VIRTUAL THOMPSON'S GROUP

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Abstract. For virtual knot theory, the virtual braid group was defined by generalizing the braid group. It was proved that any virtual link can be obtained by the closure of a virtual braid. On the other hand, due to work by Jones et al., it is known that any (oriented) link is constructed from an element of Thompson's group F. In this paper, we define the "virtual version" of Thompson's group F and prove that any virtual link is constructed from an element of the group.

1. Introduction

Virtual knot theory, introduced by Kauffman [15], is a generalization of classical knot theory. There are some motivations in this theory. One is knot theory in $\Sigma_g \times [0, 1]$, where Σ_g is a closed oriented surface of genus $g \geq 0$. Classical knot theory can be regarded as the case of $g = 0$. Another is the complete correspondence with the Gauss diagrams, which are used to define a finite type invariant [9]. As in the braid group in classical knot theory, the virtual braid group is defined and studied. Kamada [14], and Kauffman and Lambropoulou [16] introduced this notion and proved Alexander's theorem, that is, any virtual link can be obtained from the closure of a virtual braid. Moreover, they showed Markov's theorem. In other words, this theorem gives a necessary and sufficient condition for two different braids to have equivalent closures.

Recently, Jones [13] introduced a method of constructing a link from an element of Thompson's group F, and proved Alexander's theorem. It means that any link can be obtained from an element of F. For the oriented case, Jones defined a subgroup $\overrightarrow{F}$ of F whose element yields an oriented link and showed the theorem with a weaker version. After that Aiello [1] proved it completely. Golan and Sapir [8] showed the subgroup $\overrightarrow{F}$ is isomorphic to the Brown–Thompson group $F(3)$.

Thompson's group F is defined by Richard Thompson in 1965. This group is known to be related to various areas and has been studied using various definitions such as piecewise linear maps on $[0, 1]$, pairs of binary trees, and so on. We consider F as a diagram group by referring to the approach in [8]. The notion of diagram groups was suggested by Meakin and Sapir (unpublished), and then Kilibarda [17] studied the groups for the first time.

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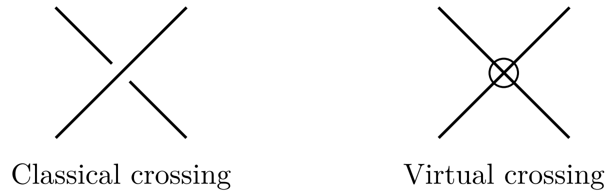


Figure 1. Classical and virtual crossings

This class of groups has been well studied not only algebraically but also geometrically. For instance, these groups are finitely presented [10], torsion-free [10], totally orderable [12], and act freely and cellularly on a CAT(0) cubical complex [6].

In this paper, we generalize Thompson's group F from the viewpoint of virtual knot theory. Namely, we define virtual Thompson's group VF as a diagram group and show the following:

Theorem 1.1. *Any virtual link can be obtained from an element in VF.*

This paper is organized as follows: In Section 2, we first summarize definitions of virtual links, diagram groups, and Thompson's group F. Then we define virtual Thompson's group VF as a diagram group. At the end of this section, we discuss some properties of diagram groups, and hence of VF. In Section 3, we introduce a method of constructing a virtual link from an element in VF. This method is a generalization of the one for F. Then we discuss the relationship between elements of VF and labeled binary trees. Some elements of VF are represented by labeled binary trees. In this sense, we can regard VF as a generalization of F. In Section 4, we show that any virtual link is obtained from some element in VF. Similar to [1, 13], this is achieved by constructing the Tait graph from a virtual link and deforming it.

Various other generalizations of Thompson's group F are also known [2, 3, 5, 18]. It is an interesting problem to study the relationship between them and VF.

2. Preliminaries

2.1. **Virtual knots and links.** In this section, we give a short description of the virtual links.

Definition 2.1. An n-component *virtual link diagram* is an immersion of n circles in the 2-sphere $\mathbb{S}^2\,(=\mathbb{R}^2 \cup \{\infty\})$ such that the multiple point set consists of finite number of transverse double points and each of them is labeled, either as a *classical crossing* or as a *virtual crossing* (see Figure 1). In particular, if $n = 1$, we also call it a *virtual knot diagram*. A virtual link diagram without virtual crossings is said to be *classical*.

Definition 2.2. An n-component *virtual link* is an equivalence class of the set of all n-component virtual link diagrams under the ambient isotopy on the plane and the

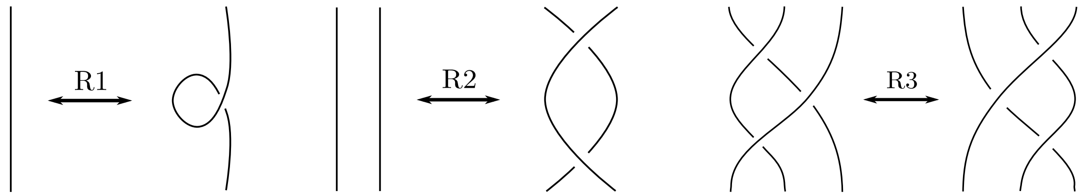


Figure 2. Classical Reidemeister moves

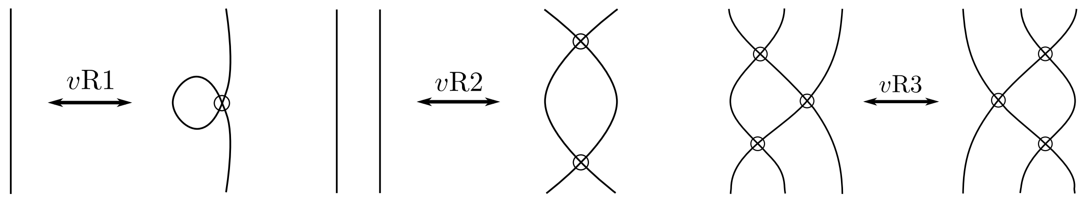


Figure 3. Virtual Reidemeister moves

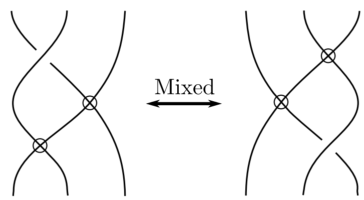


Figure 4. Mixed move

generalized Reidemeister moves, that is, the (classical) Reidemeister moves (Figure 2), the virtual Reidemeister moves (Figure 3), and the mixed move (Figure 4). If $n = 1$, we also call it a *virtual knot*.

Similarly to classical knot theory, it is natural to consider the notion of the virtual braid. Kamada [14], and Kauffman and Lambropoulou [16] defined the virtual braid group and independently proved Alexander's theorem for virtual links:

Theorem 2.3 ([14, Proposition 3], [16, Theorem 1]). *Any virtual link can be described as the closure of a virtual braid.*

2.2. **Diagram groups over semigroups.** In this section, we briefly review the definition of diagram groups. Although our purpose is to define one diagram group, we explain the formal definition of diagram groups for the reader's convenience. See [10] for details.

Let $\mathcal{P} = \langle \Sigma \mid \mathcal{R} \rangle$ be a semigroup presentation. Here, Σ is a finite set of generators, and $\mathcal{R}$ is a finite set of relations of the form $u \to v$ where u and v are finite words on Σ. We always assume that there exists no relation of the form $u \to u$, where u is a finite word on Σ. For simplicity, if $u \to v$ is in $\mathcal{R}$, then we regard $v \to u$ as also being in $\mathcal{R}$.

We fix a finite word w on Σ. Roughly speaking, for the given word w, each element (called a *diagram*) of the diagram group represents a way of rewriting by relations from w to itself again. Formally, for w, we define a diagram as a finite sequence of words on Σ

with the following form

$$w = w_1 \to w_2 \to \cdots \to w_{n-1} \to w_n = w$$

where each $w_{i-1} \to w_i$ is of the form $w'(p_{i-1})w'' \to w'(p_i)w''$ with certain words w', w'' on Σ and $p_{i-1} \to p_i$ is in $\mathcal{R}$. We call each replacement of the word in the sequence a *cell* of the diagram.

We define a reduction of a dipole as follows: Let $w = w_1 \to \cdots \to w_n = w$ be a diagram and assume that there exists i such that $w_{i-1} \to w_i \to w_{i+1}$ is of the form $w'(p_{i-1})w'' \to w'(p_i)w'' \to w'(p_{i+1})w''$ and $p_{i-1} = p_{i+1}$ holds. In this case, we obtain a new diagram by eliminating w_{i-1} and w_i, that is, by setting

$$w = w_1 \to w_2 \to \cdots \to w_{i-2} \to w_{i+1} \to \cdots \to w_n = w.$$

This operation and its inverse are called the *reduction of dipoles* and the *insertion of dipoles*, respectively.

