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Embedding of infinitely differentiable functions into the space of hyperfunctions via Čech-Dolbeault cohomology and its inverse map

By Tatsuki Nishida

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Abstract. Honda, Izawa and Suwa define Čech-Dolbeault representation of hyperfunctions and an embedding of distributions to the space of hyperfunctions. With this embedding, we can regard C^∞ functions as hyperfunctions in the framework of Čech-Dolbeault cohomology.

This article aims to characterize a Čech-Dolbeault representative which corresponds to the image of the embedding of a C^∞ function, and also to construct the inverse map of the embedding of C^∞ functions.

1. Introduction

Hyperfunctions play an essential role in the theory of linear partial differential equations. Mikio Sato invented the hyperfunction theory in 1958 (Sato[14],[15]), and this theory founded the field of mathematics known as algebraic analysis (see Sato-Kawai-Kashiwara[16], Kashiwara-Kawai-Kimura[6], Kashiwara-Schapira[7],[8]). The sheaf of hyperfunctions is defined by local cohomology with coefficients in the holomorphic functions. A section of the sheaf cohomology is usually calculated by employing the theory of Čech cohomology, and Čech representation of hyperfunctions is well-researched subject (see Kashiwara-Kawai-Kimura[6], Kaneko[5], Aoki-Kataoka-Yamazaki[1], Komori-Umeta[11]). However, to manipulate the hyperfunction theory, we often have to use sophisticated techniques of complex analysis in several variables. Recently, N. Honda, T. Izawa and T. Suwa introduce a new representation of hyperfunctions in Honda-Izawa-Suwa[2] based on the theory of Čech-Dolbeault cohomology.

Let M be a real analytic, n-dimensional, oriented manifold and X its complexification. In Čech representation, a hyperfunction is represented by a formal sum of holomorphic functions such as $\bigoplus_i (-1)^i F_{\hat{i}}$. On the other hand, the Čech-Dolbeault representative of a hyperfunction is a pair of C^∞ differential forms (μ_1, μ_{01}), where μ_1 is a $(0, n)$-form on X, and μ_{01} is a $(0, n-1)$-form on $X \setminus M$. Remarkably, only two differential forms suffice to represent a hyperfunction. Additionally, by the softness of the sheaf of C^∞ forms, we can employ helpful tools such as cutoff functions or partitions of unity.

Since Honda-Izawa-Suwa[2] defines an embedding of distributions, we can regard C^∞ functions as hyperfunctions in Čech-Dolbeault representation. This article aims to characterize a Čech-Dolbeault representative which corresponds to the image of a C^∞ function

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by the embedding map and also to construct the inverse map of the embedding of C^∞ functions.

In Čech representation, an inverse map of the embedding of C^∞ functions is defined by a sum of limits $\bigoplus_i (-1)^i F_{\widehat{i}}(x+\sqrt{-1}\,y) \to \sum_i (-1)^i F_{\widehat{i}}(x)$ $(y \to 0)$. Morimoto and Kaneko established this inverse map in Čech representation (Morimoto[13], Kaneko[5]). However, since non-trivial Čech-Dolbeault representatives always diverge on M, we cannot consider the limit $\lim\limits_{r\to 0} \mu_{01}(x+\sqrt{-1}\,r\omega)$ as the one for a Čech representative. To overcome this difficulty, we introduce a new class of forms called "quasi-Whitney", for which we can consider the limit of the form in an appropriate sense. In Lemma 4.14, we show that there always exists a representative belonging to the quasi-Whitney class for the hyperfunction image of a C^∞ function. Following this, for such a representative, we provide a simple expression of the inverse map in Theorems 4.10 and 4.11, which is our main result.

The article is organized as follows: In Section 2, we recall notations and definitions of hyperfunctions and hyperforms.

In Section 3, we recall hyperfunctions and their representations. Firstly, in Subsection 3.1, we prepare Čech-Dolbeault representation of hyperfunctions. Next, Subsection 3.2 explains the relationship between Čech representation and Čech-Dolbeault representation. Then, in Subsection 3.3, we introduce infinitesimal wedges and a twisted Radon kernel, which plays an important role in the definition of an embedding of distributions in Čech representation. Lastly, in Subsection 3.4, we define an embedding of distributions in Čech-Dolbeault representation.

Section 4 treats the main subject of our article. Subsections 4.1 and 4.2 are preparations for Subsection 4.3. Subsection 4.1 defines limits of $(n-1)$-forms. Subsection 4.2 gives an inverse map of C^∞ functions with compact support. Subsection 4.3 shows our main results, which are Theorems 4.10 and 4.11. These theorems give a simple expression of the inverse of the embedding map, where the limits of forms are essential for this expression.

2. Preliminary

2.1. Notations

Notation 2.1 Let X be a topological space. $\mathrm{Op}(X)$ denotes the set of open subsets in X. For any subset $K \subset X$, we write an interior of K as $\mathrm{Int}(K)$. For any $V, V' \in \mathrm{Op}(X)$, $V' \subset\subset V$ means that V' is a relatively compact subset of V.

Notation 2.2 For any subset K in a topological space X and for any sheaf $\mathscr{F}$ on X, we define $\mathscr{F}[K]$ by

$$\mathscr{F}[K] = \varinjlim_{K \subset U} \mathscr{F}(U),$$

where $U \subset X$ runs through open neighborhoods of K. This means that $\mathscr{F}[K]$ is a set of sections defined in open neighborhoods of K.

Let M be a real analytic manifold of dimension n and X its complexification.

Remark 2.3 In this article, we assume that manifolds are always countable at infinity, and that sheaves are always those of abelian groups.

Notation 2.4 For any $U \in \mathrm{Op}(M)$ and for any $V \in \mathrm{Op}(X)$, we say that V is a complex neighborhood of U if $U \subset V$ and U is a closed set in V. Moreover, if a complex neighborhood V of U is a Stein open set, we call V a Stein neighborhood of U.

Notation 2.5 $\mathbb{Z}_M$ (resp. $\mathbb{Z}_X$) denotes the sheaf of $\mathbb{Z}$ valued locally constant functions on M (resp. X).

Notation 2.6 We define $\mathscr{A}_M$ to be the sheaf of real analytic functions on M and $\mathscr{O}_X$ to be the sheaf of holomorphic functions on X.

Remark 2.7 For any $U \in \mathrm{Op}(M)$, we define the sections of real analytic functions on M by

$$\mathscr{A}_M(U) = \varinjlim_{V} \mathscr{O}_X(V),$$

where V runs through complex neighborhoods of U.

Definition 2.8 Let K be a closed set in M. We define $\mathscr{D}b_M$ as the sheaf of distributions on M and also define $\mathscr{D}b_K$ to be the sheaf of distributions supported by K.

Notation 2.9 For any $p, q \in \{0, 1, \cdots, n\}$, we define that $\mathscr{A}_M^{(p)}$ is the sheaf of real analytic p-forms on M, $\mathscr{O}_X^{(p)}$ is the sheaf of holomorphic p-forms on X, and $\mathscr{E}_X^{(p,q)}$ is the sheaf of C^∞ (p,q)-forms on X. For other $p, q \in \mathbb{Z}$, we set $\mathscr{A}_M^{(p)} = 0$, $\mathscr{O}_X^{(p)} = 0$ and $\mathscr{E}_X^{(p,q)} = 0$.

Remark 2.10 $\mathscr{A}_M^{(0)}$ (resp. $\mathscr{O}_X^{(0)}$) is nothing but $\mathscr{A}_M$ (resp. $\mathscr{O}_X$).

Notation 2.11 The symbol $\widehat{\bullet}$ means to omit the corresponding letter in a sequence or a family of sets, etc. For example, we employ the following notations:

- $(i_0, i_1, \cdots, \widehat{i_\ell}, \cdots, i_k) = (i_0, i_1, \cdots, i_{\ell-1}, i_{\ell+1}, \cdots, i_k)$,
- $V_0 \cap V_1 \cap \cdots \cap \widehat{V_\ell} \cap \cdots \cap V_k = V_0 \cap V_1 \cap \cdots \cap V_{\ell-1} \cap V_{\ell+1} \cap \cdots \cap V_k$,

where $k, \ell \in \mathbb{Z}_{\geq 0} = \{i \in \mathbb{Z} \mid i \geq 0\}$, $\ell \leq k$, $(i_0, i_1, \cdots, i_k) \in \{0, 1, \cdots, n\}^{k+1}$ and $V_0, V_1, \cdots, V_k \in \mathrm{Op}(X)$.

2.2. Hyperfunctions and hyperforms

Let M be a real analytic, n-dimensional, oriented manifold and X its complexification, and let $U \in \mathrm{Op}(M)$ and $V \in \mathrm{Op}(X)$ such that V is a complex neighborhood of U.

Notation 2.12 (Relative cohomology) For any sheaf $\mathscr{F}$ on X and for any $k \in \mathbb{Z}$, $H_U^k(V; \mathscr{F})$ denotes the k-th relative cohomology group of $\mathscr{F}$ with supports in U.

See Kashiwara-Kawai-Kimura[6] for the relative cohomology groups of a sheaf.

Definition 2.13 (Hyperfunctions) We define the space of hyperfunctions on U by

$$\mathscr{B}_M(U) = H_U^n(V; \mathscr{O}_X) \underset{\mathbb{Z}_M(U)}{\otimes} or_{M/X}(U),$$

where $or_{M/X}(U) = H^n_U(V; \mathbb{Z}_X)$ is the sections of the relative orientation sheaf on U.

Definition 2.14 (Hyperforms) The space of p-hyperforms on U is given by

$$\mathscr{B}^{(p)}_M(U) = H^n_U(V; \mathscr{O}^{(p)}_X) \underset{\mathbb{Z}_M(U)}{\otimes} or_{M/X}(U).$$

Note that $\mathscr{B}_M(U) = \mathscr{B}^{(0)}_M(U)$. Similarly, we define hyperforms and hyperfunctions with closed support as follows.

Definition 2.15 For any closed set K in U, the space of p-hyperforms supported by K is defined by

$$\mathscr{B}^{(p)}_K(U) = H^n_K(V; \mathscr{O}^{(p)}_X) \underset{\mathbb{Z}_M(U)}{\otimes} or_{M/X}(U).$$

The space $\mathscr{B}_K(U)$ of hyperfunctions supported by K is defined in the same way.

Remark 2.16 $\mathscr{B}^{(p)}_M = \{\mathscr{B}^{(p)}_M(U)\}_{U\in\mathrm{Op}(M)}$ and $\mathscr{B}^{(p)}_K = \{\mathscr{B}^{(p)}_{K\cap U}(U)\}_{U\in\mathrm{Op}(M)}$ form sheaves. For more details, refer to Kashiwara-Kawai-Kimura[6].

3. Čech cohomology and Čech-Dolbeault cohomology

A concrete expression of a hyperfunction is usually realized by using either the theory of Čech cohomology or that of Čech-Dolbeault cohomology. Both of these cohomology theories have their own advantages. In this section, we briefly review their definitions and establish the canonical isomorphism between them. Additionally, we consider the embedding of distributions into hyperfunctions from the viewpoint of both cohomology theories.

3.1. Čech-Dolbeault cohomology and hyperfunctions

Let M be a real analytic, n-dimensional, oriented manifold and X its complexification, and let $U \in \mathrm{Op}(M)$ and $V \in \mathrm{Op}(X)$ such that V is a complex neighborhood of U.

Definition 3.1 For any $p, q \in \mathbb{Z}$, we define

$$\mathscr{E}^{(p,q)}_X(V, V \setminus U) = \mathscr{E}^{(p,q)}_X(V) \oplus \mathscr{E}^{(p,q-1)}_X(V \setminus U),$$

and

$$\begin{array}{rccc} \bar{\vartheta} \ : & \mathscr{E}^{(p,q)}_X(V, V \setminus U) & \longrightarrow & \mathscr{E}^{(p,q+1)}_X(V, V \setminus U) \\ & \uplus & & \uplus \\ & (\mu_1, \mu_{01}) & \longmapsto & (\bar{\partial}\mu_1,\, \mu_1|_{V\setminus U} - \bar{\partial}\mu_{01}). \end{array}$$

Then, $(\mathscr{E}^{(p,\bullet)}_X(V, V \setminus U), \bar{\vartheta})$ is a complex of vector spaces. $H^{p,q}_{\bar{\vartheta}}(V, V \setminus U)$ denotes the q-th cohomology group of this complex.

If S is a closed set in V, we can define a complex $(\mathscr{E}^{(p,\bullet)}_X(V, V \setminus S), \bar{\vartheta})$ and a cohomology $H^{p,q}_{\bar{\vartheta}}(V, V \setminus S)$ in the same way.

Theorem 3.2 (Relative Dolbeault theorem) For any closed set S in V, we have the canonical isomorphism

$$H^{p,q}_{\bar{\vartheta}}(V, V \setminus S) \simeq H^q_S(V; \mathscr{O}^{(p)}_X).$$

In Theorem A.3, we provide a proof of this theorem using the framework of the derived category. Additionally, Theorem 3.23 establishes an isomorphism between Čech cohomology and Čech-Dolbeault cohomology, and its proof may offer further insights. For more details, refer to Honda-Izawa-Suwa[2] and Suwa[18].

Corollary 3.3 Let K be a closed set in U. There are isomorphisms

$$\mathscr{B}^{(p)}_M(U) \simeq H^{p,n}_{\bar{\vartheta}}(V, V \setminus U) \underset{\mathbb{Z}_M(U)}{\otimes} or_{M/X}(U)$$

and

$$\mathscr{B}^{(p)}_K(U) \simeq H^{p,n}_{\bar{\vartheta}}(V, V \setminus K) \underset{\mathbb{Z}_M(U)}{\otimes} or_{M/X}(U).$$

Definition 3.4 (Integration of n-forms) Let K be a compact set in U, and let or_M denote the orientation sheaf of M. We define the integration

$$\int_U : \mathscr{B}^{(n)}_K(U) \underset{\mathbb{Z}_M(U)}{\otimes} or_M(U) \to \mathbb{C}$$

as follows: For any $u \in \mathscr{B}^{(n)}_K(U)$ represented by $(\mu_1, \mu_{01}) \in \mathscr{E}^{(n,n)}_X(V) \oplus \mathscr{E}^{(n,n-1)}_X(V \setminus K)$, we define

$$\int_U u = \int_D \mu_1 - \int_{\partial D} \mu_{01},$$

where D is an open set with C^∞ boundary satisfying $K \subset D \subset \overline{D} \subset V$, and ∂D is the boundary of D.

Remark 3.5 We choose $(y_1, \cdots, y_n, x_1, \cdots, x_n)$ as a positive coordinate system of $\mathbb{C}^n$ with $z_i = x_i + \sqrt{-1}y_i$ $(i = 1, \cdots, n)$ through this article. Additionally, ∂D is oriented as follows: $\overline{D}$ is a manifold with C^∞ boundary ∂D. Let $p \in \partial D$. There exist a neighborhood U of p and a coordinate system $(p_1, \cdots, p_{2n})$ on U such that $\overline{D} \cap U = \{p \in U \mid p_1 \leq 0\}$. If $(p_1, p_2, p_3, \cdots, p_{2n})$ is a positive coordinate system on $\overline{D} \cap U$, we define that $(p_2, p_3, \cdots, p_{2n})$ is a positive coordinate system on $\partial D \cap U$.

Definition 3.6 (Integration of hyperfunction) Assume $M = \mathbb{R}^n$ and $X = \mathbb{C}^n$ with the coordinates $(z_1, \cdots, z_n)$, and let K be a compact set in U. Then, for any $u \in \mathscr{B}_K(U)$ represented by $(\mu_1, \mu_{01}) \in \mathscr{E}^{(0,n)}_X(V) \oplus \mathscr{E}^{(0,n-1)}_X(V \setminus K)$, we define

$$\int_U u\, dx = \int_D \mu_1 \wedge dz - \int_{\partial D} \mu_{01} \wedge dz,$$

where $dz = dz_1 \wedge \cdots \wedge dz_n$, D is an open set with C^∞ boundary satisfying $K \subset D \subset \overline{D} \subset V$, and ∂D is the boundary of D.

Theorem 3.7 ([2, Corollary 6.12]) Let $i : M \hookrightarrow X$ be the canonical embedding, and let K be a compact set in U. Then, $\mathscr{B}_K^{(p)}(U)$ and $\mathscr{A}_M^{(n-p)}[K]$ become FS and DFS spaces, respectively, and

$$\mathscr{B}_K^{(p)}(U) \times (\mathscr{A}_M^{(n-p)}[K] \otimes or_M(U)) \longrightarrow H_{\bar\vartheta}^{n,n}(V, V \setminus K) \otimes i^{-1} or_X(U) \xrightarrow{\int} \mathbb{C}$$

is topologically non-degenerate. Here, an FS space means a Fréchet-Schwartz topological vector space, and a DFS space is a dual Fréchet-Schwartz topological vector space (see Komatsu[10]). Hence, we have

$$\mathscr{B}_K^{(p)}(U) \simeq (\mathscr{A}_M^{(n-p)}[K] \otimes or_M(U))'.$$

Note that $(\bullet)'$ denotes the strong dual of a topological vector space.

