ON THE GROUP OF SELF-HOMOTOPY EQUIVALENCES OF AN F_0-SPACE

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Abstract. G. Lupton conjectured that the group of self-homotopy equivalences of an F_0-space inducing the identity on the homotopy groups is finite. Thus, the aim of this paper is to establish this conjecture.

1. Introduction

Let X be a simply connected space with finite dimensional rational homotopy, finite dimensional rational homology (i.e. a rationally elliptic space), and positive Euler characteristic. The collection of such spaces X is referred to as the class of F_0-spaces. Extensively studied by Halperin, [5], F_0-spaces are rational Poincaré duality spaces with rational cohomology $\mathbb{Q}[x_1,\dots,x_n]/(P_1,\dots,P_n)$, where the polynomials $P_1,\dots,P_n$ form a regular sequence in $\mathbb{Q}[x_1,\dots,x_n]$, i.e., $P_1\neq 0$ and for every $i\geq 1$, P_i is not a zero divisor in $\mathbb{Q}[x_1,\dots,x_n]/(P_1,\dots,P_{i-1})$. For instance, products of even spheres, complex Grassmannian manifolds and homogeneous spaces G/H such that rank G = rank H are F_0-spaces.

Let $\mathcal{E}(X)$ denote the group of self-homotopy equivalences of X and let $\mathcal{E}_{\#}(X)$ be its subgroup of the elements inducing the identity on the homotopy groups ([3],[2]).

Halperin has conjectured that the rational Serre spectral sequence collapses for any rational fibration, provided the fiber X is a F_0-space. This conjecture, which remains unsolved, can be rephrased in terms of the (graded Lie algebra of) negative-degree derivations of the rational cohomology of X (see [8] for more details). Namely:

$$Der_{<0}H^*(X;\mathbb{Q})=0 \Longleftrightarrow \text{ Halperin's conjecture holds}$$

If we look at the zero-degree derivations of the rational cohomology of X, there exists a correspondence between the decomposable derivations of $Der_0H^*(X;\mathbb{Q})$ and the subgroup $\mathcal{E}_{\#}(X)$. Hence,

$$Der_0H^*(X;\mathbb{Q}) \text{ is trivial} \Longrightarrow \mathcal{E}_{\#}(X) \text{ is finite}$$

Motivated by Halperin's conjecture and this correspondence, Lupton raises the following question:

Question([1], Problem 10): For an F_0-space X, is $\mathcal{E}_{\#}(X)$ finite?

Thus, the purpose of this paper is to settle this question in the positive using standard tools of rational homotopy theory which we refer to [4] for a general introduction to these techniques. We recall some of the notation here. By a Sullivan algebra we mean a free graded commutative algebra ΛV, for some finite-type graded vector space $V=V^{\geq 2}$, i.e., $\dim V^n<\infty$ for all $n\geq 2$, together with a differential ∂ of degree $+1$ that is decomposable, i.e., satisfies $\partial: V\to\Lambda^{\geq 2}V$. Here $\Lambda^{\geq 2}V$ denotes the graded vector space spanned by all the monomials $v_1\dots v_r$ such that $v_1,\dots,v_r\in V$ and $r\geq 2$.

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Every simply connected space X with rational cohomology of finite-type has a corresponding Sullivan algebra called the Sullivan model of X, unique up to isomorphism, that encodes the rational homotopy type of X. In particular we have

$$V^* \cong \mathrm{Hom}(\pi_*(X) \otimes \mathbb{Q}, \mathbb{Q}), \qquad H^*(\Lambda V) \cong H^*(X, \mathbb{Q}).$$

By a free differential graded Lie algebra $(\mathbb{L}(W), \delta)$ (DGL for short), we mean a free graded Lie algebra $\mathbb{L}(W)$, for some finite-type vector space $W = (W_{\geq 1})$, together with a decomposable differential δ of degree -1, i.e., $\delta(W) \subset \mathbb{L}^{\geq 2}(W)$. Here $\mathbb{L}^{\geq 2}(W)$ denotes the graded vector space spanned by all the brackets of lengths ≥ 2.