We will identify diagrams Δ with Δ' if Δ' can be obtained from Δ by applying these operations a finite number of times. In addition, we will identify two diagrams when "two cells are separated". More precisely, we define two diagrams of the following forms are equivalent:

$$w = w_1 \to \cdots \to w'p_1w''p_2w'' \to w'p_1'w''p_2w'' \to w'p_1'w''p_2'w'' \to \cdots \to w_n = w$$

and

$$w = w_1 \to \cdots \to w'p_1w''p_2w'' \to w'p_1w''p_2'w'' \to w'p_1'w''p_2'w'' \to \cdots \to w_n = w,$$

where w', w'', w''' are words on Σ, and $p_1 \to p_1', p_2 \to p_2'$ are in $\mathcal{R}$. We define the equivalence relation on the set of all diagrams as the one generated by all of the above. We write $\mathcal{D}(\mathcal{P}, w)$ for the set of all equivalence classes of diagrams.

The product on $\mathcal{D}(\mathcal{P}, w)$ is defined as follows: For two diagrams $w = a_1 \to \cdots \to a_n = w$ and $w = b_1 \to \cdots \to b_m = w$, we define their product to be the equivalence class of the "concatenation"

$$w = a_1 \to \cdots \to a_n = w = b_1 \to \cdots \to b_m = w.$$

This product is well-defined, and $\mathcal{D}(\mathcal{P}, w)$ is termed *diagram group*.

If we can not apply the reduction of dipoles, we call the diagram *reduced*. For each element of $\mathcal{D}(\mathcal{P}, w)$, there exists a unique representative with reduced [17].

EXAMPLE 2.4. Let $\mathcal{P} = \langle a, b \mid a \to ab, b \to aa, a \to aa \rangle$ and $w = a$. Then

$$(a) \to (aa) = (a)a \to (aa)a = a(aa) \to a(a) = (aa) \to (a)$$

and

$$(a) \to (ab) = a(b) \to a(aa) \to a(a) = (aa) \to (a) \tag{2.1}$$

are reduced diagrams. Their product is

$$(a) \to (aa) = (a)a \to (aa)a = a(aa) \to a(a) = (aa) \to (a)$$
$$\to (ab) = a(b) \to a(aa) \to a(a) = (aa) \to (a),$$

and this diagram is also reduced. The diagram

$$(a) \to (ab) = (a)b \to (ab)b \to (a)b = a(b) \to a(aa) \to a(a) = (aa) \to (a)$$

is not reduced since there exists $(a)b \to (ab)b \to (a)b$. If we reduce a dipole of this diagram, then we get diagram 2.1. The diagrams

$$(a) \to (ab) = a(b) \to a(aa) = (a)aa \to (aa)aa = aa(aa)$$
$$\to aa(a) = a(aa) \to a(a) = (aa) \to (a)$$

and

$$(a) \to (ab) = (a)b \to (aa)b = aa(b) \to aa(aa) \to aa(a) = a(aa) \to a(a) = (aa) \to (a)$$

are equivalent. Observe the cells $a(b) \to a(aa) \to (aa)aa$ and $(a)b \to (aa)b \to aa(aa)$.

The notions in this section can also be represented by oriented graphs. Let $w = w_1 w_2 \cdots w_n$ be a word where each w_i is in Σ. We first define the trivial geometric diagrams as follows:

Let $v_1, v_2, \ldots, v_{n+1}$ be vertices, and each v_i is connected to v_{i+1} in this orientation. Namely, this graph consists of n edges $(v_1, v_2), (v_2, v_3), \ldots, (v_n, v_{n+1})$. We label each (v_i, v_{i+1}) as w_i and omit the labels of vertices. We define a *trivial geometric diagram* of w as this graph. For an oriented graph, given a vertex v and two edges (v', v) and (v, v''), we say that (v', v) and (v, v'') are *incoming* and *outgoing edges* of v, respectively (cf. Figure 12).

Next, we define a geometric cell. Let $p \to q$ be in $\mathcal{R}$, where $p = p_1 \cdots p_n$ and $q = q_1 \cdots q_m$. Let v_{p_1} and v_{p_n} be the vertices of the trivial diagram of p such that the edges labeled by p_1 and p_n are outgoing and incoming edges of v_{p_1} and v_{p_n}, respectively. For q, we define the vertices v_{q_1} and v_{q_m} similarly. Then the graph obtained by gluing v_{p_1} and v_{p_n} to v_{q_1} and v_{q_m}, respectively, is called a *geometric (p, q)-cell*.

Generally, a geometric diagram is represented by attaching geometric cells to a trivial geometric diagram along corresponding sub-words successively. Let w be a given word on Σ and $w = z_1 \to z_2 \to \cdots z_{n-1} \to z_n = w$ be a diagram. Note that each $z_{i-1} \to z_i$ is of the form $z'(p_{i-1})z'' \to z'(p_i)z''$, where z', z'' are some words on Σ. Therefore we can attach a (p_1, p_2)-cell along the subgraph of a trivial geometric diagram of w corresponding to p_1. Then the obtained graph has two paths, $z'(p_1)z''$ and $z'(p_2)z''$. Regarding $z'(p_2)z''$ as a trivial geometric diagram, and proceeding similarly to the end, we obtain a graph. We call this graph *geometric diagram*. The trivial geometric (sub)diagrams corresponding to the top and bottom w of the geometric diagram are called the *top* and *bottom paths*,

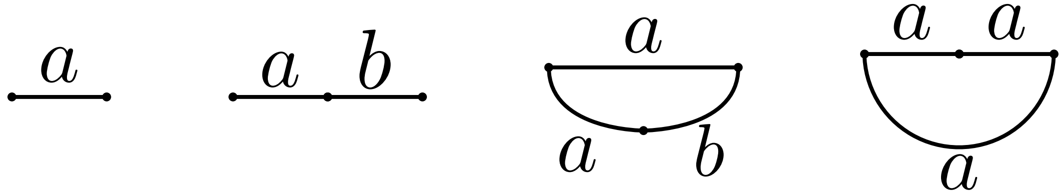


FIGURE 5. Trivial geometric diagrams of a and ab, and geometric (a, ab)-cell and (aa, a)-cell

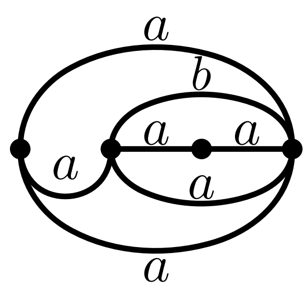


FIGURE 6. The geometric diagram corresponding to diagram 2.1

respectively. In the top path or bottom path, the vertex with only one outgoing or incoming edge is called the *initial* or *terminal vertex*, respectively. Note that for each diagram, there exists exactly one initial vertex and one terminal vertex since the top path and bottom path share them.

Similar to the equivalence relation of diagrams, we define the equivalence relation on the set of all geometric diagrams. In the rest of this paper, we do not distinguish between geometric diagrams and diagrams. See Figures 5 and 6 for examples of trivial geometric diagrams, geometric cells, and a geometric diagram corresponding to diagram 2.1 in Example 2.4.

CONVENTION 2.5. Throughout this paper, we assume that all orientations of the edges of the (geometric) diagrams illustrated in the figures are from left to right. For the sake of simplicity, we omit the illustration of the orientations unless it is important.

2.3. **Thompson's group F as a diagram group and its generalization.** In this section, we first recall Thompson's group F. Then we give the definition and some properties of virtual Thompson's group VF. This group is the most important one in this paper.

We first outline the definition of Thompson's group F. It is known that there exist various (equivalent) definitions for this group. We define this group as pairs of binary trees and then see the correspondence of the other realizations.

Let $\mathcal{T}$ be the set of all pairs of binary trees whose numbers of leaves are the same. We define the equivalence relation on $\mathcal{T}$. Let (T_+, T_-) be such two binary trees with n

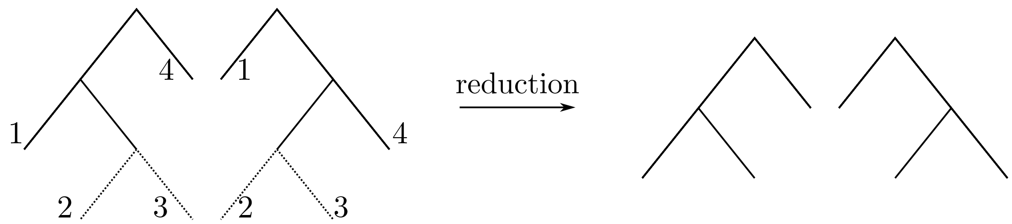


Figure 7. Example of reduction of carets

leaves. We label the leaves of the trees with the numbers $1, 2, \dots, n$ from left to right, respectively. Assume that there exists $i \in \{1, 2, \dots, n-1\}$ such that i and $i+1$ have a common parent in both T_+ and T_-. Then we can get the binary tree T'_+ by removing leaves i, $i+1$ and corresponding two edges from T_+. Similarly, we get T'_- from T_-. See figure 7 for an example of this operation. We call this operation the *reduction of carets*. We say an element in $\mathcal{T}$ is *reduced* if there exists no such i. Define the equivalence relation as the one generated by reductions and its inverses. It is known that there exists a unique reduced representative for each equivalence class [4, §2].