Proposition 3.8 (Excision, [2, Proposition 4.8]**)** For any complex neighborhood $\widetilde{V}$ of U, there is an isomorphism

$$H_{\bar\vartheta}^{p,q}(V, V \setminus U) \simeq H_{\bar\vartheta}^{p,q}(\widetilde{V}, \widetilde{V} \setminus U).$$

Let us define an embedding of real analytic functions in Čech-Dolbeault representation. Since M is orientable, we can choose a generator $e_{M/X} \in or_{M/X}(U)$ such that $e_{M/X,x}$ generates a stalk $or_{M/X,x}$ over $\mathbb{Z}$ for any $x \in U$. By considering the canonical morphism

$$or_{M/X}(U) \;=\; H_U^n(V; \mathbb{Z}_X) \longrightarrow H_U^n(V; \mathscr{O}_X),$$

we get a representative $\nu \in \mathscr{E}_X^{(0,n)}(V, V \setminus U)$ of the image of $e_{M/X} \in or_{M/X}(U)$ through the above morphism. Then, we set the constant function $1 \in \mathscr{B}_M(U)$ as $[\nu] \otimes e_{M/X}$. Let ν be such a representative.

Definition 3.9 (Embedding of $\mathscr{A}_M$ in Čech-Dolbeault representation) We define an embedding $\iota^{CD}_{\mathscr{A}_M(U)} : \mathscr{A}_M(U) \to H_{\bar\vartheta}^{0,n}(V, V \setminus U) \otimes or_{M/X}(U) \simeq \mathscr{B}_M(U)$ by

$$\begin{array}{cccccc} \iota^{CD}_{\mathscr{A}_M(U)} : \mathscr{A}_M(U) & \longrightarrow & H_{\bar\vartheta}^{0,n}(\widetilde{V}, \widetilde{V} \setminus U) \otimes or_{M/X}(U) & \simeq & H_{\bar\vartheta}^{0,n}(V, V \setminus U) \otimes or_{M/X}(U). \\ \Psi & & \Psi & & \\ f & \longmapsto & \left[\widetilde{F}\, \nu|_{\widetilde{V}}\right] \otimes e_{M/X} & & \end{array}$$

Here $\widetilde{V} \subset V$ is a complex neighborhood of U on which f can be extended to a holomorphic function $\widetilde{F} \in \mathscr{O}_X(\widetilde{V})$.

Proposition 3.10 $\iota^{CD}_{\mathscr{A}_M} = \{\iota^{CD}_{\mathscr{A}_M(U)}\}_{U \in \mathrm{Op}(M)}$ is a sheaf morphism from $\mathscr{A}_M$ to $\mathscr{B}_M$.

Let us define boundary value morphisms. If $\Omega \in \mathrm{Op}(X)$ satisfies good geometrical conditions, we can regard holomorphic functions on Ω as hyperfunctions.

Recall that V is a complex neighborhood of $U \in \mathrm{Op}(M)$. We take $\Omega \in \mathrm{Op}(X)$ satisfying the following two conditions:

(B$_1$) $\overline{\Omega} \supset U$.

(B$_2$) The inclusion $(V \setminus \Omega) \setminus M \hookrightarrow V \setminus \Omega$ gives a homotopy equivalence.

Proposition 3.11 ([2, Lemma 7.10]) We can take a representative $\nu = (\nu_1, \nu_{01}) \in \mathscr{E}_X^{(0,n)}(V, V \setminus U)$ introduced before Definition 3.9 such that $\mathrm{Supp}_V(\nu_1) \subset \Omega$ and $\mathrm{Supp}_{V\setminus U}(\nu_{01}) \subset \Omega$.

Definition 3.12 (Boundary value morphism, [2, Subsection 7.2]**)** Let $\nu \in \mathscr{E}_X^{(0,n)}(V, V \setminus U)$ be a representative given in Proposition 3.11. We define

$$b_\Omega : \mathscr{O}_X(\Omega) \to \mathscr{B}_M(U) = H_{\bar\vartheta}^{0,n}(V, V \setminus U) \otimes_{\mathbb{Z}_M(U)} or_{M/X}(U)$$

by

$$b_\Omega(F) = [F\nu] \otimes e_{M/X} \quad (F \in \mathscr{O}_X(\Omega)).$$

3.2. Čech cohomology and isomorphism b_U

This subsection aims to define a map from Čech-Dolbeault representation to Čech representation of hyperfunctions, which is given in Definition 3.22. We refer the reader to Kaneko[5] and Kashiwara-Kawai-Kimura[6] for the theory of Čech cohomology.

Assume $M = \mathbb{R}^n$ and $X = \mathbb{C}^n$. Let $U \in \mathrm{Op}(M)$ and let V be a Stein neighborhood of U. Here, by Grauert's theorem (see Theorem 1.2.3 of Kashiwara-Kawai-Kimura[6]), we can take a Stein neighborhood V of U. In what follows, we fix a section $e_{M/X} \in or_{M/X}(U)$ such that it generates $or_{M/X}$ over $\mathbb{Z}_X$ on U and determines the same relative orientation along M on each connected component of U.

Take $v_0, v_1, \cdots, v_{n-1}$ as basis vectors of $\mathbb{R}^n$, and set $v_n = -(v_0 + v_1 + \cdots + v_{n-1})$. Moreover, we assume that the orientation of the frame $v_0, v_1, \cdots, v_{n-1}$ is the same as the one determined by $e_{M/X}$. Let $H_i = \{y \in \mathbb{R}^n \mid \langle y, v_i\rangle > 0\}$ and $V_i = (\mathbb{R}^n + \sqrt{-1}\,H_i) \cap V$ $(0 \le i \le n)$. Furthermore, we define $V_{n+1} = V$, $\Lambda' = \{0, 1, \cdots, n\}$ and $\Lambda = \{0, 1, \cdots, n+1\}$. Then, $\mathcal{V}' = \{V_i\}_{i\in\Lambda'}$ is a covering of $V \setminus U$ and $\mathcal{V} = \{V_i\}_{i\in\Lambda} = \mathcal{V}' \cup \{V\}$ is a covering of V. Since $\mathbb{R}^n + \sqrt{-1}\,H_0, \cdots, \mathbb{R}^n + \sqrt{-1}\,H_n$ and V are Stein open sets, $\mathcal{V} = \{V_i\}_{i\in\Lambda}$ is a Stein covering of V, that is, each V_i is a Stein open set.

Definition 3.13 (Čech cohomology) Let $\mathscr{F}$ be a sheaf on X. We define

$$C^k(\mathcal{V}, \mathcal{V}'; \mathscr{F}) = \left\{ \{F_{i_0,\cdots,i_k}\}_{(i_0,\cdots,i_k)\in\Lambda^{k+1}} \;\middle|\; \begin{array}{l} F_{i_0,\cdots,i_k} \in \mathscr{F}(V_{i_0} \cap \cdots \cap V_{i_k}), \\ F_{i_0,\cdots,i_\ell,i_{\ell+1},\cdots,i_k} = -F_{i_0,\cdots,i_{\ell+1},i_\ell,\cdots,i_k}, \\ (i_0, \cdots, i_k) \in {\Lambda'}^{k+1} \Rightarrow F_{i_0,\cdots,i_k} = 0 \end{array} \right\},$$

and a map $\delta : C^k(\mathcal{V}, \mathcal{V}'; \mathscr{F}) \to C^{k+1}(\mathcal{V}, \mathcal{V}'; \mathscr{F})$ is defined by

$$\delta\left(\{F_{i_0,\cdots,i_k}\}_{(i_0,\cdots,i_k)\in\Lambda^{k+1}}\right) = \left\{ \sum_{\ell=0}^{k+1} (-1)^\ell \, F_{j_0,\cdots,\widehat{j_\ell},\cdots,j_{k+1}}\Big|_{V_{j_0}\cap\cdots\cap V_{j_{k+1}}} \right\}_{(j_0,\cdots,j_{k+1})\in\Lambda^{k+2}},$$

where $\widehat{\bullet}$ means to omit the corresponding letter in a sequence. $(C^k(\mathcal{V}, \mathcal{V}'; \mathscr{F}), \delta)$ is a complex, and we write its k-th cohomology group as $H^k(\mathcal{V}, \mathcal{V}'; \mathscr{F})$.

Remark 3.14 We have

$$C^k(\mathcal{V}, \mathcal{V}'; \mathscr{O}) \subset \bigoplus_{(i_0, \cdots, i_k) \in \Lambda^{k+1}} \mathscr{F}(V_{i_0} \cap \cdots \cap V_{i_k}).$$

In Čech cohomology, the sorting order of subscripts is important for the sign of representatives because of the condition $F_{i_0, \cdots, i_\ell, i_{\ell+1}, \cdots, i_k} = -F_{i_0, \cdots, i_{\ell+1}, i_\ell, \cdots, i_k}$. To emphasize this point, Kaneko[5] uses the symbol $\mathscr{F}(V_{i_0} \wedge \cdots \wedge V_{i_k})$ instead of $\mathscr{F}(V_{i_0} \cap \cdots \cap V_{i_k})$. In this article, we do not use this notation. However, we pay attention to the order of subscripts.

Theorem 3.15 (Leray) There is the canonical isomorphism

$$H^k(\mathcal{V}, \mathcal{V}'; \mathscr{O}_X) \simeq H^k_U(V; \mathscr{O}_X).$$

To prove Leray's theorem, $\mathcal{V}$ must be a Stein covering. We omit the proof of Theorem 3.15. See Theorem 5.5.6 of Kaneko[5] or Theorem 1.2.1 of Kashiwara-Kawai-Kimura[6]. Theorem 3.15 means that a formal direct sum of holomorphic functions represents a hyperfunction, i.e.,

$$\bigoplus_{0 \le i \le n} F_{\widehat{i}} \in \bigoplus_{0 \le i \le n} \mathscr{O}_X(V_{\widehat{i}})$$

gives a hyperfunction on U, where $V_{\widehat{i}} = V_0 \cap \cdots \cap \widehat{V_i} \cap \cdots \cap V_{n+1}$ and $F_{\widehat{i}} = F_{0, \cdots, \widehat{i}, \cdots, n+1}$.

Remark 3.16 We must clarify the sequence order of the $\widehat{i}$. In what follows, we write $\widehat{i}$ for the increasing sequence $0, 1, 2, \cdots, i-1, i+1, \cdots, n+1 \in \Lambda^{n+1}$.

Proposition 3.17 (Excision) For any Stein neighborhood $\widetilde{V}$ of U, we define $\widetilde{\mathcal{V}} = \{\widetilde{V}_i\}_{i \in \Lambda}$ and $\widetilde{\mathcal{V}}' = \{\widetilde{V}_i\}_{i \in \Lambda'}$ by

$$\widetilde{V}_0 = (\mathbb{R}^n + \sqrt{-1}\, H_0) \cap \widetilde{V}, \cdots, \widetilde{V}_n = (\mathbb{R}^n + \sqrt{-1}\, H_0) \cap \widetilde{V}, \text{ and } \widetilde{V}_{n+1} = \widetilde{V}.$$

Then, there exists the canonical isomorphism

$$H^n(\mathcal{V}, \mathcal{V}'; \mathscr{O}_X) \simeq H^n(\widetilde{\mathcal{V}}, \widetilde{\mathcal{V}}'; \mathscr{O}_X). \tag{3.1}$$

Furthermore, we have the canonical isomorphism

$$H^n(\mathcal{V}, \mathcal{V}'; \mathscr{O}_X) \simeq \varinjlim_{U \subset \widetilde{V}} H^n(\widetilde{\mathcal{V}}, \widetilde{\mathcal{V}}'; \mathscr{O}_X), \tag{3.2}$$

where $\widetilde{V}$ runs through complex neighborhoods of U.

Proof. We may assume $\widetilde{V} \subset V$. Now, we have a commutative diagram

$$\begin{array}{ccc} H^n_U(V; \mathscr{O}_X) & \longrightarrow & H^n_U(\widetilde{V}; \mathscr{O}_X) \\ \downarrow & & \downarrow \\ H^n(\mathcal{V}, \mathcal{V}'; \mathscr{O}_X) & \longrightarrow & H^n(\widetilde{\mathcal{V}}, \widetilde{\mathcal{V}}'; \mathscr{O}_X). \end{array}$$

By Theorem 3.15 (Leray's theorem), the vertical morphisms are isomorphic. Additionally, by the excision theorem of local cohomology, the first horizontal morphism is an isomorphism. Then, the equation (3.1) holds. By Grauert's theorem, the equation (3.2) is also valid. $\square$

Note that the excision theorem of local cohomology is explained after Definition 1.1.9 of Kashiwara-Kawai-Kimura[6], and that we refer the reader to Lemma 5.5.12 of Kaneko[5] for more details of Proposition 3.17.

Definition 3.18 (Embedding of $\mathscr{A}_M$ in Čech representation) For any $f \in \mathscr{A}_M(U)$, we define $\iota^C_{\mathscr{A}_M(U)} : \mathscr{A}_M(U) \to H^n(\mathcal{V}, \mathcal{V}'; \mathscr{O}_X) \otimes or_{M/X}(U) \simeq \mathscr{B}_M(U)$ by

$$\begin{array}{cccccc} \iota^C_{\mathscr{A}_M(U)} : \mathscr{A}_M(U) & \longrightarrow & H^n(\widetilde{\mathcal{V}}, \widetilde{\mathcal{V}}'; \mathscr{O}_X) \otimes or_{M/X}(U) & \simeq & H^n(\mathcal{V}, \mathcal{V}'; \mathscr{O}_X) \otimes or_{M/X}(U), \\ \Psi & & \Psi & & \\ f & \longmapsto & \left[\widetilde{F}|_{V_{\widehat{k}}} \right] \otimes [e_{\widehat{k}}] & & \end{array}$$

for some $k \in \{0, 1, \cdots, n\}$. Here $\widetilde{V} \subset V$ is a Stein neighborhood of U on which f can be extended to a holomorphic function $\widetilde{F} \in \mathscr{O}_X(\widetilde{V})$, we set

$$\widetilde{\mathcal{V}}' = \left\{ \widetilde{V}_0 = (\mathbb{R}^n + \sqrt{-1}H_0) \cap \widetilde{V}, \cdots, \widetilde{V}_n = (\mathbb{R}^n + \sqrt{-1}H_n) \cap \widetilde{V} \right\}$$
$$\text{and } \widetilde{\mathcal{V}} = \widetilde{\mathcal{V}}' \cup \left\{ \widetilde{V}_{n+1} = \widetilde{V} \right\}$$

as in Proposition 3.17, and $[e_{\widehat{k}}] \in or_{M/X}(U) = H^n(\mathcal{V}, \mathcal{V}'; \mathbb{Z}_X) \simeq H^n(\widetilde{\mathcal{V}}, \widetilde{\mathcal{V}}'; \mathbb{Z}_X)$ is represented by

$$e_{\widehat{k}} = \{G_{i_0, \cdots, i_n}\}_{(i_0, \cdots, i_n) \in \Lambda_k{}^{n+1}} \in \bigoplus_{(i_0, \cdots, i_n) \in \Lambda_k{}^{n+1}} \mathbb{Z}_X(V_{i_0} \cap \cdots \cap V_{i_n}),$$

where $\Lambda_k = \{0, 1, \cdots, k-1, k+1, \cdots, n+1\}$, $G_{\widehat{k}} = G_{0, \cdots, k-1, k+1, \cdots, n+1} = 1$ and $G_{i_0, \cdots, i_\ell, i_{\ell+1}, \cdots, i_n} = -G_{i_0, \cdots, i_{\ell+1}, i_\ell, \cdots, i_n}$ for any ℓ.

Note that $[(-1)^k e_{\widehat{k}}] = e_{M/X}$ holds for any $k = 0, 1, \cdots, n$.

Proposition 3.19 $\iota^C_{\mathscr{A}_M(U)}$ does not depend on the choices of $k \in \{0, 1, \cdots, n\}$ and $\iota^C_{\mathscr{A}_M} = \{\iota^C_{\mathscr{A}_M(U)}\}_{U \in \mathrm{Op}(M)}$ becomes a sheaf morphism from $\mathscr{A}_M$ to $\mathscr{B}_M$.

