Comparison of fundamental cycles and maximal ideal cycles for normal surface double points - due to Laufer decomposition -

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Abstract

For a resolution space $(\tilde{X}, E)$ of a normal complex surface singularity (X, o), the fundamental cycle Z_E and maximal ideal cycle M_E are important geometric objects associated to (X, o), which satisfy $M_E \geq Z_E$. In 1966, M. Artin proved that $M_E = Z_E$ for all resolutions of all rational singularities. However, for non-rational singularities, it is a delicate problem whether $M_E = Z_E$ or not. Any normal surface double point (i.e., multiplicity two) is a hypersurface singularity defined by $z^2 = f(x, y)$. For such singularities, we prove that $M_E > Z_E$ holds on the minimal resolution if and only if f has a canonical decomposition $f = f_{[L]} f_{[c]} f_{[o]}$ in $\mathbb{C}\{x, y\}$ called "Laufer decomposition". Moreover, we give a numerical procedure to determine whether $M_E = Z_E$ or not on the minimal resolution from the embedded topology of the branch curve singularity $(\{f = 0\}, o)$.

Keywords Normal surface double points · Fundamental cycles · Maximal ideal cycles · Laufer decompositions

Mathematics Subject Classification 32S25·14J17·32S05·32S10

1 Introduction

Let $\pi : (\tilde{X}, E) \to (X, o)$ be a good resolution of a normal complex surface singularity. Let $E = \bigcup_{i=1}^{r} E_i$ be the irreducible decomposition of the exceptional set. A divisor on $\tilde{X}$ supported in E is called a *cycle*. In [2], M. Artin defined the fundamental cycle as $Z_E := \min\{D = \sum_{i=1}^{r} a_i E_i \mid a_i > 0 \text{ and } DE_i \leqq 0 \text{ for any } i\}$. It is well-known that the value Z_E^2 is independent of the choice of a resolution, and so we put it $\mathbb{Z}_X^2$ in this paper. Therefore, $\mathbb{Z}_X^2$ is a topological invariant of (X, o). The *maximal ideal cycle* on E is defined by $M_E := \min\{(h \circ \pi)_E \mid h \in \mathfrak{m} \backslash \{0\}\}$ (see [22]), where $\mathfrak{m}$ is the maximal ideal of the local ring $\mathcal{O}_{X,o}$ and $(h \circ \pi)_E$ is a cycle $\sum_{i=1}^{r} v_{E_i}(h \circ \pi) E_i$ for vanishing order $v_{E_i}(h \circ \pi)$ of $h \circ \pi$ on E_i. Though Z_E is determined by the topological structure of (X, o), M_E

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depends on the analytic structure of (X,o). For M_E, if we take a suitable succession of blowing-ups $(\tilde{\tilde{X}}, \tilde{E}) \longrightarrow (\tilde{X}, E)$, then $-M_{\tilde{E}}^2$ is equal to $\mathrm{mult}(X,o)$ (i.e., the multiplicity of (X,o)). Namely, the maximal ideal cycle is a geometric representation of $\mathrm{mult}(X,o)$.

From the definition of M_E and Z_E, they are anti-nef cycles on E with $M_E \geq Z_E$. Moreover, Ph. Wagreich proved the following fundamental result.

Theorem 1.1 ([21, Theorem 2.7]). (i) $\mathrm{mult}(X,o) \geqq -M_E^2 \geqq -Z_E^2$.
(ii) *If* $\mathfrak{m}\mathcal{O}_{\tilde{X}}$ *is invertible, then* $\mathfrak{m}\mathcal{O}_{\tilde{X}} = \mathcal{O}_{\tilde{X}}(-M_E)$ *and* $\mathrm{mult}(X,o) = -M_E^2$.

In normal surface singularity theory, it is important to consider under what circumstances $M_E = Z_E$ holds. In [8], H. Laufer showed that $M_E > Z_E$ on the minimal resolution of a normal double point defined by $z^2 = y(x^4 + y^6)$. Regarding the importance of the condition $M_E = Z_E$, let us mention one fact. In [22], S.S.T. Yau defined maximally elliptic singularities and proved that those singularities are Gorenstein. The first named author ([16, Corollary 7.9]) proved that $M_E = Z_E$ also holds on the minimal resolutions for those singularities (also see [11,Theorem D]). Conversely, T. Okuma ([13, Theorem 5.10]) proved the converse (also see [11, Theorem A]). Namely, if (X,o) is a Gorenstein elliptic singularity and $M_E = Z_E$ on the minimal resolution, then it is a maximally elliptic singularity. About the comparison of M_E and Z_E, we can find many useful descriptions in §6 − §8 and §11 of [12]. We also refer to [7], [9], [10], [14] and [17] for many types of singularities. Here, we remark that "maximal ideal cycle" is called as "fiber cycle" in [4] .

From S.S. Abhyanker's result in [1], we can see that every normal surface double point is a hypersurface singularity defined by $z^2 = f(x,y)$ for an element f in $\mathbb{C}\{x,y\}$ (also see [5]). We remark that for a hypersurface singularity $(X,o) = \{z^n = f(x,y)\}$, (X,o) is normal if and only if f is reduced ([18, Theorem 3.2]). In the following of this section, let (X,o) be a normal double point defined by $z^2 = f(x,y)$ and $(\tilde{X}, E)$ a resolution of (X,o); also $(C,o) := (\{f = 0\}, o))$. The comparison problem of M_E and Z_E for (X,o) has been studied by H. Laufer and D.J. Dixon as follows.

Theorem 1.2 ([8, Theorem 6.3]). *If* $M_E > Z_E$ *on the minimal resolution* $(\tilde{X}, E)$ *of* (X,o), *then* $H_1(E,\mathbb{R}) \neq 0$. *Namely, if* (X,o) *has the rational homological sphere link, then* $M_E = Z_E$ *on the minimal resolution.*

Theorem 1.3 ([5, Theorem 1 and 2]). (i) *If* $ord(f)$ *is even, then* $M_E = Z_E$ *for any resolution of* (X,o).
(ii) *If* f *is irreducible and* $ord(f)$ *is odd, then* $M_E = Z_E$ *for the minimal resolution.*

Therefore, we need to consider the case that $ord(f)$ is odd and f is reducible. A. Calabri and R. Ferraro in [4] researched the comparison problem in such case. They defined a figure called the Enriques digraph for (C,o), and proved that the condition to $M_E > Z_E$ is determined in terms of the Enriques digraph (see Theorem 12.2 in [4]). They used the canonical resolution (see [4], [15]) and studied the multiplicity of strict transform of X at each step in constructing the canonical resolution. They expressed $2Z_E$ by maximum ideal cycles on such steps ([4,Theorem 11.2]). Using these results, they arrived at their main result.

Though we research the similar problem, our interests and main results are largely different from them. We study the Puiseux pairs and the resolution diagram of the branch curve. However the canonical resolution does not induce the embedded resolution of the branch curve (cf. as each process needs the condition $\deg \pi = e_p(V)$ in (1.2) [15] p.3 , see also (1.3) [15]). Hence we change the resolution method and use the covering resolution of (X, o) ((3.1) and [5]). (Though Puiseux pairs play crucial roles in our work, but they do not appear in [4] and [8]).

In the comparison problem between Z_E and M_E, most results already obtained give sufficient conditions for $Z_E = M_E$. However, it seems also interesting and important to consider the details of the situation where $Z_E < M_E$. In this paper (Theorem 3.11), for normal double points with $M_E > Z_E$, we have shown how E decomposes. Namely, the resolution graph corresponding to Laufer decomposion consists of three parts, i.e. the odd part carrying $Z_E = M_E$, the Laufer part carrying $M_E = 2Z_E$ and the contact part carrying $Z_E < M_E < 2Z_E$. The components which connect these parts are nothing but the strict transforms of exceptional curves F_1 and F_2 which appear by the first and the second blow ups of the covering resolution. Note that only the role of F_1 appears in Enriques digraph [4, § 12, Fig.2]. First, we show that if $M_E > Z_E$, the shape of E has a specific decomposition. Second, we consider a specific decomposition of f obtained from the above decomposition of E. Then we prove that the existence of the above decomposition of f is equivalent to $M_E > Z_E$ on the minimal resolution. Furthermore, we provide a numerical procedure to determine from the topology of (C, o) whether $M_E = Z_E$ or not.

In §2, we prepare some facts and terminologies on plane curve singularities. In §3, as a resolution of $(X, o) = \{z^2 = f(x, y)\}$, we explain the covering resolution over the MSGE-resolution of (C, o) (see Definition 3.1). Under the condition of $M_E > Z_E$ on a good resolution $(\tilde{X}, E)$, we prove several facts. Especially, we define the Laufer decomposition $E = E(o) \cup E(c) \cup E(L)$ (Definition 3.10), and show that E has a specific shape. (Theorem 3.11). From §4 to the end of this paper, in the topological point of view, we characterize $f \in \mathbb{C}\{x, y\}$ such that $M_E > Z_E$ holds on the minimal resolution of (X, o). For the purpose, we define elements of three types in $\mathbb{C}\{x, y\}$ (i.e., Laufer type, contact type and odd type; see Definition 4.1 and 5.1 and 6.1). When $M_E > Z_E$ holds on the minimal good resolution, such f is decomposed into a product of at most those three types of elements. Those three types of elements in $\mathbb{C}\{x,y\}$ correspond to Laufer decomposition of E as follows: [Laufer type or 1 $\Leftrightarrow E(L)$], [contact type $\Leftrightarrow E(c)$] and [odd type $\Leftrightarrow E(o)$].

In §4 (resp. §5), we characterize elements of Laufer (resp. contact) type in terms of Puiseux pair (see Theorem 4.5 and 5.4). As an improvement of Theorem 1.2, we give a lower bound for $dim_{\mathbb{R}} H_1(E(\epsilon),\mathbb{R})$ for $\epsilon = L$ or c (Theorem 4.9 (ii) and 5.9 (ii)). In §6, we characterize elements of odd type in terms of Puiseux pair (Theorem 6.4). In §7, we prove the main result of this paper (see Theorem 7.5).

Theorem 1.4. *The relation $\mathbb{M}_X > \mathbb{Z}_X$ holds on the minimal resolution if and only if f has the Laufer decomposition $f = f_{[L]}f_{[c]}f_{[o]}$ in $\mathbb{C}\{x,y\}$ (see Definition 7.1).*

Moreover, if $f = f_{[L]}f_{[c]}f_{[o]}$ is the Laufer decomposition, then the w.d.graph for (X,o) is constructed by suitable gluing of the w.d.graphs for $z^2 = \ell_\epsilon f_{[\epsilon]}$ ($\epsilon = L$ and c) and $z^2 = f_{[o]}$, where ℓ_ϵ is a linear form (see Theorem 7.3). In §8, we give a numerical procedure from the w.d.resolution graph $\Lambda(f)$ (i.e., the weighted dual graph of the exceptional set plus the strict transform $\sigma_*^{-1}C$; see (2.4)) to determine whether $M_E = Z_E$ holds or not on the minimal resolution (Procedure 8.2).

To close this section we like to state a problem. It is natural to look forward some extensions of Laufer decomposition to the cases of multiplicity ≥ 3.

Problem 1.5. *Assume that (X,o) is a normal surface singularity with multiplicity $\geqq 3$ and $M_E > Z_E$. For the irreducible decomposition $E = \bigcup_{i=1}^r E_i$ of the exceptional set, we put the ratio γ_i among the coefficients of Z_E and M_E as $\gamma_i := \operatorname{Coeff}_{E_i} M_E / \operatorname{Coeff}_{E_i} Z_E$ for any i.*

(i) *What is the maximum of $\{\gamma_i\}$?*

(ii) *For the existence of a decomposition of E similar to Laufer decomposition by means of $\{\gamma_i\}$, what conditions to (X,o) are necessary?*

(iii) *If $(X,o) = \{z^n = f(x,y)\}$ and exists a decomposition of E as* (ii), *study the Puiseux pairs of irreducible components of f(x,y) which give each subset (similar to E(L),E(c) and E(o) of the Laufer decomposition).*

Notations and terminologies.

In this paper, we always use the notations here. Let f be a reduced element in $\mathbb{C}\{x,y\}$ and $f = \prod_{j=1}^{r} f_j$ the irreducible decomposition; also we put $(C_j,o) = (\{f_j = 0\},o)$. We consider the following as the MGE (i.e., minimal good embedded)-resolution of $(C,o) = (\{f=0\},o)$:

$$(1.1) \qquad (\mathbb{C}^2,o) \xleftarrow{\sigma_1} (W_1,F(1)) \xleftarrow{\sigma_2} \cdots \xleftarrow{\sigma_N} (W_N,F(N)) \text{ and } \sigma := \sigma_1 \circ \cdots \circ \sigma_N.$$

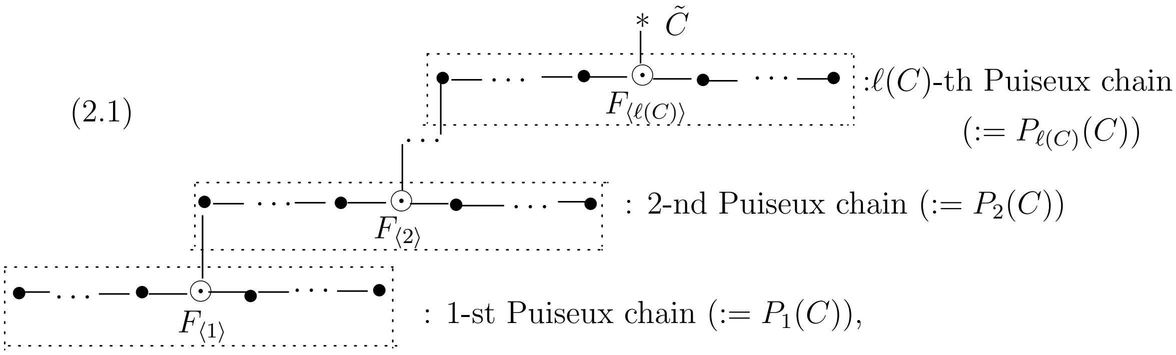
Namely, $F \cup \tilde{C}$ is simple normal crossing, where $F := F(N)$ and $\tilde{C} := \sigma_*^{-1} C$ is the strict transform of C by σ. Let $\tilde{C}_j := \sigma_*^{-1} C_j$ for any j and P_i the center of σ_i (so $P_1 = \{o\}$) and $F_i := \sigma_i^{-1}(P_i)$ for any i. The strict transform $(\sigma_{k+1} \circ \cdots \circ \sigma_N)_*^{-1} F_k$ ($\subset W_N$) is also denoted by F_k; thus $F = \bigcup_{k=1}^{N} F_k$. We represent the configuration of $F \cup \tilde{C}$ by $\Lambda(C)$ or $\Lambda(f)$, and call it the *w.d.resolution graph* (see (2.4)). For a plane curve singularity (C, o), let $T_o(C)_{red}$ (or $T_o(f)_{red}$) be the reduced form of the tangent cone of (C, o). When f is irreducible and $\{(m_1, n_1), \cdots, (m_{\ell(C)}, n_{\ell(C)})\}$ is the Puiseux pair of f, let us represent it by $Puisx(C)$ (or $Puisx(f)$).

Moreover, we remark that the *MSGE* (i.e., minimal sufficiently good embedded)-resolution defined in Definition 3.1 is used very often in this paper. If E_o is a smooth rational curve on a smooth complex surface and $E_o^2 = -1$, then it is called a (-1)-*curve.* The maximal ideal cycle and fundamental cycle on the minimal (resp. minimal good) resolution of a normal surface singularity (X, o) are denoted by $\mathbb{M}_X$ and $\mathbb{Z}_X$ (resp. $\mathbb{M}_{o,X}$ and $\mathbb{Z}_{o,X}$) respectively. In this paper, we consider that a non-singular point is a kind of rational singularities. Then, for any resolution $\sigma : (W, F) \to (\mathbb{C}^2, o)$ of a non-singular point, the maximal ideal cycle M_F and fundamental cycle Z_F conicide.

2 Preparation for plane curve singularities

[**Irreducible case**] Assume that f is irreducible and σ is the MGE-resolution in (1.1) for (C, o). The figure of the w.d.resolution graph $\Lambda(C)$ associated to $F \cup \sigma_*^{-1} C$ is given as follows (see [3, p.523]):

(2.1)



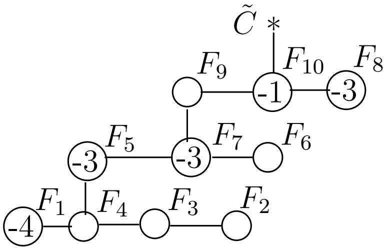
where $F_{\langle k \rangle} \subset W_{\langle k \rangle}$ for $k = 1, \cdots, \ell(C)$; $W_{\langle i \rangle} := W_{k_i}$ for k_i satisfying $\sigma_{k_i}^{-1}(P_{k_i}) = F_{\langle i \rangle}$ ($1 \leqq k_i \leqq N$). Let us call $F_{\langle k \rangle}$ the *k-th Puiseux root* of f for any k. We have $\mathrm{mult}(C, o) = ord(f) = \prod_{i=1}^{\ell(C)} m_i$, where $ord(f)$ is the order of f at the origin $\{o\} \in \mathbb{C}^2$.

Here we remark that the intersection number $CT_o(C)_{red} = n_1 m_2 \cdots m_\ell$ for a suitable coordinate of $\mathbb{C}^2$, but $CT_o(C)_{red}$ depends on the choice of a coordinate. For example, for $(C_1, o) = \{x^2 + y^5 = 0\}$ and $(C_2, o) = \{(x + y^2)^2 + y^5 = 0\}$, they are holomorphically isomorphic; but $C_1 T_o(C_1)_{red} = 5$ and $C_2 T_o(C_2)_{red} = 4$. However, we have the following:

(2.2) If $n_1 < 2m_1$, then $CT_o(C)_{red} = n_1 m_2 \cdots m_\ell$ always holds.

In fact, if $\sigma : (W, F) \to (\mathbb{C}^2, o)$ is the blowing-up at the origin and put $C_1 := \sigma_*^{-1} C$ and $T_1 := \sigma_*^{-1} T_o(C)_{red}$, then C_1 intersects T_1 transversally from $n_1 < 2m_1$; thus $CT_o(C)_{red} = (\sigma^* C)T_1 = (mult(C, o)F + C_1)T_1 = m_1 m_2 \cdots m_\ell + (n_1 - m_1)m_2 \cdots m_\ell = n_1 m_2 \cdots m_\ell$.

Example 2.1. Let (C, o) be an irreducible curve singularity given by $\{x = t^{18}, y = t^{24} + t^{33} + t^{35}\}$. Then, $Puisx(C) = \{(3, 4), (2, 11), (3, 35)\}$. The w.d. (i.e., weighted dual) resolution graph associated to the MGE-resolution of the curve singularity and the multiplicity sequence (see [3, p. 517]) are given as follows:



$$\begin{array}{l} 24 = 1 \cdot 18 + 6 \\ 18 = 3 \cdot 6 \\ \hline 9 = 1 \cdot 6 + 3 \\ 6 = 2 \cdot 3 \\ \hline 3 = 1 \cdot 2 + 1 \\ 2 = 2 \cdot 1 \end{array}$$

$F_{\langle 1 \rangle} = F_4$, $F_{\langle 2 \rangle} = F_7$, $F_{\langle 3 \rangle} = F_{10}$,

$P_1(C) = \bigcup_{i=1}^{4} F_i$,

$P_2(C) = \bigcup_{i=5}^{7} F_i$,

$P_3(C) = \bigcup_{i=8}^{10} F_i$.

Definition 2.2. (i) Assume that $\mathrm{mult}(C, o)$ $(= m_1 \cdots m_{\ell(C)})$ is even. If $m_1 \cdots m_{k-1}$ is odd and m_k is even, then we put $m_{[LP]}(f)$ (or $m_{[LP]}(C)$) $:= m_k$. Let us call $m_{[LP]}(f)$ the *LP-number (i.e., Laufer-Puiseux number)* of f or (C, o).

(ii) Assume that $n_1 m_2 \cdots m_{\ell(C)}$ is even. If n_1 is even, then we define $m_{[cP_1]}(f)$ (or $m_{[cP_1]}(C)$) $:= m_1$ and $m_{[cP_2]}(f)$ (or $m_{[cP_2]}(C)$) $:= n_1$. If $n_1 m_2 \cdots m_{k-1}$ is odd and m_k is even $(k \geqq 2)$, then we define $m_{[cP_i]}(f)$ (or $m_{[cP_i]}(C)$) $:= m_k$ for $i = 1, 2$. Hence, if n_1 is odd, then $m_{[cP_1]}(f) = m_{[cP_2]}(f)$. Let us call $m_{[cP_1]}(f)$ (resp. $m_{[cP_2]}(f)$) the *1-st (resp. 2-nd) contact-Puiseux number* of f or (C, o). We abbreviate it as the *cP_i-number* $(i = 1, 2)$ of f and use it in Theorem 5.4.

Definition 2.3. (i) $e_{P_i}(C) := \mathrm{mult}((\sigma_1 \circ \cdots \circ \sigma_{i-1})_*^{-1} C, P_i)$ for any P_i $(1 \leqq i \leqq N)$.

(ii) $mult.seq(C, o) := (e_{P_1}(C), \cdots, e_{P_N}(C))$: *multiplicity sequence* of (C, o).

(iii) If $m_{k_0} = m_{[LP]}(C)$ (resp. $m_{[cP_1]}(C)$) for $k_0 \geqq 2$, then we put

$e_{[L]}(C)$ (resp. $e_{[c]}(C)$) $:= m_{k_0+1} \cdots m_{\ell(C)}$ if $k_0 < \ell(C)$ else $e_{[L]}(C)$ (resp. $e_{[c]}(C)$) $:= 1$.

If we assume that $W_{i_k} = W_{\langle k \rangle}$ for k with $1 \leqq k \leqq \ell(C)$ and $1 \leqq i_1 < \cdots < i_{\ell(C)} \leqq N$, then we have the following (see [3, §8.4]):

$$(2.3) \quad \begin{cases} \text{(i)} \ e_{P_{i_k}}(C) = \mathrm{mult}((\sigma_1 \circ \cdots \circ \sigma_{i_k - 1})_*^{-1} C, P_{i_k}) = m_{k+1} \cdots m_{\ell(C)}, \\ \text{(ii)} \ m_{k+1} \cdots m_{\ell(C)} | \, e_{P_i}(C) \ \text{for any } i \ \text{with} \ 1 \leqq i \leqq i_k, \\ \text{(iii)} \ \text{the } k\text{-th Puiseux chain } P_k(C) \text{ in (2.1)} = \displaystyle\bigcup_{\xi=1}^{i_k - i_{k-1}} F_{i_{k-1}+\xi}. \end{cases}$$

For (C, o) in Example 2.1, we have $i_1 = 4$, $i_2 = 7$ and $i_3 = 10$ and $mult.seq(C, o) = (18, 6, 6, 6, 6, 3, 3, 2, 1, 1)$; hence $e_{P_1}(C) = 18$, $e_{P_4}(C_1) = 6$ and $e_{P_7}(C) = 3$. Moreover, $m_{[LP]}(C) = m_2 = 2$, $m_{[cP_1]}(C) = m_1 = 3$, $m_{[cP_2]}(C) = n_1 = 4$, $e_{[L]}(C) = 3$

and $e_{[c]}(C) = 6$. Since σ is a resolution of $(\mathbb{C}^2, o)$, the maximal ideal cycle M_F is equal to the fundamental cycle Z_F on F. We can see the following: $\ll \mathrm{Coeff}_{F_i} M_F \gg := (\mathrm{Coeff}_{F_1} M_F, \cdots, \mathrm{Coeff}_{F_{10}} M_F) = (1,1,2,3,3,3,6,6,12,18)$. Also, $P_1(C) = \bigcup_{i=1}^{4} F_i$, $P_2(C) = \bigcup_{i=5}^{7} F_i$ and $P_3(C) = \bigcup_{i=8}^{10} F_i$.