Let F be the set of the equivalence classes of $\mathcal{T}$. We define the product on F as follows: Let $a = (A_+, A_-)$ and $b = (B_+, B_-)$ be in $\mathcal{T}$. By the previous operations, we get two element (A'_+, A'_-) and (B'_+, B'_-) which are equivalent to a and b, respectively, and $A'_- = B'_+$ holds. Then we define the product of the equivalent class of a and that of b as that of (A'_+, B'_-). This is well-defined, and F is termed *Thompson's group F*.

The following fact is well known.

Proposition 2.6 ([4, §2]). *Thompson's group F is isomorphic to the group consisting of homeomorphisms on the closed interval $[0,1]$ satisfying the following conditions:*

(1) *they are piecewise linear and preserve the orientation,*
(2) *in each linear part, its slope is a power of 2, and*
(3) *the breakpoints are in $\mathbb{Z}[\frac{1}{2}] \times \mathbb{Z}[\frac{1}{2}]$.*

Sketch of proof. Let T be a binary tree. We decompose $[0,1]$ by assigning a subinterval to each vertex of T. First, we consider the root to be $[0,1]$. Next, if a parent has $[a,b]$, then we set its left child has $[a,(a+b)/2]$ and its right child has $[(a+b)/2, b]$, inductively. As a result, the set of leaves of T gives the decomposition $[a_1, a_2], [a_2, a_3], \dots, [a_{n-1}, a_n]$ where $a_1 = 0$ and $a_n = 1$. See also Figure 8.

Let (T_+, T_-) be in $\mathcal{T}$. Since two trees have the same number of leaves, we get two decompositions $[a_1, a_2], [a_2, a_3], \dots, [a_{n-1}, a_n]$ and $[b_1, b_2], [b_2, b_3], \dots, [b_{n-1}, b_n]$. Therefore we get a piecewise linear map on $[0,1]$ by mapping each $[a_i, a_{i+1}]$ linearly to $[b_i, b_{i+1}]$. This induces an isomorphism. $\square$

We define virtual Thompson's group as a diagram group, but the following fact is helpful for understanding where its definition comes from.

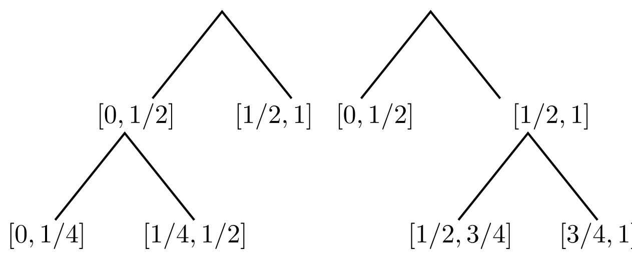


FIGURE 8. Vertices of a binary tree and subintervals

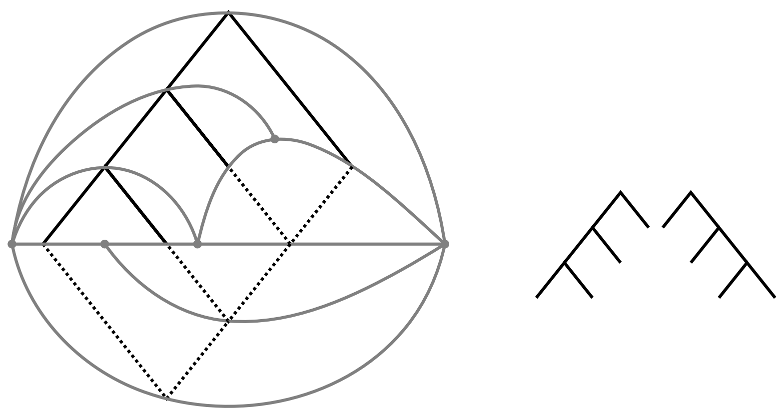


FIGURE 9. Example of the correspondence of a reduced diagram and a (reduced) pair of binary trees

PROPOSITION 2.7 ([10, Example 6.4], [6, Appendix]). *Let $\mathcal{P}_F = \langle x \mid x \to xx \rangle$ be a semigroup presentation. Then the diagram group $\mathcal{D}(\mathcal{P}_F, x)$ is isomorphic to the group F.*

Sketch of proof. Let Δ be a reduced (geometric) diagram in $\mathcal{D}(\mathcal{P}_F, x)$. We construct a pair of binary trees from Δ. This is achieved by associating each cell with a binary tree consisting of one parent and two children. Since each cell is of the form $x \to xx$ or $xx \to x$, we put a vertex on each edge and regard the vertex on the x-side (not xx-side) as the parent.

By performing the same operation for all cells, we obtain a graph. Let T_+ and T_- be the largest subgraphs whose roots are vertex on the top and bottom path, respectively. See Figure 9 and note that we omit the labels on all edges of diagrams since they are the same. Since Δ is reduced, the union of T_+ and T_- is the obtained graph, and their intersection is a set of finitely many vertices. Moreover, (T_+, T_-) is reduced.

Conversely, for given reduced (T_+, T_-), we can construct the diagram Δ by applying the inverse operation to the graph attaching corresponding leaves of T_- to T_+. It is easy to see that this operation yields an isomorphism. $\square$

In the following, we define the virtual version of Thompson's group F. The name "virtual" comes from virtual knot theory described in Section 2.1.

DEFINITION 2.8. Let $\mathcal{P}_{VF}$ be the following semigroup presentation:

$$\left\langle x, v \;\middle|\; \begin{array}{l} x \to xx, x \to vv, x \to vx, x \to xv \\ v \to vv, v \to xx, v \to vx, v \to xv \end{array} \right\rangle.$$

Then we define *virtual Thompson's group* VF to be the diagram group $\mathcal{D}(\mathcal{P}_{VF}, x)$.

REMARK 2.9. The group VF is motivated by virtual knot theory as an analogy of the virtual braid group. In this sense, this group is a "knot theoretic" Thompson's group, and seems to be algebraically different from the so-called "Thompson-like" groups.

2.4. **Properties of VF.** In this section, we list some properties of VF. The properties described below are already known to hold for diagram groups. See the respective references for details.

The following statements (1) and (2) follow from [6, Theorem 3.13, Theorem 4.1], (3) from [10, Theorem 15.25], (4) and (5) from [12, Theorem 6.1, Theorem 7.1], and (6) from [11, Theorem 9.9].

THEOREM 2.10. *Let $\mathcal{P}$ be a semigroup presentation and w be a given word.*

(1) *If $\mathcal{P}$ is a finite semigroup presentation, then $\mathcal{D}(\mathcal{P}, w)$ acts properly, cellularly, and freely by isometries on a proper* CAT(0) *cubical complex.*
(2) *If $\mathcal{P}$ is a finite semigroup presentation and the semigroup is finite, then the diagram group $\mathcal{D}(\mathcal{P}, w)$ is of type $\mathscr{F}_\infty$. Especially, $\mathcal{D}(\mathcal{P}, w)$ is finitely presented.*
(3) *The group $\mathcal{D}(\mathcal{P}, w)$ has the unique extraction of root property. Especially, $\mathcal{D}(\mathcal{P}, w)$ is torsion-free.*
(4) *The group $\mathcal{D}(\mathcal{P}, w)$ is totally orderable.*
(5) *The group $\mathcal{D}(\mathcal{P}, w)$ is residually countable.*
(6) *All integer homology groups of $\mathcal{D}(\mathcal{P}, w)$ are free abelian. Especially, the abelianization of the group $\mathcal{D}(\mathcal{P}, w)$ is free abelian.*

Here, for $n \geq 1$, a group G is of type $\mathscr{F}_n$ if there exists an aspherical CW-complex such that its fundamental group is isomorphic to G and it has finitely many n-skeleton. A group G is of type $\mathscr{F}_\infty$ if G is of type $\mathscr{F}_n$ for all $n \geq 1$.

Note that statement (1) has various corollaries such as satisfying the Haagerup property and the Baum–Connes conjecture. See [6, Section 3.4] for details.

We remark that we have $x = v$ as an element of the semigroup determined by $\mathcal{P}_{VF}$. Therefore, since $\mathcal{P}_{VF}$ is a finite presentation and $\mathcal{P}_{VF}$ determines a trivial semigroup, we have the following corollary:

COROLLARY 2.11. *The group VF has all the properties in Theorem* 2.10*.*

In addition, by using only the relation $x \to xx$ in the rewriting, we have the following:

PROPOSITION 2.12. *Thompson's group F is a subgroup of VF.*

In the rest of this section, we give an infinite presentation of VF by using the Squier complex. Let $\mathcal{P} = \langle \Sigma \mid \mathcal{R} \rangle$ be a semigroup presentation. The *Squier complex* $\mathcal{S}(\mathcal{P})$ of $\mathcal{P}$ is the 2-dimensional complex defined as follows:

- the 0-cells are the words on Σ;
- the 1-cells $e_{p,u\to v,q}$ connect two 0-cells from puq to pvq if $u \to v \in \mathcal{R}$; and
- the 2-cells $D_{p,u_1\to v_1,q,u_2\to v_2,r}$ bound the 4-cycles given by four 1-cells $e_{p,u_1\to v_1,qu_2r}$, $e_{pv_1q,u_2\to v_2,r}$, $e^{-1}_{p,u_1\to v_1,qv_2r} (= e_{p,v_1\to u_1,qv_2r})$, and $e^{-1}_{pu_1q,u_2\to v_2,r} (= e_{pu_1q,v_2\to u_2,r})$,

where $u_i \to v_i \in \mathcal{R}$ $(i = 1, 2)$ and p, q, r are words on Σ. Moreover, if $e_{p,u\to v,q}$ is an edge in $\mathcal{S}(\mathcal{P})$, then we have $e^{-1}_{p,u\to v,q} = e_{p,v\to u,q}$. For a given word w on Σ, the diagram group $\mathcal{D}(\mathcal{P}, w)$ can be regarded as the fundamental group $\pi_1(\mathcal{S}(\mathcal{P}), w)$. See [10, Section 6] or [7, Section 2] for details.