Remark 3.20 When we fix a generator $e_{M/X} \in or_{M/X}(U)$, for any hyperfunction $u = \left[\bigoplus_i (-1)^i F_{\widehat{i}}\right] \otimes e_{M/X} \in H^n(\mathcal{V}, \mathcal{V}'; \mathscr{O}_X) \otimes or_{M/X}(U) = \mathscr{B}_M(U)$, we call $\bigoplus_i (-1)^i F_{\widehat{i}} \in \bigoplus_i \mathscr{O}_X(V_{\widehat{i}})$ a Čech representative of u. This expression is different from the one given in Kaneko[5], which is explained in Remark 3.29.

Now, by the map b_Ω in Definition 3.12, we can define an isomorphism from Čech representation to Čech-Dolbeault representation. Let us construct the special representative ν whose support is contained in $V_{\widehat{i}}$. Recall that $U \in \mathrm{Op}(M)$ and that V is a Stein neighborhood of U.

Example 3.21 ([2, Example 7.14]) Let $\varphi_0, \varphi_1, \cdots, \varphi_n$ be C^∞ functions on $V \setminus U$ which satisfy

(1) $\mathrm{Supp}_{V\setminus U}(\varphi_i) \subset V_i \qquad (i = 0, 1, \cdots, n)$,

(2) $\displaystyle\sum_{i=0}^{n} \varphi_i = 1$ on $V \setminus U$.

For any $i = 0, 1, \cdots, n$, we define

$$\chi_{V_{\widehat{i}}}(z) = \begin{cases} 1 & \text{if } z \in V_{\widehat{i}}, \\ 0 & \text{otherwise}, \end{cases}$$

and

$$\nu_{i,01} = \begin{cases} (-1)^i (n-1)!\, \chi_{V_{\widehat{i}}} \bar\partial\varphi_0 \wedge \cdots \wedge \widehat{\bar\partial\varphi_i} \wedge \cdots \wedge \bar\partial\varphi_{n-1} & (i = 0, 1, \cdots, n-1), \\ (-1)^n (n-1)!\, \chi_{V_{\widehat{n}}} \bar\partial\varphi_0 \wedge \cdots \wedge \bar\partial\varphi_{n-2} & (i = n). \end{cases}$$

Then, $\nu_i = (0, \nu_{i,01})$ satisfies $[\nu_i] \otimes e_{M/X} = 1 \in \mathscr{B}_M(U)$ and $\mathrm{Supp}_{V\setminus U}(\nu_{i,01}) \subset V_{\widehat{i}}$.

Definition 3.22 For any $i = 0, 1, \cdots, n$, we take ν_i as $(0, \nu_{i,01})$ given in Example 3.21, and we set the map $b_{V_{\widehat{i}}}$ by $b_{V_{\widehat{i}}}(F_{\widehat{i}}) = [F_{\widehat{i}}\nu_i] \otimes e_{M/X}$ as in Definition 3.12. Then, we define a map $b_U : H^n(\mathcal{V}, \mathcal{V}'; \mathscr{O}_X) \otimes or_{M/X}(U) \to H^{0,n}_{\bar\vartheta}(V, V \setminus U) \otimes or_{M/X}(U)$ by

$$b_U\left(\left[\bigoplus_{i=0}^{n} (-1)^i F_{\widehat{i}}\right] \otimes e_{M/X}\right) = \sum_{i=0}^{n} b_{V_{\widehat{i}}}(F_{\widehat{i}}),$$

where $e_{M/X}$ is the section of $or_{M/X}(U)$ specified at the beginning of Subsection 3.2.

Example 7.17 of Honda-Izawa-Suwa[2] is helpful for the reader in understanding the map b_U.

Theorem 3.23 The map b_U is an isomorphism.

Proof. We have a double complex

$$\begin{array}{ccccccc}
0 \longrightarrow & C^0(\mathcal{V}, \mathcal{V}'; \mathscr{O}_X) & \hookrightarrow & C^0(\mathcal{V}, \mathcal{V}'; \mathscr{E}_X^{(0,0)}) & \xrightarrow{\bar\partial^{0,0}} & C^0(\mathcal{V}, \mathcal{V}'; \mathscr{E}_X^{(0,1)}) & \xrightarrow{\bar\partial^{0,1}} \cdots \\
& \downarrow{\scriptstyle \delta^0} & & \downarrow{\scriptstyle \delta^{0,0}} & & \downarrow{\scriptstyle \delta^{0,1}} & \\
0 \longrightarrow & C^1(\mathcal{V}, \mathcal{V}'; \mathscr{O}_X) & \hookrightarrow & C^1(\mathcal{V}, \mathcal{V}'; \mathscr{E}_X^{(0,0)}) & \xrightarrow{\bar\partial^{1,0}} & C^1(\mathcal{V}, \mathcal{V}'; \mathscr{E}_X^{(0,1)}) & \xrightarrow{\bar\partial^{1,1}} \cdots. \\
& \downarrow{\scriptstyle \delta^1} & & \downarrow{\scriptstyle \delta^{1,0}} & & \downarrow{\scriptstyle \delta^{1,1}} & \\
& \vdots & & \vdots & & \vdots &
\end{array}$$

Note that $\bar\partial^{p,q}$ is the Dolbeault operator $\bar\partial$. Since $\mathcal{V}$ is a Stein covering, each row is exact (Corollary 2.5.12 of Hörmander[4]). By the Weil procedure, the first column is quasi-isomorphic to the complex $\mathrm{tot}(C^\bullet(\mathcal{V}, \mathcal{V}'; \mathscr{E}^{(0,\bullet)}))$ (Theorem 12.5.4 of Kashiwara-Schapira[8]). The complex $\mathrm{tot}(C^\bullet(\mathcal{V}, \mathcal{V}'; \mathscr{E}^{(0,\bullet)}))$ is defined as follows: For any $p, q \in \mathbb{Z}$, we write $A^{p,q}$ for $C^p(\mathcal{V},\mathcal{V}';\mathscr{E}_X^{(0,q)})$ and let A denote a double complex $\{A^{p,q},\delta^{p,q},\bar\partial^{p,q}\}_{p,q\in\mathbb{Z}}$. For any $k\in\mathbb{Z}$, let us define

$$\mathrm{tot}(A)^k=\bigoplus_{p+q=k}A^{p,q},$$

and

$$d^k_{\mathrm{tot}(A)}\Big|_{A^{p,q}}=\delta^{p,q}+(-1)^p\bar\partial^{p,q}.$$

Here the simple complex $\big(\mathrm{tot}(A),d_{\mathrm{tot}(A)}\big)$ is called the total complex of a double complex A.

On the other hand, for any $q\in\mathbb{Z}$, we have isomorphisms

$$\mathscr{E}_X^{(0,q)}(V)\simeq C^0(\mathcal{V},\mathcal{V}';\mathscr{E}_X^{(0,q)}),$$
$$\mathscr{E}_X^{(0,q-1)}(V\setminus U)\simeq \operatorname{Ker}\delta^{1,q-1}\subset C^1(\mathcal{V},\mathcal{V}';\mathscr{E}_X^{(0,q-1)}),$$

since $\mathscr{E}_X^{(0,q-1)}$ is a sheaf. Then, a morphism

$$\begin{array}{rcl}\bar\vartheta^q\ :\ C^0(\mathcal{V},\mathcal{V}';\mathscr{E}_X^{(0,q)})\oplus\operatorname{Ker}\delta^{1,q-1} & \longrightarrow & C^0(\mathcal{V},\mathcal{V}';\mathscr{E}_X^{(0,q+1)})\oplus\operatorname{Ker}\delta^{1,q}\\ \cup & & \cup\\ (\mu_1,\mu_{01}) & \longmapsto & (\bar\partial^{0,q}\mu_1,\delta^{0,q}\mu_1-\bar\partial^{1,q-1}\mu_{01}),\end{array}$$

defines the differential of Čech-Dolbeault complex in Definition 3.1.

Now, let us prove that the diagram chasing on the double complex $C^\bullet(\mathcal{V},\mathcal{V}';\mathscr{E}^{(0,\bullet)})$ sends a cocycle of $C^n(\mathcal{V},\mathcal{V}';\mathscr{O}_X)$ to a cocycle of $C^0(\mathcal{V},\mathcal{V}';\mathscr{E}_X^{(0,n)})\oplus\operatorname{Ker}\delta^{1,n-1}$ which belongs to the same cohomology class as the one given by b_U in Definition 3.22. For any $i\in\Lambda'=\{0,1,\cdots,n\}$, let φ_i be a cutoff function defined in Example 3.21. Firstly, we regard $\bigoplus_{i=0}^{n}F_{\hat{i}}\in\operatorname{Ker}\delta^n\subset C^n(\mathcal{V},\mathcal{V}';\mathscr{O}_X)$ as an element $\omega^{n,0}$ in $C^n(\mathcal{V},\mathcal{V}';\mathscr{E}_X^{(0,0)})$. Secondly, for any $\alpha=(\alpha_0,\alpha_1,\cdots,\alpha_{n-1})\in\Lambda^n=\{0,1,\cdots,n+1\}^n$, we define

$$\omega_\alpha^{n-1,0}=\sum_{i\notin\alpha}\varphi_iF_{i\alpha_0\cdots\alpha_{n-1}}\in\mathscr{E}_X^{(0,0)}(V_{\alpha_0}\cap V_{\alpha_1}\cap\cdots\cap V_{\alpha_{n-2}}).$$

Then, $\omega^{n-1,0}=\{\omega_\alpha^{n-1,0}\}_{\alpha\in\Lambda^n}\in C^{n-1}(\mathcal{V},\mathcal{V}';\mathscr{E}_X^{(0,0)})$ satisfies $\delta^{n-1,0}(\omega^{n-1,0})=\omega^{n,0}$. Since $\bar\partial F_{\hat{i}}=0$ for any $i\in\Lambda'$, we have

$$\bar\partial\omega_\alpha^{n-1,0}=\sum_{i\notin\alpha}\bar\partial\varphi_iF_{i\alpha_0\cdots\alpha_{n-2}}.$$

Finally, we set $\omega^{n-1,1}=\bar\partial\omega^{n-1,0}\in C^{n-1}(\mathcal{V},\mathcal{V}';\mathscr{E}_X^{(0,1)})$. If we repeat this procedure, we get $\omega^{1,n-1}=\sum_{i=0}^{n}F_{\hat{i}}\,\nu_{i,01}$, where the representative $\nu_i=(0,\nu_{i,01})\in\mathscr{E}_X^{(0,n-1)}(V,V\setminus U)$ is defined in Example 3.21. Summing up, $b_U\left(\left[\bigoplus_{i=0}^{n}(-1)^iF_{\hat{i}}\right]\otimes e_{M/X}\right)=[(0,\omega^{1,n-1})]\otimes$

$e_{M/X} \in H^{0,n}_{\bar\vartheta}(V, V \setminus U) \otimes or_{M/X}(U)$ holds, and the above diagram chasing induces the same morphism from $H^n(\mathcal{V}, \mathcal{V}'; \mathscr{O}_X) \otimes or_{M/X}(U) \to H^{0,n}_{\bar\vartheta}(V, V \setminus U) \otimes or_{M/X}(U)$ as the one given by the quasi-isomorphisms

$$C^\bullet(\mathcal{V}, \mathcal{V}'; \mathscr{O}_X) \xrightarrow{\sim} \mathrm{tot}(C^\bullet(\mathcal{V}, \mathcal{V}'; \mathscr{E}_X^{(0,\bullet)})) \xleftarrow{\sim} \mathscr{E}_X^{(0,\bullet)}(V, V \setminus U).$$

The Weil procedure shows that the left morphism is quasi-isomorphic, as at the beginning of this proof. The right quasi-isomorphism is explained in Honda-Izawa-Suwa[2] and Suwa[18]. This completes the proof. □

3.3. Infinitesimal wedges and embedding of distributions via Čech cohomology

In this subsection, we define an embedding of distributions via Čech cohomology. Kaneko[5] and Aoki-Kataoka-Yamazaki[1] are helpful for readers. We use the same notations as Subsection 3.2, i.e., we keep the notations below:

- $M = \mathbb{R}^n$, $X = \mathbb{C}^n$,
- $U \in \mathrm{Op}(M)$, $V \in \mathrm{Op}(X)$ is a Stein neighborhood of U,
- $v_0, v_1, \cdots, v_{n-1}$ are basis vectors of $\mathbb{R}^n$, which has the same orientation as that of $e_{M/X}$.
- $v_n = -(v_0 + v_1 + \cdots + v_{n-1})$,
- $H_i = \{y \in \mathbb{R}^n \mid \langle y, v_i\rangle > 0\}$ $(0 \le i \le n)$,
- $V_i = (\mathbb{R}^n + \sqrt{-1}H_i) \cap V$ $(0 \le i \le n)$, $V_{n+1} = V$,
- $\Lambda' = \{0, 1, \cdots, n\}$, $\Lambda = \{0, 1, \cdots, n+1\}$,
- $\mathcal{V}' = \{V_i\}_{i \in \Lambda'}$, $\mathcal{V} = \{V_i\}_{i \in \Lambda}$,
- $V_{\widehat{i}} = V_0 \cap \cdots \cap \widehat{V_i} \cap \cdots \cap V_{n+1}$ $(0 \le i \le n)$.

We set $\Gamma_i = H_0 \cap \cdots \cap \widehat{H_i} \cap \cdots \cap H_n$ for any $i = 0, 1, \cdots, n$, and we have $V_{\widehat{i}} = (U + \sqrt{-1}\Gamma_i) \cap V$.

Definition 3.24 (Proper cone and Dual cone) A subset $\Gamma \subset \mathbb{R}^n$ is called a cone if Γ satisfies

$$\{ry \in \mathbb{R}^n \mid y \in \Gamma, r > 0\} \subset \Gamma.$$

We define a proper cone $\Gamma \subset \mathbb{R}^n$ as a cone $\Gamma \subset \mathbb{R}^n$ satisfying

$$\overline{\Gamma} \setminus \{0\} \subset \{y \in \mathbb{R}^n \mid \langle \xi, y\rangle > 0\}$$

for some $\xi \in \mathbb{R}^n$. For any cone $\Gamma \subset \mathbb{R}^n$, its dual cone Γ° is defined by

$$\Gamma^\circ = \{\xi \in \mathbb{R}^n \mid \langle \xi, y\rangle \ge 0 \ (\forall y \in \Gamma)\}.$$

Definition 3.25 (Infinitesimal wedge, [5, Definition 2.2.9]**)** Let $\Gamma \subset \mathbb{R}^n$ be an open cone and $U \in \mathrm{Op}(\mathbb{R}^n)$. An open set $W \subset \mathbb{C}^n$ is called an infinitesimal wedge of type $U + \sqrt{-1}\,\Gamma\, 0$ if it satisfies the following conditions:

(a) $W \subset U + \sqrt{-1}\,\Gamma$ holds.

(b) For any open subcone $\Gamma' \subset \Gamma$ with $\overline{\Gamma'} \setminus \{0\} \subset \Gamma$ and for any compact set $K \subset U$, there exists $\delta > 0$ such that

$$K + \sqrt{-1}\,\left(\Gamma' \cap \left\{y \in \mathbb{R}^n \;\middle|\; |y| < \delta\right\}\right) \subset W\,.$$

U is called the edge of the infinitesimal wedge W of type $U + \sqrt{-1}\,\Gamma\,0$, and Γ is called the opening of the infinitesimal wedge W of type $U + \sqrt{-1}\,\Gamma\,0$.

$F(z) \in \mathscr{O}_X(U + \sqrt{-1}\,\Gamma\,0)$ means that $F(z)$ is holomorphic in an infinitesimal wedge of type $U + \sqrt{-1}\,\Gamma\,0$. We set, for $0 \le i < j \le n$,

$$\Gamma_{ij} = H_0 \cap \cdots \cap \widehat{H_i} \cap \cdots \cap \widehat{H_j} \cap \cdots \cap H_n,$$

and define the morphism $\rho : \bigoplus\limits_{0\le i<j\le n} \mathscr{O}_X(U + \sqrt{-1}\Gamma_{ij}0) \to \bigoplus\limits_{0\le i\le n} \mathscr{O}_X(U + \sqrt{-1}\Gamma_i 0)$ by

$$\sum_{0\le i<j\le n} F_{\widehat{ij}} \longmapsto \sum_{0\le i\le n} \left(\sum_{k=0}^{i-1} (-1)^k \, F_{\widehat{ki}}\Big|_{U+\sqrt{-1}\Gamma_i 0} - \sum_{k=i+1}^{n} (-1)^k \, F_{\widehat{ik}}\Big|_{U+\sqrt{-1}\Gamma_i 0} \right).$$

Here, for any $0 \le k < i$ (resp. $i < k \le n$), $F_{\widehat{ki}} \in \mathscr{O}_X(U+\sqrt{-1}\Gamma_{ki}0)$ (resp. $F_{\widehat{ik}} \in \mathscr{O}_X(U+\sqrt{-1}\Gamma_{ki}0)$), and $\widehat{ki}$ (resp. $\widehat{ik}$) means $0, 1, \cdots, k-1, k+1, \cdots, i-1, i+1, \cdots, n+1 \in \Lambda^n$ (resp. $0, 1, \cdots, i-1, i+1, \cdots, k-1, k+1, \cdots, n+1 \in \Lambda^n$) as in Remark 3.16.

