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Asymptotic stability of depths of localizations of modules

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Abstract. Let R be a commutative noetherian ring, I an ideal of R, and M a finitely generated R-module. The asymptotic behavior of the quotient modules M/I^nM of M is an actively studied subject in commutative algebra. The main result of this paper shows that for large integers $n > 0$, the depth of the localizations of $(M/I^nM)_\mathfrak{p}$ are stable uniformly for all prime ideals $\mathfrak{p}$ of R in each of the following cases: (1) R is CM-excellent, (2) R is semi-local, (3) M or M/I^nM for some $n > 0$ is Cohen–Macaulay.

1. Introduction

Throughout the present paper, all rings are assumed to be commutative and noetherian. Let R be a ring, I an ideal of R, and M a finitely generated R-module. The asymptotic behavior of the quotient modules M/I^nM of M for large integers n is one of the most classical subjects in commutative algebra. Among other things, the asymptotic stability of the associated prime ideals and depths of M/I^nM has been actively studied. Brodmann [**1**] proved that the set of associated prime ideals of M/I^nM is stable for large n. Brodmann [**2**] also showed that for any ideal J, the grade of J on M/I^nM is stable for large n depending on J. In particular, the depth of M/I^nM attains a stable constant value for all large n when R is local. Kodiyalam [**16**] gives another proof of this result for local rings. There are a lot of studies about this subject; see [**1**], [**2**], [**16**], [**19**], [**20**] for instance.

The purpose of this paper is to proceed with the study of the above subject. In particular, we consider the existence of an integer k such that $\operatorname{depth}(M/I^tM)_\mathfrak{p} = \operatorname{depth}(M/I^kM)_\mathfrak{p}$ for all integers $t \geqslant k$ and all prime ideals $\mathfrak{p}$ of R. In this direction, by using the openness of the codepth loci of modules over excellent rings studied by Grothendieck [**8**], Rotthaus and Şega [**20**] proved that such an integer k exists if R is excellent, M is Cohen–Macaulay, and I contains an M-regular element. We aim to improve their theorem by applying the ideas of their proof. However, in our proof, we use the methods developed in [**15**] not those of Grothendieck. If such an integer k exists, then the grade of J on M/I^nM is stable for all $n \geqslant k$ and all ideals J of R; see Proposition 4.6. (Note that the integer k is independent of J.)

The main result of this paper is the following theorem; for the definition of a CM-excellent ring and an acceptable ring in the sense of Česnavičius [**5**] and Sharp [**21**], respectively see Definition 4.2. Obviously, we may replace all $\bar{R}$ in the result below with

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R. It gives a common generalization of the above mentioned theorems proved in [**16**] (that is, [**2**]) and [**20**].

Theorem 1.1 (Corollary 4.3). *Let R be a ring, I an ideal of R, and M a finitely generated R-module. Put $\bar{R} = R/(I + \mathrm{Ann}_R(M))$. Then there is $k > 0$ such that*

$$\mathrm{depth}(M/I^tM)_{\mathfrak{p}} = \mathrm{depth}(M/I^kM)_{\mathfrak{p}}$$

for all integers $t \geqslant k$ and all prime ideals $\mathfrak{p}$ of R in each of the following cases.

(1) *$\bar{R}$ is a CM-excellent ring. In particular, $\bar{R}$ is one of the following: an excellent ring, an acceptable ring, and a homomorphic image of a Cohen–Macaulay ring.*

(2) *$\bar{R}$ is a semi-local ring.*

(3) *M or M/I^nM is Cohen–Macaulay for some $n > 0$.*

The organization of this paper is as follows. In Section 2, we state the definitions of notions used in the later sections together with a couple of their basic properties. In Section 3, we study the openness of the codepth loci of graded modules. We give a sufficient condition for the codepth loci of a graded module to be open, and for the depths of localizations of homogeneous components of a graded module to be eventually stable. In Section 4, we prove Theorem 1.1 and consider some examples.

2. Definitions and lemmas

This section is devoted to preliminaries for the later sections. Rotthaus and Şega [**20**] proved the asymptotic stability of the codepth loci of homogeneous components of the associated graded module $\bigoplus_{i\geqslant 0} I^iM$, and also that of M/I^nM by using the openness of loci of a graded module. To follow their ideas, we prepare several definitions and basic lemmas about graded modules and loci.

In this section, we assume that $A = \bigoplus_{i\geqslant 0} A_i$ is a graded ring and that $M = \bigoplus_{i\in\mathbb{Z}} M_i$ is a finitely generated graded A-module. The ring A is a finitely generated A_0-algebra, and for any $i \in \mathbb{Z}$, the A_0-module M_i is finitely generated. Let S be a multiplicatively closed subset of A_0. Then $A_S = \bigoplus_{i\geqslant 0}(A_i)_S$ is also a graded ring, and $M_S = \bigoplus_{i\in\mathbb{Z}}(M_i)_S$ is a finitely generated graded A_S-module. In particular, $A_{\mathfrak{p}}$ is a graded ring having the local base ring $(A_0)_{\mathfrak{p}}$ for any prime ideal $\mathfrak{p}$ of A_0. Similarly, A/IA and M/IM are graded for any ideal I of A_0. A graded ring A which is generated as an A_0-algebra by elements of A_1 will be called *homogeneous.* Every ring R is a graded ring A with $A_0 = R$ and $A_i = 0$ for all $i \geqslant 0$.

We denote by $\mathrm{Ann}_{A_0}(M)$ the annihilator ideal of M. The *dimension* of M as an A_0-module is given by $\dim_{A_0}(M) = \dim(A_0/\mathrm{Ann}_{A_0}(M))$. Let A_0 be a local ring. In general, M is not finitely generated as an A_0-module. Here, the *depth* of M as an A_0-module is defined as follows; see [**20**, Definition 1.2.1]. Note that this coincides with the one defined in [**4**, Definition 9.1.1].

Definition 2.1. Let $(A_0, \mathfrak{m}_0)$ be local. If M is zero, then we set $\mathrm{depth}_{A_0}(M) = \infty$; otherwise, we define $\mathrm{depth}_{A_0}(M) = \sup\{n \geqslant 0 \mid$ there is an M-regular sequence

$\boldsymbol{x} = x_1, \dots, x_n$ in $\mathfrak{m}_0\}$. Also, if M is zero, then we set $\operatorname{codepth}_{A_0}(M) = -\infty$; otherwise we define $\operatorname{codepth}_{A_0}(M) = \dim_{A_0}(M) - \operatorname{depth}_{A_0}(M)$.

In this paper, the following notation is used.

Definition 2.2. Let R be a ring, I an ideal of R, and $n \geqslant 0$ an integer. We set

- $\mathrm{V}_R(I) = \{\mathfrak{p} \in \operatorname{Spec}(R) \mid I \subseteq \mathfrak{p}\}$ and $\mathrm{D}_R(I) = \{\mathfrak{p} \in \operatorname{Spec}(R) \mid I \nsubseteq \mathfrak{p}\}$.
- $\mathsf{CM}(R) = \{\mathfrak{p} \in \operatorname{Spec}(R) \mid \dim(R_\mathfrak{p}) \leqslant \operatorname{depth}(R_\mathfrak{p})\}$.
- $\mathsf{C}_n^{A_0}(M) = \{\mathfrak{p} \in \operatorname{Spec}(A_0) \mid \operatorname{codepth}_{(A_0)_\mathfrak{p}}(M_\mathfrak{p}) \leqslant n\}$.
- $\mathsf{CM}_{A_0}(M) = \{\mathfrak{p} \in \operatorname{Spec}(A_0) \mid \dim_{(A_0)_\mathfrak{p}}(M_\mathfrak{p}) \leqslant \operatorname{depth}_{(A_0)_\mathfrak{p}}(M_\mathfrak{p})\}$.

The subset $\mathsf{CM}(R)$ of $\operatorname{Spec}(R)$ is said to be the *Cohen–Macaulay locus* of R. The subset $\mathsf{C}_n^{A_0}(M)$ of $\operatorname{Spec}(A_0)$ is called the *nth codepth locus* of the A_0-module M. We call $\mathsf{CM}_{A_0}(M)$ the *Cohen–Macaulay locus* of the A_0-module M. Note that the equalities $\operatorname{Supp}_{A_0}(M) = \mathrm{V}_{A_0}(\operatorname{Ann}_{A_0}(M))$ and $\mathsf{CM}_{A_0}(M) = \mathsf{C}_0^{A_0}(M)$ hold.