THEOREM 2.13. *Virtual Thompson's group VF admits the following infinite presentation:*

Generators:

- $X_{v\to vv}, X_{v\to vx}, X_{v\to xv}$,
- $X_{x,s\to t,u}, X_{v,s\to t,u}$ *($s \in \{x, v\}, t \in \{xx, xv, vx, vv\}$ and u is a word on Σ)*,
- $X_{x\to vv,u}, X_{x\to vx,u}, X_{v\to xx,u}, X_{v\to xv,u}$ *(u is a non-empty word on Σ)*.

Relations:

- $X_{x,s_1\to t_1,ps_2q}X_{x,s_2\to t_2,q} = X_{x,s_2\to t_2,q}X_{x,s_1\to t_1,pt_2q}$,
- $X_{v,s_1\to t_1,ps_2q}X_{v,s_2\to t_2,q} = X_{v,s_2\to t_2,q}X_{v,s_1\to t_1,pt_2q}$,
- $X_{x\to vv,psq}X_{v,s\to t,q} = X_{x,s\to t,q}X_{x\to vv,ptq}$,
- $X_{x\to vx,psq}X_{v,s\to t,q} = X_{x,s\to t,q}X_{x\to vx,ptq}$,
- $X_{v\to xx,psq}X_{x,s\to t,q} = X_{v,s\to t,q}X_{v\to xx,ptq}$,
- $X_{v\to xv,psq}X_{x,s\to t,q} = X_{v,s\to t,q}X_{v\to xv,ptq}$,

where $\Sigma = \{x, v\}$, $s, s_1, s_2 \in \{x, v\}, t, t_1, t_2 \in \{xx, xv, vx, vv\}$, and p, q are words in Σ.

PROOF. First, we choose a spanning tree T of the Squier complex $\mathcal{S}(\mathcal{P})$, that is, a subtree of $\mathcal{S}(\mathcal{P})$ which contains all vertices. Then the diagram group $\mathcal{D}(\mathcal{P}, w) \cong \pi_1(\mathcal{S}(\mathcal{P}), w)$ is generated by all edges subject to the following relations:

- $e = 1$ for any 1-cell $e \in T$,
- $e_1e_2 \cdots e_k = 1$ for any 2-cell $e_1e_2 \cdots e_k$.

In this case, we define a spanning tree T of $\mathcal{S}(\mathcal{P}_{VF})$ by the following edges:

- $e_{x\to xx}, e_{x\to xv}, e_{x\to vx}, e_{x\to vv}, e_{v\to xx}$,
- $e_{x\to xx,u}, e_{x\to xv,u}, e_{v\to vx,u}, e_{v\to vv,u}$ (u is a non-empty word on Σ)

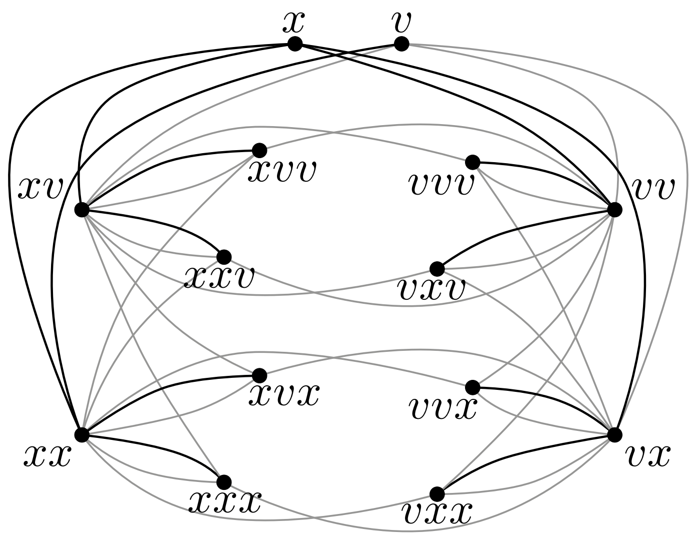


FIGURE 10. The Squier complex $\mathcal{S}(\mathcal{P}_{VF})$ corresponding to the words with up to three letters on Σ. The black edges are those of a spanning tree T.

Figure 10 shows the Squier complex corresponding to the words with up to three letters on Σ. We rewrite the letter $e_{p,s\to t,q}$ to $X_{p,s\to t,q}$. Then we obtain the generators of $\pi_1(\mathcal{S}(\mathcal{P}_{VF}),x)$ of the forms

- $X_{v\to vv}, X_{v\to vx}, X_{v\to xv}$,
- $X_{p,s\to t,u}$ $(s \in \{x,v\}, t \in \{xx,xv,vx,vv\}$, p is a non-empty word, and u is a word on $\Sigma)$,
- $X_{x\to vv,u}, X_{x\to vx,u}, X_{v\to xx,u}, X_{v\to xv,u}$ (u is a non-empty word on Σ).

By the definition of the 2-cells, we have

$$X_{p,s_1\to t_1,qs_2r}X_{pt_1q,s_2\to t_2,r} = X_{ps_1q,s_2\to t_2,r}X_{p,s_1\to t_1,qt_2r},$$

where $s_1, s_2 \in \{x,v\}, t_1, t_2 \in \{xx,xv,vx,vv\}$, and p,q,r are words on Σ. If $p = 1, s_1 = x$ ans $t_1 = xx$, then we obtain

$$X_{x\to xx,qs_2r}X_{xxq,s_2\to t_2,r} = X_{xq,s_2\to t_2,r}X_{x\to xx,qt_2r}.$$

Since $e_{x\to xx,qs_2r}$ and $e_{x\to xx,qt_2r}$ are edges of the spanning tree T, they are trivial in $\pi_1(\mathcal{S}(\mathcal{P}_{VF}),x)$, and thus $X_{xxq,s_2\to t_2,r} = X_{xq,s_2\to t_2,r}$. In general, we obtain a relation $X_{x,s\to t,r} = X_{xq,s\to t,r}$ for any word q. Similarly, we have $X_{v,s\to t,r} = X_{vq,s\to t,r}$, and they are the second generators of the presentation in the theorem. By using these relations, if $p = x$, then we are able to rewrite the relation

$$X_{x,s_1\to t_1,qs_2r}X_{xt_1q,s_2\to t_2,r} = X_{xs_1q,s_2\to t_2,r}X_{x,s_1\to t_1,qt_2r}$$

to

$$X_{x,s_1\to t_1,qs_2r}X_{x,s_2\to t_2,r} = X_{x,s_2\to t_2,r}X_{x,s_1\to t_1,qt_2r},$$

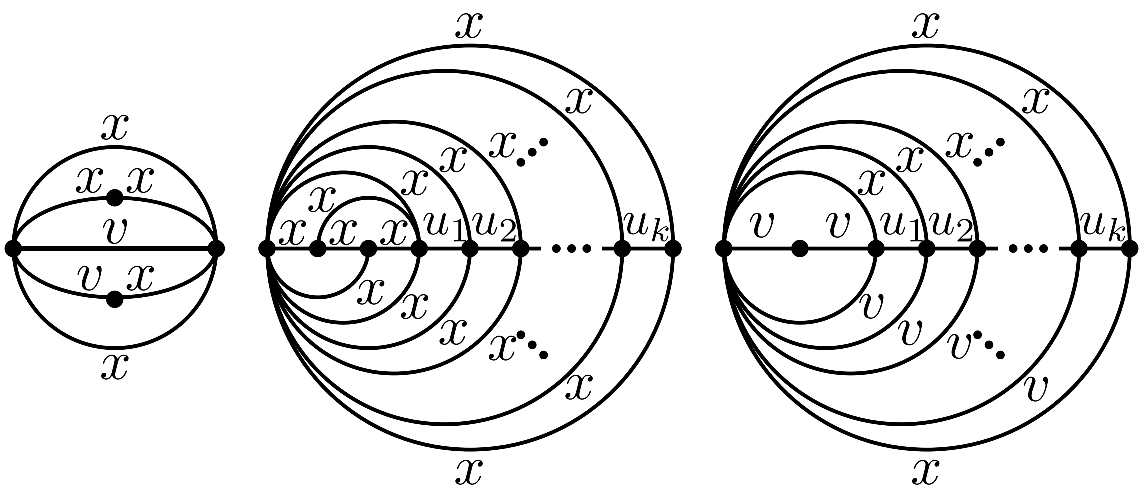


FIGURE 11. The generators $X_{v\to vx}, X_{x,x\to xx,u}$ and $X_{x\to vv,u}$ where u is a word $u_1u_2\cdots u_k$.