Definition 3.26 We define

$$\hat{H}(\{U + \sqrt{-1}\Gamma_i 0\}_{i=0}^n; \mathscr{O}_X) = \frac{\bigoplus\limits_{0\le i\le n} \mathscr{O}_X(U + \sqrt{-1}\Gamma_i 0)}{\rho\left(\bigoplus\limits_{0\le i<j\le n} \mathscr{O}_X(U + \sqrt{-1}\Gamma_{ij} 0)\right)}.$$

Remark 3.27 Since $\hat{H}(\{U + \sqrt{-1}\Gamma_i 0\}_{i=0}^n; \mathscr{O}_X)$ is a variant of Čech cohomology group, correctly speaking, we regard an element $F_{\hat{i}} \in \mathscr{O}_X(U+\sqrt{-1}\Gamma_i 0)$ as an alternative section like the definition of Čech cohomology. In this article, we identify $F_{\hat{i}}$ with the alternative section $\{\widetilde{F}_\alpha\}_{\alpha\in\{0,1,\cdots,i-1,i+1,\cdots,n+1\}^{n+1}}$ satisfying $\widetilde{F}_{\hat{i}} = \widetilde{F}_{0,1,\cdots,i-1,i+1,\cdots,n+1} = F_{\hat{i}}$.

Theorem 3.28 There exists the canonical isomorphism

$$H^n(\mathcal{V}, \mathcal{V}'; \mathscr{O}_X) \simeq \hat{H}(\{U + \sqrt{-1}\Gamma_i 0\}_{i=0}^n; \mathscr{O}_X).$$

Proof. Let $W \in \mathrm{Op}(\mathbb{C}^n)$ be a Stein neighborhood of U. For a sufficiently small $\delta > 0$, we define an open covering $\mathcal{W}' = \{W_0, W_1, \cdots, W_n\}$ of $W \setminus U$ by

$$W_i = W \cap \left\{z \in \mathbb{C}^n \;\middle|\; x \in U, \delta|y| < \langle y, v_i \rangle\right\} \quad (i = 0, 1, \cdots, n),$$

and an open covering $\mathcal{W}$ of W by

$$\mathcal{W} = \{W_0, W_1, \cdots, W_n, W_{n+1} = W\}.$$

Note that each W_i is also a Stein open set. By the definition of infinitesimal wedges, the restriction morphism induces the canonical morphisms

$$H^n(\mathcal{V}, \mathcal{V}', \mathscr{O}_X) \xrightarrow{\rho_1} \hat{H}(\{U + \sqrt{-1}\Gamma_i 0\}_{i=0}^n; \mathscr{O}_X) \xrightarrow{\rho_2} \varinjlim_{U \subset W} H^n(\mathcal{W}, \mathcal{W}', \mathscr{O}_X),$$

where W runs through Stein neighborhoods of U. Using the same arguments as in the proof of Proposition 3.17, all the morphisms in the following commutative diagram are isomorphic:

$$\begin{array}{ccc} H^n_U(V; \mathscr{O}_X) & \longrightarrow & \varinjlim_{U \subset W} H^n_U(W; \mathscr{O}_X) \\ \downarrow & & \downarrow \\ H^n(\mathcal{V}, \mathcal{V}'; \mathscr{O}_X) & \longrightarrow & \varinjlim_{U \subset W} H^n(\mathcal{W}, \mathcal{W}'; \mathscr{O}_X). \end{array}$$

This means that $\rho_2 \circ \rho_1$ is isomorphic. It follows from the theorem of the edge of the wedge by A. Martineau (see Corollary 4.6.13 [5]) that ρ_2 is injective. Thus we conclude that ρ_1 is an isomorphism. □

Note that Komori-Umeta[11] explains the general cases of Theorem 3.28.

Remark 3.29 As in the case of Remark 3.20, we write a representative of a hyperfunction $u = \left[\bigoplus_i (-1)^i F_{\widehat{i}}\right] \otimes e_{M/X} \in \hat{H}(\{U + \sqrt{-1}\Gamma_i 0\}_{i=0}^n; \mathscr{O}_X) \otimes or_{M/X}(U)$ as $\bigoplus_i (-1)^i F_{\widehat{i}} \in \bigoplus_i \mathscr{O}_X(U + \sqrt{-1}\Gamma_i 0)$ for a fixed generator $e_{M/X} \in or_{M/X}(U)$. Since the equation

$$\left[\bigoplus_{i=0}^n (-1)^i F_{\widehat{i}}\right] \otimes e_{M/X} = \sum_{i=0}^n \left[F_{\widehat{i}}\right] \otimes [e_i] \in \hat{H}(\{U + \sqrt{-1}\Gamma_i 0\}_{i=0}^n; \mathscr{O}_X) \otimes or_{M/X}(U)$$

holds, Kaneko[5] writes a Čech representative of a hyperfunction $\sum_i \left[F_{\widehat{i}}\right] \otimes [e_i]$ as $\sum_i F_{\widehat{i}}(x + \sqrt{-1}\Gamma_i 0)$. Here, $[e_i] \in or_{M/X}(U)$ is defined in Definition 3.18.

Fix a sufficiently small $\delta > 0$ and set

$$H_{\delta,i} = \left\{y \in \mathbb{R}^n \;\middle|\; \delta|y| < \langle y, v_i \rangle\right\} \qquad (i = 0, 1, \cdots, n).$$

We also define $H_{\delta,\widehat{i}}$ in the same way as $V_{\widehat{i}}$, that is,

$$H_{\delta,\widehat{i}} = H_{\delta,0} \cap \cdots \cap \widehat{H_{\delta,i}} \cap \cdots \cap H_{\delta,n}.$$

Let $\nu_i = (0, \nu_{i,01})$ $(i = 0, 1, \cdots, n)$ be the one constructed in Example 3.21, where we replace the condition (1) in Example 3.21 with the following $(1)^*$:

$(1)^*$ $\mathrm{Supp}_{V \setminus U}(\varphi_i) \subset U + \sqrt{-1} H_{\delta,i} \qquad (i = 0, 1, \cdots, n),$

Then, in this case, the resulting ν_i satisfies

$$\mathrm{Supp}_{V\setminus U}(\nu_{i,01}) \subset U + \sqrt{-1}H_{\delta,\widehat{i}} \qquad (i = 0, 1, \cdots, n).$$

Note that $H_{\delta,\widehat{i}}$ is a convex cone properly contained in Γ_i. By the definition of the infinitesimal wedge, for any infinitesimal wedge Ω of type $U + \sqrt{-1}\Gamma_i$, there exists an complex neighborhood W of U such that

$$\mathrm{Supp}_{V\setminus U}(\nu_{i,01}) \cap W \subset (U + \sqrt{-1}H_{\delta,\widehat{i}}) \cap W \subset \Omega,$$

from which we have the well-defined morphism

$$\bigoplus_{0\le i\le n} \mathscr{O}_X(U + \sqrt{-1}\Gamma_i 0) \otimes or_{M/X}(U) \longrightarrow \varinjlim_{W} H^{0,n}_{\bar\vartheta}(W, W\setminus U) \otimes or_{M/X}(U)$$

by

$$\left(\bigoplus_{0\le i\le n} (-1)^i F_{\widehat{i}}\right) \otimes e_{M/X} \longmapsto \left[\sum_{0\le i\le n} F_{\widehat{i}}\big|_W \ \nu_i|_W\right] \otimes e_{M/X},$$

where W runs through Stein neighborhoods of U. The above morphism sends $\rho\left(\bigoplus_{0\le i<j\le n} \mathscr{O}_X(U + \sqrt{-1}\Gamma_{ij} 0)\right) \otimes or_{M/X}(U)$ to 0. Hence we have the morphism

$$\hat{H}(\{U + \sqrt{-1}\Gamma_i 0\}_{i=0}^n; \mathscr{O}_X) \otimes or_{M/X}(U) \longrightarrow \varinjlim_{U\subset W} H^{0,n}_{\bar\vartheta}(W, W\setminus U) \otimes or_{M/X}(U).$$

Thanks to Theorems 3.23 and 3.28, we obtain the theorem below.

Theorem 3.30 We have the following commutative diagram whose morphisms are all isomorphic:

$$\begin{array}{ccc} H^n(\mathcal{V}, \mathcal{V}'; \mathscr{O}_X) \otimes or_{M/X}(U) & \xrightarrow{\quad b_U \quad} & H^{0,n}_{\bar\vartheta}(V, V\setminus U) \otimes or_{M/X}(U) \\ \downarrow & & \downarrow \\ \hat{H}(\{U + \sqrt{-1}\Gamma_i 0\}_{i=0}^n; \mathscr{O}_X) \otimes or_{M/X}(U) & \longrightarrow & \varinjlim_{U\subset W} H^{0,n}_{\bar\vartheta}(W, W\setminus U) \otimes or_{M/X}(U). \end{array}$$

Proof. It follows from Theorem 3.28 that the first vertical morphism is isomorphic. By Proposition 3.8, the second vertical morphism is also isomorphic. Since the map b_U is an isomorphism by Theorem 3.23, the second horizontal morphism is also isomorphic, which completes the proof. □

Definition 3.31 The second horizontal isomorphism in the above theorem

$$\hat{H}(\{U + \sqrt{-1}\Gamma_i 0\}_{i=0}^n; \mathscr{O}_X) \otimes or_{M/X}(U) \longrightarrow \varinjlim_{U\subset W} H^{0,n}_{\bar\vartheta}(W, W\setminus U) \otimes or_{M/X}(U)$$

is also denoted by b_U.

Now then, let us consider an embedding of distributions via Čech cohomology. For any $z, \zeta \in \mathbb{C}^n$, we use the notations $z\zeta = z_1\zeta_1 + \cdots + z_n\zeta_n$ and $\zeta^2 = \zeta_1^2 + \cdots + \zeta_n^2$.

Definition 3.32 For any (z, ζ), the twisted Radon kernel $W(z, \zeta)$ is defined by

$$W(z,\zeta) = \frac{(n-1)!}{(-2\pi\sqrt{-1})^n} \frac{(1-\sqrt{-1}z\zeta/\sqrt{\zeta^2})^{n-1} - (1-\sqrt{-1}z\zeta/\sqrt{\zeta^2})^{n-2}(z^2-(z\zeta)^2/\zeta^2)}{\{z\zeta + \sqrt{-1}(z^2\sqrt{\zeta^2} - (z\zeta)^2/\sqrt{\zeta^2})\}^n}.$$

Here, we define $\sqrt{\bullet} : \mathbb{C} \to \mathbb{C}$ by

$$\sqrt{z} = \sqrt{r}\, e^{\sqrt{-1}\theta/2} \quad (r > 0,\ -\pi < \theta < \pi,\ z = re^{\sqrt{-1}\theta}).$$

Lemma 3.33 ([5, Lemma 2.3.4]) We write $z = x + \sqrt{-1}y$ and $\zeta = \xi + \sqrt{-1}\eta$. Let $\Gamma \subset \mathbb{R}^n$ be an arbitrary proper convex open cone and Γ° its dual cone. There exists a $2n$ dimensional open convex cone $\Delta \subset \mathbb{R}^n_y \times \mathbb{R}^n_\eta$ such that $\Delta \cap \{\eta = 0\} \supset \Gamma$ and $W(z, \zeta) \in \mathscr{O}_{X\times X}\left(\mathbb{R}^n \times \mathrm{Int}(\Gamma^\circ) + \sqrt{-1}\Delta 0\right)$.

Moreover, $W(z, \zeta)$ can be analytically continued to a neighborhood of the real open set $(\mathbb{R}^n \setminus \{0\}) \times \mathrm{Int}(\Gamma^\circ)$. In particular, if we fix $\zeta = \xi \in \mathbb{R}^n \setminus \{0\}$, $W(z, \zeta)$ is holomorphic as a function of z in the following infinitesimal wedge with the half-space $\{y \in \mathbb{R}^n \mid y\xi > 0\}$ as the opening:

$$\mathbb{R}^n + \sqrt{-1}\left\{y \in \mathbb{R}^n \;\middle|\; y\xi > y^2|\xi| - (y\xi)^2/|\xi|\right\}.$$

Furthermore, it is analytically continued to the following complex neighborhood of $\mathbb{R}^n \setminus \{0\}$:

$$\left\{x + \sqrt{-1}\,y \in \mathbb{C}^n \;\middle|\; |y\xi| + (y^2|\xi| - (y\xi)^2/|\xi|) < \frac{1}{4}\left(|x\xi| + 2(x^2|\xi| - (x\xi)/|\xi|)\right)\right\}.$$

Lemma 3.34 ([5, Lemma 2.3.5]) Let $f(x)$ be a C^n function on U supported by a compact set $K \subset U$. We define

$$F(z,\zeta) = \int_{\mathbb{R}^n} f(w) W(z-w,\zeta)\, dw.$$

Then, $F(z, \zeta)$ satisfies the following conditions:

(1) For any proper convex open cone $\Gamma \subset \mathbb{R}^n$, F is a holomorphic function of (z, ζ) in the infinitesimal wedge of type $\mathbb{R}^n \times \mathrm{Int}(\Gamma^\circ) + \sqrt{-1}\Delta 0$ where Δ is defined in Lemma 3.33.

(2) F can be analytically continued to a neighborhood of $(U \setminus K) \times (\mathbb{R}^n \setminus \{0\})$.

(3) F can be continuously extended to $U \times (\mathbb{R}^n \setminus \{0\})$, and the continuously extended function $F(x, \xi)$ satisfies

$$f(x) = \int_{S^{n-1}} F(x,\xi)\, d\xi.$$

Lemma 3.35 ([5, Corollary 2.3.6]) We have the equation

$$\int_{S^{n-1}} W(z,\xi)\,d\xi = 0$$

in a neighborhood of $\mathbb{R}^n \setminus \{0\}$. This integral converges locally uniformly in z.

Definition 3.36 For any non-empty proper cone $\Gamma \subset \mathbb{R}^n$, we define

$$W(z,\Gamma^\circ) = \int_{\Gamma^\circ \cap S^{n-1}} W(z,\xi)\,d\xi \in \mathscr{O}(\mathbb{R}^n + \sqrt{-1}\Gamma 0).$$

Definition 3.37 (Embedding of $\mathscr{D}b_K$ in Čech representation) Let K be a compact set in U. For any $u \in \mathscr{D}b_K(U)$ and for any $i = 0, 1, \cdots, n$, we set

$$F_{\widehat{i}}(z) = \langle u(w), W(z-w, \Gamma_i^\circ)\,dw\rangle \in \mathscr{O}(U + \sqrt{-1}\Gamma_i 0),$$

and let us define $\iota^C_{\mathscr{D}b_K(U)} : \mathscr{D}b_K(U) \to \mathscr{B}_K(U)$ by $u \mapsto \left[\bigoplus_i (-1)^i F_{\widehat{i}}\right] \otimes e_{M/X}$. Here $e_{M/X}$ is the section of $or_{M/X}(U)$ specified at the beginning of Subsection 3.2.

Proposition 3.38 $\mathrm{Supp}(\iota^C_{\mathscr{D}b_K(U)}(u)) \subset \mathrm{Supp}(u)$ holds.

This proposition easily follows from Lemma 3.35.

Definition 3.39 (Embedding of $\mathscr{D}b$ in Čech representation) Let $u \in \mathscr{D}b_M(U)$. Decompose u into a locally finite sum as $u = \sum_\lambda u_\lambda$, where $u_\lambda \in \mathscr{D}b_{K_\lambda}(U)$ and K_λ is a compact set in U. We define $\iota^C_{\mathscr{D}b_M(U)}(u) = \sum_\lambda \iota^C_{\mathscr{D}b_{K_\lambda}(U)}(u_\lambda) \in \mathscr{B}_M(U)$.

Remark 3.40 Since $\langle u(w), W(z-w,\Gamma_i^\circ)\,dw\rangle \in \mathscr{O}(U + \sqrt{-1}\Gamma_i 0)$ for any $i = 0, 1, \cdots, n$, $\iota^C_{\mathscr{D}b_M(U)}$ is a map from $\mathscr{D}b_M(U)$ to $\hat{H}(\{U + \sqrt{-1}\Gamma_i 0\}_{i=0}^n; \mathscr{O}_X) \otimes or_{M/X}(U) = \mathscr{B}_M(U)$.

