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Obstructions associated with tropical curves

By Takeo NISHINOU

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Abstract. Given a complex variety X, suppose there is a degeneration $\mathfrak{X}$ of it to a union of toric varieties glued along toric divisors. Then, one can use tropical curves to study holomorphic curves on X through degenerate curves on the central fiber X_0 of $\mathfrak{X}$. In such studies, the case corresponding to so-called non-superabundant tropical curves is relatively well understood. This is the case where the curve satisfies a version of transversality. On the other hand, in the case corresponding to superabundant tropical curves, where transversality fails, not much is known. The first important step for such a study is describing the obstruction to deforming degenerate curves on X_0 to generic fibers of $\mathfrak{X}$. In this paper, we present a general formula describing such obstruction. In the study of superabundant tropical curves, allowing higher-valent vertices (i.e., those with valency greater than three) is essential, although there has been only few studies of them in this context. Our formula covers such cases.

1. Introduction

The appearance of graphs in the study of algebraic curves is quite common. In this context, usually a graph appears as the dual intersection complex of the central fiber of a suitable degenerating family of algebraic curves. Although any abstract finite graph can be realized as the dual intersection complex associated with a degenerating family of algebraic curves (see [**2**, Appendix B]), the situation is more intricate when one considers curves embedded in suitable ambient spaces. Now, the graphs have to satisfy the balancing condition at the vertices, which is the combinatorial counterpart of the harmonicity, and there are subtler conditions which correspond to the obstructions to deforming the degenerate algebraic curves.

The situation we consider is the so-called *toric degeneration* of varieties. Namely, given a complex variety X, assume that there is a degeneration $\mathfrak{X}$ of it whose central fiber X_0 is a union of toric varieties glued along toric divisors. We study curves on X via degenerate curves on X_0. When the original variety X itself is toric, the nature of the graphs associated with such degenerate algebraic curves is efficiently encoded in so-called tropical curves. G.Mikhalkin started this area of study with his pioneering work [**13**], in which he proved the correspondence between imbedded tropical curves of any genus in $\mathbb{R}^2$, and holomorphic curves in toric surfaces specified by the combinatorial data of the tropical curves. Recently, ideas from non-archimedean geometry are pushing this field forward (see for example [**3**]).

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Nevertheless, the nature of degenerate algebraic curves is not yet well understood. The main obstacle is the presence of obstructions to deforming degenerate curves to smooth (or, more generally, irreducible) curves. When X is toric, the combinatorial counterpart of such obstructed curves are superabundant tropical curves. In this paper and the sequel, we study fundamental aspects of superabundant tropical curves and their implications for algebraic curves. In particular, we investigate the role of higher-valent vertices, which has not been studied much so far in spite of its importance in the study of superabundant tropical curves. We note that higher-valent vertices are studied in the context of tropical descendant Gromov-Witten invariants (see for example [**6, 14**]), where contracted edges representing the ψ-class conditions cause the higher valency. In this paper, we study higher valency in relation to the superabundance, and there is little intersection between these studies.

It turns out that our argument works equally well when the original variety X is not necessarily toric. Namely, in such a case, there is a particular singular locus of the total space $\mathfrak{X}$ of codimension three, and when the degenerate curve is generic in the sense that it does not intersect this locus, our results remain valid with little modifications. It should be also noted that our study will play a fundamental role not only in the classical theory of tropical curves, but also in the more recent study of log Gromov-Witten invariants [**1, 4, 8**]. Namely, to compute log Gromov-Witten invariants, basically we need to consider degenerate curves corresponding to all possible tropical curves including superabundant ones. Therefore, for general study of log Gromov-Witten invariants, solid understanding of superabundant tropical curves is indispensable, and one of main reasons why there are few computations of log Gromov-Witten invariants in higher dimensions is probably the lack of such an understanding. The result of this paper will be of fundamental importance in this respect. Namely, it computes the h^{-1} and h^0 (which are the dimensions of the obstruction and the tangent spaces, respectively) of the obstruction theory used in the definition of log Gromov-Witten invariants.

The basic strategy to study curves in algebraic varieties through degeneration is two fold:

Step 1: First, construct a degenerate algebraic curve in the central fiber of a degenerating family of varieties.

Step 2: Then, deform the curve to a general fiber of the family.

Let us assume that the variety X itself is toric for the ease of exposition. Then, the dual intersection graph of the degenerate curve naturally has the structure of a tropical curve. When the tropical curve is regular (see Definition 27), the relation to algebraic curves is quite nice with respect to the both steps above.

On the other hand, when the tropical curve is superabundant, several new phenomena emerge. In this regard, Step 2 has been relatively well studied (see for example [**3, 12, 17, 23**]). However, it turns out that Step 1 is as crucial as Step 2, though this point does not seem to have been studied seriously. The importance of this point will be particularly clear when one considers tropical curves with higher-valent vertices. In this paper and the sequel, we develop a general formalism to deal with higher-valent vertices, and study the first and second steps using it.

The bridge which connects Step 1 to Step 2 is the calculation of the cohomology group in which the obstructions to deforming the degenerate algebraic curve lies. The combinatorial counterpart of the obstruction is the superabundancy of tropical curves. Therefore, describing the superabundancy effectively is an important problem in the theory of tropical curves as well as in the theory of algebraic curves. Given a tropical curve, the following questions immediately arise:

Question 1: Determine whether the tropical curve is superabundant or not.

Question 2: When the tropical curve is superabundant, calculate the number of parameters of its deformation.

Neither of the problems is easily solved when one looks only at the tropical curve itself. In Section 7, we will give a general answer to these problems by reducing them to a calculation of a certain sheaf cohomology group of the algebraic counterpart of tropical curves (Theorem 57). When the tropical curve is 3-valent and is an immersion, then this cohomology group is completely determined by the combinatorial data of the tropical curve. On the other hand, when there are higher-valent vertices in the tropical curve, this cohomology group is given as the solution space of a certain system of linear equations, which is not in general determined by the combinatorial data. Here, we give the statement for the 3-valent and immersive case to give the reader a feeling for what a description of the obstruction might look like.

Let (Γ, h) be a 3-valent, immersive tropical curve (see Definition 12 and 16). Let L be the loop part of Γ (see Definition 23) and assume for simplicity that it is connected. Then, L is a subgraph of Γ with 2- and 3-valent vertices. By cutting L at each 3-valent vertex, we obtain a set consisting of the union of edges $\{l_m\}$ of Γ. The map h embeds each l_m into $\mathbb{R}^n$, and let U_m be the linear subspace spanned by the direction vectors of the edges and $(U_m)^\perp$ be the annihilating subspace in the dual space $(\mathbb{R}^n)^\wedge$. See Example 64 for an illustrated example of this process.

Let $\{v_i\}$ be the set of 3-valent vertices of L, and $\{l_{i,j}\}$, $j = 1, 2, 3$, be the subset of $\{l_m\}$ which contains v_i. Define a linear map

$$\alpha\colon \prod_m (U_m)^\perp \to \prod_i (\mathbb{R}^n)^\wedge$$

as follows. Namely, let (u_m) be an element of $\prod_m (U_m)^\perp$. Then, the i-th component of the image $\alpha((u_m))$ is given by the sum

$$u_{i,1} + u_{i,2} + u_{i,3}.$$

Let H be the kernel of the ap α. Then, we have the following, which is equivalent to Corollary 59.

THEOREM 1. *The tropical curve (Γ, h) is superabundant if and only if the vector space H is not zero. Moreover, the space of obstructions to deforming a degenerate holomorphic curve of type (Γ, h) (see Section 3) is given by $H \otimes \mathbb{C}$.* □

See Theorem 57 for the claim in the general case.

With this description, we can study correspondence theorems for superabundant tropical curves [**17**]. It also gives a basis for studies in various other situations [**17, 18, 20**].

The basics of the study of higher-valent vertices are given in Sections 5 and 6. The key point is that although algebraic geometric counterpart of a general higher-valent vertex is not necessarily easy to handle, such a vertex can be obtained from a standard higher-valent vertex, whose algebraic geometric counterpart is a line in a projective space. Using this, we can reduce various calculation on degenerate curves to calculation on a line in a projective space, which can be done very explicitly. In particular, this enables us to calculate the obstruction cohomology group in various situations.

So far, we have explained our results in terms of tropical curves. On the side of algebraic curves, this corresponds to the study of curves on toric varieties through toric degenerations in the sense of [**22**]. More general varieties also degenerate into a union of toric varieties, and we can associate dual intersection graphs with degenerate algebraic curves on the central fiber of such a degeneration. These graphs also satisfy balancing conditions at vertices, but they are not globally imbedded in an affine space, unlike the case of usual tropical curves.

However, most of our calculation concerns the directions normal to the space spanned by the edges in the loop of the graph, and these directions make sense even if the graph is not globally embedded in an affine space. Using this fact, when the degenerate curve does not intersect the singular locus of the total space $\mathfrak{X}$, our results still hold with little change, see Section 9. When the degenerate curve intersects the singular locus of $\mathfrak{X}$, we need different argument, see [**18**]. Combining these results, given any variety, once we have a reasonable degeneration so that we can study degenerate curves on it, we will be able to deduce large amount of information about curves on the original variety.

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2. Tropical curves

Although most results in this paper are valid in more general situations, we usually deal with the case of toric degenerations of toric varieties first, and later provide complements necessary to cover general cases. This is because in the case of toric degenerations of toric varieties, we can make use of the well established languages in the theory of tropical curves, and modifications necessary to deal with general cases are usually small. Thus, we start with introducing some notions in the theory of tropical curves.

First, we recall some definitions about tropical curves, see [**13, 22**] for more information. Let $\overline{\Gamma}$ be a weighted, connected finite graph without loops. We allow parallel edges. Its sets of vertices and edges are denoted by $\overline{\Gamma}^{[0]}$, $\overline{\Gamma}^{[1]}$, respectively. We write by $w_{\overline{\Gamma}} \colon \overline{\Gamma}^{[1]} \to \mathbb{N} \setminus \{0\}$ the weight function. An edge $E \in \overline{\Gamma}^{[1]}$ has adjacent vertices $\partial E = \{V_1, V_2\}$. When $\overline{\Gamma}$ is seen as a topological space (see below), we think of V_1, V_2 as subsets of E. Let $\overline{\Gamma}^{[0]}_{\infty} \subset \overline{\Gamma}^{[0]}$ be the set of all 1-valent vertices. We write $\Gamma = \overline{\Gamma} \setminus \overline{\Gamma}^{[0]}_{\infty}$. Non-compact edges of Γ are often called *unbounded edges*. Let $\Gamma^{[1]}_{\infty}$ be the set of all unbounded edges. Let $\Gamma^{[0]}, \Gamma^{[1]}$ and w_{Γ} be the sets of vertices and edges of Γ and the weight function of Γ (induced from $w_{\overline{\Gamma}}$ in the obvious way), respectively. Note that $\Gamma^{[1]} = \overline{\Gamma}^{[1]}$. The set of *flags* of Γ is

$$F\Gamma = \{(V, E) \big| V \in \partial E\}.$$

The graphs $\overline{\Gamma}$ and Γ have natural topologies as a finite CW complex and its open subset. If E is an edge of Γ, we call the complement of its adjacent vertices the interior of E and write it as E°. Let N be a free abelian group of rank $n \geq 2$ and we write $N_{\mathbb{K}} = N \otimes_{\mathbb{Z}} \mathbb{K}$, where $\mathbb{K} = \mathbb{Q}, \mathbb{R}, \mathbb{C}$.

Definition 2. We call a continuous map $h \colon \Gamma \to N_{\mathbb{R}}$ an *affine semi-imbedding* if for every edge $E \in \Gamma^{[1]}$, the restriction $h\big|_E$ is either an embedding with the image $h(E)$ contained in an affine line, or a contraction where $h(E)$ is a point. Moreover, if an unbounded edge is not contracted, then its image must be unbounded, too.

We note the following observation.

Lemma 3. *Let $h \colon \Gamma \to N_{\mathbb{R}}$ be an affine semi-imbedding. Let $p \in h(\Gamma)$ be any point on the image of h. Then, a connected component of $h^{-1}(p)$ is a closed subgraph of Γ, or a point in the interior of some edge.*

Proof. Let B be a connected component of $h^{-1}(p)$. Assume that two points q_1, q_2 on an edge E of Γ is contained in B. Then, by definition of affine semi-imbedding, the entire edge E must be contained in B. In this case, by an inductive argument, it is easy to see that B is a closed subgraph of Γ. If there are no such pair of points, then B is a vertex or a point in the interior of an edge. In the former case, B is also a closed subgraph. This proves the claim. $\square$

Let h be an affine semi-imbedding. Let $\mathcal{S} \subset h(\Gamma)$ be the set of points with the property that $p \in \mathcal{S}$ if and only if for any neighborhood O_p of p in $N_{\mathbb{R}}$, the intersection $O_p \cap h(\Gamma)$ is not homeomorphic to an open interval. The following is easy to see.

Lemma 4. *Let h be an affine semi-imbedding. Then, the following statements hold.*

1. *$\mathcal{S}$ is a finite set.*
2. *By adding finite number of vertices to Γ, we can assume that the inverse image $h^{-1}(p)$ of each $p \in \mathcal{S}$ consists of closed subgraphs of Γ.*

Proof. The first claim is obvious since $\overline{\Gamma}$ is a finite graph. As for the second claim, we take the added vertices to be the points in the interior of some edges mentioned in Lemma 3. □

Under the condition of Lemma 4 (2), the image $h(\Gamma)$ has a natural structure of a graph as follows.