which is the first relation of the presentation. Similarly, we have

$$X_{v,s_1\to t_1,qs_2r}X_{v,s_2\to t_2,r} = X_{v,s_2\to t_2,r}X_{v,s_1\to t_1,qt_2r},$$

which is the second relation. Moreover, if $p = 1, s_1 = x$, and $t_1 = vv$, then

$$X_{x\to vv,qs_2r}X_{vvq,s_2\to t_2,r} = X_{xq,s_2\to t_2,r}X_{x\to vv,qt_2r},$$

and thus

$$X_{x\to vv,qs_2r}X_{v,s_2\to t_2,r} = X_{x,s_2\to t_2,r}X_{x\to vv,qt_2r},$$

which is the third relation. Similarly, we obtain the fourth to sixth relations. □

For example, substituting $s_i = x, t_i = xx$ $(i = 1, 2)$, $p = x^j$, and $q = x^k$ $(j, k \geq 0)$ in the first relation, we have

$$X_{x,x\to xx,x^jxx^k}X_{x,x\to xx,x^k} = X_{x,x\to xx,x^k}X_{x,x\to xx,x^jxxx^k}.$$

Set $x_k := X_{x,x\to xx,x^k}$, then we rewrite this relation as follows:

$$x_{j+k+1}x_k = x_kx_{j+k+2},$$

that is,

$$x_nx_k = x_kx_{n+1} \ (0 \leq k < n),$$

which is exactly the relation for Thompson's group F.

From the presentation in Theorem 2.13, three generators $X_{v\to vv}, X_{v\to vx}$ and $X_{v\to xv}$ have no relations. Therefore, the virtual Thompson's group VF is the free product of the free group of rank 3 generated by these generators and the remaining part of VF.

Finally, the generators of VF can be described as the geometric diagrams shown in Figure 11.

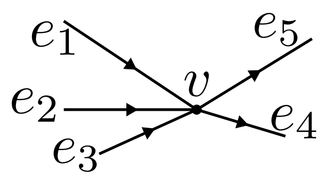


FIGURE 12. The edge e_1 is the first incoming edge and e_3 is the last incoming edge of v.

3. THE CONSTRUCTION OF VIRTUAL LINKS FROM VIRTUAL THOMPSON'S GROUP

3.1. **The construction.** In this section, we explain the construction of a virtual link from an element of virtual Thompson's group VF with an example. This construction is based on [13] and [8].

Step 1: Construct the Thompson graph $T(\Delta)$.

Let Δ be a reduced diagram in $VF = \mathcal{D}(\mathcal{P}_{VF}, x)$. We define the *Thompson graph* $T(\Delta)$ as a "subgraph" of the diagram Δ as follows (cf. [8, Definition 3.2]): the vertices of $T(\Delta)$ are all vertices of Δ except the terminal vertex. In order to define the edges of $T(\Delta)$, we use the following lemma.

LEMMA 3.1 ([10, Lemma 3.7]). *For any inner vertex v of Δ, that is, the vertex which does not coincide with the initial vertex nor the terminal vertex, there uniquely exists a sequence $e_1, \dots, e_n$ of edges with endpoint v in the counterclockwise order such that for some k $(1 \leq k < n)$, edges $e_1, \dots, e_k$ are incoming and edges $e_{k+1}, \dots, e_n$ are outgoing (see Figure 12).*

For any inner vertex v of Δ, we assign numbers to edges with endpoint v as in Lemma 3.1. The edges of $T(\Delta)$ are the first and the last incoming edges with respect to the order for each inner vertex of Δ. If the first and the last edges of v coincide, that is, v has exactly one incoming edge, then we make a copy of the incoming edge labeled by the same letter. Therefore, any inner vertex of $T(\Delta)$ has two incoming edges. Figure 13 is an example of this step.

Step 2: Construct the medial graph $M(T(\Delta))$.

The medial graph is defined for any connected plane graph. Let G be a connected plane graph, and then its medial graph $M(G)$ is obtained as follows: we put a vertex of $M(G)$ on every edge of G, and join two new vertices by an edge if the corresponding edges of G are adjacent on a face of G. Figure 14 is an example of this step.

Step 3: Construct the virtual link diagram $L(\Delta)$.

In general, because the medial graph is 4-valent, we are able to obtain a virtual link diagram $L(\Delta)$ by turning all vertices of $M(T(\Delta))$ into classical or virtual crossings: for a

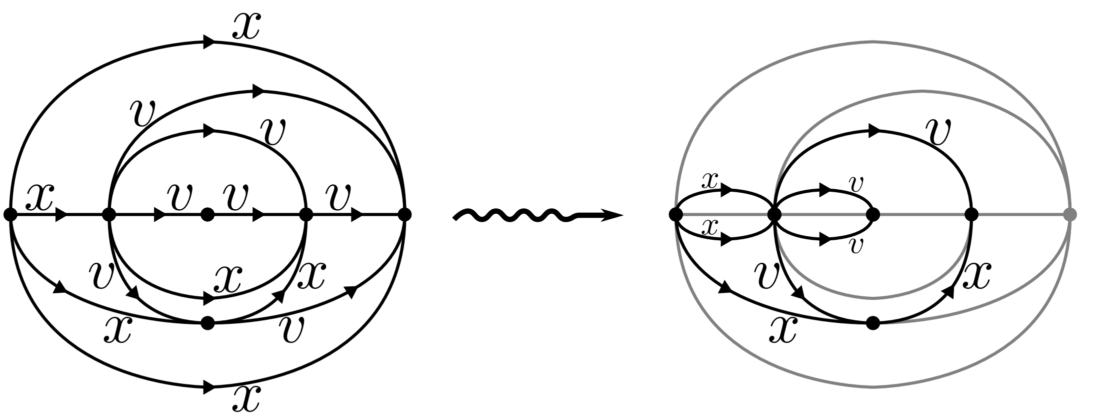


FIGURE 13. An example of the Thompson graph $T(\Delta)$

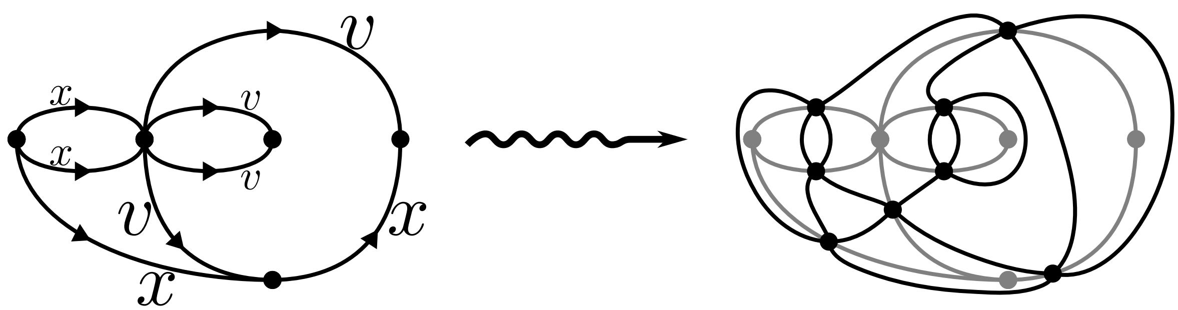


FIGURE 14. An example of the medial graph $M(T(\Delta))$

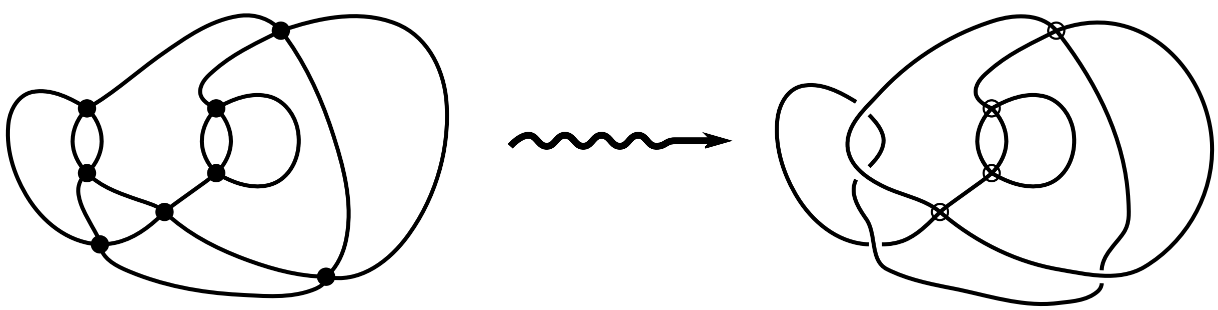


FIGURE 15. An example of the virtual link diagram $L(\Delta)$

vertex of $M(T(\Delta))$, if the corresponding edge in $T(\Delta)$ is

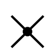
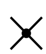
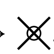
the first and labeled by x, then  $\to$ ,
the last and labeled by x, then  $\to$ , or (3.1)
labeled by v, then  $\to$ .