We now prepare several lemmas about graded modules, which are needed to prove the results of the next section. Some of the results below are proved in [**15**] and [**20**]. The ring is assumed to be homogeneous throughout [**20**], but some lemmas for which that assumption is not required in the proof are cited without proof.

Lemma 2.3. [**20**, Lemma 1.1.1] *Suppose that A is homogeneous. Then there exists an integer k such that* $\operatorname{Ann}_{A_0}(M_t) = \operatorname{Ann}_{A_0}(M_k)$ *for all integers $t \geqslant k$.*

Lemma 2.4. [**20**, Lemma 1.1.2] *The function $F : \operatorname{Ass}_A(M) \to \operatorname{Ass}_{A_0}(M)$ defined by $F(P) = P \cap A_0$ is well defined and surjective. In particular, $\operatorname{Ass}_{A_0}(M)$ is a finite set.*

Lemma 2.5. [**20**, Lemma 1.1.3(2)] *Put $I = \operatorname{Ann}_{A_0}(M)$. For any prime ideal $\mathfrak{p}$ of A_0, there is an equality* $\operatorname{Ann}_{(A_0)_\mathfrak{p}}(M_\mathfrak{p}) = I(A_0)_\mathfrak{p}$, *and thus* $\dim_{(A_0)_\mathfrak{p}}(M_\mathfrak{p}) = \operatorname{ht}(\mathfrak{p}/I)$. *In particular, we have* $\operatorname{Supp}_{A_0}(M) = \mathrm{V}_{A_0}(I)$.

Lemma 2.6. *Let $(A_0, \mathfrak{m}_0)$ be a local ring.*

(1) [**20**, Lemma 1.2.2(1)] *There is an equality* $\dim_{A_0}(M) = \sup\{\dim_{A_0}(M_i) \mid i \in \mathbb{Z}\}$.

(2) [**20**, Lemma 1.2.2(2)] *One has the equality* $\operatorname{depth}_{A_0}(M) = \inf\{\operatorname{depth}_{A_0}(M_i) \mid i \in \mathbb{Z}\}$.

(3) *Let $0 \to N \to M \to L \to 0$ be an exact sequence of finitely generated graded A-modules. Then*

$$\operatorname{depth}_{A_0}(M) \geqslant \min\{\operatorname{depth}_{A_0}(N),\ \operatorname{depth}_{A_0}(L)\}.$$

(4) *Suppose that a sequence $\boldsymbol{x} = x_1, \dots, x_n$ of elements in $\mathfrak{m}_0$ is an M-regular sequence. Then we have*

$$\dim_{A_0}(M) = \dim_{A_0}(M/\boldsymbol{x}M) + n, \text{ and } \operatorname{depth}_{A_0}(M) = \operatorname{depth}_{A_0}(M/\boldsymbol{x}M) + n.$$

Proof. (1): We have $\mathrm{V}_{A_0}(\mathrm{Ann}_{A_0}(M)) = \mathrm{Supp}_{A_0}(M) = \bigcup_{i\in\mathbb{Z}} \mathrm{Supp}_{A_0}(M_i)$.

(3): We can choose $i \in \mathbb{Z}$ such that $\mathrm{depth}_{A_0}(M) = \mathrm{depth}_{A_0}(M_i)$ by (2). There is an exact sequence $0 \to N_i \to M_i \to L_i \to 0$ of A_0-modules. It follows from (2) and [**4**, Proposition 1.2.9] that

$$\mathrm{depth}_{A_0}(M_i) \geqslant \min\{\mathrm{depth}_{A_0}(N_i),\ \mathrm{depth}_{A_0}(L_i)\} \geqslant \min\{\mathrm{depth}_{A_0}(N),\ \mathrm{depth}_{A_0}(L)\}.$$

(4): The assertion follows from (1) and (2). ■

Lemma 2.7. *Let $\mathfrak{p}$ be a prime ideal of A_0, and let $I = \mathrm{Ann}_{A_0}(M)$.*

(1) *Suppose that a sequence $\boldsymbol{x} = x_1, \dots, x_n$ of elements in $\mathfrak{p}$ is an $M_{\mathfrak{p}}$-regular sequence. Then there exists $f \in A_0 \setminus \mathfrak{p}$ such that $\boldsymbol{x}$ is an M_f-regular sequence.*

(2) *The prime ideal $\mathfrak{p}$ belongs to $\mathrm{Ass}_{A_0}(M)$ if and only if $\mathrm{depth}_{(A_0)_{\mathfrak{p}}}(M_{\mathfrak{p}}) = 0$.*

(3) *Suppose that $n \leqslant \mathrm{depth}_{(A_0)_{\mathfrak{p}}}(M_{\mathfrak{p}}) < \infty$. Then there is an $M_{\mathfrak{p}}$-regular sequence $\boldsymbol{x} = x_1, \dots, x_n$ of elements in $\mathfrak{p}$ such that $\mathrm{ht}\boldsymbol{x}A_0 = n$.*

(4) *If $\mathfrak{p}M = 0$, then M_f is a free $(A_0/\mathfrak{p})_f$-module for some $f \in A_0 \setminus \mathfrak{p}$.*

(5) [**15**, Lemma 2.7(4)] *If $\mathfrak{p}$ is a minimal prime ideal of I, then $\sqrt{I(A_0)_f} = \mathfrak{p}(A_0)_f$ for some $f \in A_0 \setminus \mathfrak{p}$.*

Proof. (1): We may assume $n = 1$. Let φ be the multiplication map of M by x_1. The submodule $\ker\varphi$ of M is a finitely generated A-module, and $(\ker\varphi)_{\mathfrak{p}} = 0$. We have $(\ker\varphi)_f = 0$ for some $f \in A_0 \setminus \mathfrak{p}$, which means that x_1 is an M_f-regular element.

(2): It follows from [**18**, Theorem 6.2] that $\mathfrak{p}$ is in $\mathrm{Ass}_{A_0}(M)$ if and only if $\mathfrak{p}(A_0)_{\mathfrak{p}}$ is in $\mathrm{Ass}_{(A_0)_{\mathfrak{p}}}(M_{\mathfrak{p}})$. Hence the "only if" part is trivial. In order to prove the "if" part, suppose $\mathrm{depth}_{(A_0)_{\mathfrak{p}}}(M_{\mathfrak{p}}) = 0$. There is $i \in \mathbb{Z}$ such that $\mathrm{depth}_{(A_0)_{\mathfrak{p}}}(M_i)_{\mathfrak{p}} = \mathrm{depth}_{(A_0)_{\mathfrak{p}}}(M_{\mathfrak{p}}) = 0$ by Lemma 2.6(2). The prime ideal $\mathfrak{p}$ belongs to the subset $\mathrm{Ass}_{A_0}(M_i)$ of $\mathrm{Ass}_{A_0}(M)$.

(3): We prove the lemma by induction on n. There exists an $M_{\mathfrak{p}}$-regular sequence $\boldsymbol{x}' = x_1, \dots, x_{n-1}$ of elements in $\mathfrak{p}$ such that $\mathrm{ht}\boldsymbol{x}'A_0 = n-1$ by the induction hypothesis. Note that $n \leqslant \mathrm{depth}_{(A_0)_{\mathfrak{p}}}(M_{\mathfrak{p}}) \leqslant \mathrm{ht}\mathfrak{p}$. Since $\mathrm{Ass}_{A_0}(M/\boldsymbol{x}'M)$ is a finite set, we can choose $x_n \in \mathfrak{p}$ such that it does not belong to all prime ideals belonging to $\mathrm{Ass}_{A_0}(M/\boldsymbol{x}'M) \setminus \mathrm{V}_{A_0}(\mathfrak{p})$ and all minimal prime ideals of $\boldsymbol{x}'A_0$. It follows from (2) and Lemma 2.6(4) that the inequality $\mathrm{depth}_{(A_0)_{\mathfrak{p}}}(M/\boldsymbol{x}'M)_{\mathfrak{p}} > 0$ holds, and thus $\mathfrak{p}$ is not in $\mathrm{Ass}_{A_0}(M/\boldsymbol{x}'M)$. It is easy to verify that $\boldsymbol{x} = x_1, \dots, x_n$ is as desired; see [**18**, Theorems 6.1(ii) and 6.2].