Definition 5. Let $h\colon \Gamma \to N_{\mathbb{R}}$ be an affine semi-imbedding satisfying the condition of Lemma 4 (2). We call a point in $h(\Gamma)$ a vertex if it is the image of some vertex of Γ. Similarly, we call a subset of $h(\Gamma)$ an edge if it is the image of some edge of Γ which is not contracted.

We also have the following.

Lemma 6. *In the situation of Definition 5, by further adding finite number of vertices to* Γ*, we can assume that the inverse image* $h^{-1}(p)$ *of a vertex* p *of* $h(\Gamma)$ *is a union of closed subgraphs.*

Proof. By Lemma 3, a connected component of $h^{-1}(p)$ is either a closed subgraph or a point in the interior of an edge. By taking the latter points to be 2-valent vertices, the claim follows. □

In this case, the inverse image $h^{-1}(\mathfrak{E}^{\circ})$ of the interior $\mathfrak{E}^{\circ}$ of an edge $\mathfrak{E}$ of $h(\Gamma)$ is the union of the interior of the edges of Γ each of which is mapped to $\mathfrak{E}$ homeomorphically.

Definition 7. We call an affine semi-imbedding $h\colon \Gamma \to N_{\mathbb{R}}$ *well-segmented* if it satisfies the condition of Lemma 6. Given an affine semi-imbedding h, we can make another well-segmented affine semi-imbedding by applying Lemma 4 (2) and Lemma 6. We call it the *well-segmented completion* of h, or the well-segmented completion of Γ with respect to h.

Let $h\colon \Gamma \to N_{\mathbb{R}}$ be an affine semi-imbedding. Let p be a vertex of $h(\Gamma)$ and Γ_1 be one of the connected components of $h^{-1}(p)$ which is a subgraph of Γ. Note that since we do not assume h to be well-segmented, there may be other connected components which are not subgraphs. Then, Γ_1 contains several vertices $q_1, \dots, q_a$. Let $E_1, \dots, E_b$ be the edges of $\Gamma \setminus \Gamma_1$ emanating from some of $q_1, \dots, q_a$.

Definition 8. A *parametrized tropical curve* in $N_{\mathbb{R}}$ is an affine semi-imbedding $h\colon \Gamma \to N_{\mathbb{R}}$ which satisfies the following conditions.

(i) For every edge $E \in \Gamma^{[1]}$, the image $h(E)$ is either contained in an affine line with a rational slope, or a point.

(ii) For every vertex $V \in \Gamma^{[0]}$, $h(V) \in N_{\mathbb{Q}}$.

(iii) The following *balancing condition* holds. Namely, for each vertex p of $h(\Gamma)$ and each connected component Γ_1 of $h^{-1}(p)$ which is a subgraph, the equality

$$\sum_{j=1}^{b} w_{\Gamma}(E_j) u_j = 0 \tag{1}$$

holds, using the notation in the above paragraph. Here, u_j is the primitive integral vector of N in the direction of the edge $h(E_j)$ emanating from p.

REMARK 9. *In* [**22**], *$h|_E$ is assumed to be an embedding (see* [**22**, Definition 1.1]*) for every edge E. The reason why we adopt the above definition is that these general cases appear naturally when we consider superabundant tropical curves. It is also natural in view of partially compactifying the parameter space of tropical curves, see Lemma 18.*

LEMMA 10. *Let $h\colon \Gamma \to N_{\mathbb{R}}$ be a parametrized tropical curve. Then, its well-segmented completion is also a parametrized tropical curve.*

PROOF. Note that when we take the well-segmented completion of an affine semi-imbedding, we only add 2-valent vertices to Γ. By definition of affine semi-imbedding, the balancing condition clearly holds at these vertices. □

REMARK 11. *When dealing with tropical curves, usually we do not need to take the well-segmented completion. In fact, the property of being well-segmented is often incompatible with the claim as Proposition 17 below when there is a map h which is not an embedding. However, when considering the relationship to holomorphic curves in toric varieties, it is natural to assume that any connected component of the inverse image of a vertex of $h(\Gamma)$ is a subgraph. In general, the property of being well-segmented is not typically suited to the study of a family of tropical curves, but it provides a useful setup for studying a fixed tropical curve.*

An isomorphism between parametrized tropical curves $h\colon \Gamma \to N_{\mathbb{R}}$ and $h'\colon \Gamma' \to N_{\mathbb{R}}$ is a homeomorphism $\Phi\colon \Gamma \to \Gamma'$ respecting the graph structures and the weights such that $h = h' \circ \Phi$.

DEFINITION 12. A *tropical curve* is an isomorphism class of parametrized tropical curves. A tropical curve is *3-valent* if any vertex of Γ is at most 3-valent. The *genus* of a tropical curve is the first Betti number of Γ. When the first Betti number of Γ is zero, it is often called a tree.

In this paper, we usually assume that a tropical curve (Γ, h) is 3-valent in the sense above unless otherwise noted. Note that in our definition, there can be 2-valent vertices in the abstract graph Γ underlying a 3-valent tropical curve. By (i) of Definition 8, we have a map $u\colon F\Gamma \to N$ sending a flag (V, E) to the primitive integral vector $u_{(V,E)} \in N$ emanating from $h(V)$ in the direction of $h(E)$ or to the zero vector. The following definition offers a slightly abstracted representation of this situation.

DEFINITION 13. A *combinatorial type* (or simply the *type*) on a graph Γ is a map $u\colon F\Gamma \to N$. For a type u and a flag (V, E), we write $u(V, E)$ as $u(E)$ when V is clear from the context, and refer to it as the type of (V, E) or, of E. A tropical curve (Γ, h) is of type u if for each flag (V, E) of Γ such that E is not contracted, the primitive integral vector emanating from $h(V)$ in the direction of $h(E)$ equals $u(V, E)$. In this case, we also call u the combinatorial type of the tropical curve (Γ, h).

REMARK 14. *Note that given a tropical curve of type u, even if a flag (V,E) has non-zero image by u, the edge E can be contracted, while if $u(E)=0$, then E must be contracted. In general, if we consider a fixed tropical curve, it is useful to assume $u(E)=0$ for contracted edges, while if we consider a family of tropical curves, it is often useful to adopt the general definition.*

DEFINITION 15. The *degree* of a tropical curve (Γ,h) of type u is the function $\Delta(\Gamma,h)=\Delta\colon N\setminus\{0\}\to\mathbb{N}$ with finite support defined by

$$\Delta(\Gamma,h)(v):=\sharp\{(V,E)\in F\Gamma | E\in\Gamma_\infty^{[1]} \text{ is not contracted by } h, w(E)u_{(V,E)}=v,\}.$$

Let $e=|\Delta|=\sum_{v\in N\setminus\{0\}}\Delta(v)$. This is the same as the number of unbounded edges of Γ not contracted by h (not necessarily equal to the number of $h(\Gamma)$ since some of the edges may have the same image).

DEFINITION 16. We call a tropical curve (Γ,h) *immersive* if for any $E\in\Gamma^{[1]}$, the restriction of h to E is an embedding. Note that even if (Γ,h) is immersive, some of the edges of Γ (even those emanating from the same vertex) can have the same image.

PROPOSITION 17 ([**13**, Proposition 2.13]). *The space parametrizing immersive purely 3-valent tropical curves (that is, without 2-valent vertices) of a given combinatorial type is given by the set of rational points of an open convex polyhedral domain in the real affine k-dimensional space, where $k\geq e+(n-3)(1-g)$, if it is non-empty. Here, e is the number of unbounded edges of Γ as in Definition 15, n is the dimension of the target space $N_{\mathbb{R}}$, and g is the genus of Γ. If there are r 2-valent vertices, then we have $k\geq e+r+(n-3)(1-g)$.*

PROOF. For the reader's convenience, we recall the outline of the proof given in [**13**, Proposition 2.13]. Assume Γ is purely 3-valent. Take a compact connected subgraph $\Gamma'\subset\Gamma$ of genus zero which contains all the vertices. Γ' has $e-3+2g$ edges (note that all the edges are bounded), and there are g bounded edges in Γ not contained in Γ'. Let $E_1,\dots,E_{e-3+2g}$ be these edges. Take a vertex $V\in\Gamma'$ and assume it is mapped to the origin of $N_{\mathbb{R}}$. Then, the immersions of Γ' into $N_{\mathbb{R}}$ compatible with the given combinatorial type are parametrized by the strictly positive orthant

$$P=\{(x_1,\dots,x_{e-3+2g})\in\mathbb{R}^{e-3+2g}\,|\,\forall x_i>0\}$$

of $\mathbb{R}^{e-3+2g}$, whose coordinates correspond to the lengths of the images of E_i, $i=1,\dots,e-3+2g$. Tropical curves correspond to the rational points of it, due to the condition that the vertices are contained in $N_{\mathbb{Q}}$.

Such an immersion can be extended to a tropical curve from Γ of the given type if the pairs of vertices attached to the remaining g bounded edges define the lines of correct slopes. Each of the g edges imposes at most $n-1$ linear conditions. That is, each edge F_i in these g edges determines a linear subspace $L_{F_i}\subset\mathbb{R}^{e-3+2g}$ of codimension at most $n-1$, and points in P which can be extended to a tropical curve must lie in L_{F_i}. Further, those points in P at which the end points ∂F_i of F_i are mapped to the same point in $N_{\mathbb{R}}$ is given by the intersection of P and a hyperplane H_{F_i} in $\mathbb{R}^{e-3+2g}$. H_{F_i} divides $\mathbb{R}^{e-3+2g}$

into two pieces. The interior of one of them, which we write by $H_{F_i}^+$, has the property that the points of the intersection $P \cap L_{F_i} \cap H_{F_i}^+$ correspond to those immersions of Γ' which can be extended to immersions of $\Gamma' \cup F_i$ compatible with the given type.

Therefore, the points of P which can be extended to a tropical curve from Γ is given by the intersection

$$P \cap (\bigcap_{i=1}^{g} L_{F_i}) \cap (\bigcap_{i=1}^{g} H_{F_i}^+),$$

which is an open convex polyhedral domain. It has dimension at least $e - 3 + 2g - g(n-1) = e - 3 - g(n-3)$. There is an additional n-dimensional freedom of parallel transformation of the image of the vertex V. This proves the claim for the purely 3-valent case. If there are 2-valent vertices, each one increases the number of bounded edges by one, and the length of the new edge contributes to the dimension of the parameter space. □

According to the proof, the rational points in the boundary of the open convex domain parametrizing immersive tropical curves have explicit geometric meaning. Namely, these points correspond to maps from Γ to $\mathbb{R}^n$ where some of the bounded edges of Γ are contracted, while the restriction of the maps to non-contracted edges is still compatible with the given combinatorial type. Let Γ be a 3-valent graph and we fix a type for it.

Lemma 18. *The map h from Γ to $\mathbb{R}^n$ corresponding to a rational point in the boundary of the open convex domain parametrizing immersive tropical curves of the given type is a tropical curve.*

Proof. It suffices to prove the balancing condition at each vertex of the image. Let p be a vertex of $h(\Gamma)$. Let Γ_1 be a connected component of $h^{-1}(p)$ which is a subgraph of Γ. It is a closed subgraph of Γ. Let $\{V_1, \dots, V_k\}$ be the set of vertices of Γ_1. Let $\{E_{i,j}\}$, $i = 1, \dots, k$, be the set of edges emanating from V_i, and not contained in Γ_1. Note that this set may be empty for some i, and that $E_{i,j}$ is not contracted by h. Then, the balancing condition at the image of Γ_1 means the equality

$$\sum_{i,j} w_\Gamma(E_{i,j}) u(V_i, E_{i,j}) = 0, \tag{2}$$

here $u\colon F\Gamma \to N$ is the map associated with the given type. In other words, $u(V_i, E_{i,j})$ is the primitive integral vector in the direction of $h(E_{i,j})$ emanating from $h(V_i) = p$.

Let $h'\colon \Gamma \to \mathbb{R}^n$ be an immersive tropical curve of the given type corresponding to a point near the rational boundary point we are dealing with. Let $\{V_1, \dots, V_k, V_{k+1}, \dots, V_l\}$ be the set of all the vertices of Γ, At each vertex V_i, the balancing condition holds for the map h'. Let B_i be the equation as in Definition 8 (iii) associated with V_i. Then, the equation (2) is nothing but the sum of all the equations B_i, $i = 1, \dots, k$. Here, note that if a bounded edge E is contained in Γ_1 and if we write its end points by V and V', the contributions $w_\Gamma(E)u(V, E)$ and $w_\Gamma(E)u(V', E)$ cancel since $u(V, E) = -u(V', E)$. Thus, the map h also satisfies the balancing condition. □

Definition 19. Fix a combinatorial type of 3-valent tropical curves whose parameter space of immersive curves is non-empty. Then, we define the parameter space of *all* 3-valent tropical curves of the given combinatorial type as the closure of the parameter space of immersive curves.

Remark 20. *In other words, an element of the parameter space of tropical curves of a given combinatorial type is a tropical curve which can be deformed into an immersive tropical curve of that combinatorial type. If there is no immersive tropical curve of the given combinatorial type, then the corresponding parameter space is defined to be empty in this paper (see Example 24). Note that if an unbounded edge is contracted, then it can never be deformed into an immersive tropical curve. Therefore, there is no such a curve in our parameter space.*

The following is an immediate consequence of Proposition 17.