Figure 15 is an example of this step.

3.2. **Labeled binary trees.** In this section, we discuss the relationship between elements of VF and labeled binary trees. Suppose that the diagram $\Delta : x = w_1 \to w_2 \to \cdots \to$

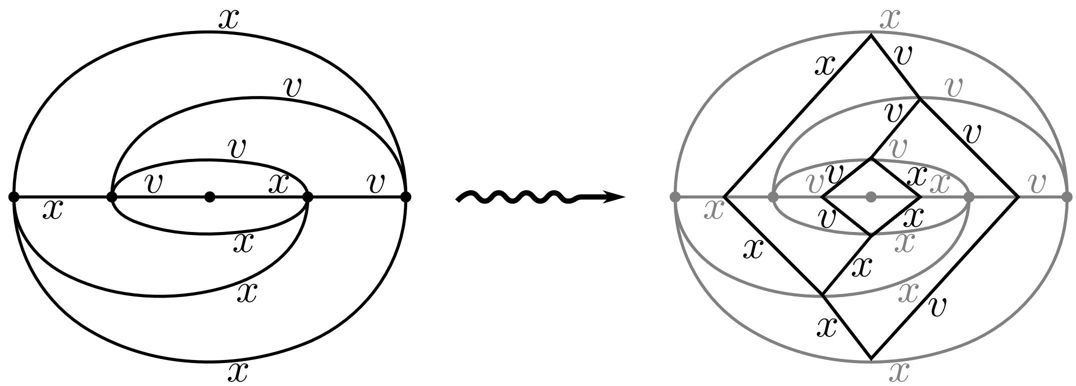


FIGURE 16. An example of the correspondence of a diagram and a pair of labeled binary trees.

$w_{n-1} \to w_n = x$ satisfies the following condition:

$$\text{There uniquely exists } i \in \mathbb{Z} \cap [1, n] \text{ such that } \begin{cases} |w_j| < |w_{j+1}| & (1 \le j < i) \\ |w_j| > |w_{j+1}| & (i \le j < n) \end{cases} \text{ hold,} \quad (3.2)$$

where $|\cdot|$ denotes the length of a word. Geometrically, this condition implies that all vertices of Δ can be placed on a straight line, and every vertex except the initial vertex has an incoming edge connected to its immediate left one. The path of Δ on the straight line connecting all the vertices is exactly the trivial geometric diagram of w_i. Then the cell $w_j \to w_{j+1}$ for $1 \le j < i$ is the (s, t)-cell and the cell $w_j \to w_{j+1}$ for $i \le j < n$ is the (t, s)-cell, where $s \in \{x, v\}$ and $t \in \{xx, xv, vx, vv\}$. In this case, the first and last edges coincide with the top-most and bottom-most incoming edges of [8, Definition 3.2], respectively. Similarly to the proof of Proposition 2.7, the diagram Δ can be described as a pair (T_+, T_-) of labeled binary trees with the same number of leaves. The label of each edge is determined by the one of the corresponding "child" edge of the cell in Δ. We give an example in Figure 16.

On the other hand, Jones [13] introduced a method of constructing a link diagram from an element of F by using a pair of binary trees. In the case above, this construction can be extended naturally. Let (T_+, T_-) be a pair of reduced labeled binary trees with $n+1$ leaves obtained from an element of VF, and place its leaves at $\left(\frac{1}{2}, 0\right), \left(\frac{3}{2}, 0\right), \ldots, \left(\frac{2n+1}{2}, 0\right)$. Note that the tree T_+ is in the upper half-plane, and T_- is in the lower half-plane. The plane graph $\Gamma(T_+, T_-)$, which is called the Γ-graph of (T_+, T_-), is defined uniquely up to ambient isotopy on the 2-sphere $\mathbb{S}^2 (= \mathbb{R}^2 \cup \{\infty\})$ as follows: the vertices of $\Gamma(T_+, T_-)$ are put at $(0, 0), (1, 0), \ldots, (n, 0)$. An edge of $\Gamma(T_+, T_-)$ passes transversely just once an edge $/$ of T_+ (i.e., an edge from top right to bottom left) or an edge $\backslash$ of T_- (i.e., an edge from top left to bottom right) and does not do the other edges of (T_+, T_-). Every edge is labeled by x or v corresponding to the label of an edge of (T_+, T_-). We illustrate an example in Figure 17.

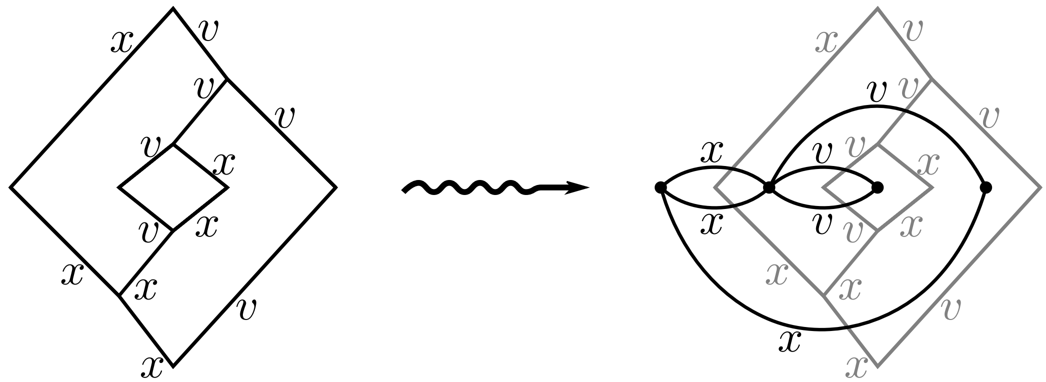


FIGURE 17. An example of the correspondence of a pair of labeled binary trees and a Γ-graph.

For the two constructions, the following holds:

PROPOSITION 3.2 (cf. [8, Proposition 3.5]). *Let* Δ *be a diagram of* VF *satisfying condition* (3.2) *and* (T_+, T_-) *the pair of labeled binary trees obtained from* Δ. *Then the Thompson graph* $T(\Delta)$ *is isomorphic to the* Γ*-graph* $\Gamma(T_+, T_-)$.

PROOF. Let Δ be a diagram in VF. By forgetting the labels x and v of the edges in Δ, we obtain the (possibly non-reduced) diagram $\widetilde{\Delta}$ in F. When the diagram $\widetilde{\Delta}$ is reduced, this proposition is already proved by Golan and Sapir [8] by stretching the edges of $T(\widetilde{\Delta})$ upward. In general, $\widetilde{\Delta}$ is not reduced but satisfies condition (3.2), and thus this diagram can also be described as a pair $(\widetilde{T_+}, \widetilde{T_-})$ of (non-labeled) binary trees with the same number of leaves. Then, we can use the argument of Golan and Sapir, and prove this proposition. Putting the labels on the edges of $(\widetilde{T_+}, \widetilde{T_-})$ (see Figure 16), we obtain the pair (T_+, T_-) of labeled binary trees of Δ. Moreover, the correspondence of the labels of the edges of $T(\Delta)$ and $\Gamma(T_+, T_-)$ is clear from the construction. We illustrate an example in Figure 18. □

4. PROOF OF THEOREM 1.1

Theorem 1.1 states that every virtual link can be described as the virtual link diagram $L(\Delta)$ for an element Δ of VF. In this section, we prove this theorem. The procedure of the proof is based on [13]. In fact, we are always able to choose an element of VF representing a given virtual link which satisfies condition (3.2).

Let L be a virtual link diagram.

Step 1: Construct the Tait graph $T(L)$.

We apply the checkerboard coloring to the diagram L, that is, we paint regions of L with black or white so that adjacent regions are different colors. By convention, the color of the unbounded region is white. The vertices of the Tait graph $T(L)$ correspond to the

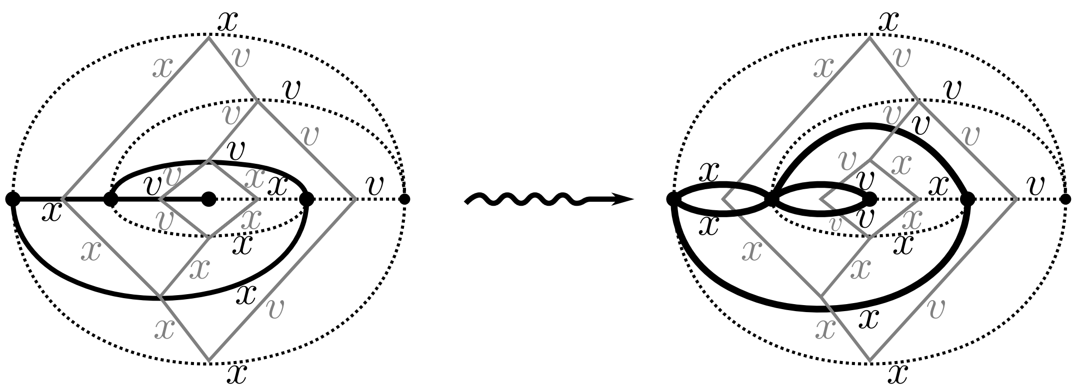


Figure 18. An example of the correspondence in Proposition 3.2. By stretching the edges of $T(\Delta)$ upward, it is isomorphic to the graph $\Gamma(T_+, T_-)$. The gray letters are labels of binary trees.