Every simply connected space X with rational cohomology of finite-type has a corresponding DGL $(\mathbb{L}(W), \delta)$ called the Quillen model of X, unique up to isomorphism, which determines completely the rational homotopy type of X. In particular we have

$$W_* \cong H_{*+1}(X; \mathbb{Q}), \qquad H_*(\mathbb{L}(W)) \cong \pi_{*+1}(X) \otimes \mathbb{Q}.$$

This work consists of five sections, the first one being the introduction. Section 2 is devoted to state some results on the notion of DGL-homotopy as well as the properties of an F_0-space X notably, if $(\mathbb{L}(W), \delta)$ is its Quillen model, then we introduce the group $\mathcal{E}_\#(\mathbb{L}(W))$ of the self-homotopy equivalences of $(\mathbb{L}(W), \delta)$ constituting on the elements $[\alpha]$ satisfying $H_*(\alpha) = \mathrm{id}$. In Section 3, we prove that if $[\alpha] \in \mathcal{E}_\#(\mathbb{L}(W))$, then α is homotopic to DGL-map $\tilde{\alpha}$ satisfying $\tilde{\alpha}(W) = W$. In Section 4 and 5, we focus on studying the properties of $(\mathbb{L}(W), \delta)$ to show that $\mathcal{E}_\#(\mathbb{L}(W))$ is trivial. Consequently, by virtue of the localization theorem of Maruyama [7], we derive that $\mathcal{E}_\#(X)$ is finite.

2. Preliminaries

2.1. **Homotopy between DGL-maps (see [4, §21]).** Let $(\mathbb{L}(W), \delta)$ be a DGL. Define the DGL $\mathbb{L}(W, sW, W'), D)$ with $W \cong W'$ and $(sW)_i = W_{i-1}$. The differential D is given by

$$D(w) = \partial(w), \qquad D(sw) = w', \qquad D(w') = 0. \tag{1}$$

Define S as the derivation of degree $+1$ on $\mathbb{L}(W, sW, W')$ given by

$$S(w) = sw, \qquad S(sw) = S(w') = 0.$$

A homotopy between two DGL-maps $\alpha, \alpha' : (\mathbb{L}(W), \delta) \to (\mathbb{L}(W), \delta)$ is DGL-map

$$F \colon (\mathbb{L}(W, sW, W'), D) \to (\mathbb{L}(W), \delta),$$

such that $F(w) = \alpha(w)$ and $F \circ e^\theta(w) = \alpha'(w)$, where

$$e^\theta(w) = w + w' + \sum_{n \geq 1} \frac{1}{n!}(S \circ D)^n(w), \quad \text{and} \quad \theta = D \circ S + S \circ D.$$

Thus, the notion of DGL-homotopy allows us to define the following group.

Definition 2.1. Let $\mathcal{E}_\#(\mathbb{L}(W))$ denote the group of self-homotopy equivalences of $(\mathbb{L}(W), \delta)$ constituting with the elements $[\alpha]$ satisfying $H_*(\alpha) = \mathrm{id}$, where

$$H_*(\alpha) : H_*(\mathbb{L}(W)) \to H_*(\mathbb{L}(W)).$$

By virtue of the properties of the model of Quillen and the localization theorem of Maruyama [7], we deduce that if X is an F_0-space, then we have

$$\mathcal{E}_\#(X) \otimes \mathbb{Q} \cong \mathcal{E}_\#(\mathbb{L}(W)). \tag{2}$$

Thus, the group $\mathcal{E}_\#(X)$ is finite if and only if the group $\mathcal{E}_\#(\mathbb{L}(W))$ is trivial.

Later on we will need the following two lemmas.

Lemma 2.2. *Let $\alpha, \tilde{\alpha}\colon (\mathbb{L}(W_{\leq n}), \delta) \to (\mathbb{L}(W_{\leq n}), \delta)$ be two DGL-maps such that*

$$\alpha(w) = \tilde{\alpha}(w) + y \ \text{ on } W_n \quad \text{and} \quad \alpha = \tilde{\alpha} \ \text{ on } W_{\leq n-1}.$$

Assume that $y = \delta(z)$, where $z \in \mathbb{L}(W_{\leq n})$. Then α and $\tilde{\alpha}$ are homotopic.