Corollary 21. *Let (Γ, h) be a purely 3-valent tropical curve. Then, the parameter space of 3-valent tropical curves of the given combinatorial type is, if it is not empty, a closed convex polyhedral domain in the real affine k-dimensional space, where $k \geq e + (n-3)(1-g)$.* □

Remark 22. *The term 'closed' in the statement of the corollary does not mean 'compact'. Since we can parallel transport any tropical curve, the parameter space is always non-compact.*

Before stating Assumption A below, we prepare some terminology. Let Γ be a non-compact finite graph as above.

Definition 23. (i) An edge $E \in \Gamma^{[1]}$ is said to be a *part of a loop* of Γ if the graph $\Gamma \setminus E^\circ$ has the smaller first Betti number than Γ. Here, $E^\circ = E \setminus \partial E$.

(ii) The *loop part* of Γ is the subgraph of Γ composed of the union of all parts of the loops of Γ.

(iii) A *bouquet* of Γ is a connected component of the loop part of Γ. A subset of a bouquet which is homeomorphic to a circle is called a *loop*.

In particular, a bouquet or a loop does not contain unbounded edges.

Example 24. Proposition 17 fails to hold for non-immersive tropical curves. For example, consider an abstract 3-valent graph Γ which has three unbounded edges E_1, E_2, E_3 of weight one. Assume that the set $\Gamma \setminus \{E_1^\circ, E_2^\circ, E_3^\circ\}$ is a bouquet. The following map $h\colon \Gamma \to \mathbb{R}^2$ gives a tropical curve.

- h maps the ends of E_1, E_2, E_3 to the origin $(0,0) \in \mathbb{R}^2$.
- h maps the edges E_1, E_2, E_3 onto the half lines

$$\{(x,0) \mid x \geq 0\},\ \ \{(0,y) \mid y \geq 0\},\ \ \{(x,y) \mid x = y \leq 0\},$$

respectively.

- h contracts the other part of Γ to $(0,0) \in \mathbb{R}^2$.

Then, it is easy to see that there is no deformation of (Γ, h) other than parallel transports. Therefore, if the genus of Γ is positive and h is not immersive, Proposition 17 does not always hold. This tropical curve is not contained in the parameter space in the sense of Definition 19 for any combinatorial type.

Also, even when Γ is a tree, (Γ, h) need not deform into an immersive tropical curve, see Figure 1.

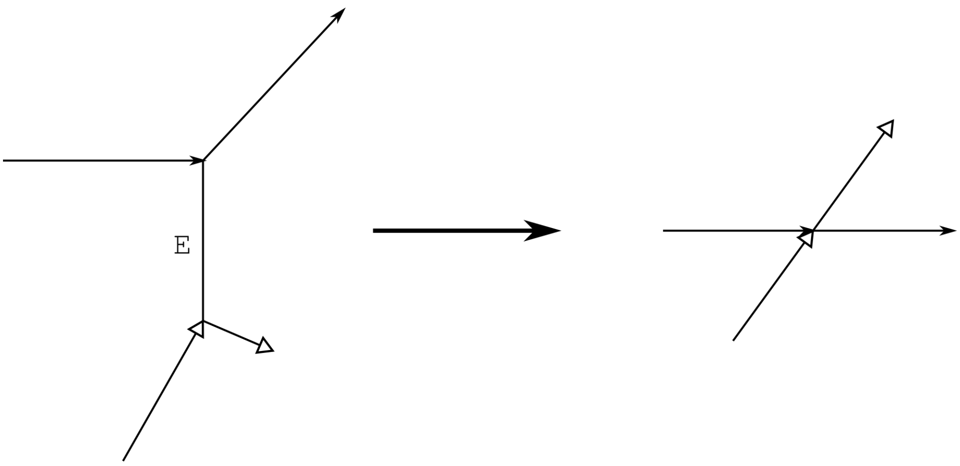

Figure 1. A tropical curve in $\mathbb{R}^2$ which cannot be deformed into an immersive curve, even though the domain abstract graph is a tree. The edge E in the abstract graph is contracted to the unique vertex of the image. The two upper edges of the abstract graph are mapped into the horizontal line in $\mathbb{R}^2$. Similarly, the two lower edges are mapped into the other line.

As this example shows, when we take into consideration those maps which cannot be deformed into immersive ones, it becomes difficult to give a unified treatment of tropical curves. Therefore, we introduce the following assumption. Let (Γ, h) be a 3-valent tropical curve in the sense of Definition 12. We fix a type of Γ.

Assumption A.

(i) The map h does not contract a loop to a point.

(ii) (Γ, h) can be deformed into an immersive tropical curve of the given type.

REMARK 25. 1. *When (Γ, h) satisfies Assumption A, we define a part of a loop of the image $h(\Gamma)$ as an edge E of Γ which is a part of a loop in Γ and not contracted by h. We sometimes identify E and its image $h(E)$ if no confusion could occur. The loop part, a bouquet and a loop of $h(\Gamma)$ are defined similarly.*

2. *By* (ii) *of Assumption A, when a tropical curve (Γ, h) satisfies it, the parameter space of the given type is a non-empty closed convex polyhedral domain in the real*

affine k-dimensional space, where $k \geq e+(n-3)(1-g)$, by Corollary 21. The subset of tropical curves satisfying Assumption A is neither closed nor open in general, since some of the edges can be contracted while edges in a loop cannot be contracted simultaneously.

Now, we define the superabundancy of tropical curves. We follow [**13**, Definition 2.22] so that the definition covers cases where Γ is not necessarily 3-valent. In such a case, if $V \in \Gamma^{[0]}$ is a k-valent vertex ($k \geq 2$), then we define the overvalence $\mathrm{ov}(v)$ of v to be $k-3$. In particular, if v is 2-valent, its overvalence is -1. We define the overvalence $\mathrm{ov}(\Gamma)$ of Γ to be

$$\mathrm{ov}(\Gamma) = \sum_{v \in \Gamma^{[0]}} \mathrm{ov}(v).$$

Let $h \colon \Gamma \to \mathbb{R}^n$ be any tropical curve. Let c be the number of edges of Γ contracted by the map h.

We fix a type of Γ. In the proposition below, we assume that if u is the type of (Γ, h), then $u(E) = 0$ for any edge E of Γ that is contracted. Then, we have the following generalization of Proposition 17.

PROPOSITION 26. *(*[**13**, Proposition 2.14]*) Let (Γ, h) be a tropical curve. Then, the space parameterizing tropical curves of the given combinatorial type is, if it is not empty, an open convex polyhedral domain in the real affine k-dimensional space, where $k \geq e + (n-3)(1-g) - \mathrm{ov}(\Gamma) - c$.* □

DEFINITION 27 ([**13**, Definition 2.22]). Using the notation in the above paragraph, we call a tropical curve (Γ, h) of a fixed type *regular* if the dimension of the convex cone in Proposition 26 is equal to $e + (n-3)(1-g) - \mathrm{ov}(\Gamma) - c$. If the dimension is larger than that, then (Γ, h) is called *superabundant*.

REMARK 28. 1. *In* [**21**]*, it was established that any regular tropical curve satisfies Assumption A. More precisely, after changing the weights of contracted edges of Γ if necessary, there is a regular immersive 3-valent tropical curve $(\widetilde{\Gamma}, \widetilde{h})$ of a suitable type $\widetilde{u}$ with the following property. Namely, there is a tropical curve $(\widetilde{\Gamma}, \overline{h})$ in the boundary of the parameter space of tropical curves containing $(\widetilde{\Gamma}, \widetilde{h})$ in the sense of Definition 19, such that $\overline{h}$ can be expressed as a contraction of $(\widetilde{\Gamma}, \widetilde{h})$. This means that there is a contraction of some edges $q \colon \widetilde{\Gamma} \to \Gamma$ compatible with the weights, satisfying $\overline{h} = h \circ q$. Thus, Assumption A provides a natural generalization of the regularity condition, particularly in more general settings.*

2. *In this paper, we primarily consider tropical curves satisfying Assumption A. In particular, the type u is not zero on any edge of Γ, which implies $c = 0$. Furthermore, $-\mathrm{ov}(\Gamma)$ is equal to the number of 2-valent vertices.*

To see whether a tropical curve satisfying Assumption A of a given combinatorial type is superabundant or not, it is enough to check it for an immersive tropical curve obtained by deforming the original curve. On the other hand, we will see below that the superabundancy of an immersive tropical curve can be effectively calculated via algebraic geometry.

3. Toric varieties associated with tropical curves and pre-log curves on them

In this section, we recall some notions from algebraic geometry relevant to our purpose. From now on, we assume tropical curves are well-segmented (see Definition 7) unless otherwise noted. Also, we assume that unbounded edges are not contracted.

Definition 29. A toric variety X defined by a fan Σ is called *associated with a tropical curve* (Γ, h) if the set of the rays of Σ contains the set of the rays (that is, one dimensional cones) spanned by the vectors in the support of the degree map $\Delta\colon N \setminus \{0\} \to \mathbb{N}$ of (Γ, h).

If $\mathfrak{E}$ is an unbounded edge of $h(\Gamma)$, there is a unique divisor of X corresponding to it. We write it by $D_{\mathfrak{E}}$ and call it the *divisor associated with the edge* $\mathfrak{E}$.

Given a tropical curve (Γ, h) in $N_{\mathbb{R}}$, recall that in our definition, the vertices have rational coordinates. We can construct a polyhedral decomposition $\mathscr{P}$ of $N_{\mathbb{R}}$ defined over $\mathbb{Q}$ such that $h(\Gamma)$ is contained in the 1-skeleton of $\mathscr{P}$ in a way that the set of vertices of $h(\Gamma)$ is contained in that of $\mathscr{P}$. ([**22**, Proposition 3.9]).

Definition 30. Given such $\mathscr{P}$, we construct a degenerating family $\mathfrak{X} \to \mathbb{A}^1_{\mathbb{C}}$ of a toric variety X associated with (Γ, h) ([**22**, Section 3], see also Convention 31 below). We call such a family a *degeneration of X defined respecting* (Γ, h). Let X_0 be the central fiber. It is a union $X_0 = \cup_{v \in \mathscr{P}^{[0]}} X_{0,v}$ of toric varieties intersecting along toric strata. Here, $\mathscr{P}^{[0]}$ is the set of the vertices of $\mathscr{P}$.

Convention 31. From now on, we assume that the vertices of $h(\Gamma)$ are contained in N, and that if E is a bounded edge of Γ which is not contracted by h, the integral length of $h(E)$ is a positive integer multiple of its weight.

These conditions can be met by a homothetic expansion. Under these conditions, the central fiber X_0 will be reduced, and we can apply a result in log deformation theory relevant to us [**22**, Proposition 7.1].

Definition 32 ([**22**, Definition 4.1]). Let X be a toric variety. A holomorphic curve $C \subset X$ is *torically transverse* if it is disjoint from all toric strata of codimension greater than one. A stable map $\phi\colon C \to X$ is torically transverse if $\phi^{-1}(\mathrm{int}X) \subset C$ is dense and $\phi(C) \subset X$ is a torically transverse curve. Here, $\mathrm{int}X$ is the complement of the union of toric divisors.

Definition 33. Let C_0 be a prestable curve. A *pre-log curve* on X_0 is a stable map $\varphi_0\colon C_0 \to X_0$ with the following properties.

(i) For any $v \in \mathcal{P}^{[0]}$, the restriction $C_0 \times_{X_0} X_{0,v} \to X_{0,v}$ is a torically transverse stable map.

(ii) Let $P \in C_0$ be a point which maps to the singular locus of X_0. Then, C_0 has a node at P, and φ_0 maps the two branches $(C_0', P), (C_0'', P)$ of C_0 at P to different irreducible components $X_{0,v'}, X_{0,v''} \subset X_0$. Moreover, if w' is the intersection multiplicity of the restriction $(C_0', P) \to (X_{0,v'}, D')$ with the toric divisor $D' \subset X_{0,v'}$, and w'' is the similar intersection multiplicity for $(C_0'', P) \to (X_{0,v''}, D'')$, then $w' = w''$.

Let X be a toric variety and D be the union of toric divisors. In [**22**, Definition 5.2], a non-constant torically transverse map $\phi\colon \mathbb{P}^1 \to X$ is called a *line* if

$$\sharp\phi^{-1}(D) \leq 3.$$

In this case, the image of ϕ is contained in the closure of the orbit of the action of a subtorus of dimension at most two of the big torus acting on X ([**22**, Lemma 5.2]). Because we consider more general tropical curves, we need to extend this notion.

Let Γ be a weighted 3-valent tree and $h\colon \Gamma \to N_{\mathbb{R}}$ be a map which gives Γ a structure of a tropical curve, and assume that the image $h(\Gamma)$ has only one vertex v. Let $E_1, \dots, E_s$ be the unbounded edges of Γ, and $\mathfrak{E}_1, \dots, \mathfrak{E}_s$ be their images (it can happen that $\mathfrak{E}_i = \mathfrak{E}_j$ for $i \neq j$). Note that we are assuming unbounded edges are not contracted. Let X be a toric variety associated with (Γ, h).

Definition 34. A non-constant torically transverse map $\phi\colon \mathbb{P}^1 \to X$ is called *of type* (Γ, h), or *of type* v when h is clear from the context, if ϕ satisfies the following property:

- Let w_i be the weight of E_i. Then, $\phi(\mathbb{P}^1)$ has an intersection with the divisor $D_{\mathfrak{E}_i}$ with intersection multiplicity w_i, and there is no intersection between $\phi(\mathbb{P}^1)$ and toric divisors other than these.

Note that when some of $\mathfrak{E}_1, \dots, \mathfrak{E}_s$ coincide, then there are several intersections between $\phi(\mathbb{P}^1)$ and the corresponding toric divisor.