[Reducible case]

Proposition 2.4 (see [3, p.414, p.535]). (i) *For two plane curve singularities (C_1, o) and (C_2, o), they are equivalent in embedded topology if and only if $\Lambda(C_1) = \Lambda(C_2)$, where $\Lambda(C_i)$ is the w.d.resolution graph for $i = 1, 2$.*

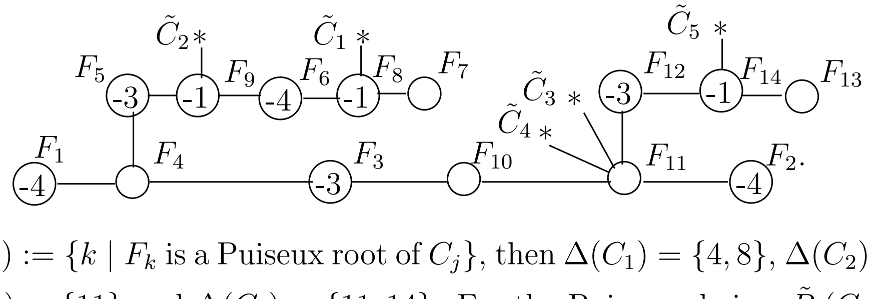
(ii) *The multiplicity sequence is determined by the w.d.resolution graph.*

Let $\sigma_{\langle j \rangle}$ be the MGE-resolution of (C_j, o) for $j = 1, \cdots, r$. Moreover, let $\breve{\sigma}_{\langle j \rangle}$ be a succession of blowing-ups such that $\sigma_{\langle j \rangle} \circ \breve{\sigma}_{\langle j \rangle}$ gives the MGE-resolution σ of (C, o). From the uniqueness of MGE-resolution of (C, o), we have $\sigma_{\langle 1 \rangle} \circ \breve{\sigma}_{\langle 1 \rangle} = \cdots = \sigma_{\langle r \rangle} \circ \breve{\sigma}_{\langle r \rangle} = \sigma$.

Definition 2.5. Under the situation above, if F_k $(\subset W_{(j)})$ is a Puiseux root of $(C_j, o) := (\{f_j = 0\}, o)$ $(1 \leqq j \leqq r)$, then the strict transform $(\breve{\sigma}_{\langle j \rangle})_*^{-1} F_k$ $(\subset W_N)$ is also called a *Puiseux root of* (C_j, o). For the i-the Puiseux chain $P_i(C_j)$ of (C_j, o) in $W_{(j)}$, put $\bar{P}_i(C_j)$ (or $\bar{P}_i(f_j)$) $:= (\breve{\sigma}_{\langle j \rangle})_*^{-1} P_i(C_j)$ $(\subset W_N)$ and $I_i(C_j) := \{\xi \in I \mid \bar{P}_i(C_\xi) \cap \bar{P}_i(C_j) \neq \emptyset\}$ for $I := \{1, \cdots, r\}$. Let us define $\tilde{P}_i(C_j)$ (or $\tilde{P}_i(f_j)$) $:= \bigcup_{\xi \in I_i(C_j)} \bar{P}_i(C_\xi)$ for $i = 1, \cdots, \ell_j$. We call $\tilde{P}_i(C_j)$ (or $\tilde{P}_i(f_j)$) the *i-th Puiseux chain* of (C_j, o) in $(W_N, F(N))$. If $F_{i_0} \subset \tilde{P}_{i-1}(C_j)$ and $F_{i_0} \cap \tilde{P}_i(C_j) \neq \emptyset$, then we call F_{i_0} the *Puiseux root of* $\tilde{P}_i(C_j)$.

Example 2.6. Assume that $(C, o) := \bigcup_{j=1}^{5} (C_j, o)$ is given by local parametrizations: $(C_1, o) := \{x = t^6, y = t^8 + t^{13}\}$, $(C_2, o) := \{x = t^6, y = t^8 + t^{11}\}$, $(C_i, o) := \{x = (-1)^i t^4, y = t^7\}$ for $i = 3, 4$ and $(C_5, o) := \{x = t^8, y = t^{14} + t^{17}\}$. Then, we can see that $Puisx(C_1) = \{(3,4), (2,13)\}$, $Puisx(C_2) = \{(3,4), (2,11)\}$, $Puisx(C_3) = Puisx(C_4) = \{(4,7)\}$ and $Puisx(C_5) = \{(4,7), (2,17)\}$. Using the parametrization of (C_j, o) $(j = 1, \cdots, 5)$, we can compute the MGE-resolution $\sigma : (W, F) \to (\mathbb{C}^2, o)$ of (C, o), and obtain the w.d.resolution graph $\Lambda(C)$ as follows:

(2.4)



If we put $\Delta(C_j) := \{k \mid F_k \text{ is a Puiseux root of } C_j\}$, then $\Delta(C_1) = \{4, 8\}$, $\Delta(C_2) = \{4, 9\}$, $\Delta(C_3) = \Delta(C_4) = \{11\}$ and $\Delta(C_5) = \{11, 14\}$. For the Puiseux chains, $\tilde{P}_1(C_1) = \cdots =$

$\tilde{P}_1(C_5) = (\bigcup_{i=1}^{4} F_i) \cup F_{10} \cup F_{11}$, $\tilde{P}_2(C_1) = \tilde{P}_2(C_2) = \bigcup_{i=5}^{9} F_i$ and $\tilde{P}_2(C_5) = \bigcup_{i=12}^{14} F_i$. In addition, the root of $\tilde{P}_2(C_1)$ (resp. $\tilde{P}_2(C_5)$) is equal to F_4 (resp. F_{11}) and so on.

Definition 2.7. (see [3, p.506]). If f is irreducible, then there exists a unique total order among $\{F_1, \cdots, F_N\}$ according to the order of blowing-ups. If f is reducible (i.e., $r > 1$), then the strict transforms of all irreducible components of (C, o) are separated by σ. For $\tilde{C}_j := \sigma_*^{-1} C_j$ $(1 \leqq j \leqq r)$, we define the following.

(i) Define the partial order among $\{F_1, \cdots, F_N\}$ according to the order of blowing-ups. Moreover, if $\tilde{C}_j \cap F_{k_j} \neq \emptyset$ in W_N, then define the order by $F_{k_j} \prec \tilde{C}_j$. The partial order among $\{F_1, \cdots, F_N, \tilde{C}_1, \cdots, \tilde{C}_r\}$ is called *the standard order*, which is indicated by "$\prec$". This partially ordered set has only one minimal element F_1 $(= \sigma_1^{-1}(\{o\}))$.

(ii) $mult.seq(f)$ (or $mult.seq(C)$) $:= (e_{P_1}(C), \cdots, e_{P_N}(C))$, where $e_{P_k}(C)$ means the multiplicity of $(\sigma_1 \circ \cdots \circ \sigma_{k-1})_*^{-1} C$ at P_k $(2 \leqq k \leqq N)$ and $e_{P_1}(C) := \mathrm{mult}(C, o)$.

(iii) $\ll\!\mathrm{Coeff}_{F_i}(g)\!\gg := [\mathrm{Coeff}_{F_1}(g \circ \sigma)_{F(N)}, \cdots, \mathrm{Coeff}_{F_N}(g \circ \sigma)_{F(N)}]$ for any $g \in \mathbb{C}\{x, y\}$.

Example 2.8. For (C, o) of Example 2.6, the *standard order* among $\{F_i\}_{1 \leqq i \leqq 24} \cup \{C_j\}_{1 \leqq j \leqq 5}$ is given as follows:

$$\begin{array}{l}
F_1 \prec F_2 \prec F_3 \prec F_4 \prec F_5 \prec F_6 \prec F_9 \prec \tilde{C}_2 \\
F_6 \prec F_7 \prec F_8 \prec \tilde{C}_1 \\
F_3 \prec F_{10} \prec F_{11} \prec F_{12} \prec F_{13} \prec F_{14} \prec \tilde{C}_5. \\
F_{10} \prec \tilde{C}_3, \quad F_{11} \prec \tilde{C}_4
\end{array}$$

Then, $mult.seq(C_j)$ are given as follows: $(6, 2, 2, 2, 2, 2, 1, 1, 0, \cdots, 0)$ for $j = 1$; $(6, 2, 2, 2, 2, 1, 0, 0, 1, 0, \cdots, 0)$ for $j = 2$; $(4, 3, 1, 0, \cdots, 0, 1, 1, 0, 0, 0)$ for $j = 3, 4$; $(8, 6, 2, 0, \cdots, 0, 2, 2, 2, 1, 1)$ for $j = 5$; also $mult.seq(C) = (28, 16, 8, 4, 4, 3, 1, 1, 1, 4, 4, 2, 1, 1)$. For general $\alpha, \beta \in \mathbb{C}$ and $f \in \mathbb{C}\{x, y\}$,

$$\ll\!\mathrm{Coeff}_{F_i}(\alpha x + \beta y)\!\gg = [1, 1, 2, 3, 3, 3, 3, 6, 6, 3, 4, 4, 4, 8] \text{ and}$$
$$\ll\!\mathrm{Coeff}_{F_i}(f)\!\gg = [28, 44, 80, 112, 116, 119, 120, 240, 236, 128, 176, 178, 179, 358].$$

Assume the situation of Definition 2.5. Let F_ξ be a Puiseux root of an irreducible component of $(C, o) = \bigcup_{i=1}^{r} (C_i, o)$ and define

$$(2.5) \qquad J_{F_\xi} := \{j \in I \mid F_\xi \text{ is a Puiseux root of } C_j\} \text{ and } D(F_\xi) := \bigcup_{j \in J_{F_\xi}} C_j,$$

where $I := \{1, \cdots, r\}$. Then we can easily see the following (see [3, p.512-529]).

Lemma 2.9. *Under the situation above, if $F_k \preceq F_\xi$, then $e_{P_k}(D(F_\xi)) = \sum_{j \in J_{F_\xi}} e_{P_k}(C_j)$ is divisible by $e_{P_\xi}(D(F_\xi))$.*

In (2.4), F_{11} is a Puiseux root of C_3 and $J_{F_{11}} = \{3,4,5\}$; also $e_{P_{11}}(D(F_{11})) = \sum_{j=3}^{5} e_{P_{11}}(C_j) = 4$. From the computation in Example 2.8, we have $e_{P_1}(D(F_{11})) = 16$, $e_{P_2}(D(F_{11})) = 12$, $e_{P_3}(D(F_{11})) = 8$ and $e_{P_{10}}(D(F_{11})) = 4$.

Definition 2.10. Under the situation of Definition 2.7, let F_{i_0} ($\subset W_N$) be a Puiseux root of C_j ($1 \leqq j \leqq r$) and η a linear form in $\mathbb{C}\{x,y\}$.

(i) F_{i_0} is called *the η-root* of C_j if $\mathrm{Coeff}_{F_{i_0}}(\eta \circ \sigma)_{F(N)}$ is even and $\mathrm{Coeff}_{F_k}(\eta \circ \sigma)_{F(N)}$ is odd for any Puiseux root F_k of C_j with $F_k \prec F_{i_0}$.

(ii) F_{i_0} is called *the Laufer root* of C_j if F_{i_0} is the $(\alpha x + \beta y)$-root of C_j, where α, β are general elements in $\mathbb{C}$.

(iii) F_{i_0} is called *the contact root* of C_j if F_{i_0} is the ℓ-root of C_j, where ℓ is a linear form with $T_o(C_j)_{red} = \{\ell = 0\}$.

(iv) The η-root of C_j is denoted by $F_{[\eta]}(C_j)$; also the Laufer (resp. contact) root of C_j is denoted by $F_{[L]}(C_j)$ (resp. $F_{[c]}(C_j)$).

In Example 2.6, we have $T_o(C)_{red} = \{y = 0\}$ and $\ll\mathrm{Coeff}_{F_i}(y)\gg = [1,2,3,4,4,4,4,8,8,5,7,7,7,14]$. From this and computations in Example 2.8, we can easily check that $F_{[L]}(C_1) = F_8$, $F_{[L]}(C_2) = F_9$ and $F_{[c]}(C_1) = F_{[c]}(C_2) = F_4$; also $F_{[L]}(C_j) = F_{11}$ for $j = 3,4,5$ and $F_{[c]}(C_5) = F_{14}$. However, C_3 and C_4 have not the contact roots.

Definition 2.11. Under the same condition as Definition 2.10, consider two different irreducible plane curve singularities $(C_{i_1}, o) = (\{f_{i_1} = 0\}, o)$ and $(C_{i_2}, o) = (\{f_{i_2} = 0\}, o)$.

(i) If $F_{[\eta]}(C_{i_1}) = F_{[\eta]}(C_{i_2})$ for a linear form η, then we say that C_{i_1} and C_{i_2} are *η-equivalent* and denote it by $C_{i_1} \overset{[\eta]}{\sim} C_{i_2}$ (or $f_{i_1} \overset{[\eta]}{\sim} f_{i_2}$).

(ii) If $F_{[L]}(C_{i_1}) = F_{[L]}(C_{i_2})$ (resp. $F_{[c]}(C_{i_1}) = F_{[c]}(C_{i_2})$), then we say that C_{i_1} and C_{i_2} are *Laufer (resp. contact) equivalent* and denote it by $C_{i_1} \overset{[L]}{\sim} C_{i_2}$ or $f_{i_1} \overset{[L]}{\sim} f_{i_2}$ (resp. $C_{i_1} \overset{[c]}{\sim} C_{i_2}$ or $f_{i_1} \overset{[c]}{\sim} f_{i_2}$).

(iii) For any j in $I := \{1, \cdots, r\}$, we define the following:

$$I_{[\eta]}(C_j) \text{ (or } I_{[\eta]}(f_j)) := \{\xi \in I \mid C_\xi \overset{[\eta]}{\sim} C_j\} \text{ and } D_{[\eta]}(C_j) \text{ (or } D_{[\eta]}(f_j)) := \bigcup_{\xi \in I_{[\eta]}(C_j)} C_\xi,$$

$$I_{[\epsilon]}(C_j) \text{ (or } I_{[\epsilon]}(f_j)) := \{\xi \in I \mid C_\xi \overset{[\epsilon]}{\sim} C_j\}$$

$$\text{and } D_{[\epsilon]}(C_j) \text{ (or } D_{[\epsilon]}(f_j)) : = \bigcup_{\xi \in I_{[\epsilon]}(C_j)} C_\xi \text{ for } \epsilon = L \text{ or } c.$$

In Example 2.6, $C_3 \overset{[L]}{\sim} C_4 \overset{[L]}{\sim} C_5$ and the Laufer root is F_{11}. In addition, $C_1 \overset{[c]}{\sim} C_2$ and the contact root is F_4. From (2.3) and Definitions 2.3 and 2.11, we can see the following.

Lemma 2.12. *Let η be a linear form with $\eta \nmid f$ and $L_\eta := \{\eta = 0\}$. For any j, we have the following (see Definition 2.3* (iii) *for $e_{[L]}(C_j)$ and $e_{[c]}(C_j)$).*

(i) *If $L_\eta \not\subset T_o(C)_{red}$, then*

$$D_{[L]}(C_j)L_\eta = \sum_{\xi \in I_{[L]}(C_j)} \mathrm{mult}(C_\xi, o) = m_1(C_j)\cdots m_{[LP]}(C_j) \sum_{\xi \in I_{[L]}(C_j)} e_{[L]}(C_\xi).$$

(ii) *If $L_\eta = T_o(C)_{red}$ and $n_1(C_j) < 2m_1(C_j)$ for any j, then*

$$D_{[c]}(C_j)L_\eta = \begin{cases} n_1(C_j) \sum_{\xi \in I_{[c]}(C_j)} e_{[c]}(C_\xi) \text{ if } n_1(C_j) \text{ is even,} \\ n_1(C_j)m_2(C_j)\cdots m_{[cP_i]}(C_j) \sum_{\xi \in I_{[c]}(C_j)} e_{[c]}(C_\xi) \;\text{ if } n_1(C_j)m_2(C_j)\cdots m_{k-1}(C_j) \\ \text{is odd and } m_k(C_j)\, (= m_{[cP_i]}(C_j)) \text{ is even, where } k \geqq 2 \text{ and } i = 1, 2. \end{cases}$$

Let $h = \prod_{j=1}^{r} h_j$ be the irreducible decomposition of a reduced element h of $\mathbb{C}\{x, y\}$. Let ℓ be a linear form with $\ell \nmid h$ and $L_\ell := \{\ell = 0\}$. Let (C, o) (resp. $(\bar{C}, o)$) be the curve singularity defined by $\ell h = 0$ (resp. $h = 0$) and $(C_j, o) := (\{h_j = 0\}, o)$ for any j. Let $(\mathbb{C}^2, o) \xleftarrow{\sigma_1} \cdots \xleftarrow{\sigma_N} (W_N, F(N))$ be the MGE-resolution of (C, o).

Definition 2.13. Under the situation above, let P_k be the center of σ_k and $F_k := \sigma_k^{-1}(P_k) \subset W_k$ $(1 \leqq k \leqq N)$. For simplicity, $(\sigma_{k+1} \circ \cdots \circ \sigma_i)_*^{-1} F_k$ is also denoted by F_k for any i with $k < i \leqq N$; thus $F(N) = \bigcup_{i=1}^{N} F_i$. For any k with $1 \leqq k \leqq N$ and $\sigma := \sigma_1 \circ \cdots \circ \sigma_N$, we define following integers:

$\lambda[F_k] := \mathrm{Coeff}_{F_k}((\ell h) \circ \sigma)_{F(N)}$, $\ell[F_k] := \mathrm{Coeff}_{F_k}(\ell \circ \sigma)_{F(N)}$ and

$\gamma[F_k] := \mathrm{Coeff}_{F_k}((\alpha x + \beta y) \circ \sigma)_{F(N)}$, where α, β are general elements in $\mathbb{C}$.

If $L_\ell \not\subset T_o(C)_{red}$, then $\ell[F_k] = \gamma[F_k]$ for any $F_k \subset F(N)$. However, we remark that if $(W_N, F(N)) \xleftarrow{\tau} (W', F')$ is the blow-up at $Q := F_1 \cap \sigma_*^{-1} L_\ell$ and $F_{k+1} := \tau^{-1}(Q)$, then $\ell[F_{k+1}] = 2$ and $\gamma[F_{k+1}] = 1$.

We have $\lambda[F_1] = ord(h) + 1$ and $\ell[F_1] = 1$. Also, we can easily see the following:

$$(2.6) \qquad \lambda[F_k] = \sum_{P_k \in F_i} \lambda[F_i] + e_{P_k}(\bar{C}) + e_{P_k}(L_\ell) \text{ and } \ell[F_k] = \sum_{P_k \in F_i} \ell[F_i] + e_{P_k}(L_\ell),$$

where $e_{P_k}(\bar{C}) := \sum_{j=1}^{r} e_{P_k}(C_j)$; also $e_{P_k}(L_\ell) := 1$ if $P_k \in (\sigma_1 \circ \cdots \circ \sigma_{k-1})_*^{-1} L_\ell$ else $e_{P_k}(L_\ell) := 0$. For the i-th Puiseux chain $\tilde{P}_i(C_j)$, we can easily see the following:

$$(2.7) \qquad \begin{array}{l} \text{if } F_{\xi_1}, F_{\xi_2} \subset \tilde{P}_i(C_j) \text{ satisfy } F_{\xi_1} \cap F_{\xi_2} \neq \emptyset \text{ and } F_{\xi_1} \neq F_{\xi_2}, \\ \text{then } \gamma[F_{\xi_1}] \text{ or } \gamma[F_{\xi_2}] \text{ is odd.} \end{array}$$

Throughout of this paper, we need several discussions on the relations between the multiplicities of branch locus and orders λ, ℓ and γ in the resolution process.

Let $F_{k(1)}, \cdots, F_{k(r)}$ be irreducible components of $F(N)$ with $F_{k(j)} \prec \tilde{C}_j$ for any j. In general, it might be happend that there exists different i, j with $k(i) = k(j)$. Hence, for

$F_{k(1)}, \cdots, F_{k(r)}$, not all of them are different. The strict transform of $\bar{C}$ onto W_k is also represented by $\bar{C}$. Now we shall state the following simple one.

Lemma 2.14. *Under the situation of Definition 2.13, let j and $F_{k(j)}$ be as above. For this j, the following two conditions are equivalent.*

(i) *The relation $\lambda[F_k] \equiv \ell[F_k]$ mod 2 holds for any k with $F_k \preceq F_{k(j)}$.*

(ii) *The multiplicity $e_{P_k}(\bar{C})$ is even for any k with $F_k \preceq F_{k(j)}$.*

Proof. As seen in Definition 2.7 (i), F_1 is the minimal for the order $\preceq$. By definition, we have $\lambda[F_1] = ord(h) + 1$ and $\ell[F_1] = 1$. Hence it follows that $\lambda[F_1] \equiv \ell[F_1]$ mod 2 if and only if $e_{P_1}(\bar{C})$ is even. The assertion is induced from the induction on the standard partial order by using the relations (2.6). Q.E.D.

Proposition 2.15. *Under the situation of Definition 2.13, if $ord(h_j)$ is even and $F_{[\ell]}(h_j)$ exists for any j, then the following three conditions are equivalent.*

(i) *For every h_j, $\lambda[F_k] \equiv \ell[F_k]$ mod 2 for any k with $F_k \preceq F_{[\ell]}(h_j)$ on the standard order and the ℓ-root $F_{[\ell]}(h_j)$ of h_j.*

(ii) *For every h_j, $e_{P_k}(\bar{C})$ is even for any k with $F_k \preceq F_{[\ell]}(h_j)$.*

(iii) *For every h_j, $e_{P_k}(D_{[\ell]}(C_j))$ is even for any k with $F_k \preceq F_{[\ell]}(h_j)$.*

Proof. (i) $\Leftrightarrow$ (ii) If we put $F_{k(j)} := F_{[\ell]}(h_j)$, then the assertion holds by Lemma 2.14.

(ii) $\Rightarrow$ (iii) We put $g_\xi := \prod_{j \in I_{[\ell]}(h_\xi)} h_j$ and so $h = \prod_{\xi=1}^{s} g_\xi$, where $I = I_{[\ell]}(h_{i_1}) \sqcup \cdots \sqcup I_{[\ell]}(h_{i_s})$ (disjoint union). We prove the assertion by induction on s. If $s = 1$, then $h = g_1$ and thus $e_{P_k}(D_{[\ell]}(h_1)) = e_{P_k}(\bar{C})$ is even by (ii). Assume that the assertion is correct in the $(s-1)$-th step, and consider the s-th step. Let $F_{[\ell]}(h_{i_1})$ be a maximal ℓ-root with respect to "$\preceq$ ". It means that there does not exist h_j with $F_{[\ell]}(h_{i_1}) \prec F_{[\ell]}(h_j)$. It always exsists, even if it is not unique. Put $F_{\ell(i_1)} := F_{[\ell]}(h_{i_1})$ and $P_{\ell(i_1)} := \sigma_{\ell(i_1)}(F_{\ell(i_1)})$. Then, $e_{P_{\ell(i_1)}}(D_{[\ell]}(h_{i_1})) = e_{P_{\ell(i_1)}}(\bar{C})$ because $F_{\ell(i_1)}$ is a maximal ℓ-root. Hence, $e_{P_{\ell(i_1)}}(D_{[\ell]}(h_{i_1}))$ is even by (ii). From Lemma 2.9, $e_{P_k}(D_{[\ell]}(h_{i_1}))$ is even for any k with $F_k \preceq F_{[\ell]}(h_{i_1})$. If we put $(C', o) := (\{g_2 \cdots g_s = 0\}, o)$, then $e_{P_k}(C') = e_{P_k}(\bar{C}) - e_{P_k}(D_{[\ell]}(h_{i_1}))$ is even from (ii) and the above. Namely, for any $j \in \bigcup_{\xi=2}^{s} I_{[\ell]}(h_{i_\xi})$, $e_{P_k}(C')$ is even for any k with $F_k \preceq F_{[\ell]}(h_j)$. From the hypothesis of induction, $e_{P_k}(D_{[\ell]}(h_{i_\xi}))$ is even for any ξ and k with $F_k \preceq F_{[\ell]}(h_{i_\xi})$ $(2 \leqq \xi \leqq s)$.

