Normal subgroups and support τ-tilting modules *†

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Abstract

Let $\tilde{G}$ be a finite group, G a normal subgroup of $\tilde{G}$ and k an algebraically closed field of characteristic $p > 0$. The first main result in this paper is to show that support τ-tilting $k\tilde{G}$-modules with some properties are support τ-tilting modules as kG-modules, too. As the second main result, we give equivalent conditions for support τ-tilting $k\tilde{G}$-modules to satisfy the above properties, and show that the set of the support τ-tilting $k\tilde{G}$-modules with the properties is isomorphic to the set of $\tilde{G}$-invariant support τ-tilting kG-modules as posets. As an application, we show that the set of $\tilde{G}$-invariant support τ-tilting kG-modules is isomorphic to the set of support τ-tilting $k\tilde{G}$-modules in the case that the index of G in $\tilde{G}$ is a p-power. As a further application, we give a feature of vertices of indecomposable τ-rigid $k\tilde{G}$-modules. Finally, we give block versions of the above results.

1 Introduction

Since 2014 when τ-tilting theory was introduced by T. Adachi, O. Iyama, and I. Reiten [1], the theory continues to develop rapidly. The main theme of the theory is to study of support τ-tilting modules, and many researchers have already contributed to it. In fact, the support τ-tilting modules are in one-to-one correspondences with the various representation-theoretically important objects including two-term silting complexes [1], functorially finite torsion classes [1], left-finite semibricks [3], two-term simple-minded collections [3, 15], and so on. In particular, the theory is expected to be helpful in solving Broué's abelian defect group conjecture because the theory is useful for the classification of two-term tilting complexes over group algebras or block algebras of finite groups. Even though the studies on the τ-tilting theory related to the modular representation theory of finite groups are very important for these reasons, there are few such studies. Therefore, it should be quite important to study modular representation theory in relation to τ-tilting theory. Indeed, the authors started such a study in [16, 18, 17, 19]. However, the results obtained so far are all concerned with the induction functor; they tell under some distinct situations that the induction functor $\mathrm{Ind}_G^{\tilde{G}}$ from kG-modules to $k\tilde{G}$-modules induces a poset homomorphisms with some nice property from the support τ-tilting modules over kG to those over $k\tilde{G}$, where G is a normal subgroup of a finite group $\tilde{G}$ and k an algebraically closed field of characteristic $p > 0$. (Poset means partially ordered set, as usual.) Naturally, there arises the following question.

Question 1.1. When does the restriction functor from $k\tilde{G}$-modules to kG-modules give the maps from the support τ-tilting modules over $k\tilde{G}$ to those over kG?

Regarding this question, in [5], S. Breaz, A. Marcus, and G. C. Modoi gave a positive answer in case that the quotient group $\tilde{G}/G$ is a p-prime group (i.e. the prime number p does not divide the order of the factor group $\tilde{G}/G$). Therefore, we consider the case that $\tilde{G}/G$ is not necessarily a p-prime group, and get the following positive answer to the question.

Theorem 1.2 (Theorem 3.4 and Corollary 3.5)**.** Let $\tilde{G}$ be a finite group, G a normal subgroup of $\tilde{G}$, $\tilde{M}$ a relatively G-projective support τ-tilting $k\tilde{G}$-module, and $(\tilde{M}, \tilde{P})$ a corresponding support τ-tilting pair. If it holds that $\mathrm{Ind}_G^{\tilde{G}}\mathrm{Res}_G^{\tilde{G}}\tilde{M} \in \mathrm{add}\,\tilde{M}$, then $\mathrm{Res}_G^{\tilde{G}}\tilde{M}$ is a support τ-tilting kG-module, and $(\mathrm{Res}_G^{\tilde{G}}\tilde{M}, \mathrm{Res}_G^{\tilde{G}}\tilde{P})$ a corresponding support τ-tilting pair. Moreover, for relatively G-projective support τ-tilting $k\tilde{G}$-modules $\tilde{M}_1$ and $\tilde{M}_2$ with the property that $\mathrm{Ind}_G^{\tilde{G}}\mathrm{Res}_G^{\tilde{G}}\tilde{M}_i \in \mathrm{add}\,\tilde{M}_i$ for $i = 1, 2$, if $\tilde{M}_1 \geq \tilde{M}_2$ in $\mathrm{s}\tau\text{-tilt}\,k\tilde{G}$, then $\mathrm{Res}_G^{\tilde{G}}\tilde{M}_1 \geq \mathrm{Res}_G^{\tilde{G}}\tilde{M}_2$ in $\mathrm{s}\tau\text{-tilt}\,kG$.

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Let M be a $\tilde{G}$-invariant support τ-tilting kG-module. The first author showed that $\mathrm{Ind}_G^{\tilde{G}}M$ is a support τ-tilting $k\tilde{G}$-module [16, Theorem 3.2]. We are interested in what is the image of the set of $\tilde{G}$-invariant support τ-tilting kG-modules under the map induced by the induction functor $\mathrm{Ind}_G^{\tilde{G}}$. Therefore, we give equivalent conditions to the assumption of Theorem 1.2, and finally we clarify what the image of the map induced by $\mathrm{Ind}_G^{\tilde{G}}$ is in the following theorem.

Theorem 1.3 (Theorem 3.8 and Corollary 3.9)**.** Let $\tilde{M}$ be a support τ-tilting $k\tilde{G}$-module. Then the following conditions are equivalent:

(1) $\tilde{M} =_{\mathrm{add}} \mathrm{Ind}_G^{\tilde{G}}M$ for some $\tilde{G}$-invariant support τ-tilting kG-module M.

(2) $\mathrm{Ind}_G^{\tilde{G}}\mathrm{Res}_G^{\tilde{G}}\tilde{M} \in \mathrm{add}\,\tilde{M}$ and $\tilde{M}$ is relatively G-projective.

(3) $S \otimes_k \tilde{M} \in \mathrm{add}\,\tilde{M}$ for each simple $k(\tilde{G}/G)$-module S.

Moreover, denoting by $(\mathrm{s}\tau\text{-tilt}\,kG)^{\tilde{G}}$ the subset of $\mathrm{s}\tau\text{-tilt}\,kG$ consisting of $\tilde{G}$-invariant support τ-tilting kG-modules and by $(\mathrm{s}\tau\text{-tilt}\,k\tilde{G})^{\star}$ the subset of $\mathrm{s}\tau\text{-tilt}\,k\tilde{G}$ consisting of support τ-tilting $k\tilde{G}$-modules satisfying the above equivalent conditions, the induction functor $\mathrm{Ind}_G^{\tilde{G}}$ induces a poset isomorphism

$$\begin{aligned} (\mathrm{s}\tau\text{-tilt}\,kG)^{\tilde{G}} &\xrightarrow{\sim} (\mathrm{s}\tau\text{-tilt}\,k\tilde{G})^{\star} \\ M &\longmapsto \mathrm{Ind}_G^{\tilde{G}}M. \end{aligned}$$

The studies on the vertices of indecomposable modules over group algebras have been done by many researchers for a long time, for example, see [6, 7, 8, 11, 14, 23]. On the other hand, τ-rigid modules over finite-dimensional algebras are important classes and there are many studies on the modules. They have nice properties and are essential objects for the representation theory, for example, see [1, 4, 9, 10, 13]. Therefore, one of our interests is to give a feature of the vertices of indecomposable τ-rigid modules over group algebras. As a further application of Theorem 1.3, we give a feature of vertices of indecomposable τ-rigid modules.

Theorem 1.4 (Theorem 3.14)**.** Let $\tilde{G}$ be a finite group. Then any indecomposable τ-rigid $k\tilde{G}$-module has a vertex contained in a Sylow p-subgroup of $\tilde{G}$ properly if and only if $\tilde{G}$ has a normal subgroup of p-power index in $\tilde{G}$.

As a natural question, we wonder if we get a block version of our theorems for group algebras. In particular, we are interested in how we give block versions of Theorems 1.2 and 1.3. As the results, we get block versions of the theorems. Let G be a normal subgroup of a finite group $\tilde{G}$ and B a block of kG. We denote by $I_{\tilde{G}}(B) := \left\{ \tilde{g} \in \tilde{G} \;\middle|\; \tilde{g}B\tilde{g}^{-1} = B \right\}$ the inertial group of B in $\tilde{G}$.

Theorem 1.5 (Theorem 5.6)**.** Let G be a normal subgroup of a finite group $\tilde{G}$, B a block of kG, $\tilde{B}$ a block of $k\tilde{G}$ covering B, β the block of $kI_{\tilde{G}}(B)$ satisfying

$$\sum_{x \in [\tilde{G}/I_{\tilde{G}}(B)]} x 1_\beta x^{-1} = 1_{\tilde{B}}$$

and $\tilde{M}$ a support τ-tilting $\tilde{B}$-module. If it holds that $\beta\mathrm{Ind}_G^{I_{\tilde{G}}(B)}\mathrm{Res}_G^{I_{\tilde{G}}(B)}\beta\mathrm{Res}_{I_{\tilde{G}}(B)}^{\tilde{G}}\tilde{M} \in \mathrm{add}\,\beta\mathrm{Res}_{I_{\tilde{G}}(B)}^{\tilde{G}}\tilde{M}$ and $\beta\mathrm{Res}_{I_{\tilde{G}}(B)}^{\tilde{G}}\tilde{M}$ is relatively G-projective, then we have that $\mathrm{Res}_G^{I_{\tilde{G}}(B)}\beta\mathrm{Res}_{I_{\tilde{G}}(B)}^{\tilde{G}}\tilde{M}$ is a support τ-tilting B-module. Moreover, if $(\tilde{M}, \tilde{P})$ is a support τ-tilting pair for $\tilde{B}$ corresponding to $\tilde{M}$, then the pair

$$(\mathrm{Res}_G^{I_{\tilde{G}}(B)}\beta\mathrm{Res}_{I_{\tilde{G}}(B)}^{\tilde{G}}\tilde{M}, \mathrm{Res}_G^{I_{\tilde{G}}(B)}\beta\mathrm{Res}_{I_{\tilde{G}}(B)}^{\tilde{G}}\tilde{P})$$

is a support τ-tilting pair for B corresponding to $\mathrm{Res}_G^{I_{\tilde{G}}(B)}\beta\mathrm{Res}_{I_{\tilde{G}}(B)}^{\tilde{G}}\tilde{M}$.

Theorem 1.6 (Theorem 5.7 and Corollary 5.8)**.** Let G be a normal subgroup of a finite group $\tilde{G}$, B a block of kG, $\tilde{B}$ a block of $k\tilde{G}$ covering B, β the block of $kI_{\tilde{G}}(B)$ satisfying

$$\sum_{x \in [\tilde{G}/I_{\tilde{G}}(B)]} x 1_\beta x^{-1} = 1_{\tilde{B}}$$

and $\tilde{M}$ a support τ-tilting $\tilde{B}$-module. Then the following conditions are equivalent:

(1) $\tilde{M} =_{\mathrm{add}} \tilde{B}\mathrm{Ind}_G^{\tilde{G}} M$ for some $I_{\tilde{G}}(B)$-invariant support τ-tilting B-module M.

(2) $\beta\mathrm{Ind}_G^{I_{\tilde{G}}(B)}\mathrm{Res}_G^{I_{\tilde{G}}(B)}\beta\mathrm{Res}_{I_{\tilde{G}}(B)}^{\tilde{G}}\tilde{M} \in \mathrm{add}\,\beta\mathrm{Res}_{I_{\tilde{G}}(B)}^{\tilde{G}}\tilde{M}$ and $\beta\mathrm{Res}_{I_{\tilde{G}}(B)}^{\tilde{G}}\tilde{M}$ is relatively G-projective.

(3) $\beta(S \otimes_k \beta\mathrm{Res}_{I_{\tilde{G}}(B)}^{\tilde{G}}\tilde{M}) \in \mathrm{add}\,\beta\mathrm{Res}_{I_{\tilde{G}}(B)}^{\tilde{G}}\tilde{M}$ for each simple $k(I_{\tilde{G}}(B)/G)$-module S.

Moreover, denoting by $(\mathrm{s}\tau\text{-tilt}\,B)^{I_{\tilde{G}}(B)}$ the subset of $\mathrm{s}\tau\text{-tilt}\,B$ consisting of $I_{\tilde{G}}(B)$-invariant support τ-tilting B-modules and by $(\mathrm{s}\tau\text{-tilt}\,\tilde{B})^{\star\star\star}$ the subset of $\mathrm{s}\tau\text{-tilt}\,\tilde{B}$ consisting of support τ-tilting $\tilde{B}$-modules satisfying the above equivalent conditions, the functor $\tilde{B}\mathrm{Ind}_G^{\tilde{G}}$ induces a poset isomorphism

$$\begin{aligned} (\mathrm{s}\tau\text{-tilt}\,B)^{I_{\tilde{G}}(B)} &\xrightarrow{\sim} (\mathrm{s}\tau\text{-tilt}\,\tilde{B})^{\star\star\star} \\ M &\longmapsto \tilde{B}\mathrm{Ind}_G^{\tilde{G}} M. \end{aligned}$$

Our particular interest is in the case that the index of G in $\tilde{G}$ is a p-power. In fact, under some assumptions, it is expected that tilting complexes over the block B of kG give those over the unique block $\tilde{B}$ of $k\tilde{G}$ covering B (for example, see [12, 20, 26]). In this regard, the authors showed the following result in [17].

Theorem 1.7 ([17, Theorem 1.2])**.** *Let G be a normal subgroup of a finite group $\tilde{G}$ of p-power index in $\tilde{G}$, B a block of kG, and $\tilde{B}$ the unique block of $k\tilde{G}$ covering B. Assume that the following two conditions are satisfied:*

(1) *Any indecomposable B-module is $I_{\tilde{G}}(B)$-invariant.*

(2) *The set of isomorphism classes of basic support τ-tilting B-modules is a finite set.*

Then the induction functor $\mathrm{Ind}_G^{\tilde{G}}$ induces an isomorphism from $\mathrm{s}\tau\text{-tilt}\,B$ to $\mathrm{s}\tau\text{-tilt}\,\tilde{B}$ of posets.