Let (Γ, h) be a tropical curve satisfying Assumption A. Let $v \in h(\Gamma)^{[0]}$ be a vertex. The inverse image $h^{-1}(v)$ consists of closed subgraphs of Γ, since we are assuming (Γ, h) is well-segmented. We write the set of connected components of $h^{-1}(v)$ by $\{\gamma_{v(i)}\}$, $i = 1, \dots, k$, where k is the number of connected components of $h^{-1}(v)$.

Definition 35. Let $\{\Gamma_{v(i)}\}$ be the set of open subgraphs of Γ such that $\Gamma_{v(i)}$ is the union of $\gamma_{v(i)}$ and the open parts of the edges emanating from its vertices. See Figure 2 for an illustration. Let Γ_h be the graph obtained from Γ by contracting each $\gamma_{v(i)}$ to a vertex, for all $v \in h(\Gamma)^{[0]}$.

The following is obvious.

Lemma 36. *There are natural maps*

$$\pi\colon \Gamma \to \Gamma_h$$

and

$$\overline{h}\colon \Gamma_h \to h(\Gamma)$$

satisfying $h = \overline{h} \circ \pi$. □

Note that for each point q of $h(\Gamma)$, the inverse image $\overline{h}^{-1}(q)$ is a finite subset of Γ_h. In particular, if q is a vertex $v \in h(\Gamma)^{[0]}$, then there is one-to-one correspondence between the set $\overline{h}^{-1}(v)$ and the set of connected components of $h^{-1}(v)$.

Let X be a toric variety associated with (Γ, h), and $\mathfrak{X} \to \mathbb{A}^1_{\mathbb{C}}$ be a degeneration of X defined respecting (Γ, h). Let X_0 be the central fiber of this degeneration. Each vertex v of $h(\Gamma)$ determines an irreducible component $X_{0,v}$ of X_0. Given a pre-log curve $\varphi_0\colon C_0 \to X_0$, let $\{C_{0,v(i)}\}$ be the set of irreducible components of C_0 whose images by φ_0 is contained in $X_{0,v}$.

Definition 37. A pre-log curve $\varphi_0\colon C_0 \to X_0$ is called *of type* (Γ, h) if the following conditions are satisfied:

- The dual intersection graph of C_0 is Γ_h. In particular, for each $v \in h(\Gamma)^{[0]}$, if we write $\overline{h}^{1}(v) = \{v(i)\}$, then there exist irreducible components corresponding to these points, which we write by $C_{0,v(i)}$.
- The restriction of φ_0 to each $C_{0,v(i)}$ is a torically transverse rational curve of type $(\Gamma_{v(i)}, h|_{\Gamma_{v(i)}})$.

The set of unbounded edges of Γ_h is in bijection with the finite subset of C_0 consisting of the inverse images of toric divisors of irreducible components of X_0 which lift to toric divisors of X. Elements of this set are considered as marked points on C_0.

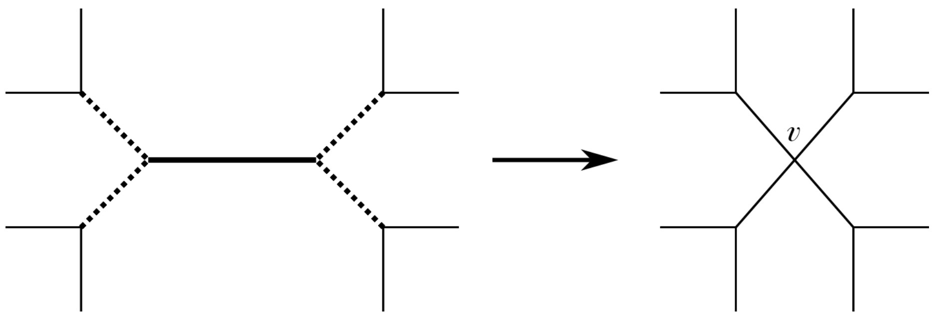

Figure 2. The abstract graph Γ (the picture on the left) is mapped to a graph in $\mathbb{R}^n$ (the picture on the right). The bold line segment in Γ is the inverse image of the vertex v and the union of the bold line segment and the dotted line segments in Γ is the subgraph $\Gamma_{v(1)}$ (in this case, $\{v(i)\} = \{v(1)\}$). In this example, the graph Γ_h is the same as the picture on the right as an abstract graph.

The components $\{C_{0,v(i)}\}$ are disjoint, since otherwise the graphs $\Gamma_{v(i)}$ and $\Gamma_{v(i')}$ have a non-contracting common (open) edge E for some i and i', but since both of the ends of E are mapped to the vertex v, the edge E must be contracted. Note that each $C_{0,v(i)}$ is an irreducible rational curve.

Remark 38 ([**22**, Definition 5.6]). *A pre-log curve* $\varphi_0 \colon C_0 \to X_0$ *is* maximally degenerate *if it is of type* (Γ, h) *for some* immersive *tropical curve* (Γ, h).

Definition 39. A 3-valent tropical curve (Γ, h) is *smoothable* if there is a pre-log curve $\varphi_0 \colon C_0 \to X_0$ of type (Γ, h) with the following property. Namely, there exists a family of stable maps over $\mathbb{A}^1_{\mathbb{C}}$

$$\Phi \colon \mathfrak{C}/\mathbb{A}^1_{\mathbb{C}} \to \mathfrak{X}/\mathbb{A}^1_{\mathbb{C}}$$

such that $\mathfrak{C}/\mathbb{A}^1_{\mathbb{C}}$ is a flat family of pre-stable curves whose fiber over 0 is isomorphic to C_0, and the restriction of Φ to C_0 is a stable map equivalent to φ_0. We also call such a pre-log curve *smoothable*.

4. Cohomology groups containing the obstruction

Our goal will be describing the (dual) obstruction cohomology group associated with degenerate algebraic curves corresponding to a tropical curve satisfying Assumption A (in particular, it need not be immersive), see Theorem 57.

Contrary to the immersive case (see Corollary 59), the description in a general case is not purely combinatorial reflecting the fact that higher-valent vertices correspond to rational curves with $k(\geq 4)$ special points (that is, nodes and marked points), which have their own moduli. The study of such curves owes a lot to the degeneration technique. Namely, given a curve we want to investigate, the structure of its degeneration is not necessarily simple. However, in the degenerate situation, we can reduce various calculation to the standard case (typically curves of degree one in projective spaces). Thus, we can obtain important quantities such as obstruction classes rather explicitly.

Let (Γ, h), $h \colon \Gamma \to N_{\mathbb{R}} \cong \mathbb{R}^n$, be a tropical curve satisfying Assumption A and X be a toric variety associated with it. Let $\mathfrak{X} \to \mathbb{A}^1_{\mathbb{C}}$ be a degeneration of X defined respecting (Γ, h). Let $\mathscr{P}$ be a polyhedral decomposition of $N_{\mathbb{R}}$ defining $\mathfrak{X}$. Assume there is a pre-log curve $\varphi_0 \colon C_0 \to X_0$ of type (Γ, h) (in fact, the result in this section makes sense even if such a curve does not exist, see Remark 61).

Remark 40. *It is in general necessary to allow 2-valent vertices to* Γ *in order to assure the property* $h^{-1}(\mathscr{P}^{[0]}) = \Gamma^{[0]}$. *Then, there are components of* C_0 *corresponding to these 2-valent vertices. However, as in the proof of* [**22**, Proposition 7.1]*, these components do not play an essential role in the argument below. Therefore, we usually neglect them and regard* Γ *as if it has only 3-valent vertices for the simplicity of presentation.*

We can give log structures to C_0 and $\mathfrak{X}$ with the following properties. Namely, we introduce a natural log structure on $\mathfrak{X}$ induced naturally from the toric structure of it. Then, we put a log structure on C_0 which is log smooth over the standard log point, and strict on $\varphi_0^{-1}(\mathrm{int}X)$. We can choose it so that the map φ_0 naturally lifts to a map between log schemes. See [**22**, Section 7] for more details. There are log tangent sheaves associated with these log structures. There may be several log structures on C_0 with the above properties, however, they have isomorphic log tangent sheaves and the choice does not affect the argument below. The tangent space and the obstruction space of the deformation of φ_0 are calculated in terms of these sheaves. See [**11**] for fundamental properties of log structures.

Suppose that a lift

$$\varphi_{k-1}\colon C_{k-1}/O_{k-1} \to \mathfrak{X}$$

of φ_0 is constructed. Here,

$$O_{k-1} = \operatorname{Spec}\mathbb{C}[t]/t^k$$

and its log structure is given by

$$\mathbb{C}^* \oplus \mathbb{N} \to \mathbb{C}[t]/t^k,$$

which is the identity map on $\mathbb{C}^*$ and sends $1 \in \mathbb{N}$ to t. Then, as in the proof of [**22**, Lemma 7.2], an extension C_k/O_k of C_{k-1}/O_{k-1} exists and such extensions are parametrized by the space $H^1(C_0, \Theta_{C_0/O_0})$, here Θ_{C_0/O_0} is the log tangent sheaf.

On the other hand, the obstruction to lifting the map φ_{k-1} to the next order lies in $H^1(C_0, \varphi_0^*\Theta_{\mathfrak{X}/\mathbb{A}^1_{\mathbb{C}}})$, here $\Theta_{\mathfrak{X}/\mathbb{A}^1_{\mathbb{C}}}$ is the log tangent sheaf relative to the base. As in usual deformation theory of smooth varieties, there is a following standard result in the log smooth deformation theory ([**11**, Proposition 3.9], see also the proof of [**22**, Lemma 7.2]).

Proposition 41. *If $H^1(C_0, \varphi_0^*\Theta_{\mathfrak{X}/\mathbb{A}^1_{\mathbb{C}}})$ vanishes, then the pre-log curve φ_0 is smoothable.* □

On the other hand, the sheaf $\varphi_0^*\Theta_{\mathfrak{X}/\mathbb{A}^1_{\mathbb{C}}}$ fits in the exact sequence

$$0 \to \Theta_{C_0/O_0} \to \varphi_0^*\Theta_{\mathfrak{X}/\mathbb{A}^1_{\mathbb{C}}} \to \mathcal{N}_{\varphi_0} \to 0. \tag{3}$$

Here,

$$\mathcal{N}_{\varphi_0} = \varphi_0^*\Theta_{\mathfrak{X}/\mathbb{A}^1_{\mathbb{C}}}/\Theta_{C_0/O_0}$$

is the log normal sheaf. The restriction of $\mathcal{N}_{\varphi_0}$ to any component $C_{0,v}$ of C_0 is isomorphic to the usual (non-log) normal sheaf of it. Note that we have the natural isomorphism

$$\Theta_{\mathfrak{X}/\mathbb{A}^1_{\mathbb{C}}} \simeq N \otimes_{\mathbb{Z}} \mathcal{O}_{\mathfrak{X}},$$

here N is the free abelian group such that the fan defining a general fiber of $\mathfrak{X}$ lies in $N_{\mathbb{R}} = N \otimes \mathbb{R}$.

We have a map between cohomology groups

$$H^1(C_0, \Theta_{C_0/O_0}) \to H^1(C_0, \varphi_0^*\Theta_{\mathfrak{X}/\mathbb{A}^1_{\mathbb{C}}}).$$

The group $H^1(C_0, \Theta_{C_0/O_0})$ is the tangent space of the moduli space of deformations of C_0, and the obstruction classes in $H^1(C_0, \varphi_0^*\Theta_{\mathfrak{X}/\mathbb{A}^1_{\mathbb{C}}})$ which are in the image of the above map can be cancelled when we deform the moduli of the domain of the stable maps. Namely, we have the following standard fact.

Proposition 42. *If the map $H^1(C_0, \Theta_{C_0/O_0}) \to H^1(C_0, \varphi_0^*\Theta_{\mathfrak{X}/\mathbb{A}^1_{\mathbb{C}}})$ is a surjection, then the map φ_0 is smoothable.*

Proof. Assume we have constructed a log deformation $\varphi_k \colon C_k \to \mathfrak{X}$, of the map φ_0, where k is a non-negative integer. We fix a log deformation C'_{k+1} of the curve C_k as a reference. Note that such C'_{k+1} exists since the obstruction $H^2(C_0, \Theta_{C_0/O_0})$ to the existence of the deformation vanishes. The obstruction to deforming φ_k to a map from C_{k+1} is calculated as follows.

Take a suitable open covering $\{U_i\}$ of C_0 so that there is a lift $\varphi'_{k+1}|_{U_i} \colon U_{i,k+1} \to \mathfrak{X}$ of the restriction of φ_k to each of the open set U_i. Here, $U_{i,k+1}$ is the restriction of the structure of the log curve C'_{k+1} to the open subset U_i. The existence of such a covering follows from the general theory of log smooth deformations.

The set of lifts on U_i forms a torsor over the abelian group of sections $\Gamma(U_i, \varphi_0^*\Theta_{\mathfrak{X}/\mathbb{A}^1_{\mathbb{C}}})$, and the differences of the lifts on the intersections $U_i \cap U_j$ determine a $\varphi_0^*\Theta_{\mathfrak{X}/\mathbb{A}^1_{\mathbb{C}}}$-valued Čech 1-cocycle, which represents the class α.