(iii) $\Rightarrow$ (ii) Put $A(F_k) := \{j \mid F_k \prec \hat{C}_j\}$ and $\bar{A}(F_k) := \{j_1, \cdots, j_m \in A(F_k) \mid C_{j_\xi} \overset{[\ell]}{\not\sim} C_{j_\zeta}$ if $1 \leqq \xi < \zeta \leqq m\}$. Then, $e_{P_k}(\bar{C}) = \sum_{j_\xi \in \bar{A}(F_k)} e_{P_k}(D_{[\ell]}(C_{j_\xi}))$. Hence, (ii) is obtained from (iii). Q.E.D.

3 Covering resolutions for normal surface double points

In this section, we explain covering resolutions over the MSGE-resolutions for normal surface double points. Moreover, we describe some facts under the condition of $\mathbb{Z}_X^2 = -1$.

Definition 3.1. Under the situation of (1.1), when D_i and D_j are different two irreducible components of $F(N) \cup (\bigcup_{j=1}^{r} \hat{C}_j)$, we put $P(D_i, D_j) := D_i \cap D_j$ if $D_i \cap D_j \neq \emptyset$. Put $\Sigma := \{P(D_i, D_j) \in W_N \mid \mathrm{Coeff}_{D_i}(f \circ \sigma)_{W_N}$ and $\mathrm{Coeff}_{D_j}(f \circ \sigma)_{W_N}$ are odd$\}$. Let $(W_N, F(N)) \overset{\sigma_{N+1}}{\longleftarrow} \cdots \overset{\sigma_{\hat{N}}}{\longleftarrow} (W_{\hat{N}}, F(\hat{N}))$ be the succession of one time blowing-ups at all points in Σ. Though there are many choices on successions of blowing-ups $(\mathbb{C}^2, o) \overset{\sigma_1}{\longleftarrow} \cdots \overset{\sigma_{\hat{N}}}{\longleftarrow} (W_{\hat{N}}, F(\hat{N}))$, the composition map $\hat{\sigma} := \sigma_1 \circ \cdots \circ \sigma_{\hat{N}} : (W_{\hat{N}}, F(\hat{N})) \longrightarrow (\mathbb{C}^2, o)$ is uniquely determined by f. We call $\hat{\sigma}$ the *MSGE (i.e., minimal sufficiently good embedded)-resolution* of $(C, o) := (\{f = 0\}, o)$. In [5], such resolution plays a very important role.

For $\hat{\sigma}$ above, consider the covering resolution of a normal double point $(X, o) = \{z^2 = f(x, y)\}$. Let $p : X \longrightarrow \mathbb{C}^2$ be a double covering map induced from $\mathbb{C}^3 \to \mathbb{C}^2$ given by $(x, y, z) \mapsto (x, y)$. We have the following diagram (see [5], [19, p.139]):

(3.1)
$$\begin{tikzcd}
\mathbb{C}^3 & & W_{\hat{N}} \times \mathbb{C} \arrow[ll, "\hat{\sigma}\times id"'] & \\
\cup & & \cup & \\
(X, o) \arrow[d, "p"'] & & X' \arrow[ll, "\phi_1"'] \arrow[d, "\varphi'_X"'] & (\hat{X}, \hat{E}), \arrow[l, "\phi_2"'] \arrow[dl, "\varphi_X"] \\
(\mathbb{C}^2, o) & & (W_{\hat{N}}, F(\hat{N})) \arrow[ll, "\hat{\sigma}"'] &
\end{tikzcd}$$

where X' is the fiber product $X \times_{\mathbb{C}^2} W_{\hat{N}}$ and $\phi_1 := \hat{\sigma} \times id|_{X'}$ is a birational morphism and ϕ_2 is the normalization map. Since $\hat{\sigma}$ is the MSGE-resolution, $\hat{X}$ is non-singular and φ_X is a holomorphic double covering map. Also, $\hat{\pi} := \phi_1 \circ \phi_2 : (\hat{X}, \hat{E}) \to (X, o)$ is a good resolution and we call it *the covering resolution* over $\hat{\sigma}$; put $\hat{C} := (\hat{\sigma})_*^{-1} C$ for $(C, o) = (\{f = 0\}, o)$.

Lemma 3.2. ([5, p.107-108]). *Put $E_i := (\varphi_X)_*^{-1} F_i$ for the map φ_X in (3.1). Also, put $\lambda_i := \mathrm{Coeff}_{F_i}(f \circ \hat{\sigma})_{\hat{F}}$ for any i and $A := \bigcup_{\lambda_i : odd} F_i$. Then*

(i) *if λ_j is odd, then $E_i^2 = \dfrac{1}{2} F_i^2$;*

(ii) *if λ_j is even and F_i does not meet $A \cup \hat{C}$, then, $E_i^2 = F_i^2$;*

(iii) *if λ_j is even and F_i meets $A \cup \hat{C}$, then, $E_i^2 = 2F_i^2$.*

Lemma 3.3. (see [19, Lemma 3.1) *Let G be an irreducible component F_i or $\bar{C}_j$ in* $\mathrm{supp}((f \circ \hat{\sigma})_{W_{\hat{N}}})$ *and G_o an irreducible component of the strict transform $(\varphi_X)_*^{-1} G$ in (3.1). Let M_F (resp. $M_{\hat{E}}$) be the maximal ideal cycle on F (resp. $\hat{E}$). Then*

(i) $v_{G_o}(z \circ \hat{\pi}) = v_G(f \circ \hat{\sigma}) / \gcd(2, v_G(f \circ \hat{\sigma}))$ *and*
$v_{G_o}(g \circ \hat{\sigma} \circ \varphi_X) = 2\, v_G(g \circ \hat{\sigma}) / \gcd(2, v_G(f \circ \hat{\sigma}))$ *for any $g \in \mathbb{C}\{x, y\}$;*

(ii) $\mathrm{Coeff}_{E_i} M_{\hat{E}} = 2\ \mathrm{Coeff}_{F_i} M_F / \gcd(2, v_{F_i}(f \circ \hat{\sigma}))$ *for* $E_i := (\varphi_X)_*^{-1} F_i$.

Lemma 3.4. ([21, p.426]). *Let* $\pi : (\tilde{X}, E) \longrightarrow (X, o)$ *be a resolution of a normal surface singularity. Let* D_1 *and* D_2 *be two anti-nef cycles on* E *(i.e.,* $D_j E_i \leqq 0$ *for any irreducible component* E_i *of* E *and* $j = 1, 2$*) with* $D_1 \leqq D_2$*. Then* $D_1^2 \geqq D_2^2$*, and* $D_1 = D_2$ *if and only if* $D_1^2 = D_2^2$*.*

In the following, let (X, o) be a normal surface singularity with $\mathbb{Z}_X^2 = -1$. Let $\pi : (\tilde{X}, E) \longrightarrow (X, o)$ be a resolution and $E = \bigcup_{j=1}^{r} E_j$ the irreducible decomposition.

Definition 3.5. Let D be an anti-nef effective cycle on E. If E_i satisfies $DE_i < 0$, then we call E_i *a D-negative component.* Since $\mathbb{Z}_X^2 = -1$, there exists a unique Z_E-negative component E_α on E satisfying $\mathrm{Coeff}_{E_\alpha} Z_E = 1$ and $Z_E E_\alpha = -1$.

Lemma 3.6. *Assume that there is an anti-nef effective cycle* M *with* $M^2 = -2$ *on* E*. If we put* $Z_1 := M - Z_E$ *and* $A := \mathrm{supp}(Z_1)$*, then we have the following.*

(i) $Z_1^2 = -1$, $Z_E Z_1 = 0$, $MZ_E = MZ_1 = -1$ *and* $Z_1 E_i \leqq 0$ *for any* i $(\neq \alpha)$.

(ii) $E_\alpha \not\subset A$, $\mathrm{Coeff}_{E_\alpha} M = 1$, $ME_\alpha = 0$ *and* $Z_1 E_\alpha = 1$.

(iii) A *is a connected set with* $Z_A = Z_1$*, and* $M < 2Z_E$.

Proof. (i) Since $-2 = M^2 = Z_E^2 + 2Z_E Z_1 + Z_1^2$, we have $Z_1^2 = -1$ and $Z_E Z_1 = 0$. Since $0 = Z_E Z_1 = Z_E(M - Z_E) = MZ_E - \mathbb{Z}_X^2 = MZ_E + 1$, we have $MZ_E = -1$. Hence, $MZ_1 = -1$ and $Z_1 E_i = ME_i \leqq 0$ for any i $(\neq \alpha)$.

(ii) If $E_\alpha \subset A$, then $0 = Z_E Z_1 \leqq Z_E E_\alpha = -1$ (contradiction). Thus, $E_\alpha \not\subset A$ and so $\mathrm{Coeff}_{E_\alpha} M = \mathrm{Coeff}_{E_\alpha} Z_E = 1$; thus $Z_1 E_\alpha \geqq 0$. If $Z_1 E_\alpha = 0$, then Z_1 is an anti-nef cycle on E from (i). Then, $Z_E \leqq Z_1$ and so $E_\alpha \subset A$. This contradicts to the above. Hence, $Z_1 E_\alpha > 0$ and so $-1 = Z_E E_\alpha = ME_\alpha - Z_1 E_\alpha$; thus $ME_\alpha = 0$ and $Z_1 E_\alpha = 1$.

(iii) Let $A = \bigcup_{k=1}^{s} A(k)$ be the disjoint union of connected components. Then, $\sum_{k=1}^{s} Z_{A(k)} \leqq Z_1$ from (i) and (ii); thus $-s \geqq \sum_{k=1}^{s} Z_{A(k)}^2 \geq Z_1^2 = -1$. Then, $s = 1$ and A is connected. From (i), $Z_1 E_i = ME_i \leqq 0$ for any E_i with $i \neq \alpha$. Since $E_\alpha \not\subset A$ by (ii), Z_1 is an anti-nef cycle on A and so $Z_A \leqq Z_1$. From (i), we have $-1 = Z_1^2 \leqq Z_A^2 < 0$ and $Z_1 = Z_A \leqq Z_E$ by Lemma 3.4, and so $M \leqq 2Z_E$. If $M = 2Z_E$, then it yields a contradiction: $-2 = M^2 = 4Z_E^2 = -4$. Hence, $M < 2Z_E$. Q.E.D.

Proposition 3.7. *Assume that* π *is the minimal or minimal good resolution.*

(i) *If there exists an anti-nef effective cycle* M *on* E *with* $M^2 = -2$*, then* $A := \mathrm{supp}(M - Z_E)$ *is not contracted to a rational singularity. Here, a nonsingular point is considered as a kind of rational singularity.*

(ii) *When* (X, o) *is a normal double point,* $\mathbb{M}_X = \mathbb{Z}_X$ *if and only if* $\mathbb{M}_{o,X} = \mathbb{Z}_{o,X}$.

Proof. (i) For Z_1 and A in Lemma 3.6, assume that A is contracted to a rational singularity (Y,o). Thus, $\mathrm{mult}(Y,o) = -Z_A^2 = -Z_1^2 = 1$ by Artin' result ([2]). By a succession of contractions of (-1)-curves $\tau : (\tilde{X}, E) \longrightarrow (\bar{X}, \bar{E})$ and Lemma 3.6 (ii), $Q := \tau(A)$ is a non-singular point on $\bar{X}$ with $Q \in \tau(E_\alpha)$. As a corollary of intersection theory (for example, see [6, p.386-395]), we can see that if $\sigma : (W,F) \longrightarrow (\mathbb{C}^2, o)$ is a succession of blowing-ups, we have $\mathrm{mult}(C,o) = Z_F \sigma_*^{-1} C$ for a curve singularity $(C,o) = (\tau(E_\alpha), Q)$. Therefore, from $\mathrm{mult}_Q(\bar{E}) = Z_A E_\alpha = Z_1 E_\alpha = 1$, $\bar{E}$ is simple normal crossing. This contradicts to the assumption that π is the minimal or minimal good resolution.

(ii) Since "if "part is obvious, we prove "only if "part. Assume that $\mathbb{M}_X = \mathbb{Z}_X$. If $\mathbb{M}_{o,X} > \mathbb{Z}_{o,X}$, then $\mathbb{M}_{o,X}^2 = -\mathrm{mult}(X,o) = -2 < \mathbb{Z}_{o,X}^2 = -1$ from Theorem 1.1 and [21, Theorem 2.7]. From (i), A is not contracted to a non-singular point on the minimal resolution space. This contradicts to $\mathbb{M}_X = \mathbb{Z}_X$. Q.E.D.

Lemma 3.8. *Assume the same situation as Lemma 3.6.*

(i) *There exists a unique irreducible component E_β of E with $ME_\beta = -1$ and $\mathrm{Coeff}_{E_\beta} M = 2$. Especially, E_β is the M-negative component of E.*

(ii) *$E_\beta \neq E_\alpha$, $Z_E E_\beta = 0$ and $\mathrm{Coeff}_{E_\beta} Z_E = 1$.*

Proof. (i) Assume that there exist different two M-negative components E_{β_1} and E_{β_2}. Then, $ME_{\beta_i} = -1$ and $\mathrm{Coeff}_{E_{\beta_i}} M = 1$ for $i = 1, 2$. From $\mathrm{Coeff}_{E_{\beta_i}} Z_E = \mathrm{Coeff}_{E_{\beta_i}} M = 1$, we have $Z_E E_{\beta_i} = E_{\beta_i}^2 + (Z_E - E_{\beta_i})E_{\beta_i} \leqq E_{\beta_i}^2 + (M - E_{\beta_i})E_{\beta_i} = ME_{\beta_i}$. This yields a contradiction: $-1 = Z_E^2 \leqq Z_E(E_{\beta_1} + E_{\beta_2}) \leqq ME_{\beta_1} + ME_{\beta_2} = -2$. Hence the M-negative component E_β exists uniquely. If $ME_\beta = -2$, then $\mathrm{Coeff}_{E_\beta} M = \mathrm{Coeff}_{E_\beta} Z_E = 1$ and so it yields a contradiction $-1 \leqq Z_E E_\beta \leqq ME_\beta = -2$. Hence, $ME_\beta = -1$ and $\mathrm{Coeff}_{E_\beta} M = 2$.

(ii) From Lemma 3.6, we have $\mathrm{Coeff}_{E_\alpha} M = 1$. Then, $E_\alpha \neq E_\beta$ from (i) and so $Z_E E_\beta = 0$. Here, assume that $\mathrm{Coeff}_{E_\beta} Z_E \geqq 2$. From (i), we have $\mathrm{Coeff}_{E_\beta} M = \mathrm{Coeff}_{E_\beta} Z_E = 2$. This yields a contradiction : $0 = Z_E E_\beta = 2E_\beta^2 + (Z_E - 2E_\beta)E_\beta \leqq 2E_\beta^2 + (M - 2E_\beta)E_\beta = ME_\beta = -1$. Thus, we have $\mathrm{Coeff}_{E_\beta} Z_E = 1$. Q.E.D.

Lemma 3.9. *Put $Z_2 := 2Z_E - M$ on E (so $Z_E = Z_1 + Z_2$) and $B := \mathrm{supp}(Z_2)$.*

(i) *$Z_2 E_\beta = 1$, $Z_2 E_\alpha = -2$, $E_\beta \not\subset B$ and $Z_2^2 = -2$.*

(ii) *B is a connected set and $Z_B = Z_2$.*

Proof. (i) By Lemma 3.8, $Z_2 E_\beta = 2Z_E E_\beta - ME_\beta = 1$ and $\mathrm{Coeff}_{E_\beta} M = 2$ and $\mathrm{Coeff}_{E_\beta} Z_E = 1$; thus $E_\beta \not\subset B$ from the definition of Z_2. From Lemma 3.6, $Z_2 E_\alpha = 2Z_E E_\alpha - ME_\alpha = -2$. Since $MZ_E = -1$ by Lemma 3.6 (i), $Z_2^2 = -4MZ_E - 6 = -2$.

(ii) Let $B = \bigcup_{j=1}^{s} B(j)$ be the decomposition of connected components such that $E_\alpha \subset B(1)$. Since $Z_2^2 = -2$ by (i), we have that $s \leqq 2$ and $Z_B = \sum_{j=1}^{2} Z_{B(j)}$. Assume that $B(2) \neq \emptyset$. Since $Z_1 E_i = 0$ for any irreducible component E_i of $B(2)$, we have $0 \geqq Z_E E_i = (Z_1 + Z_2)E_i = Z_2 E_i = (2Z_E - M)E_i = -ME_i \geqq 0$ and then $Z_2 E_i = 0$ for any $E_i \subset B(2)$. This is a contradiction and so $s = 1$. Thus B is connected. If $Z_B^2 = -1$, then $B = E$. This is also a contradiction and so $Z_B^2 = -2$. Hence, we have $-2 = Z_2^2 \leqq Z_B^2 = -2$ and so $Z_2 = Z_B$. Q.E.D.

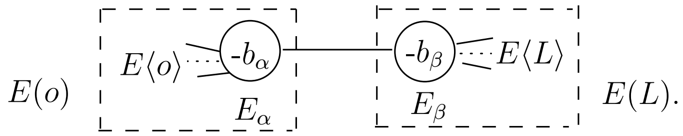
Definition 3.10. From Lemma 3.6 (iii), we have $Z_E < M_E < 2Z_E$. For the irreducible decomposition $E = \bigcup_{i=1}^{n} E_i$ and $I := \{1, \cdots, n\}$, we put $I(L) := \{i \in I \mid \mathrm{Coeff}_{E_i} M = 2\,\mathrm{Coeff}_{E_i} Z_E\}$, $I(o) := \{i \in I \mid \mathrm{Coeff}_{E_i} M = \mathrm{Coeff}_{E_i} Z_E\}$ and $I(c) := \{i \in I \mid \mathrm{Coeff}_{E_i} Z_E < \mathrm{Coeff}_{E_i} M < 2\,\mathrm{Coeff}_{E_i} Z_E\}$, where $I = I(L) \sqcup I(c) \sqcup I(o)$ (disjoint union). Let us define the three subsets of E as follows: $E(L) := \bigcup_{i \in I(L)} E_i$, $E(c) := \bigcup_{i \in I(c)} E_i$ and $E(o) := \bigcup_{i \in I(o)} E_i$. Though $E_\beta \subset E(L)$ and $E_\alpha \subset E(o)$, $E(c)$ may happen to be empty. In this paper, $E(o)$, $E(c)$ and $E(L)$ are called the *odd part* of E, the *contact part* of E and the *Laufer part* of E respectively. Also, we call $E = E(o) \cup E(c) \cup E(L)$ *the Laufer decomposition of* E.

Theorem 3.11. *Under the situation above, if we put* $E\langle \epsilon \rangle := E(\epsilon) \backslash E_\beta$ *for* $\epsilon = L$ *and* o, *then we have the following.*

(i) $E(L)$ *and* $E(o)$ *are connected sets; also* $E(c)$ *is a connected set if* $E(c) \neq \emptyset$.

(ii) *If* $E(c) = \emptyset$, *then* $E_\alpha E(L) = E_\beta E(o) = 1$; *hence, the rough shape of the w.d.graph of* E *is given as follows:*

(3.2)

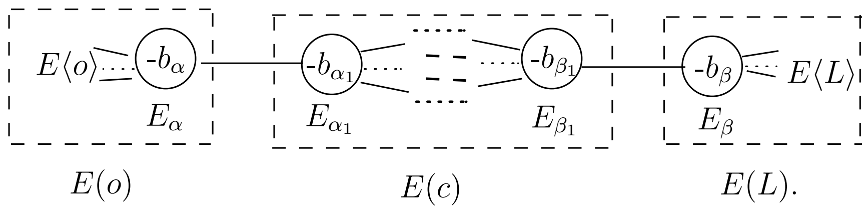


(iii) *If* $E(c) \neq \emptyset$, *then there exist irreducible components* $E_{\alpha_1}, E_{\beta_1} \subset E(c)$ *satisfying the following conditions:*

$$\begin{cases} \text{(iii-1)}\ \ E_\alpha E_{\alpha_1} = E_\beta E_{\beta_1} = 1; \textit{ also it may happen to be } E_{\alpha_1} = E_{\beta_1}. \\ \text{(iii-2)}\ \ E(o) \cap E(c) = E_\alpha \cap E_{\alpha_1} \textit{ and } E(c) \cap E(L) = E_\beta \cap E_{\beta_1}. \\ \text{(iii-3)}\ \ \mathrm{Coeff}_{E_{\alpha_1}} M = \mathrm{Coeff}_{E_{\alpha_1}} Z_E + 1 \textit{ and } \mathrm{Coeff}_{E_{\beta_1}} M = 2\,\mathrm{Coeff}_{E_{\beta_1}} Z_E - 1. \end{cases}$$

(iv) *If* $E(c) \neq \emptyset$, *then the rough shape of the w.d.graph of* E *is given as follows:*

(3.3)



Proof. Let $Z_1 := M_E - Z_E$, $Z_2 := 2Z_E - M_E$, $A := \mathrm{supp}(Z_1)$ and $B := \mathrm{supp}(Z_2)$ as Lemmata 3.6 and 3.9.

(i) By Lemmata 3.6 and 3.9, $E(o) \cap A = E_\alpha \cap A$ (resp. $E(L) \cap B = E_\beta \cap B$) is one point; also A and B are connected. Hence $E(L)$ and $E(o)$ are connected. If $E(c) \neq \emptyset$ and $E(c)$ is not connected, then A and B are not connected. This contradicts to Lemmata 3.6 and 3.9.

(ii) If $E_i \subset E(o)$ with $i \neq \alpha$, then $E_i \neq E_\beta$ from $E_i \not\subset E(L)$ and so $Z_1 E_i = 0$. Thus, $Z_1 E(o) = Z_1 E_\alpha = 1$ from Lemma 3.6 (ii) and so $E_\alpha E(L) = 1$. If $E_i \subset E(L)$ and $E_i \neq E_\beta$, then $E_i \neq E_\alpha$ from $E_\alpha \not\subset E(L)$ and so $Z_2 E_i = 0$. Then, $Z_2 E(L) = Z_2 E_\beta = 1$ from Lemma 3.9 (i) and so $E_\beta E(o) = 1$.

(iii) Since $E(L) = \bigcup\limits_{E_i \not\subset B} E_i$, $E(c) = \bigcup\limits_{E_i \subset A \cap B} E_i$ and $E(o) = \bigcup\limits_{E_i \not\subset A} E_i$, we have $Z_1 E_i = 0$ for any $E_i \subset E(o) \backslash E_\alpha$. Since $Z_1 E_\alpha = 1$ by Lemma 3.6 (ii), there exists a unique irreducible component E_{α_1} of A satisfying $\mathrm{Coeff}_{E_{\alpha_1}} Z_1 = 1$ and $E_\alpha E_{\alpha_1} = 1$. Hence, $E(c) \cap E(o) = A \cap E(o) = E_{\alpha_1} \cap E_\alpha$ is one point. Therefore, $\mathrm{Coeff}_{E_{\alpha_1}} M = \mathrm{Coeff}_{E_{\alpha_1}} Z_E + \mathrm{Coeff}_{E_{\alpha_1}} Z_1 = \mathrm{Coeff}_{E_{\alpha_1}} Z_E + 1$. If $E_j \subset E(L) \backslash E_\beta$, then $E_j \neq E_\alpha$ from $E_\alpha \subset E(o)$, and so $Z_2 E_j = 0$. Since $Z_2 E_\beta = 1$ by Lemma 3.9 (i), there exists a unique irreducible component E_{β_1} of B satisfying $\mathrm{Coeff}_{E_{\beta_1}} Z_2 = 1$ and $E_\beta E_{\beta_1} = 1$; hence $E(c) \cap E(L) = B \cap E(L) = E_\beta \cap E_{\beta_1}$ is one point. Therefore, $\mathrm{Coeff}_{E_{\beta_1}} M = 2\,\mathrm{Coeff}_{E_{\beta_1}} Z_E - \mathrm{Coeff}_{E_{\beta_1}} Z_2 = 2\,\mathrm{Coeff}_{E_{\beta_1}} Z_E - 1$.

