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Brownian Hitting to Spheres

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Abstract. The joint distribution of the first hitting time of a Brownian motion with constant drift to a sphere and the hitting place is studied. Explicit formulae by means of spherical harmonics for the joint density are presented. The result is applied to a study on the asymptotic behavior of the distribution function.

1. Introduction and main results

For $d \geqq 2$, let $B = \{B(t)\}_{t \geqq 0}$ be a standard d-dimensional Brownian motion. Setting $B^{(v)}(t) = B(t) + vt$ for a fixed $v \in \mathbf{R}^d$, we consider a Brownian motion $B^{(v)} = \{B^{(v)}(t)\}_{t \geqq 0}$ with the constant drift v. Letting S_r^{d-1} be the sphere in $\mathbf{R}^d$ with radius r and centered at the origin, we are concerned with the joint distribution of the first hitting time $\sigma_r^{(v)}$ of $B^{(v)}$ to S_r^{d-1} and the hitting place $B^{(v)}(\sigma_r^{(v)})$.

The aim of this article is to show an explicit expression for the density of the joint distribution by means of the spherical harmonics, that is, the Gegenbauer and the Chebyshev polynomials. As an application, we study the asymptotic behavior of the tail probability $P_a(t < \sigma_r^{(v)} < \infty, B^{(v)}(\sigma_r^{(v)}) \in A), A \subset S_r^{d-1}$ when the Brownian motion starts from outside the ball.

Several authors have studied the joint distribution for the standard Brownian motion. It should be noted at first that in the exit problem, that is the case of $a < r$, the joint density is given by a solution for a heat equation with the Dirichlet boundary condition. See Aizenman-Simon [**1**] for general discussion and Hsu [**11**] for an explicit expression in the case of the spheres. Wendel [**23**] has shown a nice result on the expectations of functions of $(\sigma_r^{(0)}, B(\sigma_r^{(0)})$ by using the spherical harmonics. See Gzyl [**5**] and references therein for a recent study on this direction and Uchiyama [**21, 22**] on the asymptotic behavior of the distribution functions and its application to the Wiener sausage. A similar problem for a Brownian motion with drift has been discussed in Yin-Wang [**24**].

We proceed to a different way from the above cited works. Starting from the skew-product representation of Brownian motion, we use the fact that the one-dimensional projection of the Brownian motion on the sphere $S^{d-1} = S_1^{d-1}$ defines a diffusion process and that the eigenvalues and the eigenfunctions for the infinitesimal generator are explicitly given by the spherical harmonics.

By the rotation invariance of the probability of Brownian motion, it is sufficient to consider the case where $B(0) = (a, 0, ..., 0)$ and $a > 0$. As usual we denote by I_ν and K_ν

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the modified Bessel functions and also by C_n^ν and T_n the Gegenbauer and the Chebyshev polynomials, respectively. The inner product in $\mathbf{R}^d$ is written as $\langle \cdot, \cdot \rangle$.

THEOREM 1.1. *Denote by E_a the expectation with respect to the probability law P_a of the Brownian motion B. Then, for all $\lambda > 0$ and $u \in \mathbf{R}^d$, we have*

$$E_a\Big[e^{-\lambda\sigma_r^{(v)}} e^{\langle u, B^{(v)}(\sigma_r^{(v)})\rangle} I_{\{\sigma_r^{(v)}<\infty\}}\Big] = e^{-av_1}\bigg\{\frac{\mathcal{L}_0(a\sqrt{2\lambda+|v|^2})}{\mathcal{L}_0(r\sqrt{2\lambda+|v|^2})}\int_{S^1} e^{r\langle u+v,z\rangle} ds(z)$$
$$+ 2\sum_{n=1}^\infty \frac{\mathcal{L}_n(a\sqrt{2\lambda+|v|^2})}{\mathcal{L}_n(r\sqrt{2\lambda+|v|^2})}\int_{S^1} e^{r\langle u+v,z\rangle} T_n(z_1) ds(z)\bigg\}$$

when $d = 2$ and, when $d \geqq 3$,

$$E_a\Big[e^{-\lambda\sigma_r^{(v)}} e^{\langle u, B^{(v)}(\sigma_r^{(v)})\rangle} I_{\{\sigma_r^{(v)}<\infty\}}\Big]$$
$$= \frac{1}{\nu} e^{-av_1} \sum_{n=0}^\infty (n+\nu)\frac{a^{-\nu}\mathcal{L}_{n+\nu}(a\sqrt{2\lambda+|v|^2})}{r^{-\nu}\mathcal{L}_{n+\nu}(r\sqrt{2\lambda+|v|^2})}\int_{S^{d-1}} e^{r\langle u+v,z\rangle} C_n^\nu(z_1) ds(z),$$

where ds is the uniform probability measure on S^{d-1}, and $\mathcal{L} = I$ for $a < r$ and $\mathcal{L} = K$ for $a > r$.