Proposition 3.41 $\iota^C_{\mathscr{D}b_M(U)}$ is well-defined, and $\iota^C_{\mathscr{D}b_M} = \{\iota^C_{\mathscr{D}b_M(U)}\}_{U \in \mathrm{Op}(M)}$ is a sheaf morphism from $\mathscr{D}b_M$ to $\mathscr{B}_M$.

Proposition 3.42 ([5, Theorem 3.5.5]) The diagram below commutes.

$$\begin{array}{ccc} \mathscr{A}_M(U) & \hookrightarrow & \mathscr{D}b_M(U), \\ \downarrow{\scriptstyle \iota^C_{\mathscr{A}_M(U)}} & \swarrow{\scriptstyle \iota^C_{\mathscr{D}b_M(U)}} & \\ H^n(\mathcal{V},\mathcal{V}';\mathscr{O}_X) \otimes or_{M/X}(U) & & \end{array}$$

where $\mathscr{A}_M(U) \hookrightarrow \mathscr{D}b_M(U)$ is the canonical embedding, and $\iota^C_{\mathscr{A}_M(U)}$ is defined in Definition 3.18. Note that we identify $H^n(\mathcal{V},\mathcal{V}';\mathscr{O}_X)$ with $\hat{H}(\{U + \sqrt{-1}\Gamma_i 0\}_{i=0}^n; \mathscr{O}_X)$ by Theorem 3.28.

3.4. Embedding of distributions via Čech-Dolbeault cohomology

Honda-Izawa-Suwa[2] defines an embedding $\mathscr{D}b_M \hookrightarrow \mathscr{B}_M$ in the framework of Čech-Dolbeault cohomology. Let $M = \mathbb{R}^n$, $X = \mathbb{C}^n$ and $U \in \mathrm{Op}(M)$. Since $M = \mathbb{R}^n$ is oriented, there is an isomorphism $or_M \simeq \mathbb{Z}_M$ which also gives the isomorphism

$$\mathscr{B}_M(U) \simeq H^n_U(V;\mathscr{O}_X),$$

where V is a complex neighborhood of U. By this isomorphism, we omit the section of the relative orientation sheaf in the definition of $\mathscr{B}_M(U)$. Now, let $\theta(z,t)$ be a C^∞ function on $X \times U$ which satisfies the following conditions:

(1) $\theta(z,t)$ is identically 1 in a neighborhood of $\{0\} \times U$.

(2) $\mathrm{Supp}(\theta) \subset T$, where T is an open set in $X \times U$ such that

$$\{0\} \times U \subset T \subset \left\{(z,t) \in X \times U \;\middle|\; |z| < 3^{-1}\min\{1\,,\,\mathrm{dist}(t, M \setminus U)\}\right\}.$$

Here, $\mathrm{dist}(t, M\backslash U)$ means the distance from t to $M\backslash U$ in M, and we set $\mathrm{dist}(t,\varnothing) = +\infty$.

Definition 3.43 ([2, Theorem 8.1]) Let $\tau = (\tau_1, \tau_{01}) \in \mathscr{E}^{(0,n)}_X(X, X \setminus M)$ be a Čech-Dolbeault representative of Dirac's delta function. For any $u \in \mathscr{D}b_M(U)$, we define

$$\iota^{CD}_{\mathscr{D}b_M(U)}(u) = \left[\left(\int_U (\theta\tau_1 + \bar{\partial}_z\theta \wedge \tau_{01})(z-t,t)u(t)dt, \int_U (\theta\tau_{01})(z-t,t)u(t)dt\right)\right].$$

Proposition 3.44 ([2, Lemma 8.19]) $\iota^{CD}_{\mathscr{D}b_M} = \{\iota^{CD}_{\mathscr{D}b_M(U)}\}_{U \in \mathrm{Op}(M)}$ is a sheaf morphism from $\mathscr{D}b_M$ to $\mathscr{B}_M$.

We omit the proof of this proposition. For more details, see Section 8 of Honda-Izawa-Suwa[2]. Thanks to Proposition 3.44, by gluing the embedding morphisms on each local chart, we can obtain the embedding morphism $\iota^{CD}_{\mathscr{D}b_M} : \mathscr{D}b_M \to \mathscr{B}_M$ for a real analytic manifold M.

4. Embedding of C^∞ functions

4.1. Limits of $(n-1)$-forms

Let M be a real analytic, n-dimensional, oriented manifold and X its complexification. We define a normal bundle $T_M X$ by $\mathrm{Coker}\,(TM \to M \underset{X}{\times} TX)$ as a vector bundle, and we write the coordinate of $T_M X$ as (x,η). Moreover, we equip $T_M X$ with a bundle metric and fix it in what follows. By the tubular neighborhood theorem, we can find an isomorphism $T_M X \simeq X$ with the following commutative diagram:

$$\begin{array}{ccc} T_M X & \xrightarrow{\sim} & X. \\ \uparrow & \nearrow & \\ M & & \end{array}$$

Here $M \to T_M X$ is the embedding which regards M as the zero section of $T_M X$ and $M \to X$ is the closed embedding. Note that we replace X with a small open neighborhood of M in X if necessary, and that we identify X with $T_M X$ in this subsection.

We define $B = \left\{(x,\eta) \in T_M X = X \;\middle|\; |\eta| \le 1\right\}$ and $S = \partial B$. Let $\tau \colon S \to M$ be the restriction of the canonical projection $X \simeq T_M X \to M$, and let i be an embedding

$S \hookrightarrow X \setminus M$. Set $S_x = S \cap \tau^{-1}(x)$ for $x \in M$. The projection $\tau : S \to M$ induces maps

$$\tau_* : TS \to S \underset{M}{\times} TM \quad \text{and} \quad \tau^* : S \underset{M}{\times} T^*M \to T^*S.$$

For any $\lambda > 0$ and for any $(x, \eta) \in T_M X \simeq X$, let us define a map $\phi_\lambda : X \setminus M \to X \setminus M$ by $\phi_\lambda(x, \eta) = (x, \lambda\eta)$. We note that the $\operatorname{Ker}\tau_*$ consists of a vector field on S which is tangent to S_x for any $x \in M$.

Definition 4.1 For any $\mu \in \bigwedge^{n-1} T^*(X \setminus M)$ and for any $v \in \bigwedge^{n-1} \operatorname{Ker}\tau_*$, if the limit

$$\lim_{\lambda \to +0} \langle i^* \phi_\lambda^* \mu, v \rangle$$

exists and converges locally uniformly on S, we say that μ converges locally uniformly on S. Then, there exists a form $\tilde{\mu} \in \bigwedge^{n-1} \operatorname{Coker}\tau^*$ such that

$$\langle \tilde{\mu}, v \rangle = \lim_{\lambda \to +0} \langle i^* \phi_\lambda^* \mu, v \rangle$$

for any $v \in \bigwedge^{n-1} \operatorname{Ker}\tau_*$. We define $L(\mu) = -2^{n-1}\tilde{\mu}$.

Since a form in $\operatorname{Im}\tau^* \subset T^*S$ becomes zero on $\operatorname{Ker}\tau_* \subset TS$, we can regard $L(\mu)|_{S_x}$ as an $(n-1)$-form on S_x. Hence we may consider the integration of $L(\mu)$ on S_x and $I(\mu)(x) = \displaystyle\int_{S_x} L(\mu)$ is a continuous function on M. By a system of local coordinates, we can describe Definition 4.1 concretely as follows:

Remark 4.2 We identify S with $M \times \sqrt{-1}S^{n-1}$ and take $\omega = (\omega_1, \cdots, \omega_n)$ as a system of homogeneous coordinates of S^{n-1}. We define $i_x : S_x \to X \setminus M$ by

$$\begin{array}{cccc} i_x : & S_x & \longrightarrow & X \setminus M \\ & \Psi & & \Psi \\ & \omega & \longmapsto & x + \sqrt{-1}\omega \end{array}$$

for any $x \in M$. Let us take

$$\mu(z) = \sum_{|I|+|J|=n-1} f_{I,J}(z)\, dz_I \wedge d\bar{z}_J \in \bigwedge^{n-1} T^*(X \setminus M)\,,$$

and we define $\varepsilon_{I,J}$ by

$$\varepsilon_{I,J} = \begin{cases} \operatorname{sgn}\begin{pmatrix} i_1 \cdots i_p & j_1 & \cdots & j_q \\ k_1 \cdots k_p & k_{p+1} & \cdots & k_{n-1} \end{pmatrix} & \text{if } i_1, \cdots, i_p, j_1, \cdots, j_q \text{ are mutually distinct,} \\ 0 & \text{otherwise,} \end{cases}$$

where $p, q \in \{0, 1, 2, \cdots, n\}$, $p + q = n - 1$, $I = (i_1, \cdots, i_p) \in \{1, 2, \cdots, n\}^p$, $J = (j_1, \cdots, j_q) \in \{1, 2, \cdots, n\}^q$, $dz_I = dz_{i_1} \wedge \cdots \wedge dz_{i_p}$, $d\bar{z}_J = d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_q}$, $\{k_1, \cdots, k_{n-1}\} = I \cup J$ and $k_1 < k_2 < \cdots < k_{n-1}$.

If the limit

$$\begin{aligned}&\lim_{\lambda\to+0}(i_x^*\circ\phi_\lambda^*(\mu))(\omega)\\&=\lim_{\lambda\to+0}(\sqrt{-1}\,\lambda)^{n-1}\sum_{\substack{|I|+|J|=n-1,\\ k\in\{1,\cdots,n\}\setminus(I\cup J)}}(-1)^{|J|+k+1}\,\varepsilon_{I,J}\,\omega_k\,f_{I,J}(x+\sqrt{-1}\lambda\omega)\end{aligned}$$

exists and converges to a continuous function locally uniformly on S for any $(x,\sqrt{-1}\omega)\in M\times\sqrt{-1}S^{n-1}$, we say that μ converges locally uniformly on S. Then $L(\mu)$ exists and it is given by

$$\begin{aligned}L(\mu)(x,\omega)&=-2^{n-1}\lim_{\lambda\to+0}(i_x^*\circ\phi_\lambda^*(\mu))(\omega)\\&=\lim_{\lambda\to+0}(2\sqrt{-1}\,\lambda)^{n-1}\sum_{\substack{|I|+|J|=n-1,\\ k\in\{1,\cdots,n\}\setminus(I\cup J)}}(-1)^{|J|+k}\,\varepsilon_{I,J}\,\omega_k\,f_{I,J}(x+\sqrt{-1}\lambda\omega)\,ds.\end{aligned}$$

Here $ds=\sum_{i=1}^{n}(-1)^{i+1}\omega_i\,d\omega_1\wedge\cdots\wedge\widehat{d\omega_i}\wedge\cdots\wedge d\omega_n$.

4.2. Reconstruction of C^∞ functions with compact support

Let $M=\mathbb{R}^n$ and $X=\mathbb{C}^n$, and let $U\in\mathrm{Op}(M)$ and $V\in\mathrm{Op}(X)$ such that V is a complex neighborhood of U. Since $M=\mathbb{R}^n$ is oriented, we omit the section of the relative orientation sheaf in the definition of $\mathscr{B}_M(U)$.

Theorem 4.3 Let K be a compact set in U, $f(x)\in C^n(U)$ whose support is contained in K, and let $\mu=(\mu_1,\mu_{01})\in\mathscr{E}_X^{(0,n)}(V,V\setminus K)$ be a representative of $\iota^{CD}_{\mathscr{D}b_M(U)}(f)$. We define

$$\widetilde{\mathcal{R}}(z,\zeta)=\int_D W(z-w,\zeta)\,\mu_1(w)\wedge dw-\int_{\partial D}W(z-w,\zeta)\,\mu_{01}(w)\wedge dw,$$

where D is an open set with C^∞ boundary satisfying $K\subset D\subset\overline{D}\subset V$ and $z\notin\overline{D}$. Then, $\widetilde{\mathcal{R}}$ does not depend on the choices of a representative μ and a domain D, and satisfies the following conditions:

(1) For any proper convex open cone $\Gamma\subset\mathbb{R}^n$, $\widetilde{\mathcal{R}}$ is a holomorphic function of (z,ζ) in the infinitesimal wedge of type $\mathbb{R}^n\times\mathrm{Int}(\Gamma^\circ)+\sqrt{-1}\Delta 0$ where Δ is defined in Lemma 3.33.

(2) $\widetilde{\mathcal{R}}$ can be analytically continued to a neighborhood of $(U\setminus K)\times(\mathbb{R}^n\setminus\{0\})$.

(3) $\widetilde{\mathcal{R}}$ can be continuously extended to $U\times(\mathbb{R}^n\setminus\{0\})$, and the continuously extended function $\widetilde{\mathcal{R}}(x,\xi)$ satisfies

$$f(x)=\int_{S^{n-1}}\widetilde{\mathcal{R}}(x,\xi)\,d\xi.$$

Proof. Firstly, we show that $\mu\in\mathrm{Im}\,\bar\vartheta\Rightarrow\widetilde{\mathcal{R}}=0$. By the assumption, there exists (τ_1,τ_{01}) such that $\mu_1=\bar\partial\tau_1$ on V, $\mu_{01}=\tau_1-\bar\partial\tau_{01}$ on $V\setminus K$. Then, we have the following calculation:

$$\begin{aligned}\widetilde{\mathcal{R}}(z,\zeta) &= \int_D W(z-w,\zeta)\,\bar\partial\tau_1(w)\wedge dw - \int_{\partial D} W(z-w,\zeta)\,(\tau_1(w)-\bar\partial\tau_{01}(w))\wedge dw\\ &= \int_{\partial D} W(z-w,\zeta)\,\bar\partial\tau_{01}(w)\wedge dw \qquad\qquad (\text{the Stokes formula and } d=\partial+\bar\partial)\\ &= 0 \qquad\qquad (\partial\partial D=\varnothing).\end{aligned}$$

Hence $\widetilde{\mathcal{R}}$ does not depend on the choice of representatives.

Secondly, we prove that $\widetilde{\mathcal{R}}$ does not depend on a domain D. Let $K\subset D'\subset D\subset V$ and $z\notin \overline{D}$. Since

$$\begin{aligned}\widetilde{\mathcal{R}}(z,\zeta) &= \int_D W(z-w,\zeta)\,\mu_1(w)\wedge dw - \int_{\partial D} W(z-w,\zeta)\,\mu_{01}(w)\wedge dw\\ &= \int_{D'} W(z-w,\zeta)\,\mu_1(w)\wedge dw - \int_{\partial D'} W(z-w,\zeta)\,\mu_{01}(w)\wedge dw\\ &\quad + \int_{D\setminus D'} W(z-w,\zeta)\,\mu_1(w)\wedge dw - \int_{\partial(D\setminus D')} W(z-w,\zeta)\,\mu_{01}(w)\wedge dw,\end{aligned}$$

it is sufficient to prove that

$$\int_{D\setminus D'} W(z-w,\zeta)\,\mu_1(w)\wedge dw = \int_{\partial(D\setminus D')} W(z-w,\zeta)\,\mu_{01}(w)\wedge dw.$$

This equation follows from $\bar\partial\mu_{01}=\mu_1$ on $D\setminus D'\subset V\setminus K$ and the Stokes formula.

Finally, let us prove

$$\widetilde{\mathcal{R}}(z,\zeta) = \int_U W(z-t,\zeta)f(t)dt\,.$$

We choose Bochner-Martinelli type $(0,-(-1)^{\frac{n(n+1)}{2}}\beta)$ as a representative of Dirac's δ, where β is defined by

$$\beta(z) = (-1)^{\frac{n(n-1)}{2}}\frac{(n-1)!}{(2\pi\sqrt{-1})^n}\frac{1}{\|z\|^{2n}}\sum_{i=1}^n(-1)^{i-1}\bar z_i\,d\bar z_1\wedge\cdots\wedge\widehat{d\bar z_i}\wedge\cdots\wedge d\bar z_n.$$

We take $\theta(z,t)\in C^\infty(V\times U)$ satisfying the conditions of Definition 3.43. By the definition of $\iota^{CD}_{\mathscr{D}b_M(U)}(f)$, we can choose a representative μ as follows:

$$\mu = -(-1)^{\frac{n(n+1)}{2}}\left(\int_U f(t)\bar\partial_z\theta(z-t,t)\wedge\beta(z-t)\,dt\,,\ \int_U f(t)\theta(z-t,t)\beta(z-t)\,dt\right).$$

Using the Bochner-Martinelli formula (Lemma A.5), we get

$$\begin{aligned}\widetilde{\mathcal{R}}(z,\zeta) &= -(-1)^{\frac{n(n+1)}{2}}\int_D W(z-w,\zeta)\left(\int_U f(t)\bar\partial_w\theta(w-t,t)\wedge\beta(w-t)\,dt\right)\wedge dw\\ &\quad + (-1)^{\frac{n(n+1)}{2}}\int_{\partial D} W(z-w,\zeta)\left(\int_U f(t)\theta(w-t,t)\beta(w-t)\,dt\right)\wedge dw\end{aligned}$$

$$= \int_U f(t)\,dt \left((-1)^{\frac{n(n+1)}{2}} \int_{\partial D} W(z-w,\zeta)\theta(w-t,t)\beta(w-t) \wedge dw \right.$$
$$\left. -(-1)^{\frac{n(n+1)}{2}} \int_D W(z-w,\zeta)\bar{\partial}_w \theta(w-t,t) \wedge \beta(w-t) \wedge dw \right)$$
$$= \int_U f(t) W(z-t,\zeta)\theta(0,t)\,dt \quad = \int_U f(t) W(z-t,\zeta)\,dt.$$

By Lemma 3.34, $f(x) = \displaystyle\int_{S^{n-1}} \widetilde{\mathcal{R}}(x,\xi)\,d\xi$ holds. □

Definition 4.4 Let K be a compact set in U. For any $u = [\mu] = [(\mu_1, \mu_{01})] \in \mathscr{B}_K(U)$, we define

$$\widetilde{\mathcal{R}}[\mu](z,\zeta) = \int_D W(z-w,\zeta)\,\mu_1(w) \wedge dw - \int_{\partial D} W(z-w,\zeta)\,\mu_{01}(w) \wedge dw,$$

where D is an open set with C^∞ boundary satisfying $K \subset D \subset \overline{D} \subset V$ and $z \notin \overline{D}$.