Proof. Define $F\colon (\mathbb{L}(W_{\leq n}, sW_{\leq n}, W'_{\leq n}), D) \to (\mathbb{L}(W_{\leq n}), \delta)$ by setting

$$\begin{aligned} F(w) &= \alpha(w), \quad F(w') = -y \ \text{ and } \ F(sw) = -z \ \text{ for } \ w \in W_n, \\ F(w) &= \alpha(w), \quad F(w') = 0 \ \text{ and } \ F(sw) = 0 \ \text{ for } \ w \in W_{\leq n-1}. \end{aligned} \tag{3}$$

Let $w \in W_n$, by considering the relations (1), (3) and as $\delta(w) \in \mathbb{L}(W_{\leq n-1})$, we get

$$\delta F(w) = \delta\alpha(w), \qquad FD(w) = F(\delta(w)) = \alpha\delta(w).$$

Moreover, a straightforward computation shows

$$\begin{aligned} \delta F(w') &= \delta(-y) = -\delta(\delta(z)) = 0, \qquad & FD(w') = F(0) = 0, \\ \delta F(sw) &= \delta(-z) = -y, & FD(sw) = F(w') = -y, \end{aligned}$$

implying that F is a DGL-map. Next, on the one hand, from (3), we have $F(w) = \alpha(w)$ for every $w \in W$. On the other hand, by expanding the expression $(S \circ \partial)^n(w)$ leads to linear combinations of brackets involving the generators sw, where $w \in W_{\leq n-1}$. Since in this case $F(sw) = 0$, it follows that $\sum_{n\geq 1} \frac{1}{n!} F(S \circ D)^n(w) = 0$. Consequently, we obtain

$$\begin{aligned} F \circ e^{\theta}(w) &= F(w) + F(w') = \alpha(w) - y = \tilde{\alpha}(w)\ , \qquad & \text{if } w \in W_n, \\ F \circ e^{\theta}(w) &= F(w) + F(w') = \alpha(w), & \text{if } w \in W_{\leq n-1}. \end{aligned}$$

But by hypothesis we have $\alpha(w) = \tilde{\alpha}(w)$ on $W_{\leq n-1}$, so for all $w \in W$ we have $F \circ e^{\theta}(w) = \tilde{\alpha}(w)$ implying that F is the needed homotopy. □

Lemma 2.3. *Let $\alpha, \beta\colon (\mathbb{L}(W_{\leq n}), \delta) \to (\mathbb{L}(W_{\leq n}), \delta)$ be two DGL-maps such that*

$$\begin{aligned} \alpha(w) &= \beta(w) + y, \quad w \in W_n, \quad y \in \mathbb{L}_n(W_{\leq n-1}), \\ \alpha &\simeq \beta, \qquad \text{on } \mathbb{L}(W_{\leq n-1}). \end{aligned}$$

There is a cycle $y' \in \mathbb{L}_n(W_{\leq n-1})$ such that α is homotopic to the following DGL-map

$$\begin{aligned} \alpha'(w) &= \beta(w) + y', \quad w \in W_n, \\ \alpha' &= \beta, \qquad \text{on } \mathbb{L}(W_{\leq n-1}). \end{aligned} \tag{4}$$

Proof. Since α and β are homotopic on $\mathbb{L}(W_{\leq n-1})$, there exits a homotopy

$$F\colon (\mathbb{L}(W_{\leq n-1}, sW_{\leq n-1}, W'_{\leq n-1}), D) \to (\mathbb{L}(W_{\leq n-1}), \delta),$$

such that

$$F(w) = \beta(w), \qquad F \circ e^{\theta}(w) = \alpha(w), \qquad \forall w \in W_{\leq n-1}. \tag{5}$$

Therefore for $w \in W_n$, the element $F\Big(\sum_{n\geq 1} \frac{1}{n!}(S \circ D)^n(w)\Big)$ is a well-defined element in $\mathbb{L}_n(W_{\leq n-1})$. Thus we define

$$y' = y - F\Big(\sum_{n\geq 1} \frac{1}{n!}(S \circ D)^n(w)\Big). \tag{6}$$

Now, by hypothesis we have

$$\delta(\beta(w)) + \delta(y) = \delta\alpha(w) = \alpha(\delta(w)) = F \circ e^{\theta}(\delta(w)) = F \circ e^{\theta}(D(w)). \tag{7}$$

But e^θ is a DGL-automorphism, so

$$\begin{aligned}
F\circ e^\theta(D(w)) &= F\circ D(e^\theta(w)) = F\circ D\Big(w+w'+\sum_{n\geq 1}\frac{1}{n!}(S\circ D)^n(w)\Big)\\
&= F(D(w))+F(D(w'))+F\circ D\Big(\sum_{n\geq 1}\frac{1}{n!}(S\circ D)^n(w)\Big)\\
&= F(\delta(w))+\delta\circ F\Big(\sum_{n\geq 1}\frac{1}{n!}(S\circ D)^n(w)\Big)\\
&= \beta(\delta(w))+\delta\Big(\sum_{n\geq 1}\frac{1}{n!}F(S\circ D)^n(w)\Big)\\
&= \delta(\beta(w))+\delta\Big(\sum_{n\geq 1}\frac{1}{n!}F(S\circ D)^n(w)\Big). \qquad (8)
\end{aligned}$$