Setting $v = 0, u = 0$ and noting that the surface integrals of $T_n(z_1)$ and $C_n^\nu(z_1)$ vanish for $n \geqq 1$, we recover the well-known formula for $E_a[e^{-\lambda\sigma_r^{(0)}}]$ (cf. [**2**]).

We can invert the joint Laplace transforms and obtain the following. We let $\rho_{a,r}^{(\mu)}(t)$ be the probability density of the first hitting time to r of a Bessel process with index μ starting from a.

THEOREM 1.2. *For $t > 0$ and $z \in \mathbf{R}^d$ with $|z| = r$, we have*

$$P_a(\sigma_r^{(v)} \in dt, B^{(v)}(\sigma_r^{(v)}) \in dz) = e^{-av_1+\langle v,z\rangle} e^{-\frac{|v|^2}{2}t}\rho_{a,r}^{(0)}(t) dt ds_r(z)$$
$$+ 2e^{-av_1+\langle v,z\rangle} e^{-\frac{|v|^2}{2}t}\sum_{n=1}^\infty \Big(\frac{a}{r}\Big)^n \rho_{a,r}^{(n)}(t) T_n\Big(\frac{z_1}{r}\Big) dt ds_r(z)$$

when $d = 2$ and, when $d \geqq 3$

$$P_a(\sigma_r^{(v)} \in dt, B_{\sigma_r}^{(v)} \in dz)$$
$$= \frac{1}{\nu} e^{-av_1+\langle v,z\rangle} e^{-\frac{|v|^2}{2}t}\sum_{n=0}^\infty (n+\nu)\Big(\frac{a}{r}\Big)^n \rho_{a,r}^{(n+\nu)}(t) C_n^\nu\Big(\frac{z_1}{r}\Big) dt ds_r(z),$$

where $\nu = \frac{d-2}{2}$ and ds_r is the uniform probability measure on S_r^{d-1}.

When $v = 0$, the authors [**9**] have shown another expression for the joint Laplace transform, from which we can prove Theorem 1.2 by using the Cameron-Martin theorem (see (4.1) below).

The rest of this article is organized as follows. In the next Section 2, admitting Theorem 1.2 as proved, we investigate the asymptotic behavior of the distribution function $P_a(t < \sigma_r^{(v)} < \infty, B^{(v)}(\sigma_r^{(v)}) \in A)$, $A \subset S_r^{d-1}$, as $t \to \infty$. In Section 3 we study the first coordinate or the one-dimensional projection of the Brownian motion on S^{d-1}. The results play important roles in the final Section 4, where we give proofs of Theorems 1.1 and 1.2.

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2. Asymptotic behavior of distribution function

In this section, assuming $a > r$ and applying Theorem 1.2, we study the asymptotic behavior of the distribution function $P_a(t < \sigma_r^{(v)} < \infty, B^{(v)}(\sigma_r^{(v)}) \in A)$ as $t \to \infty$ for a fixed Borel subset A of the sphere S_r^{d-1}. We use the same notation as in the previous section.

We need the following facts on the tail probabilities for the first hitting times of Bessel processes. For details, see [**7, 8**] on the basic results, [**3**] and [**10**] on the estimates for the remainder terms in the cases of $d = 2$ and $d \geqq 3$, respectively.

Consider a Bessel process with index ν and starting from a defined on some probability space $(\Omega, \mathcal{F}, Q_a^{(\nu)})$ and let τ_r be its hitting time to r. Then the asymptotic behavior of $Q_a^{(\nu)}(t < \tau_r < \infty)$ when $a > r$ is given by

$$Q_a^{(0)}(t < \tau_r < \infty) = \frac{2\log(a/r)}{\log t} + O\big(\frac{1}{(\log t)^2}\big) \tag{2.1}$$

when $d = 2$ and

$$Q_a^{(\nu)}(t < \tau_r < \infty) = \kappa^{(\nu)} t^{-\nu} + \begin{cases} O(t^{-2\nu}) & d = 3, \\ O(t^{-2}\log t) & d = 4, \\ O(t^{-\nu-1}) & d \geqq 5, \end{cases} \tag{2.2}$$

where the constant $\kappa^{(\nu)}$ is given by

$$\kappa^{(\nu)} = \frac{1}{\Gamma(\nu+1)} \Big(\frac{r^3}{2a}\Big)^{\nu} \Big\{ \Big(\frac{a}{r}\Big)^{\nu} - \Big(\frac{a}{r}\Big)^{-\nu} \Big\}.$$

Applying these results with some estimates for the remainder terms, we show the results in this section.