Since $\widetilde{\mathcal{R}}[\mu]$ does not depend on the choice of μ by the same arguments as in the proof of Theorem 4.3, for any $u = [\mu] \in \mathscr{B}_K(U)$, we write $\widetilde{\mathcal{R}}[u] = \widetilde{\mathcal{R}}[\mu]$. Recall that $M = \mathbb{R}^n$, $X = \mathbb{C}^n$ and $U \in \mathrm{Op}(M)$.

Lemma 4.5 We assume that Γ_i is defined as in Subsection 3.3 for any $i = 0, 1, \cdots, n$. Let K be a compact set in U, V a Stein neighborhood of U and $u = [\mu] = [(\mu_1, \mu_{01})] \in \mathscr{B}_K(U)$ ($\mu \in \mathscr{E}_X^{(0,n)}(V, V \setminus K)$). We define $\widetilde{\mathcal{R}}[\mu](z, \Gamma_i^\circ)$ by

$$\widetilde{\mathcal{R}}[\mu](z,\Gamma_i^\circ) = \int_{\Gamma_i^\circ \cap S^{n-1}} \widetilde{\mathcal{R}}[\mu](z,\xi)\,d\xi \in \mathscr{O}_X(U + \sqrt{-1}\Gamma_i 0).$$

Then, $\displaystyle\bigoplus_{i=0}^{n} (-1)^i \widetilde{\mathcal{R}}[\mu](z,\Gamma_i^\circ)$ is a Čech representative of u, and the morphism

$$\mathcal{R} : H^{0,n}_{\bar\vartheta}(V, V \setminus K) \otimes or_{M/X}(U) \to \hat{H}(\{U + \sqrt{-1}\Gamma_i 0\}_{i=0}^n; \mathscr{O}_X) \otimes or_{M/X}(U)$$
$$\simeq H^n(\mathcal{V}, \mathcal{V}'; \mathscr{O}_X) \otimes or_{M/X}(U)$$

defined by

$$\mathcal{R}(u) = \mathcal{R}(\mu) = \left[\bigoplus_{i=0}^{n} (-1)^i \widetilde{\mathcal{R}}[\mu](z,\Gamma_i^\circ) \right]$$

satisfies $b_U \circ \mathcal{R} = \mathrm{id}$ on $\mathscr{B}_K(U)$, where b_U is given in Definition 3.31.

Proof. Since $\widetilde{\mathcal{R}}[\mu](z,\Gamma_i^\circ)$ can be analytically continued to a complex neighborhood V' of $U \setminus K$ and $\displaystyle\sum_{i=0}^{n} (-1)^i \widetilde{\mathcal{R}}[\mu](z,\Gamma_i^\circ) = 0$ holds in V', for any $\psi \in \mathscr{A}_M[K]$, we can perform the integration of the hyperfunction $\psi\mathcal{R}(u)$ as the one defined by Kaneko (see Section 3.4 of [5]). Let us show

$$\langle \mathcal{R}(\mu), \psi dx \rangle = \langle \mu, \psi dx \rangle \quad (\psi \in \mathscr{A}_M[K]).$$

Here, the above symbols $\langle \bullet, \bullet \rangle$ on the left-hand side and the right-hand side are cohomological pairings realized by the integrations of Čech cohomology and Čech-Dolbeault cohomology, respectively.

Let $V'' \in \mathrm{Op}(X)$ be the intersection of V and the domain of $\psi \in \mathscr{A}_M[K]$, and let W_i ($i = 0, \cdots, n$) be the infinitesimal wedge of type $U + \sqrt{-1}\Gamma_i 0$ on which $\widetilde{\mathcal{R}}[\mu](z, \Gamma_i^\circ)$ is defined. By the definition of the integration of Čech representation, we have

$$
\begin{aligned}
&\langle \mathcal{R}(\mu), \psi dx \rangle \\
&= \sum_i \int_{(-1)^i (U' + \sqrt{-1}\,\varepsilon_i)} \psi(z)\, (-1)^i\, \widetilde{\mathcal{R}}[\mu](z, \Gamma_i^\circ)\, dz \\
&= \sum_i \int_{U' + \sqrt{-1}\,\varepsilon_i} \psi(z)\, \widetilde{\mathcal{R}}[\mu](z, \Gamma_i^\circ)\, dz \\
&= \sum_i \int_{U' + \sqrt{-1}\,\varepsilon_i} \psi(z) \int_{\Gamma_i^\circ \cap S^{n-1}} \Big(\int_D W(z - w, \xi)\, \mu_1(w) \wedge dw \\
&\qquad\qquad\qquad\qquad - \int_{\partial D} W(z - w, \xi)\, \mu_{01}(w) \wedge dw \Big)\, d\xi\, dz \\
&= \int_D \Big(\sum_i \int_{\Gamma_i^\circ \cap S^{n-1}} \int_{U' + \sqrt{-1}\,\varepsilon_i} \psi(z)\, W(z - w, \xi)\, dz\, d\xi \Big)\, \mu_1(w) \wedge dw \\
&\quad - \int_{\partial D} \Big(\sum_i \int_{\Gamma_i^\circ \cap S^{n-1}} \int_{U' + \sqrt{-1}\,\varepsilon_i} \psi(z)\, W(z - w, \xi)\, dz\, d\xi \Big)\, \mu_{01}(w) \wedge dw \\
&= \int_D \psi(w)\, \mu_1(w) \wedge dw - \int_{\partial D} \psi(w)\, \mu_{01}(w) \wedge dw \\
&= \langle \mu, \psi dx \rangle\,,
\end{aligned}
$$

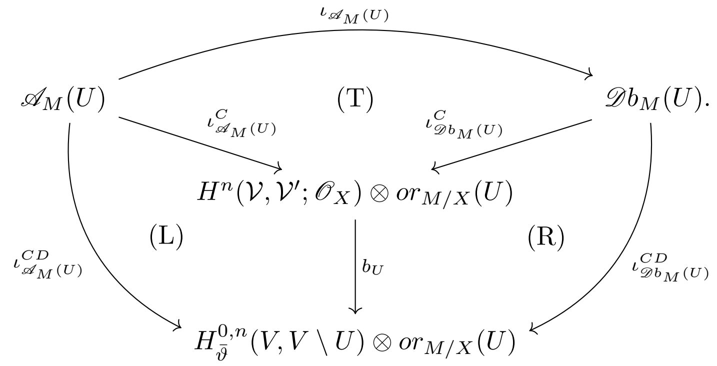
where $U' \,(\supset K\,)$ is an open set in U such that $U' \subset\subset U$, $\varepsilon_i : U \to \mathbb{R}^n$ is a smooth mapping such that

$$
\begin{cases}
\varepsilon_i(t) = 0 & (t \in U \setminus U'), \\
t + \sqrt{-1}\, r\, \varepsilon_i(t) \in (U' + \sqrt{-1}\Gamma_i) \cap W_i \cap V'' & (t \in U',\ 0 < r \le 1),
\end{cases}
$$

$U' + \sqrt{-1}\,\varepsilon_i = \{t + \sqrt{-1}\,\varepsilon_i(t) \mid t \in U'\}$ and $-(U' + \sqrt{-1}\,\varepsilon_i)$ is just $U' + \sqrt{-1}\,\varepsilon_i$ with the opposite orientation, and $D \in \mathrm{Op}(X)$ satisfies $K \subset D \subset \overline{D} \subset V''$ and $(U' + \sqrt{-1}\,\varepsilon_i) \cap D = \emptyset$.

Since $\mathscr{B}_K(U)$ is the dual topological vector space of $\mathscr{A}_M[K]$ by Theorem 3.7, $\langle \mathcal{R}(\mu), \psi dx \rangle = \langle \mu, \psi dx \rangle$ means that $\mathcal{R}(\mu)$ and μ represent the same hyperfunction. Because b_U is isomorphic, we get $b_U \circ \mathcal{R} = \mathrm{id}$ on $\mathscr{B}_K(U)$. □

Corollary 4.6 Let $\iota_{\mathscr{A}_M(U)} : \mathscr{A}_M(U) \to \mathscr{D}b_M(U)$ be the canonical embedding. The diagram below commutes.



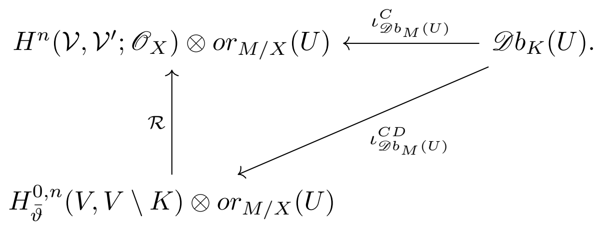
Note that we identify $H^n(\mathcal{V},\mathcal{V}';\mathscr{O}_X)$ with $\hat{H}(\{U+\sqrt{-1}\Gamma_i 0\}_{i=0}^n;\mathscr{O}_X)$ by Theorem 3.28.

Proof. By Proposition 3.42, (T) commutes. The commutativity of (L) is obvious from the definition. Let us prove that (R) commutes. Since all morphisms are induced from the sheaf morphisms and $\mathscr{D}b_M$ is a soft sheaf, it is enough to prove the claim locally. Hence, we may consider only sections with compact support.

Let U be an open set in M, K a compact set in U and u a section of $\mathscr{D}b_K(U)$. Using the same arguments as in the proof of Theorem 4.3, we have

$$\widetilde{\mathcal{R}}[\mu](z,\Gamma_i^\circ)=\langle u(t),W(z-t,\Gamma_i^\circ)dt\rangle \quad (i=0,1,\cdots,n),$$

where μ is a Čech-Dolbeault representative of $\iota^{CD}_{\mathscr{D}b_M(U)}(u)$. By Definition 3.37, the diagram below commutes.



Then, we obtain the commutativity of (R) by Lemma 4.5. □

By Corollary 4.6, $\iota^{CD}_{\mathscr{D}b_M}\big|_{\mathscr{A}_M}$ coincides with $\iota^{CD}_{\mathscr{A}_M}$.

4.3. Inverse map of embedding of C^∞ functions

Let us consider the characterization of $\iota^{CD}_{\mathscr{D}b_M}(C^\infty)$ and the inverse map from $\iota^{CD}_{\mathscr{D}b_M}(C^\infty)$ to C^∞.

Let M be a real analytic, n-dimensional, oriented manifold and X its complexification, and let $U\in \mathrm{Op}(M)$ and $V\in\mathrm{Op}(X)$ such that V is a complex neighborhood of U. Since M is oriented, we omit the section of the relative orientation sheaf in the definition of $\mathscr{B}_M(U)$.

Throughout this subsection, we use the same notations as in Subsection 4.1: We write $B=\{(x,\eta)\in X \mid |\eta|\le 1\}$, $S=\partial B$ and $S_x=\partial B\cap\tau^{-1}(x)$ for any $x\in U$. We may assume $B \subset V$. Remember that the projection $\tau : S \to U$ is associated with the projection $V \simeq T_U V \to U$ which is given by the tubular neighborhood theorem. The projection $\tau : S \to U$ induces maps

$$\tau_* : TS \to S \underset{U}{\times} TU \quad \text{and} \quad \tau^* : S \underset{U}{\times} T^*U \to T^*S.$$

In what follows, since the problem is local, we regard V (resp. U) as X (resp. M). Note that we can regard S_x and S as S^{n-1} and $U \times \sqrt{-1}S^{n-1}$, respectively.

Definition 4.7 A form $\mu_{01} \in \mathscr{E}_X^{(0,n-1)}(V \setminus U)$ is said to belong to $\mathscr{E}_X^{\mathrm{qw},(0,n-1)}(V \setminus U)$ if the following conditions are satisfied.

(1) $\mu_{01} \in \bigwedge^{n-1} T^*(V \setminus U)$ converges to a continuous form $L(\mu_{01}) \in \bigwedge^{n-1} \operatorname{Coker} \tau^*$ locally uniformly on S.

(2) $L(\mu_{01})$ is a C^∞ form on S.

Here, $L(\mu_{01})$ is defined in Subsection 4.1.

Note that we call $\mathscr{E}_X^{\mathrm{qw},(0,n-1)}(V \setminus U)$ a space of quasi-Whitney $(0, n-1)$-forms. By Remark 4.2, we can rephrase Definition 4.7 as follows:

Remark 4.8 A form $\mu_{01} \in \mathscr{E}_X^{(0,n-1)}(V \setminus U)$ is said to belong to $\mathscr{E}_X^{\mathrm{qw},(0,n-1)}(V \setminus U)$ if the following conditions are satisfied.

(1) For any $x \in U$ and $\omega \in S^{n-1}$, the limit

$$\lim_{\lambda \to +0} (-2\sqrt{-1}\,\lambda)^{n-1} \sum_{i=1}^{n} (-1)^i \omega_i f_i(x + \sqrt{-1}\lambda\omega)$$

exists and converges to a continuous function locally uniformly with respect to $(x, \sqrt{-1}\omega) \in U \times \sqrt{-1}S^{n-1}$, where $f_1, \cdots, f_n$ are coefficients of μ_{01}, that is,

$$\mu_{01} = \sum_{i=1}^{n} f_i \, d\bar{z}_1 \wedge \cdots \wedge \widehat{d\bar{z}_i} \wedge \cdots \wedge d\bar{z}_n.$$

(2) $L(\mu_{01})$ is a C^∞ form on $U \times \sqrt{-1}S^{n-1}$.

Recall that we can write

$$L(\mu_{01})(x, \omega) = \lim_{\lambda \to +0} (-2\sqrt{-1}\,\lambda)^{n-1} \sum_{i=1}^{n} (-1)^i \, \omega_i \, f_i(x + \sqrt{-1}\lambda\omega) \, ds, \tag{4.1}$$

and $ds = \sum_{i=1}^{n} (-1)^{i+1} \omega_i \, d\omega_1 \wedge \cdots \wedge \widehat{d\omega_i} \wedge \cdots \wedge d\omega_n$.

Definition 4.9 Let us define

$$\mathscr{E}_X^{\mathrm{qw},(0,n)}(V, V \setminus U) = \left\{ \mu = (\mu_1, \mu_{01}) \in \mathscr{E}_X^{(0,n)}(V, V \setminus U) \;\middle|\; \mu_{01} \in \mathscr{E}_X^{\mathrm{qw},(0,n-1)}(V \setminus U) \right\}.$$

The following theorem is our main result.

Theorem 4.10 Let $x_0 \in M$. There exist an open neighborhood U of x_0 and a complex neighborhood V of U for which the following 2 conditions are equivalent. For any $u \in \mathscr{B}_M(U)$,

(1) There is a C^∞ function f on U such that $u = \iota^{CD}_{\mathscr{D}b_M(U)}(f)$.

(2) There is a Čech-Dolbeault representative $\mu = (\mu_1, \mu_{01})$ of u such that $\mu \in \mathscr{E}_X^{\mathrm{qw},(0,n)}(V, V \setminus U)$.

Additionally,

$$I(\mu)(x) = \int_{S_x} L(\mu_{01})(x, \omega) \in C^\infty(U)$$

is well-defined and we get $I(\mu) = f$.

We give a global version of Theorem 4.10 for $M = \mathbb{R}^n$.

Theorem 4.11 Assume $M = \mathbb{R}^n$ and $X = \mathbb{C}^n$. For any $U \in \mathrm{Op}(M)$, for any $V \in \mathrm{Op}(X)$ which is a complex neighborhood of U and for any $u \in \mathscr{B}_M(U)$, the conditions (1) and (2) as in Theorem 4.10 are equivalent. Additionally, we define $I(\mu)$ as the same integration in Theorem 4.10. Then, $I(\mu)$ is well-defined and $I(\mu) = f$ holds.