This theorem can be applied to the case that the block B has a cyclic defect group, but the two conditions limit the scope of its use. For example, the theorem cannot be applied to the case that $p = 2$, G is the alternating group A_4 of degree 4 and that $\tilde{G}$ is the symmetric group S_4 of degree 4, because the nontrivial simple kA_4-modules are not S_4-invariant. Indeed, $\mathrm{s}\tau\text{-tilt}\,kA_4$ is not isomorphic to $\mathrm{s}\tau\text{-tilt}\,kS_4$ because the number of isomorphism classes of simple kA_4-modules is three and that of kS_4 is two. However, we wonder if the induction functor might give some kinds of good relation between the special subsets of the two, and finally, as an application of Theorem 1.6, we could get the following theorem which can be applied to the case of kA_4 and kS_4. The following theorem is a significant generalization of Theorem 1.7 and enables us to explain the phenomenon occurred in [16, Example 3.9] (see Example 3.11).

Theorem 1.8 (Theorem 5.9)**.** *Let G be a normal subgroup of a finite group $\tilde{G}$, B a block of kG and $\tilde{B}$ a block of $k\tilde{G}$ covering B. If the quotient group $I_{\tilde{G}}(B)/G$ is a p-group, then the functor $\mathrm{Ind}_G^{\tilde{G}}$ induces an isomorphism as posets between $(\mathrm{s}\tau\text{-tilt}\,B)^{I_{\tilde{G}}(B)}$ and $\mathrm{s}\tau\text{-tilt}\,\tilde{B}$, where $(\mathrm{s}\tau\text{-tilt}\,B)^{I_{\tilde{G}}(B)}$ is the subset of $\mathrm{s}\tau\text{-tilt}\,B$ consisting of $I_{\tilde{G}}(B)$-invariant support τ-tilting B-modules.*

Throughout this paper, we fix the following notation:

Let k be an algebraically closed field of characteristic $p > 0$. An algebra means a k-algebra. For a finite-dimensional algebra Λ, a Λ-module means a finite-dimensional left Λ-module. For a Λ-module M, we denote the Auslander-Reiten translate of M by τM. In case that Λ is a symmetric algebra, τM is isomorphic to $\Omega^2 M$. We denote the category of all direct summands of finite direct sums of copies of M by $\mathrm{add}\,M$. For Λ-modules M and N, we write $M =_{\mathrm{add}} N$ if $\mathrm{add}\,M = \mathrm{add}\,N$. This relation is an equivalence relation. We denote by $\mathrm{s}\tau\text{-tilt}\,\Lambda$ the set of equivalence classes of support τ-tilting Λ-modules under the equivalence relation $=_{\mathrm{add}}$.

Let G be a finite group and H a subgroup of G. We denote the restriction functor from kG-modules to kH-modules by Res_H^G and the induction functor $kG \otimes_{kH} -$ from kH-modules to kG-modules by Ind_H^G. We denote the trivial kG-module by k_G.

Let $\tilde{G}$ be a finite group, G a normal subgroup of $\tilde{G}$. We denote a set of coset representatives of G in $\tilde{G}$ by $[\tilde{G}/G]$. For a kG-module M and $\tilde{g} \in \tilde{G}$, we define a kG-module $\tilde{g}M$ consisting of symbols $\tilde{g}m$ as a set, where $m \in M$, and its kG-module structure is given by $\tilde{g}m + \tilde{g}m' := \tilde{g}(m + m'), g(\tilde{g}m) := \tilde{g}(\tilde{g}^{-1}g\tilde{g}m)$ and $\lambda(\tilde{g}m) = \tilde{g}(\lambda m)$ for $m, m' \in M, g \in G$ and $\lambda \in k$. For a kG-module M, we say that M is $\tilde{G}$-invariant if M is isomorphic to $\tilde{g}M$ for any $\tilde{g} \in \tilde{G}$.

2 Preliminaries

In this section, we give elementary facts on the modular representation theory which are helpful to prove our results.

Proposition 2.1 ([2, Lemma 8.5, Lemma 8.6])**.** Let G be a finite group, K a subgroup of G, H a subgroup of K. For any kG-module U and kH-module V, the following hold:

(1) $\mathrm{Res}^G_H U \cong \mathrm{Res}^K_H \mathrm{Res}^G_K U$.

(2) $\mathrm{Ind}^G_H V \cong \mathrm{Ind}^G_K \mathrm{Ind}^K_H V$.

(3) $\mathrm{Ind}^G_H (V \otimes_k \mathrm{Res}^G_H U) \cong (\mathrm{Ind}^G_H V) \otimes_k U$.

(4) $\mathrm{Hom}_{kG}(U, \mathrm{Ind}^G_H V) \cong \mathrm{Hom}_{kH}(\mathrm{Res}^G_H U, V)$.

(5) $\mathrm{Hom}_{kG}(\mathrm{Ind}^G_H V, U) \cong \mathrm{Hom}_{kH}(V, \mathrm{Res}^G_H U)$.

(6) The functors Res^G_H and Ind^G_H send free modules (projective modules) to free modules (projective modules, respectively).

In the modular representation theory of finite groups, Mackey's decomposition formula is well-known and important. We recall Mackey's decomposition formula for normal subgroups.

Proposition 2.2 ([2, Lemma 8.7])**.** Let G be a normal subgroup of a finite group $\tilde{G}$ and M a kG-module. Then the following isomorphism as kG-modules holds:

$$\mathrm{Res}^{\tilde{G}}_G \mathrm{Ind}^{\tilde{G}}_G M \cong \bigoplus_{x \in [\tilde{G}/G]} xM.$$

The following is known as Eckmann-Shapiro Lemma.

Lemma 2.3 ([22, Proposition 2.20.7])**.** Let H be a finite group of a finite group G, M a kH-module and N a kG-module. Then for all $n \in \mathbb{N}$, there exists an isomorphism of k-vector spaces:

$$\mathrm{Ext}^n_{kH}(M, \mathrm{Res}^G_H N) \cong \mathrm{Ext}^n_{kG}(\mathrm{Ind}^G_H M, N)$$

The following lemma is a refinement of [16, Lemma 3.1] which requires the $\tilde{G}$-invariance for the kG-module.

Lemma 2.4. Let G be a normal subgroup of $\tilde{G}$ and M a kG-module. Then the following hold:

(1) $\mathrm{Ind}^{\tilde{G}}_G(\Omega M) \cong \Omega(\mathrm{Ind}^{\tilde{G}}_G M)$.

(2) $\mathrm{Ind}^{\tilde{G}}_G(\tau M) \cong \tau(\mathrm{Ind}^{\tilde{G}}_G M)$.

Proof. It is enough to show that the statement (1) holds since $\tau \cong \Omega^2$ for symmetric algebras. There exists a projective kG-module Q such that $\mathrm{Ind}^{\tilde{G}}_G(\Omega M) \cong \Omega(\mathrm{Ind}^{\tilde{G}}_G M) \oplus Q$ and that $\mathrm{Ind}^{\tilde{G}}_G P(M) \cong P(\mathrm{Ind}^{\tilde{G}}_G M) \oplus Q$. Hence, we have that

$$\mathrm{Res}^{\tilde{G}}_G \mathrm{Ind}^{\tilde{G}}_G(\Omega M) \cong \mathrm{Res}^{\tilde{G}}_G \Omega(\mathrm{Ind}^{\tilde{G}}_G M) \oplus \mathrm{Res}^{\tilde{G}}_G Q,$$

and the left-hand side is isomorphic to $\bigoplus_{x \in [\tilde{G}/G]} x\Omega M$ by Proposition 2.2. However, each $x\Omega M$ has no projective summands and the restricted module $\mathrm{Res}^{\tilde{G}}_G Q$ is a projective kG-module by Proposition 2.1 (6), which implies that $Q = 0$. Therefore, we conclude that $\mathrm{Ind}^{\tilde{G}}_G(\Omega M) \cong \Omega(\mathrm{Ind}^{\tilde{G}}_G M)$. □

Lemma 2.5. Let G be a normal subgroup of a finite group $\tilde{G}$ and $\tilde{M}$ be a $k\tilde{G}$-module. Then $\mathrm{Res}^{\tilde{G}}_G \tilde{M}$ is a $\tilde{G}$-invariant kG-module.

Proof. Take $\tilde{g} \in \tilde{G}$ arbitrarily. We consider the map

$$\begin{aligned} f\colon \mathrm{Res}^{\tilde{G}}_{G}\tilde{M} &\longrightarrow \tilde{g}\mathrm{Res}^{\tilde{G}}_{G}\tilde{M} \\ m &\longmapsto \tilde{g}m. \end{aligned}$$

Clearly, this map is linear and bijective. We only show that the map is kG-homomorphism, but for any $g \in G$ and $m \in \mathrm{Res}^{\tilde{G}}_{G}\tilde{M}$, it holds that

$$f(gm) = \tilde{g}gm = \tilde{g}g\tilde{g}^{-1}\tilde{g}m = g \cdot \tilde{g}m = g \cdot f(m).$$

□

3 Main Theorems

In this section we give theorems stated in Section 1 and their proofs. Throughout this section, $\tilde{G}$ denotes a finite group and G a normal subgroup of $\tilde{G}$.

First, we start with a consideration on restricted modules of rigid modules and τ-rigid modules. Let Λ be a finite-dimensional algebra. We recall that a Λ-module M is rigid (resp. τ-rigid) if $\mathrm{Ext}^1_\Lambda(M, M) = 0$ (resp. $\mathrm{Hom}_\Lambda(M, \tau M) = 0$). We remark that τ-rigid modules are rigid modules by the Auslander-Reiten duality $\overline{\mathrm{Hom}}_\Lambda(X, Y) \cong D\,\mathrm{Ext}^1_\Lambda(Y, \tau X)$, where $\overline{\mathrm{Hom}}_\Lambda(X, Y)$ denotes the quotient of $\mathrm{Hom}_\Lambda(X, Y)$ by the subspace of the homomorphisms factoring through projective Λ-modules.

Lemma 3.1. Let $\tilde{M}$ be a $k\tilde{G}$-module with the property that $\mathrm{Ind}^{\tilde{G}}_{G}\mathrm{Res}^{\tilde{G}}_{G}\tilde{M} \in \mathrm{add}\,\tilde{M}$. Then the following hold:

(1) If $\tilde{M}$ is a rigid $k\tilde{G}$-module, then the restricted module $\mathrm{Res}^{\tilde{G}}_{G}\tilde{M}$ is a rigid kG-module.

(2) If $\tilde{M}$ is a τ-rigid $k\tilde{G}$-module, then the restricted module $\mathrm{Res}^{\tilde{G}}_{G}\tilde{M}$ is a τ-rigid kG-module.

Proof. (1) Let $\tilde{M}$ be a rigid $k\tilde{G}$-module. Then, by Lemma 2.3, we have that

$$\mathrm{Ext}^1_{kG}(\mathrm{Res}^{\tilde{G}}_{G}\tilde{M}, \mathrm{Res}^{\tilde{G}}_{G}\tilde{M}) \cong \mathrm{Ext}^1_{k\tilde{G}}(\tilde{M}, \mathrm{Ind}^{\tilde{G}}_{G}\mathrm{Res}^{\tilde{G}}_{G}\tilde{M}).$$

By the assumption that $\mathrm{Ind}^{\tilde{G}}_{G}\mathrm{Res}^{\tilde{G}}_{G}\tilde{M} \in \mathrm{add}\,\tilde{M}$ and the rigidity of $\tilde{M}$, we have that the right-hand side is 0. Hence, $\mathrm{Res}^{\tilde{G}}_{G}\tilde{M}$ is a rigid kG-module.
(2) Let $\tilde{M}$ be a τ-rigid $k\tilde{G}$-module. Then we have that

$$\mathrm{Hom}_{kG}(\mathrm{Res}^{\tilde{G}}_{G}\tilde{M}, \tau\mathrm{Res}^{\tilde{G}}_{G}\tilde{M}) \cong \mathrm{Hom}_{k\tilde{G}}(\tilde{M}, \mathrm{Ind}^{\tilde{G}}_{G}\tau\mathrm{Res}^{\tilde{G}}_{G}\tilde{M}) \cong \mathrm{Hom}_{k\tilde{G}}(\tilde{M}, \tau\mathrm{Ind}^{\tilde{G}}_{G}\mathrm{Res}^{\tilde{G}}_{G}\tilde{M}),$$

where the last isomorphism comes from Lemma 2.4. By the assumption that $\mathrm{Ind}^{\tilde{G}}_{G}\mathrm{Res}^{\tilde{G}}_{G}\tilde{M} \in \mathrm{add}\,\tilde{M}$ and the τ-rigidity of $\tilde{M}$, we have that $\mathrm{Hom}_{k\tilde{G}}(\tilde{M}, \tau\mathrm{Ind}^{\tilde{G}}_{G}\mathrm{Res}^{\tilde{G}}_{G}\tilde{M}) = 0$, which implies that $\mathrm{Res}^{\tilde{G}}_{G}\tilde{M}$ is a τ-rigid kG-module. □

For a finite group H and a subgroup K of H, we recall that a kH-module M is relatively K-projective if M is a direct summand of $\mathrm{Ind}^H_K\mathrm{Res}^H_K M$.

Lemma 3.2. Let $\tilde{M}$ be a relatively G-projective $k\tilde{G}$-module. Then $\mathrm{Res}^{\tilde{G}}_{G}(\Omega\tilde{M}) \cong \Omega(\mathrm{Res}^{\tilde{G}}_{G}\tilde{M})$. In particular, it holds that $\tau(\mathrm{Res}^{\tilde{G}}_{G}\tilde{M}) \cong \mathrm{Res}^{\tilde{G}}_{G}(\tau\tilde{M})$.

Proof. There exists a projective kG-module P such that $\mathrm{Res}^{\tilde{G}}_{G}(\Omega\tilde{M}) \cong \Omega(\mathrm{Res}^{\tilde{G}}_{G}\tilde{M}) \oplus P$. Hence, it is enough to show that $P = 0$. It is clear in the case that $\tilde{M}$ is a projective $k\tilde{G}$-module.