First we consider the usual Brownian motion. We write σ_r for $\sigma_r^{(0)}$. As is easily guessed, the leading term is given by the first terms of the right hand sides in Theorem 1.2.

Theorem 2.1. *For any Borel subset A of S_r^{d-1}, we have, as $t \to \infty$,*

$$P_a(t < \sigma_r < \infty, B(\sigma_r) \in A) = \frac{2\log(a/r)}{\log t} s_r(A) + O\big(\frac{1}{(\log t)^2}\big)$$

when $d = 2$ and

$$P_a(t<\sigma_r<\infty, B(\sigma_r)\in A)=\kappa^{(\nu)}s_r(A)t^{-\nu}+\begin{cases}O(t^{-2\nu}) & d=3,\\ O(t^{-2}\log t) & d=4,\\ O(t^{-\nu-1}) & d\geqq 5,\end{cases}$$

Remark 2.2. The leading terms of the tail probabilities were obtained by Hunt, Spitzer and Port for the hitting times to general bounded sets. For the details, see Port-Stone [**19**, pp. 64 and 81].

Remark 2.3. For the distribution function $Q_a^{(\nu)}(t<\tau_r<\infty)$ of the first hitting time of the Bessel process, a precise asymptotic expansion has been shown in Hamana et al. [**6**]. Using the results, we can show asymptotic expansions for our joint distribution functions. The details will be published elsewhere.

For a proof of Theorem 2.1, we show the following estimate for the tail probability of σ_r.

Lemma 2.4. *Assume $d\geqq 3$. Then, for $t>0$, we have*

$$P_a(t<\sigma_r<\infty)\leqq \frac{r^{2\nu}}{2^\nu\Gamma(\nu+1)t^\nu}.$$

Proof. Let L_r be the last hitting time of the Brownian motion B to the sphere S_r^{d-1}:

$$L_r=\sup\{s>0:|B(s)|=r\}.$$

As usual we set $L_r=0$ when B does not hit S_r^{d-1}. Then we have

$$P_a(t<\sigma_r<\infty)\leqq P_a(t<L_r<\infty).$$

Let μ_r be the equilibrium measure of the ball $\mathbf{B}_r$ with radius r and centered at the origin. Then it is well known ([**19**]) that

$$P_a(t<L_r<\infty)=\int_t^\infty ds\int_{\mathbf{R}^d}\frac{1}{(2\pi s)^{d/2}}e^{-\frac{|x-a|^2}{2s}}d\mu_r(x).$$

Recalling now that the capacity of $\mathbf{B}_r$ is $\mu_r(\mathbf{R}^d)=2\pi^{\frac{d}{2}}r^{d-2}/\Gamma(\frac{d}{2}-1)$, we see

$$P_a(t<L_r<\infty)\leqq\int_t^\infty ds\int_{\mathbf{R}^d}\frac{1}{(2\pi s)^{d/2}}d\mu_r(x)=\frac{r^{2\nu}}{2^\nu\Gamma(\nu+1)t^\nu}. \qquad \square$$

Remark 2.5. For transient one-dimensional diffusion processes, the densities of the last hitting times are written by means of the transition densities. This is the case of the Bessel processes with dimensions $d\geqq 3$ and, moreover, we have explicit expressions for the transition densities. We can give another proof for Lemma 2.4 by using these facts.

We can now give a proof of Theorem 2.1. Note that the infinite sum below for the expression for the joint distribution is absolutely convergent.

For $d = 2$, we have by Theorem 1.2

$$P_a(t < \sigma_r < \infty, B(\sigma_r) \in A) = Q_a^{(0)}(\tau_r > t)s_b(A) + I_t,$$

where

$$I_t = 2\sum_{n=1}^{\infty} \big(\frac{a}{r}\big)^n Q_a^{(n)}(t < \tau_r < \infty) \int_A T_n\big(\frac{z_1}{r}\big) ds_r(z).$$

Assume $t > 1$ and note $|T_n(x)| = |\cos(n \arccos x)| \leqq 1$. Then, by Lemma 2.4, we get

$$|I_t| \leqq 2\sum_{n=1}^{\infty} \Big(\frac{a}{r}\Big)^n \frac{r^{2n}}{2^n \Gamma(n+1)t^n} \leqq \frac{2}{t}\sum_{n=1}^{\infty}\Big(\frac{ar}{2}\Big)^n \frac{1}{n!} \leqq \frac{2}{t} e^{\frac{ar}{2}}$$

and, by (2.1), the assertion of Theorem 2.1.