Considering a local coordinate system, it is enough to prove Theorem 4.11 for Theorem 4.10. To show Theorem 4.11, we prove the following 3 lemmas (Lemma 4.12, Lemma 4.13 and Lemma 4.14). Hereafter, let us assume $M = \mathbb{R}^n$, $X = \mathbb{C}^n$, and let $U \in \mathrm{Op}(M)$ and $V \in \mathrm{Op}(X)$ such that V is a complex neighborhood of U.

Lemma 4.12 Let u be a hyperfunction on U, and let μ and μ' be representatives of u. If μ, $\mu' \in \mathscr{E}_X^{\mathrm{qw},(0,n)}(V, V \setminus U)$, then $I(\mu) = I(\mu')$.

Proof. Fix $x_0 \in U$. Let us prove that there exists an open neighborhood U_0 of x_0 such that $I(\mu) = I(\mu')$ on U_0. Let $x_0 \in V'' \subset\subset V' \subset\subset V$, $U'' = V'' \cap U$, and $U' = V' \cap U$. We can choose representatives $\tilde{\mu}$ and $\tilde{\mu}'$ satisfying the following conditions (see Lemma A.4):

(1) $\tilde{\mu}$ and $\tilde{\mu}'$ determine the same hyperfunction on U.

(2) $\tilde{\mu} = \mu$ on V'' and $\tilde{\mu}' = \mu'$ on V''.

(3) $\mathrm{Supp}\,\tilde{\mu} \subset V'$ and $\mathrm{Supp}\,\tilde{\mu}' \subset V'$.

It is clear that $\tilde{\mu}|_{V''}, \tilde{\mu}'|_{V''} \in \mathscr{E}_X^{\mathrm{qw},(0,n)}(V'', V'' \setminus U'')$. Then, we set

$$\widetilde{\mathcal{R}}[\tilde{\mu}](z, \zeta) = \int_D W(z-w, \zeta)\, \tilde{\mu}_1(w) \wedge dw - \int_{\partial D} W(z-w, \zeta)\, \tilde{\mu}_{01}(w) \wedge dw,$$
$$\widetilde{\mathcal{R}}[\tilde{\mu}'](z, \zeta) = \int_D W(z-w, \zeta)\, \tilde{\mu}'_1(w) \wedge dw - \int_{\partial D} W(z-w, \zeta)\, \tilde{\mu}'_{01}(w) \wedge dw,$$

with the notations of Theorem 4.3. We may assume $V'' \subset\subset D$. Since $\tilde{\mu}$ and $\tilde{\mu}'$ satisfy the condition (1), we can prove $\widetilde{\mathcal{R}}[\tilde{\mu}](z, \zeta) = \widetilde{\mathcal{R}}[\tilde{\mu}'](z, \zeta)$ in the same way as the proof of Theorem 4.3. Hence, it is enough to prove that there exists an open neighborhood

$U_0 \subset U''$ of x_0 such that $\widetilde{\mathcal{R}}[\tilde{\mu}](z,\zeta)$ can be continuously extended to $U_0 \times (\mathbb{R}^n \setminus \{0\})$ and we have

$$\int_{S^{n-1}} \widetilde{\mathcal{R}}[\tilde{\mu}](x,\xi) d\xi = I(\mu)(x) \qquad (x \in U_0).$$

Let D' be an open subset of V such that $D \cap (U \setminus U'') \subset D'$ and $x_0 \in \mathrm{Int}(U'' \setminus D')$. We define D'' as an open subset of U'' such that $D'' = \mathrm{Int}(U'' \setminus D')$. By deforming D, we get

$$\begin{aligned}\widetilde{\mathcal{R}}[\tilde{\mu}](z,\zeta) = &\int_{D'} W(z-w,\zeta)\, \tilde{\mu}_1(w) \wedge dw - \int_{\partial D'} W(z-w,\zeta)\, \tilde{\mu}_{01}(w) \wedge dw \\ &+ \int_{D''} W(z-t,\zeta) \left(\int_{S^{n-1}} L(\mu_{01})(t,\omega) \right) dt,\end{aligned}$$

for any $z \notin \overline{D' \cup D''}$. Note that, by $\tilde{\mu} = \mu$ on V'', we have $L(\tilde{\mu}_{01})(t,\omega) = L(\mu_{01})(t,\omega)$ $(t \in D'',\ \omega \in S^{n-1})$. The deformation of D is illustrated in Figure 1.

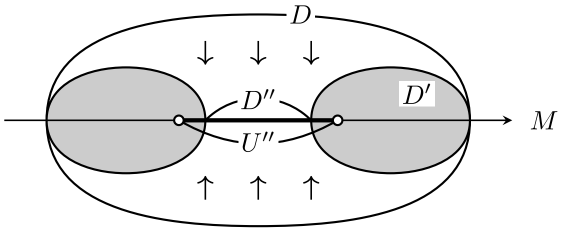


Figure 1. Deformation of the integration domain D

From Lemma 3.33, the first and the second integrals can be analytically continued to a complex neighborhood of D'', and the third integral can be continuously extended to $D'' \times (\mathbb{R}^n \setminus \{0\})$ by Theorem 4.3 (3). Hence, for any $x \in D''$, the equation

$$\begin{aligned}\int_{S^{n-1}} \widetilde{\mathcal{R}}[\tilde{\mu}](x,\xi)\, d\xi = &\int_{S^{n-1}} \left(\int_{D'} W(x-w,\xi)\, \tilde{\mu}_1(w) \wedge dw \right) d\xi \\ &- \int_{S^{n-1}} \left(\int_{\partial D'} W(x-w,\xi)\, \tilde{\mu}_{01}(w) \wedge dw \right) d\xi \\ &+ \int_{S^{n-1}} \int_{D''} W(x-t,\xi) \left(\int_{S^{n-1}} L(\mu_{01})(t,\omega) \right) dt\, d\xi\end{aligned}$$

holds. By Lemma 3.35, the first and the second integrals become 0. Then, we get the equation

$$\int_{S^{n-1}} \widetilde{\mathcal{R}}[\tilde{\mu}](x,\xi)\, d\xi = \int_{S^{n-1}} \int_{D''} W(x-t,\xi)\, I(\mu)(t)\, dt\, d\xi \qquad (x \in D'').$$

By applying Lemmas 3.34 and 3.35 to the case $f = I(\mu)$, we obtain

$$\int_{S^{n-1}} \int_{D''} W(x-t,\xi)\, I(\mu)(t)\, dt\, d\xi = I(\mu)(x) \qquad (x \in D'').$$

This completes the proof. □

By Lemma 4.12, $I(\mu)$ does not depend on the choice of $\mu \in \mathscr{E}_X^{\mathrm{qw},(0,n)}(V, V \setminus U)$. If a hyperfunction u is represented by an element μ of $\mathscr{E}_X^{\mathrm{qw},(0,n)}(V, V \setminus U)$, we write $I(u)$ for $I(\mu)$ hereafter. We can regard I as a map

$$I : \frac{\mathscr{E}_X^{\mathrm{qw},(0,n)}(V, V \setminus U)}{\mathscr{E}_X^{\mathrm{qw},(0,n)}(V, V \setminus U) \cap \bar{\vartheta}(\mathscr{E}_X^{(0,n-1)}(V, V \setminus U))} \longrightarrow C^\infty(U) \ .$$

Lemma 4.13 Let $f \in C^\infty(U)$. If $\iota^{CD}_{\mathscr{D}b_M(U)}(f)$ has a representative $\mu \in \mathscr{E}_X^{\mathrm{qw},(0,n)}(V, V \setminus U)$, then $I \circ \iota^{CD}_{\mathscr{D}b_M(U)}(f) = f$.

Proof. Let $x \in U'' \subset\subset U' \subset\subset U$ and $\psi \in C_0^\infty(U)$ such that

$$\psi(t) = \begin{cases} 1 & \text{on } U'', \\ 0 & \text{on } U \setminus U'. \end{cases}$$

We write complex neighborhoods of U'', U', and U as $V'' \subset\subset V' \subset\subset V$, respectively. By Lemma A.4, there exists a representative $\tilde{\mu}$ of $\iota^{CD}_{\mathscr{D}b_M(U)}(\psi f)$ such that

$$\tilde{\mu} = \mu \text{ on } V'' \quad \text{and} \quad \tilde{\mu} = 0 \text{ on } V \setminus V' .$$

Since $\mathrm{Supp}(\iota^{CD}_{\mathscr{D}b_M(U)}(\psi f)) \subset \overline{U'}$, it follows from Theorem 4.3 and Lemma 4.12 that

$$I(\mu)(x) = \int_{S^{n-1}} \widetilde{\mathcal{R}}[\tilde{\mu}](x, \xi)\, d\xi = \psi(x) f(x) = f(x)$$

hold for any $x \in U''$. This completes the proof. □

Remember that $U \in \mathrm{Op}(\mathbb{R}^n)$ and that $V \in \mathrm{Op}(\mathbb{C}^n)$ is a complex neighborhood of U.

Lemma 4.14 Let u be a hyperfunction on U. The following conditions are equivalent.

(i) There exists $f \in C^\infty(U)$ such that $u = \iota^{CD}_{\mathscr{D}b_M(U)}(f)$.

(ii) There exists a representative μ of u such that $\mu \in \mathscr{E}_X^{\mathrm{qw},(0,n)}(V, V \setminus U)$.

Proof. First, we prove (i) ⇒ (ii). We define $(\tau_1, \tau_{01}) = \left(0, \dfrac{(n-1)!\, \bar{\partial}\varphi_1 \wedge \cdots \wedge \bar{\partial}\varphi_{n-1}}{(2\pi\sqrt{-1})^n\, z_1 \cdots z_n}\right)$ as a representative of Dirac's delta function. Here $\varphi_i \in C^\infty(X \setminus M)$ $(i = 1, \cdots, n)$ is defined as follows:

> Let $\{\widetilde{\varphi}_i\}_{i=1}^n$ be the partition of unity of S^{n-1} such that $\widetilde{\varphi}_i = 0$ on a neighborhood of $S^{n-1} \cap \{y_i = 0\}$ $(i = 1, \cdots, n)$. Then, we define
>
> $$\varphi_i(x + \sqrt{-1} r\omega) = \widetilde{\varphi}_i(\omega)$$
>
> for any $x \in M$, $r > 0$, $\omega \in S^{n-1}$ such that $x + \sqrt{-1} r\omega \in X$.

This representative is explained in Example 7.26 of Honda-Izawa-Suwa[2]. By the definition of $\iota^{CD}_{\mathscr{D}b_M(U)}(f) = [\mu] = [(\mu_1, \mu_{01})]$, we obtain

$$\mu = \left(\int_U (\theta\tau_1 + \bar\partial_z \theta \wedge \tau_{01})(z-t,t) f(t) dt, \int_U (\theta\tau_{01})(z-t,t) f(t) dt \right).$$

Then, it is sufficient to prove that $\mu = (\mu_1, \mu_{01})$ satisfies $\mu_{01} \in \mathscr{E}_X^{\mathrm{qw},(0,n-1)}(V \setminus U)$. Note that we have

$$\mu_{01} = \frac{(n-1)!\, \bar\partial\varphi_1 \wedge \cdots \wedge \bar\partial\varphi_{n-1}}{(2\pi\sqrt{-1})^n} \int_U \frac{\theta(z-t,t) f(t)}{(z_1 - t_1)\cdots(z_n - t_n)} dt.$$

Let $z = x + \sqrt{-1}y$, $r = \sqrt{y_1^2 + \cdots + y_n^2}$ and

$$(\omega_1, \cdots, \omega_n) = \left(\frac{y_1}{r}, \cdots, \frac{y_n}{r}\right) \in S^{n-1} \subset \mathbb{R}^n.$$

The embedding $i : S^{n-1} \to \mathbb{R}^n$ induces the map

$$i_* : TS^{n-1} \to S^{n-1} \underset{\mathbb{R}^n}{\times} T\mathbb{R}^n,$$

that is, by considering the $(n-1)$-sphere S^{n-1} as a subset of $\mathbb{R}^n$, a vector field on S^{n-1} can be described by a vector field on $\mathbb{R}^n$. The space $\operatorname{Im} i_*$ is generated by

$$\frac{\partial}{\partial\omega_i} = \frac{\partial}{\partial y_i} - \frac{y_i}{r^2}\sum_{j=1}^n y_j \frac{\partial}{\partial y_j} \quad (i = 1, 2, \cdots, n)$$

with the relation

$$\omega_1 \frac{\partial}{\partial\omega_1} + \cdots + \omega_n \frac{\partial}{\partial\omega_n} = 0,$$

and we have $\rho_*\left(\dfrac{\partial}{\partial y_i}\right) = r^{-1}\dfrac{\partial}{\partial\omega_i}$, where $\rho : Y' = \mathbb{R}^n \setminus \{0\} \ni y \mapsto y/r \in S^{n-1}$ and $\rho_* : TY' \to Y' \underset{S^{n-1}}{\times} TS^{n-1}$ is induced by ρ. Under the above consideration, we define

$$J_n = \begin{pmatrix} \dfrac{\partial\varphi_1}{\partial\bar z_1} & \cdots & \dfrac{\partial\varphi_1}{\partial\bar z_n} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial\varphi_{n-1}}{\partial\bar z_1} & \cdots & \dfrac{\partial\varphi_{n-1}}{\partial\bar z_n} \end{pmatrix} \text{ and } \widetilde{J}_n = \begin{pmatrix} \dfrac{\partial\widetilde\varphi_1}{\partial\omega_1} & \cdots & \dfrac{\partial\widetilde\varphi_1}{\partial\omega_n} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial\widetilde\varphi_{n-1}}{\partial\omega_1} & \cdots & \dfrac{\partial\widetilde\varphi_{n-1}}{\partial\omega_n} \end{pmatrix}.$$

Note that $\varphi_i(z) = \widetilde\varphi_i(\omega)$. Additionally, an $(n-1)\times(n-1)$ matrix J_{ni} (resp. $\widetilde{J}_{ni}$) denotes the submatrix of J_n (resp. $\widetilde{J}_n$) from which the i-th column is omitted. The $(0, n-1)$-form $\bar\partial\varphi_1 \wedge \cdots \wedge \bar\partial\varphi_{n-1}$ is written as

$$\left(\sum_{i=1}^n \frac{\partial\varphi_1}{\partial\bar z_i} d\bar z_i\right) \wedge \cdots \wedge \left(\sum_{i=1}^n \frac{\partial\varphi_{n-1}}{\partial\bar z_i} d\bar z_i\right) = \sum_{i=1}^n \det J_{ni}\, d\bar z_1 \wedge \cdots \wedge \widehat{d\bar z_i} \wedge \cdots \wedge d\bar z_n.$$

Since $\det J_{ni} = (-2\sqrt{-1}r)^{-n+1} \det \widetilde{J}_{ni}$, the coefficient of $L(\mu_{01})$ is represented by

$$\lim_{r\to+0} \frac{(n-1)!}{(2\pi\sqrt{-1})^n} \sum_{i=1}^{n} (-1)^i \omega_i \det \widetilde{J}_{ni} \int_U \frac{\theta(x+\sqrt{-1}r\omega - t, t) f(t)}{(x_1+\sqrt{-1}r\omega_1 - t_1)\cdots(x_n+\sqrt{-1}r\omega_n - t_n)} dt.$$

Here, $L(\mu_{01})$ is defined by the equation (4.1) in Remark 4.8.

Let us prove that the coefficient of $L(\mu_{01})$ converges to a continuous function locally uniformly on $U \times \sqrt{-1}S^{n-1}$ (the condition (1) of Definition 4.7). The $\det \widetilde{J}_{ni}$ vanishes in a neighborhood of the set $\{y_1 y_2 \cdots y_n = 0\}$, since $\bar{\partial}\varphi_1 \wedge \cdots \wedge \bar{\partial}\varphi_{n-1}$ is 0 in a conic open neighborhood of the set $\{y_1 y_2 \cdots y_n = 0\}$. Therefore, we consider the above limit on $\{\omega_1 \cdots \omega_n \neq 0\}$. By Lebesgue's dominated convergence theorem, the integral

$$\int_U \frac{\partial^n(\theta(x-t,t)f(t))}{\partial t_1 \cdots \partial t_n} \log(x_1 - t_1) \cdots \log(x_n - t_n) dt$$

is continuous on $x \in U$. Then, as was in the proof of Corollary 2.3.2 of Kaneko[5], by employing the partial integration with respect to the variables $t_1, t_2, \cdots, t_n$ respectively, the coefficient of $L(\mu_{01})$ converges to a continuous function

$$\frac{(n-1)!}{(2\pi\sqrt{-1})^n} \sum_{i=1}^{n} (-1)^i \omega_i \det \widetilde{J}_{ni} \int_U \frac{\partial^n(\theta(x-t,t)f(t))}{\partial t_1 \cdots \partial t_n} \log(x_1 - t_1) \cdots \log(x_n - t_n) dt$$

with respect to $(x, \sqrt{-1}\omega) \in U \times \sqrt{-1}S^{n-1}$, which implies the condition (1) of Definition 4.7. By the repetition of partial integrations several times in the same way as above, we can also show that $L(\mu_{01})$ is a C^∞ form on $U \times \sqrt{-1}S^{n-1}$ (the condition (2) of Definition 4.7).