We may assume that $\tilde{M}$ has no projective summands. Since $\tilde{M}$ is relatively G-projective, $\Omega\tilde{M}$ is relatively G-projective too (for example see [2, Proposition 20.7]). Hence, $\Omega\tilde{M}$ is a direct summand of $\mathrm{Ind}^{\tilde{G}}_{G}\mathrm{Res}^{\tilde{G}}_{G}\Omega\tilde{M}$. On the other hand, by the isomorphism $\mathrm{Res}^{\tilde{G}}_{G}\Omega\tilde{M} \cong \Omega(\mathrm{Res}^{\tilde{G}}_{G}\tilde{M}) \oplus P$, we have that

$\mathrm{Ind}_G^{\tilde{G}}\mathrm{Res}_G^{\tilde{G}}\Omega\tilde{M} \cong \mathrm{Ind}_G^{\tilde{G}}(\Omega(\mathrm{Res}_G^{\tilde{G}}\tilde{M})) \oplus \mathrm{Ind}_G^{\tilde{G}}P$. Here, since $\mathrm{Ind}_G^{\tilde{G}}P$ is a projective $k\tilde{G}$-module by Proposition 2.1 (6) and $\Omega\tilde{M}$ has no projective summands by the self-injectivity of $k\tilde{G}$, we have that $\Omega\tilde{M}$ is a direct summand of $\mathrm{Ind}_G^{\tilde{G}}(\Omega(\mathrm{Res}_G^{\tilde{G}}\tilde{M}))$. Therefore, $\mathrm{Res}_G^{\tilde{G}}(\Omega\tilde{M})$ is a direct summand of

$$\mathrm{Res}_G^{\tilde{G}}\mathrm{Ind}_G^{\tilde{G}}(\Omega(\mathrm{Res}_G^{\tilde{G}}\tilde{M})) \cong \bigoplus_{\tilde{g}\in[\tilde{G}/G]} \tilde{g}\Omega(\mathrm{Res}_G^{\tilde{G}}\tilde{M})$$

by Proposition 2.2, which implies that $\mathrm{Res}_G^{\tilde{G}}(\Omega\tilde{M})$ is has no projective summands because each $\tilde{g}\Omega(\mathrm{Res}_G^{\tilde{G}}\tilde{M})$ has no projective summands by the self-injectivity of kG. Thus, we conclude that $P=0$ and $\mathrm{Res}_G^{\tilde{G}}(\Omega\tilde{M}) \cong \Omega(\mathrm{Res}_G^{\tilde{G}}\tilde{M})$.

The later assertion follows from the fact that $\tau \cong \Omega^2$ and the relative G-projectivity of $\Omega\tilde{M}$. $\square$

The following is important for the proof of Theorem 3.4.

Proposition 3.3 ([1, Corollary 2.13])**.** Let Λ be a finite-dimensional algebra. For a τ-rigid pair (M,P) for Λ the following are equivalent:

(1) (M,P) is a support τ-tilting pair for Λ.

(2) If $\mathrm{Hom}_\Lambda(M,\tau X)=0, \mathrm{Hom}_\Lambda(X,\tau M)=0$ and $\mathrm{Hom}_\Lambda(P,X)=0$, then $X\in \mathrm{add}\, M$.

Theorem 3.4. Let $\tilde{M}$ be a support τ-tilting $k\tilde{G}$-module. If it holds that $\mathrm{Ind}_G^{\tilde{G}}\mathrm{Res}_G^{\tilde{G}}\tilde{M}\in \mathrm{add}\,\tilde{M}$ and $\tilde{M}$ is relatively G-projective, then we have that $\mathrm{Res}_G^{\tilde{G}}\tilde{M}$ is a support τ-tilting kG-module. Moreover, if $(\tilde{M},\tilde{P})$ is a support τ-tilting pair for $k\tilde{G}$ corresponding to $\tilde{M}$, then $(\mathrm{Res}_G^{\tilde{G}}\tilde{M},\mathrm{Res}_G^{\tilde{G}}\tilde{P})$ is a support τ-tilting pair for kG corresponding to $\mathrm{Res}_G^{\tilde{G}}\tilde{M}$.

Proof. Let $(\tilde{M},\tilde{P})$ be a support τ-tilting pair for $k\tilde{G}$ corresponding to the support τ-tilting $k\tilde{G}$-module $\tilde{M}$.

First, we show that $(\mathrm{Res}_G^{\tilde{G}}\tilde{M},\mathrm{Res}_G^{\tilde{G}}\tilde{P})$ is a τ-rigid pair for kG. Since the $k\tilde{G}$-module $\tilde{M}$ is a support τ-tilting module, it is a τ-rigid module. Hence, we have that $\mathrm{Res}_G^{\tilde{G}}\tilde{M}$ is a τ-rigid $k\tilde{G}$-module by Lemma 3.1. On the other hand, by Proposition 2.1 we have that

$$\mathrm{Hom}_{kG}(\mathrm{Res}_G^{\tilde{G}}\tilde{P},\mathrm{Res}_G^{\tilde{G}}\tilde{M}) \cong \mathrm{Hom}_{k\tilde{G}}(\tilde{P},\mathrm{Ind}_G^{\tilde{G}}\mathrm{Res}_G^{\tilde{G}}\tilde{M}).$$

Now, by the assumption that $\mathrm{Ind}_G^{\tilde{G}}\mathrm{Res}_G^{\tilde{G}}\tilde{M}\in\mathrm{add}\,\tilde{M}$, we have that $\mathrm{Hom}_{kG}(\mathrm{Res}_G^{\tilde{G}}\tilde{P},\mathrm{Res}_G^{\tilde{G}}\tilde{M})=0$ because $(\tilde{M},\tilde{P})$ is a support τ-tilting pair for $k\tilde{G}$. Therefore, we conclude that $(\mathrm{Res}_G^{\tilde{G}}\tilde{M},\mathrm{Res}_G^{\tilde{G}}\tilde{P})$ is a τ-rigid pair for kG.

Next, we show that the τ-rigid pair $(\mathrm{Res}_G^{\tilde{G}}\tilde{M},\mathrm{Res}_G^{\tilde{G}}\tilde{P})$ is a support τ-tilting pair for kG. We show that $X\in\mathrm{add}\,\mathrm{Res}_G^{\tilde{G}}\tilde{M}$ under the assumption that

$$\mathrm{Hom}_{kG}(X,\tau(\mathrm{Res}_G^{\tilde{G}}\tilde{M}))=\mathrm{Hom}_{kG}(\mathrm{Res}_G^{\tilde{G}}\tilde{M},\tau X)=\mathrm{Hom}_{kG}(\mathrm{Res}_G^{\tilde{G}}\tilde{P},X)=0,$$

which implies that the pair $(\mathrm{Res}_G^{\tilde{G}}\tilde{M},\mathrm{Res}_G^{\tilde{G}}\tilde{P})$ is a support τ-tilting pair for kG by Proposition 3.3. Under these assumptions, we have the following:

$$\begin{aligned}
\mathrm{Hom}_{k\tilde{G}}(\mathrm{Ind}_G^{\tilde{G}}X,\tau\tilde{M}) &\cong \mathrm{Hom}_{kG}(X,\mathrm{Res}_G^{\tilde{G}}(\tau\tilde{M})) && \text{(Proposition 2.1)}\\
&\cong \mathrm{Hom}_{kG}(X,\tau(\mathrm{Res}_G^{\tilde{G}}\tilde{M})) && \text{(Lemma 3.2)}\\
&=0.
\end{aligned}$$

$$\begin{aligned}
\mathrm{Hom}_{k\tilde{G}}(\tilde{M},\tau(\mathrm{Ind}_G^{\tilde{G}}X)) &\cong \mathrm{Hom}_{k\tilde{G}}(\tilde{M},\mathrm{Ind}_G^{\tilde{G}}(\tau X)) && \text{(Lemma 2.4)}\\
&\cong \mathrm{Hom}_{kG}(\mathrm{Res}_G^{\tilde{G}}\tilde{M},\tau X) && \text{(Proposition 2.1)}\\
&=0.
\end{aligned}$$

$$\begin{aligned}\operatorname{Hom}_{k\tilde{G}}(\tilde{P}, \operatorname{Ind}_G^{\tilde{G}} X) &\cong \operatorname{Hom}_{k\tilde{G}}(\operatorname{Res}_G^{\tilde{G}} \tilde{P}, X) \qquad \text{(Proposition 2.1)}\\ &= 0.\end{aligned}$$

By these three isomorphisms and the fact that $(\tilde{M}, \tilde{P})$ is a support τ-tilting pair for $k\tilde{G}$, applying Proposition 3.3, we have that $\operatorname{Ind}_G^{\tilde{G}} X \in \operatorname{add} \tilde{M}$. Also, X is a direct summand of $\operatorname{Res}_G^{\tilde{G}} \operatorname{Ind}_G^{\tilde{G}} X$ by Proposition 2.2. Therefore, we have that $X \in \operatorname{add} \operatorname{Res}_G^{\tilde{G}} \tilde{M}$. □

Corollary 3.5. Let $\tilde{M}_1$ and $\tilde{M}_2$ be relatively G-projective support τ-tilting $k\tilde{G}$-modules such that $\operatorname{Ind}_G^{\tilde{G}} \operatorname{Res}_G^{\tilde{G}} \tilde{M}_i \in \operatorname{add} \tilde{M}_i$ for $i = 1, 2$. Then $\tilde{M}_1 \geq \tilde{M}_2$ in $\text{s}\tau\text{-tilt}\, k\tilde{G}$ implies that $\operatorname{Res}_G^{\tilde{G}} \tilde{M}_1 \geq \operatorname{Res}_G^{\tilde{G}} \tilde{M}_2$ in $\text{s}\tau\text{-tilt}\, kG$.

Proof. The consequence immediately follows from Theorem 3.4 and the exactness of the functor $\operatorname{Res}_G^{\tilde{G}}$. □

We wish to restate the assumption of Theorem 3.4 with some equivalent conditions. First, we give the lemmas which can be applied in case of rigid $k\tilde{G}$-modules not only support τ-tilting $k\tilde{G}$-modules.

Lemma 3.6. Let $\tilde{M}$ be a rigid $k\tilde{G}$-module and L a $k\tilde{G}$-module. If it holds that $S \otimes_k \tilde{M} \in \operatorname{add} \tilde{M}$ for any composition factor S of L, then the following isomorphism as $k\tilde{G}$-modules holds:

$$L \otimes_k \tilde{M} \cong \bigoplus_S S \otimes_k \tilde{M},$$

where S runs over all composition factors of L.

Proof. Let L be an arbitrary $k\tilde{G}$-module and $\tilde{M}$ a rigid $k\tilde{G}$-module satisfying that

$$S \otimes_k \tilde{M} \in \operatorname{add} \tilde{M} \text{ for any composition factor of } S \text{ of } L. \tag{3.1}$$

We use induction on the composition length $\ell(L)$ of L. If $\ell(L) = 1$, there is nothing to prove. Hence, we assume that $\ell(L) \geq 2$ and that the statement for any $k\tilde{G}$-module L' satisfying $\ell(L') < \ell(L)$ is true. Let T be a simple submodule of L. We get the exact sequence

$$0 \longrightarrow T \otimes_k \tilde{M} \longrightarrow L \otimes_k \tilde{M} \longrightarrow L/T \otimes_k \tilde{M} \longrightarrow 0 \tag{3.2}$$

obtained by applying the exact functor $- \otimes_k \tilde{M}$ to the exact sequence

$$0 \longrightarrow T \longrightarrow L \longrightarrow L/T \longrightarrow 0.$$

By the rigidity of $\tilde{M}$, the assumption (3.1) and the assumption of this induction, the sequence (3.2) splits, and we get that

$$L \otimes_k \tilde{M} \cong T \otimes_k \tilde{M} \oplus L/T \otimes_k \tilde{M} \cong T \otimes_k \tilde{M} \oplus \bigoplus_{S'} S' \otimes_k \tilde{M} \cong \bigoplus_S S \otimes_k \tilde{M},$$

where S' and S run over all composition factors of L/T and L, respectively. □

Lemma 3.7. Let $\tilde{M}$ be a rigid $k\tilde{G}$-module. Then the following conditions are equivalent:

(1) $\operatorname{Ind}_G^{\tilde{G}} \operatorname{Res}_G^{\tilde{G}} \tilde{M} \in \operatorname{add} \tilde{M}$ and $\tilde{M}$ is relatively G-projective.

(2) $S \otimes_k \tilde{M} \in \operatorname{add} \tilde{M}$ for each simple $k(\tilde{G}/G)$-module S.

Proof. By Proposition 2.1, we have that

$$\operatorname{Ind}_G^{\tilde{G}} \operatorname{Res}_G^{\tilde{G}} \tilde{M} \cong \operatorname{Ind}_G^{\tilde{G}}(k_G \otimes_k \operatorname{Res}_G^{\tilde{G}} \tilde{M}) \cong (\operatorname{Ind}_G^{\tilde{G}} k_G) \otimes_k \tilde{M} \cong k(\tilde{G}/G) \otimes_k \tilde{M}.$$

(1) $\Rightarrow$ (2). By the assumptions, we have that $\mathrm{Ind}_G^{\tilde{G}}\mathrm{Res}_G^{\tilde{G}}\tilde{M} =_{\mathrm{add}} \tilde{M}$. Hence, by Proposition 2.1 we get that

$$\begin{aligned} S \otimes_k \tilde{M} &=_{\mathrm{add}} S \otimes_k \mathrm{Ind}_G^{\tilde{G}}\mathrm{Res}_G^{\tilde{G}}\tilde{M} \\ &\cong \mathrm{Ind}_G^{\tilde{G}}(\mathrm{Res}_G^{\tilde{G}}S \otimes_k \mathrm{Res}_G^{\tilde{G}}\tilde{M}) \\ &\cong \mathrm{Ind}_G^{\tilde{G}}(k_G^{\oplus \dim_k S} \otimes_k \mathrm{Res}_G^{\tilde{G}}\tilde{M}) \\ &=_{\mathrm{add}} \mathrm{Ind}_G^{\tilde{G}}\mathrm{Res}_G^{\tilde{G}}\tilde{M} \\ &=_{\mathrm{add}} \tilde{M}, \end{aligned}$$

for any simple $k(\tilde{G}/G)$-module S, which implies that $S \otimes_k \tilde{M} \in \mathrm{add}\,\tilde{M}$.
(2) $\Rightarrow$ (1). By Lemma 3.6, we have that

$$\mathrm{Ind}_G^{\tilde{G}}\mathrm{Res}_G^{\tilde{G}}\tilde{M} \cong k(\tilde{G}/G) \otimes_k \tilde{M} \cong \bigoplus_S S \otimes_k \tilde{M},$$

where S runs over all composition factors of the $k\tilde{G}$-module $k(\tilde{G}/G)$. Therefore, the assumption implies that $\mathrm{Ind}_G^{\tilde{G}}\mathrm{Res}_G^{\tilde{G}}\tilde{M} \in \mathrm{add}\,\tilde{M}$. Moreover, since the trivial $k\tilde{G}$-module $k_{\tilde{G}}$ appears as a composition factor of $k(\tilde{G}/G)$, we have that the module $\tilde{M}$ appears as a direct summand of $\mathrm{Ind}_G^{\tilde{G}}\mathrm{Res}_G^{\tilde{G}}\tilde{M}$, that is $\tilde{M}$ is a relatively G-projective $k\tilde{G}$-module. □

We obtain below some conditions that are equivalent to the assumption of Theorem 3.4.