For $d \geqq 3$, we have

$$P_a(t < \sigma_r < \infty, B(\sigma_r) \in A) = Q_a^{(\nu)}(t < \tau_r < \infty)s_r(A) + J_t,$$

where

$$J_t = \frac{1}{\nu}\sum_{n=1}^{\infty}(n+\nu)\big(\frac{a}{r}\big)^n Q_a^{(n+\nu)}(t < \tau_r < \infty) \int_A C_n^{\nu}\big(\frac{z_1}{r}\big) ds_r(z).$$

Now recall the estimate

$$\max_{|y| \leqq 1} |C_n^{\nu}(y)| = C_n^{\nu}(1) \leqq Cn^{2\nu-1} \tag{2.3}$$

for the Gegenbauer polynomial C_n^{ν}, where C is some constant (see, e.g., [**15**, pp.218, 225]). Then, combining this estimate with Lemma 2.4 and (2.2), we see $J_t = O(t^{-1-\nu})$ and the assertion of Theorem 2.1.

Secondly we consider Brownian motion $B^{(v)}$ with drift v and prove the following.

Theorem 2.6. *For any Borel subset A of S_r^{d-1}, we have*

$$\begin{aligned} &P_a(t < \sigma_r^{(v)} < \infty, B^{(v)}(\sigma_r^{(v)}) \in A) \\ &\quad = \frac{4}{|v|^2} \log\big(\frac{a}{r}\big) e^{-av_1} \int_A e^{\langle v,z\rangle} ds_r(z) \times \frac{1}{t(\log t)^2} e^{-\frac{|v|^2}{2}t}(1+o(1)) \end{aligned}$$

when $d = 2$ and

$$\begin{aligned} &P_a(t < \sigma_r^{(v)} < \infty, B^{(v)}(\sigma_r^{(v)}) \in A) \\ &\quad = \frac{2\nu\kappa^{(\nu)}}{|v|^2} e^{-av_1} \int_A e^{\langle v,z\rangle} ds_r(z) \times t^{-\nu-1} e^{-\frac{|v|^2}{2}t}(1+o(1)) \end{aligned}$$

when $d \geqq 3$.

For a proof of the theorem, we recall from [**9**] the asymptotic result for the integral

$$H^{(\nu)}(t) := \int_t^\infty e^{-\frac{|v|^2}{2}s}\rho_{a,r}^{(\nu)}(s)ds.$$

We have, when $d = 2$,

$$H^{(0)}(t) = \frac{4\log(a/r)}{|v|^2 t(\log t)^2} e^{-\frac{|v|^2}{2}t}(1 + o(1)) \tag{2.4}$$

and, when $d \geqq 3$,

$$H^{(\nu)}(t) = \frac{2L(\nu)}{|v|^2 t^{\nu+1}} e^{-\frac{|v|^2}{2}t}(1 + o(1)). \tag{2.5}$$

Note that there is a mistake in [**9**] for (2.4).

The main term of the first assertion is given by (2.4). Formula (2.5) is used to give the main term when $d \geqq 3$ and also to show estimates for the higher order terms.

The assertion of Theorem 2.6 follows from the following estimate for $H^{(\nu)}(t)$, which is uniform in the index ν.

LEMMA 2.7. *There exists a constant C, depending on $|v|$ and r, such that*

$$H^{(\nu)}(t) \leqq \frac{Cr^{2\nu}}{\Gamma(\nu)} \frac{1}{(2t)^{\nu+1}} e^{-\frac{|v|^2}{2}t}$$

holds for all $d \geqq 3$.

PROOF. We use (2.5) when $d = 3$ and $d = 4$, and assume $d \geqq 5$ in the following.

Let P_y be the d-dimensional Wiener measure with starting point y and use the same notation σ_r for the first hitting time to S_r^{d-1} of the corresponding Brownian motion. Moreover, let $p(t, x, y) = (2\pi t)^{-d/2}\exp(-|y - x|^2/2t)$ be the Gaussian kernel and set $\alpha = |v|^2/2$ for simplicity. Then we have

$$H^{(\nu)}(t) = \alpha \int_t^\infty e^{-\alpha s} P_a(t < \sigma_r \leqq s)ds$$

and, setting $e = (1, 0, ..., 0)$,

$$\begin{aligned} P_a(t < \sigma_r \leqq s) &\leqq \int_{\mathbf{R}^d} p(t, ae, y)P_y(\sigma_r \leqq s - t)dy \\ &\leqq \frac{1}{(2\pi t)^{d/2}} \int_{\mathbf{R}^d} P_y(\sigma_r \leqq s - t)dy \end{aligned}$$

by the Markov property of Brownian motion. Hence we get, after a simple change of variables,

$$H^{(\nu)}(t) \leqq \frac{\alpha e^{-\alpha t}}{(2\pi t)^{d/2}} \int_0^\infty e^{-\alpha s}ds \int_{\mathbf{R}^d} P_y(\sigma_r \leqq s)dy.$$

