverings $\phi : \mathbb{P} \to \mathbb{P}$ satisfying the following conditions:

- The set of ramification points of ϕ coincides with $\{[0], [1], [\infty]\}$;
- If λ_x $(x = 0, 1, \infty)$ denotes the ramification index of ϕ at $[x]$, then λ_0, λ_1, λ_∞ satisfy the inequality $\lambda_0 + \lambda_1 + \lambda_\infty < 2 \cdot p^N$ (resp., λ_0, λ_1, λ_∞ are all odd and satisfy the inequality $\lambda_0 + \lambda_1 + \lambda_\infty < 2 \cdot p^N$).

Here, the equivalence relation is defined in such a way that two coverings $\phi_1, \phi_2 : \mathbb{P} \to \mathbb{P}$ are equivalent if there exists an element $h \in \mathrm{PGL}_2(k)\,(= \mathrm{Aut}_k(\mathbb{P}))$ with $\phi_2 = h \circ \phi_1$. Since the identity morphism $\mathrm{id}_{\mathbb{P}}$ of $\mathbb{P}$ defines a tamely ramified covering with ramification indices $(1, 1, 1)$, the set Cov_+ (resp., Cov) is nonempty. By applying some of the results proved so far, we obtain the following assertion.

Theorem 7.4.3 (cf. Theorem C)**.** *The assignment $\phi \mapsto \mathscr{F}_\phi^\spadesuit$ gives a 4-to-1 (resp., a 1-to-1, i.e., bijective) correspondence*

$$
\Upsilon_+ : \mathrm{Cov}_+ \twoheadrightarrow \overset{\text{Zzz...}}{\mathrm{Op}}_{N,2,\mathscr{P}} \ \left(\textit{resp., } \Upsilon : \mathrm{Cov} \xrightarrow{\sim} \overset{\text{Zzz...}}{\mathrm{Op}}_{N,2,\mathscr{P}} \right).
$$

In particular, the set $\overset{\text{Zzz...}}{\mathrm{Op}}_{N,2,\mathscr{P}}$ *is finite and admits an inclusion into the following set:*

$$\mathbb{B}_N^{\circledast} := \left\{ (\lambda_0, \lambda_1, \lambda_\infty) \in \mathbb{B}^3 \,\middle|\, \lambda_0 + \lambda_1 + \lambda_\infty < 2 \cdot p^N \text{ and } |\lambda_0 - \lambda_1| \leq \lambda_\infty \leq \lambda_0 + \lambda_1 \right\},$$

where $\mathbb{B}$ *denotes the set of positive odd integers* a *with* $p \nmid a$, $a < p^N$.

Proof. First, we shall consider the former assertion. Let us take a dormant $\mathrm{PGL}_2^{(N)}$-oper $\mathscr{F}^\spadesuit$ classified by $\overset{\text{Zzz...}}{\mathrm{Op}}_{N,2,\mathscr{P}}$, and denote by $(\rho_0, \rho_1, \rho_\infty)$ the radii of $\mathscr{F}^\spadesuit$. The result of Proposition 7.4.1 shows that $\mathscr{F}^\spadesuit$ is the unique dormant $\mathrm{PGL}_2^{(N)}$-oper on $\mathscr{P}$ of radii $(\rho_0, \rho_1, \rho_\infty)$. Now, let us choose a triple of integers $\vec{\lambda} := (\lambda_0, \lambda_1, \lambda_\infty)$ associated to $(\rho_0, \rho_1, \rho_\infty)$ as defined in Section 7.2. As observed in Remark 7.2.2, this triple satisfies the inequalities in (7.2). It follows that the triples

$$\begin{aligned} \vec{\lambda}_0 &:= (\lambda_0, p^N - \lambda_1, p^N - \lambda_\infty), \\ \vec{\lambda}_1 &:= (p^N - \lambda_0, \lambda_1, p^N - \lambda_\infty), \\ \vec{\lambda}_\infty &:= (p^N - \lambda_0, p^N - \lambda_1, \lambda_\infty), \end{aligned}$$