Here we use the facts that $D(w')=0$ by (1), $F\circ D=\delta\circ F$ and $F(\delta(w))=\beta(\delta(w))$ because $\delta(w)\in\mathbb{L}(W_{\leq n-1})$ and $F=\beta$ on $W_{\leq n-1}$ by (5). Comparing (7) and (8) we get

$$\delta(y)=\delta\Big(\sum_{n\geq 1}\frac{1}{n!}F(S\circ D)^n(w)\Big),$$

which implies according to (6) that $\delta(y')=0$.

Now define $G\colon(\mathbb{L}(W_{\leq n},sW_{\leq n},W'_{\leq n}),D)\to(\mathbb{L}(W_{\leq n}),\delta)$ by setting

$$\begin{aligned}
G(w) &= \alpha'(w), \qquad G(w')=G(sw)=0, \qquad \text{for } w\in W_n,\\
G &= F, \qquad \text{on } W_{\leq n-1}.
\end{aligned}$$

Let us consider the relations (1). A simple computation shows that

$$\delta(G(w))=\delta(\alpha'(w)), \qquad G(D(w))=G(\delta(w)).$$

As $\delta(w)\in\mathbb{L}_n(W_{\leq n-1})$, it follows that $G(\delta(w))=F(\delta(w))$ and by (4), (5) we get $F(\delta(w))=\beta(\delta(w))=\alpha'(\delta(w))$. As a result $\delta(G(w))=G(D(w))$. Also by taking into consideration the relations (1), we obtain

$$\delta(G(w'))=GD(w')=0, \qquad \delta(G(sw))=0, \qquad GD(sw)=G(w')=0,$$

proving that G is a DGL-map satisfying $G(w)=\alpha'(w)$ for all $w\in W_{\leq n}$. Moreover, we have

$$G\circ e^\theta(w)=G\Big(w+w'+\sum_{n\geq 1}\frac{1}{n!}(S\circ D)^n(w)\Big)=G(w)+G\Big(\sum_{n\geq 1}\frac{1}{n!}(S\circ D)^n(w)\Big).$$

As $\sum\limits_{n\geq 1}\frac{1}{n!}(S\circ D)^n(w)\in\mathbb{L}_n(W_{\leq n-1})$ and $F=G$ on $W_{\leq n-1}$, it follows that

$$G\circ e^\theta(w)=\alpha'(w)+F\Big(\sum_{n\geq 1}\frac{1}{n!}(S\circ D)^n(w)\Big)=(\beta(w)+y')+(y-y')=\alpha(w).$$

Here we use (6). Consequently, α and α' are homotopic. □

2.2. **Whitehead exact sequence of a DGL.** Let $(\mathbb{L}(W),\delta)$ be a DGL. If

$$j_n : H_n(\mathbb{L}(W_{\leq n})) \to W_n, \qquad\qquad j_n(\{w+y\}) = w,$$

where $w \in W_n$, $y \in \mathbb{L}_n(W_{\leq n-1})$ and where $\{w+y\}$ denote the homology class of the cycle $w+y$, then we define the graded vector space Γ_* by setting

$$\Gamma_n = \ker\big(H_n(\mathbb{L}(W_{\leq n})) \xrightarrow{j_n} W_n\big), \qquad\qquad \forall n \geq 2. \tag{9}$$

To every DGL $(\mathbb{L}(W),\delta)$, we can assign (see [2, 3] for more details) the following long exact sequence

$$\cdots \to W_{n+1} \xrightarrow{b_{n+1}} \Gamma_n \to H_n(\mathbb{L}(W)) \to W_n \xrightarrow{b_n} \cdots \tag{10}$$

called the Whitehead exact sequence of $(\mathbb{L}(W),\delta)$. Here $b_n(w) = \{\delta(w)\}$, where $\{\delta(w)\}$ denotes the homology class of $\delta(w)$ in $\mathbb{L}_{n-1}(W_{\leq n-1})$.