Theorem 3.8. Let $\tilde{M}$ be a support τ-tilting $k\tilde{G}$-module. Then the following conditions are equivalent:

(1) $\tilde{M} =_{\mathrm{add}} \mathrm{Ind}_G^{\tilde{G}}M$ for some $\tilde{G}$-invariant support τ-tilting kG-module M.

(2) $\mathrm{Ind}_G^{\tilde{G}}\mathrm{Res}_G^{\tilde{G}}\tilde{M} \in \mathrm{add}\,\tilde{M}$ and $\tilde{M}$ is relatively G-projective.

(3) $S \otimes_k \tilde{M} \in \mathrm{add}\,\tilde{M}$ for each simple $k(\tilde{G}/G)$-module S.

Proof. (1) $\Rightarrow$ (2). Assume that $\tilde{M} =_{\mathrm{add}} \mathrm{Ind}_G^{\tilde{G}}M$ for some $\tilde{G}$-invariant support τ-tilting kG-module M. Then clearly $\tilde{M}$ is a relatively G-projective $k\tilde{G}$-module (see [2, 3.9.1]), and by Proposition 2.2, we have that

$$\mathrm{Ind}_G^{\tilde{G}}\mathrm{Res}_G^{\tilde{G}}\tilde{M} =_{\mathrm{add}} \mathrm{Ind}_G^{\tilde{G}}\mathrm{Res}_G^{\tilde{G}}\mathrm{Ind}_G^{\tilde{G}}M \cong \mathrm{Ind}_G^{\tilde{G}}\Big(\bigoplus_{\tilde{g}\in[\tilde{G}/G]} \tilde{g}M\Big) \cong \bigoplus_{\tilde{g}\in[\tilde{G}/G]} \mathrm{Ind}_G^{\tilde{G}}M \in \mathrm{add}\,\tilde{M}.$$

(2) $\Rightarrow$ (1). Assume that $\mathrm{Ind}_G^{\tilde{G}}\mathrm{Res}_G^{\tilde{G}}\tilde{M} \in \mathrm{add}\,\tilde{M}$ and that $\tilde{M}$ is relatively G-projective. Put $M := \mathrm{Res}_G^{\tilde{G}}\tilde{M}$. Then by Lemma 2.5 and Theorem 3.4, M is a $\tilde{G}$-invariant support τ-tilting kG-module. We show that $\mathrm{Ind}_G^{\tilde{G}}M =_{\mathrm{add}} \tilde{M}$, that is $\mathrm{add}(\mathrm{Ind}_G^{\tilde{G}}M) = \mathrm{add}\,\tilde{M}$. By the assumption that $\mathrm{Ind}_G^{\tilde{G}}\mathrm{Res}_G^{\tilde{G}}\tilde{M} \in \mathrm{add}\,\tilde{M}$, we have $\mathrm{add}(\mathrm{Ind}_G^{\tilde{G}}M) \subset \mathrm{add}\,\tilde{M}$. On the other hand, since $\tilde{M}$ is relatively G-projective, $\tilde{M}$ is a direct summand of $\mathrm{Ind}_G^{\tilde{G}}\mathrm{Res}_G^{\tilde{G}}\tilde{M} = \mathrm{Ind}_G^{\tilde{G}}M$. Hence, we have $\mathrm{add}\,\tilde{M} \subset \mathrm{add}(\mathrm{Ind}_G^{\tilde{G}}M)$.
(2) $\Leftrightarrow$ (3). Since support τ-tilting $k\tilde{G}$-modules are rigid $k\tilde{G}$-modules, the equivalence follows from Lemma 3.7. □

Corollary 3.9. Let $(\mathrm{s}\tau\text{-tilt}\,kG)^{\tilde{G}}$ be the subset of $\mathrm{s}\tau\text{-tilt}\,kG$ consisting of $\tilde{G}$-invariant support τ-tilting kG-modules and $(\mathrm{s}\tau\text{-tilt}\,k\tilde{G})^{\star}$ the subset of $\mathrm{s}\tau\text{-tilt}\,k\tilde{G}$ consisting of support τ-tilting $k\tilde{G}$-modules satisfying the equivalent conditions of Theorem 3.8. Then the induction functor $\mathrm{Ind}_G^{\tilde{G}}$ induces a poset isomorphism

$$\begin{aligned} (\mathrm{s}\tau\text{-tilt}\,kG)^{\tilde{G}} &\xrightarrow{\sim} (\mathrm{s}\tau\text{-tilt}\,k\tilde{G})^{\star} \\ M &\longmapsto \mathrm{Ind}_G^{\tilde{G}}M. \end{aligned} \tag{3.3}$$

In particular, the induction functor $\mathrm{Ind}_G^{\tilde{G}}$ induces the poset monomorphism

$$\begin{aligned} (\mathrm{s}\tau\text{-tilt}\,kG)^{\tilde{G}} &\longrightarrow \mathrm{s}\tau\text{-tilt}\,k\tilde{G} \\ M &\longmapsto \mathrm{Ind}_G^{\tilde{G}}M. \end{aligned} \tag{3.4}$$

Proof. By [16, Theorem 3.2], the map (3.4) is well-defined. Moreover, by the exactness of the functor $\mathrm{Ind}_G^{\tilde{G}}$, if $N \leq M$ in $\mathrm{s}\tau\text{-tilt}\, kG$ then $\mathrm{Ind}_G^{\tilde{G}} N \leq \mathrm{Ind}_G^{\tilde{G}} M$ in $\mathrm{s}\tau\text{-tilt}\, k\tilde{G}$ for any support τ-tilting kG-modules N and M. Therefore, the map (3.4) is a poset homomorphism. By the exactness of the functor $\mathrm{Res}_G^{\tilde{G}}$, Lemma 2.5 and Theorem 3.4, the map

$$\begin{aligned} (\mathrm{s}\tau\text{-tilt}\, k\tilde{G})^{\star} &\longrightarrow (\mathrm{s}\tau\text{-tilt}\, kG)^{\tilde{G}} \\ \tilde{M} &\longmapsto \mathrm{Res}_G^{\tilde{G}} \tilde{M} \end{aligned} \tag{3.5}$$

is well-defined and a poset homomorphism.

We show that the map (3.4) restricts to a bijection (3.3). By the definition of $(\mathrm{s}\tau\text{-tilt}\, k\tilde{G})^{\star}$ and the above argument, the map (3.3) is well-defined and a poset homomorphism. For any relatively G-projective support τ-tilting $k\tilde{G}$-module $\tilde{M}$ with $\mathrm{Ind}_G^{\tilde{G}} \mathrm{Res}_G^{\tilde{G}} \tilde{M} \in \mathrm{add}\, \tilde{M}$, we have that

$$\mathrm{Ind}_G^{\tilde{G}} \mathrm{Res}_G^{\tilde{G}} \tilde{M} =_{\mathrm{add}} \tilde{M}.$$

Moreover, for any $\tilde{G}$-invariant support τ-tilting kG-module M, we have that

$$\mathrm{Res}_G^{\tilde{G}} \mathrm{Ind}_G^{\tilde{G}} M \cong \bigoplus_{x \in [\tilde{G}/G]} xM =_{\mathrm{add}} M.$$

Therefore, the maps (3.3) and (3.5) are inverse to each other.

The latter assertion immediately follows from the fact that the map (3.4) is the composition of the poset isomorphism (3.3) and the inclusion map $(\mathrm{s}\tau\text{-tilt}\, k\tilde{G})^{\star} \hookrightarrow \mathrm{s}\tau\text{-tilt}\, k\tilde{G}$. □

As an application of Theorem 3.8, we consider the case that $\tilde{G}/G$ is a p-group.

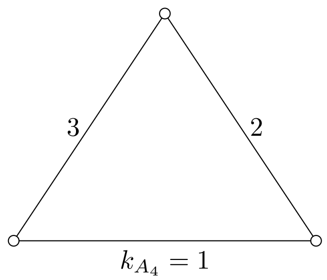
Theorem 3.10. Let $\tilde{G}$ be a finite group and G a normal subgroup of $\tilde{G}$ of p-power index in $\tilde{G}$. Then the induction functor $\mathrm{Ind}_G^{\tilde{G}}$ induces an isomorphism as posets between $(\mathrm{s}\tau\text{-tilt}\, kG)^{\tilde{G}}$ and $\mathrm{s}\tau\text{-tilt}\, k\tilde{G}$, where $(\mathrm{s}\tau\text{-tilt}\, kG)^{\tilde{G}}$ is the subset of $\mathrm{s}\tau\text{-tilt}\, kG$ consisting of $\tilde{G}$-invariant support τ-tilting kG-module.

Proof. By Corollary 3.9, the map (3.3) is a poset isomorphism. It is enough to show that $(\mathrm{s}\tau\text{-tilt}\, k\tilde{G})^{\star} = \mathrm{s}\tau\text{-tilt}\, k\tilde{G}$. It is clear that $(\mathrm{s}\tau\text{-tilt}\, k\tilde{G})^{\star} \subset \mathrm{s}\tau\text{-tilt}\, k\tilde{G}$. To prove the reverse inclusion, take an arbitrary support τ-tilting $k\tilde{G}$-module $\tilde{M}$. Since $\tilde{G}/G$ is a p-group, the only simple $k(\tilde{G}/G)$-module is the trivial $k(\tilde{G}/G)$-module. Hence, the condition (3) of Theorem 3.8 is satisfied in our situation because $k_{\tilde{G}/G} \otimes_k \tilde{M}$ is isomorphic to $\tilde{M}$. Therefore, we conclude that $\mathrm{s}\tau\text{-tilt}\, k\tilde{G} \subset (\mathrm{s}\tau\text{-tilt}\, k\tilde{G})^{\star}$. □

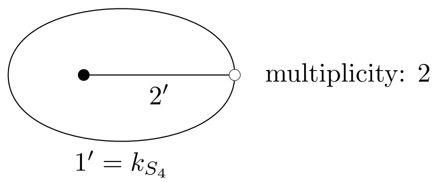
Here we reproduce from [16] an example with an error corrected; in Figure 2 below, the centered direct-sum of three non-projectives is corrected.

Example 3.11. Let k be an algebraically closed field of characteristic $p = 2$. We consider that the case that G is the alternating group A_4 of degree 4 and $\tilde{G}$ is the symmetric group S_4 of degree 4. The algebras kA_4 and kS_4 are Brauer graph algebras associated to the Brauer graphs in Figure 1(a) and Figure 1(b), respectively:

Now we draw the Hasse diagram $\mathcal{H}(\mathrm{s}\tau\text{-tilt}\, kA_4)$ of the poset $\mathrm{s}\tau\text{-tilt}\, kA_4$ as follows:



(a) The Brauer graph of kA_4



(b) The Brauer graph of kS_4

Figure 1: Brauer graphs

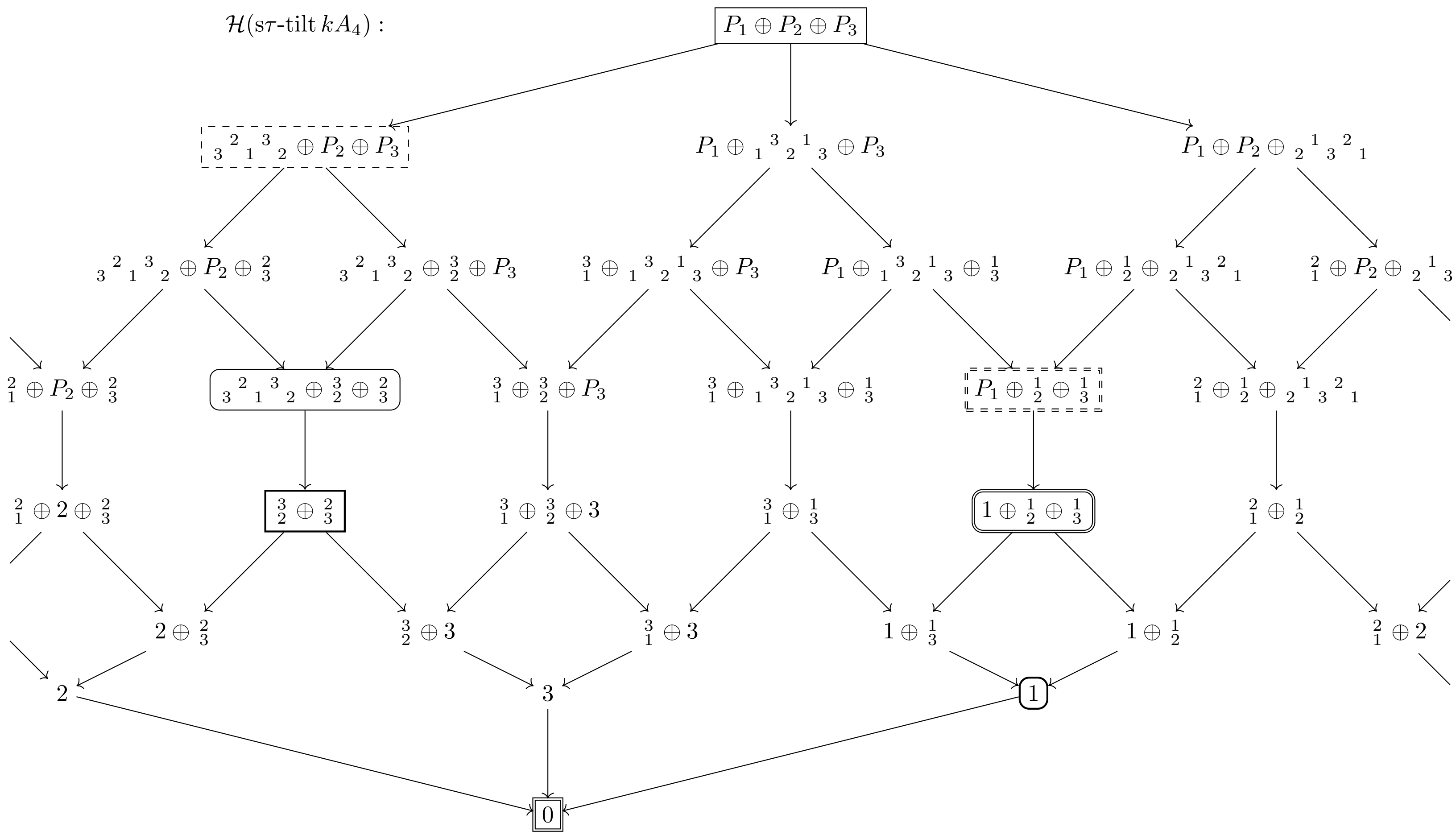


Figure 2: The Hasse diagram of sτ-tilt kA_4

The enclosed support τ-tilting modules in Figure 2 are all the invariant support τ-tilting modules under the action of S_4. Next, we draw the Hasse diagram $\mathcal{H}(\mathrm{s}\tau\text{-tilt}\, kS_4)$ of poset $\mathrm{s}\tau\text{-tilt}\, kS_4$ as follows:

$\mathcal{H}(\mathrm{s}\tau\text{-tilt}(kS_4))$:

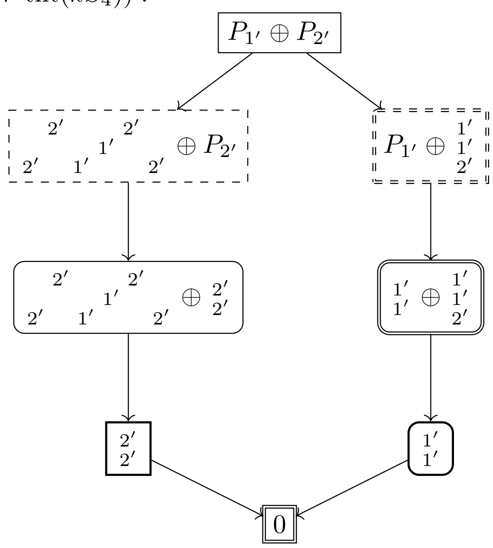


Figure 3: The Hasse diagram of $\mathrm{s}\tau\text{-tilt}\, kS_4$

The functor $\mathrm{Ind}_{A_4}^{S_4}$ takes each enclosed S_4-invariant support τ-tilting kA_4-module in Figure 2 to that in Figure 3 with the same square.