As in the proof of Lemma 2.4, let L_r be the last hitting time of the Brownian motion to S_r^{d-1}. Then we have

$$\int_{\mathbf{R}^d} P_y(\sigma_r \leqq s)dy = \int_{\mathbf{R}^d} P_y(0 < L_r \leqq s)dy + \int_{\mathbf{R}^d} P_y(\sigma_r \leqq s < L_r)dy. \tag{2.6}$$

For the second term of the right hand side, Le Gall [**14**] has shown

$$\int_{\mathbf{R}^d} P_y(\sigma_r \leqq s < L_r)dy = \int_{\mathbf{R}^d} P_y(\sigma_r \leqq s)P_y(\sigma_r < \infty)dy, \tag{2.7}$$

which implies

$$\int_{\mathbf{R}^d} P_y(\sigma_r \leqq s < L_r)dy \leqq \int_{\mathbf{R}^d} P_y(\sigma_r < \infty)^2 dy. \tag{2.8}$$

This estimate is sufficient for our purpose. We give another elementary proof of (2.7) after completing the proof of Lemma 2.7.

Letting μ_r denote the equilibrium measure of the ball $\mathbf{B}_r$, we have, for the first term of (2.6),

$$\begin{aligned}\int_0^\infty e^{-\alpha s}ds \int_{\mathbf{R}^d} P_y(0 < L_r \leqq s)dy &= \int_0^\infty e^{-\alpha s}ds \int_{\mathbf{R}^d} dy \int_0^s du \int_{\mathbf{R}^d} p(u,y,z)d\mu_r(z)\\ &= \int_0^\infty e^{-\alpha s}ds \int_0^s d\tau \int_{\mathbf{R}^d} d\mu_r(z)\\ &= \frac{2\pi^{d/2}r^{d-2}}{\alpha^2\Gamma(\frac{d}{2}-1)}.\end{aligned}$$

For the second term, we recall

$$P_y(\sigma_r < \infty) = 1 \wedge \Big(\frac{r}{|y|}\Big)^{d-2}.$$

Then, by (2.8), we get

$$\begin{aligned}\int_0^\infty e^{-\alpha s}ds \int_{\mathbf{R}^d} P_y(\sigma_r < \infty)^2 dy &= \frac{1}{\alpha}\Big\{\int_{|y|\leqq r} dy + \int_{|y|\geqq r} \big(\frac{r}{|y|}\big)^{2(d-2)}dy\Big\}\\ &= \frac{2\pi^{\frac{d}{2}}r^d}{\alpha\Gamma(\frac{d}{2})}\Big(\frac{1}{d} + \frac{1}{d-4}\Big).\end{aligned}$$

Combining the above inequalities, we obtain the assertion of the lemma. □

Proof of (2.7). By the Markov property of Brownian motion, we have

$$\begin{aligned}\int_{\mathbf{R}^d} P_y(\sigma_r \leqq s < L_r)dy &= \int_{\mathbf{R}^d} E_y[1_{\{\sigma_r\leqq s\}}1_{\{L_r>s\}}]dy\\ &= \int_{\mathbf{R}^d} E_y[1_{\{\sigma_r\leqq s\}}E_{B(s)}[1_{\{L_r>0\}}]]dy\end{aligned}$$

$$
\begin{aligned}
&= \int_{\mathbf{R}^d} dy \int_{\mathbf{R}^d} E_y[1_{\{\sigma_r \leqq s\}} P_{B(s)}(L_r > 0) | B(s) = x] p(s, y, x) dx \\
&= \int_{\mathbf{R}^d} dx \int_{\mathbf{R}^d} P_x(L_r > 0) P_y(\sigma_r \leqq s | B(s) = x) p(s, y, x) dy.
\end{aligned}
$$

Note here that $P_x(L_r > 0) = P_x(\sigma_r < \infty)$ and that the time reversal of a pinned Brownian motion is again a pinned Brownian motion. Then we obtain

$$
\begin{aligned}
\int_{\mathbf{R}^d} P_y(\sigma_r \leqq s < L_r) dy &= \int_{\mathbf{R}^d} P_x(\sigma_r < \infty) dx \int_{\mathbf{R}^d} P_x(\sigma_r \leqq s | B(s) = y) p(s, x, y) dy \\
&= \int_{\mathbf{R}^d} P_x(\sigma_r < \infty) P_x(\sigma_r \leqq s) dx. \qquad \square
\end{aligned}
$$

3. Projection of Brownian motion on sphere

In the proof of Theorem 1.1, we use the fact that the one-dimensional projections of a Brownian motion on the sphere define diffusion processes and that the eigenfunctions are given by the spherical harmonics. We here present the results which might be of independent interest.