(iv) follows easily from (i)-(iii). Q.E.D.

In the following, let $(X, o) = \{z^2 = f(x, y)\}$ be a normal double point and $f = \prod\limits_{j=1}^{r} f_j$ the irreducible decomposition. Assume that $ord(f_{j_0})$ is odd ($1 \leqq j_0 \leqq r$). The next construction of the MSGE-resolution (see Definition 3.1) of $(C, o) := (\{f = 0\}, o)$ is useful in the following.

$$
(3.4)\quad \begin{cases} \text{(i)}\quad F_1 := \sigma_1^{-1}(\{0\}) \ \text{ and } \ P_2 =: F_1 \cap (\sigma_1)_*^{-1} C_{j_0} \ \text{ for } \ (C_{j_0}, o) := (\{f_{j_0} = 0\}, o), \\ \text{(ii)}\quad (W_1, F(1)) \xleftarrow{\sigma_2} (W_2, F(2)) \ \text{ is the blowing-up at } P_2 \ \text{ and } F_2 := \sigma_2^{-1}(P_2), \\ \text{(iii)}\quad (W_2, F(2)) \xleftarrow{\sigma_3} \cdots \xleftarrow{\sigma_{\hat{N}}} (W_{\hat{N}}, F(\hat{N})) \ \text{is a succession of blowing-ups such} \\ \qquad\qquad \text{that } \ \hat{\sigma} := \sigma_1 \circ \cdots \circ \sigma_{\hat{N}} \ \text{ is the MSGE-resolution of } (C, o). \end{cases}
$$

Let us put $F_k := \sigma_k^{-1}(P_k)$ for the center P_k of σ_k, and $(\sigma_k \circ \cdots \circ \sigma_{\hat{N}})_*^{-1} F_k$ is also denoted by F_k ($1 \leqq k \leqq \hat{N}$) and $\hat{F} := F(\hat{N})$; hence $\hat{F} = \bigcup\limits_{k=1}^{\hat{N}} F_k$. Let $\hat{\pi} : (\hat{X}, \hat{E}) \longrightarrow (X, o)$ be the covering resolution over $\hat{\sigma}$. Put $E_i := (\varphi_X)_*^{-1} F_i$ for any $F_i \subset F(\hat{N})$ and φ_X of (3.1); also $(C_j, o) := (\{f_j = 0\}, 0)$ for any j.

Proposition 3.12. *Under the situation above, if we put $I_o := \{j \in \{1, \cdots, r\} \mid$ $ord(f_j)$ is odd$\}$ and $f_{\langle\langle o \rangle\rangle} := \prod\limits_{j \in I_o} f_j$, then we have the following.*

(i) *E_1 is a smooth irreducible rational curve which is the $M_{\hat{E}}$-negative component of $\hat{E}$ and $E_1 \subset \hat{E}(L)$. Moreover, we have $M_{\hat{E}}E_1 = -1$ and $Z_{\hat{E}}E_1 = 0$.*

(ii) *$T_0(f_{\langle\langle o\rangle\rangle})_{red}$ is a single line. In particular, σ_2 and F_2 are uniquely determined by $f(x,y)$.*

Proof. (i) From [19, Proposition 3.3], we have $M_{\hat{E}} = (\alpha x + \beta y)_{\hat{E}}$ for general elements α, β of $\mathbb{C}$ and E_1 satisfies $M_{\hat{E}}E_1 < 0$. Since $\mathrm{Coeff}_{F_1}(f \circ \hat{\sigma})_{\hat{F}} = ord(f)$ is odd and $\mathrm{Coeff}_{F_1}((\alpha x+\beta y)\circ\hat{\sigma})_{\hat{F}} = 1$, E_1 is a smooth irreducible rational curve and $\mathrm{Coeff}_{E_1}M_{\hat{E}} = 2$ from Lemma 3.3 (ii). Thus we have $M_{\hat{E}}^2 = -2$ and $M_{\hat{E}}E_1 = -1$. Hence, E_1 is the $M_{\hat{E}}$-negative component. From Theorem 3.11, we have $E_1 \subset \hat{E}(L)$. Hence, E_1 is not the $Z_{\hat{E}}$-negative component and so $Z_{\hat{E}}E_1 = 0$.

(ii) Let f_{j_0}be any irreducible factor of $f_{\langle\langle o\rangle\rangle}$. Let F_{j_0} be the irreducible component of $F(\hat{N})$ with $E_{j_0} \cap (\varphi_X)_*^{-1}\hat{C}_{j_0} \neq \emptyset$ for $E_{j_0} := (\varphi_X)_*^{-1}F_{j_0}$, where $\hat{C}_{j_0} := (\hat{\sigma})_*^{-1}C_{j_0}$. From M. Noether's Theorem ([3, p.518]) and the assumption, we have

$$\mathrm{Coeff}_{F_{j_0}}((\alpha x + \beta y) \circ \hat{\sigma})_{\hat{F}} = ord(f_{j_0}) \equiv 1 \pmod 2$$

and

$$\mathrm{Coeff}_{F_{j_0}}(f \circ \hat{\sigma})_{\hat{F}} = ord(f \circ f_{j_0}) \equiv 0 \pmod 2.$$

Therefore, $\mathrm{Coeff}_{E_{j_0}} M_{\hat{E}}$ should be odd by Lemma 3.3 (ii). Since $\mathrm{Coeff}_{E_i} M_{\hat{E}}$ is even for any $E_i \subset \hat{E}(L)$, the component E_{j_0} should be contained in $\hat{E}(c) \cup \hat{E}(o)$. Now we assume that there exist mutually distinct functions f_1 and f_2 in $f_{\langle\langle o\rangle\rangle}$ such that $T_o(f_1)_{red} \neq T_o(f_2)_{red}$. Then the component $E_1 = E_\beta$ intersects at least two points with the component in $\hat{E}(c) \cup \hat{E}(o)$. This contradicts the configuration (3.2) or (3.3). Hence, we have the assertion. Q.E.D.

From Proposition 3.12 (ii), we can easily see the following.

Theorem 3.13. *For a normal double point $(X, o) = \{z^2 = f(x, y)\}$ and $f = \prod_{j=1}^{r} f_j$ the irreducible decomposition, we assume that there are at least two irreducible factors f_1 and f_2 of f such that $ord(f_1)$ and $ord(f_2)$ are odd and $T_o(f_1)_{red} \neq T_o(f_2)_{red}$. Then, $\mathbb{Z}_X^2 = -2$ and so $M_E = Z_E$ for any resolution space $(\tilde{X}, E)$ of (X, o).*

Proposition 3.14. *Assume $\mathbb{Z}_X^2 = -1$. Let $(\hat{X}, \hat{E}) \xrightarrow{\hat{\pi}} (X, o)$ be the covering resolution over the MSGE-resolution $\hat{\sigma}$ of (C, o) constructed in (3.4). Then,*

(i) $\mathrm{Coeff}_{F_2}(f \circ \hat{\sigma})_{\hat{F}}$ *is even,*

(ii) *E_2 is the $Z_{\hat{E}}$-negative component of $\hat{E}$ and so E_2 is an irreducible curve,*

(iii) *if we put $K(c) := \{k \in \{1, \cdots, \hat{N}\} \mid (\sigma_3 \circ \cdots \circ \sigma_{\hat{N}})(F_k) = F_1 \cap F_2$ in $W_2\}$*, then $\hat{E}(c) = \bigcup_{k \in K(c)} E_k$.

Proof. Let us note that F_2 is uniquely determined from f by Proposition 3.12 (ii).

(i) Suppose that $\mathrm{Coeff}_{F_2}(f\circ\hat{\sigma})_{\hat{F}}$ is odd. Consider the diagram (3.1) for this case. Since $\mathrm{Coeff}_{F_2}M_{\hat{F}}=1$, we have $\mathrm{Coeff}_{E'_2}M_{\hat{E}}=2$ from Lemma 3.3 (ii), where E'_2 is any irreducible component of E_2. First, assume that $\mathrm{Coeff}_{E'_2}Z_{\hat{E}}=2$. From the definition of $\hat{E}(o)$, we have $E'_2\subset\hat{E}(o)$. Let $\bigcup_{i=3}^{\ell_0}F_i$ be the union of all components which are contracted to $F_1\cap F_2$ by $\sigma_3\circ\cdots\circ\sigma_{\ell_0}$ ($\ell_0\leqq\hat{N}$). Then we have $\mathrm{Coeff}_{F_i}M_{\hat{F}}\geqq 2$ for any i with $3\leqq i\leqq\ell_0$. From Lemma 3.3 (ii), we have

$$\mathrm{Coeff}_{E_i}M_{\hat{E}}\geqq 2 \text{ for any } i \text{ with } 3\leqq i\leqq\ell_0. \tag{3.5}$$

Since $E_1\subset\hat{E}(L)$ and $E'_2\subset\hat{E}(o)$, there is the $Z_{\hat{E}}$-negative component E_{k_0} in $\bigcup_{i=3}^{\ell_0}E_i$ by Theorem 3.11. Hence, $\mathrm{Coeff}_{E_{k_0}}M_{\hat{E}}=1$ from Lemma 3.6 (ii); this contradicts to (3.5). Second, assume that $\mathrm{Coeff}_{E'_2}Z_{\hat{E}}=1$; thus $E'_2\subset\hat{E}(L)$. Let f_{j_0} be an irreducible factor of f such that $ord(f_{j_0})$ is odd, and $C_{j_0}:=\{f_{j_0}=0\}$. Since $(W_{\hat{N}},\hat{F})$ is the MSGE-resolution space of (C,o), there exists an irreducible component F_{j_0} of $\hat{F}$ with $F_{j_0}\cap\hat{C}_{j_0}\neq\emptyset$, where $\hat{C}_{j_0}:=\hat{\sigma}_*^{-1}C_{j_0}$. Since $\mathrm{Coeff}_{\hat{C}_{j_0}}(f\circ\hat{\sigma})_{W_{\hat{N}}}=1$, $\mathrm{Coeff}_{F_{j_0}}(f\circ\hat{\sigma})_{\hat{F}}$ is even. Hence, $\mathrm{Coeff}_{E_{j_0}}M_{\hat{E}}=\mathrm{Coeff}_{F_{j_0}}M_{\hat{F}}$ from Lemma 3.3 and $\mathrm{Coeff}_{F_{j_0}}M_{\hat{F}}=ord(f_{j_0})$ from M. Noether's theorem; thus $\mathrm{Coeff}_{E_{j_0}}M_{\hat{E}}$ is odd. On the other hand, E'_2 and E_{j_0} are contained in a same connected component of $\hat{E}\backslash E_1$ from Theorem 3.11, and $E'_2\subset\hat{E}(L)$. Thus, we have $E_{j_0}\subset\hat{E}(L)$ by Theorem 3.11. Hence, $\mathrm{Coeff}_{E_{j_0}}M_{\hat{E}}$ is even. This yields a contradiction. Therefore, $\mathrm{Coeff}_{F_2}(f\circ\hat{\sigma})_{\hat{F}}$ is even.

(ii) Since $\mathrm{Coeff}_{F_2}M_{\hat{F}}=1$ and (i), we have $\mathrm{Coeff}_{E'_2}M_{\hat{E}}=\mathrm{Coeff}_{E'_2}Z_{\hat{E}}=1$ for $E'_2:=(\varphi_X)_*^{-1}F_2$ from Lemma 3.3 (ii). Hence, we have $E'_2\subset\hat{E}(o)$. Assume $F_1\cap F_2\cap(\sigma_1\circ\sigma_2)_*^{-1}C=\emptyset$ on W_2. Then, $E_1\cap E'_2\neq\emptyset$ and $E_1\subset\hat{E}(L)$ from Proposition 3.12 (i). Since $\mathrm{Coeff}_{F_1}(f\circ\hat{\sigma})_{\hat{F}}$ is odd and $F_1\cap F_2\neq\emptyset$ in $W_{\hat{N}}$, E'_2 is irreducible (i.e., $E'_2=E_2$) in this case. From (i) and $\mathrm{Coeff}_{F_2}M_{\hat{F}}=1$, we have $\mathrm{Coeff}_{E_2}M_{\hat{E}}=1$ and so $E_2\subset\hat{E}(o)$. Hence, $\hat{E}(c)=\emptyset$ and E_2 is the $Z_{\hat{E}}$-negative component by Theorem 3.11 (ii). Assume $F_1\cap F_2\cap(\sigma_1\circ\sigma_2)_*^{-1}C\neq\emptyset$ in W_2. If we take a succession of blowing-ups $\sigma_3,\cdots,\sigma_{\ell_0}$ in (i), then $\bigcup_{i=3}^{\ell_0}F_i$ intersects at one point of F_j for $j=1,2$. From the construction, we can easily see that $\mathrm{Coeff}_{F_i}M_{\hat{F}}\geqq 2$ for any i ($3\leqq i\leqq\ell_0$). Hence, $\mathrm{Coeff}_{E_i}M_{\hat{E}}\geqq 2$ for any i ($3\leqq i\leqq\ell_0$) by Lemma 3.3 (ii). If E'_2 is not the $Z_{\hat{E}}$-negative component, then $\bigcup_{i=3}^{\ell_0}E_i$ contains the $Z_{\hat{E}}$-negative component E_{i_0} from $E'_2\subset\hat{E}(o)$ and Theorem 3.11 ($3\leqq i_0\leqq\ell_0$). Thus, $\mathrm{Coeff}_{E_{i_0}}M_{\hat{E}}=1$ from Lemma 3.6 (ii) and this contradicts to $\mathrm{Coeff}_{E_{i_0}}M_{\hat{E}}\geqq 2$. Hence, E'_2 is the $Z_{\hat{E}}$-negative component. By the uniqueness of $Z_{\hat{E}}$-negative component, E_2 is irreducible (i.e., $E'_2=E_2$).

(iii) Since E_1 (resp. E_2) is the $M_{\hat{E}}$ (resp. $Z_{\hat{E}}$)-negative component by Proposition 3.12 (i) (resp. Proposition 3.14 (ii)), (iii) follows easily from Theorem 3.11 (iv) and the construction of $(\hat{X},\hat{E})$. Q.E.D.

Assume that $\mathbb{Z}_X^2 = -1$. From Theorem 1.3 (i), $ord(f)$ is odd and there exists f_{j_0} such that $ord(f_{j_0})$ is odd. Let $L_0 := T_o(f_{j_0})_{red}$ and ℓ a linear form with $L_0 = \{\ell = 0\}$. Let $\hat{\sigma} : (W_{\hat{N}}, F(\hat{N})) \longrightarrow (\mathbb{C}^2, o)$ be the MSGE-resolution of $(C, o) := (\{f = 0\}, o)$ constructed in (3.4) and $\hat{\pi} : (\hat{X}, \hat{E}) \longrightarrow (X, o)$ the covering resolution over $\hat{\sigma}$.

Definition 3.15. Under the situation above, let $\hat{E} = \hat{E}(L) \cup \hat{E}(c) \cup \hat{E}(o)$ be the Laufer decomposition of $\hat{E}$ with respect to $M_{\hat{E}}$, where $\hat{E}(L) \neq \emptyset$ and $\hat{E}(o) \neq \emptyset$ and $\hat{E}(c)$ may be empty. Put $\hat{F}(\epsilon) := \varphi_X(\hat{E}(\epsilon))$ for $\epsilon = L$, c or o, where φ_X is the map in (3.1). Let $I(\epsilon) := \{j \in I \mid (\hat{\sigma})_*^{-1} C_j \cap \hat{F}(\epsilon) \neq \emptyset\}$. Let us define elements of three types in $\mathbb{C}\{x, y\}$ as follows: $f_{[o]} := \prod_{j \in I(o)} f_j$ and $f_{[\epsilon]} := \prod_{j \in I(\epsilon)} f_j$ if $I(\epsilon) \neq \emptyset$ else $f_{[\epsilon]} := 1$ for $\epsilon = L$ or c. Since $I = I(L) \sqcup I(c) \sqcup I(o)$, we have $f = f_{[L]} f_{[c]} f_{[o]}$. We call it *the weak Laufer decomposition of* f. Further, $f_{[L]}$, $f_{[c]}$ and $f_{[o]}$ are called the *Laufer factor of* f, *the contact factor of* f and *the odd factor of* f respectively.

Proposition 3.16. *Under the same situation as Definition 3.15, let* $f = f_{[L]} f_{[c]} f_{[o]}$ *be the weak Laufer decomposition of* f. *Then, we have the following.*

(i) *For any irreducible factor* f_j *of* $f_{[L]}$, $ord(f_j)$ *is even.*

(ii) $ord(f_{[c]})$ *is even and* $ord(f_{[o]})$ *is odd.*

Proof. (i) Assume that there is an irreducible factor f_{j_0} of $f_{[L]}$ such that $ord(f_{j_0})$ is odd. Put $\hat{C}_{j_0} := (\hat{\sigma})_*^{-1}\{f_{j_0} = 0\}$. Let F_{j_0} be an irreducible component of $\hat{F}$ satisfying $F_{j_0} \cap \hat{C}_{j_0} \neq \emptyset$. Since $\mathrm{Coeff}_{F_{j_0}} M_{\hat{F}} = ord(f_{j_0})$ is odd and $\mathrm{Coeff}_{F_{j_0}}(f \circ \hat{\sigma})_{\hat{F}}$ is even, $\mathrm{Coeff}_{E_{j_0}} M_{\hat{E}}$ is odd by Lemma 3.3. This yields a contradiction because of $E_{j_0} \subset \hat{E}(L)$.

(ii) If $f_{[c]} = 1$, then $ord(f_{[o]})$ is odd from the discussion in (i). Thus, assume that $f_{[c]} \neq 1$. Put $(C_{[\epsilon]}, o) := (\{f_{[\epsilon]} = 0\}, o)$ and $\hat{C}_{[\epsilon]} := (\hat{\sigma})_*^{-1} C_{[\epsilon]}$ for $\epsilon = L$, c and o. Also, put $d := ord(f)$, $d_{[\epsilon]} := e_{P_2}((\sigma_1)_*^{-1} C_{[\epsilon]})$ for $\epsilon = c$ and o. Then, $\mathrm{Coeff}_{F_1}(f \circ \sigma_1 \circ \sigma_2)_{W_2} = \mathrm{Coeff}_{F_1}(f \circ \hat{\sigma})_{W_{\hat{N}}} = d$, and $\mathrm{Coeff}_{F_2}(f \circ \sigma_1 \circ \sigma_2)_{W_2} = \mathrm{Coeff}_{F_2}(f \circ \hat{\sigma})_{W_{\hat{N}}} = d + d_{[c]} + d_{[o]}$ is even from Proposition 3.14 (i). From Theorem 3.11, $E_2 \cap (\hat{E}(L) \cup \hat{E}(c))$ is one point and put it $\bar{Q}$. Thus, $\varphi_X|_{E_2} : E_2 \longrightarrow F_2$ is a double covering map which ramified at $\bar{Q}$. Put $\tilde{\varphi} := \sigma_3 \circ \cdots \circ \sigma_{\hat{N}} \circ \varphi_X$ and so $Q := \tilde{\varphi}(\bar{Q}) = F_1 \cap F_2 \in W_2$. Then, Q is a branch point of a double covering map $\tilde{\varphi}|_{E_2} : E_2 \longrightarrow F_2$. From Proposition 3.14 (iii), $d_{[c]}$ is equal to the intersection number $I_Q(F_2, (\sigma_1 \circ \sigma_2)_*^{-1} C_{[c]})$. Since Q is a branch point of $\tilde{\varphi}|_{E_2}$, we have the following:

$$d + d_{[c]} = I_Q(F_2, dF_1 + (\sigma_1 \circ \sigma_2)_*^{-1} C_{[c]}) \text{ is odd.} \tag{3.6}$$

In fact, if it is even, $\tilde{\varphi}|_{E_2}$ does not ramify at $\bar{Q}$; it yields a contradiction. Hence, $d_{[c]}$ is even and $d_{[o]}$ is odd. Q.E.D.

From the shape of the exceptional set $\hat{E}$ in Theorem 3.11, we derive the following properties for the branch curve singularity $(\{f = 0\}, o)$.

Proposition 3.17. *Assume the same situation as Proposition 3.16, and let* $f = \prod_{j=1}^{r} f_j$ *be irreducible decomposition. Then, we have the following.*

(i) $T_o(f_{[o]})_{red}$ *is a line* (=: L_o), *and* $2m_1(f_j) \leqq n_1(f_j)$ *if* $f_j \mid f_{[o]}$ *with* $ord(f_j) \geqq 2$.

(ii) $\mathrm{mult}_{P_2}(\sigma_1)_*^{-1}C_{[o]} = \mathrm{mult}_{P_1} C_{[o]} = ord(f_{[o]})$, *where* $P_1 = \{o\} \in \mathbb{C}^2$.

(iii) *If* $f_{[L]} \neq 1$, *then* $T_o(f_{[L]})_{red} \not\supset L_o$.

(iv) *If* $f_{[c]} \neq 1$, *then* $T_o(f_{[c]})_{red} = L_o$ *and* $\mathrm{mult}_{P_2}(\sigma_1)_*^{-1}C_{[c]}$ *is even.*

(v) *If* $f_{[o]}$ *is a linear form, then* $\hat{E}(o) = E_2$ *and* $\mathbb{M}_{o,X} > \mathbb{Z}_{o,X}$ ($\Longleftrightarrow$ $\mathbb{M}_X > \mathbb{Z}_X$).

Proof. (i) and (ii). From Proposition 3.14 (ii), $E_2 := (\varphi_X)_*^{-1}F_2$ is the $Z_{\hat{E}}$-negative component of $\hat{E}$. From Theorem 3.11 (iv) and the construction of $\hat{E} = (\varphi_X)_*^{-1}\hat{F}$, we have $(\sigma_1 \circ \sigma_2)_*^{-1}C_{[o]} \not\supset F_1 \cap F_2$ in W_2. Hence, the assertions of (i) and (ii) follows from standard arguments on Puiseux pairs.

(iii) From Proposition 3.12 (i), $E_1 := (\varphi_X)_*^{-1}F_1$ is the $M_{\hat{E}}$-negative component of $\hat{E}$. From Theorem 3.11 (iv) and (3.3), we have $E\langle L\rangle \cap E(c) = \emptyset$; thus $(\sigma_1)_*^{-1}C_{[L]} \cap (\sigma_1)_*^{-1}C_{[o]} = \emptyset$ from the construction of $\hat{\sigma}$. Therefore, $T_o(f_{[L]})_{red} \not\supset L_o$.

(iv) From Proposition 3.14 (iii), $(\sigma_1)_*^{-1}C_{[o]} \cap (\sigma_1)_*^{-1}C_{[c]} \neq \emptyset$ and so $T_o(f_{[c]})_{red} = L_o$. Also, $\mathrm{Coeff}_{F_2}(f \circ \hat{\sigma})_{\hat{F}}$ is even by Proposition 3.14 (i). Since $\mathrm{Coeff}_{F_2}(f \circ \hat{\sigma})_{\hat{F}} = ord(f) + ord(f_{[o]}) + \mathrm{mult}_{P_2}(\sigma_1)_*^{-1}C_{[c]}$ and $ord(f_{[o]})$ is odd by Proposition 3.16, $\mathrm{mult}_{P_2}(\sigma_1)_*^{-1}C_{[c]}$ is even.