On the other hand, by assumption, the class α is mapped to $0 \in H^1(C_0, \mathcal{N}_{\varphi_0})$. This implies that we can perturb each lift $\varphi'_{k+1}|_{U_i}$ by a section in $\Gamma(U_i, \varphi_0^*\Theta_{\mathfrak{X}/\mathbb{A}^1_{\mathbb{C}}})$ so that the differences of the lifts map to zero in $\mathcal{N}_{\varphi_0}$ not only cohomologically, but also at the level of cocycles. Let $\varphi_{k+1}|_{U_i}$ be the perturbed map of $\varphi'_{k+1}|_{U_i}$ satisfying this property. By the exact sequence (3), it follows that the differences of the lifts give Θ_{C_0/O_0}-valued Čech 1-cocycle, where we see Θ_{C_0/O_0} as a subsheaf of $\varphi_0^*\Theta_{\mathfrak{X}/\mathbb{A}^1_{\mathbb{C}}}$ in the natural way.

In particular, the class α can be seen as a class of $H^1(C_0, \Theta_{C_0/O_0})$. Note that the set of log deformations of C_k is a torsor over $H^1(C_0, \Theta_{C_0/O_0})$. Since we have fixed the reference deformation C'_{k+1}, it follows that if we take C_{k+1} to be the deformation of C_k corresponding to the class α relative to C'_{k+1}, the maps $\varphi_{k+1}|_{U_i}$ naturally glues into a global map φ_{k+1} on C_{k+1}. This proves the claim. $\square$

In other words, the obstruction to smoothing φ_0 in fact lies in the cohomology group $H^1(C_0, \mathcal{N}_{\varphi_0})$. By the Serre duality for nodal curves, we have

$$H^1(C_0, \mathcal{N}_{\varphi_0}) \cong H^0(C_0, \mathcal{N}_{\varphi_0}^\vee \otimes \omega_{C_0})^\vee.$$

Here, ω_{C_0} is the dualizing sheaf of the nodal curve C_0, which is the sheaf of meromorphic 1-forms with logarithmic poles allowed at the nodes. It is known that ω_{C_0} is an invertible sheaf.

The purpose of this paper is to calculate these groups. The calculation is based on gluing local data. We begin with the study of such local pieces.

5. Computation of the dual obstruction cohomology groups associated with higher-valent vertices

In the following three sections, we prove the main result, Theorem 57, which describes the group $H^0(C_0, \mathcal{N}_{\varphi_0}^\vee \otimes \omega_{C_0})^\vee$ introduced at the end of the previous section. Theorem 57 also provides, on the tropical side, the answer to the Questions 1 and 2 in the introduction for tropical curves satisfying Assumption A. Since the proof includes several steps, we summarize them here.

As we have just mentioned, our purpose is to compute the group $H^0(C_0, \mathcal{N}_{\varphi_0}^\vee \otimes \omega_{C_0})^\vee$, where $\varphi_0\colon C_0 \to X_0$ is a pre-log curve of some type (Γ, h), see Definition 37. Each irreducible component $C_{0,v}$ of C_0 corresponds to a vertex v of Γ_h, and the restriction of φ_0 to $C_{0,v}$ gives a torically transverse curve of type $(\Gamma_v, h|_{\Gamma_v})$, using the notation of Definitions 35 and 37. Our strategy to compute $H^0(C_0, \mathcal{N}_{\varphi_0}^\vee \otimes \omega_{C_0})^\vee$ is to first study the group $H^0(C_{0,v}, (\mathcal{N}_{\varphi_0}^\vee \otimes \omega_{C_0})|_{C_{0,v}})^\vee$ for each component, and then glue these pieces together.

When (Γ, h) is a 3-valent, immersive tropical curve, the description of $H^0(C_0, \mathcal{N}_{\varphi_0}^\vee \otimes \omega_{C_0})^\vee$ is given by the combinatorics of (Γ, h) (see Corollary 59). However, in general, it is described as the solution space to a system of linear equations, see Theorem 57. To write down these equations explicitly, we need detailed information about each $H^0(C_{0,v}, (\mathcal{N}_{\varphi_0}^\vee \otimes \omega_{C_0})|_{C_{0,v}})^\vee$. Dealing directly with the restriction $\varphi_0|_{C_{0,v}}$ is not easy, however. For example, if $X_{0,v}$ is the component to which $C_{0,v}$ is mapped, $X_{0,v}$ is often singular, and there may be no canonical desingularization or nice coordinate systems. Additionally, the image of $C_{0,v}$ itself is also often singular.

To resolve this difficulty, we note that the map $\varphi_0|_{C_{0,v}}$ factors through a map to an open toric subvariety $\mathbb{P}_{\Gamma_0, h_0}$ of a projective space:

$$\varphi_0|_{C_{0,v}} = f \circ \varphi,$$

where

$$\varphi\colon C_{0,v} \to \mathbb{P}_{\Gamma_0, h_0}$$

is a map whose image is a torically transverse line (i.e., a curve of degree one), and $f\colon \mathbb{P}_{\Gamma_0, h_0} \to X_{0,v}$ is a toric map, see Proposition 51. When the map $\varphi_0|_{C_{0,v}}$ itself is isomorphic to the map φ, then we can describe the group $H^0(C_{0,v}, (\mathcal{N}_{\varphi_0}^\vee \otimes \omega_{C_0})|_{C_{0,v}})^\vee$ in detail. We do this in this section, and Lemmas 46, 47, and Remark 49 provide the desired result. The general case is treated in Section 6, where we combine the results in this section and the combinatorial data associated to the toric map f. The result is Lemma 53.

Finally, in Section 7, we glue the local pieces studied so far, applying the compatibility condition from Definition 54. Then, based on a combinatorial argument (see Lemma 56) we derive the main result, Theorem 57.

Now, let us begin the local study. Let Γ_0 be a 3-valent tree graph with unbounded edges (see the beginning of Section 2). Let r be the number of vertices of Γ_0. Then, Γ_0 has $r+2$ unbounded edges. We assume all the edge weights of Γ_0 are 1. We write by

$$E_1, \dots, E_{r+2}$$

the unbounded edges. Let

$$h_0 \colon \Gamma_0 \to \mathbb{R}^n, \quad n \geq r+1,$$

be the map which contracts all the bounded edges and maps the edges E_i onto specific rays as follows:

$$E_i \to \mathbb{R}_{\geq 0} \cdot (0, \dots, \overset{i}{\check{1}}, \dots, 0), \;\; i = 1, \dots, r+1,$$

$$E_{r+2} \to \mathbb{R}_{\geq 0} \cdot (-1, \dots, -1, 0, \dots, 0),$$

where, in the latter, the first $r+1$ components are -1. In particular, h_0 maps all the vertices of Γ_0 to the origin. Clearly this satisfies the balancing condition and gives Γ_0 a structure of a tropical curve.

Let (Γ, h), $h \colon \Gamma \to \mathbb{R}^n$, be a tropical curve satisfying Assumption A which contains a subgraph such that the restriction of h to it is isomorphic to (Γ_0, h_0) restricted to a suitable open connected subset containing all the vertices. We regard the image of this subset by h_0 as a subgraph of $h(\Gamma)$. Let X be a toric variety associated with (Γ, h) and $\mathfrak{X}$ be a toric degeneration of X defined respecting (Γ, h). Let $\varphi_0 \colon C_0 \to X_0$ be a pre-log curve of type (Γ, h). We put a log structure on C_0 so that the map φ_0 extends to a morphism between log schemes (with the log structure on $\mathfrak{X}$ coming from the toric structure) and the composition of φ_0 with the projection $\mathfrak{X} \to \mathbb{A}^1_{\mathbb{C}}$ is log smooth, see [**22**, Section 7].

REMARK 43. *In general, there exist tropical curves such that there are no degenerate holomorphic curves of those types, even if we restrict our attention to embedded tropical curves. Here, we assume there is a degenerate holomorphic curve of type* (Γ, h).

Our purpose is to calculate the cohomology group $H^1(C_0, \mathcal{N}_{\varphi_0})$ or its dual $H^0(C_0, \mathcal{N}_{\varphi_0}^{\vee} \otimes \omega_{C_0})$. The sheaf $\mathcal{N}_{\varphi_0}$ is locally free, and elements of $H^0(C_0, \mathcal{N}_{\varphi_0}^{\vee} \otimes \omega_{C_0})$ can be described by gluing sections of $\mathcal{N}_{\varphi_0}^{\vee} \otimes \omega_{C_0}$ restricted to irreducible components of C_0. Therefore, we first concentrate on the study of the restriction of $\mathcal{N}_{\varphi_0}^{\vee} \otimes \omega_{C_0}$ to the component $C_{0,v}$ of C_0 corresponding to a vertex v of $h(\Gamma)$ which is modeled on the unique vertex of $h_0(\Gamma_0)$. It is easy to see the following.

LEMMA 44. *The restriction of* $\mathcal{N}_{\varphi_0}$ *to* $C_{0,v}$ *is isomorphic to*

$$\mathcal{O}_{\mathbb{P}^1}(1)^r \oplus \mathcal{O}_{\mathbb{P}^1}^{n-r-1}.$$

PROOF. It suffices to compute the normal sheaf of a line (that is, a curve of degree one) on $\mathbb{P}^{r+1} \times (\mathbb{C}^*)^{n-r-1}$. The normal sheaf of a hyperplane in $\mathbb{P}^{r+1}$ is isomorphic to the restriction of $\mathcal{O}_{\mathbb{P}^{r+1}}(1)$ to it. By repeating this, the claim follows. □

Let s be the number of nodes of C_0 contained in $C_{0,v}$. This is the same as the number of the edges among $h_0(E_1), \ldots, h_0(E_{r+2})$ which are the restrictions of the bounded edges of $h(\Gamma)$.

LEMMA 45. *We have*

$$\dim H^0(C_{0,v}, \mathcal{N}_{\varphi_0}^\vee \otimes \omega_{C_0}|_{C_{0,v}}) = \begin{cases} 0, & (s = 0, 1) \\ r(s-2) + (n-r-1)(s-1), & (s \geq 2). \end{cases}$$

PROOF. Note that the restriction $\omega_{C_{0,v}} = \omega_{C_0}|_{C_{0,v}}$ is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(-2+s)$. Thus,

$$\begin{aligned} H^0(C_{0,v}, \mathcal{N}_{\varphi_0}^\vee \otimes \omega_{C_0}|_{C_{0,v}}) &\cong H^0(C_{0,v}, (\mathcal{O}_{\mathbb{P}^1}(-1)^r \oplus \mathcal{O}_{\mathbb{P}^1}^{n-r-1}) \otimes \mathcal{O}_{\mathbb{P}^1}(-2+s)) \\ &\cong H^0(C_{0,v}, \mathcal{O}_{\mathbb{P}^1}(-3+s)^r \oplus \mathcal{O}_{\mathbb{P}^1}(-2+s)^{n-r-1}). \end{aligned}$$

The result follows from this. $\square$

For notational simplicity, we assume all the edges $E_1, \ldots, E_{r+2}$ are the restrictions of bounded edges of Γ, so that $s = r+2$ (see Remark 49 for general cases). The sheaf $\omega_{C_{0,v}}$ is the sheaf of meromorphic 1-forms such that they can have logarithmic poles at the points of $C_{0,v}$ which are nodes of C_0. These nodes bijectively correspond to the edges $E_1, \ldots, E_{r+2}$. By the residue theorem, the residues $a_1, \ldots, a_s$ at these poles sum up to zero:

$$a_1 + \cdots + a_s = 0.$$

Let us fix an affine coordinate ζ on $C_{0,v}$ such that the coordinates of the points corresponding to the edges $E_1, \ldots, E_{r+1}, E_{r+2}$ are

$$p_1, \ldots, p_{r+1}, \infty.$$

Then, we can take

$$\sigma_i = \frac{d\zeta}{\zeta - p_i}, \quad i = 1, \ldots, r+1$$

as a basis of the space of sections of $\omega_{C_{0,v}}$.

Recall that the log normal sheaf $\mathcal{N}_{\varphi_0}$ is the quotient $\varphi_0^*\Theta_{\mathfrak{X}/\mathbb{A}^1_{\mathbb{C}}}/\Theta_{C_0/O_0}$ and the sheaf $\varphi_0^*\Theta_{\mathfrak{X}/\mathbb{A}^1_{\mathbb{C}}}$ is naturally isomorphic to the sheaf $N \otimes_{\mathbb{Z}} \mathcal{O}_{C_0}$ [**10**, Example 5.6]. Here, N is the free abelian group of rank n.

We note that $\mathcal{N}_{\varphi_0}$ is locally free. Then, the sheaf $\mathcal{N}_{\varphi_0}^\vee \otimes \omega_{C_0}|_{C_{0,v}}$ can be seen as a subsheaf of the sheaf of $N_{\mathbb{C}}^\vee$-valued meromorphic 1-forms. In particular, a section of the restriction of $\mathcal{N}_{\varphi_0}^\vee \otimes \omega_{C_0}$ to the component $C_{0,v}$ can be written in the form

$$\sum_{i,j=1}^{r+1} a_{i,j} e_i^\vee \otimes \sigma_j + \sum_{i=r+2}^{n} \sum_{j=1}^{r+1} c_{i,j} e_i^\vee \otimes \sigma_j, \tag{4}$$

where $a_{i,j}$ and $c_{i,j}$ are complex numbers and $\{e_i\}$, $i = 1, \ldots, n$, is a basis of N such that

$\{e_i\}$, $i = 1, \ldots, r+1$, is a basis of $N \cap N_v$, where $N_v \subset N_{\mathbb{R}}$ is the subspace spanned by the directions of the edges emanating from the vertex v. Note that $\{e_i\}$, $i = 1, \ldots, r+1$, can be taken so that e_i is the direction vector of $h(E_i)$. The set of vectors $\{e_i^{\vee}\}$ is the dual basis of $\{e_i\}$.