Let $\theta = \{\theta(t)\}_{t \geqq 0}$ be a Brownian motion on S^{d-1}, which corresponds to the Laplace-Beltrami operator on S^{d-1}, endowed with the usual Euclidean metric. θ may be given by solving Stroock's stochastic differential equation ([**12, 20**])

$$
d\theta_t^i = \sum_{j=1}^d (\delta_{ij} - \theta_t^i \theta_t^j) \circ dw_t^j, \qquad i = 1, ..., d,
$$

where $\{w_t\}_{t \geqq 0}$ is a d-dimensional standard Brownian motion. From this equation we can easily show the following for the one-dimensional projections. The result is fundamental in our argument. It should be noted that Mijatovic-Mramor-Uribe Bravo [**16**] has shown that all the projections of θ are diffusion processes which are realized as unique solutions of stochastic differential equations.

Proposition 3.1. *The first coordinate $\{\theta_1(t)\}_{t \geqq 0}$ of θ is a diffusion process on $(-1, 1)$ whose generator is*

$$
\mathcal{G}_d = \frac{1}{2}(1 - x^2)\frac{d^2}{dx^2} - \frac{d-1}{2} x \frac{d}{dx}.
$$

We see easily that the boundaries ± 1 are regular and reflecting when $d = 2$ and they are entrance ones when $d \geqq 3$. The eigenvalues and the eigenfunctions of $\mathcal{G}_d$ are explicitly given and we have the eigenfunction expansion for the transition densities. Since these play important roles in the next section, we now recall some fundamental facts. For details of the Chebyshev and the Gegenbauer polynomials, we refer to [**4, 15, 17**].

Write

$$
\mathcal{G}_d = \frac{1}{2(1 - x^2)^{\frac{d-3}{2}}} \frac{d}{dx}\Big(\frac{1}{(1 - x^2)^{-\frac{d-1}{2}}} \frac{d}{dx} \Big)
$$

and let $dm(x) = 2(1-x^2)^{\frac{d-3}{2}}dx$ be the canonical (speed) measure. Note that dm is a finite measure on $(-1,1)$. Moreover, we take

$$s(x) = \int_0^x (1-y^2)^{-\frac{d-1}{2}}dy$$

as the scale function.

When $d=2$, $s(\pm 1)$ are both finite and the boundaries are regular. The Chebyshev polynomial $T_n(x) = \cos(n \arccos x)$ satisfies

$$\mathcal{G}_2 T_n = -\frac{n^2}{2}T_n \qquad \text{and} \qquad \frac{1}{s'(x)}\frac{dT_n(x)}{dx}\bigg|_{x=\pm 1} = 0.$$

Moreover the orthogonality relation is given by

$$\int_{-1}^1 T_m(x)T_n(x)\frac{dx}{\sqrt{1-x^2}} = \begin{cases} 0 & m \neq n \\ \frac{\pi}{2} & m = n \neq 0 \\ \pi & m = n = 0. \end{cases}$$

Hence, setting

$$\phi_0^0(x) = \frac{1}{\sqrt{2\pi}}, \qquad \phi_n^0(x) = \frac{1}{\sqrt{\pi}}T_n(x) \quad (n \geqq 1),$$

we see that $\{\phi_n^0\}_{n=0}^\infty$ gives rise to an orthonormal basis of $L^2(dm)$ and that the transition density $p_2(t,x,y)$ of $\{\theta_1(t)\}_{t\geqq 0}$ with respect to dm is given by

$$p_2(t,x,y) = \frac{1}{2\pi} + \frac{1}{\pi}\sum_{n=1}^\infty e^{-\frac{1}{2}n^2 t}T_n(x)T_n(y). \tag{3.1}$$

For $d \geqq 3$, the eigenfunctions are given by the Gegenbauer polynomials C_n^ν defined by

$$\sum_{n=0}^\infty u^n C_n^\nu(x) = \frac{1}{(1+u^2-2ux)^\nu}, \qquad |u| < 1.$$

In fact, recalling $\nu = (d-2)/2$, we have

$$\mathcal{G}_d C_n^\nu = -\frac{1}{2}n(n+2\nu)C_n^\nu \qquad \text{and} \qquad \frac{1}{s'(x)}\frac{dC_n^\nu(x)}{dx}\bigg|_{x=\pm 1} = 0$$

and the orthogonality relation

$$\int_{-1}^1 C_m^\nu(x)C_n^\nu(x)(1-x^2)^{\nu-\frac{1}{2}}dx = \delta_{m,n}\frac{\pi\Gamma(n+2\nu)}{2^{2\nu-1}(n+\nu)n!(\Gamma(\nu))^2}.$$