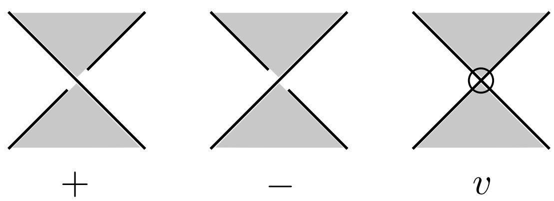


Figure 19. The labels of crossings

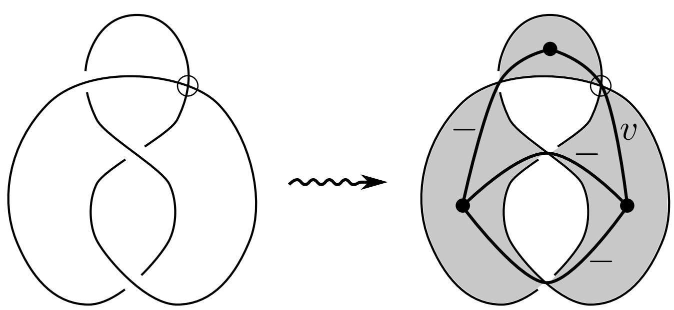


Figure 20. An example of the checkerboard coloring and the Tait graph

black regions of L, and the edges correspond to the crossings and are labeled by $+$, $-$, or v according to the rule in Figure 19. We give an example of the Tait graph in Figure 20.

Step 2: Deform the graph $T(L)$.

Jones [13] gave a sufficient condition for a connected plane graph to be obtained from an element (T_+, T_-) of F. Due to the work of Golan and Sapir [8], this is interpreted in terms of diagrams (cf. Proposition 2.7). Combining these two, the following holds:

Lemma 4.1 ([13, Lemma 4.1.4], [8, Proposition 3.5]). *Let Γ be a connected plane graph. Suppose that Γ consists of two trees, Γ_+ in the upper half-plane and Γ_- in the lower half-plane, and these two trees satisfy the following properties:*

(1) *the vertices are* $(0,0),(1,0),\ldots,(n,0)$,
(2) *each vertex other than* $(0,0)$ *is connected to exactly one vertex to its left one, and*
(3) *each edge can be parametrized by a smooth curve* $(x(t),y(t))$ *for* $t\in[0,1]$ *with* $x'(t)>0$ *and either* $y(t)>0$ *or* $y(t)<0$ *for* $t\in(0,1)$.

Then there exists an reduced element (T_+,T_-) *of* F *such that* $\Gamma(T_+,T_-)$ *is isomorphic to* Γ. *Equivalently, there exists a reduced diagram* Δ *of* F *such that* $T(\Delta)$ *is isomorphic to* Γ.

For the virtual case, the Thompson graph has labels x or v. In particular, if there exists a vertex of a diagram Δ with exactly one incoming edge, then it has two incoming edges with the same labels in the Thompson graph $T(\Delta)$. Hence, we obtain the condition for the virtual version:

Lemma 4.2. *Let* Γ *be a connected plane graph with each edge labeled by* x *or* v. *Suppose that* Γ *consists of two trees,* Γ_+ *in the upper half-plane and* Γ_- *in the lower half-plane, and these two trees satisfy the properties* (1), (2) *and* (3) *in Lemma* 4.1. *Moreover, assume that* Γ *satisfies the following condition:*

(4) *Two edges connecting adjacent two vertices have the same labels.*

Then there exists a pair (T_+,T_-) *of labeled binary trees in* VF *such that* $\Gamma(T_+,T_-)$ *is isomorphic to* Γ. *Equivalently, there exists an element* Δ *of* VF *satisfying condition* (3.2) *such that* $T(\Delta)$ *is isomorphic to* Γ.

For a diagram Δ satisfying condition (3.2), the Tait graph of $L(\Delta)$ (i.e., the Thompson graph $T(\Delta)$) satisfies the conditions of Lemma 4.2, with edges in the upper half-plane labeled by $+$ or v and edges in the lower half-plane labeled by $-$ or v. Therefore, in order to prove the main theorem, we apply the Reidemeister moves on the given Tait graph so that the deformed graph satisfies the condition of Lemma 4.2.

We recall some local moves on the labeled plane graph corresponding to the Reidemeister moves R1 and R2 (see Figure 21).

Definition 4.3 ([13, Definition 5.3.4]). Two labeled plane graphs are *2-equivalent* if they differ by planar isotopies and any of the moves R1, R2a, and R2b.

The moves R1, R2a, and R2b on the labeled plane graph correspond to the Reidemeister moves R1 and R2 on the virtual link diagram, respectively. Therefore, let L and L' be virtual link diagrams. If the Tait graphs $T(L)$ and $T(L')$ are 2-equivalent, then L and L' are equivalent.

Lemma 4.4 ([13, Lemma 5.3.6]). *Any Tait graph is* 2*-equivalent to a plane graph satisfying conditions* (1) *and* (3) *in Lemma* 4.1.

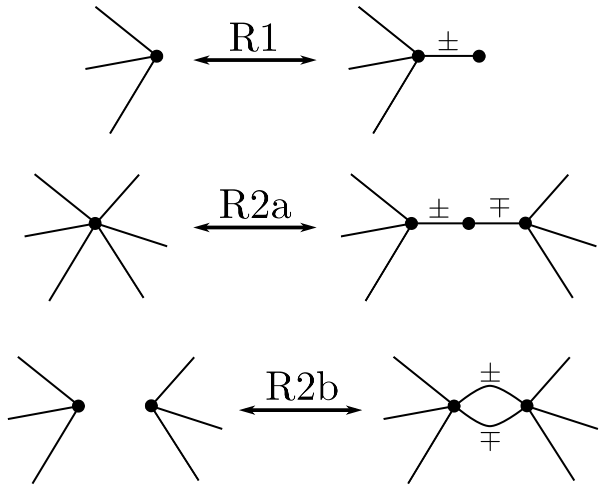


FIGURE 21. The Reidemeister moves R1 and R2 on the plane graph

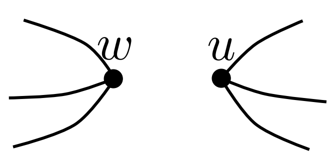
Therefore, we may assume that the Tait graph $T(L)$ satisfies the above conditions. Such plane graph is said to be *standard*. Suppose that the edges of a standard plane graph are oriented from left to right. We recall some notations in [13].

DEFINITION 4.5 ([13, Definition 5.3.7 and 5.3.8]). For a vertex u of $T(L)$, we set

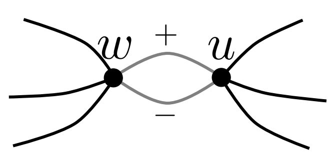
$$\begin{aligned} e^{\mathrm{up}} &:= \{e \in E(T(L)) \mid e \text{ lies in the upper half-plane}\}, \\ e^{\mathrm{down}} &:= \{e \in E(T(L)) \mid e \text{ lies in the lower half-plane}\}, \\ e_u^{\mathrm{in}} &:= \{e \in E(T(L)) \mid \tau(e) = u\}, \\ e_u^{\mathrm{out}} &:= \{e \in E(T(L)) \mid \iota(e) = u\}, \end{aligned}$$

where $E(T(L))$ is the set of all edges of $T(L)$, and $\tau(e)$ and $\iota(e)$ are the terminal and initial vertices of e, respectively.

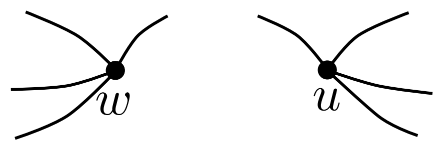
Case 1. There exists a vertex u different from $(0,0)$ with $e_u^{\mathrm{in}} = \emptyset$. Let w be the vertex immediately to the left of u as below:



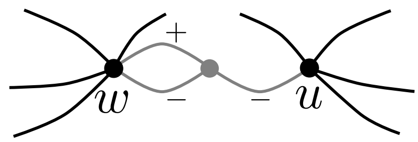
We add two edges connecting w and u so that the deformed graph is 2-equivalent to the original graph:



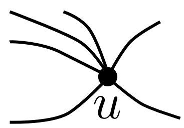
Case 2. There exists a vertex u with $|e_u^{\mathrm{in}}| = 1$. We may assume that the incoming edge of u is in the upper half-plane. This is labeled by $+$, $-$, or v. The situation near u is as below:



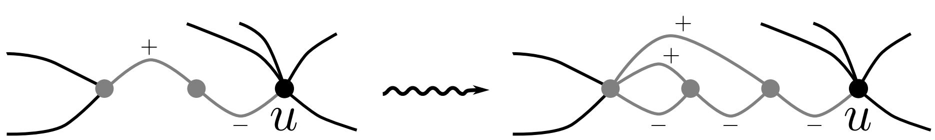
Then we add one vertex and three edges as below:



Case 3. There exists a vertex u with $|e_u^{\mathrm{in}} \cap e^{\mathrm{up}}| > 1$ or $|e_u^{\mathrm{in}} \cap e^{\mathrm{down}}| > 1$. We may show only the first case, and the other case is similar. The situation near u is below:



Then we add three vertices and five edges as below:

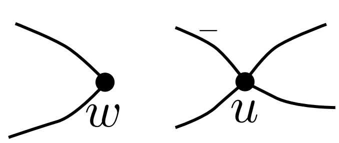


We can repeat this operation until the vertex u satisfies $|e_u^{\mathrm{in}} \cap e^{\mathrm{up}}| = 1$.