The stalk of Θ_{C_0/O_0} at the point p_i corresponding to the edge E_i is spanned by the slope of the image $h(E_i)$, considered as a subset of $N \otimes_{\mathbb{Z}} \mathcal{O}_{C_{0,v}}$. Sections of $\Gamma(C_{0,v}, (\varphi_0^*\Theta_{\mathfrak{X}/\mathbb{A}^1_{\mathbb{C}}}/\Theta_{C_0/O_0})^{\vee} \otimes \omega_{C_0})$ must annihilate these vectors, and this condition implies the following.

LEMMA 46. *The coefficients of a section* $\sum_{i,j=1}^{r+1} a_{i,j} e_i^{\vee} \otimes \sigma_j + \sum_{i=r+2}^{n} \sum_{j=1}^{r+1} c_{i,j} e_i^{\vee} \otimes \sigma_j$ *of* $\mathcal{N}_{\varphi_0}^{\vee} \otimes \omega_{C_0}|_{C_{0,v}}$ *satisfy*

$$a_{1,1} = a_{2,2} = \cdots = a_{r+1,r+1} = 0.$$

□

A general torically transverse curve of degree one in $\mathbb{P}^{r+1}$ is defined by the equations of the following form:

$$b_2 X_1 + X_2 + c_2 = 0, \ \ b_3 X_1 + X_3 + c_3 = 0, \ \ \cdots, \ \ b_{r+1} X_1 + X_{r+1} + c_{r+1} = 0,$$

where $X_i = \frac{x_i}{x_{r+2}}$ are affine coordinates of $\mathbb{P}^{r+1}$ corresponding to $e_i^{\vee}$, and x_i are homogeneous coordinates of $\mathbb{P}^{r+1}$.

Note that Θ_{C_0/O_0} is an invertible sheaf on C_0. The image of a local section of it in $\varphi_0^*\Theta_{\mathfrak{X}/\mathbb{A}^1_{\mathbb{C}}}$ is characterized by the property that it is annihilated by sections of $\mathcal{N}_{\varphi_0}^{\vee}$. When

$$X_2 X_3 \cdots X_{r+1} \neq 0,$$

there is a local section of Θ_{C_0/O_0} whose image in $\varphi_0^*\Theta_{\mathfrak{X}/\mathbb{A}^1_{\mathbb{C}}}$ is given by

$$X_1 \partial_{X_1} + \frac{X_2 + c_2}{X_2} X_2 \partial_{X_2} + \cdots + \frac{X_{r+1} + c_{r+1}}{X_{r+1}} X_{r+1} \partial_{X_{r+1}}.$$

Note that the vector $e_i^{\vee}$ is naturally identified with the section $\frac{dX_i}{X_i}$ of the pull back of the log cotangent sheaf

$$\varphi_0^*\Omega_{\mathfrak{X}/\mathbb{A}^1_{\mathbb{C}}} \cong N^{\vee} \otimes \mathcal{O}_{C_0}.$$

Let ζ be the parameter of $C_{0,v}$ given by $\varphi_0^* X_1$. Taking ζ in this way, the coordinate p_i of the pole of σ_i is given by

$$p_i = -\frac{c_i}{b_i}, \ \ i = 1, \ldots, r+1,$$

here we take $c_1 = 0, b_1 = -1$ (thus, $p_1 = 0$).

Then, the above annihilating condition implies

$$\sum_{i,j=1}^{r+1} a_{i,j}\frac{\zeta}{\zeta-p_i}\cdot\frac{1}{\zeta-p_j}=\zeta\sum_{i,j=1}^{r+1} a_{i,j}\frac{\prod_{l\neq i,j}(\zeta-p_l)}{\prod_{l=1}^{r+1}(\zeta-p_l)}=0.$$

Thus, we have the following. Let us define a polynomial $P(\zeta)$ of ζ by

$$P(\zeta)=\sum_{i,j=1}^{r+1} a_{i,j}\prod_{l\neq i,j}(\zeta-p_l)=\sum_{k=0}^{r-1} A_k(\{a_{i,j}\})\zeta^k,$$

where

$$A_k(\{a_{i,j}\})=(-1)^{r-1-k}\sum_{i,j=1}^{r+1}\sum_{\substack{J\subset I\setminus\{i,j\},\\ |J|=r-1-k}} p_{j_1}\cdots p_{j_{r-1-k}}a_{i,j}$$

is a linear polynomial of $a_{i,j}$. Here, $I=\{1,\dots,r+1\}$, and in the summation the set $J=\{j_1,\dots,j_{r-1-k}\}$ runs through all subsets of $I\setminus\{i,j\}$ of cardinality $r-1-k$.

LEMMA 47. *The coefficients $a_{i,j}$ of a section $\sum_{i,j=1}^{r+1} a_{i,j}e_i^\vee\otimes\sigma_j+\sum_{i=r+2}^{n}\sum_{j=1}^{r+1} c_{i,j}e_i^\vee\otimes\sigma_j$ of $\mathcal{N}_{\varphi_0}^\vee\otimes\omega_{C_0}|_{C_{0,v}}$ satisfy*

$$A_k(\{a_{i,j}\})=0,\quad k=0,\dots,r-1.$$

There is no constraint to the constants $c_{i,j}$. □

REMARK 48. *Note that Lemmas 46 and 47 give $2r+1$ linear conditions to the coefficients $\{a_{i,j}\}$. These are all the conditions imposed on these coefficients, giving $(r+1)^2-(2r+1)=r^2$ freedom to them. This is compatible with the dimension calculated in Lemma 45 (with $s=r+2, n=r+1$).*

REMARK 49. *When there are unbounded edges of Γ_0 which are also unbounded in Γ, then the numbers $a_{i,j}$ associated with these edges should be set to zero. Explicitly, when the edge of direction e_j is also unbounded in Γ, then $a_{i,j}$, $i=1,\dots,r+1$ should be all zero.*

6. Dual obstruction spaces for general vertices

So far, we considered tropical curves with a higher-valent vertex whose image is locally isomorphic to the image of the tropical curve (Γ_0,h_0) introduced at the beginning of Section 5. Such a vertex corresponds to a line in a projective space. In this section we consider more general higher-valent vertices.

Namely, let us take a 3-valent abstract tree Γ_1 with $r+2$ unbounded edges so that it is combinatorially isomorphic to Γ_0. However, now we allow the edges of Γ_1 to have various weights (both on bounded and unbounded edges). Then, consider a proper map

$$h_1\colon\Gamma_1\to\mathbb{R}^n=N_1\otimes\mathbb{R},$$

where N_1 is a free abelian group of rank n, with the following properties:

- The map h_1 contracts all the bounded edges of Γ_1.
- The map h_1 gives a structure of a tropical curve to Γ_1.

As in the previous section, we assume that (Γ_1, h_1) is a restriction to an open subset of a larger tropical curve (Γ, h) and all the unbounded edges of Γ_1 are restrictions of bounded edges of Γ.

REMARK 50. *For applications, it is important to note that the map h_1 may send some of the unbounded edges to the same image. Although the image of these edges is a half line which corresponds to a toric divisor via the construction in Section 3, a pre-log curve of type (Γ_1, h_1) should intersect this toric divisor at several different points of the domain curve (their images on the toric divisor can be the same).*

Let us take a standard basis $\{e_1, \dots, e_{r+1}\}$ of $\mathbb{R}^{r+1}$. Recall that the directions of the edges of the image $h_0(\Gamma_0)$ are given by $e_1, \dots, e_{r+1}$ and $f = -e_1 - \cdots - e_{r+1}$. We write by $E_1, \dots, E_{r+1}$ and F the unbounded edges of Γ_0 which are mapped to the edges of these directions by h_0.

We fix an identification ι of the abstract graphs Γ_0 and Γ_1. Let

$$n_1, \dots, n_{r+1} \in N_1$$

be the primitive integral vectors of the directions of $h_1 \circ \iota(E_1), \dots, h_1 \circ \iota(E_{r+1})$. Let $w_1, \dots, w_{r+1}$ be the weights of these edges.

Then, define a linear map

$$\Xi \colon \mathbb{R}^{r+1} \to \mathbb{R}^n$$

by extending the map

$$e_i \mapsto w_i n_i, \;\; i = 1, \dots, r+1$$

linearly. It is easy to see that $h_0(\Gamma_0)$ is mapped by Ξ onto $h_1(\Gamma_1)$. When the map Ξ is not surjective, we add an $n - \dim \mathrm{Im}\Xi$ dimensional vector space V to $\mathbb{R}^{r+1}$ and define a map $g \colon V \to \mathbb{R}^n$ so that the map

$$\Xi \oplus g \colon \mathbb{R}^{r+1} \oplus V \to \mathbb{R}^n$$

is an surjection. The choice of g does not affect the following argument. Let us write

$$\mathbb{R}^m := \mathbb{R}^{r+1} \oplus V$$

and choose a basis $\{e_1^\vee, \dots, e_m^\vee\}$ of the dual space extending a dual basis $\{e_1^\vee, \dots, e_{r+1}^\vee\}$ of the above basis of $\mathbb{R}^{r+1}$ so that $\{e_{r+2}^\vee, \dots, e_m^\vee\}$ is a basis of $(\mathbb{R}^{r+1})^\perp$. For notational simplicity, we write $\Xi \oplus g$ by $\widetilde{\Xi}$.

The images of the tropical curves (Γ_0, h_0) and (Γ_1, h_1) can be seen as incomplete fans consisting of one dimensional cones in $\mathbb{R}^m$ and $\mathbb{R}^n$, respectively. Let $\mathbb{P}_{(\Gamma_0, h_0)}$ and

$\mathbb{P}_{(\Gamma_1,h_1)}$ be the toric varieties defined by these fans. The map $\widetilde{\Xi}$ induces a map $f_{\widetilde{\Xi}}$ between toric varieties from $\mathbb{P}_{(\Gamma_0,h_0)}$ to $\mathbb{P}_{(\Gamma_1,h_1)}$.

PROPOSITION 51. *The map $f_{\widetilde{\Xi}}$ sends torically transverse curves of type (Γ_0, h_0) to torically transverse curves of type (Γ_1, h_1), and all pre-log curves of type (Γ_1, h_1) can be obtained in this way.*

PROOF. The former claim is obvious. For the latter, let $\varphi_1 \colon \mathbb{P}^1 \to \mathbb{P}_{(\Gamma_1,h_1)}$ be a pre-log curve of type (Γ_1, h_1). Then, the inverses image $\varphi_1^{-1}(\varphi_1(\mathbb{P}^1))$ may have several irreducible components, but it is easy to see that each of them intersects the toric boundary of $\mathbb{P}_{(\Gamma_0,h_0)}$ with multiplicity one. Then, each of these irreducible components is a line. $\square$

Let

$$\psi : \mathbb{P}^1 \to \mathbb{P}_{(\Gamma_1,h_1)}$$

be a torically transverse curve of type (Γ_1, h_1). We think of it as the restriction of a pre-log curve $C_0 \to X_0$ of type (Γ, h) to a suitable irreducible component of C_0. Assume that ψ is obtained from a line $\varphi \colon \mathbb{P}^1 \to \mathbb{P}^{r+1} \times (\mathbb{C}^*)^{\dim V}$ by composing with $f_{\widetilde{\Xi}}$ (note that $\mathbb{P}_{(\Gamma_0,h_0)}$ is naturally an open subvariety of $\mathbb{P}^{r+1} \times (\mathbb{C}^*)^{\dim V}$). Note that a section of the sheaf $\mathcal{N}_\psi^\vee \otimes \omega_{C_0}|_{\mathbb{P}^1}$ is a meromorphic 1-form on $\mathbb{P}^1$ with values in $(N_1 \otimes \mathbb{C})^\vee$. Here, $\omega_{C_0}|_{\mathbb{P}^1}$ is the restriction of ω_{C_0} to the relevant component of C_0.

The map $\widetilde{\Xi}$ induces a map between the dual spaces

$$\widetilde{\Xi}^* \colon (N_1 \otimes \mathbb{C})^\vee \to (\mathbb{C}^m)^\vee.$$

Then, the pull back $\widetilde{\Xi}^* \mathcal{N}_\psi^\vee$ is a sheaf on $\mathbb{P}^1$ with values in $(\mathbb{C}^m)^\vee$. Since the sheaf $\widetilde{\Xi}^* \mathcal{N}_\psi^\vee$ is naturally a subsheaf of $\mathcal{N}_\varphi^\vee$, sections of $\widetilde{\Xi}^*(\mathcal{N}_\psi^\vee \otimes \omega_{C_0}|_{\mathbb{P}^1})$ can also be described by the numbers $\{a_{i,j}\}$ and $\{c_{i,j}\}$ given in (4) in Section 5.

Let y_i be the point of $\mathbb{P}^1$ which is mapped by φ to the toric divisor of $\mathbb{P}^{r+1} \times (\mathbb{C}^*)^{\dim V}$ corresponding to the edge E_i of Γ_0 $(i = 1, \ldots, r+1)$. The fiber of $\mathcal{N}_\varphi^\vee$ at each y_i can be identified with the annihilator subspace $(E_i)^\perp$ in $(\mathbb{C}^m)^\vee$ of the direction of the edge E_i.

DEFINITION 52. Let F_i be the subspace of $(E_i)^\perp$ of codimension $r + 1 - \dim \mathrm{Im}\Xi$ which annihilates the kernel of the map

$$\widetilde{\Xi}_{\mathbb{C}} \colon \mathbb{C}^m \to N_1 \otimes \mathbb{C}.$$

It is clear that the fiber of $\widetilde{\Xi}^* \mathcal{N}_\psi^\vee$ at y_i is canonically isomorphic to F_i.