Hence, setting

$$\phi_n^\nu(x) = \Big(\frac{(n+\nu)n!}{\pi\Gamma(n+2\nu)}\Big)^{\frac{1}{2}} 2^{\nu-1}\Gamma(\nu)C_n^\nu(x),$$

we obtain an orthonormal basis $\{\phi_n^\nu\}_{n=0}^\infty$ of $L^2(dm)$ and an eigenfunction expansion for the transition density $p_d(t,x,y)$ of $\{\theta_1(t)\}_{t\geqq 0}$ with respect to dm,

$$p_d(t,x,y) = \sum_{n=0}^\infty e^{-\frac{1}{2}n(n+2\nu)t}\phi_n^\nu(x)\phi_n^\nu(y). \tag{3.2}$$

4. Proof of Theorems 1.1 and 1.2

First of all we note that the Cameron-Martin theorem and the strong Markov property of Brownian motion imply

$$E_a\Big[e^{-\lambda\sigma_r^{(v)}} e^{\langle u, B^{(v)}(\sigma_r^{(v)})\rangle} I_{\{\sigma_r^{(v)}<\infty\}}\Big] = e^{-av_1}E_a\Big[e^{-(\lambda+\frac{|v|^2}{2})\sigma_r} e^{\langle u+v, B(\sigma_r)\rangle} I_{\{\sigma_r<\infty\}}\Big]. \tag{4.1}$$

We have written σ_r for $\sigma_r^{(0)}$ as before. Hence we have only to consider the usual Brownian motion.

We start the argument from the skew-product representation of the standard Brownian motion $B=\{B(t)\}_{t\geqq 0}$: there exists a d-dimensional Bessel process $R=\{R_t\}_{t\geqq 0}$ (with index $\nu=(d-2)/2$) and a Brownian motion $\theta=\{\theta(t)\}_{t\geqq 0}$ on S^{d-1}, independent of R, such that

$$B_t = R_t\theta(\Xi_t), \qquad \Xi_t = \int_0^t \frac{ds}{R_s^2}.$$

$B_0=(a,0,...,0)$ means $R_0=a$ and $\theta(0)=(1,0,...,0)$.

We let τ_r be the first hitting time of R to r. Then, denoting by $E_a^{(\nu)}[\,\cdot\,]$ and $E_\theta[\,\cdot\,]$ the expectations with respect to the probability laws of R and θ, respectively, we have by the independence of R and θ

$$E_a[e^{-\lambda\sigma_r}e^{\langle u,B(\sigma_r)\rangle}] = E_a^{(\nu)}[e^{-\lambda\tau_r}E_\theta[e^{r\langle u,\theta(t)\rangle}]\Big|_{t=\Xi_{\tau_r}}].$$

First we prove the theorems when $d=2$. Writing $\theta(t)=(\theta_1(t),\theta_2(t))$ and $u=(u_1,u_2)$, we have by the rotation invariance of the law of standard Brownian motion

$$\begin{aligned} E_\theta[e^{r\langle u,\theta(t)\rangle}] &= E_\theta[e^{ru_1\theta_1(t)}E_\theta[e^{ru_2\theta_2(t)}|\theta_1(t)]] \\ &= \int_{-1}^1 e^{ru_1y}\frac{1}{2}\big(e^{ru_2\sqrt{1-y^2}}+e^{-ru_2\sqrt{1-y^2}}\big)P(\theta_1(t)\in dy), \end{aligned}$$

since the distribution of $\theta_2(t)$ given $\theta_1(t)=y$ is the uniform distribution on the set of the two points $\pm\sqrt{1-y^2}$. Hence formula (3.1) implies

$$E_\theta[e^{r\langle u,\theta(t)\rangle}] = \frac{1}{2\pi}\int_{-1}^1 e^{ru_1y}\frac{1}{2}\big(e^{ru_2\sqrt{1-y^2}}+e^{-ru_2\sqrt{1-y^2}}\big)\frac{2dy}{\sqrt{1-y^2}}$$

$$+\frac{1}{\pi}\sum_{n=1}^{\infty}e^{-\frac{1}{2}n^2t}\int_{-1}^{1}e^{ru_1y}\frac{1}{2}\big(e^{ru_2\sqrt{1-y^2}}+e^{-ru_2\sqrt{1-y^2}}\big)T_n(y)\frac{2dy}{\sqrt{1-y^2}}$$

since $T_n(1)=1$. The change of order of the integral and the sum is easily justified because $|T_n(y)| \leqq 1$. We can write the integrals on the right hand side as surface integrals and obtain

$$E_\theta[e^{r\langle u,\theta(t)\rangle}] = \int_{S^1} e^{r\langle u,z\rangle}ds(z) + 2\sum_{n=1}^{\infty}e^{-\frac{1}{2}n^2t}\int_{S^1}e^{r\langle u,z\rangle}T_n(z_1)ds(z).$$