(v) Let L be the line defined by $f_{[o]} = 0$. If we put $Q_1 := F_1 \cap F_2$ and $Q_2 := F_2 \cap L$, then $\bar{\varphi} := (\sigma_3 \circ \cdots \circ \sigma_{\hat{N}} \circ \varphi_X)|_{E_2} : E_2 \to F_2$ is a double covering map which is ramified only at $\{Q_1, Q_2\}$ from (3.6). Then, E_2 is a non-singular rational curve in $\hat{X}$. Further, we can see that $\bar{\varphi}(\hat{E}(L) \cup \hat{E}(c)) = F_1$ and $\bar{\varphi}(\hat{E}(o)) = F_2$; also $\hat{E}(o) = E_2$. Assume that $\mathbb{M}_{o,X} = \mathbb{Z}_{o,X}$. Let $\pi : (\tilde{X}, E) \to (X, o)$ be the minimal good resolution and $\tau : (\hat{X}, \hat{E}) \to (\tilde{X}, E)$ a succession of contractions of (-1)-curves with $\hat{\pi} = \pi \circ \tau$. Then, $\tau(\hat{E}(L) \cup \hat{E}(c))$ is a non-singular point on $\tilde{X}$. Thus, $E = \tau(E_2)$ and it is a (-1)-curve from $\mathbb{Z}_X^2 = -1$. Hence, (X, o) is a non-singular point (contradiction). Q.E.D.

The following result is implicitly described in [4, §11] as a key fact. We describe it according to our argument.

Theorem 3.18. *Let* (X, o) *be a normal double point defined by* $z^2 = f(x, y)$ *such that* $ord(f)$ *is odd.*

(i) *If* $\mathbb{Z}_X^2 = -1$, *then* $2\,\mathbb{Z}_{o,X} = (\theta \circ \pi)_E$ *for an element* $\theta \in \mathbb{C}\{x, y\}$ *on the minimal good resolution* $\hat{\pi} : (\tilde{X}, E) \longrightarrow (X, o)$ *and the fundamental cycle* $\mathbb{Z}_{o,X}$ (:=Z_E) *on* E.

(ii) $\mathbb{Z}_X^2 = -1$ *if and only if there exists an element* $\theta \in \mathbb{C}\{x, y\}$ *satisfying the following three conditions:* (I) $ord(\theta) = 1$, (II) $2 \mid \mathrm{Coeff}_{E_i}(\theta \circ \hat{\pi})_{\hat{E}}$ *for any* i, (III) $T_o(\theta)_{red} =$

$T_o(f_0)_{red}$ and $T_{P_2}(\Theta)_{red} \not\subset T_{P_2}(C_{(1)})_{red}$, where f_0 is an irreducible factor such that $ord(f_0)$ is odd; also $\Theta := (\sigma_1)_*^{-1}\{\theta = 0\}$ and $C_{(1)} := (\sigma_1)_*^{-1}C$ for $C = \{f = 0\}$.

Proof. First, we prove (ii). Let us prove "only if "part. Let (u, v) be a local coordinate of $\mathbb{C}^2$ near by $\{o\}$ such that $T_o(f_0)_{red} = \{v = 0\}$. Put $\theta_a := v + au^2$ for a constant $a \in \mathbb{C}$ and $L_a := \{\theta_a = 0\}$ and $L'_a := (\sigma_1)_*^{-1}L_a$. Let a_0 be an element of $\mathbb{C}$ such that $L'_{a_0} \not\subset T_{P_2}(C_{(1)})_{red}$. If we put $\theta := \theta_{a_0}$ and $\Theta_2 := (\sigma_2)_*^{-1}L_{a_0}$, then θ satisfies (I) and (III). In addition, from $T_{P_2}(\Theta_2)_{red} \not\subset T_{P_2}(C_{(1)})_{red}$, we have $F_2 \perp \Theta_2$ (i.e., intersects transversally) on W_2, and so $F_2 \perp \hat{\Theta}$ on $W_{\hat{N}}$ for $\hat{\Theta} := (\hat{\sigma})_*^{-1}L_{a_0}$ in $W_{\hat{N}}$. Thus, $\mathrm{Coeff}_{F_2}(\theta \circ \hat{\sigma})_{F(\hat{N})} = 2$ and $E_2 \perp (\varphi_X)_*^{-1}\hat{\Theta}$ in $\hat{X}$ since $\mathrm{Coeff}_{F_2}(f \circ \hat{\sigma})_{F(\hat{N})}$ is even by Proposition 3.14 (i). We have $\mathrm{Coeff}_{E_2}(\theta \circ \hat{\pi})_{\hat{E}} = 2$ from Lemma 3.3 (i). Furthermore, $(\varphi_X)_*^{-1}\hat{\Theta}$ is a smooth curve on $\hat{X}$ and it has two disjoint connected components intersecting E_2 transversally. We have $(\theta \circ \hat{\pi})_{\hat{X}} = (\theta \circ \hat{\pi})_{\hat{E}} + (\varphi_X)_*^{-1}\hat{\Theta} \sim 0$ on $\hat{X}$ and E_2 is the only irreducible component of $\hat{E}$ satisfying $(\theta \circ \hat{\pi})_{\hat{E}}E_k < 0$. Therefore, we have the following:

$$A := (\theta \circ \hat{\pi})_{\hat{E}}E_k = \begin{cases} -2 & \text{if } k = 2 \\ 0 & \text{if } k \neq 2 \end{cases} \quad \text{and} \quad B := Z_{\hat{E}}E_k = \begin{cases} -1 & \text{if } k = 2 \\ 0 & \text{if } k \neq 2. \end{cases}$$

Here, if we put $(\theta \circ \hat{\pi})_{\hat{E}} = \sum_{i=1}^{\hat{N}} a_i E_i$, $Z_E = \sum_{i=1}^{\hat{N}} b_i E_i$, then we have two linear equations on $\{a_1, \dots, a_{\hat{N}}\}$ and $\{b_1, \dots, b_{\hat{N}}\}$ as follows:

$$A = \textstyle\sum_{i=1}^{\hat{N}}(E_kE_i)a_i = -2 \text{ for } k = 2 \text{ else } A = 0, \text{ and}$$

$$B = \textstyle\sum_{i=1}^{\hat{N}}(E_kE_i)b_i = -1 \text{ for } k = 1 \text{ else B=0} \quad (k = 1, \dots, \hat{N}).$$

Applying Cramer's formula, we have

$$(3.7) \qquad (\theta \circ \hat{\pi})_{\hat{E}} = 2Z_{\hat{E}} \text{ on } (\hat{X}, \hat{E}).$$

Hence, θ satisfies (II).

Next, let us consider "if "part. If we put $\Theta_o := (\hat{\sigma})_*^{-1}\{\theta = 0\}$, then $(\theta \circ \hat{\sigma})_{F(\hat{N})} + \Theta_o \sim 0$ on $W_{\hat{N}}$ and $\mathrm{Coeff}_{F_2}(\theta \circ \hat{\sigma})_{F(\hat{N})} = 2$ from (I) and (III). Hence, $(\theta \circ \hat{\sigma})^2_{F(\hat{N})} = -2$; thus we have $(\theta \circ \hat{\pi})^2_{\hat{E}} = -4$. If we put $D := (\theta \circ \hat{\pi})_{\hat{E}}/2$ by (II), then D is an anti-nef cycle on E and $D^2 = -1$. Hence, $D = Z_{\hat{E}}$ and so $\mathbb{Z}_X^2 = -1$.

(i) In (3.7), we showed $2Z_{\hat{E}} = (\theta \circ \hat{\pi})_{\hat{E}}$ and so $2\,\mathbb{Z}_{o,X} = (\theta \circ \pi)_E$. Q.E.D.

4 Elements of Laufer type in $\mathbb{C}\{x, y\}$

In [8], H. Laufer showed that the normal double point (X, o) defined by $z^2 = x(y^4 + x^6)$ satisfies $\mathbb{M}_{o,X} > \mathbb{Z}_{o,X}$ (i.e., $\mathbb{M}^2_{o,X} = -2 < \mathbb{Z}^2_X = -1$ from Lemma 3.4). If (Y, o) is a normal double point defined by $z^2 = (ax + by)(y^4 + x^6)$ with $a \neq 0$, then $\mathbb{M}_{o,Y} > \mathbb{Z}_{o,Y}$. Generalizing such property of $y^4 + x^6$, we give the following definition.

Definition 4.1. Let h be a reduced element of $\mathbb{C}\{x,y\}$ and $\ell \in \mathbb{C}\{x,y\}$ a linear form with $\{\ell = 0\} \not\subset T_0(h)_{red}$. Let (X, o) be a normal double point defined by $z^2 = \ell(x,y)h(x,y)$. From Proposition 3.7 (ii) and 3.17 (v), we have the following equivalences:

$$\mathbb{Z}_X^2 = -1 \quad \Leftrightarrow \quad \mathbb{M}_{o,X} > \mathbb{Z}_{o,X} \quad \Leftrightarrow \quad \mathbb{M}_X > \mathbb{Z}_X.$$

If $h \in \mathbb{C}\{x,y\}$ satisfies conditions above, then we call h an element of Laufer type. If h is of Laufer type, then $ord(h)$ is even from Theorem 1.3 (i).

Lemma 4.2. *If $h = \prod_{j=1}^{r} h_j$ is the irreducible decomposition of an element h of Laufer type, then $ord(h_j)$ is even for any j.*

Proof. Put $(X,o) := \{z^2 = \ell h\}$ for a linear form with $\{\ell = 0\} \not\subset T_o(h)_{red}$. Thus, we have $T_o(\ell)_{red} \neq T_o(h_j)_{red}$ for any j. If there is j_0 $(1 \leqq j_0 \leqq r)$ such that $ord(h_{j_0})$ is odd, then $\mathbb{Z}_X^2 = -2$ from Theorem 3.13. This contradicts to $\mathbb{Z}_X^2 = -1$. Q.E.D.

From now on, we characterize elements of Laufer type by the w.d.resolution graph $\Lambda(h)$ (see Theorem 4.5). Let $h = \prod_{j=1}^{r} h_j$ be the irreducible decomposition of a reduced element $h \in \mathbb{C}\{x,y\}$ whose order is even. Let ℓ be a linear form with $\{\ell = 0\} \not\subset T_0(h)_{red}$. Let (X,o) be the normal double point defined by $z^2 = \ell h$. Let $\hat{\pi}\colon (\hat{X}, \hat{E}) \longrightarrow (X, o)$ be the covering resolution over $\hat{\sigma}$ as (3.4), and we also put $E_i := (\varphi_X)_*^{-1} F_i$ for φ_X in (3.1).

Lemma 4.3. *Under the conditions above, the following three conditions are equivalent:*

(i) *h is of Laufer type,* (ii) $M_{\hat{E}}|_{\hat{E}\setminus E_2} = 2Z_{\hat{E}}|_{\hat{E}\setminus E_2}$,

(iii) $\mathrm{Coeff}_{E_i} M_{\hat{E}}$ is even for any $E_i \subset \hat{E}$ with $i \neq 2$.

Proof. (i) $\Rightarrow$ (ii). By Proposition 3.17 (v), we have $\hat{E}(o) = E_2$. Since ℓ satisfies the conditions of θ in Theorem 3.18 (ii), the assertion holds. Also, (ii) $\Rightarrow$ (iii) is obvious and (iii) $\Rightarrow$ (i) is proved by Theorem 3.18 (ii). Q.E.D.

Under the situation above, put $(\bar{C}, o) := (\{h = 0\}, o) \subset (\mathbb{C}^2, o)$ and let $\tau : (W_{\hat{N}}, F(\hat{N})) \to (W_N, F(N))$ be a succession of contractions of (-1)-curves satisfying $\hat{\sigma} = \sigma \circ \tau$, where $\sigma : (W_N, F(N)) \longrightarrow (\mathbb{C}^2, o)$ is the MGE-resolution of $(\bar{C}, o)$. For $\lambda[F_k]$ and $\gamma[F_k]$ in Definition 2.13, we have the following.

Proposition 4.4. *The following three conditions are equivalent.*

(i) *h is of Laufer type.*

(ii) *For any h_j, $ord(h_j)$ is even and $\lambda[F_k] \equiv \gamma[F_k]$ mod 2 for any k with $F_k \preceq F_{[L]}(h_j)$ on the standard order and the Laufer root $F_{[L]}(h_j)$ of h_j (Definition 2.7 and 2.10).*

(iii) *For any h_j, $ord(h_j)$ is even and $e_{P_k}(D_{[L]}(C_j))$ is even for any k with $F_k \preceq F_{[L]}(h_j)$, where $P_k = \sigma_k(F_k)$.*

Proof. (i) $\Rightarrow$ (ii) By Lemma 4.2, $ord(h_j)$ is even for any j. Assume that there exists h_j and F_k satisfying $\lambda[F_k] \not\equiv \gamma[F_k] \bmod 2$ and $F_k \preceq F_{[L]}(h_j)$. If $\lambda[F_k]$ is even and $\gamma[F_k]$ is odd, then $\mathrm{Coeff}_{E_k} M_{\hat{E}}$ is odd by Lemma 3.3. This yields a contradiction by Lemma 4.3. Next, assume that $\lambda[F_k]$ is odd and $\gamma[F_k]$ is even; thus $F_1 \prec F_k$ from $\gamma[F_1] = 1$. In addition, assume that F_k is included in the i-th Puiseux chain $\tilde{P}_i(C_j)$ (Definition 2.5) with $i \geqq 2$. Let F_s be the Puiseux root of $\tilde{P}_i(C_j)$. From $F_s \prec F_{[L]}(h_j)$, $\gamma[F_s]$ is odd. If $\lambda[F_s]$ is even, then $\mathrm{Coeff}_{E_s} M_{\hat{E}}$ is odd by Lemma 3.3. This contradicts to (i) by Lemma 4.3. Therefore, $\lambda[F_s]$ is odd. If $F_k \cap F_s \neq \emptyset$ in W_N and τ_1 is the blowing-up at $F_k \cap F_s$, then $F_{\zeta_0} := {\tau_1}^{-1}(F_k \cap F_s)$ $(\subset F(\hat{N}))$ satisfies that $\lambda[F_{\zeta_0}]$ is even and $\gamma[F_{\zeta_0}]$ is odd. Thus, $\mathrm{Coeff}_{E_{\zeta_0}} M_{\hat{E}}$ is odd as above, and this contradicts to (i). Therefore, we have $F_k \cap F_s = \emptyset$ in W_N. Hence, there exists $F_{\xi_0} \subset \tilde{P}_i(C_j)$ satisfying $F_{\xi_0} \cap F_k \neq \emptyset$ in W_N. From (2.7), $\gamma[F_{\xi_0}]$ is odd and so $\lambda[F_{\xi_0}]$ is odd from (i) and Lemma 3.3. If τ_2 is the blowing-up at $F_k \cap F_{\xi_0}$ and $F_{\zeta_0} := {\tau_2}^{-1}(F_k \cap F_{\xi_0})$, then $\lambda[F_{\zeta_0}]$ is even and $\gamma[F_{\zeta_0}]$ is odd. This yields a contradiction from (i). If $F_k \subset \tilde{P}_1(C_j)$, then (2.7) holds for $\tilde{P}_1(C_j)$ and so it yields a contradiction as above.

(ii) $\Rightarrow$ (i) From Lemma 3.3 (ii), $\mathrm{Coeff}_{E_k} M_{\hat{E}}$ is even for any k, j with $F_k \preceq F_{[L]}(h_j)$. On the other hand, we can easily see that $\mathrm{Coeff}_{F_k} M_F$ is even for any k with $F_{[L]}(h_j) \preceq F_k$. Let F_k be an irreducible component of $F(\hat{N}) \backslash \tau_*^{-1} F(N)$ such that $\sigma_k(F_k) = F_{i_1} \cap F_{i_2}$ and F_{i_1}, $F_{i_2} \prec F_{[L]}(h_j)$, where $\lambda[F_{i_\ell}]$ is odd for $\ell = 1, 2$. From the hypothesis of (ii), $\gamma[F_{i_\ell}]$ is odd for $\ell = 1, 2$. Hence, $\lambda[F_k]$ and $\gamma[F_k]$ are even. Therefore, $\mathrm{Coeff}_{E_k} M_{\hat{E}}$ is even for any $E_k \subset \hat{E}$ with $k \neq 2$. Hence, h is of Laufer type by Lemma 4.3; hence (i) $\Leftrightarrow$ (ii) holds.

(ii) $\Leftrightarrow$ (iii) is proved by Proposition 2.15. Q.E.D.

The following is the main result of this section.

Theorem 4.5. *Let* $h = \prod_{j=1}^{r} h_j$ *be the irreducible decomposition of a reduced element* h *of* $\mathbb{C}\{x, y\}$. *Then,* h *is of Laufer type if and only if every* h_j *satisfies the following two conditions:*

$$\text{(i)} \quad ord(h_j) \text{ is even}, \qquad \text{(ii)} \quad 2m_{[LP]}(h_j) \mid \sum_{k \in I_{[L]}(h_j)} ord(h_k),$$

where $m_{[LP]}(h_j)$ *is the LP-number (Definition* 2.2 (i)) *and* $I_{[L]}(h_j)$ *is the set defined in Definition* 2.11 (iii).

Proof. Put $Puisx(h_j) = \{(m_1(h_j), n_1(h_j)), \cdots, (m_{\ell_j}(h_j), n_{\ell_j}(h_j))\}$ for any j. Assume that $ord(h_j)$ is even for any j. If we put $m_{L_j}(h_j) = m_{[LP]}(h_j)$, then we have

$$\sum_{k \in I_{[L]}(h_j)} ord(h_k) = \left(\prod_{i=1}^{L_j} m_i(h_j)\right) e_{P_{L(j)}}(D_{[L]}(h_j)), \tag{4.1}$$

where $P_{L(j)} := \sigma_{L(j)}(F_{[L]}(h_j))$ for the Laufer root $F_{[L]}(h_j)$.

Let us prove “if” part. From (4.1) and (ii), $e_{P_{L(j)}}(D_{[L]}(h_j))$ is even, because $\prod_{i=1}^{L_j-1} m_i(h_j)$ is odd. From Lemma 2.9, we have $e_{P_{L(j)}}(D_{[L]}(h_j)) \mid e_{P_k}(D_{[L]}(h_j))$ for any k with $F_k \preceq F_{[L]}(h_j)$, and so $e_{P_k}(D_{[L]}(h_j))$ is even. Therefore, h is of Laufer type from Proposition 4.4 ((iii) $\Rightarrow$ (i)). Next, let us prove “ only if ” part. From Lemma 4.2, (i) is proved. Since $\prod_{i=1}^{L_j-1} m_i(h_j)$ is odd and $e_{P_{L(j)}}(D_{[L]}(h_j))$ is even by Proposition 4.4 ((i) $\Rightarrow$ (iii)), we have $2m_{[LP]}(h_j) = 2m_{L_j}(h_j)$ divides $\sum_{k\in I_{[L]}(h_j)} ord(h_k)$ from (4.1). Q.E.D.

Corollary 4.6. *Let h be an irreducible and reduced element of $\mathbb{C}\{x,y\}$. Then, h is of Laufer type if and only if there are at least two even integers $m_i(h)$, $m_j(h)$ $(i<j)$ in the Puiseux pair of h.*

Corollary 4.7. *If h is of Laufer type, then $ord(h) \equiv 0$ mod 4.*

Proof. If h is irreducible, then the assertion is proved by Corollary 4.6. If h is reducible, then h is decomposed as $h = \prod_{\xi=1}^{s} h_{[\xi]}$ for $h_{[\xi]} := \prod_{i\in I_{[L]}(h_{j_\xi})} h_i$, where $\bigcup_{\xi=1}^{s} I_{[L]}(h_{j_\xi}) = \{1,\cdots,r\}$. From Theorem 4.5 (ii), $ord(h_{[\xi]})$ is divided by $2m_{[LP]}(h_{j_\xi})$ and $m_{[LP]}(h_{j_\xi})$ is even for any ξ. Q.E.D.

Corollary 4.8. *Let g and h be relatively prime two elements in $\mathbb{C}\{x,y\}$. If g and h are of Laufer type, then gh is also of Laufer type.*

Let $\pi : (\tilde{X}, E) \longrightarrow (X, o)$ be any resolution of a normal double point defined by $z^2 = f(x,y)$ and assume $\mathbb{M}^2_{o,X} = -2 < \mathbb{Z}^2_X = -1$. In [8, Theorem 6.3], H. Laufer proved that $H_1(E,\mathbb{R}) \neq 0$. We consider a lower bound of $\dim_{\mathbb{R}} H_1(E,\mathbb{R})$ (see Theorem 4.9 and 5.9). Let $\hat{\pi} : (\hat{X}, \hat{E}) \longrightarrow (X,o)$ be the covering resolution over the MSGE-resolution $\hat{\sigma} : (W_{\hat{N}}, F(\hat{N})) \longrightarrow (\mathbb{C}^2, o)$ of the $(\{f=0\}, o)$. Let $\hat{\hat{\pi}} : (\hat{\hat{X}}, \hat{\hat{E}}) \to (X,o)$ be a resolution such that there exist two succesions of blowing-ups $\tau_1 : (\hat{\hat{X}}, \hat{\hat{E}}) \to (\tilde{X}, E)$ and $\tau_2 : (\hat{\hat{X}}, \hat{\hat{E}}) \to (\hat{X}, \hat{E})$ satisfying $\hat{\hat{\pi}} = \pi \circ \tau_1 = \hat{\pi} \circ \tau_2$. For $\epsilon = L$ or c, if we put

$$E\langle\epsilon\rangle := \tau_1(\tau_2^* \hat{E}(\epsilon)), \qquad (4.2)$$

then it is easy to see that $\dim_{\mathbb{R}} H_1(E\langle\epsilon\rangle, \mathbb{R})$ is independent of the choice of $(\tilde{X}, E)$. Let us call $E\langle L\rangle$ the Laufer part of E and so on.

Theorem 4.9. *Under the situation above, if $f_{[L]} \neq 1$ (Definition 3.15), then we have the following.*

(i) *Let F_{k_0} be the Laufer root $F_{[L]}(f_j)$ of f_j with $f_j \mid f_{[L]}$. Then, the Laufer part $E\langle L\rangle$ contains at least one $\mathbb{P}^1$-cycle containing E_{k_0} or the genus $g(E_{k_0})$ of E_{k_0} is greater than or equal to 1, where $E_{k_0} := \tau((\varphi_X)_*^{-1} F_{k_0})$ for a holomorphic map φ_X of (3.1).*

(ii) *Let $s_{[L]}(f)$ be the number of all Laufer roots for $f_{[L]}$ (i.e., all Laufer roots of irreducible factors of $f_{[L]}$). Then, $\dim_{\mathbb{R}} H_1(E\langle L\rangle, \mathbb{R}) \geq s_{[L]}(f)$.*

Proof. Since (ii) is obvious from (i), we prove (i). Let f_{j_0} be any irreducible factor of $f_{[L]}$. Since $ord(f_{j_0})$ is even by Lemma 4.2, let $F_{k_0} := F_{[L]}(f_{j_0})$ and $\hat{P}_{i_0}(f_{j_0})$ the i_0-th Puiseux chain (see Definition 2.5) containing F_{k_0}. Here, we prove the following:

$$(4.3)\quad \begin{array}{l} \text{there exist different two irreducible components } F_{i_1} \text{ and } F_{i_2} \text{ in} \\ \hat{P}_{i_0}(f_{j_0}) \cup \hat{P}_{i_0-1}(f_{j_0}) \quad \text{such that} \quad F_{i_\xi} \cap F_{k_0} \neq \emptyset \text{ and } \lambda[F_{i_\xi}] \text{ is odd } (\xi = 1, 2), \end{array}$$

where $\hat{P}_{i_0}(f_{j_0}) := \eta^{-1}(\tilde{P}_{i_0}(f_{j_0}))$ for the succession of blowing-ups $\eta := \sigma_{N+1} \circ \cdots \circ \sigma_{\hat{N}} : (W_{\hat{N}}, F(\hat{N})) \longrightarrow (W_N, F(N))$ such that $\sigma \circ \eta$ is the MSGE-resolution.