Remark 3.12. Let $(\mathrm{Ind}_G^{\tilde{G}})^{-1}(\mathrm{s}\tau\text{-tilt}\, k\tilde{G}) := \{M \in \mathrm{s}\tau\text{-tilt}\, kG \mid \mathrm{Ind}_G^{\tilde{G}} M \in \mathrm{s}\tau\text{-tilt}\, k\tilde{G}\}$. Then $(\mathrm{s}\tau\text{-tilt}\, kG)^{\tilde{G}}$ is contained in $(\mathrm{Ind}_G^{\tilde{G}})^{-1}(\mathrm{s}\tau\text{-tilt}\, k\tilde{G})$ by [16, Theorem 3.2]. On the other hand, they do not coincide in general. Moreover, though the poset homomorphism

$$\begin{aligned} (\mathrm{s}\tau\text{-tilt}\, kG)^{\tilde{G}} &\longrightarrow \mathrm{s}\tau\text{-tilt}\, k\tilde{G} \\ M &\longmapsto \mathrm{Ind}_G^{\tilde{G}} M \end{aligned}$$

is a monomorphism by Corollary 3.9, the one

$$\begin{aligned} (\mathrm{Ind}_G^{\tilde{G}})^{-1}(\mathrm{s}\tau\text{-tilt}\, k\tilde{G}) &\longrightarrow \mathrm{s}\tau\text{-tilt}\, k\tilde{G} \\ M &\longmapsto \mathrm{Ind}_G^{\tilde{G}} M \end{aligned}$$

is not a monomorphism in general.

For example, for $p = 2$, the alternating group A_4 of degree 4 and the symmetric group S_4 of degree 4, a kA_4-module $M := 1 \oplus \begin{smallmatrix} 1 \\ 2 \end{smallmatrix}$ is a support τ-tilting kA_4-module, where 1 denotes the trivial kA_4-module and 2 a non-trivial kA_4-module. Also, it holds that $\sigma M \cong 1 \oplus \begin{smallmatrix} 1 \\ 3 \end{smallmatrix}$ for $\sigma \in S_4 \setminus A_4$, where 3 denotes the non-trivial simple kA_4-module not isomorphic to 2. Therefore, we have that $M \notin (\mathrm{s}\tau\text{-tilt}\, kA_4)^{S_4}$. However, $\mathrm{Ind}_{A_4}^{S_4} M \cong \begin{smallmatrix} 1' \\ 1' \end{smallmatrix} \oplus \begin{smallmatrix} 1' \\ 1' \\ 2' \end{smallmatrix}$ is a support τ-tilting kS_4-module, where $1'$ denotes the trivial kS_4-module and $2'$ the simple kS_4-module of dimension 2. This implies that $M \in (\mathrm{Ind}_{A_4}^{S_4})^{-1}(\mathrm{s}\tau\text{-tilt}\, kS_4)$. Moreover, for $N := 1 \oplus \begin{smallmatrix} 1 \\ 2 \end{smallmatrix} \oplus \begin{smallmatrix} 1 \\ 3 \end{smallmatrix} \in (\mathrm{s}\tau\text{-tilt}\, kA_4)^{S_4}$, it holds that $\mathrm{Ind}_{A_4}^{S_4} N \cong \begin{smallmatrix} 1' \\ 1' \end{smallmatrix} \oplus \begin{smallmatrix} 1' \\ 1' \\ 2' \end{smallmatrix} \oplus \begin{smallmatrix} 1' \\ 1' \\ 2' \end{smallmatrix} =_{\mathrm{add}} \begin{smallmatrix} 1' \\ 1' \end{smallmatrix} \oplus \begin{smallmatrix} 1' \\ 1' \\ 2' \end{smallmatrix} (=_{\mathrm{add}} \mathrm{Ind}_{A_4}^{S_4} M)$. Therefore, the map

$$\begin{aligned} (\mathrm{Ind}_{A_4}^{S_4})^{-1}(\mathrm{s}\tau\text{-tilt}\, kS_4) &\longrightarrow \mathrm{s}\tau\text{-tilt}\, kS_4 \\ M &\longmapsto \mathrm{Ind}_{A_4}^{S_4} M \end{aligned}$$

is not a monomorphism.

At the end of this section, we discuss a feature of vertices of indecomposable τ-rigid $k\tilde{G}$-modules.

Lemma 3.13. Let $\tilde{G}$ be a finite group. Then the trivial $k\tilde{G}$-module $k_{\tilde{G}}$ is a τ-rigid if and only if $\tilde{G}$ has no normal subgroups of index p.

Proof. By [21, Chap. I, Corollary 10.13], there exists a normal subgroup of $\tilde{G}$ of index p if and only if $\mathrm{Ext}^1_{k\tilde{G}}(k_{\tilde{G}}, k_{\tilde{G}}) \neq 0$. Also, by the simplicity of the trivial $k\tilde{G}$-module and Auslander-Reiten duality, we have that

$$\mathrm{Hom}_{k\tilde{G}}(k_{\tilde{G}}, \tau k_{\tilde{G}}) \cong \overline{\mathrm{Hom}}_{k\tilde{G}}(k_{\tilde{G}}, \tau k_{\tilde{G}}) \cong D\,\mathrm{Ext}^1_{k\tilde{G}}(k_{\tilde{G}}, k_{\tilde{G}}).$$

Therefore, we get the result. □

Theorem 3.14. Let $\tilde{G}$ be a finite group. Then any indecomposable τ-rigid $k\tilde{G}$-module has a vertex contained in a Sylow p-subgroup of $\tilde{G}$ properly if and only if $\tilde{G}$ has a proper normal subgroup of p-power index.

Proof. Assume that $\tilde{G}$ has no proper normal subgroup of p-power index. Then by Lemma 3.13, the trivial $k\tilde{G}$-module, whose vertex is a Sylow p-subgroup of $\tilde{G}$, is a τ-rigid module.

Conversely, assume that $\tilde{G}$ has normal subgroup of p-power index. In this case, there exists a normal subgroup G of $\tilde{G}$ of index p. Let $\tilde{X}$ be an arbitrary τ-rigid $k\tilde{G}$-module. Then, $\tilde{X}$ is a direct summand of a support τ-tilting $k\tilde{G}$-module $\tilde{M}$ by [1, Theorem 2.10], that is, $\tilde{X}$ is relatively G-projective. Also, there exists a $\tilde{G}$-invariant support τ-tilting kG-module M such that $\tilde{M} =_{\mathrm{add}} \mathrm{Ind}_G^{\tilde{G}} M$ by Theorem 3.10. Hence, $\tilde{X}$ is a direct summand of $\mathrm{Ind}_G^{\tilde{G}} M$. Therefore, $\tilde{X}$ has a vertex contained in a Sylow p-subgroup of $\tilde{G}$ properly. □

4 Preliminaries for block versions of the main results

We recall the definition of blocks of group algebras. Let G be a finite group. The group algebra kG has a unique decomposition

$$kG = B_0 \times \cdots \times B_l$$

into the direct product of indecomposable algebras. We call each indecomposable direct product component B_i a block of kG and the decomposition above the block decomposition. We remark that any block B_i is a two-sided ideal of kG.

For any indecomposable kG-module U, there exists a unique block B_i of kG such that $U = B_iU$ and $B_jU = 0$ for all $j \neq i$. Then we say that U lies in the block B_i or simply U is a B_i-module. We denote by $B_0(kG)$ the principal block of kG, in which the trivial kG-module k_G lies.

Let G be a normal subgroup of a finite group $\tilde{G}$, B a block of kG and $\tilde{B}$ a block of $k\tilde{G}$. We say that $\tilde{B}$ covers B (or that B is covered by $\tilde{B}$) if $1_B 1_{\tilde{B}} \neq 0$, where 1_B and $1_{\tilde{B}}$ denote the respective identity element of B and $\tilde{B}$.

Proposition 4.1 ([2, Theorem 15.1, Lemma 15.3])**.** With the notation above, the following are equivalent:

(1) The block $\tilde{B}$ covers B.

(2) There exists a non-zero $\tilde{B}$-module $\tilde{U}$ such that $\mathrm{Res}_G^{\tilde{G}}\tilde{U}$ has a non-zero direct summand lying in B.

(3) For any non-zero $\tilde{B}$-module $\tilde{U}$, there exists a non-zero direct summand of $\mathrm{Res}_G^{\tilde{G}}\tilde{U}$ lying in B.

(4) For any non-zero $\tilde{B}$-module $\tilde{U}$ and indecomposable direct summand V of $\mathrm{Res}_G^{\tilde{G}}\tilde{U}$, there exists $\tilde{g} \in \tilde{G}$ such that V lies in the block $\tilde{g}B\tilde{g}^{-1}$.

(5) The block B is a direct summand of $\tilde{B}$ as a (kG, kG)-bimodule.

(6) The block $\tilde{B}$ is a direct summand of $k\tilde{G}B\tilde{G}$ as a $(k\tilde{G}, k\tilde{G})$-bimodule.

We denote by $I_{\tilde{G}}(B)$ the inertial group of B in $\tilde{G}$, that is

$$I_{\tilde{G}}(B) := \left\{ \tilde{g} \in \tilde{G} \;\middle|\; \tilde{g}B\tilde{g}^{-1} = B \right\}.$$

Remark 4.2. For a block $\tilde{B}$ of $k\tilde{G}$ and a block B of kG, the block $\tilde{B}$ covers only B if and only if $I_{\tilde{G}}(B) = \tilde{G}$ by [2, Theorem 15.1 (1)]. Since $\mathrm{Res}_G^{\tilde{G}} k_{\tilde{G}} \cong k_G$, the principal block $B_0(kG)$ of kG is the only block of kG covered by the principal block $B_0(k\tilde{G})$ of $k\tilde{G}$ by the equivalence of Proposition 4.1 (1), (3). Therefore, we have that $I_{\tilde{G}}(B_0(kG)) = \tilde{G}$.

Proposition 4.3. Let G be a normal subgroup of a finite group $\tilde{G}$, B a block of kG and U an indecomposable B-module. Then the following hold:

(1) For a block $\tilde{B}$ of $k\tilde{G}$ covering the block B, the module $\mathrm{Res}_G^{\tilde{G}} \tilde{B} \mathrm{Ind}_G^{\tilde{G}} U$ has a direct summand isomorphic to U. In particular, the $\tilde{B}$-module $\tilde{B}\mathrm{Ind}_G^{\tilde{G}} U$ is non-zero.

(2) Any indecomposable direct summand $\tilde{V}$ of $\mathrm{Ind}_G^{\tilde{G}} U$ lies in a block of $k\tilde{G}$ covering B.

Proof. Let U be an indecomposable B-module. By the equivalence of Proposition 4.1 (1), (5), the block $\tilde{B}$ has a direct summand B as a (kG, kG)-bimodule. Hence, there exists a (kG, kG)-bimodule B' such that $\tilde{B} \cong B \oplus B'$ as a (kG, kG)-bimodule. Therefore, we have that

$$\mathrm{Res}_G^{\tilde{G}} \tilde{B} \mathrm{Ind}_G^{\tilde{G}} U \cong \mathrm{Res}_G^{\tilde{G}} \tilde{B}(k\tilde{G} \otimes_{kG} U) \cong \mathrm{Res}_G^{\tilde{G}} \tilde{B} \otimes_{kG} U \cong (B \oplus B') \otimes_{kG} U \cong U \oplus (B' \otimes_{kG} U),$$

which prove (1).

Let $\tilde{V}$ be an indecomposable direct summand of $\mathrm{Ind}_G^{\tilde{G}} U$ lying in a block $\tilde{A}$ of $k\tilde{G}$. Since the restricted module $\mathrm{Res}_G^{\tilde{G}} \tilde{V}$ is a direct summand of the kG-module $\mathrm{Res}_G^{\tilde{G}} \mathrm{Ind}_G^{\tilde{G}} U$, we have that the block $\tilde{A}$ covers B by Proposition 2.2 and the equivalences of Proposition 4.1 (1), (2), (4). Hence, we get that (2). □

The following is a generalization of [24, Corollary 5.5.6] (or [25, Corollary 9.9.6]).

Proposition 4.4. Let G be a normal subgroup of a finite group $\tilde{G}$ and B a block of kG. If there exists an indecomposable B-module X such that $\mathrm{Ind}_G^{\tilde{G}} X$ is an indecomposable $k\tilde{G}$-module, then there exists only one block of $k\tilde{G}$ covering B.