(4): We see that M is a finitely generated $A/\mathfrak{p}A$-module. Since $A/\mathfrak{p}A$ is a finitely generated $A_0/\mathfrak{p}$-algebra, the assertion follows from [**10**, Lemma 8.1]. ■

Next, we state some basic lemmas about open subsets of the spectrum of rings. Below is a helpful lemma to see if the locus is open, called the *topological Nagata criterion.*

Lemma 2.8. [**18**, Theorem 24.2] *Let R be a ring and U a subset of $\mathrm{Spec}(R)$. Then U is open if and only if the following two statements hold true.*

(1) *U is stable under generalization, that is, if $\mathfrak{p} \in U$ and $\mathfrak{q} \in \mathrm{Spec}(R)$ with $\mathfrak{q} \subseteq \mathfrak{p}$, then $\mathfrak{q} \in U$.*

(2) *U contains a nonempty open subset of* $\mathrm{V}(\mathfrak{p})$ *for all* $\mathfrak{p} \in U$.

In general, it is not difficult to check that a given locus satisfies (1) of the above lemma. Indeed, for any $n \geqslant 0$, the nth codepth locus of a module is stable under generalization; see Lemmas 3.1 and 3.2. The following two lemmas and remark are useful to verify whether a subset of $\mathrm{Spec}(R)$ satisfies (2).

LEMMA 2.9. [**15**, Lemmas 2.5] *Let R be a ring, $\mathfrak{p} \in \mathrm{Spec}(R)$, and $f \in R \setminus \mathfrak{p}$. Let $F : \mathrm{D}(f) \to \mathrm{Spec}(R_f)$ be the natural homeomorphism, S a subset of $\mathrm{Spec}(R)$, and T the image of $\mathrm{D}(f) \cap S$ by F. Then S contains a nonempty open subset of $\mathrm{V}(\mathfrak{p})$ if and only if T contains a nonempty open subset of $\mathrm{V}(\mathfrak{p}R_f)$.*

LEMMA 2.10. [**15**, Lemmas 2.6] *Let R be a ring, I an ideal of R, and $\mathfrak{p} \in \mathrm{V}(I)$. Let $F : \mathrm{V}(I) \to \mathrm{Spec}(R/I)$ be the natural homeomorphism, S a subset of $\mathrm{Spec}(R)$, and T the image of $\mathrm{V}(I) \cap S$ by F. Then S contains a nonempty open subset of $\mathrm{V}(\mathfrak{p})$ if and only if T contains a nonempty open subset of $\mathrm{V}(\mathfrak{p}/I)$.*

REMARK 2.11. The subset $\mathrm{Supp}_{A_0}(M) = \mathrm{V}_{A_0}(\mathrm{Ann}_{A_0}(M))$ of $\mathrm{Spec}(A_0)$ is closed. Therefore, if a prime ideal $\mathfrak{p}$ of A_0 is not in $\mathrm{Supp}_{A_0}(M)$, then the codepth locus $\mathsf{C}_n^{A_0}(M)$ contains a nonempty open subset $\mathrm{D}_{A_0}(\mathrm{Ann}_{A_0}(M)) \cap \mathrm{V}_{A_0}(\mathfrak{p})$ of $\mathrm{V}_{A_0}(\mathfrak{p})$ for any $n \geqslant 0$.

We close this section by stating an elementary lemma about open subsets.

LEMMA 2.12. *Let R be a ring, and let $\{U_n^t\}_{n \geqslant 0, t \in \mathbb{Z}}$ be a family of open subsets of $\mathrm{Spec}(R)$. Suppose that U_n^t is contained in both U_n^{t+1} and U_{n+1}^t for all $t \in \mathbb{Z}$ and all $n \geqslant 0$. Then there is an integer k such that $U_n^t = U_n^k$ for all $t \geqslant k$ and all $n \geqslant 0$.*

PROOF. There is an integer k_1 such that $U_t^t = U_{k_1}^{k_1}$ for all $t \geqslant k_1$ since R is noetherian. Also, there is an integer k_2 such that $U_n^t = U_n^{k_2}$ for all $t \geqslant k_2$ and all $0 \leqslant n < k_1$. Put $k = \max\{k_1, k_2\}$. For all $t \geqslant k$ and all $n \geqslant 0$, we have $U_n^t = U_n^k$ by considering the case $0 \leqslant n < k_1$ and the case $k_1 \leqslant n$ separately. ∎

3. The openness of the codepth loci of graded modules

In this section, we study the openness of the codepth loci of graded modules. The purpose of this section is to give a sufficient condition for the depths of localizations of homogeneous components of a graded module to be eventually stable. As in the previous section, we assume in this section that $A = \bigoplus_{i \geqslant 0} A_i$ is a graded ring and that $M = \bigoplus_{i \in \mathbb{Z}} M_i$ is a finitely generated graded A-module.

The following two lemmas are known results of Grothendieck [**8**, (6.11.5)] and Rotthaus–Şega [**20**, Lemma 2.5], respectively. However, our proofs are simpler than theirs.

LEMMA 3.1. *Let $(R, \mathfrak{m})$ be a local ring, $\mathfrak{p}$ a prime ideal of R, and N a finitely generated R-module. Then we have*

$$\mathrm{codepth}_{R_\mathfrak{p}}(N_\mathfrak{p}) \leqslant \mathrm{codepth}_R(N).$$

PROOF. We may assume that $\mathfrak{p}$ belongs to $\mathrm{Supp}_R(N)$. Let $\boldsymbol{x} = x_1, \dots, x_n$ be a maximal N-regular sequence in $\mathfrak{p}$. There exists an associated prime ideal $\mathfrak{q}$ of $N/\boldsymbol{x}N$ containing $\mathfrak{p}$. By [**18**, Theorem 17.2], we obtain

$$\begin{aligned}\mathrm{depth}_R(N) - \mathrm{depth}_{R_\mathfrak{p}}(N_\mathfrak{p}) &\leqslant \mathrm{depth}_R(N) - n = \mathrm{depth}_R(N/\boldsymbol{x}N) \leqslant \dim(R/\mathfrak{q})\\ &\leqslant \dim(R/\mathfrak{p}) \leqslant \dim_R(N) - \dim_{R_\mathfrak{p}}(N_\mathfrak{p}).\end{aligned}$$

This says that the assertion holds. ■

LEMMA 3.2. *Let $\mathfrak{p}$ and $\mathfrak{q}$ be prime ideals of A_0 with $\mathfrak{p} \subseteq \mathfrak{q}$. Then we have*

$$\mathrm{codepth}_{(A_0)_\mathfrak{p}}(M_\mathfrak{p}) \leqslant \mathrm{codepth}_{(A_0)_\mathfrak{q}}(M_\mathfrak{q}).$$

PROOF. By (1) and (2) of Lemma 2.6, we can take integers $i, j, k, l \in \mathbb{Z}$ such that

$$\begin{aligned}&\dim_{(A_0)_\mathfrak{p}}(M_\mathfrak{p}) = \dim_{(A_0)_\mathfrak{p}}(M_i)_\mathfrak{p},\ \mathrm{depth}_{(A_0)_\mathfrak{p}}(M_\mathfrak{p}) = \mathrm{depth}_{(A_0)_\mathfrak{p}}(M_j)_\mathfrak{p},\\ &\dim_{(A_0)_\mathfrak{q}}(M_\mathfrak{q}) = \dim_{(A_0)_\mathfrak{q}}(M_k)_\mathfrak{q},\ \text{and } \mathrm{depth}_{(A_0)_\mathfrak{q}}(M_\mathfrak{q}) = \mathrm{depth}_{(A_0)_\mathfrak{q}}(M_l)_\mathfrak{q}.\end{aligned}$$

It follows from Lemma 3.1 that

$$\mathrm{codepth}_{(A_0)_\mathfrak{p}}(M_\mathfrak{p}) = \mathrm{codepth}_{(A_0)_\mathfrak{q}}(N_\mathfrak{p}) \leqslant \mathrm{codepth}_{(A_0)_\mathfrak{p}}(N_\mathfrak{q}) = \mathrm{codepth}_{(A_0)_\mathfrak{q}}(M_\mathfrak{q})$$

as $N := M_i \oplus M_j \oplus M_k \oplus M_l$ is a finitely generated A_0-module. ■

We consider the openness of the codepth loci of a graded module to state the main result of this paper. The following theorem is a graded version of [**15**, Theorem 5.4].