First, assume that there are different two irreducible components F_{i_1} and F_{i_2} in $\hat{P}_{i_0}(f_{j_0})$ such that $F_{i_\xi} \cap F_{k_0} \neq \emptyset$ $(\xi = 1, 2)$. From (2.7), $\gamma[F_{i_\xi}]$ is odd for $\xi = 1, 2$; thus $\lambda[F_{i_\xi}]$ is odd by Proposition 4.4 (ii). Second, assume that $F_{k_0} \cap \hat{P}_{i_0-1}(f_{j_0}) \neq \emptyset$. Then, there is a root F_{i_1} of $\hat{P}_{i_0}(f_{j_0})$ such that $F_{k_0} \cap F_{i_1} \neq \emptyset$. Hence, $\gamma[F_{i_1}]$ is odd by (2.7) and so $\lambda[F_{i_1}]$ is odd by Proposition 4.4 (ii). Therefore, there is F_{i_2} $(\neq F_{k_0})$ in $\hat{P}_{i_0}(f_{j_0})$ such that $F_{i_2} \cap F_{k_0} \neq \emptyset$. Since $\gamma[F_{k_0}]$ is even, $\gamma[F_{i_2}]$ is odd by (2.7). Hence, $\lambda[F_{i_2}]$ is odd by Proposition 4.4. Thus, (4.3) is proved.

If $F_{k_0} \cap \hat{C}_{j_0} \neq \emptyset$ or there is F_{i_3} $(\subset \hat{P}_{i_0+1}(f_{j_0}))$ such that $\lambda[F_{i_3}]$ is odd and $F_{i_3} \cap F_{k_0} \neq \emptyset$, then $g(E_{k_0}) \geqq 1$ from (4.3). Therefore, we may assume that $F_{k_0} \cap \hat{C} = \emptyset$ and there is not F_{i_3} $(\subset \hat{P}_{i_0}(f_{j_0})$ and $F_{i_3} \neq F_{i_1}, F_{i_2})$ such that $\lambda[F_{i_3}]$ is odd and $F_{i_3} \cap F_{k_0} \neq \emptyset$. Then, there is F_{k_1} in $\hat{P}_{i_0+1}(f_{j_0})$ such that $F_{k_0} \cap F_{k_1} \neq \emptyset$ and $\lambda[F_{k_1}]$ is even. Thus, there exists a $\mathbb{P}^1$-chain $\bigcup_{\zeta=1}^{s} F_{k_\zeta}$ $(s \geqq 1)$ in $F(\hat{N})$ satisfying the following two conditions:

$$\begin{cases} (1)\ \ F_{k_\zeta} \cap F_{k_{\zeta+1}} \neq \emptyset \ \text{ for } 0 \leqq \zeta \leqq s-1 \ \text{ and } \ \lambda[F_{k_\zeta}] \text{ is even for } \ 0 \leqq \zeta \leqq s, \\ (2)\ \ (\bigcup_{\zeta=1}^{s-1} F_{k_\zeta}) \cap (F(o) \cup \hat{C}) = \emptyset \text{ and } F_{k_s} \cap (F(o) \cup \hat{C}) \neq \emptyset, \ \text{where } F(o) := \bigcup_{\lambda[F_i]:odd} F_i. \end{cases}$$

Then, $\bigcup_{\zeta=0}^{s} E_{k_\zeta}$ makes a $\mathbb{P}^1$-cycle in $\hat{E}(L)$ of $\hat{E}$. Any irreducible component E_k with $g(E_k) \geqq 1$ and $\mathbb{P}^1$-cycle is not contracted on $\tilde{X}$ (see the correspondence (4.2)). Thus (i) is proven. Q.E.D.

5 Elements of contact type in $\mathbb{C}\{x, y\}$

In this section, we characterize elements of $\mathbb{C}\{x, y\}$ which give contact factors for the weak Laufer decompositions (see Definition 3.15).

Definition 5.1. Let h be a reduced element of $\mathbb{C}\{x, y\}$ and $h = \prod_{j=1}^{r} h_j$ the irreducible decomposition. Put $Puisx(h_j) = \{(m_1(h_j), n_1(h_j)), \cdots, (m_{\ell_j}(h_j), n_{\ell_j}(h_j))\}$ for each j. Assume that $T_o(h)_{red} = \{\ell = 0\}$ for a linear form ℓ with $\ell \nmid h$ and $n_1(h_j) < 2m_1(h_j)$ for any j. Let (X, o) be a normal double point defined by $z^2 = \ell h$. From Proposition 3.7 (ii) and 3.17 (v), we have the following equivalences:

$$\mathbb{Z}_X^2 = -1 \quad \Leftrightarrow \quad \mathbb{M}_{o,X} > \mathbb{Z}_{o,X} \quad \Leftrightarrow \quad \mathbb{M}_X > \mathbb{Z}_X.$$

If h satisfies the conditions above, then we call h an element of contact type. If h is of contact type, then $ord(h)$ is even from Theorem 1.3.

For example, if $(X, o) = \{z^2 = y(y^6 + x^8)\}$, then the w.d.graph and M_E and Z_E on the minimal resolution are given as follows:

$$\underset{E_2}{(-3)} - \underset{E_3\ [1]}{(-1)} - \underset{E_1}{\bigcirc}\ ; \qquad M_E = 2E_1 + E_2 + 3E_3 \text{ and } Z_E = E_1 + E_2 + 2E_3.$$

Then $M_E^2 = -2 < \mathbb{Z}_X{}^2 = -1$ and so $y^6 + x^8$ is of contact type.

In the following, we characterize elements of contact type in terms of Puiseux pair (see Theorem 5.4). Let $h = \prod_{j=1}^{r} h_j$ be the irreducible decomposition of a reduced element of $\mathbb{C}\{x, y\}$ such that $ord(h)$ is even. Let ℓ be a linear form in $\mathbb{C}\{x, y\}$ such that $L := T_o(h)_{red} = \{\ell = 0\}$ and $\ell \nmid h$. Put $(X, o) = \{z^2 = \ell h\}$ and assume the following:

(5.1) $(\mathbb{C}^2, o) \xleftarrow{\sigma_1} (W_1, F(1)) \xleftarrow{\sigma_2} \cdots \xleftarrow{\sigma_{\hat{N}}} (\hat{W}, \hat{F}) := (W_{\hat{N}}, F(\hat{N}))$ is the MSGE-resolution of $(\{\ell h = 0\}, o))$ constructed as (3.4). Put $\hat{\sigma} := \sigma_1 \circ \cdots \circ \sigma_{\hat{N}}$.

Then, σ_2 is the blowing-up at $P_2 := F_1 \cap (\sigma_1)_*^{-1} L$; also $F(1) = F_1 = \sigma_1^{-1}(\{o\})$ and $F_2 = \sigma_2^{-1}(P_2)$. Further, let $(\hat{X}, \hat{E}) \xrightarrow{\hat{\pi}} (X, o)$ be the covering resolution over $\hat{\sigma}$.

Lemma 5.2. *Under the condition above, if $n_1(h_j) < 2m_1(h_j)$ for any j, then the following three conditions are equivalent.*
(i) *h is of contact type,* (ii) $(\ell \circ \hat{\pi})_{\hat{E}} = 2Z_{\hat{E}}$, (iii) $\mathrm{Coeff}_{E_i}(\ell \circ \hat{\pi})_{\hat{E}}$ *is even for any $E_i \subset E$ $(i \neq 2)$.*

Proof. (ii) $\Rightarrow$ (iii) is obvious. Also, since ℓ satisfies the conditions of θ in Theorem 3.18, (i) $\Rightarrow$ (ii) and (iii) $\Rightarrow$ (i) follow directly from Theorem 3.18 (i) and 3.18 (ii) respectively. Q.E.D.

Proposition 5.3. *Under the situation of (5.1), assume that $n_1(h_j) < 2m_1(h_j)$ for any j. If we put $L := \{\ell = 0\}$, then the following three conditions are equivalent.*

(i) *h is of contact type.*

(ii) *For any h_j, $C_j L$ is even and $\lambda[F_k] \equiv \ell[F_k]$ mod 2 for any k with $F_k \preceq F_{[c]}(h_j)$ for the contact root of (C_j, o) defined in Definition 2.10 (iii), where $\lambda[F_k]$ and $\ell[F_k]$ are integers defined in Definition 2.13.*

(iii) *For any h_j, $C_j L$ is even and $e_{P_k}(D_{[c]}(h_j))$ is even for any k with $F_k \preceq F_{[c]}(h_j)$, where $D_{[c]}(h_j)$ is defined in Definition 2.11 (iii).*

Proof. First we prove that if h is of contact type, then $C_j L$ is even for any j. Let F_{j_0} $(\subset F(\hat{N}))$ be the irreducible component which intersects the strict transform $\hat{C}_j := (\hat{\sigma})_*^{-1} C_j$. Since $\mathrm{Coeff}_{\hat{C}_j}((\ell h) \circ \hat{\sigma})_{\hat{W}} = 1$ and $\hat{\sigma}$ is the MSGE-resolution, $\mathrm{Coeff}_{F_{j_0}}((\ell h) \circ \hat{\sigma})_{\hat{W}}$ is even. In fact, if $\mathrm{Coeff}_{F_{j_0}}(\ell \circ \hat{\sigma})_{\hat{W}}$ is odd, then $\mathrm{Coeff}_{E_{j_0}}(\ell \circ \hat{\pi})_{\hat{X}}$ is odd from Lemma

3.3 (i) and this contradicts to Lemma 5.2. Therefore, $C_jL = n_1(h_j)m_2(h_j)\cdots m_{\ell_j}(h_j)$ $= \mathrm{Coeff}_{F_{j_0}}(\ell\circ\hat{\sigma})_{\hat{W}}$ is even.

By replacing $\gamma[F_k]$ with $\ell[F_k]$ and using Lemma 5.2 instead of Lemma 4.3, we can prove (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) by the same way as Proposition 4.4, so we omit the detail. Q.E.D.

Now we characterize elements of contact type in terms of Puiseux pair.

Theorem 5.4. *Let $h = \prod_{j=1}^{r} h_j$ be the irreducible decomposition of a reduced element h of $\mathbb{C}\{x,y\}$ and $(C_j,o) := (\{h_j = 0\},o)$ for any j. Assume that $n_1(h_j) < 2m_1(h_j)$ for any j and $T_o(h)_{red}$ is a line L. Consider the following three conditions:*

(i) C_jL $(= n_1(h_j)m_2(h_j)\cdots m_{\ell_j}(h_j))$ *is even for any j,*

(ii) $2m_{[cP_1]}(h_j) \mid \sum_{i\in I_{[c]}(h_j)} ord(h_i)$ *for any j,*

(iii) $2m_{[cP_2]}(h_j) \mid \sum_{i\in I_{[c]}(h_j)} C_iL$ *for any j,*

where $m_{[cP_i]}(h_j)$ is the cP_i-number for $i=1,2$ (see Definition 2.2 (ii)). Then,

h is an element of contact type $\Leftrightarrow$ (i) *and* (ii) *hold $\Leftrightarrow$* (i) *and* (iii) *hold.*

Proof. Let ℓ be a linear form with $L = \{\ell = 0\}$. Let $(W_{\hat{N}}, F(\hat{N})) \xrightarrow{\hat{\sigma}} (\mathbb{C}^2,o)$ be the MSGE-resolution of $(C,o) := (\{\ell h = 0\},o)$. Assume that (i) holds. Let F_{N_0} be the contact root $F_{[c]}(h_j)$ and F_{N_0} the ξ_0-th Puiseux root of (C_j,o) (see (2.1)). Let $I_{[c]}(h_j)$ and $D_{[c]}(h_j)$ be the sets defined in Definition 2.11 (iii). From Lemma 2.12 (ii), we have

$$\sum_{i\in I_{[c]}(h_j)} C_iL = D_{[c]}(h_j)L = n_1(h_j)m_2(h_j)\cdots m_{\xi_o}(h_j)\sum_{i\in I_{[c]}(h_j)} e_{P_{N_0}}(C_i)$$
$$\text{and}\ \sum_{i\in I_{[c]}(h_j)} ord(h_i) = m_1(h_j)\cdots m_{\xi_o}(h_j)\sum_{i\in I_{[c]}(h_j)} e_{P_{N_0}}(C_i),$$

where $P_{N_0} := \sigma_{N_0}(F_{N_0})$. Therefore, we have the following equivalences:

$$(5.2)\qquad 2m_{[cP_1]}(h_j) \mid \sum_{i\in I_{[c]}(h_j)} ord(h_i) \ \Leftrightarrow\ 2 \mid e_{P_{N_0}}(D([c](h_j)) \Leftrightarrow 2m_{[cP_2]}(h_j) \mid \sum_{i\in I_{[c]}(h_j)} C_iL.$$

Assume that h is of contact type. Then, C_jL is even for any j from Proposition 5.3. Thus (i) holds. In addition, $e_{P_{N_0}}(D_{[c]}(h_j)) = \sum_{i\in I_{[c]}(h_j)} e_{P_{N_0}}(C_i)$ is even for any j from Proposition 5.3 (iii). Then, (ii) and (iii) hold from (5.2). Hence, if h is of contact type, then [(i)+(ii)] and [(i)+(iii)] hold. Also, we have [(i)+(ii)] $\Leftrightarrow$ [(i)+(iii)].

We prove that if (i) and (iii) hold, then h is of contact type. We put $\bar{m}_1(h_j) := n_1(h_j)$ and $\bar{m}_i(h_j) := m_i(h_j)$ for any $i \in \{2,\cdots,\ell_j\}$; also $\bar{m}_{\xi_k}(h_j) := m_{[cP_k]}(h_j)$ for $k=1,2$. From (i), $C_jL = \bar{m}_1(h_j)\cdots\bar{m}_{\ell_j}(h_j)$ is even for any j. Let ξ_1 be a positive integer satisfying $\bar{m}_{\xi_1}(h_j) = m_{[cP_2]}(h_j)$ and so $\mathrm{Coeff}_{F_{[c]}(h_j)}(\ell\circ\hat{\sigma})_{W_{\hat{N}}} = \bar{m}_1(h_j)\cdots\bar{m}_{\xi_1}(h_j)$. Thus, we have $C_jL = \bar{m}(h_j)\cdots\bar{m}_{\xi_1}(h_j)e_{P_{N_1}}(C_j)$, where $P_{N_1} := \sigma_{N_1}(F_{[c]}(h_j))$. Since $\bar{m}_1(h_j)\cdots\bar{m}_{\xi_1-1}(h_j)$ is odd, $e_{P_{N_1}}(D_{[c]}(h_j)) = \sum_{i\in I_{[c]}(h_j)} e_{P_{N_1}}(C_i)$ is even by (iii) and (5.2). From Lemma 2.9, $e_{P_k}(D_{[c]}(h_j)) = \sum_{i\in I_{[c]}(h_j)} e_{P_k}(C_i)$ is even for any k with $F_k \preceq F_{N_1} = F_{[c]}(h_j)$. Therefore, h

is of contact type from Proposition 5.3 ((iii)⇒ (i)). Henceforth, the assertion is proved. Q.E.D.

Corollary 5.5. *Let h be an irreducible and reduced element of $\mathbb{C}\{x,y\}$. Then h is of contact type if and only if $n_1(h) < 2m_1(h)$ holds and there are at least two even integers in $\{n_1(h), m_2(h), \cdots, m_\ell(h)\}$.*

Corollary 5.6. *Let h be an irreducible and reduced element of $\mathbb{C}\{x,y\}$. Then h is of contact type and Laufer type if and only if $n_1(h) < 2m_1(h)$ holds and there are at least two even integers in $\{m_2(h), \cdots, m_\ell(h)\}$.*

Proof. Since "if "part is obvious from Corollary 4.6 and 5.5, we prove "only if "part. If $\#\{i \in \{2, \cdots, \ell\} \mid m_i(h) \text{ is even }\} = 1$, then $m_1(h)$ is even from Corollary 4.6. Hence, $n_1(h)$ is odd and so $m_{[cP_1]}(h) = m_{i_0}(h)$ $(i_0 \geqq 2)$ is even. From Theorem 5.4, $2m_{i_0}(h)|ord(h)$ and so $m_{i_0+1}(h) \cdots m_\ell(h)$ is even. Q.E.D.

Corollary 5.7. *Let g and h be two relatively prime elements in $\mathbb{C}\{x,y\}$ such that $T_o(g)_{red} = T_o(h)_{red}$. If g and h are of contact type, then gh is also of contact type.*

Corollary 5.8. *If h is of contact, then $ord(h)$ is even and greater or equal to* 6.

Proof. First, assume that h is irreducible and $Puisx(h) = \{(m_1, n_1), \cdots, (m_\ell, n_\ell)\}$. If n_1 is even, then $m_1 \geqq 3$ and there is even m_i $(i \geqq 2)$; so $ord(h) = m_1 \cdots m_\ell \geqq 6$. If n_1 is odd, then there are at least two m_{i_1} and m_{i_2} which are even $(2 \leqq i_1 < i_2 \leqq \ell)$. Since $m_1 \geqq 2$, we have $ord(h) = m_1 \cdots m_\ell \geqq 8$. Next we consider the irreducible decomposition $h = \prod_{j=1}^{r} h_j$ $(r \geqq 2)$ such that any h_j is not of contact type. From the condition (iii) of Theorem 5.4, there are h_{i_0} such that $h_1 h_{i_0}$ is of contact type and $F_{[c]}(h_1) = F_{[c]}(h_{i_0})$. If $F_{[c]}(h_1)$ appear in the 1-st Puiseux chain, then $n_1(h_1)$ is even with $n_1(h_1) \geqq 4$; hence $m_1(h_1) \geqq 3$. Then we have $ord(h) \geqq ord(h_1 h_{i_0}) \geqq 6$. If $F_{[c]}(h_1)$ appear in the k-th Puiseux chain $(k \geqq 2)$, then $ord(h) \geqq ord(h_1 h_{i_0}) \geqq 8$. Q.E.D.

For elements of contact type, we describe the result corresponding to Theorem 4.9.

Theorem 5.9. *For the irreducible decomposition $f = \prod_{i=1}^{r} f_i$ of a reduced element f in $\mathbb{C}\{x,y\}$, assume the same situation as (4.2). If $f_{[c]} \neq 1$ for the contact factor $f_{[c]}$ of f (Definition 3.15), then we have the following.*

(i) *Let F_{k_0} be the contact root $F_{[c]}(f_j)$ of f_j with $f_j \mid f_{[c]}$. Then, the contact part $E\langle c\rangle$ contains at least one $\mathbb{P}^1$-cycle containing E_{k_0} or $g(E_{k_0}) \geqq 1$, where $E_{k_0} := \tau((\varphi_X)_*^{-1} F_{k_0})$ for a holomorphic map φ_X of (3.1).*

(ii) *Let $s_{[c]}(f)$ be the number of all contact roots for $f_{[c]}$ (i.e., all contact roots of irreducible factors of $f_{[c]}$). Then, $\dim_{\mathbb{R}} H_1(E\langle c\rangle, \mathbb{R}) \geq s_{[c]}(f)$.*

Proof. This is proved according to exactly the same argument as Theorem 4.9. If we exchange as follows: (1) $F_{[L]}(h_j) \Rightarrow F_{[c]}(h_j)$, (2) $\gamma(F_k) \Rightarrow \ell(F_k)$, (3) Proposition 4.4 (ii) $\Rightarrow$ Proposition 5.3 (ii). The assertion is proved similarly to Theorem 4.9. Q.E.D.

6 Elements of odd type in $\mathbb{C}\{x, y\}$

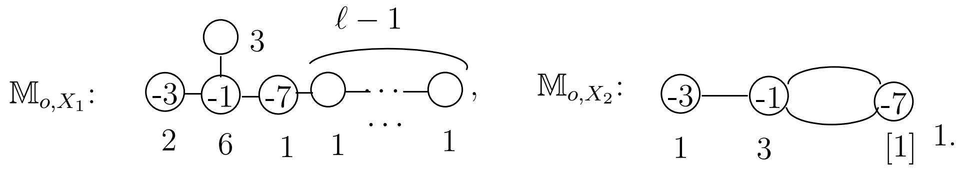
In this section, we characterize elements of $\mathbb{C}\{x, y\}$ which give odd factors for the weak Laufer decompositions (see Definition 3.15).

Definition 6.1. Let h be a reduced element of $\mathbb{C}\{x, y\}$ such that $ord(h)$ is odd. If h satisfies one of the following two conditions, then h is called an *element of odd type.*

(i) $ord(h) = 1$,

(ii) If $ord(h) \geqq 3$, then $\mathbb{M}_X^2 = -1$ ($\Leftrightarrow \mathbb{M}_{o,X}^2 = -1$ from Proposition 3.7 (ii)) for a normal double point defined by $z^2 = h(x, y)$.

Example 6.2. Let $h_1 := y^3 + x^{6\ell+1}$ ($\ell \geqq 1$) and $h_2 := y(y^3 + x^6)(y^3 + x^7)$. Put $(X_i, o) = \{z^2 = h_i(x, y)\}$ $(i = 1, 2)$. Their maximal ideal cycles on the minimal good resolutions are respectively given as follows. Then we can see that they are of odd type.



Theorem 6.3. Let (X, o) be a normal double point defined by $z^2 = f(x, y)$ with $\mathbb{Z}_X^2 = -1$. Let $f = f_{[L]} f_{[c]} f_{[o]}$ be the weak Laufer decomposition (Definition 3.15). Let L_o be the line $T_o(f_{[o]})_{red}$ (see Proposition 3.17 (i)).

(i) *If $f_{[L]} \neq 1$, then $f_{[L]}$ is of Laufer type and $T_o(f_{[L]})_{red} \not\supset L_o$.*

(ii) *If $f_{[c]} \neq 1$, then $f_{[c]}$ is of contact type and $T_o(f_{[c]})_{red} = L_o$.*

Proof. Let $\ell(x, y)$ be a linear form with $L_o = \{\ell = 0\}$. Put $g_{[L]} := \ell f_{[L]}$ (resp. $g_{[c]} := \ell f_{[c]}$) if $f_{[L]} \neq 1$ (resp. $f_{[c]} \neq 1$); also $g_{[o]} := f_{[o]}$ if $ord(f_{[o]}) \geqq 2$. By Proposition 3.16, $ord(g_{[\epsilon]})$ is odd for any ϵ. By Proposition 3.17 (iii) and (iv), we have $L_o \not\subset T_o(f_{[L]})_{red}$ if $f_{[L]} \neq 1$ else $L_o = T_o(f_{[c]})_{red}$. For $\epsilon = L$, c or o, let $(Y_{[\epsilon]}, o)$ be a normal double point defined by $z^2 = g_{[\epsilon]}$. Let $(\mathbb{C}^2, o) \xleftarrow{\sigma_1} (W_1, F(1)) \xleftarrow{\sigma_2} (W_2, F(2)) \xleftarrow{\sigma_{\epsilon,3}} (W_{\epsilon,3}, F_{[\epsilon]}(3)) \xleftarrow{\sigma_{\epsilon,4}} \cdots \xleftarrow{\sigma_{\epsilon,N_\epsilon}} (W_{\epsilon,N_\epsilon}, F_{[\epsilon]}(N_\epsilon))$ $(=: (\hat{W}_{[\epsilon]}, \hat{F}_{[\epsilon]}))$ be the MSGE-resolution of $(\{g_{[\epsilon]} = 0\}, o)$ constructed as (3.4) and put $\hat{\sigma}_{[\epsilon]} := \sigma_1 \circ \sigma_2 \circ \sigma_{\epsilon,3} \circ \cdots \circ \sigma_{\epsilon,N_\epsilon}$. Let $(\hat{W}_{[\epsilon]}, \hat{F}_{[\epsilon]}) \xleftarrow{\sigma_{\epsilon,N_\epsilon+1}} \cdots \xleftarrow{\sigma_{\epsilon,\hat{N}_\epsilon}} (W_{\hat{N}_\epsilon}, F_{[\epsilon]}(\hat{N}_\epsilon))$ be a succession of blowing-ups such that $\hat{\sigma}_{[\epsilon]} \circ \sigma_{\epsilon,N_{\epsilon+1}} \circ \cdots \circ \sigma_{\epsilon,\hat{N}_\epsilon}$ is the MSGE-resolution of $(C, o) := (\{f = 0\}, o)$. Since the MSGE-resolution of (C, o) is uniquely determined by f, we have $\hat{N}_L = \hat{N}_c = \hat{N}_o$ $(=: \hat{N})$ and $\hat{\sigma}_{[L]} \circ \sigma_{L,N_L+1} \circ \cdots \circ \sigma_{L,\hat{N}} = \hat{\sigma}_{[c]} \circ \sigma_{c,N_c+1} \circ \cdots \circ \sigma_{c,\hat{N}} = \hat{\sigma}_{[o]} \circ \sigma_{o,N_o+1} \circ \cdots \circ \sigma_{o,\hat{N}}$ $(=: \hat{\sigma})$. Hence, we can put $(\hat{W}, \hat{F}) := (W_{\hat{N}_\epsilon}, F_{[\epsilon]}(\hat{N}_\epsilon))$ for any ϵ. Let $F_{X,i}$ (resp. $F_{\epsilon,i}$) be the strict transform of $\sigma_i^{-1}(P_i)$ onto $\hat{W}$ (resp. $\hat{W}_{[\epsilon]}$) for $i=1,2$. Let $(\hat{Y}_{[\epsilon]},\hat{E}_{[\epsilon]}) \xrightarrow{\hat{\pi}_{[\epsilon]}} (Y_{[\epsilon]},o)$ (resp. $(\hat{X},\hat{E}) \xrightarrow{\hat{\pi}} (X,o)$) be the covering resolution over $\hat{\sigma}_{[\epsilon]}$ (resp. $\hat{\sigma}$). Then, $E_{X,1} := (\varphi_X)_*^{-1}F_1$ (resp. $E_{X,2} := (\varphi_X)_*^{-1}F_2$) is the $M_{\hat{E}}$ (resp. $Z_{\hat{E}}$)-negative component from Propositions 3.12 (i) and 3.14 (ii). Hence we have $\mathrm{Coeff}_{E_{X,2}}M_{\hat{E}}=1$ by Lemma 3.6 (ii).