Proof. Let $\tilde{A}$ and $\tilde{B}$ be a block of $k\tilde{G}$ covering B. The modules $\tilde{A}\mathrm{Ind}_G^{\tilde{G}} X$ and $\tilde{B}\mathrm{Ind}_G^{\tilde{G}} X$ are non-zero direct summands of the indecomposable $k\tilde{G}$-module $\mathrm{Ind}_G^{\tilde{G}} X$ by Proposition 4.3 (1). Hence, we get that $\tilde{B}\mathrm{Ind}_G^{\tilde{G}} X \cong \tilde{A}\mathrm{Ind}_G^{\tilde{G}} X \cong \mathrm{Ind}_G^{\tilde{G}} X$ by the indecomposability of $\mathrm{Ind}_G^{\tilde{G}} X$. Since the non-zero $k\tilde{G}$-module $\mathrm{Ind}_G^{\tilde{G}} X$ lies in the blocks $\tilde{A}$ and $\tilde{B}$, we get that $\tilde{A} = \tilde{B}$. □

Corollary 4.5 ([24, Corollary 5.5.6] or [25, Corollary 9.9.6])**.** If $\tilde{G}/G$ is a p-group, then there exists only one block of $k\tilde{G}$ covering B.

Proof. It immediately follows from Proposition 4.4 and Green's indecomposability theorem (for example, see [2, 11, 24]). □

Proposition 4.6 ([22, Theorem 6.8.3] or [24, Theorem 5.5.10, Theorem 5.5.12])**.** Let G be a normal subgroup of a finite group $\tilde{G}$ and B a block of kG. Then the following hold:

(1) For any block β of $kI_{\tilde{G}}(B)$ covering B, there exists a block $\tilde{B}$ of $k\tilde{G}$ such that

$$\sum_{x \in [\tilde{G}/I_{\tilde{G}}(B)]} x 1_\beta x^{-1} = 1_{\tilde{B}},$$

and then $\tilde{B}$ covers B. Moreover, the correspondence sending β to $\tilde{B}$ induces a bijection between the set of blocks of $kI_{\tilde{G}}(B)$ covering B and those of $k\tilde{G}$ covering B.

(2) If $\tilde{B}$ corresponds to β under the bijection of (1), then the induction functor

$$\mathrm{Ind}_{I_{\tilde{G}}(B)}^{\tilde{G}} \colon kI_{\tilde{G}}(B)\text{-mod} \longrightarrow k\tilde{G}\text{-mod}$$

restricts to a Morita equivalence

$$\mathrm{Ind}_{I_{\tilde{G}}(B)}^{\tilde{G}} \colon \beta\text{-mod} \longrightarrow \tilde{B}\text{-mod}$$

and its inverse functor is given by

$$\beta \mathrm{Res}^{\tilde{G}}_{I_{\tilde{G}}(B)} \colon \tilde{B}\text{-mod} \longrightarrow \beta\text{-mod}\,.$$

Proposition 4.7. Let G be a normal subgroup of a finite group $\tilde{G}$, B a block of kG, U a B-module, β a block of $kI_{\tilde{G}}(B)$ covering B and $\tilde{B}$ a block of $k\tilde{G}$ covering B such that

$$\sum_{x\in[\tilde{G}/I_{\tilde{G}}(B)]} x1_\beta x^{-1} = 1_{\tilde{B}}.$$

Then $\tilde{B}\mathrm{Ind}^{\tilde{G}}_G U \cong \mathrm{Ind}^{\tilde{G}}_{I_{\tilde{G}}(B)}\beta\mathrm{Ind}^{I_{\tilde{G}}(B)}_G U$.

Proof. Let $\tilde{B}_1 = \tilde{B}, \dots, \tilde{B}_e$ be the all blocks of $k\tilde{G}$ covering B. By Proposition 4.6, we can take $\beta_1 = \beta, \dots, \beta_e$ the blocks of $kI_{\tilde{G}}(B)$ satisfying the induction functor $\mathrm{Ind}^{\tilde{G}}_{I_{\tilde{G}}(B)}$ restricts to a Morita equivalence

$$\mathrm{Ind}^{\tilde{G}}_{I_{\tilde{G}}(B)} \colon \beta_i\text{-mod} \longrightarrow \tilde{B}_i\text{-mod}$$

for any $i = 1, \dots, e$. By Proposition 4.3 (2), we get the following isomorphism:

$$\mathrm{Ind}^{I_{\tilde{G}}(B)}_G U \cong \beta_1\mathrm{Ind}^{I_{\tilde{G}}(B)}_G U \oplus \cdots \oplus \beta_e\mathrm{Ind}^{I_{\tilde{G}}(B)}_G U.$$

Moreover, by Proposition 2.1 (2), we have that

$$\mathrm{Ind}^{\tilde{G}}_G U \cong \mathrm{Ind}^{\tilde{G}}_{I_{\tilde{G}}(B)}\mathrm{Ind}^{I_{\tilde{G}}(B)}_G U \cong \mathrm{Ind}^{\tilde{G}}_{I_{\tilde{G}}(B)}\beta_1\mathrm{Ind}^{I_{\tilde{G}}(B)}_G U \oplus \cdots \oplus \mathrm{Ind}^{\tilde{G}}_{I_{\tilde{G}}(B)}\beta_e\mathrm{Ind}^{I_{\tilde{G}}(B)}_G U.$$

Since the $k\tilde{G}$-module $\mathrm{Ind}^{\tilde{G}}_{I_{\tilde{G}}(B)}\beta_i\mathrm{Ind}^{I_{\tilde{G}}(B)}_G U$ lies in the block $\tilde{B}_i$ for any $i = 1, \dots, e$, we get that

$$\tilde{B}_i\mathrm{Ind}^{\tilde{G}}_G U \cong \mathrm{Ind}^{\tilde{G}}_{I_{\tilde{G}}(B)}\beta_i\mathrm{Ind}^{I_{\tilde{G}}(B)}_G U.$$

Therefore, we complete the proof. ◻

5 Block version of main results

In this section we give block versions of our theorems stated in Section 3. Let Λ be a finite-dimensional algebra. For Λ-modules M and N, we write $M \leq_{\mathrm{add}} N$ if $\mathrm{add}\, M \subset \mathrm{add}\, N$. This relation is clearly reflexive and transitive. Moreover, if $M \leq_{\mathrm{add}} N$ and $N \leq_{\mathrm{add}} M$ then $M =_{\mathrm{add}} N$ for any Λ-modules M and N. The following is the special case of the block version of Theorem 3.4.

Theorem 5.1. Let G be a normal subgroup of a finite group $\tilde{G}$, B a block of kG satisfying $I_{\tilde{G}}(B) = \tilde{G}$, $\tilde{B}$ a block of $k\tilde{G}$ covering B and $\tilde{M}$ a support τ-tilting $\tilde{B}$-module. If it holds that $\tilde{B}\mathrm{Ind}^{\tilde{G}}_G\mathrm{Res}^{\tilde{G}}_G\tilde{M} \in \mathrm{add}\,\tilde{M}$ and $\tilde{M}$ is relatively G-projective, then we have that $\mathrm{Res}^{\tilde{G}}_G\tilde{M}$ is a support τ-tilting B-module. Moreover, if $(\tilde{M}, \tilde{P})$ is a support τ-tilting pair for $\tilde{B}$ corresponding to $\tilde{M}$, then $(\mathrm{Res}^{\tilde{G}}_G\tilde{M}, \mathrm{Res}^{\tilde{G}}_G\tilde{P})$ is a support τ-tilting pair for B corresponding to $\mathrm{Res}^{\tilde{G}}_G\tilde{M}$.

Proof. Let $(\tilde{M}, \tilde{P})$ be a support τ-tilting pair for $\tilde{B}$ corresponding to the support τ-tilting $\tilde{B}$-module $\tilde{M}$. Our assumption $I_{\tilde{G}}(B) = \tilde{G}$ implies that the block B is the only block of kG covered by $\tilde{B}$ by Remark 4.2. Hence, we have that the restricted modules $\mathrm{Res}^{\tilde{G}}_G\tilde{M}$ and $\mathrm{Res}^{\tilde{G}}_G\tilde{P}$ are B-modules by Proposition 4.1 (4).

First, we show that $(\mathrm{Res}^{\tilde{G}}_G\tilde{M}, \mathrm{Res}^{\tilde{G}}_G\tilde{P})$ is a τ-rigid pair for B. Since the $\tilde{B}$-module $\tilde{M}$ is a support τ-tilting $\tilde{B}$-module, it is a τ-rigid $\tilde{B}$-module. Hence, we have that $\mathrm{Res}^{\tilde{G}}_G\tilde{M}$ is a τ-rigid B-module by Lemma 3.1. On the other hand, by Proposition 2.1 (4) we have that

$$\begin{aligned}\mathrm{Hom}_B(\mathrm{Res}^{\tilde{G}}_G\tilde{P}, \mathrm{Res}^{\tilde{G}}_G\tilde{M}) &\cong \mathrm{Hom}_{kG}(\mathrm{Res}^{\tilde{G}}_G\tilde{P}, \mathrm{Res}^{\tilde{G}}_G\tilde{M})\\ &\cong \mathrm{Hom}_{k\tilde{G}}(\tilde{P}, \mathrm{Ind}^{\tilde{G}}_G\mathrm{Res}^{\tilde{G}}_G\tilde{M})\\ &\cong \mathrm{Hom}_{\tilde{B}}(\tilde{P}, \tilde{B}\mathrm{Ind}^{\tilde{G}}_G\mathrm{Res}^{\tilde{G}}_G\tilde{M}).\end{aligned}$$

Now, by the assumption that $\tilde{B}\mathrm{Ind}_G^{\tilde{G}}\mathrm{Res}_G^{\tilde{G}}\tilde{M} \in \operatorname{add}\tilde{M}$, we have that $\mathrm{Hom}_{\tilde{B}}(\tilde{P}, \tilde{B}\mathrm{Ind}_G^{\tilde{G}}\mathrm{Res}_G^{\tilde{G}}\tilde{M}) = 0$ because $(\tilde{M}, \tilde{P})$ is a support τ-tilting pair for $\tilde{B}$. Therefore, we conclude that $(\mathrm{Res}_G^{\tilde{G}}\tilde{M}, \mathrm{Res}_G^{\tilde{G}}\tilde{P})$ is a τ-rigid pair for B.

Next, we show that the τ-rigid pair $(\mathrm{Res}_G^{\tilde{G}}\tilde{M}, \mathrm{Res}_G^{\tilde{G}}\tilde{P})$ is a support τ-tilting pair for B. We show that $X \in \operatorname{add}\mathrm{Res}_G^{\tilde{G}}\tilde{M}$ under the assumption that

$$\mathrm{Hom}_B(X, \tau(\mathrm{Res}_G^{\tilde{G}}\tilde{M})) = \mathrm{Hom}_B(\mathrm{Res}_G^{\tilde{G}}\tilde{M}, \tau X) = \mathrm{Hom}_B(\mathrm{Res}_G^{\tilde{G}}\tilde{P}, X) = 0,$$

which implies that the pair $(\mathrm{Res}_G^{\tilde{G}}\tilde{M}, \mathrm{Res}_G^{\tilde{G}}\tilde{P})$ is a support τ-tilting pair for B by Proposition 3.3. Under these assumptions, we have the following:

$$\begin{aligned}
\mathrm{Hom}_{\tilde{B}}(\tilde{B}\mathrm{Ind}_G^{\tilde{G}}X, \tau\tilde{M}) &\cong \mathrm{Hom}_{k\tilde{G}}(\mathrm{Ind}_G^{\tilde{G}}X, \tau\tilde{M}) && (\tau\tilde{M} \text{ is a } \tilde{B}\text{-module})\\
&\cong \mathrm{Hom}_{kG}(X, \mathrm{Res}_G^{\tilde{G}}(\tau\tilde{M})) && (\text{Proposition 2.1 (5)})\\
&\cong \mathrm{Hom}_{kG}(X, \tau(\mathrm{Res}_G^{\tilde{G}}\tilde{M})) && (\text{Lemma 3.2})\\
&\cong \mathrm{Hom}_B(X, \tau(\mathrm{Res}_G^{\tilde{G}}\tilde{M})) && (X \text{ and } \tau(\mathrm{Res}_G^{\tilde{G}}\tilde{M}) \text{ are } B\text{-modules})\\
&= 0.
\end{aligned}$$

$$\begin{aligned}
\mathrm{Hom}_{\tilde{B}}(\tilde{M}, \tau(\tilde{B}\mathrm{Ind}_G^{\tilde{G}}X)) &\cong \mathrm{Hom}_{\tilde{B}}(\tilde{M}, \tilde{B}\mathrm{Ind}_G^{\tilde{G}}(\tau X)) && (\text{Lemma 2.4})\\
&\cong \mathrm{Hom}_{k\tilde{G}}(\tilde{M}, \mathrm{Ind}_G^{\tilde{G}}(\tau X)) && (\tilde{M} \text{ is the } \tilde{B}\text{-module})\\
&\cong \mathrm{Hom}_{kG}(\mathrm{Res}_G^{\tilde{G}}\tilde{M}, \tau X) && (\text{Proposition 2.1 (4)})\\
&\cong \mathrm{Hom}_B(\mathrm{Res}_G^{\tilde{G}}\tilde{M}, \tau X) && (\mathrm{Res}_G^{\tilde{G}}\tilde{M} \text{ and } \tau X \text{ are } B\text{-modules})\\
&= 0.
\end{aligned}$$

$$\begin{aligned}
\mathrm{Hom}_{\tilde{B}}(\tilde{P}, \tilde{B}\mathrm{Ind}_G^{\tilde{G}}X) &\cong \mathrm{Hom}_{k\tilde{G}}(\tilde{P}, \mathrm{Ind}_G^{\tilde{G}}X) && (\tilde{P} \text{ is a } \tilde{B}\text{-module})\\
&\cong \mathrm{Hom}_{kG}(\mathrm{Res}_G^{\tilde{G}}\tilde{P}, X) && (\text{Proposition 2.1 (4)})\\
&\cong \mathrm{Hom}_B(\mathrm{Res}_G^{\tilde{G}}\tilde{P}, X) && (\mathrm{Res}_G^{\tilde{G}}\tilde{P} \text{ and } X \text{ are } B\text{-modules})\\
&= 0.
\end{aligned}$$

By these three isomorphisms and the fact that $(\tilde{M}, \tilde{P})$ is a support τ-tilting pair for $\tilde{B}$, applying Proposition 3.3, we have that $\tilde{B}\mathrm{Ind}_G^{\tilde{G}}X \in \operatorname{add}\tilde{M}$. Also, since the block $\tilde{B}$ covers B, the B-module X is a direct summand of $\mathrm{Res}_G^{\tilde{G}}\tilde{B}\mathrm{Ind}_G^{\tilde{G}}X$ by Proposition 4.3 (1). Therefore, we have that $X \in \operatorname{add}\mathrm{Res}_G^{\tilde{G}}\tilde{M}$. $\square$

We wish to restate the assumption of Theorem 5.1 with some equivalent conditions. First, we give the lemmas which can be applied in case of rigid $\tilde{B}$-modules not only support τ-tilting $\tilde{B}$-modules. The following lemma is a block version of Lemma 3.6, which is helpful to prove Theorem 5.4.