Now, recalling the formula ([**2**, p.407])

$$E_a^{(0)}[e^{-\lambda\tau_r-\frac{1}{2}n^2\Xi_{\tau_r}}] = \frac{\mathcal{L}_n(a\sqrt{2\lambda})}{\mathcal{L}_n(r\sqrt{2\lambda})}, \tag{4.2}$$

we obtain the assertion of Theorem 1.1 when $d=2$ and $v=0$.

To prove Theorem 1.2, note another formula ([**2**, p.398])

$$E_a^{(\mu)}[e^{-\lambda\tau_r}] = \int_0^\infty e^{-\lambda t}\rho_{a,r}^{(\mu)}(t)dt = \frac{a^{-\mu}\mathcal{L}_\mu(a\sqrt{2\lambda})}{r^{-\mu}\mathcal{L}_\mu(r\sqrt{2\lambda})}.$$

Then we obtain the density when $d=2$. Again we can easily show the absolute convergence and justify the change of the sum and the integrals in t.

For the Brownian motion $B^{(v)}$ with drift v, we have only to use (4.1) and replace λ and u with $\lambda+|v|^2/2$ and $u+v$, respectively.

Next we prove the theorems in the case of $d \geqq 3$, when, for the spherical Brownian motion θ, the conditional distribution of $(\theta_2(t),...,\theta_d(t))$ given $\theta_1(t)=\xi_1$ is the uniform distribution on the sphere $S^{d-2}_{\sqrt{1-\xi_1^2}}$ with radius $\sqrt{1-\xi_1^2}$. Hence, writing $u=(u_1,u'),\theta=(\theta_1,\theta')\in\mathbf{R}\times\mathbf{R}^{d-1}$, we have

$$E_\theta[e^{\langle u,r\theta(t)\rangle}] = E_\theta\Big[e^{ru_1\theta_1(t)}\int_{S^{d-2}}e^{r\sqrt{1-\theta_1(t)^2}\langle u',\xi'\rangle}\ ds(\xi')\Big].$$

By using the facts on the Gegenbauer polynomials given in the previous section and writing the double integral as a surface integral, we obtain, from (3.2)

$$\begin{aligned}
&E_\theta[e^{\langle u,r\theta(t)\rangle}]\\
&=\sum_{n=0}^{\infty}e^{-\frac{1}{2}n(n+2\nu)t}\phi_n^\nu(1)\int_{-1}^{1}\phi_n^\nu(\xi_1)e^{ru_1\xi_1}2(1-\xi_1^2)^{\frac{d-3}{2}}d\xi_1\\
&\qquad\qquad\times\int_{S^{d-2}}e^{r\sqrt{1-\xi_1^2}\langle u',\xi'\rangle}\frac{\mathrm{vol}(d\xi')}{\mathrm{vol}(S^{d-2})}\\
&=\sum_{n=0}^{\infty}\frac{(n+\nu)2^{2\nu-1}\Gamma(\nu)^2\,\mathrm{vol}(S^{d-1})}{\pi\Gamma(2\nu)\,\mathrm{vol}(S^{d-2})}e^{-\frac{1}{2}n(n+2\nu)t}\int_{S^{d-1}}C_n^\nu(w_1)e^{r\langle u,w\rangle}ds(w).
\end{aligned}$$

We have used the formula $C_n^\nu(1)=\Gamma(n+2\nu)/(n!\Gamma(2\nu))$, and also the estimate (2.3) to justify the change of the order of the sum and the integration.

Moreover, recalling the formulae

$$\mathrm{vol}(S^{d-1}) = \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})} \qquad \text{and} \qquad \Gamma(2\nu) = \frac{2^{2\nu}}{2\sqrt{\pi}}\Gamma(\nu)\Gamma(\nu+\frac{1}{2}),$$

we obtain

$$E_\theta[e^{\langle u, r\theta(t)\rangle}] = \frac{1}{\nu}\sum_{n=0}^{\infty}(n+\nu)e^{-\frac{1}{2}n(n+2\nu)t}\int_{S^{d-1}} C_n^\nu(w_1)e^{r\langle u,w\rangle}ds(w).$$

Now, using (4.2), we obtain the assertion of Theorem 1.1 when $v=0$. Theorem 1.2 and the results for $B^{(v)}$ are proven in the same way as in the case of $d=2$.

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