THEOREM 3.3. *Let $\mathfrak{p} \in \mathsf{CM}_{A_0}(M) \cap \mathrm{Supp}_{A_0}(M)$. If $\mathsf{CM}(A_0/\mathfrak{p})$ contains a nonempty open subset of $\mathrm{Spec}(A_0/\mathfrak{p})$, then $\mathsf{CM}_{A_0}(M)$ contains a nonempty open subset of $\mathrm{V}_{A_0}(\mathfrak{p})$.*

PROOF. Note we may replace A with $A/\mathrm{Ann}_A(M)$ to assume $\mathrm{Supp}_{A_0}(M) = \mathrm{Spec}(A_0)$; see Lemma 2.10. Since $\mathfrak{p}$ belongs to $\mathsf{CM}_{A_0}(M)$ and $\mathrm{Supp}_{A_0}(M) = \mathrm{Spec}(A_0)$, we have $d := \mathrm{depth}_{(A_0)_\mathfrak{p}}(M_\mathfrak{p}) = \dim_{(A_0)_\mathfrak{p}}(M_\mathfrak{p}) = \mathrm{ht}\mathfrak{p}$. We can choose a sequence $\boldsymbol{x} = x_1, \dots, x_d$ in $\mathfrak{p}$ such that it is an $M_\mathfrak{p}$-regular sequence and $\mathrm{ht}\boldsymbol{x}A_0 = d$ by Lemma 2.7(3). It follows from Lemma 2.7(5) that $\mathfrak{p}^r(A_0)_a$ is contained in $\boldsymbol{x}(A_0)_a$ for some $a \in A_0 \setminus \mathfrak{p}$ and some $r > 0$. Lemma 2.7(1) yields that for some $b \in A_0 \setminus \mathfrak{p}$, $\boldsymbol{x}$ is an M_b-regular sequence. By assumption, $(A_0/\mathfrak{p})_c$ is Cohen–Macaulay for some $c \in A_0 \setminus \mathfrak{p}$. Set $N = M/\boldsymbol{x}M$. Thanks to Lemma 2.7(4), for each $1 \leqslant i \leqslant r$, there is $d_i \in A_0 \setminus \mathfrak{p}$ such that $(\mathfrak{p}^{i-1}N/\mathfrak{p}^iN)_{d_i}$ is free as an $(A_0/\mathfrak{p})_{d_i}$-module. Put $f = abcd_1 \cdots d_r \in A_0 \setminus \mathfrak{p}$. We can replace our ring A with its localization A_f to prove the theorem; see Lemma 2.9. Therefore, we may assume that the following conditions are satisfied.

(a) $\mathfrak{p}^r$ is contained in $\boldsymbol{x}A_0$. (In particular, $\mathfrak{p}$ is the unique minimal prime ideal of $\boldsymbol{x}A_0$.)

(b) $\boldsymbol{x}$ is an M-regular sequence.

(c) $A_0/\mathfrak{p}$ is Cohen–Macaulay.

(d) $\mathfrak{p}^{i-1}N/\mathfrak{p}^iN$ is free as an $A_0/\mathfrak{p}$-module for each $1 \leqslant i \leqslant r$.

We claim that $\mathsf{CM}_{A_0}(M)$ contains $\mathrm{V}_{A_0}(\mathfrak{p})$. Let $\mathfrak{q} \in \mathrm{V}_{A_0}(\mathfrak{p})$. We obtain

$$\mathrm{ht}(\mathfrak{q}/\boldsymbol{x}A_0) = \mathrm{ht}(\mathfrak{q}/\mathfrak{p}) = \mathrm{depth}(A_0/\mathfrak{p})_{\mathfrak{q}} = \mathrm{depth}_{(A_0)_{\mathfrak{q}}}(\mathfrak{p}^{i-1}N/\mathfrak{p}^iN)_{\mathfrak{q}}$$

for each $1 \leqslant i \leqslant r$ by (a), (c), and (d). Using (a) and Lemma 2.6(3), we see by induction on i that for any $1 \leqslant i \leqslant r$, $\mathrm{depth}_{(A_0)_{\mathfrak{q}}}(\mathfrak{p}^{r-i}N)_{\mathfrak{q}} \geqslant \mathrm{ht}(\mathfrak{q}/\boldsymbol{x}A_0)$. It follows from Lemmas 2.5 and 2.6(4) that

$$\mathrm{depth}(M_{\mathfrak{q}}) = \mathrm{depth}(N_{\mathfrak{q}}) + d \geqslant \mathrm{ht}(\mathfrak{q}/\boldsymbol{x}A_0) + d = \mathrm{ht}\mathfrak{q} \geqslant \dim(M_{\mathfrak{q}})$$

as $\mathrm{ht}\boldsymbol{x}A_0 = d$. This means that $\mathfrak{q}$ belongs to $\mathsf{CM}_{A_0}(M)$. ■

Below is a direct corollary of Theorem 3.3.

Corollary 3.4. *Suppose that $\mathsf{CM}(A_0/\mathfrak{p})$ contains a nonempty open subset of $\mathrm{Spec}(A_0/\mathfrak{p})$ for any $\mathfrak{p} \in \mathrm{Supp}_{A_0}(M) \cap \mathsf{CM}_{A_0}(M)$. Then $\mathsf{CM}_{A_0}(M)$ is open.*

Proof. It suffices to verify that $\mathsf{CM}_{A_0}(M) = \mathsf{C}_0^{A_0}(M)$ satisfies the two conditions of Lemma 2.8. Lemma 3.2 yields that it is stable under generalization. Let $\mathfrak{p} \in \mathsf{CM}_{A_0}(M)$. If $\mathfrak{p}$ belongs to $\mathrm{Supp}_{A_0}(M)$, then $\mathsf{CM}_{A_0}(M)$ contains a nonempty open subset of $\mathrm{V}_{A_0}(\mathfrak{p})$ by Theorem 3.3. Otherwise, by Remark 2.11, $\mathsf{CM}_{A_0}(M)$ contains a nonempty open subset $\mathrm{D}_{A_0}(\mathrm{Ann}_{A_0}(M)) \cap \mathrm{V}_{A_0}(\mathfrak{p})$ of $\mathrm{V}_{A_0}(\mathfrak{p})$. ■

Corollary 3.4 is a graded version of [**15**, Corollary 5.5(1)]. When the base ring of $A/\mathrm{Ann}_A(M)$ is catenary, Theorem 3.3 can be extended as follows.

Theorem 3.5. *Let $n \geqslant 0$ be an integer and let $\mathfrak{p} \in \mathsf{C}_n^{A_0}(M) \cap \mathrm{Supp}_{A_0}(M)$. Suppose that that the ring $A_0/\mathrm{Ann}_{A_0}(M)$ is catenary. If $\mathsf{CM}(A_0/\mathfrak{p})$ contains a nonempty open subset of $\mathrm{Spec}(A_0/\mathfrak{p})$, then $\mathsf{C}_n^{A_0}(M)$ contains a nonempty open subset of $\mathrm{V}_{A_0}(\mathfrak{p})$.*

Proof. We prove the theorem by induction on n. We have already shown the case where $n = 0$ in Theorem 3.3. Let $n > 0$ and $d = \mathrm{depth}_{(A_0)_{\mathfrak{p}}}(M_{\mathfrak{p}})$. We may assume $\mathrm{codepth}_{(A_0)_{\mathfrak{p}}}(M_{\mathfrak{p}}) = n$ by the induction hypothesis. Thanks to Lemma 2.7(1), there is a sequence $\boldsymbol{x} = x_1, \ldots, x_d$ in $\mathfrak{p}$ and $a \in A_0 \setminus \mathfrak{p}$ such that $\boldsymbol{x}$ is an M_a-regular sequence. We can choose $b \in A_0 \setminus \mathfrak{p}$ such that $b \in \mathfrak{q}$ for any minimal prime ideal $\mathfrak{q}$ of $\mathrm{Ann}_{A_0}(M)$, which is not contained in $\mathfrak{p}$. Then $\mathfrak{p}(A_0)_b$ contains any minimal prime ideal of $\mathrm{Ann}_{(A_0)_b}(M_b)$.

Set $N = M/\boldsymbol{x}M$. Note that the $\mathfrak{p}$-torsion submodule $\Gamma_{\mathfrak{p}}(N)$ of N is finitely generated and graded as an A-module. We easily see that $\mathrm{Supp}_{A_0}(\Gamma_{\mathfrak{p}}(N)) = \mathrm{V}_{A_0}(\mathfrak{p})$ as $\mathfrak{p}$ belongs to $\mathrm{Ass}_{A_0}(N)$; see Lemmas 2.6(4) and 2.7(2). Lemma 2.6(4) yields the equalities $n = \mathrm{codepth}(N_{\mathfrak{p}}) = \dim(N_{\mathfrak{p}})$. Since there is an inequality

$$\dim_{(A_0)_{\mathfrak{p}}}(\Gamma_{\mathfrak{p}}(N))_{\mathfrak{p}} = 0 < n = \dim(N_{\mathfrak{p}}),$$

it is seen that $\mathfrak{p}$ is in $\mathsf{C}_0^{A_0}(\Gamma_{\mathfrak{p}}(N))$ and $\dim(N/\Gamma_{\mathfrak{p}}(N))_{\mathfrak{p}} = \dim(N_{\mathfrak{p}}) = n$. On the other hand, Lemma 2.7(2) implies $\mathrm{depth}(N/\Gamma_{\mathfrak{p}}(N))_{\mathfrak{p}} > 0$. Thus $\mathfrak{p}$ belongs to $\mathsf{C}_{n-1}^{A_0}(N/\Gamma_{\mathfrak{p}}(N))$. We can take $c, d \in A_0 \setminus \mathfrak{p}$ such that $\mathsf{C}_0^{A_0}(\Gamma_{\mathfrak{p}}(N))$ contains $\mathrm{D}_{A_0}(cA_0) \cap \mathrm{V}_{A_0}(\mathfrak{p})$ and $\mathsf{C}_{n-1}^{A_0}(N/\Gamma_{\mathfrak{p}}(N))$ contains $\mathrm{D}_{A_0}(dA_0) \cap \mathrm{V}_{A_0}(\mathfrak{p})$ by the induction hypothesis.