For any F_i in $\hat{F}_{[\epsilon]}$, let us represent $(\sigma_{\epsilon,N_{\epsilon+1}}\circ\cdots\circ\sigma_{\epsilon,\hat{N}_\epsilon})_*^{-1}F_i$ by the same notation F_i. For $\epsilon=L$, c, o and any $F_i\subset\hat{F}_{[\epsilon]}$, we can easily see the following:

$$(6.1)\quad \begin{cases} \text{(A)}\ \mathrm{Coeff}_{F_i}((\alpha x+\beta y)\circ\hat{\sigma}_{[\epsilon]})_{\hat{F}_{[\epsilon]}} = \mathrm{Coeff}_{F_i}((\alpha x+\beta y)\circ\hat{\sigma})_{\hat{F}} \ \text{for general}\quad \alpha,\beta\in\mathbb{C}, \\ \text{(B)}\ \mathrm{Coeff}_{F_i}(\ell\circ\hat{\sigma}_{[\epsilon]})_{\hat{F}_{[\epsilon]}} = \mathrm{Coeff}_{F_i}(\ell\circ\hat{\sigma})_{\hat{F}}, \\ \text{(C)}\ \mathrm{Coeff}_{F_i}(g_{[\epsilon]}\circ\hat{\sigma}_{[\epsilon]})_{\hat{F}_{[\epsilon]}} \equiv \mathrm{Coeff}_{F_i}(f\circ\hat{\sigma})_{\hat{F}} \mod 2. \end{cases}$$

For $\epsilon=L$ and c, let $\varphi_{Y_{[\epsilon]}}:(\hat{Y}_{[\epsilon]},\hat{E}_{[\epsilon]})\longrightarrow(\hat{W}_{[\epsilon]},\hat{F}_{[\epsilon]})$ be a holomorphic map given as (3.1) for $(Y_{[\epsilon]},o)$; and put $E_{\epsilon,i} := (\varphi_{Y_{[\epsilon]}})_*^{-1}F_i$ $(i=3,\cdots,N_\epsilon)$. From (6.1)-B for $i=1,2$, the w.d.resolution graphs of $(\hat{F}_{[\epsilon]}\backslash(F_1\cup F_2))\cup(\hat{\sigma}_{[\epsilon]})_*^{-1}C_{[\epsilon]}$ and $(\hat{F}(\epsilon)\backslash F_\delta)\cup(\hat{\sigma})_*^{-1}C_{[\epsilon]}$ coincide, where $\hat{F}(\epsilon) := \varphi_X(\hat{E}(\epsilon))$ (see Definition 3.15); also $F_\delta := F_1$, $\emptyset$ and $F_1\cup F_2$ for $\epsilon=L$, c and o respectively. From (6.1)-(B), we have the following for $E_\delta := (\varphi_X)_*^{-1}F_\delta$.

$$(6.2)\qquad \text{the w.d.graph of } \hat{E}_{[\epsilon]}\backslash(E_{\epsilon,1}\cup E_{\epsilon,2}) = \text{the w.d.graph of } \hat{E}(\epsilon)\backslash E_\delta.$$

Further, by (6.1)-(A) and Lemma 3.3 (ii), we have

$$(6.3)\qquad \mathrm{Coeff}_{E_{\epsilon,i}}M_{\hat{E}_{[\epsilon]}} = \mathrm{Coeff}_{E_{X,i}}M_{\hat{E}(\epsilon)} \quad \text{for any } i \text{ with } E_{\epsilon,i}\subset\hat{E}_{[\epsilon]}.$$

(i) Assume that $f_{[L]}\neq 1$. From (6.1)-(6.3), we have $\mathrm{Coeff}_{E_{L,i}}M_{\hat{E}_{[L]}} = \mathrm{Coeff}_{E_{X,i}}M_{\hat{E}(L)}$ and this is even for any i with $E_{L,i}\subset\hat{E}_{[L]}\backslash E_{L,2}$. From Lemma 4.3, $f_{[L]}$ is of Laufer type.

(ii) Assume that $f_{[c]}\neq 1$. From (6.1)-(6.3), we have $\mathrm{Coeff}_{E_{c,i}}(\ell\circ\hat{\pi}_{[c]})_{\hat{E}_{[c]}} = \mathrm{Coeff}_{E_{X,i}}(\ell\circ\hat{\pi})_{\hat{E}(c)}$ and this is even for any $E_{c,i}\subset\hat{E}_{[c]}$. From Lemma 5.2, $f_{[c]}$ is of contact type. Q.E.D.

In the following, we characterize elements of odd type in terms of Puiseux pair.

Theorem 6.4. *Let* $f=\prod_{j=1}^{r}f_j$ *be the irreducible decomposition of a reduced element* f *in* $\mathbb{C}\{x,y\}$*. Then, we have the following.*

(i) *f is of odd type if and only if* $\begin{cases}\text{(I)}\ \ ord(f)\ \text{is odd},\ \ \text{(II)}\ \ T_o(f)_{red}\ \text{is a line}, \\ \text{(III)}\ \ 2m_1(f_j)\leqq n_1(f_j)\ \text{for any}\ f_j\ \text{with}\ ord(f_j)\geqq 2.\end{cases}$

(ii) *Under the situation of Theorem 6.3, $f_{[o]}$ is of odd type.*

Proof. (i) Let $\hat{\pi}:(\hat{X},\hat{E})\to(X,o)=\{z^2=f(x,y)\}$ be the covering resolution over the MSGE-resolution $\hat{\sigma}$ of $(C,o)=(\{f=0\},o)$ constructed as (3.4).

($\Rightarrow$) Since $M_{\hat{E}}^2=-2<\mathbb{Z}_X^2=-1$, (I) is obvious from Theorem 1.3 (i). We prove (II) and (III). Let $f=f_{[L]}f_{[c]}f_{[o]}$ be as in Theorem 6.3, and $\tau:(\hat{X},\hat{E})\longrightarrow(\tilde{X},E)$ be a succession of contractions of (-1)-curves onto the minimal good resolution. If $f_{[L]}f_{[c]}\neq 1$, then $\hat{E}(L)\cup\hat{E}(c)$ is not contracted to a non-singular point through τ from Theorems 4.9

and 5.9. Since $\mathrm{Coeff}_{\hat{E}_i} M_{\hat{E}} > \mathrm{Coeff}_{\hat{E}_i} \mathbb{Z}_{\hat{E}}$ for $\hat{E}_i \subset \hat{E}(L) \cup \hat{E}(c)$, we have $\mathbb{M}_{o,X} > \mathbb{Z}_{o,X}$. As the contraposition, $\mathbb{M}_{o,X}^2 = -1$ implies $f_{[L]}f_{[c]} = 1$ and so $f = f_{[o]}$. Hence, (II) and (III) are proved by Proposition 3.17 (i).

($\Leftarrow$) From the conditions (I)-(III), $\mathrm{Coeff}_{F_1}(f \circ \hat{\sigma})_{F(\hat{N})}$ ($= ord(f)$) is odd and $\mathrm{Coeff}_{F_2}(f \circ \hat{\sigma})_{F(\hat{N})} = 2\,ord(f)$ and $\mathrm{Coeff}_{F_i} M_{F(\hat{N})} = 1$ for $i = 1, 2$. Hence, for $E_i := (\varphi_X)_*^{-1} F_i$, we have $\mathrm{Coeff}_{E_1} M_{\hat{E}} = 2$ and $\mathrm{Coeff}_{E_2} M_{\hat{E}} = 1$. For a linear form $\ell(x,y) = \alpha x + \beta y$ for general α, $\beta \in \mathbb{C}$, we have $M_{\hat{E}} = (\ell \circ \hat{\pi})_{\hat{E}}$ (see [19, Proposition 3.3]). Since $0 \sim (\ell \circ \hat{\pi})_{\hat{X}} = M_{\hat{E}} + L$ and $E_1 \perp L$ and $E_2 \cap L = \emptyset$, we have $0 = (\ell \circ \hat{\pi})_{\hat{X}} E_1 = (2E_1 + E_2 + L)E_1 = 2E_1^2 + 2$ and so E_1 is a (-1)-curve. If $\eta : (\hat{X}, \hat{E}) \longrightarrow (\bar{X}, \bar{E})$ is the contraction map of E_1, then $(\bar{X}, \bar{E})$ is a good resolution of (X, o). Then, $\bar{L} := \eta(L)$ intersects E_2 transversally at $E_2 \cap \bar{L}$. Since $M_{\bar{E}} + \bar{L} \sim 0$, $M_{\bar{E}} E_2 = -\bar{L} E_2 = -1$ and $M_{\bar{E}} E_i = 0$ for any i ($i \neq 2$). From $\mathrm{Coeff}_{E_2} M_{\bar{E}} = \mathrm{Coeff}_{E_2} M_{\hat{E}} = 1$, we have $M_{\bar{E}}^2 = -1$; thus $\mathbb{M}_{o,X}^2 = -1$.

The assertion (ii) is proved by (i) and Proposition 3.17 (i). Q.E.D.

Corollary 6.5. *Let* $f = \prod_{j=1}^{r} f_j$ *be the irreducible decomposition of a reduced element* f *in* $\mathbb{C}\{x,y\}$. *Assume that* $f_1, \cdots, f_r$ *are of odd type and* r *is odd. Then,* f *is of odd type if and only if* $T_o(f_1)_{red} = \cdots = T_o(f_r)_{red}$.

The above result is induced from Theorem 3.13 and 6.4. The following is a refinement of Theorem 1.3 (ii) .

Corollary 6.6. *Suppose that* f *is irreducible and* $ord(f)$ *is odd* ($\geqq 3$). *Then,* (i) $\mathbb{M}_{o,X}^2 = \mathbb{Z}_X^2 = -1$ *if and only if* $2m_1(f) \leqq n_1(f)$, *and* (ii) $\mathbb{M}_{o,X}^2 = \mathbb{Z}_X^2 = -2$ *if and only if* $2m_1(f) > n_1(f)$.

Corollary 6.7. *Let* $(X, o) := \{z^2 = f(x,y)\}$ *be a normal double point and* $f = \prod_{j=1}^{r} f_j$ *the irreducible decomposition in* $\mathbb{C}\{x,y\}$. *If* $2 \leqq ord(f_j)$ *and* $n_1(f_j) < 2m_1(f_j)$ *for any* j, *then* $\mathbb{Z}_X^2 = -2$.

Proof. From Theorem 1.3 (i), we assume that $ord(f)$ is odd. If $\mathbb{M}_{o,X}^2 = -1$, then f is of odd type. Hence, $2m_1(f_j) \leqq n_1(f_j)$ for any j and this is a contradiction. If $\mathbb{M}_{o,X} > \mathbb{Z}_{o,X}$, we have $ord(f_{[o]}) \geqq 3$ from Theorem 6.3 and the assumption. Hence, from Theorem 6.4, $2m_1(f_j) \leqq n_1(f_j)$ if $f_j \mid f_{[o]}$. This contradicts to $n_1(f_j) < 2m_1(f_j)$. Q.E.D.

7 Comparison of $\mathbb{M}_X$ and $\mathbb{Z}_X$ due to Laufer decomposition for $z^2 = f(x,y)$, and the gluing of weighted dual graphs

Let (X, o) be a normal double point defined by $z^2 = f(x,y)$. In Theorems 6.3 and 6.4, we proved that if $\mathbb{M}_{o,X} > \mathbb{Z}_{o,X}$ on the minimal good resolution, then f has the weak Laufer decomposition (Definition 3.15) with $f_{[L]}f_{[c]} \neq 1$. In this section, we will prove its converse by introducing the following notion of the Laufer decomposition of f (Corollary 7.4, Theorem 7.5).

Definition 7.1. Let $f, f_{(L)}, f_{(c)}$ and $f_{(o)}$ be reduced elements of $\mathbb{C}\{x,y\}$ with $f = f_{(L)}f_{(c)}f_{(o)}$. Then, we say that f has the *Laufer decomposition* if it satisfies the following three conditions:

$$\begin{cases} \text{(i)} \;\; f_{(o)} \text{ is of odd type and } f_{(L)}f_{(c)} \neq 1. \\ \text{(ii)} \;\; \text{If } \; f_{(L)} \neq 1, \text{then } \; f_{(L)} \text{ is of Laufer type and } T_o(f_{(o)})_{red} \not\subset T_o(f_{(L)})_{red}. \\ \text{(iii)} \;\; \text{If } f_{(c)} \neq 1, \; \text{ then } f_{(c)} \text{ is of contact type and } T_o(f_{(o)})_{red} = T_o(f_{(c)})_{red}. \end{cases}$$

When the Laufer decomposition of f exists, from the properties of each types (Theorems 4.5 and 5.4 and 6.4), it is determined uniquely up to units of $\mathbb{C}\{x,y\}$. Moreover, after showing $\mathbb{Z}_{o,X}^2 = -1$ under the assumption of the existence of Laufer decomposition (Corollary 7.4), the components $f_{(L)}, f_{(c)}$ and $f_{(o)}$ are naturally recognized to those of Definition 3.15 and Theorem 6.3. From now on, let $f_{(L)}$, $f_{(c)}$ and $f_{(o)}$ be elements of Laufer type, contact type and odd type respectively such that $T_o(f_{(o)})_{red} = T_o(f_{(c)})_{red} \not\subset T_o(f_{(L)})_{red}$ as in Definition 7.1. Let ℓ_o be a linear form with $T_o(f_{(o)})_{red} = \{\ell_o = 0\}$. Here, we define the following:

$$h_\epsilon := \ell_o f_{(\epsilon)} \quad \text{for} \quad \epsilon = L \text{ and } \; c; \text{ also } \; h_o := f_{(o)} \text{ if } ord(f_{(o)}) \geqq 3. \tag{7.1}$$

At first, we show that the w.d.graph associated to $z^2 = f$ is constructed from w.d.graphs associated to $z^2 = h_\epsilon$ for $\epsilon = L, c$ and o (see Theorem 7.3). Let (Y_ϵ, o) be the normal double point defined by $z^2 = h_\epsilon$ for any ϵ. Let $\sigma_{[\epsilon]} : (W_{\hat{N}_\epsilon}, F(\hat{N}_\epsilon)) \to (\mathbb{C}^2, o)$ be the MSGE-resolution of $(C_{(\epsilon)}, o) = \{h_\epsilon = 0\}$ constructed in (3.4) and $\hat{\pi}_{[\epsilon]} : (\hat{Y}_{[\epsilon]}, \hat{E}_{[\epsilon]}) \to (Y_\epsilon, o)$ be the covering resolution over $\sigma_{[\epsilon]}$. Let $\varphi_{Y_\epsilon} : (\hat{Y}_{[\epsilon]}, \hat{E}_{[\epsilon]}) \to (W_{\hat{N}_\epsilon}, F(\hat{N}_\epsilon))$ be the holomorphic map in (3.1). We set $E_{\epsilon,i} = (\varphi_{Y_\epsilon})_*^{-1} F_{\epsilon,i}$ for any $F_{\epsilon,i} \subset F(\hat{N}_\epsilon)$. Further, let $\pi_{[\epsilon]} : (\tilde{Y}_\epsilon, E_{[\epsilon]}) \longrightarrow (Y_\epsilon, o)$ be the minimal good resolution. From the definitions of Laufer, contact and odd type, we have the conditions $M_{E_{[\epsilon]}}^2 = -2 < Z_{E_{[\epsilon]}}^2 = -1$ for $\epsilon = L$ and c, and $M_{E_{[o]}}^2 = Z_{E_{[o]}}^2 = -1$. Put $E[[\epsilon]] := E_{[\epsilon]} \backslash (E_{\epsilon,1} \cup E_{\epsilon,2})$ for $\epsilon = L$ and c; also put $E[[o]] := E_{[o]} \backslash E_{o,2}$. By Theorem 3.11, the w.d.graph of $E_{[\epsilon]}$ ($\epsilon = L$, c and o) is given as follows:

$$(7.2) \quad \begin{cases} \text{(i)} \;\; \Gamma(E_{[L]}) := \underset{E_{L,2}}{\bigcirc} - \underset{E_{L,1}}{(-b_{L,1})} \cdots \Gamma(E[[L]]) \quad \text{and } b_{L,1} \geqq 2, \\ \text{(ii)} \;\; \Gamma(E_{[c]}) := \underset{E_{c,2}}{(-b_{c,2})} - \Gamma(E[[c]]) - \underset{E_{c,1}}{(-b_{c,1})} \quad \text{and } b_{c,i} \geqq 2 \text{ for } i = 1,2, \\ \text{(iii)} \;\; \Gamma(E_{[o]}) := \Gamma(E[[o]]) \cdots \underset{[g_o] \;\; E_{o,2}}{(-b_{o,2})} \quad \text{and } b_{o,2} \geqq 1. \end{cases}$$

In (7.2)-(i), since $0 = M_{E_{[L]}}E_{L,2} = E_{L,2}^2 + 2$ from $\mathrm{Coeff}_{E_{L,1}} M_E = 2$, we have $E_{L,2}^2 = -2$. Further we have $E_{\epsilon,2} \simeq \mathbb{P}^1$ ($\epsilon = L, c$) by the proof of Proposition 3.17 (v), and $E_{\epsilon,1} \simeq \mathbb{P}^1$ ($\epsilon = L, c$) by Proposition 3.12 (i).

Lemma 7.2. *In the figures of (7.2), we have the following.*

(i) *If we put* $m_\epsilon := (M_{E_{[\epsilon]}}|_{E[[\epsilon]]})E_{\epsilon,1}$ *and* $z_\epsilon := (Z_{E_{[\epsilon]}}|_{E[[\epsilon]]})E_{\epsilon,1}$ *for* $\epsilon = L$ *and* c*, then* $m_L = 2b_{L,1} - 2$, $z_L = b_{L,1} - 1$, $m_c = 2b_{c,1} - 1$ *and* $z_c = b_{c,1}$.

(ii) *If we put* $\bar{m}_\epsilon := (M_{E_{[\epsilon]}}|_{E[[\epsilon]]})E_{\epsilon,2}$ *and* $\bar{z}_\epsilon := (Z_{E_{[\epsilon]}}|_{E[[\epsilon]]})E_{\epsilon,2}$ *for* $\epsilon = c$ *and* o*, then* $\bar{m}_c = b_{c,2}$, $\bar{z}_c = b_{c,2} - 1$ *and* $\bar{m}_o = \bar{z}_o = b_{o,2} - 1$.

Proof. (i) is induced easily from $M_{E_{[\epsilon]}}E_{\epsilon,1} = -1$ and $Z_{E_{[\epsilon]}}E_{\epsilon,1} = 0$ for L and c. (ii) is induced easily from $M_{E_{[c]}}E_{c,2} = 0$ and $Z_{E_{[c]}}E_{c,2} = Z_{E_{[o]}}E_{o,2} = -1$. Q.E.D.

Under the situation above, we describe how to obtain $\Gamma(E)$ by gluing $\Gamma(E_{[L]})$, $\Gamma(E_{[c]})$ and $\Gamma(E_{[o]})$.

Theorem 7.3. *Let* $f = f_{(L)}f_{(c)}f_{(o)}$ *be the Laufer decomposition and* $\pi : (\tilde{X}, E) \to (X, o)$ *the minimal good resolution of* $(X, o) = \{z^2 = f(x, y)\}$*. Then the w.d.graph* $\Gamma(E)$ *is given as follows:*

(i) $\Gamma(E[[o]]) \cdots\; \underset{[g_o]\ \ E_2}{-(b_{c,2} + b_{o,2} - 1)} \;—\; \Gamma(E[[c]]) \;—\; \underset{E_1}{-(b_{L,1} + b_{c,1} - 1)} \;\cdots\; \Gamma(E[[L]])$

if $f_{[L]} \neq 1$ and $f_{[c]} \neq 1$ and $ord(f_{[o]}) \geqq 3$.

(ii) $\Gamma(E[[o]]) \cdots\; \underset{[g_o]\ \ E_2}{-(b_{c,2} + b_{o,2} - 1)} \;—\; \Gamma(E[[c]]) \;—\; \underset{E_1}{-b_{c,1}}$

if $f_{[L]} = 1$ and $f_{[c]} \neq 1$ and $ord(f_{[o]}) \geqq 3$.

(iii) $\Gamma(E[[o]]) \cdots\; \underset{[g_o]\ \ E_2}{-(b_{o,2} + 1)} \;—\; \underset{E_1}{-b_{L,1}} \;\cdots\; \Gamma(E[[L]])$

if $f_{[L]} \neq 1$ and $f_{[c]} = 1$ and $ord(f_{[o]}) \geqq 3$.

(iv) $\underset{E_2}{-b_{c,2}} \;—\; \Gamma(E[[c]]) \;—\; \underset{E_1}{-(b_{L,1} + b_{c,1} - 1)} \;\cdots\; \Gamma(E[[L]])$

if $f_{[L]} \neq 1$ and $f_{[c]} \neq 1$ and $ord(f_{[o]}) = 1$.