Next, let us prove (ii) $\Rightarrow$ (i). It suffices to show $\iota^{CD}_{\mathscr{D}b_M(U)} \circ I(\mu)$ and μ give the same equivalent class. Let U' be a sufficiently small open neighborhood of x, V' a complex neighborhood of U' such that $V' \subset V$, and $\chi_{U'}$ the characteristic function of U'. By considering $\chi_{U'} I(\mu)$ instead of $I(\mu)$ and noticing that $\iota^{CD}_{\mathscr{D}b_M(U)}$ is induced from a sheaf morphism, we have

$$\begin{aligned} \iota^{CD}_{\mathscr{D}b_M(U)}(I(\mu))\Big|_{U'} &= \iota^{CD}_{\mathscr{D}b_M(U)}(\chi_{U'} I(\mu))\Big|_{U'} \\ &= \left[\left(\int_{U'} (\theta\tau_1 + \bar{\partial}_z\theta \wedge \tau_{01})(z-t,t) I(\mu)(t) dt, \int_{U'} (\theta\tau_{01})(z-t,t) I(\mu)(t) dt\right)\right]\Bigg|_{U'}. \end{aligned}$$

Now we take (τ_1, τ_{01}) to be a representative of $b_M\left(\bigoplus_i (-1)^i \int_{\Gamma_i^\circ \cap S^{n-1}} W(z,\xi) d\xi\right)$. Note that b_M is defined in Definition 3.31 and that $\bigoplus_i (-1)^i \int_{\Gamma_i^\circ \cap S^{n-1}} W(z,\xi) d\xi$ is a Čech representative of Dirac's delta. Since V' is small enough, we can take the θ so that

$$(\tau_1(z), \tau_{01}(z)) = ((\theta\tau_1 + \bar{\partial}_z\theta \wedge \tau_{01})(z,t), \theta\tau_{01}(z,t)) \quad ((z,t) \in V' \times U')$$

holds. Then, in the definition of b_U (Definitions 3.22 and 3.31), we take the specific φ_i's which are independent of the variables x such as the ones given in the first part of this proof. For such a b_U, we can interchange the order of b_U and the integrations with respect to the variable t, and hence, we have

$$\iota^{CD}_{\mathscr{D}b_M(U)} \circ I(\mu)\Big|_{U'} = \left[b_{U'}\left(\bigoplus_i (-1)^i \int_{\Gamma_i^\circ \cap S^{n-1}} \int_{U'} W(z-t,\xi) I(\mu)(t)\, dt\, d\xi \right)\right].$$

By Lemma 4.5, it is enough to prove that

$$\left[\bigoplus_i (-1)^i \widetilde{\mathcal{R}}[\tilde{\mu}](z,\Gamma_i^\circ)\right] = \left[\bigoplus_i (-1)^i \int_{\Gamma_i^\circ \cap S^{n-1}} \int_{U'} W(z-t,\xi) I(\mu)(t)\, dt\, d\xi \right] \text{ on } U' \quad (4.2)$$

holds as Čech representation, where $\tilde{\mu}$ is a Čech-Dolbeault representative such that $\tilde{\mu}|_{V'} = \mu|_{V'}$ and $\tilde{\mu} \in \mathscr{E}_X^{(0,n)}(V, V\setminus K)$ for a compact set K with $\overline{U'} \subset \mathrm{Int}(K) \subset K \subset U$. $\widetilde{\mathcal{R}}[\tilde{\mu}](z,\Gamma_i^\circ)$ is defined by

$$\widetilde{\mathcal{R}}[\tilde{\mu}](z,\Gamma_i^\circ) = \int_{\Gamma_i^\circ \cap S^{n-1}} \left(\int_D W(z-w,\xi)\tilde{\mu}_1(w) \wedge dw - \int_{\partial D} W(z-w,\xi)\tilde{\mu}_{01}(w)\wedge dw \right) d\xi\,.$$

By deforming D suitably and using the same arguments as in the proof of Lemma 4.12, we can prove the equation (4.2). This completes the proof. □

A. Appendix

Let X be a complex manifold, V an open set in X and S a closed set in V.

Definition A.1 For any complex $(\mathscr{F}^\bullet, d^\bullet)$ of sheaves on X, we define

$$\mathscr{F}^q(V, V\setminus S) = \mathscr{F}^q(V) \oplus \mathscr{F}^{q-1}(V\setminus S),$$

and

$$\begin{array}{cccc} \vartheta^q : & \mathscr{F}^q(V, V\setminus S) & \longrightarrow & \mathscr{F}^{q+1}(V, V\setminus S) \\ & \cup & & \cup \\ & (s_1, s_{01}) & \longmapsto & (d^q(s_1),\, s_1|_{V\setminus S} - d^{q-1}(s_{01})). \end{array}$$

Then, $(\mathscr{F}^\bullet(V, V\setminus S), \vartheta^\bullet)$ is a complex of abelian groups. $H^q(V, V\setminus S; \mathscr{F}^\bullet)$ denotes the q-th cohomology group of this complex.

By the definition, we can also write $\mathscr{F}^q(V, V\setminus S) = \Gamma(V;\mathscr{F}^q) \oplus \Gamma(V\setminus S; \mathscr{F}^{q-1})$.

Lemma A.2 Let $\mathscr{F}^\bullet$ and $\mathscr{G}^\bullet$ be complexes bounded below consisting of soft sheaves on X. If $\mathscr{F}^\bullet$ is quasi-isomorphic to $\mathscr{G}^\bullet$, then $\mathscr{F}^\bullet(V, V\setminus S)$ is quasi-isomorphic to $\mathscr{G}^\bullet(V, V\setminus S)$.

Proof. By Lemma 1 and Lemma 2 of Kashiwara-Kawai-Kimura[6], we may assume that there is a morphism from $\mathscr{F}^\bullet$ to $\mathscr{G}^\bullet$. Here, we note that flabby sheaves are soft. Let us consider the following commutative diagram of short exact sequences of complexes.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (\mathscr{F}^\bullet[-1])^q(V\setminus S) & \xrightarrow{i_2} & \mathscr{F}^q(V)\oplus\mathscr{F}^{q-1}(V\setminus S) & \xrightarrow{p_1} & \mathscr{F}^q(V) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & (\mathscr{G}^\bullet[-1])^q(V\setminus S) & \xrightarrow{i_2} & \mathscr{G}^q(V)\oplus\mathscr{G}^{q-1}(V\setminus S) & \xrightarrow{p_1} & \mathscr{G}^q(V) & \longrightarrow & 0, \end{array}$$

where $(\bullet)[-1]$ is the shift functor, $q\in\mathbb{Z}$, i_2 is the inclusion map into the second component and p_1 is the projection onto the first component. Then, we get the following commutative diagram of long exact sequences.

$$\begin{array}{ccccccccc} \cdots\to & H^{q-1}(V\setminus S;\mathscr{F}^\bullet) & \to & H^q(V,V\setminus S;\mathscr{F}^\bullet) & \to & H^q(V;\mathscr{F}^\bullet) & \xrightarrow{\iota} & H^q(V\setminus S;\mathscr{F}^\bullet) & \to\cdots \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ \cdots\to & H^{q-1}(V\setminus S;\mathscr{G}^\bullet) & \to & H^q(V,V\setminus S;\mathscr{G}^\bullet) & \to & H^q(V;\mathscr{G}^\bullet) & \xrightarrow{\iota} & H^q(V\setminus S;\mathscr{G}^\bullet) & \to\cdots, \end{array}$$

where $q\in\mathbb{Z}$ and ι is induced by restriction. These long exact sequences are discussed in Suwa[17]. Since soft sheaves are $\Gamma(W;\bullet)$-injective for any $W\in\mathrm{Op}(X)$ (see Definition 1.8.2, Proposition 2.5.10 and Exercise II.6 of Kashiwara-Schapira[7]), we have $H^q(W;\mathscr{F}^\bullet)\simeq H^q(W;\mathscr{G}^\bullet)$ for any $W\in\mathrm{Op}(X)$ and $q\in\mathbb{Z}$. Applying the 5-lemma to the above diagram, we obtain $H^q(V,V\setminus S;\mathscr{F}^\bullet)\simeq H^q(V,V\setminus S;\mathscr{G}^\bullet)$ for any $q\in\mathbb{Z}$. □

Theorem A.3 For any $p\in\mathbb{Z}$, we have the quasi-isomorphisms

$$\mathscr{E}_X^{(p,\bullet)}(V,V\setminus S)\ \simeq\ \mathscr{I}^\bullet(V,V\setminus S)\ \simeq\ \mathbf{R}\Gamma_S(V;\mathscr{O}_X^{(p)}),$$

where $(\mathscr{I}^\bullet,d^\bullet)$ is a flabby resolution of $\mathscr{O}_X^{(p)}$.

Proof. Since $\mathscr{I}^\bullet$ is a flabby resolution of $\mathscr{O}_X^{(p)}$, $\mathbf{R}\Gamma_S(V;\mathscr{O}_X^{(p)})$ is quasi-isomorphic to $\Gamma_S(V;\mathscr{I}^\bullet)$. Moreover, by Lemma A.2, $\mathscr{E}_X^{(p,\bullet)}(V,V\setminus S)$ is quasi-isomorphic to $\mathscr{I}^\bullet(V,V\setminus S)$. Then, it suffices to show that $\Gamma_S(V;\mathscr{I}^\bullet)$ is quasi-isomorphic to $\mathscr{I}^\bullet(V,V\setminus S)$.

The canonical morphism $\phi:\Gamma_S(V;\mathscr{I}^\bullet)\to\mathscr{I}^\bullet(V,V\setminus S)$ is defined by

$$\begin{array}{ccc} \Gamma_S(V;\mathscr{I}^q) & \longrightarrow & \mathscr{I}^q(V,V\setminus S) \\ \cup\!\!\!\!\!\! \shortmid & & \cup\!\!\!\!\!\! \shortmid \\ s & \longmapsto & (s,0) \end{array} \qquad (q\in\mathbb{Z}).$$

By $\mathrm{supp}(s)\subset S$, ϕ forms a complex morphism and it induces morphisms between the cohomology groups. It is easy to see that each $\phi^q: H_S^q(V;\mathscr{I}^\bullet)\to H^q(V,V\setminus S;\mathscr{I}^\bullet)$ is injective. Hence, let us show that for any $q\in\mathbb{Z}$, ϕ^q is surjective.

Since each $\mathscr{I}^q$ is flabby, for any $[(s_1,s_{01})]\in H^q(V,V\setminus S;\mathscr{I}^\bullet)$, there exists $s'\in\Gamma(V,\mathscr{I}^{q-1})$ such that $s'|_{V\setminus S}=s_{01}$. Then, we get

$$[(s_1,s_{01})]=[(s_1,s_{01})-\vartheta(s',0)]=[(s_1-d^{q-1}(s'),0)]$$

and we define $\tilde{s}=s_1-d^{q-1}(s')$. By $(s_1,s_{01})\in\operatorname{Ker}\vartheta^q$, we have $\tilde{s}|_{V\setminus S}=s_1|_{V\setminus S}-d^{q-1}(s_{01})=0$ and $[\tilde{s}]\in H_S^q(V;\mathscr{I}^\bullet)$. This means that ϕ^q is surjective, and it has been shown that $H_S^q(V;\mathscr{I}^\bullet)\simeq H^q(V,V\setminus S;\mathscr{I}^\bullet)$ for any $q\in\mathbb{Z}$. This completes the proof. □

For the proof above, Honda-Komori[3] is a useful reference. If the readers are interested in the specific flabby resolution $\mathscr{B}^{(p,\bullet)}$ of $\mathscr{O}_X^{(p)}$, Komatsu[9] will be a useful reference.

Let M be a real analytic, n-dimensional, oriented manifold and X its complexification, and let $U \in \mathrm{Op}(M)$ and V a complex neighborhood of U. Since M is oriented, we omit the section of the relative orientation sheaf in the definition of $\mathscr{B}_M(U)$.

Lemma A.4 (a representative with compact support) Assume $U \subset\subset M$. Let $\mu \in \mathscr{E}_X^{(0,n)}(V, V \setminus U)$ be a representative of a hyperfunction and $V', V'' \in \mathrm{Op}(X)$ such that $V' \subset\subset V \subset\subset V''$. Then, there exists a representative $\tilde{\mu} \in \mathscr{E}_X^{(0,n)}(X, X \setminus M)$ such that

$$\tilde{\mu} = \begin{cases} \mu \text{ on } V', \\ 0 \text{ on } X \setminus \overline{V''}. \end{cases}$$

Proof. Since $\mathscr{B}_M$ is flabby, we can find a hyperfunction $\tilde{u} \in \mathscr{B}_M(M)$ such that $\tilde{u}|_U = [\mu]$, $\tilde{u}|_{M \setminus \overline{U}} = 0$. Hence, there exist $\mu_0 \in \mathscr{E}_X^{(0,n)}(X, X \setminus M)$, $\tau' \in \mathscr{E}_X^{(0,n-1)}(V, V \setminus U)$ and $\tau'' \in \mathscr{E}_X^{(0,n-1)}(X \setminus \overline{V}, (X \setminus \overline{V}) \setminus (M \setminus \overline{U}))$ such that

$$\begin{aligned} \mu_0|_V + \bar{\vartheta}\tau' &= \mu\,, \\ \mu_0|_{X \setminus \overline{V}} + \bar{\vartheta}\tau'' &= 0\,. \end{aligned}$$

Let V_0' and V_0'' be satisfying $V' \subset\subset V_0' \subset\subset V \subset\subset V_0'' \subset\subset V''$. We define $\varphi', \varphi'' \in C_0^\infty(X)$ by

$$\varphi' = \begin{cases} 1 \text{ on } V', \\ 0 \text{ on } X \setminus \overline{V_0'}, \end{cases} \qquad \varphi'' = \begin{cases} 1 \text{ on } X \setminus \overline{V''}, \\ 0 \text{ on } V_0''. \end{cases}$$

Then, if we set $\tilde{\mu} = \mu_0 + \bar{\vartheta}(\varphi'\tau') + \bar{\vartheta}(\varphi''\tau'')$, we get

$$\begin{aligned} \tilde{\mu}|_{V'} &= \mu_0|_{V'} + \bar{\vartheta}\,\tau'|_{V'} &&= \mu|_{V'}\,, \\ \tilde{\mu}|_{X \setminus \overline{V''}} &= \mu_0|_{X \setminus \overline{V''}} + \bar{\vartheta}\,\tau''|_{X \setminus \overline{V''}} &&= 0\,. \end{aligned}$$

This completes the proof. □

Lemma A.5 (Bochner-Martinelli formula) Let $D \subset \mathbb{C}^n$ be a bounded domain with C^1 boundary. For any $z \in D$ and for any $g \in C^1(\overline{D})$, we have

$$g(z) = (-1)^{\frac{n(n+1)}{2}} \int_{\partial D} g(\zeta)\beta(\zeta - z) \wedge d\zeta - (-1)^{\frac{n(n+1)}{2}} \int_D \bar{\partial} g(\zeta) \wedge \beta(\zeta - z) \wedge d\zeta,$$

where

$$\beta(z) = (-1)^{\frac{n(n-1)}{2}} \frac{(n-1)!}{(2\pi\sqrt{-1})^n} \frac{1}{\|z\|^{2n}} \sum_{i=1}^n (-1)^{i-1} \bar{z}_i\, d\bar{z}_1 \wedge \cdots \wedge \widehat{d\bar{z}_i} \wedge \cdots \wedge d\bar{z}_n.$$

We omit the proof of Lemma A.5 (see Krantz[12]). Note that we choose $(y_1, \cdots, y_n, x_1, \cdots, x_n)$ as a positive coordinate system on $\mathbb{C}^n$. Since the conventional positive coordinate system is $(x_1, y_1, \cdots, x_n, y_n)$, we have to multiply the formula by

$(-1)^{\frac{n(n+1)}{2}}$.

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Tatsuki NISHIDA
Department of Mathematics
Graduate School of Science
Hokkaido University
Sapporo 060-0810, Japan
E-mail: nishida@math.sci.hokudai.ac.jp