Put $f = abcd \in A_0 \setminus \mathfrak{p}$. We can replace our ring A with its localization A_f to prove the theorem; see Lemma 2.9. We may assume that the following conditions are satisfied.

(a) There is an M-regular sequence $\boldsymbol{x} = x_1, \dots, x_d$ in $\mathfrak{p}$.

(b) The prime ideal $\mathfrak{p}$ contains any minimal prime ideal of $\operatorname{Ann}_{A_0}(M)$.

(c) The set $\mathrm{V}_{A_0}(\mathfrak{p})$ is contained in both $\mathsf{C}_0^{A_0}(\Gamma_{\mathfrak{p}}(N))$ and $\mathsf{C}_{n-1}^{A_0}(N/\Gamma_{\mathfrak{p}}(N))$.

We prove that $\mathrm{V}_{A_0}(\mathfrak{p})$ is contained in $\mathsf{C}_n^{A_0}(M)$. Let $\mathfrak{q} \in \mathrm{V}_{A_0}(\mathfrak{p})$. By (c), we have

$$\operatorname{depth}(\Gamma_{\mathfrak{p}}(N))_{\mathfrak{q}} \geqslant \dim(\Gamma_{\mathfrak{p}}(N))_{\mathfrak{q}} \text{ and } \operatorname{depth}(N/\Gamma_{\mathfrak{p}}(N))_{\mathfrak{q}} \geqslant \dim(N/\Gamma_{\mathfrak{p}}(N))_{\mathfrak{q}} - (n-1).$$

The ring $A_0/\operatorname{Ann}(M)$ is catenary. So, by (b) and Lemma 2.5, we obtain equalities

$$\operatorname{ht}(\mathfrak{q}/\mathfrak{p}) = \operatorname{ht}(\mathfrak{q}/\operatorname{Ann}(M)) - \operatorname{ht}(\mathfrak{p}/\operatorname{Ann}(M)) = \dim(M_{\mathfrak{q}}) - \dim(M_{\mathfrak{p}}) = \dim(M_{\mathfrak{q}}) - (n+d).$$

Note that $\operatorname{Supp}(N/\Gamma_{\mathfrak{p}}(N)) = \operatorname{Supp}(N)$ since $\operatorname{Supp}(N/\Gamma_{\mathfrak{p}}(N))$ contains $\mathrm{V}_{A_0}(\mathfrak{p})$. We get

$$\dim(N/\Gamma_{\mathfrak{p}}(N))_{\mathfrak{q}} = \dim(N_{\mathfrak{q}}) = \dim(M_{\mathfrak{q}}) - d.$$

From the above, the inequalities

$$\begin{aligned} &\operatorname{depth}(\Gamma_{\mathfrak{p}}(N))_{\mathfrak{q}} \geqslant \dim(\Gamma_{\mathfrak{p}}(N))_{\mathfrak{q}} = \operatorname{ht}(\mathfrak{q}/\mathfrak{p}) = \dim(M_{\mathfrak{q}}) - (n+d) \text{ and} \\ &\operatorname{depth}(N/\Gamma_{\mathfrak{p}}(N))_{\mathfrak{q}} \geqslant \dim(N/\Gamma_{\mathfrak{p}}(N))_{\mathfrak{q}} - (n-1) = \dim(M_{\mathfrak{q}}) - (n+d) + 1 \end{aligned}$$

hold. Therefore, we observe that

$$\begin{aligned} \operatorname{depth}(M_{\mathfrak{q}}) = \operatorname{depth}(N_{\mathfrak{q}}) + d &\geqslant \min\{\operatorname{depth}(\Gamma_{\mathfrak{p}}(N))_{\mathfrak{q}},\ \operatorname{depth}(N/\Gamma_{\mathfrak{p}}(N))_{\mathfrak{q}}\} + d \\ &\geqslant \dim(M_{\mathfrak{q}}) - n \end{aligned}$$

by Lemma 2.6(3), which means that $\mathfrak{q}$ belongs to $\mathsf{C}_n^{A_0}(M)$. ■

For the codepth loci, the same proof yields the analogous result as Corollary 3.4.

COROLLARY 3.6. *Suppose that the ring $A_0/\operatorname{Ann}_{A_0}(M)$ is catenary.*

(1) *Let $n \geqslant 0$ be an integer. Suppose that $\mathsf{CM}(A_0/\mathfrak{p})$ contains a nonempty open subset of $\operatorname{Spec}(A_0/\mathfrak{p})$ for any $\mathfrak{p} \in \operatorname{Supp}_{A_0}(M) \cap \mathsf{C}_n^{A_0}(M)$. Then $\mathsf{C}_n^{A_0}(M)$ is open.*

(2) *Suppose that $\mathsf{CM}(A_0/\mathfrak{p})$ contains a nonempty open subset of $\operatorname{Spec}(A_0/\mathfrak{p})$ for any $\mathfrak{p} \in \operatorname{Supp}_{A_0}(M)$. Then $\mathsf{C}_n^{A_0}(M)$ is open for any integer $n \geqslant 0$.*

We study the asymptotic behavior of the depths of localizations of homogeneous components of a graded module. We prepare the following basic lemma to state Lemma 3.8. When the ring is excellent, Lemma 3.7 is proved in Section 4 of [**20**]. Here we give a proof that does not require that assumption.

LEMMA 3.7. *Suppose that A is homogeneous and that $(A_0, \mathfrak{m}_0, k_0)$ is local. Then there exists an integer k such that $\operatorname{depth}_{A_0}(M_t) = \operatorname{depth}_{A_0}(M_k)$ for all integers $t \geqslant k$.*

Proof. For all integers $t \in \mathbb{Z}$, the equalities

$$\operatorname{depth}_{A_0}(M_t) = \inf\{i \mid \operatorname{Ext}^i_{A_0}(k_0, M_t) \neq 0\} = \inf\{i \mid \operatorname{Ann}_{A_0}(\operatorname{Ext}^i_{A_0}(k_0, M_t)) \neq A_0\}$$

hold. It is seen that $\operatorname{Ext}^i_{A_0}(k_0, M) \simeq \bigoplus_{t\in\mathbb{Z}} \operatorname{Ext}^i_{A_0}(k_0, M_t)$ is a finitely generated graded A-module for all $i \geqslant 0$. (Compute them using a minimal free resolution of the A_0-module k_0.) Lemma 2.3 implies that there is k such that the equalities $\operatorname{Ann}_{A_0}(\operatorname{Ext}^i_{A_0}(k_0, M_t)) = \operatorname{Ann}_{A_0}(\operatorname{Ext}^i_{A_0}(k_0, M_k))$ hold for all $t \geqslant k$ and all $0 \leqslant i \leqslant \dim(A_0)$, which means that the assertion holds. ■

Applying the ideas of the proof of [**20**, Theorem 4.2], we can prove the result below, which extends it.