LEMMA 53. *A section of $\widetilde{\Xi}^*(\mathcal{N}_\psi^\vee \otimes \omega_{C_0}|_{\mathbb{P}^1})$ is described by the set of numbers $\{a_{i,j}\}$ and $\{c_{i,j}\}$ with the following properties.*

- *The vector valued residue*

$$\sum_{i=1}^{r+1} a_{i,j} e_i^\vee + \sum_{i=r+2}^{m} c_{i,j} e_i^\vee$$

at y_j is contained in F_j.

- *The conditions given in Lemmas 46 and 47, namely*

$$a_{1,1} = \cdots = a_{r+1,r+1} = 0$$

and

$$A_k(\{a_{i,j}\}) = 0, \quad k = 0, \dots, r-1$$

hold.

PROOF. It is clear that the conditions for $\{a_{i,j}\}$ and $\{c_{i,j}\}$ in the statement are necessary to define a section of $\widetilde{\Xi}^*(\mathcal{N}_{\psi}^{\vee} \otimes \omega_{C_0}|_{\mathbb{P}^1})$.

For a general point z on $\mathbb{P}^1$, the fiber of $\mathcal{N}_{\varphi}^{\vee}$ is the annihilator of the (log) tangent space of $\varphi(z)$, which can be identified with a subspace of $(\mathbb{C}^m)^{\vee}$, and the fiber of $\widetilde{\Xi}^*\mathcal{N}_{\psi}^{\vee}$ is the subspace $F_z \subset (\mathbb{C}^m)^{\vee}$ consisting of the vectors which also annihilate the kernel of the above map $\widetilde{\Xi}_{\mathbb{C}}$. Therefore, we need to show that if the numbers $\{a_{i,j}\}$ and $\{c_{i,j}\}$ satisfy the conditions in the statement, the value at z of the section of $\mathcal{N}_{\varphi}^{\vee} \otimes \omega_{C_0}|_{\mathbb{P}^1}$ determined by $\{a_{i,j}\}$ and $\{c_{i,j}\}$ is contained in F_z.

Namely, given the vector valued residue $\alpha_j = \sum_{i=1}^{r+1} a_{i,j} e_i^{\vee} + \sum_{i=r+2}^{m} c_{i,j} e_i^{\vee}$ at y_j as in the statement, the corresponding section of $\mathcal{N}_{\varphi}^{\vee} \otimes \omega_{C_0}|_{\mathbb{P}^1}$ is uniquely given by

$$\sum_{j=1}^{r+1} \alpha_j \sigma_j$$

as in (4) in the previous section. Since the vectors α_j all annihilate the kernel of $\widetilde{\Xi}_{\mathbb{C}}$, the value of the section at any point of $\mathbb{P}^1$ also annihilates it. This proves the lemma. □

7. Description of the dual obstruction spaces for general global pre-log curves

The argument so far established the description of the space of sections of the sheaf $\mathcal{N}_{\varphi_0}^{\vee} \otimes \omega_{C_0}$ restricted to a component of a pre-log curve. Now, we consider the global version of the problem.

Let (Γ, h) be a tropical curve satisfying Assumption A. Let $\varphi_0 \colon C_0 \to \mathfrak{X}$ be a pre-log curve of type (Γ, h) in a suitable toric degeneration defined respecting (Γ, h). We would like to give a description of elements of the group

$$H = H^0(C_0, \mathcal{N}_{\varphi_0}^{\vee} \otimes \omega_{C_0}).$$

We will do this by gluing local data. Consider a vertex v of the image $h(\Gamma)$. In general, the inverse image of v has several connected components. Let us take a subgraph $\Gamma_{v(i)}$ of Γ as in Definition 37. Then, the restriction of h to $\Gamma_{v(i)}$ is modeled on a map h_1 in the previous section.

To describe H, it is useful to take the graph Γ_h instead of Γ, see Definition 35. Recall that the map $h\colon \Gamma \to N_{\mathbb{R}}$ factors through Γ_h. We write the corresponding map by $\overline{h}\colon \Gamma_h \to N_{\mathbb{R}}$. Also, let $\pi\colon \Gamma \to \Gamma_h$ be the natural map, as in Definition 35. Then, we can identify the images $\pi(\Gamma_{v(i)})$ and $h(\Gamma_{v(i)})$ as abstract topological spaces. We write by $v(i)$ the unique vertex of Γ_h in $\pi(\Gamma_{v(i)})$.

Using Lemma 53 and Remark 49, we attach data to the subgraph $\pi(\Gamma_{v(i)})$ of Γ_h as follows. Namely, we attach values $a_{j,l}$ and $c_{j,l}$ to each edge E_l of $\pi(\Gamma_{v(i)})$ in the following way.

Construction:

1. If E_l is an unbounded edge of Γ_h, then all $a_{j,l}$ and $c_{j,l}$ with l fixed is zero, by Remark 49.
2. Otherwise, the values $a_{j,l}$ and $c_{j,l}$ can take any value satisfying the conditions in Lemma 53.

This determines a vector

$$\sum_{i=1}^{r+1} a_{i,j} e_i^\vee + \sum_{i=r+2}^{m} c_{i,j} e_i^\vee$$

in $(N_{\mathbb{C}})^\vee \subset (\mathbb{C}^m)^\vee$ at the point on C_0 corresponding to a flag $(v(i), E_l)$ of Γ_h (which is a node or a marked point, depending on whether E_l is bounded or unbounded). Although the integer m and the embedding $(N_{\mathbb{C}})^\vee \subset (\mathbb{C}^m)^\vee$ depend on the choice of vertices of Γ_h, the space $(N_{\mathbb{C}})^\vee$ makes sense globally. Note that at each vertex $v(i)$, the values $a_{j,l}$ and $c_{j,l}$ are assigned to all but one edge emanating from it, corresponding to F in the notation of Section 6. The vector associated with this flag is determined by the residue theorem.

Definition 54. We say that the set of the values attached to the flags of Γ_h by Construction above is a *compatible labeling* when the sum of the values attached to the two flags associated with

1. each bounded edge, and
2. each 2-valent vertex

is zero.

This reflects the residue theorem and the relation of the frames

$$\frac{dz_1}{z_1} + \frac{dz_2}{z_2} = 0$$

of ω_{C_0} at the corresponding node of C_0. Here, z_1, z_2 are coordinates of the two branches at the node.

By the argument so far, we have the following.

PROPOSITION 55. *The set of elements of the group $H^0(C_0, \mathcal{N}_{\varphi_0}{}^\vee \otimes \omega_{C_0})$ is naturally bijective to the set of compatible labelings of the flags of Γ_h.* □

Now, we study the set of compatible labelings. Let $L = \cup_i L_i$ be the loop part of Γ_h (Definition 23), where each L_i is a bouquet. This is a closed subgraph of Γ_h. Let us write $\Gamma_T = \Gamma_h \setminus L$. The closure of a connected component of Γ_T is a tree. There are two types of these trees, namely:

(U) It contains only one flag whose vertex is contained in a loop.

(B) Otherwise.

By inductive argument, it is easy to see that all the flags in a component of type (U) must have the value zero, including the unique flag whose vertex is contained in a loop. For the type (B), we have the following result.

LEMMA 56. *All the flags of a component of type (B), including the flags whose vertices are contained in the loops, must have the value $0 \in N_{\mathbb{C}}^\vee$.*

PROOF. Note that Γ_h can be written in the form illustrated in the following figure (Figure 3).

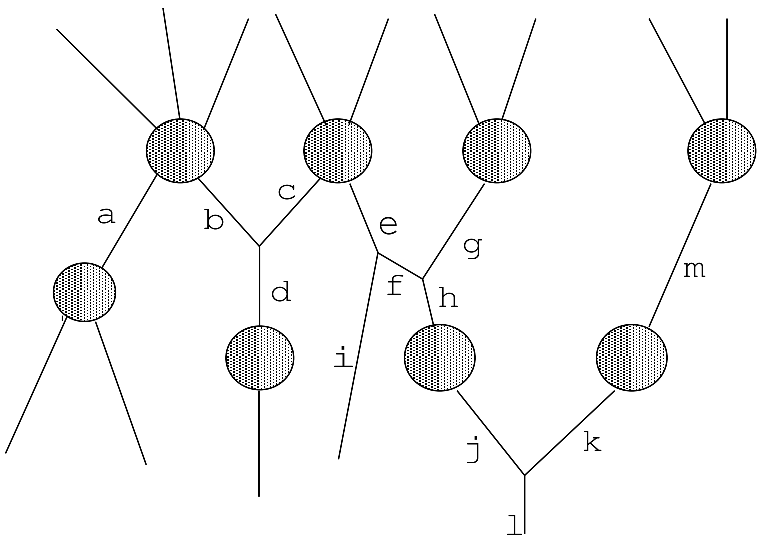

Figure 3.

Here, each shaded disk corresponds to some component L_i of the loop part of Γ_h. By definition of $\{L_i\}$, if we regard these disks as vertices, we obtain another tree Γ'_{φ_0}.

As we noted above, all the edges contained in the components of type (U) have the value zero. In Figure 3, this means that all the edges (outside the shaded disks) except the ones labeled by $a, b, c, \cdots, m$ have the value zero. We call the edges outside the shaded disks the *bridges*.

Now, by the fact that Γ'_{φ_0} is a tree, it is easy to see that there is a shaded disk such that the bridges emanating from it have the value zero except one bridge. Let us call this bridge r and label the remaining bridges emanating from the shaded disc $a_1, \dots, a_s$. By the condition that the sum of the values of the edges emanating from each vertex is zero (the residue theorem), we see that the sum of the values attached to $a_1, \dots, a_s, r$ is zero. Since $a_1, \dots, a_s$ have the value zero, it follows that r has also the value zero. By induction on the number of shaded disks, we see that all the bridges, and so all the edges of the components of type (B) also have the value zero. □

Let us describe the space $H = H^1(C_0, \mathcal{N}_{\varphi_0})^\vee$. According to this lemma, we only need to consider the flags contained in some bouquet in Γ_h. Let L be a bouquet. Let $\{v_i\}$ be the set of vertices of L with the valence at least three. Cutting L at each v_i, we obtain a set of the union of segments $\{l_m\}$ in Γ_h. Let U_m be the linear subspace of $N_\mathbb{C}$ spanned by the direction vectors of the images of the edges contained in the segments of l_m. The following theorem follows from the argument so far.

THEOREM 57. *Elements of the space H are described by the following data.*

(I) Give the value zero to all the flags not contained in any bouquet.

(II) Give a value in $(U_m)^\perp \subset N_\mathbb{C}^\vee$ to each of the flags associated with the edges of l_m.

(III) The data in (I) and (II) give an element of H if and only if the following conditions are satisfied.

  (a) *At each $r+2$-valent vertex q of Γ_h $(r \geq 1)$, the data in (II) attached to the flags whose vertex is q satisfy the condition in Lemma 53.*

  (b) *The data in (II) is compatible in the sense of Definition 54.* □

There are a few remarks for the notation in this theorem.

REMARK 58. 1. *Note that the numbers $\{a_{i,j}\}$ and $\{c_{i,j}\}$ in Lemma 53 are associated with edges of the tropical curve (Γ_0, h_0) in $\mathbb{R}^{r+1}$, and the data in (II) determine these numbers through the projection argument in Section 6.*

2. *As we noted before, the numbers $\{a_{i,j}\}$ and $\{c_{i,j}\}$ are associated with $r+1$ edges among the $r+2$ edges emanating from q. The residue at the point corresponding to the remaining edge is determined from them by the residue theorem. For some choice of $r+1$ edges out of the total $r+2$ edges, the numbers $\{a_{i,j}\}$ and $\{c_{i,j}\}$ constructed from the data in (I), (II) satisfy the condition $(III)(a)$ above if and only if the numbers $\{a_{i,j}\}$ and $\{c_{i,j}\}$ satisfy it for any choice of $r+1$ edges.*

This is the most general form of the description of the space H for pre-log curves corresponding to tropical curves satisfying Assumption A. When the tropical curve (Γ, h) is immersive, so that only the case $r = 1$ appears, it specializes to more explicit description. Namely, the condition $(III)(a)$ becomes the following form.

Corollary 59. *In Theorem 57, assume* (Γ, h) *is immersive, so that* $\Gamma_h = \Gamma$. *Then, the data* $(I), (II)$ *give an element of* H *if and only if it is compatible in the sense of Definition 54 and at each 3-valent vertex* v *of a bouquet* L*, the identity*

$$u_1 + u_2 + u_3 = 0$$

holds. Here, u_i *are elements of* $(U_m)^{\perp}$ *attached to the flags whose vertex is* v. □

This assertion, together with the following claim, provides a description of superabundancy, which answers Questions 1 and 2 in the introduction, for tropical curves satisfying Assumption A.

Corollary 60. *Let* (Γ, h) *be a 3-valent immersive tropical curve. Then, the number of parameters to deform it is given by*

$$(n-3)(1-g) + e + \dim H.$$

Proof. Let us write the dimension of $H^0(C_0, \mathcal{N}_{\varphi_0})$ by d_1 and the dimension of $H^1(C_0, \mathcal{N}_{\varphi_0})$ by d_2. In the proof of [**22**, Proposition 5.7], it is shown that when the tropical curve (Γ, h) is immersive, d_1 is the same as the dimension of the parameter space of the corresponding tropical curve. In [**22**, Proposition 5.7], this claim was proved for rational tropical curves, but the same proof applies to curves with higher genus, including superabundant ones. Also, it is known that the dimension of the cohomology group $H^0(C_0, \omega_0)$ equals g (see [**9**, Exercise 3-4]). Therefore, the dimension of the cohomology group $H^1(C_0, \varphi_0^*\Theta_{\mathfrak{X}/\mathbb{A}^1_{\mathbb{C}}}) \cong H^0(C_0, \omega_{C_0}^{\oplus n})^{\vee}$ equals ng.