Lemma 5.2. Let $\tilde{G}$ be a finite group, $\tilde{B}$ a block of $k\tilde{G}$, $\tilde{M}$ a rigid $\tilde{B}$-module and L a $k\tilde{G}$-module. If it holds that $\tilde{B}(S \otimes_k \tilde{M}) \in \operatorname{add}\tilde{M}$ for any composition factor S of L, then the following isomorphism as $\tilde{B}$-modules holds:

$$\tilde{B}(L \otimes_k \tilde{M}) \cong \bigoplus_S \tilde{B}(S \otimes_k \tilde{M}),$$

where S runs over all composition factors of L.

Proof. A similar proof of Lemma 3.6 works in this setting. Let L be an arbitrarily $k\tilde{G}$-module and $\tilde{M}$ a rigid $\tilde{B}$-module satisfying that

$$\tilde{B}(S \otimes_k \tilde{M}) \in \operatorname{add}\tilde{M} \text{ for any composition factors } S \text{ of } L. \tag{5.1}$$

We use induction on the composition length $\ell(L)$ of L. If $\ell(L)=1$, there is nothing to prove. Hence, we assume that $\ell(L)\geq 2$ and that the statement for any $k\tilde{G}$-module L' satisfying $\ell(L')<\ell(L)$ is true. Let T be a simple submodule of L. We get the exact sequence

$$0 \longrightarrow \tilde{B}(T\otimes_k \tilde{M}) \longrightarrow \tilde{B}(L\otimes_k \tilde{M}) \longrightarrow \tilde{B}((L/T)\otimes_k \tilde{M}) \longrightarrow 0 \tag{5.2}$$

obtained by applying the exact functor $\tilde{B}(-\otimes_k \tilde{M})$ to the exact sequence

$$0 \longrightarrow T \longrightarrow L \longrightarrow L/T \longrightarrow 0.$$

By the rigidity of $\tilde{M}$, the assumption (5.1) and the assumption of the induction, the sequence (5.2) splits, and we get that

$$\tilde{B}(L\otimes_k \tilde{M}) \cong \tilde{B}(T\otimes_k \tilde{M}) \oplus \tilde{B}((L/T)\otimes_k \tilde{M}) \cong \tilde{B}(T\otimes_k \tilde{M}) \oplus \bigoplus_{S'} \tilde{B}(S'\otimes_k \tilde{M}) \cong \bigoplus_{S} \tilde{B}(S\otimes_k \tilde{M}),$$

where S' and S run over all composition factors of L/T and L, respectively. $\square$

Lemma 5.3. Let G be a normal subgroup of a finite group $\tilde{G}$, $\tilde{B}$ a block of $k\tilde{G}$ and $\tilde{M}$ a rigid $\tilde{B}$-module. Then the following conditions are equivalent:

(1) $\tilde{B}\mathrm{Ind}_G^{\tilde{G}}\mathrm{Res}_G^{\tilde{G}}\tilde{M} \in \mathrm{add}\,\tilde{M}$ and $\tilde{M}$ is relatively G-projective.

(2) $\tilde{B}(S\otimes_k \tilde{M}) \in \mathrm{add}\,\tilde{M}$ for each simple $k(\tilde{G}/G)$-module S.

Proof. By Proposition 2.1, we have that

$$\tilde{B}\mathrm{Ind}_G^{\tilde{G}}\mathrm{Res}_G^{\tilde{G}}\tilde{M} \cong \tilde{B}\mathrm{Ind}_G^{\tilde{G}}(k_G\otimes_k \mathrm{Res}_G^{\tilde{G}}\tilde{M}) \cong \tilde{B}((\mathrm{Ind}_G^{\tilde{G}}k_G)\otimes_k \tilde{M}) \cong \tilde{B}(k(\tilde{G}/G)\otimes_k \tilde{M}).$$

(1) $\Rightarrow$ (2). By the assumptions, we have that $\tilde{B}\mathrm{Ind}_G^{\tilde{G}}\mathrm{Res}_G^{\tilde{G}}\tilde{M} =_{\mathrm{add}} \tilde{M}$. Hence, by Proposition 2.1 we get that

$$\begin{aligned}
\tilde{B}(S\otimes_k \tilde{M}) &=_{\mathrm{add}} \tilde{B}(S\otimes_k \tilde{B}\mathrm{Ind}_G^{\tilde{G}}\mathrm{Res}_G^{\tilde{G}}\tilde{M})\\
&\leq_{\mathrm{add}} \tilde{B}(S\otimes_k \mathrm{Ind}_G^{\tilde{G}}\mathrm{Res}_G^{\tilde{G}}\tilde{M})\\
&\cong \tilde{B}\mathrm{Ind}_G^{\tilde{G}}(\mathrm{Res}_G^{\tilde{G}}S\otimes_k \mathrm{Res}_G^{\tilde{G}}\tilde{M})\\
&\cong \tilde{B}\mathrm{Ind}_G^{\tilde{G}}(k_G^{\oplus \dim_k S}\otimes_k \mathrm{Res}_G^{\tilde{G}}\tilde{M})\\
&=_{\mathrm{add}} \tilde{B}\mathrm{Ind}_G^{\tilde{G}}\mathrm{Res}_G^{\tilde{G}}\tilde{M}\\
&=_{\mathrm{add}} \tilde{M},
\end{aligned}$$

for any simple $k(\tilde{G}/G)$-module S, which implies that $\tilde{B}(S\otimes_k \tilde{M}) \in \mathrm{add}\,\tilde{M}$.
(2) $\Rightarrow$ (1). By Lemma 5.2, we have that

$$\tilde{B}\mathrm{Ind}_G^{\tilde{G}}\mathrm{Res}_G^{\tilde{G}}\tilde{M} \cong \tilde{B}(k(\tilde{G}/G)\otimes_k \tilde{M}) \cong \bigoplus_{S} \tilde{B}(S\otimes_k \tilde{M}),$$

where S runs over all composition factors of the $k\tilde{G}$-module $k(\tilde{G}/G)$. Therefore, the assumption implies that $\tilde{B}\mathrm{Ind}_G^{\tilde{G}}\mathrm{Res}_G^{\tilde{G}}\tilde{M} \in \mathrm{add}\,\tilde{M}$. Moreover, since the trivial $k\tilde{G}$-module $k_{\tilde{G}}$ appears as a composition factor of $k(\tilde{G}/G)$, we have that the $\tilde{B}$-module $\tilde{M}$ appears as a direct summand of $\tilde{B}\mathrm{Ind}_G^{\tilde{G}}\mathrm{Res}_G^{\tilde{G}}\tilde{M}$, that is $\tilde{M}$ is a relatively G-projective $k\tilde{G}$-module. $\square$

We obtain below some conditions that are equivalent to the assumption of Theorem 5.1.

Theorem 5.4. Let G be a normal subgroup of a finite group, B a block of kG satisfying $I_{\tilde{G}}(B)=\tilde{G}$ and $\tilde{B}$ a block of $k\tilde{G}$ covering B. Let $\tilde{M}$ be a support τ-tilting $\tilde{B}$-module. Then the following conditions are equivalent:

(1) $\tilde{M} =_{\mathrm{add}} \tilde{B}\mathrm{Ind}_G^{\tilde{G}} M$ for some $\tilde{G}$-invariant support τ-tilting B-module M.

(2) $\tilde{B}\mathrm{Ind}_G^{\tilde{G}}\mathrm{Res}_G^{\tilde{G}}\tilde{M} \in \mathrm{add}\,\tilde{M}$ and $\tilde{M}$ is relatively G-projective.

(3) $\tilde{B}(S \otimes_k \tilde{M}) \in \mathrm{add}\,\tilde{M}$ for each simple $k(\tilde{G}/G)$-module S.

Proof. (1) $\Rightarrow$ (2). Assume that $\tilde{M} =_{\mathrm{add}} \tilde{B}\mathrm{Ind}_G^{\tilde{G}} M$ for some $\tilde{G}$-invariant support τ-tilting B-module M. Then clearly $\tilde{M}$ is a relatively G-projective $\tilde{B}$-module, and we get that

$$\begin{aligned}
\tilde{B}\mathrm{Ind}_G^{\tilde{G}}\mathrm{Res}_G^{\tilde{G}}\tilde{M} &=_{\mathrm{add}} \tilde{B}\mathrm{Ind}_G^{\tilde{G}}\mathrm{Res}_G^{\tilde{G}}\tilde{B}\mathrm{Ind}_G^{\tilde{G}} M \\
&\leq_{\mathrm{add}} \tilde{B}\mathrm{Ind}_G^{\tilde{G}}\mathrm{Res}_G^{\tilde{G}}\mathrm{Ind}_G^{\tilde{G}} M \\
&\cong \tilde{B}\mathrm{Ind}_G^{\tilde{G}}\Big(\bigoplus_{\tilde{g}\in[\tilde{G}/G]} \tilde{g}M\Big) \\
&\cong \bigoplus_{\tilde{g}\in[\tilde{G}/G]} \tilde{B}\mathrm{Ind}_G^{\tilde{G}} M \\
&=_{\mathrm{add}} \tilde{M}.
\end{aligned}$$

Hence, we get $\tilde{B}\mathrm{Ind}_G^{\tilde{G}}\mathrm{Res}_G^{\tilde{G}}\tilde{M} \in \mathrm{add}\,\tilde{M}$.
(2) $\Rightarrow$ (1). Assume that $\tilde{B}\mathrm{Ind}_G^{\tilde{G}}\mathrm{Res}_G^{\tilde{G}}\tilde{M} \in \mathrm{add}\,\tilde{M}$ and that $\tilde{M}$ is relatively G-projective. Put $M := \mathrm{Res}_G^{\tilde{G}}\tilde{M}$. Then by Lemma 2.5, Proposition 4.1 (4), Remark 4.2 and Theorem 5.1, M is a $\tilde{G}$-invariant support τ-tilting B-module. We show that $\tilde{B}\mathrm{Ind}_G^{\tilde{G}} M =_{\mathrm{add}} \tilde{M}$, that is $\mathrm{add}(\tilde{B}\mathrm{Ind}_G^{\tilde{G}} M) = \mathrm{add}\,\tilde{M}$. By the assumption that $\tilde{B}\mathrm{Ind}_G^{\tilde{G}} M = \tilde{B}\mathrm{Ind}_G^{\tilde{G}}\mathrm{Res}_G^{\tilde{G}}\tilde{M} \in \mathrm{add}\,\tilde{M}$, we have that $\mathrm{add}(\tilde{B}\mathrm{Ind}_G^{\tilde{G}} M) \subset \mathrm{add}\,\tilde{M}$. On the other hand, since $\tilde{M}$ is relatively G-projective, $\tilde{M}$ is a direct summand of $\mathrm{Ind}_G^{\tilde{G}}\mathrm{Res}_G^{\tilde{G}}\tilde{M} = \mathrm{Ind}_G^{\tilde{G}} M$. Moreover, since $\tilde{M}$ lies in $\tilde{B}$, $\tilde{M}$ is a direct summand of $\tilde{B}\mathrm{Ind}_G^{\tilde{G}} M$. Hence, we have $\mathrm{add}\,\tilde{M} \subset \mathrm{add}(\tilde{B}\mathrm{Ind}_G^{\tilde{G}} M)$.
(2) $\Leftrightarrow$ (3) Since support τ-tilting $\tilde{B}$-modules are rigid $\tilde{B}$-modules, the equivalence follows from Lemma 5.3. □

Corollary 5.5. Let G be a normal subgroup of a finite group $\tilde{G}$, B a block of kG satisfying $I_{\tilde{G}}(B) = \tilde{G}$ and $\tilde{B}$ a block of $k\tilde{G}$ covering B. We denote by $(\mathrm{s}\tau\text{-tilt}\,B)^{\tilde{G}}$ the subset of $\mathrm{s}\tau\text{-tilt}\,B$ consisting of $\tilde{G}$-invariant support τ-tilting B-modules and by $(\mathrm{s}\tau\text{-tilt}\,\tilde{B})^{\star\star}$ the subset of $\mathrm{s}\tau\text{-tilt}\,\tilde{B}$ consisting of support τ-tilting $\tilde{B}$-modules satisfying the equivalent conditions of Theorem 5.4. Then the functor $\tilde{B}\mathrm{Ind}_G^{\tilde{G}}$ induces a poset isomorphism

$$\begin{aligned}
(\mathrm{s}\tau\text{-tilt}\,B)^{\tilde{G}} &\xrightarrow{\sim} (\mathrm{s}\tau\text{-tilt}\,\tilde{B})^{\star\star} \\
M &\longmapsto \tilde{B}\mathrm{Ind}_G^{\tilde{G}} M.
\end{aligned} \tag{5.3}$$

In particular, the functor $\tilde{B}\mathrm{Ind}_G^{\tilde{G}}$ induces the poset monomorphism

$$\begin{aligned}
(\mathrm{s}\tau\text{-tilt}\,B)^{\tilde{G}} &\longrightarrow \mathrm{s}\tau\text{-tilt}\,\tilde{B} \\
M &\longmapsto \tilde{B}\mathrm{Ind}_G^{\tilde{G}} M.
\end{aligned} \tag{5.4}$$

Proof. By [16, Theorem 3.3], the map (5.4) is well-defined. Moreover, by the exactness of the functor $\tilde{B}\mathrm{Ind}_G^{\tilde{G}}$, if $N \leq M$ in $\mathrm{s}\tau\text{-tilt}\,B$ then $\tilde{B}\mathrm{Ind}_G^{\tilde{G}} N \leq \tilde{B}\mathrm{Ind}_G^{\tilde{G}} M$ in $\mathrm{s}\tau\text{-tilt}\,\tilde{B}$. Therefore, the map (5.4) is a poset homomorphism. By the exactness of the functor $\mathrm{Res}_G^{\tilde{G}}$, Lemma 2.5 and Theorem 5.1, the map

$$\begin{aligned}
(\mathrm{s}\tau\text{-tilt}\,\tilde{B})^{\star\star} &\longrightarrow (\mathrm{s}\tau\text{-tilt}\,B)^{\tilde{G}} \\
\tilde{M} &\longmapsto \mathrm{Res}_G^{\tilde{G}}\tilde{M}
\end{aligned} \tag{5.5}$$

is well-defined and a poset homomorphism.