Lemma 3.8. *Suppose that A is homogeneous. Denote by N_t the graded A-module $\bigoplus_{i\geqslant t} M_i$ for each $t \in \mathbb{Z}$. If $\mathsf{C}^{A_0}_n(N_t)$ is open for all $t \in \mathbb{Z}$ and all $n \geqslant 0$, then there is an integer k such that for all integers $t \geqslant k$ and all prime ideals $\mathfrak{p}$ of A_0,*

$$\operatorname{depth}_{(A_0)_\mathfrak{p}}(M_t)_\mathfrak{p} = \operatorname{depth}_{(A_0)_\mathfrak{p}}(M_k)_\mathfrak{p}.$$

Proof. It follows from (1) and (2) of Lemma 2.6 that $\mathsf{C}^{A_0}_n(N_t)$ is contained in both $\mathsf{C}^{A_0}_n(N_{t+1})$ and $\mathsf{C}^{A_0}_{n+1}(N_t)$ for all $t \in \mathbb{Z}$ and all $n \geqslant 0$. By Lemmas 2.3 and 2.12, we can choose an integer $l \in \mathbb{Z}$ such that

$$J := \operatorname{Ann}_{A_0}(M_t) = \operatorname{Ann}_{A_0}(M_l) \quad \text{and} \quad U_n := \mathsf{C}^{A_0}_n(N_t) = \mathsf{C}^{A_0}_n(N_l)$$

for all $t \geqslant l$ and $n \geqslant 0$. Note that any prime ideal of A_0 belongs to U_n for some $n \geqslant 0$. Since $U_0 \subseteq U_1 \subseteq \cdots$ is an ascending chain of open subsets, there exists $m \geqslant 0$ such that $U_m = \bigcup_{n\geqslant 0} U_n = \operatorname{Spec}(A_0)$. For each $0 \leqslant n \leqslant m-1$, we can write $\mathrm{V}_{A_0}(I_n) = \operatorname{Spec}(A_0) \setminus U_n$ for some ideal I_n of A_0. The subset $\bigcup_{n=0}^{m-1} \operatorname{Ass}_{A_0}(A_0/I_n)$ of $\operatorname{Spec}(A_0)$ is finite. It follows from Lemma 3.7 that we can take $k \geqslant l$ such that

$$\operatorname{depth}(M_t)_\mathfrak{q} = \operatorname{depth}(M_k)_\mathfrak{q} \tag{3.8.1}$$

for any $t \geqslant k$, and any $\mathfrak{q} \in \bigcup_{n=0}^{m-1} \operatorname{Ass}_{A_0}(A_0/I_n)$. For any $\mathfrak{p} \in \mathrm{V}_{A_0}(J)$ and any $t \geqslant k$, we have the equality

$$\dim(M_t)_\mathfrak{p} = \dim(M_k)_\mathfrak{p}. \tag{3.8.2}$$

Let $\mathfrak{p}$ be a prime ideal of A_0. We claim that $\operatorname{depth}(M_t)_\mathfrak{p} = \operatorname{depth}(M_k)_\mathfrak{p}$ for all $t \geqslant k$. We may assume that $\mathfrak{p}$ contains J. If $\mathfrak{p}$ belongs to U_0, then we have

$$\operatorname{depth}(M_t)_\mathfrak{p} \leqslant \dim(M_t)_\mathfrak{p} \leqslant \dim_{(A_0)_\mathfrak{p}}(N_k)_\mathfrak{p} \leqslant \operatorname{depth}_{(A_0)_\mathfrak{p}}(N_k)_\mathfrak{p} \leqslant \operatorname{depth}(M_t)_\mathfrak{p}$$

for all $t \geqslant k$ by (1) and (2) of Lemma 2.6. This means that the claim holds. If $\mathfrak{p}$ does not belong to U_0, then $\operatorname{codepth}_{(A_0)_\mathfrak{p}}(N_k)_\mathfrak{p} = n+1$ for some $0 \leqslant n \leqslant m-1$. As $\mathfrak{p}$ is in $\mathrm{V}_{A_0}(I_n)$, we see that $\mathfrak{q} \subseteq \mathfrak{p}$ for some $\mathfrak{q} \in \operatorname{Ass}_{A_0}(A_0/I_n)$. Since $\mathfrak{q}$ is not in U_n, we get $n+1 \leqslant \operatorname{codepth}(N_k)_\mathfrak{q}$, and in particular, $\mathfrak{q}$ contains J. On the other hand, by (1) and (2) of Lemma 2.6, (3.8.1), and (3.8.2), it is seen that for all $t \geqslant k$,

$$\operatorname{codepth}(N_k)_{\mathfrak{q}} = \operatorname{codepth}(M_t)_{\mathfrak{q}}.$$

Also, it is follows (1) and (2) of Lemma 2.6 and Lemma 3.1 that for all $t \geqslant k$,

$$\operatorname{codepth}(M_t)_{\mathfrak{q}} \leqslant \operatorname{codepth}(M_t)_{\mathfrak{p}} \leqslant \operatorname{codepth}(N_k)_{\mathfrak{p}} = n+1.$$

Hence we get $\operatorname{codepth}(M_t)_{\mathfrak{p}} = n+1$ for all $t \geqslant k$. The claim follows from (3.8.2). ■

4. Asymptotic stability of depths of localizations of modules

In this section, we prove the main result of this paper. All of the results of Theorem 1.1 are given as corollaries of the theorem below.

THEOREM 4.1. *Let R be a ring, I an ideal of R, and M a finitely generated R-module. Suppose that that the ring $\bar{R} := R/(I+\operatorname{Ann}_R(M))$ is catenary and that $\mathsf{CM}(R/\mathfrak{p})$ contains a nonempty open subset of $\operatorname{Spec}(R/\mathfrak{p})$ for any $\mathfrak{p} \in \operatorname{Supp}_R(\bar{R})$. Then there is an integer $k > 0$ such that for all integers $t \geqslant k$ and all prime ideals $\mathfrak{p}$ of R,*

$$\operatorname{depth}(M/I^tM)_{\mathfrak{p}} = \operatorname{depth}(M/I^kM)_{\mathfrak{p}}.$$

PROOF. The associated graded ring $A = \bigoplus_{i \geqslant 0} I^i/I^{i+1}$ is homogeneous. Then $\bigoplus_{i \geqslant 0} I^iM/I^{i+1}M$ is a finitely generated graded A-module. By Corollary 3.6(2) and Lemma 3.8, we find an integer $m > 0$ such that

$$\operatorname{depth}(I^tM/I^{t+1}M)_{\mathfrak{p}} = \operatorname{depth}(I^mM/I^{m+1}M)_{\mathfrak{p}} \tag{4.1.1}$$

for all integers $t \geqslant m$ and all prime ideals $\mathfrak{p}$ of R. For any $i > 0$ and any $\mathfrak{p} \in \operatorname{Spec}(R)$,

$$X := \operatorname{Supp}_R(\bar{R}) = \operatorname{Supp}_R(M/I^iM) \text{ and } \dim(M/I^iM)_{\mathfrak{p}} = \dim(\bar{R})_{\mathfrak{p}}. \tag{4.1.2}$$

Applying Corollary 3.6(2) to $A = A_0 = R$, we see that $U_n^t := \bigcup_{m \leqslant i \leqslant t} \mathsf{C}_n^R(M/I^iM)$ is open for any $t \geqslant m$ and any $n \geqslant 0$. Lemma 2.12 implies that there is an integer $l \geqslant m$ such that $U_n^t = U_n^l$ for all $t \geqslant l$ and all $n \geqslant 0$. Put $k = l+1$. By (4.1.2), we have only to show that the following claim holds. Indeed, it means that $\operatorname{codepth}(M/I^tM)_{\mathfrak{p}} = \operatorname{codepth}(M/I^kM)_{\mathfrak{p}}$ for all $t \geqslant k$ and all prime ideals $\mathfrak{p}$ of R.

CLAIM. $\mathsf{C}_n^R(M/I^tM) = \mathsf{C}_n^R(M/I^kM)$ for all $t \geqslant k$ and all $n \geqslant 0$.

Fix an integer $n \geqslant 0$. Let $\mathfrak{p}$ be a prime ideal of R belonging to $\mathsf{C}_n^R(M/I^tM)$ for some $t \geqslant k$. We prove that $\mathfrak{p}$ is in $\mathsf{C}_n^R(M/I^iM)$ for all $i \geqslant k$. We may assume that $\mathfrak{p}$ is in X. By (4.1.1) and (4.1.2), we obtain $r := \operatorname{depth}(I^kM/I^{k+1}M)_{\mathfrak{p}} = \operatorname{depth}(I^iM/I^{i+1}M)_{\mathfrak{p}}$ and $d := \dim(M/I^kM)_{\mathfrak{p}} = \dim(M/I^iM)_{\mathfrak{p}}$ for all $i \geqslant m$. The prime ideal $\mathfrak{p}$ belongs to $\mathsf{C}_n^R(M/I^sM)$ for some $m \leqslant s \leqslant l$ since $U_n^t = U_n^l$, which means $\operatorname{depth}(M/I^sM)_{\mathfrak{p}} \geqslant d-n$. For each integer $i \geqslant s$, there is an exact sequence

$$0 \to (I^iM/I^{i+1}M)_{\mathfrak{p}} \to (M/I^{i+1}M)_{\mathfrak{p}} \to (M/I^iM)_{\mathfrak{p}} \to 0.$$

Suppose $r < d-n$. It follows from [**4**, Proposition 1.2.9] and by induction on i that for all

$i \geqslant s$, $\mathrm{depth}(M/I^{i+1}M)_{\mathfrak{p}} = r$. In particular, we have $\mathrm{depth}(M/I^tM)_{\mathfrak{p}} = r < d-n$, that is, $\mathfrak{p}$ is not in $\mathsf{C}_n^R(M/I^tM)$. This is a contradiction. Hence, we get $r \geqslant d-n$. Similarly, we see by induction on i that $\mathrm{depth}(M/I^iM)_{\mathfrak{p}} \geqslant d-n$ for any $i \geqslant s$. This means $\mathfrak{p}$ belongs to $\mathsf{C}_n^R(M/I^iM)$ for all integers $i \geqslant s$. The proof of claim is now completed. ■

We recall a few definitions of notions used in our next result.