respectively, satisfy conditions (a) and (b) in Section 7.2, and conversely, each triple of integers satisfying (a) and (b) is one of the four tripes $\vec{\lambda}$, $\vec{\lambda}_0$, $\vec{\lambda}_1$, $\vec{\lambda}_\infty$. Thus, by Proposition 7.3.5, the preimage of the element defined by $\mathscr{F}^\spadesuit$ via Υ_+ coincides with $\{[\phi_{\vec{\lambda}}], [\phi_{\vec{\lambda}_0}], [\phi_{\vec{\lambda}_1}], [\phi_{\vec{\lambda}_\infty}]\}$, where, for a triple $\vec{\lambda}' := (\lambda'_0, \lambda'_1, \lambda'_\infty)$, $[\phi_{\vec{\lambda}'}]$ denotes a unique (up to equivalence) covering classified by Cov_+ whose ramification index at $[x]$ $(x = 0, 1, \infty)$ is λ'_x. This proves the non-resp'd assertion. Also, the resp'd assertion follows from the fact that only one of the four triples $\vec{\lambda}, \vec{\lambda}_0, \vec{\lambda}_1, \vec{\lambda}_\infty$ satisfies the condition that all factors are odd.

Finally, the latter assertion follows from the resp'd portion of the former assertion (and its proof). This completes the proof of this theorem. □

Remark 7.4.4. In the case of $N = 1$, we know that the embedding $\overset{\text{Zzz...}}{\mathrm{Op}}_{1,2,\mathscr{P}} \hookrightarrow \mathbb{B}_1^{\circledast}$ resulting from Theorem 7.4.3 is bijective (cf. [Moc, Introduction, Theorem 1.3, (2)]). The resulting correspondence $\overset{\text{Zzz...}}{\mathrm{Op}}_{1,2,\mathscr{P}} \cong \mathbb{B}_1^{\circledast}$ allows us to translate dormant PGL_2-opers into edge-colorings on trivalent graphs, as well as lattice points of a rational polytope (cf. [LiOs1], [Wak2]). However, at the time of writing the present paper, the author does not know much about the image of this map for a general N.

Remark 7.4.5. In [Moc, Chapter II, Definition 2.2], S. Mochizuki introduced the notion of a(n) *(dormant) m-connection* (for each nonnegative integer m) on a flat $\mathbb{P}^1$-bundle. Here, we recall its definition briefly. Let $\mathscr{X} := (X, \{\sigma_i\}_i)$ be as in (5.1). Also, let $(\mathcal{P}, \nabla)$ be a flat $\mathbb{P}$-bundle on $X^{\log}$, i.e., a $\mathbb{P}$-bundle $\mathcal{P}$ on X equipped with a logarithmic connection ∇ (with respect to the log structure of $X^{\log}$). Denote by W_{m+1} the ring of Witt vectors with coefficients in k of length $m+1$. Then, a dormant m-connection on $(\mathcal{P}, \nabla)$ (of prescribed radii) is defined as a crystal in $\mathbb{P}$-bundles on the log crystalline site $\mathrm{Crys}(X^{\log}/W_{m+1})$ inducing $(\mathcal{P}, \nabla)$ via reduction module p and satisfying some other conditions. The condition of being a dormant m-connection is described in terms of p^{m+1}-curvature in the sense of [Moc, Chapter II, the discussion in Section 2.1], which is different from (but closely related to) our definition of p^{m+1}-curvature because it relies, at least a priori, upon the crystalline structure over W_{m+1}. According to [Moc, Chapter IV, Theorem 2.3], there exists a bijective correspondence between the elements in Cov and the set of dormant $(N-1)$-connections on a torally indigenous bundle (i.e., a $\mathrm{PGL}_2^{(1)}$-oper) on $\mathscr{P}$. By combining this fact with Theorem 7.4.3, we see that each dormant $\mathrm{PGL}_2^{(N)}$-oper on $\mathscr{P}$ of radii $\vec{\rho} \in ((\mathbb{Z}/p^N\mathbb{Z})^\times/\{\pm 1\})^3$ may be uniquely extended to a dormant $(N-1)$-connection of the same radii.

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