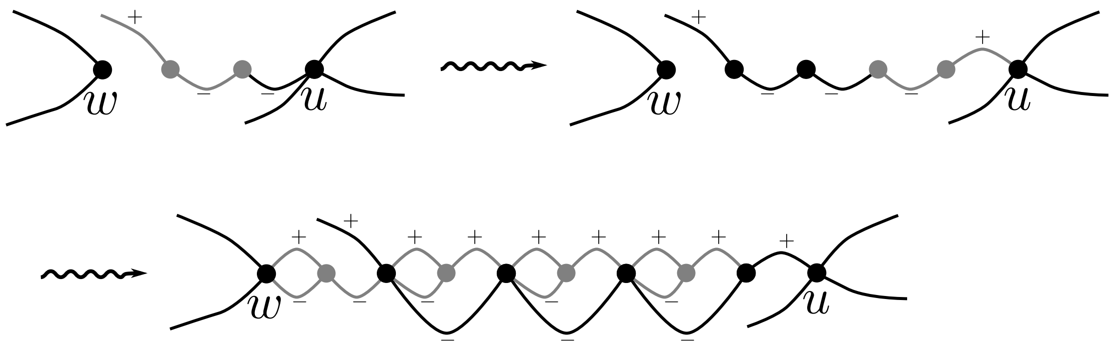
Case 4. After applying the previous deformations, all vertices, except the vertex $(0, 0)$, have two incoming edges, one in the upper half-plane and the other in the lower half-plane. Hence, this graph satisfies condition (2) in Lemma 4.1. Then we may have two problems that

(i) a $-$-labeled edge is in the upper half-plane or a $+$-labeled edge is in the lower half-plane, and
(ii) two edges connecting the adjacent two vertices have labels $+$ and v, or v and $-$, respectively.

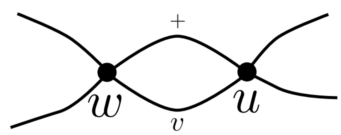
We consider the first problem, and we may show only the case of $-$-labeled edge in the upper half-plane. This situation looks like:



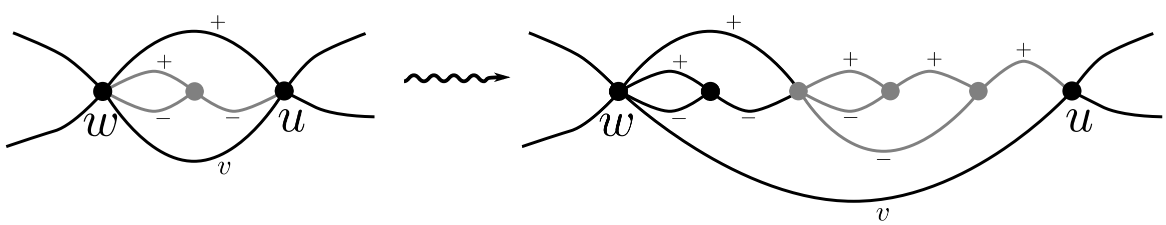
Then we apply the following deformation



Next, we consider the second problem. We may show only the case that two labels are $+$ and v, respectively. Suppose that such two edges connect the adjacent vertices w and u, then the situation looks like:



Then we apply the following deformations:



From the above, the proof of Theorem 1.1 is complete.

Example 4.6. Figure 22 shows the application of the algorithm to the virtual knot 3.1 in the list[1] by Jeremy Green. Its last figure is a diagram of VF representing 3.1. The top and bottom paths must be labeled by x from the definition. However, other than those edges, gray edges can be labeled by either x or v.

Since a virtual link is an immersion of circles, its orientation is induced from the one of each circle. Jones defined a subgroup $\overrightarrow{F}$ of F which is called oriented Thompson's group. This group consists of all pairs of binary trees whose Γ-graphs are 2-colorable, and its element yields an oriented link. Aiello [1] proved Alexander's theorem for the oriented case by using another local move. From [8, Lemma 4.1], we are able to define a subgroup $\overrightarrow{VF}$ of VF consisting of all diagrams whose Thompson graphs are 2-colorable. Moreover, by using Aiello's move, the oriented version of Theorem 1.1 can be proved similarly.

Theorem 4.7. *Any oriented virtual link can be obtained from an element in* $\overrightarrow{VF}$.

[1] `http://www.math.toronto.edu/~drorbn/Students/GreenJ/`

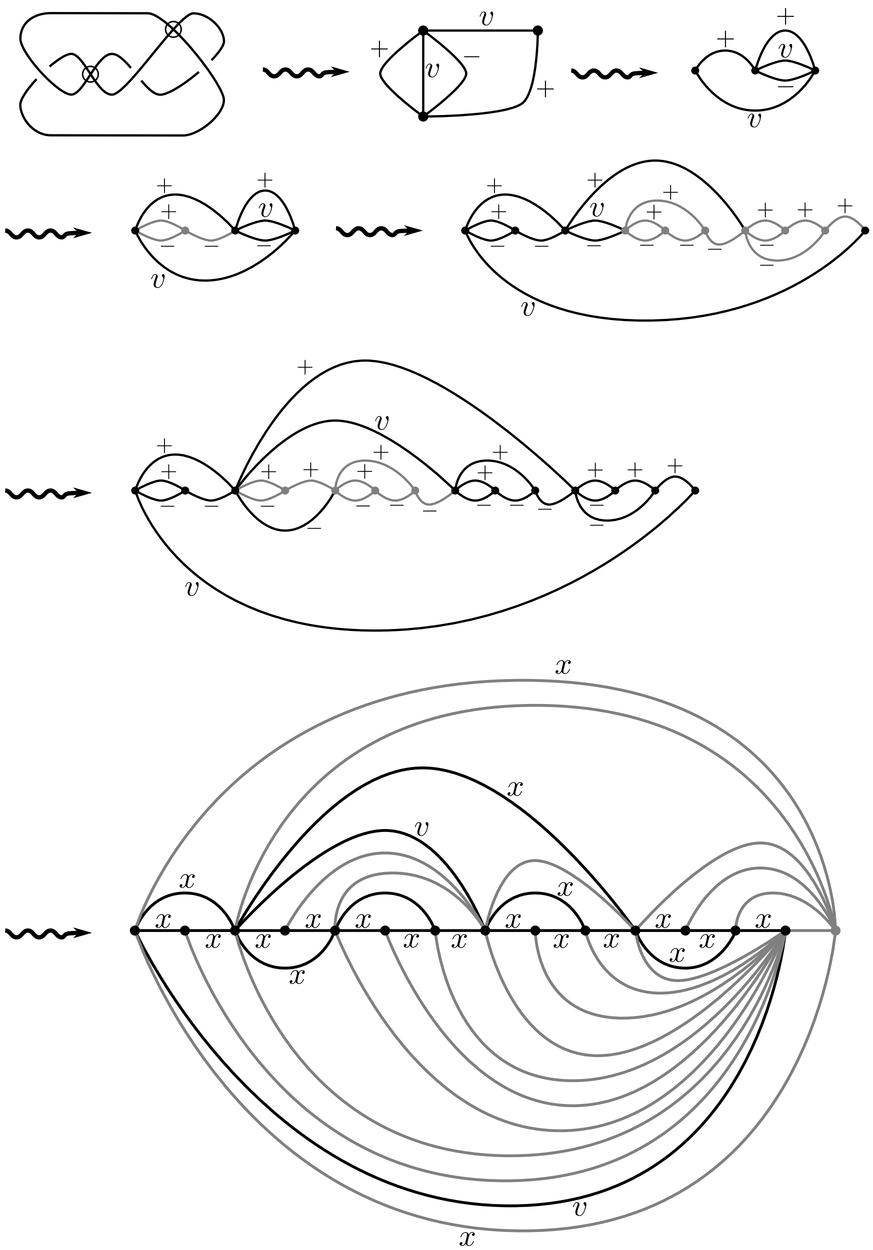


Figure 22

Finally, Golan and Sapir [8] showed that oriented Thompson's group $\overrightarrow{F}$ is isomorphic to the Brown–Thompson group $F(3)$, which is a diagram group especially. In general, a subgroup of the diagram group is not always a diagram group, and thus there is a natural problem whether $\overrightarrow{VF}$ is a diagram group or not.

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