Definition 4.2. A ring R is said to be *quasi-excellent* if the following two conditions are satisfied.

(1) The regular locus $\mathsf{Reg}(S) = \{\mathfrak{p} \in \mathrm{Spec}(S) \mid \text{the local ring } S_{\mathfrak{p}} \text{ is regular}\}$ of S is open for all finitely generated R-algebras S.

(2) All the formal fibers of $R_{\mathfrak{p}}$ are regular for all prime ideals $\mathfrak{p}$ of R.

A ring R is said to be *excellent* if it is quasi-excellent and universally catenary. A ring in which "regular" is replaced with "Cohen–Macaulay" and "Gorenstein" in both conditions (1) and (2) in the definition of an excellent ring is called a *CM-excellent* ring and an *acceptable* ring, respectively.

Typical examples of a CM-excellent ring include an excellent ring, an acceptable ring, and a homomorphic image of a Cohen–Macaulay ring; see [**14**], [**17**] for instance. Applying the above theorem, we can prove the main result of this paper.

Corollary 4.3. *Let R be a ring and I an ideal of R. Let M be a finitely generated R-module. Put $\bar{R} = R/(I + \mathrm{Ann}_R(M))$. Then there is an integer $k > 0$ such that*

$$\mathrm{depth}(M/I^tM)_{\mathfrak{p}} = \mathrm{depth}(M/I^kM)_{\mathfrak{p}}$$

for all integers $t \geqslant k$ and all prime ideals $\mathfrak{p}$ of R in each of the following cases.
(1) $\bar{R}$ is CM-excellent. (2) $\bar{R}$ is semi-local. (3) M or M/I^nM is Cohen–Macaulay for some $n > 0$.

Proof. (1): The assertion follows immediately from Theorem 4.1.

(2): Consider first the case when R is local. Let $\hat{R}$ be the completion of R and $\hat{M}$ the completion of M. For any prime ideal $\mathfrak{p}$ of R, there is a prime ideal $\mathfrak{q}$ of $\hat{R}$ such that $\mathfrak{p} = \mathfrak{q} \cap R$ as $\hat{R}$ is faithfully flat over R. It follows from [**4**, Proposition 1.2.16(a)] that

$$\mathrm{depth}_{R_{\mathfrak{p}}}(M/I^tM)_{\mathfrak{p}} = \mathrm{depth}_{\hat{R}_{\mathfrak{q}}}(\hat{M}/I^t\hat{M})_{\mathfrak{q}} - \mathrm{depth}_{\hat{R}_{\mathfrak{q}}}(\hat{R}_{\mathfrak{q}}/\mathfrak{p}\hat{R}_{\mathfrak{q}})$$

for any $t > 0$. The assertion follows from Theorem 4.1 since $\hat{R}$ is (CM-)excellent.

Next, we handle the case where R is general. Let $\mathfrak{p}_1, \cdots \mathfrak{p}_r$ be all maximal ideals that are in $\mathrm{Supp}_R(\bar{R})$. For all $1 \leqslant i \leqslant r$, there is an integer $k_i > 0$ such that $\mathrm{depth}(M/I^tM)_{\mathfrak{p}} = \mathrm{depth}(M/I^{k_i}M)_{\mathfrak{p}}$ for all $t \geqslant k_i$ and all prime ideals $\mathfrak{p}$ which is contained in $\mathfrak{p}_i$ because $R_{\mathfrak{p}_i}$ is local. Setting $k = \max\{k_i \mid 1 \leqslant i \leqslant r\}$ completes the proof.

(3): Suppose that there is a Cohen–Macaulay module N such that $\mathrm{Supp}_R(N)$ contains $\mathrm{Supp}_R(\bar{R})$. It is seen from [**4**, Theorem 2.1.3(b)] and the proof of [**4**, Theorem 2.1.12] that $\bar{R}$ is catenary. Since $\mathsf{CM}_R(N) = \mathrm{Spec}(R)$ is open, [**15**, Theorem 5.4] implies that for any $\mathfrak{p} \in \mathrm{Supp}_R(\bar{R})$, $\mathsf{CM}(R/\mathfrak{p})$ contains a nonempty open subset of $\mathrm{Spec}(R/\mathfrak{p})$. Thus the assertion follows from Theorem 4.1. ■

The assumptions about the ring $\bar{R}$ in Theorem 4.1 and Corollary 4.3 are satisfied if they hold for R. The above corollary recovers [**20**, Theorem 5.3]. Indeed, in that result, it was assumed that R is excellent, M is Cohen–Macaulay, and I is not contained in any minimal prime ideal of M.

By using the technique of the proof of Theorem 4.1, the depths of localizations of $M/I^{n+1}M$ can be measured by those of $I^nM/I^{n+1}M$ for each integer n. We provide two examples where Corollary 4.3 is applicable, but [**20**, Theorem 5.3] is not.

EXAMPLE 4.4. Let $R = K[\![x, y, z, w]\!]/(xy - zw)$ be a quotient of a formal power series ring over a field K. Take the ideal $I = (x)$ of R and the finitely generated R-module $M = R/(w)$. The module M is Cohen–Macaulay, and all elements of I are zero-divisors of M. Then M is also a module over $A = K[\![x, y, z]\!]$. We see that $M \simeq A/(xy)$, $M/I^nM \simeq A/(x^n, xy)$ and $I^nM/I^{n+1}M \simeq A/(x, y)$. Let $\mathfrak{p}$ be a prime ideal of A. A similar argument to the latter part of the proof of Theorem 4.1 shows that $\mathrm{depth}(M/I^nM)_\mathfrak{p} = \mathrm{ht}\mathfrak{p} - 2$ for any integer $n \geqslant 2$ if $\mathfrak{p}$ contains the ideal (x, y) of A; otherwise, we have $(M/I^{n+1}M)_\mathfrak{p} \simeq (M/I^nM)_\mathfrak{p}$ for any integer $n \geqslant 1$. This says that the integer $k = 2$ satisfies the assertion of Corollary 4.3.

EXAMPLE 4.5. Let $R = K[x, y, z]$ be a polynomial ring over a field K. Take the ideal $I = (x)$ of R and the finitely generated R-module $M = R/(x^my, x^mz)$, where $m > 0$. The ring R is regular but not local. All elements of I are zero-divisors of M. The R-module M is not Cohen–Macaulay; see [**4**, Theorem 2.1.2(a)]. We have

$$M/I^nM \simeq R/(x^n, x^my, x^mz), \quad I^nM/I^{n+1}M \simeq \begin{cases} R/(x) & (n < m) \\ R/(x, y, z) & (n \geqslant m). \end{cases}$$

Let $\mathfrak{p}$ be a prime ideal of R. Suppose $\mathfrak{p} = (x, y, z)$. We get $\mathrm{depth}(M/I^nM)_\mathfrak{p} = 2$ for any $0 \leqslant n \leqslant m$. On the other hand, we obtain $\mathrm{depth}(M/I^nM)_\mathfrak{p} = 0$ for any $n > m$ since the submodule $I^{n-1}M/I^nM$ of M/I^nM is isomorphic to $R/\mathfrak{p}$. It is seen that $(M/I^{n+1}M)_\mathfrak{p} \simeq (M/I^nM)_\mathfrak{p}$ for any integer $n \geqslant m$ if $\mathfrak{p} \neq (x, y, z)$. This says that the integer $k = m + 1$ satisfies the assertion of Corollary 4.3.

For modules all of whose localizations have the same depth, the notion of a regular sequence is consistent.