Let v be an s-valent vertex of Γ_h. Then, the restriction $\Theta_{C_0/O_0}|_{C_{0,v}}$ is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(2-s)$. From this, it is easy to see $H^0(C_0, \Theta_{C_0/O_0}) = 0$. On the other hand, by a straightforward calculation, we obtain $\dim H^1(C_0, \Theta_{C_0/O_0}) = e + 3g - 3$.

Then, by the long exact sequence associated with the sequence (3) after Proposition 41, we have the equality

$$d_1 - d_2 = (n-3)(1-g) + e.$$

Note that $\dim H = d_2$. These observations prove the theorem. □

Remark 61. *An important remark concerning Theorem 57 is that the sheaves* $\varphi_0^*\Theta_{\mathfrak{X}/\mathbb{A}^1_{\mathbb{C}}}$ *and* $\varphi_0^*\Theta_{\mathfrak{X}/\mathbb{A}^1_{\mathbb{C}}}/\Theta_{C_0/O_0}$ *exist even when the map* φ_0 *does not exist globally. They are constructed by gluing locally free sheaves on each irreducible component at the nodes. At each node* $p \in C_0$*, the pullback of* $\Theta_{\mathfrak{X}/\mathbb{A}^1_{\mathbb{C}}}$ *and* $\Theta_{\mathfrak{X}/\mathbb{A}^1_{\mathbb{C}}}/\Theta_{C_0/O_0}$ *are canonically isomorphic to* $N_{\mathbb{C}}$ *and* $N_{\mathbb{C}}/\mathbb{C}u_E$*, respectively. Here,* u_E *is the direction vector of the image of the edge* E *of* Γ *corresponding to* p *by the map* h*. In particular, these isomorphisms do not depend on the choice of the map* $\varphi_0|_{C_{0,v}}$ *on the irreducible components of* C_0*. More specifically, let* $\{v_1, v_2\}$ *be the boundary of the edge* E *and* p_i*, i=1, 2, be the points of* C_{0,v_i} *corresponding to the node* p*. Then, by the above observation, we can identify the fibers of the pullback of the sheaves at* p_1 *and* p_2 *even if the images* $\varphi_0|_{C_{0,v_1}}(p_1)$ *and* $\varphi_0|_{C_{0,v_2}}(p_2)$ *do not coincide. In particular, the space* H *also makes sense and Corollary 60 gives us the degree of freedom to deform the given immersive superabundant tropical* *curve even if a pre-log curve of type* (Γ, h) *does not exist.*

The following is immediate, because when the genus of Γ is one, there is no r-valent vertex in L with $r \geq 3$. See also [**23**].

COROLLARY 62. *When* (Γ, h) *is a tropical curve of genus one satisfying Assumption A, then* $H \cong U^{\perp}$*, here* U *is the linear subspace of* $N_{\mathbb{C}}$ *spanned by the direction vectors of the segments of the loop of* Γ*.* □

In view of Theorem 57, we give the following definition.

DEFINITION 63. Let (Γ, h) be a tropical curve satisfying Assumption A. Then, the *support of superabundancy* of (Γ, h) is the closed subgraph Γ_{ss} of Γ_h with the following property: For any edge $\mathfrak{E}$ of Γ_{ss}, there is an element of H such that the value of it on the flags associated with $\mathfrak{E}$ is not zero.

Large part of results which are valid for superabundant curves of genus one can be extended to those tropical curves whose support of superabundancy is the disjoint union of loops.

8. Examples

EXAMPLE 64. Let us consider immersive tropical curves Γ_1 and Γ_2 of genus two in $\mathbb{R}^3$ given in Figure 4.

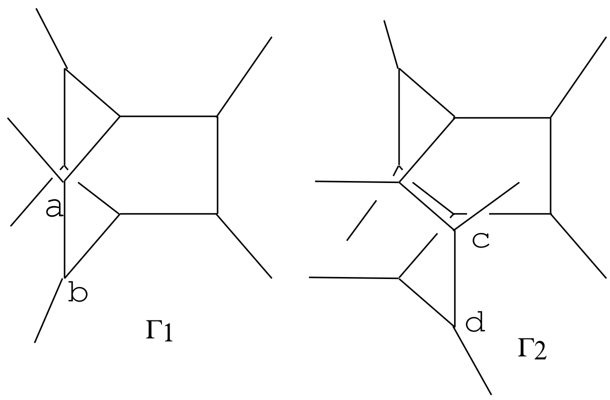

Figure 4.

The curve Γ_1 has 6 unbounded edges of the directions

$$(1, 0, 1), \ (1, 0, -1), \ (-1, -1, 1), \ (-1, -1, -1), \ (0, 1, 1), \ (0, 1, -1).$$

The bounded edges are:

- Three parallel vertical edges of the direction $(0, 0, 1)$.
- Three pairs of parallel edges of the directions

$$(1,0,0),\ \ (-1,-1,0),\ \ (0,1,0),$$

respectively.

The curve Γ_2 is a modification of Γ_1 at the vertices a and b. Namely:

1. Delete the edge $\overline{ab}$ as well as the neighboring unbounded edges.
2. Add a pair of parallel unbounded edges of the direction $(-1,0,0)$, and a pair of parallel bounded edges of the direction $(1,1,0)$ of the same length.
3. Connect the end points c, d of the bounded edges added in (2) by a segment of the direction $(0,0,1)$.
4. Add unbounded edges of the directions $(1,1,1),(1,1,-1)$ at the vertices c, d, respectively.

Using Corollary 59, it is easy to see that Γ_1 is superabundant, while Γ_2 is non-superabundant. Namely, the sets of piecewise linear segments $\{l_m\}$ of these tropical curves are given by the following three components, respectively (Figure 5).

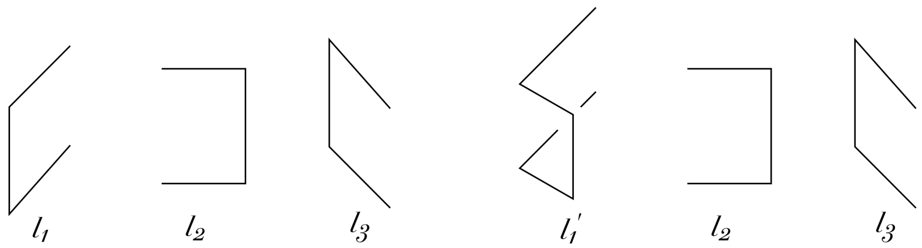

Figure 5.

We write the corresponding linear subspaces of $N_{\mathbb{C}} \cong \mathbb{C}^3$ by U_{l_1}, U_{l_2}, etc.. Then, using the standard nondegenerate quadratic form on $\mathbb{C}^3$ to identify it with its dual,

$$(U_{l_1})^{\perp} \cong \mathbb{C}\cdot(1,0,0),\ \ (U_{l_2})^{\perp} \cong \mathbb{C}\cdot(0,1,0),\ \ (U_{l_3})^{\perp} \cong \mathbb{C}\cdot(1,-1,0).$$

It is easy to see that the space H for Γ_1 is a one dimensional vector space. Thus, Γ_1 is superabundant.

On the other hand, since $U_{l_1'} \cong \mathbb{C}^3$, $(U_{l_1'})^{\perp} = \{0\}$. From this, it is easy to see that the space H for Γ_2 is $\{0\}$. Therefore, Γ_2 is non-superabundant.

Example 65. Plane tropical curves were studied by Mikhalkin [**13**] in great detail. There it was shown that any immersive plane tropical curve without multiple edges is non-superabundant and smoothable. On the other hand, if we only assume Assumption A, then even 3-valent immersive (in the sense of Definition 16) plane tropical curves can be superabundant and non-smoothable. The simplest example is given by the one in Figure 6.

In Figure 6, all the edges except E and F have weight two, while the edges E and F have weight one. The loop part is the union of E and F, and the space H is one

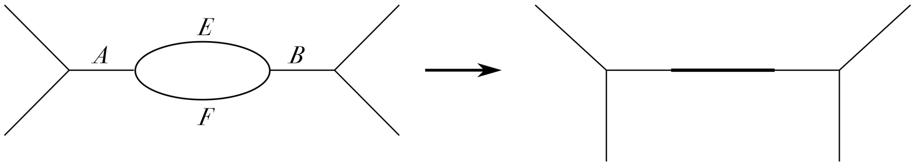

Figure 6. The abstract graph (the picture on the left) is mapped into $\mathbb{R}^2$. The edges E and F have the same image (the bold line in the picture on the right).

dimensional. According to the study in [**17**], this tropical curve is smoothable if and only if the lengths of the images of the edges A and B are the same.

9. Toric degenerations of non-toric varieties

So far, our ambient space $\mathfrak{X}$ is a degeneration of a toric variety X. However, our argument is valid when a not necessarily toric variety degenerates in a special way. Namely, suppose a complex variety X of dimension n (we assume it to be reduced and irreducible, but not necessarily regular or complete) has a degeneration

$$\pi\colon \mathfrak{X} \to \mathbb{A}^1_{\mathbb{C}}.$$

Here, $\pi^{-1}(1)$ is isomorphic to X and the central fiber $X_0 := \pi^{-1}(0)$ is a union of reduced toric varieties glued along toric divisors. A typical example is the degeneration given by the equation

$$f(x_0, x_1, \cdots, x_n) + t x_0 x_1 \cdots x_k = 0,$$

in $\mathbb{P}^n \times \mathbb{C}$, where $t \in \mathbb{C}$ is the parameter of degeneration, and f is a homogeneous polynomial of degree $k+1$, $k \le n$.

Let $X_0 = \cup_i X_{0,i}$ be the irreducible components of X_0. Each $X_{0,i}$ is a toric variety and we assume the intersection $X_{0,ij} = X_{0,i} \cap X_{0,j}$ is a toric stratum in both $X_{0,i}$ and $X_{0,j}$. Moreover, we assume the following. Namely, suppose the intersection $X_{0,ij}$ is a toric divisor. Let D be the union of toric divisors of $X_{0,ij}$. Then, there is a hypersurface $\mathcal{H}$ of $X_{0,ij}$ with the property that at any point p in

$$X_{0,ij} \setminus (\mathcal{H} \cup D),$$

there is a neighborhood U in the total space $\mathfrak{X}$ such that the restriction of π to U is analytically isomorphic to an open neighborhood of the origin of the variety defined by

$$xy = t^m \subset \mathbb{C}^n \times \mathbb{C}.$$

Here, m is a positive integer independent of p (but it can depend on the pair i, j), and x, y are parts of standard linear coordinates of $\mathbb{C}^n$.

We consider a curve $\varphi_0\colon C_0 \to X_0$ such that the restriction of φ_0 to each component of C_0 is a torically transverse map of some type v_i in the sense of Definition 34. Let Γ_h be the dual intersection graph of C_0. The vertex v_i is naturally considered as a vertex of Γ_h. Assume that $\varphi_0(C_0)$ does not intersect $\mathcal{H}$ for any $X_{0,ij}$. Then, the local calculation in Section 6 and the argument in Proposition 55 are valid. Therefore, a result similar to Theorem 57 holds in this case, too. To provide details, although there is no torus $N\otimes_{\mathbb{Z}}\mathbb{C}^*$ acting on the whole X_0, there is such N_i (and a torus $N_i\otimes_{\mathbb{Z}}\mathbb{C}^*$) associated with each component $X_{0,i}$. The type v_i specifies a tropical curve with one vertex in $(N_i)_{\mathbb{R}}$, and the annihilating subspaces of the directions of the edges make sense. If $X_{0,ij}$ is a toric divisor of $X_{0,i}$ and $X_{0,j}$, then the vertices v_i and v_j are connected by an edge E_{ij} of Γ_h. The edge E_{ij} determines rays in $(N_i)_{\mathbb{R}}$ and $(N_j)_{\mathbb{R}}$. Although we do not have a natural isomorphism between $(N_i)_{\mathbb{R}}$ and $(N_j)_{\mathbb{R}}$, we do have one between the annihilating subspaces

$$E_{ij}^{\perp}\subset (N_i)_{\mathbb{R}}^{\perp} \quad \text{and} \quad E_{ij}^{\perp}\subset (N_j)_{\mathbb{R}}^{\perp}.$$

These data are enough for the construction of the space $(U_m)^{\perp}$ in the statement of Theorem 57. Then, elements of the space H are given by attaching vectors of $E_{ij}^{\perp}$ to the flags of the graph Γ_h precisely in the way prescribed in Theorem 57.

Remark 66. *Although we do not need it in this paper, one often attaches an integral affine manifold with singularity to degenerations of a variety like the one considered above, see for example* [**7**]. *A loop around the singular locus in general gives a non-trivial monodromy of the affine structure, which does not exist in the case of toric degenerations of toric varieties. The monodromy fixes the directions corresponding to the conormal vectors of the curve C_0, so this picture is compatible with the argument above, where there was no appearance of a monodromy.*

Remark 67. *One can also study curves which intersect the singular locus $\mathcal{H}$. These curves are important in relation to mirror symmetry. Such an intersection contributes to the space H (so also to the obstruction), but its effect can be calculated in a simple way at least in a generic situation. See* [**18**].

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Takeo Nishinou

Department of Mathematics, Rikkyo University, 3-34-1, Nishi-Ikebukuro, Toshima, Tokyo, Japan

E-mail: nishinou@rikkyo.ac.jp