We show that the map (5.4) restricts to a bijection (5.3). By the definition of $(\mathrm{s}\tau\text{-tilt}\, k\tilde{G})^{\star\star}$ and the above argument, the map (5.3) is well-defined and a poset homomorphism. For any relatively G-projective support τ-tilting $\tilde{B}$-module $\tilde{M}$ with $\tilde{B}\mathrm{Ind}_G^{\tilde{G}}\mathrm{Res}_G^{\tilde{G}}\tilde{M} \in \mathrm{add}\,\tilde{M}$, we have that

$$\tilde{M} =_{\mathrm{add}} \tilde{B}\mathrm{Ind}_G^{\tilde{G}}\mathrm{Res}_G^{\tilde{G}}\tilde{M}.$$

Moreover, for any $\tilde{G}$-invariant support τ-tilting B-module M, we have that

$$M \leq_{\mathrm{add}} \mathrm{Res}_G^{\tilde{G}}\tilde{B}\mathrm{Ind}_G^{\tilde{G}}M \leq_{\mathrm{add}} \mathrm{Res}_G^{\tilde{G}}\mathrm{Ind}_G^{\tilde{G}}M \cong \bigoplus_{\tilde{g}\in[\tilde{G}/G]} \tilde{g}M =_{\mathrm{add}} M$$

by Proposition 4.3 (1), Proposition 2.2 and the $\tilde{G}$-invariance of M. Therefore, we have that $M =_{\mathrm{add}} \mathrm{Res}_G^{\tilde{G}}\tilde{B}\mathrm{Ind}_G^{\tilde{G}}M$. Therefore, the maps (5.3) and (5.5) are inverse to each other.

The latter assertion immediately follows from the fact that the map (5.4) is the composition of the poset isomorphism (5.3) and the inclusion map $(\mathrm{s}\tau\text{-tilt}\,\tilde{B})^{\star\star} \hookrightarrow \mathrm{s}\tau\text{-tilt}\,\tilde{B}$. $\square$

The following is a block version of Theorem 3.4.

Theorem 5.6. Let G be a normal subgroup of a finite group $\tilde{G}$, B a block of kG, $\tilde{B}$ a block of $k\tilde{G}$ covering B, β the block of $kI_{\tilde{G}}(B)$ satisfying

$$\sum_{x\in[\tilde{G}/I_{\tilde{G}}(B)]} x1_\beta x^{-1} = 1_{\tilde{B}}$$

and $\tilde{M}$ a support τ-tilting $\tilde{B}$-module. If it holds that $\beta\mathrm{Ind}_G^{I_{\tilde{G}}(B)}\mathrm{Res}_G^{I_{\tilde{G}}(B)}\beta\mathrm{Res}_{I_{\tilde{G}}(B)}^{\tilde{G}}\tilde{M} \in \mathrm{add}\,\beta\mathrm{Res}_{I_{\tilde{G}}(B)}^{\tilde{G}}\tilde{M}$ and $\beta\mathrm{Res}_{I_{\tilde{G}}(B)}^{\tilde{G}}\tilde{M}$ is relatively G-projective, then we have that $\mathrm{Res}_G^{I_{\tilde{G}}(B)}\beta\mathrm{Res}_{I_{\tilde{G}}(B)}^{\tilde{G}}\tilde{M}$ is a support τ-tilting B-module. Moreover, if $(\tilde{M}, \tilde{P})$ is a support τ-tilting pair for $\tilde{B}$ corresponding to $\tilde{M}$, then the pair

$$(\mathrm{Res}_G^{I_{\tilde{G}}(B)}\beta\mathrm{Res}_{I_{\tilde{G}}(B)}^{\tilde{G}}\tilde{M}, \mathrm{Res}_G^{I_{\tilde{G}}(B)}\beta\mathrm{Res}_{I_{\tilde{G}}(B)}^{\tilde{G}}\tilde{P})$$

is a support τ-tilting pair for B corresponding to $\mathrm{Res}_G^{I_{\tilde{G}}(B)}\beta\mathrm{Res}_{I_{\tilde{G}}(B)}^{\tilde{G}}\tilde{M}$.

Proof. Since the functor

$$\beta\mathrm{Res}_{I_{\tilde{G}}(B)}^{\tilde{G}} \colon \tilde{B}\text{-mod} \longrightarrow \beta\text{-mod}$$

is a Morita equivalence by Proposition 4.6, the module $\beta\mathrm{Res}_{I_{\tilde{G}}(B)}^{\tilde{G}}\tilde{M}$ is a support τ-tilting β-module and

$$(\beta\mathrm{Res}_{I_{\tilde{G}}(B)}^{\tilde{G}}\tilde{M}, \beta\mathrm{Res}_{I_{\tilde{G}}(B)}^{\tilde{G}}\tilde{P})$$

is a corresponding support τ-tilting pair for β. Hence, by Theorem 5.1 it immediately follows the consequence. $\square$

The following is a block version of Theorem 3.8.

Theorem 5.7. Let G be a normal subgroup of a finite group $\tilde{G}$, B a block of kG, $\tilde{B}$ a block of $k\tilde{G}$ covering B, β the block of $kI_{\tilde{G}}(B)$ satisfying

$$\sum_{x\in[\tilde{G}/I_{\tilde{G}}(B)]} x1_\beta x^{-1} = 1_{\tilde{B}}$$

and $\tilde{M}$ a support τ-tilting $\tilde{B}$-module. Then the following conditions are equivalent:

(1) $\tilde{M} =_{\mathrm{add}} \tilde{B}\mathrm{Ind}_G^{\tilde{G}}M$ for some $I_{\tilde{G}}(B)$-invariant support τ-tilting B-module M.

(2) $\beta\mathrm{Ind}_G^{I_{\tilde{G}}(B)}\mathrm{Res}_G^{I_{\tilde{G}}(B)}\beta\mathrm{Res}_{I_{\tilde{G}}(B)}^{\tilde{G}}\tilde{M} \in \mathrm{add}\,\beta\mathrm{Res}_{I_{\tilde{G}}(B)}^{\tilde{G}}\tilde{M}$ and $\beta\mathrm{Res}_{I_{\tilde{G}}(B)}^{\tilde{G}}\tilde{M}$ is relatively G-projective.

(3) $\beta(S\otimes_k \beta\mathrm{Res}^{\tilde G}_{I_{\tilde G}(B)}\tilde M)\in \mathrm{add}\,\beta\mathrm{Res}^{\tilde G}_{I_{\tilde G}(B)}\tilde M$ for each simple $k(I_{\tilde G}(B)/G)$-module S.

Proof. We remark that the module $\beta\mathrm{Res}^{\tilde G}_{I_{\tilde G}(B)}\tilde M$ is a support τ-tilting β-module since the functor

$$\beta\mathrm{Res}^{\tilde G}_{I_{\tilde G}(B)}\colon \tilde B\text{-mod}\longrightarrow \beta\text{-mod} \tag{5.6}$$

is a Morita equivalence by Proposition 4.6.

(1) $\Rightarrow$ (2). Assume that $\tilde M =_{\mathrm{add}} \tilde B\mathrm{Ind}^{\tilde G}_G M$ for some $I_{\tilde G}(B)$-invariant support τ-tilting B-module M. By [16, Theorem 3.3], the module $\beta\mathrm{Ind}^{I_{\tilde G}(B)}_G M$ is a support τ-tilting β-module. Since the functor

$$\mathrm{Ind}^{\tilde G}_{I_{\tilde G}(B)}\colon \beta\text{-mod}\longrightarrow \tilde B\text{-mod} \tag{5.7}$$

is a Morita equivalence with the inverse functor (5.6). we have $\mathrm{Ind}^{\tilde G}_{I_{\tilde G}(B)}\beta\mathrm{Res}^{\tilde G}_{I_{\tilde G}(B)}\tilde M\cong\tilde M$. Also, by the assumption and Proposition 4.7, we get that $\tilde M=_{\mathrm{add}}\tilde B\mathrm{Ind}^{\tilde G}_G M\cong \mathrm{Ind}^{\tilde G}_{I_{\tilde G}(B)}\beta\mathrm{Ind}^{I_{\tilde G}(B)}_G M$. Therefore, we have that $\mathrm{Ind}^{\tilde G}_{I_{\tilde G}(B)}\beta\mathrm{Res}^{\tilde G}_{I_{\tilde G}(B)}\tilde M=_{\mathrm{add}}\mathrm{Ind}^{\tilde G}_{I_{\tilde G}(B)}\beta\mathrm{Ind}^{I_{\tilde G}(B)}_G M$. Hence, by the fact that the functor (5.7) is a Morita equivalence again, we have that $\beta\mathrm{Res}^{\tilde G}_{I_{\tilde G}(B)}\tilde M=_{\mathrm{add}}\beta\mathrm{Ind}^{I_{\tilde G}(B)}_G M$. Therefore, we get the consequence (2) by the equivalence of Theorem 5.4. (1) and (2).

(2) $\Rightarrow$ (1). Since $\beta\mathrm{Res}^{\tilde G}_{I_{\tilde G}(B)}\tilde M$ is a support τ-tilting β-module, there exists an $I_{\tilde G}(B)$-invariant support τ-tilting B-module M such that $\beta\mathrm{Res}^{\tilde G}_{I_{\tilde G}(B)}\tilde M=_{\mathrm{add}}\beta\mathrm{Ind}^{I_{\tilde G}(B)}_G M$ by the assumptions and Theorem 5.4. Therefore, by Proposition 4.7, we get that

$$\tilde M\cong \mathrm{Ind}^{\tilde G}_{I_{\tilde G}(B)}\beta\mathrm{Res}^{\tilde G}_{I_{\tilde G}(B)}\tilde M=_{\mathrm{add}}\mathrm{Ind}^{\tilde G}_{I_{\tilde G}(B)}\beta\mathrm{Ind}^{I_{\tilde G}(B)}_G M\cong\tilde B\mathrm{Ind}^{\tilde G}_G M.$$

(2) $\Leftrightarrow$ (3). Since the support τ-tilting β-module $\beta\mathrm{Res}^{\tilde G}_{I_{\tilde G}(B)}\tilde M$ is the rigid β-module, the equivalence follows from Lemma 5.3. $\square$

Corollary 5.8. Let $(\mathrm{s}\tau\text{-tilt}\,B)^{I_{\tilde G}(B)}$ be the subset of $\mathrm{s}\tau\text{-tilt}\,B$ consisting of $I_{\tilde G}(B)$-invariant support τ-tilting B-modules and $(\mathrm{s}\tau\text{-tilt}\,\tilde B)^{\star\star\star}$ the subset of $\mathrm{s}\tau\text{-tilt}\,\tilde B$ consisting of support τ-tilting $\tilde B$-modules satisfying the equivalent conditions of Theorem 5.7. Then the functor $\tilde B\mathrm{Ind}^{\tilde G}_G$ induces a poset isomorphism

$$\begin{aligned}(\mathrm{s}\tau\text{-tilt}\,B)^{I_{\tilde G}(B)}&\longrightarrow(\mathrm{s}\tau\text{-tilt}\,\tilde B)^{\star\star\star}\\ M&\longmapsto \tilde B\mathrm{Ind}^{\tilde G}_G M.\end{aligned}\tag{5.8}$$

In particular, the functor $\tilde B\mathrm{Ind}^{\tilde G}_G$ induces the poset monomorphism

$$\begin{aligned}(\mathrm{s}\tau\text{-tilt}\,B)^{I_{\tilde G}(B)}&\longrightarrow\mathrm{s}\tau\text{-tilt}\,\tilde B\\ M&\longmapsto \tilde B\mathrm{Ind}^{\tilde G}_G M.\end{aligned}$$

Proof. Let $(\mathrm{s}\tau\text{-tilt}\,\beta)^{\star\star}$ be the subset of $\mathrm{s}\tau\text{-tilt}\,\beta$ consisting of support τ-tilting β-modules satisfying the equivalent conditions of Theorem 5.4. Since the functor

$$\mathrm{Ind}^{\tilde G}_{I_{\tilde G}(B)}\colon \beta\text{-mod}\longrightarrow \tilde B\text{-mod}$$

is a Morita equivalence, we have poset isomorphisms

$$\begin{aligned}\mathrm{s}\tau\text{-tilt}\,\beta&\longrightarrow\mathrm{s}\tau\text{-tilt}\,\tilde B\\ M&\longmapsto \mathrm{Ind}^{\tilde G}_{I_{\tilde G}(B)}M\end{aligned}$$

and

$$\begin{aligned} (s\tau\text{-tilt}\,\beta)^{\star\star} &\longrightarrow (s\tau\text{-tilt}\,\tilde{B})^{\star\star\star} \\ M &\longmapsto \operatorname{Ind}_{I_{\tilde{G}}(B)}^{\tilde{G}} M. \end{aligned} \tag{5.9}$$

By Theorem 5.4, we get the poset isomorphism

$$\begin{aligned} (s\tau\text{-tilt}\,B)^{I_{\tilde{G}}(B)} &\longrightarrow (s\tau\text{-tilt}\,\beta)^{\star\star} \\ M &\longmapsto \beta\operatorname{Ind}_{G}^{I_{\tilde{G}}(B)} M. \end{aligned} \tag{5.10}$$

By Proposition 4.7, the map (5.8) is the composition of the poset isomorphisms (5.10) and (5.9). Hence, we complete the proof. □

As an application of Corollary 5.8, we consider the case that $I_{\tilde{G}}(B)/G$ is a p-group. The following theorem is a significant generalization of [17, Theorem 1.2] and [10, Theorem 15].

Theorem 5.9. Let G be a normal subgroup of a finite group $\tilde{G}$, B a block of kG and $\tilde{B}$ a block of $k\tilde{G}$ covering B. If the quotient group $I_{\tilde{G}}(B)/G$ is a p-group, then the functor $\operatorname{Ind}_G^{\tilde{G}}$ induces an isomorphism as posets between $(s\tau\text{-tilt}\,B)^{I_{\tilde{G}}(B)}$ and $s\tau\text{-tilt}\,\tilde{B}$, where $(s\tau\text{-tilt}\,B)^{I_{\tilde{G}}(B)}$ is the subset of $s\tau\text{-tilt}\,B$ consisting of $I_{\tilde{G}}(B)$-invariant support τ-tilting B-modules.

Proof. This immediately follows from Corollary 4.5, Theorem 5.7 (3), Corollary 5.8 and the fact that the trivial $k(I_{\tilde{G}}(B)/G)$-module is a unique simple $k(I_{\tilde{G}}(B)/G)$-module. □

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