Proof. Let $(\hat{X}, \hat{E}) \xrightarrow{\hat{\pi}} (X, o)$ be the covering resolution over the MSGE-resolution $\hat{\sigma}$ of (C, o) constructed in (3.4). If $f_{(\epsilon)} \neq 1$ (resp. $f_{(\epsilon)} = 1$), then we put $(C_{(\epsilon)}, o) := (\{f_{(\epsilon)} = 0\}, o)$ (resp. $:= \emptyset$) for $\epsilon = L$, c. When $C_{(\epsilon)} \neq \emptyset$, from the conditions of $T_o(f_{(\epsilon)})_{red}$ in Definition 7.1, we can see the following:

$$(7.3)\quad \begin{cases} F_1 \cap (\sigma_1 \circ \sigma_2)_*^{-1} C_{(\epsilon)} \neq \emptyset \text{ if } \epsilon = L \text{ or } c;\ F_2 \cap (\sigma_1 \circ \sigma_2)_*^{-1} C_{(\epsilon)} \neq \emptyset \text{ if } \epsilon = c \text{ or } o; \\ F_1 \cap F_2 \cap (\sigma_1 \circ \sigma_2)_*^{-1} C_{(\epsilon)} = \emptyset \text{ (resp. } \neq \emptyset) \text{ if } \epsilon = L \text{ or } o \text{ (resp. } \epsilon = c). \end{cases}$$

We remark that $ord(f_{(o)})$ is odd and $ord(f_{(\epsilon)}) \geqq 4$ if $f_{(\epsilon)} \neq 1$ for $\epsilon = L$ or c (see Corollary 4.7 and 5.8). When $ord(f_{(\epsilon)}) \geqq 2$, let $(W_2, F(2)) \xleftarrow{\sigma_{\epsilon,3}} (W_{\epsilon,3}, F_\epsilon(3)) \xleftarrow{\sigma_{\epsilon,4}} \cdots \xleftarrow{\sigma_{\epsilon,N_\epsilon}}$

$(W_{\epsilon,N_\epsilon}, F_\epsilon(N_\epsilon))$ $(:= (\hat{W}_{Y_\epsilon}, \hat{F}(Y_\epsilon)))$ be a succession of blowing-ups such that $\hat{\sigma}_{Y_\epsilon} := \sigma_1 \circ \sigma_2 \circ \sigma_{Y_\epsilon}$ is the MSGE-resolution of $(\{h_\epsilon = 0\}, o)$ for h_ϵ defined in (7.1), where $\sigma_{Y_\epsilon} := \sigma_{\epsilon,3} \circ \cdots \circ \sigma_{\epsilon,N_\epsilon}$ for $\epsilon = L, c$ and o. Let us put $\hat{G}(Y_\epsilon) := \hat{F}(Y_\epsilon) \cup (\hat{\sigma}_{Y_\epsilon})_*^{-1} C_{(\epsilon)}$ $(\subset \hat{Y}_\epsilon)$ and $\hat{G}(\epsilon) := \hat{F}(\epsilon) \cup \bigcup_{j \in I_{[\epsilon]}} (\hat{\sigma})_*^{-1} C_j$ $(\subset \hat{Y})$ for $\epsilon = L$, c and o, where $\hat{F}(\epsilon) := \varphi_X(\hat{E}(\epsilon))$ and $I_{[\epsilon]} := \{j \in \{1, \cdots, r\} \mid (\hat{\sigma})_*^{-1} C_j \cap \hat{F}(\epsilon) \neq \emptyset\}$. Then, we have the equality about the resolution diagrams $\Lambda(\hat{G}(\epsilon)) = \Lambda(\hat{G}(Y_\epsilon) \backslash (F_1 \cup F_2))$ for any ϵ. Thus $\Lambda(\hat{G})$ is given as follows:

$$\text{(7.4)} \qquad \Lambda(\hat{G}(o)) \cdots \underset{F_2}{\boxed{-c_2}} \text{———} \Lambda(\hat{G}(c)) \text{———} \underset{F_1}{\boxed{-c_1}} \cdots \Lambda(\hat{G}(L)).$$

For $\epsilon = L$ or c, $\Lambda(\hat{G}(\epsilon)) = \emptyset$ if and only if $f_{(\epsilon)} = 1$; also $\Lambda(\hat{G}(o)) = \emptyset$ if and only if $ord(f_{(o)}) = 1$. However, $\Lambda(\hat{G}(L)) \cup \Lambda(\hat{G}(c)) \neq \emptyset$ because of $f_{(L)} f_{(c)} \neq 1$.

Let σ_{Y_ϵ} be the identity map if $0 \leqq ord(f_{(\epsilon)}) \leqq 1$. Since the MSGE-resolution exists uniquely, we have $\sigma_1 \circ \sigma_2 \circ \sigma_{Y_L} \circ \sigma_{Y_c} \circ \sigma_{Y_o} = \hat{\sigma}$. We show the following:

$$\text{(7.5)} \qquad \mathrm{Coeff}_{F_2}(f \circ \hat{\sigma})_{\hat{F}} \text{ is even, and } \mathrm{Coeff}_{\hat{E}_2} M_{\hat{E}} = 1.$$

If we put $d_\epsilon := \mathrm{mult}_{P_2}((\sigma_1)_*^{-1} C_{(\epsilon)})$ for $\epsilon = c$ and o, then $\mathrm{Coeff}_{F_2}(h_c \circ \sigma_{Y_\epsilon})_{\hat{F}(Y_c)} = d_c + ord(f_{(c)}) + 2$ is even by Proposition 3.14 (i). Since $ord(f_{(c)})$ is even by Definition 5.1, d_c is even. Since $d_o = ord(f_{(o)})$ is odd from Theorem 6.4, we have $\mathrm{Coeff}_{F_2}(f \circ \hat{\sigma})_{\hat{F}} = d_c + ord(f) + d_o$ is even. Since $\mathrm{Coeff}_{F_2} M_{\hat{F}} = 1$, $\mathrm{Coeff}_{\hat{E}_2} M_{\hat{E}} = 1$ follows from Lemma 3.3. Now we show the following assertion:

$$\text{(7.6)} \qquad \hat{E}_2 \text{ is irreducible, and } g(\hat{E}_2) = g_o.$$

In fact, the double covering $\varphi_X|_{\hat{E}_2} : \hat{E}_2 \longrightarrow F_2$ has the ramification point $F_1 \cap F_2$ by the odd-ness of $d_c +$ $ord(f)$ as in arguments after (3.6), and so $\hat{E}_2$ is irreducible. Further the remaining ramification points are exactly same as for the case $f_{(o)}$ as seen in (7.4), hence we have the relation $g(\hat{E}_2) = g_o$.

In this situation, consider the following:

(7.7) $\tau : (\hat{X}, \hat{E}) \to (\tilde{X}, E)$ is a succession of contractions of (-1)-curves with $\hat{\pi} = \pi \circ \tau$, and $\hat{E}_i := (\varphi_X)_*^{-1} F_i$ for the map φ_X in (3.1) and $E_i := \tau(\hat{E}_i)$.

From our assumption $f_{(L)} f_{(c)} \neq 1$, we have $M_E^2 = -2$ by Theorem 6.4. Hence we obtain the relations $M_{\hat{E}} \hat{E}_i = M_E \tau(\hat{E}_i)$ for all i. Now, we show the following assertions:

$$\text{(7.8)} \qquad \hat{E}_1 \text{ and } \hat{E}_2 \text{ are not contracted to points by } \tau.$$

Since $ord(f)$ is odd, we have $M_{\hat{E}} \hat{E}_1 < 0$ and $M_{\hat{E}} \hat{E}_2 = 0$ in (3.1). Hence, $M_E \tau(\hat{E}_1) < 0$ holds, so $\tau(\hat{E}_1)$ is not contracted to a point. For $\hat{E}_2$, let $\eta : (\hat{X}, \hat{E}) \to (\bar{X}, \bar{E})$ be a succession of contractions of (-1)-curves, and set $\bar{E}_2 := \eta(\hat{E}_2)$. As same as τ, we have $0 = M_{\hat{E}} \hat{E}_2 = M_{\bar{E}} \bar{E}_2$. Since $\mathrm{Coeff}_{\bar{E}_2} M_{\bar{E}} = 1$, we have $\bar{E}_2^2 = -M_{\bar{E}}|_{\bar{E} - \bar{E}_2} \bar{E}_2$. Since $\mathrm{Coeff}_{\hat{E}_i} M_{\hat{E}} \geq 2$ for $\hat{E}_i \subset \hat{E}_1 \cup \hat{E}(c)$ by Theorem 3.11 and (6.1)(A),(B), we conclude the relation $\bar{E}_2^2 \leq -2$. Hence $\hat{E}_2$ is not contracted to a point by τ.

Now we have the following:

(7.9) $\quad \mathrm{Coeff}_{E_1} M_E = 2$ and $\mathrm{Coeff}_{E_2} M_E = 1$ for $E_i := \tau(\hat{E}_i)$ $(i = 1, 2)$.

From now on, we prove the case of (i). By (7.4), we have the decomposition $\hat{G}\backslash(F_1 \cup F_2) = \hat{G}(L) \sqcup \hat{G}(c) \sqcup \hat{G}(o)$ satisfying $\Lambda(\hat{G}(\epsilon)) = \Lambda(\hat{G}(Y_\epsilon))\backslash(F_1 \cup F_2))$. Hence we may consider that $\Lambda(\hat{G}(Y_\epsilon))$ is embedded into $\Lambda(\hat{G})$. We have the following:

$$\begin{cases} \mathrm{Coeff}_{F_1}(f \circ \hat{\sigma})_{\hat{F}} \ \text{ and } \ \mathrm{Coeff}_{F_1}(h_\epsilon \circ \hat{\sigma}_{Y_\epsilon})_{\hat{F}(Y_\epsilon)} \ \text{ are obviously odd,} \\ \mathrm{Coeff}_{F_2}(f \circ \hat{\sigma})_{\hat{F}} \ \text{ and } \ \mathrm{Coeff}_{F_2}(h_\epsilon \circ \hat{\sigma}_{Y_\epsilon})_{\hat{F}(Y_\epsilon)} \ \text{ are even by Proposition 3.14 and (7.5),} \\ \mathrm{Coeff}_{F_{\epsilon,i}}(f \circ \hat{\sigma})_{\hat{F}} \equiv \mathrm{Coeff}_{F_{\epsilon,i}}(h_\epsilon \circ \hat{\sigma}_{Y_\epsilon})_{\hat{F}(Y_\epsilon)} \bmod 2 \ \text{ for any } \ F_{\epsilon,i} \subset \hat{F}(Y_\epsilon) \ \text{ by (6.1)-(B).} \end{cases}$$

Therefore, we have $\Gamma(\hat{E}(\epsilon)) = \Gamma(\hat{E}(Y_\epsilon)\backslash(\hat{E}_1 \cup \hat{E}_2))$ from $\Lambda(\hat{G}(\epsilon)) = \Lambda(\hat{G}(Y_\epsilon)\backslash(F_1 \cup F_2))$ and (7.4). Moreover, except for the values of $\hat{E}_i^2$ and $E_{\epsilon,i}^2$ $(i = 1, 2)$, we have $\Gamma(\tau(\hat{E}(L) \cup \hat{E}_1)) = \Gamma(E[[L]] \cup E_{L,1})$, $\Gamma(\tau(\hat{E}(c) \cup \hat{E}_1 \cup \hat{E}_2)) = \Gamma(E[[c]] \cup E_{c,1} \cup E_{c,2})$ and $\Gamma(\tau(\hat{E}(o) \cup \hat{E}_2)) = \Gamma(E[[o]] \cup E_{o,2})$. Hence, from (7.4), $\Gamma(E)$ is given as follows:

$$\Gamma(E[[o]]) \cdots \underset{[g_1]\, E_2}{(-b_2)} \text{———} \Gamma(E[[c]]) \text{———} \underset{E_1}{(-b_1)} \cdots \Gamma(E[[L]]).$$

From $-1 = M_E E_1 = -2b_1 + m_L + m_c$ in Lemma 7.2 (i), we have $b_1 = b_{L,1} + b_{c,1} - 1$. Also, $0 = M_E E_2 = -b_2 + \bar{m}_c + \bar{m}_o$ and so $b_2 = b_{c,2} + b_{o,2} - 1$ in Lemma 7.2 (ii). Thus $\Gamma(E)$ is given by (i). We can prove (ii)-(iv) more easily as well, so omit them. Q.E.D.

Corollary 7.4. *Assume the same condition as Theorem 7.3. Then,* $\mathbb{M}_X > \mathbb{Z}_X$ *and* $E_1 := \tau(\hat{E}_1)$ *(resp.* $E_2 := \tau(\hat{E}_2)$*) is the* M_E *(resp.* Z_E*)-negative component of* E *in the w.d.graphs of* (i)-(iv). *Hence, the figures of* (i)-(iv) *give the Laufer decomposition of* E.

Proof. In the figure of Theorem 7.3, if $f_\epsilon \neq 1$, then we put $D_{[\epsilon]} := Z_{E_{[\epsilon]}}|_{E[[\epsilon]]}$ else $D_{[\epsilon]} := 0$. Also, we define a cycle D on E by $D := D_{[L]} + D_{[c]} + D_{[o]} + E_1 + E_2$. Then, $DE_{\epsilon,i} = Z_{E_{[\epsilon]}} E_{\epsilon,i} = 0$ for any $E_{\epsilon,i} \subset E[[\epsilon]]$. For each case of (i)-(iv) in Theorem 7.3, we can see that $DE_1 = 0$ and $DE_2 = -1$ from Lemma 7.2. Hence, D is an anti-nef cycle on E and $D^2 = -1$. Then, we have $D = Z_E$ and $\mathbb{Z}_X^2 = -1$. Since $\mathrm{Coeff}_{E_1} M_E = 2 > \mathrm{Coeff}_{E_1} Z_E = 1$ and (7.8), E_1 is not contracted to a point on the minimal good resolution. Thus, E_1 is the M_E-negative component and $\mathbb{M}_{o,X} > \mathbb{Z}_{o,X}$. Thus, $\mathbb{M}_X > \mathbb{Z}_X$ from Proposition 3.7 (ii). Further, since $Z_E E_2 = DE_2 = -1$, E_2 is the Z_E-negative component. Q.E.D.

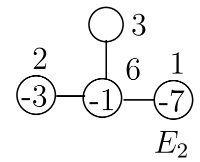
Using all results of this paper, we show the following our main result.

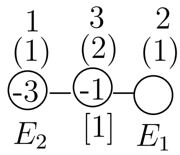
Theorem 7.5. Let (X, o) be a normal double point defined by $z^2 = f(x, y)$. Then, the following three conditions are equivalent;

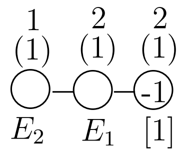
(i) $\mathbb{M}_X > \mathbb{Z}_X$, (ii) $\mathbb{M}_{o,X} > \mathbb{Z}_{o,X}$, (iii) f *has the Laufer decomposition.*

Proof. (i) $\Leftrightarrow$ (ii) is proved in Proposition 3.7 (ii). (ii) $\Rightarrow$ (iii) is proved by Theorems 6.3 and 6.4 as remarked in the beginning of this section. And finally (iii) $\Rightarrow$ (i) is proved by Corollary 7.4. Q.E.D.

Example 7.6. Let us consider $f_{(L)} := x^4 + y^6$ (Laufer type), $f_{(c)} := y^6 + x^8$ (contact type), and $f(o) := y^3 + x^7$ (odd type). Then we have the following:

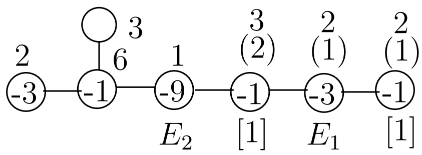
$z^2 = f_3$: 

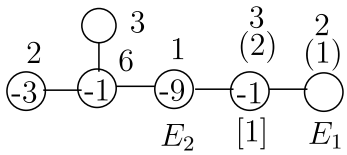
, $z^2 = yf_2$: 

and $z^2 = yf_1$: 

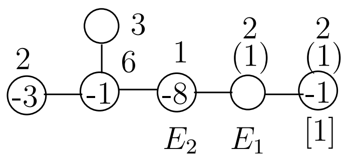
.

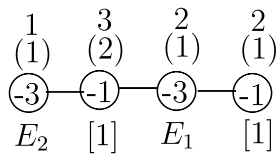
Also, put $(X_1, o) := \{z^2 = f_{(L)}f_{(c)}f_{(o)}\}$, $(X_2, o) := \{z^2 = f_{(c)}f_{(o)}\}$, $(X_3, o) := \{z^2 = f_{(L)}f_{o)}\}$ and $(X_4, o) := \{z^2 = yf_{(L)}f_{(c)}\}$. If $\pi_\ell : (\tilde{X}_\ell, E\langle\ell\rangle) \to (X_\ell, o)$ is the minimal good resolution ($\ell = 1, \cdots, 4$), then we have the following:

$\Gamma(E\langle 1\rangle)$: 

, $\Gamma(E\langle 2\rangle)$: 

,

$\Gamma(E\langle 3\rangle)$: 

, $\Gamma(E\langle 4\rangle)$: 

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8 A numerical procedure to determine whether $\mathbb{M}_X = \mathbb{Z}_X$ or not from the topology of the branch curve singularity

For a normal double point $(X, o) = \{z^2 = f(x, y)\}$, consider the following three types (see Definition 6.1 in [20]):

(8.1) m.i.type I: $\mathbb{Z}_X^2 = -2$, m.i.type II: $\mathbb{M}_X > \mathbb{Z}_X$, m.i.type III: $\mathbb{M}_X^2 = -1$.

We call it *the maximal ideal type* for (X, o) and it is abbreviated as *m.i.type*. Then, any normal double point belongs to one of those m.i.types. Let $f = \prod_{j=1}^{r} f_j$ be the irreducible decomposition in $\mathbb{C}\{x, y\}$ and $(W, F) \longrightarrow (\mathbb{C}^2, o)$ the MGE-resolution of $(\{f = 0\}, o)$. Also, put $(C_j, o) := (\{f_j = 0\}, o)$ for any j.

Remark 8.1. (i) *m.i.type of* (X, o) *is* I *if and only if* f *is not of odd type and has not the Laufer decomposition.*

(ii) *m.i.type of* (X, o) *is* II *if and only if* f *has the Laufer decomposition.*

(iii) *m.i.type of* (X, o) *is* III *if and only if* f *is of odd type.*

Procedure 8.2. We give a procedure to determine the m.i.type of (X, o) from the w.d.resolution graph $\Lambda(f)$ of the plane curve singularity $(\{f = 0\}, o)$.

(1) Compute the fundamental cycle Z_F and $\mathrm{mult}(C_j, o)$ and $Puisx(C_j)$ for any j.

(2) If $ord(f)$ is even, then (X, o) is of m.i.type I from Theorem 1.3. Thus, let us assume that $ord(f)$ is an odd integer ($\geqq 3$) in the following.

(3) Put $g := \prod_{ord(f_j):odd} f_j$. If $T_o(g)_{red}$ is not a line, then (X, o) is of m.i.type I from Theorem 3.13. Thus, let us assume that $T_o(g)_{red}$ is a line ($=: L_o$) in the following.

(4) Define three sets as follows:

$$\Delta(o) := \{j \in I \mid 2m_1(f_j) \leqq n_1(f_j) \text{ and } T_o(f_j)_{red} = L_o\},$$
$$\Delta(c) := \{j \in I \mid 2m_1(f_j) > n_1(f_j) \text{ and } T_o(f_j)_{red} = L_o\} \text{ and}$$
$$\Delta(L) := \{j \in I \mid T_o(f_j)_{red} \neq L_o\}, \text{ where } I := \{1, \cdots, r\}.$$

Put $f_{(o)} := \prod_{j \in \Delta(o)} f_j$. For $\epsilon := L$ or c, put $f_{(\epsilon)} := \prod_{j \in \Delta(\epsilon)} f_j$ if $\Delta(\epsilon) \neq \emptyset$, otherwise $f_{(\epsilon)} := 1$. Since $I = \Delta(o) \sqcup \Delta(c) \sqcup \Delta(L)$ (i.e., disjoint union), we have $f = f_{(L)} f_{(c)} f_{(o)}$.

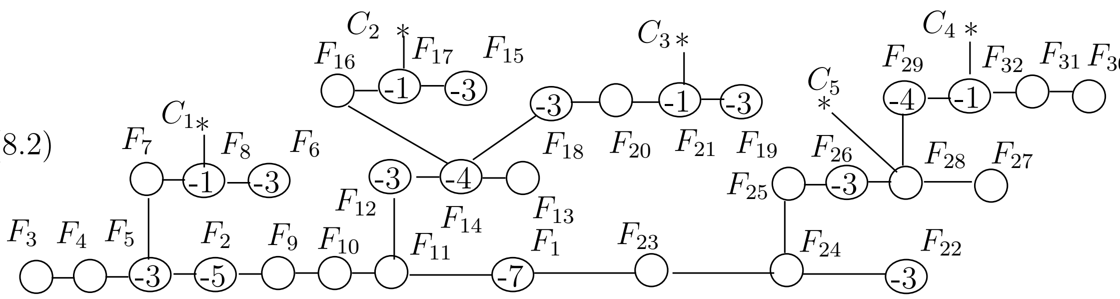
(5) If $f_{(L)} \neq 1$ and $f_{(L)}$ is not of Laufer type, then (X, o) is of m.i.type I. If $f_{(c)} \neq 1$ and $f_{(c)}$ is not of contact type, then (X, o) is of m.i.type I.

(6) Assume that $f_{(L)} f_{(c)} \neq 1$. If $f_{(L)} \neq 1$ (resp. $f_{(c)} \neq 1$) and $f_{(L)}$ (resp. $f_{(c)}$) is of Laufer (resp. contact) type, then (X, o) is of m.i.type II.

(7) If $f_{(L)} f_{(c)} = 1$, then (X, o) is of m.i.type III.

Example 8.3. For the irreducible decomposition $f = \prod_{i=1}^{5} f_j$ of f in $\mathbb{C}\{x, y\}$, assume that the w.d.resolution graph $\Lambda(f)$ of $(C, o) = \{f = 0\}$ is given as follows:

(8.2)



where $(C_j, o) := (\{f_j = 0\}, o)$.

(1) The fundamental cycle Z_F is given by $\ll \mathrm{Coeff}_{F_i} Z_F \gg =$

$$[1, 1, 1, 2, 3, 3, 6, 9, 2, 3, 4, 4, 4, 8, 8, 16, 24, 8, 8, 16, 24, 1, 2, 3, 3, 3, 3, 6, 6, 6, 12, 18].$$

From this, we can see that F_1 is the Z_F-negative component. Hence, we have

$\mathrm{mult}(C_1, o) = \mathrm{Coeff}_{F_8} Z_F = 9$, $\mathrm{mult}(C_2, o) = \mathrm{mult}(C_3, o) = \mathrm{Coeff}_{F_{21}} Z_F = 24$,

$\mathrm{mult}(C_4, o) = \mathrm{Coeff}_{F_{32}} Z_F = 18$, $\mathrm{mult}(C_5, o) = \mathrm{Coeff}_{F_{28}} Z_F = 6$; also

$Puisx(C_1) = \{(3, 7), (3, 23)\}$, $Puisx(C_2) = \{(4, 5), (2, 13), (3, 41)\}$,

$Puisx(C_3) = \{(4, 5), (2, 13), (3, 44)\}$, $Puisx(C_4) = \{(3, 5), (2, 15), (3, 49)\}$

and $Puisx(C_5) = \{(3, 5), (2, 15)\}$.

(2) $ord(f) = \mathrm{mult}(C, o) = 81$ is odd.

(3) From $\Lambda(f)$, $T_o(f_1)_{red} = T_o(f_2)_{red} = T_o(f_3)_{red}$ and it is a line ($=: L_o$); also $L_o \not\subset T_o(f_i)_{red}$ for $i = 4, 5$.

(4) $f_{(o)} := f_1$, $f_{(c)} := f_2 f_3$ and $f_{(L)} := f_4 f_5$.

(5) $f_{(L)}$ is of Laufer type from Theorem 4.5, and $F_{[L]}(f_{(L)}) = F_{28}$.

(6) Let ℓ be a linear form with $T_o(C_1)_{red} = \{\ell = 0\}$. Then, $\ll \mathrm{Coeff}_{F_i}(\ell \circ \sigma)_F \gg = [1, 2, 3, 5, 7, 7, 14, 21, 3, 4, 5, 5, 5, 10, 10, 20, 30, 10, 10, 20, 30]$ for $1 \leqq i \leqq 21$ and $\mathrm{Coeff}_{F_i}(\ell \circ \sigma)_F = \mathrm{Coeff}_{F_i} Z_F$ $(22 \leqq i \leqq 32)$. From Theorem 5.4, $f_{(c)}$ is of contact type and $F_{[c]}(f_{(c)}) = F_{14}$. Then, $f = f_{(L)} f_{(c)} f_{(o)}$ is a Laufer decomposition. Hence, if $\Lambda(f)$ is given by (8.2), then $(X, o) = \{z^2 = f(x, y)\}$ is always of m.i.type II.

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