PROPOSITION 4.6. *Let R be a ring. Let M and N be finitely generated R-modules. Suppose that* $\mathrm{depth}(M_\mathfrak{p}) = \mathrm{depth}(N_\mathfrak{p})$ *for all prime ideals $\mathfrak{p}$ of R. Then, for any sequence $\boldsymbol{x} = x_1, \ldots, x_n$ in R, $\boldsymbol{x}$ is an M-regular sequence if and only if it is an N-regular sequence. In particular,* $\mathrm{grade}(J, M) = \mathrm{grade}(J, N)$ *for any ideal J of R.*

PROOF. We observe that $\mathrm{Supp}_R(M) = \mathrm{Supp}_R(N)$. We prove the proposition by induction on n. It is seen by assumption that for any $x \in R$, $\mathrm{Supp}_R(M/xM) = \mathrm{Supp}_R(N/xN)$ and $\mathrm{Ass}_R(M) = \mathrm{Ass}_R(N)$. This says that the assertion of the proposition holds in the case $n = 1$. Suppose $n > 1$. We may assume that $\boldsymbol{x}' = x_1, \dots, x_{n-1}$ is a regular sequence on both M and N. Then we see that $\operatorname{depth}(M/\boldsymbol{x}'M)_{\mathfrak{p}} = \operatorname{depth}(N/\boldsymbol{x}'N)_{\mathfrak{p}}$ for all prime ideals $\mathfrak{p}$ of R. Applying the case $n = 1$ shows the assertion. ∎

The following result is a direct corollary of Corollary 4.3 and Proposition 4.6. It improves [**16**, Corollary 9] and partly refines [**2**, Theorem (2)(i)]. Indeed, unlike those results, the integer k does not depend on the ideal J in Corollary 4.7.

Corollary 4.7. *Let R be a ring, I an ideal of R, and M a finitely generated R-module. Suppose that we are in one of the cases of* Corollary 4.3*. Then there is $k > 0$ such that for all $t \geqslant k$ and all sequences $\boldsymbol{x} = x_1, \dots, x_n$ in R, $\boldsymbol{x}$ is an M/I^tM-regular sequence if and only if it is an M/I^kM-regular sequence. In particular,* $\operatorname{grade}(J, M/I^tM) = \operatorname{grade}(J, M/I^kM)$ *for all integers $t \geqslant k$ and all ideals J of R.*

We remark that the theorem proved by Brodmann [**1**] is recovered from this corollary.

Remark 4.8. Let R be a ring, I an ideal of R, and M a finitely generated R-module. Lemma 2.4 asserts that $\bigcup_{i \geqslant 0} \operatorname{Ass}_R(I^iM/I^{i+1}M)$ is a finite set. By induction on $n > 0$, it is seen that $\operatorname{Ass}_R(M/I^nM)$ is contained in $\bigcup_{i=0}^{n-1} \operatorname{Ass}_R(I^iM/I^{i+1}M)$; see [**18**, Theorem 6.3]. The set $X := \bigcup_{n>0} \operatorname{Ass}_R(M/I^nM)$ is also a finite set. It follows from Corollary 4.7 that there is an integer $k > 0$ such that $\operatorname{depth}(M/I^tM)_{\mathfrak{p}} = \operatorname{depth}(M/I^kM)_{\mathfrak{p}}$ for all integers $t \geqslant k$ and all prime ideals $\mathfrak{p}$ of R belonging to X. This says that for all integers $t \geqslant k$, $\operatorname{Ass}_R(M/I^tM) = \operatorname{Ass}_R(M/I^kM)$.

Finally, one application of the main result is described.

Remark 4.9. Let $(R, \mathfrak{m})$ be a local ring or standard graded ring (in which case $\mathfrak{m}$ is the irrelevant ideal), I an ideal of R, and $d = \dim(R)$. The symbol λ denotes length. The question of when the limit

$$\lim_{n \to \infty} \frac{\lambda(\mathrm{H}^i_{\mathfrak{m}}(R/I^n))}{n^d}$$

exists and what value its limit takes has been actively studied; see [**6**], [**7**], [**9**], [**11**], [**12**] for instance. In particular, as in [**7**, Question 1.1], one of the most important problems is when does $\mathrm{H}^i_{\mathfrak{m}}(R/I^n)$ have finite length for all large n? Suppose that R is CM-excellent. It follows from [**3**, Proposition 9.1.2] and [**13**, Theorem 1.1] that for any n, the equality

$$\begin{aligned} &\inf\{i \mid \mathrm{H}^i_{\mathfrak{m}}(R/I^n) \text{ is not finitely generated}\} \\ &= \inf\{\operatorname{depth}(R/I^n)_{\mathfrak{p}} + \operatorname{ht}(\mathfrak{m}/\mathfrak{p}) \mid \mathfrak{p} \in \operatorname{Spec}(R) \setminus \{\mathfrak{m}\}\} \end{aligned}$$

holds. Corollary 4.3 deduces that the right side of the equation attains a stable constant value for all large n. Therefore, we see that for large n, the smallest integer t such that $\lambda(\mathrm{H}^t_{\mathfrak{m}}(R/I^n)) = \infty$ is a constant value independent of n. This means that for any $i < t$, $\mathrm{H}^i_{\mathfrak{m}}(R/I^n)$ has finite length for all large n.

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References

[1] M. Brodmann, Asymptotic stability of Ass(M/I^nM), Proc. Amer. Math. Soc. **74** (1979), no. 1, 16–18.

[2] M. Brodmann, The asymptotic nature of the analytic spread, Math. Proc. Cambridge Philos. Soc. **86** (1979), no. 1, 35–39.

[3] M. P. Brodmann; R. Y. Sharp, Local cohomology. An algebraic introduction with geometric applications. Second edition. Cambridge Stud. Adv. Math. **136**, Cambridge University Press, Cambridge, 2013.

[4] W. Bruns; J. Herzog, Cohen–Macaulay rings, revised edition, Cambridge Stud. Adv. Math. **39**, Cambridge University Press, Cambridge, 1998.

[5] K. Česnavičius, Macaulayfication of Noetherian schemes, Duke Math. J. **170** (2021), no. 7, 1419–1455.

[6] S. D. Cutkosky, Asymptotic multiplicities of graded families of ideals and linear series, Adv. Math. **264** (2014), 55–113.

[7] H. Dao; J. Montaño, Length of local cohomology of powers of ideals, Trans. Amer. Math. Soc. **371** (2019), no. 5, 3483–3503.

[8] A. Grothendieck, Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas. II, Inst. Hautes Études Sci. Publ. Math. **24**, (1965).

[9] J. Herzog; T. J. Puthenpurakal; J. K. Verma, Hilbert polynomials and powers of ideals, Math. Proc. Cambridge Philos. Soc. **145** (2008), no. 3, 623–642.

[10] M. Hochster; J. L. Roberts, Rings of invariants of reductive groups acting on regular rings are Cohen–Macaulay, Adv. Math. **13** (1974), 115–175.

[11] J. Jeffries; J. Montaño; M. Varbaro, Multiplicities of classical varieties, Proc. Lond. Math. Soc. (3) **110** (2015), no. 4, 1033–1055.

[12] D. Katz; J. Validashti, Asymptotic multiplicities of graded families of ideals and linear series, Collect. Math. **61** (2010), no. 1, 1–24.

[13] T. Kawasaki, On Faltings' annihilator theorem, Proc. Amer. Math. Soc. **136** (2008), no. 4, 1205–1211.

[14] T. Kawasaki, Finiteness of Cousin cohomologies, Trans. Amer. Math. Soc. **360** (2008), no. 5, 2709–2739.

[15] K. Kimura, Openness of various loci over Noetherian rings, J. Algebra **633** (2023), 403–424.

[16] V. Kodiyalam, Homological invariants of powers of an ideal, Proc. Amer. Math. Soc. **118** (1993), no. 3, 757–764.

[17] C. Massaza; P. Valabrega, Sull'apertura di luoghi in uno schema localmente noetheriano, Boll. Unione Mat. Ital. A (5) **14** (1977), no. 3, 564–574.

[18] H. Matsumura, Commutative ring theory, Translated from the Japanese by M. Reid, Second edition, Cambridge Stud. Adv. Math. **8**, Cambridge University Press, Cambridge, 1989.

[19] S. McAdam, Asymptotic prime divisors, Lecture Notes in Math. vol. 1023, Springer-Verlag, Berlin, 1983.

[20] C. Rotthaus; L. M. Şega, Open loci of graded modules, Trans. Amer. Math. Soc. **358** (2006), no. 11, 4959–4980.

[21] R. Y. Sharp, Acceptable rings and homomorphic images of Gorenstein rings, J. Algebra **44** (1977), no. 1, 246–261.

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