2.3. **Elliptic spaces.** Recall that a simply connected space X is called rationally elliptic if it satisfies $\dim(\pi_*(X)\otimes\mathbb{Q}) < \infty$ and $\dim H^*(X,\mathbb{Q}) < \infty$ ([4], §32). The following result mentions some important properties of rationally elliptic spaces.

Proposition 2.4. *(*[4] *Proposition* 32.6 and 32.10). If $(\mathbb{L}(W),\delta)$ is the Quillen model of a rationally elliptic space of formal dimension M, then

- $\dim W_{M-1} = 1$ and $W_i = 0$ for all $i \geq M$.
- $\sum_{i\geq 1}(2i+1)\dim H_{2i}(\mathbb{L}(W)) - \sum_{i\geq 1}(2i)\,(\dim H_{2i-1}(\mathbb{L}(W)) - 1) = M$.

Furthermore, the following statements are equivalent

(1) X is an F_0-space.
(2) $\dim H_{\text{even}}(\mathbb{L}(W)) = \dim H_{\text{even}}(\mathbb{L}(W))$.
(3) $W_{\text{even}} = 0$.

Remark 2.5. According to Proposition 2.4, the formal dimension of an F_0-space must be an even integer.

3. Properties of the group $\mathcal{E}_{\#}(\mathbb{L}(W))$

The purpose of this section is to study the properties of the group $\mathcal{E}_{\#}(\mathbb{L}(W))$, introduced in definition 2.1, in the case where the DGL $(\mathbb{L}(W),\delta)$ is the Quillen model of an F_0-space.

As it is stated in the introduction, an F_0-space is an elliptic space such that its rational cohomology is a graded algebra on the form $\mathbb{Q}[x_1,\ldots,x_n]/(P_1,\ldots,P_n)$, where the polynomials $P_1,\ldots,P_n$ form a regular sequence in $\mathbb{Q}[x_1,\ldots,x_n]$.
In [5], it is shown that the Sullivan model of an F_0-space is given by

$$(\Lambda V,\partial) = (\Lambda(x_1,\ldots,x_n;y_1,\ldots,y_n),\partial)\ ,\ \partial(x_i) = 0\ ,\ \partial(y_i) = P_i\ ,\ 1 \leq i \leq n,$$

where the generator $x_1,\ldots,x_n$ are of even degrees and $y_1,\ldots,y_n$ are of odd degrees.

It well-known that F_0-spaces are formal (see [5], theorem 5), i.e., there exists a quasi-isomorphism $\mathcal{M}(X) \to (H_*(X,\mathbb{Q}),0)$. As a result, the differential of the Quillen model $(\mathbb{L}(W),\delta)$ is purely quadratic, i.e., $\delta(W) \subset [W,W]$ (see [9], proposition 3.2). Moreover, taking into account that $W_{\text{even}} = 0$, we deduce that $W = W_{\text{odd}}$.

Remark 3.1. Recall that we have $V^{\text{even}} \cong H_{\text{odd}}(\mathbb{L}(W))$, therefore, to each $x_i \in V^{\text{even}}$ corresponds a homology class $\{w_i + q_i\} \in H_{\text{odd}}(\mathbb{L}(W))$ such that w_i is indecomposable and q_i is decomposable. Since $\delta(w_i) = -\delta(q_i)$, it follows that $\delta(w_i)$ has bracket length greater or equal than 3. But δ is purely quadratic, it follows that $q_i = 0$. As a result, $H_{\text{odd}}(\mathbb{L}(W))$ is generated by $w_1,\ldots,w_n$.

Proposition 3.2. *Let $(\mathbb{L}(W),\delta)$ be the Quillen model of an F_0-space X. Then the graded vector space Γ_{odd}, defined in* (9), is trivial.

Proof. Assume there is $\{z\}\neq 0\in\Gamma_{\text{odd}}$. Since $W_{\text{even}}=0$, the exact sequence (10)implies that $\{z\}\in H_{\text{odd}}(\mathbb{L}(W))$ which is impossible as z is decomposable due to Remark 3.1. □

Let us consider the Quillen model $(\mathbb{L}(W),\delta)$ of an F_0-space X of formal dimension M. By virtue of Proposition 2.4, we can write

$$\begin{aligned} W &= W_{r_1}\oplus\cdots\oplus W_{r_m}\oplus W_{M-1}, && r_1<\cdots<r_m<M-1,\\ W_{r_i} &= \langle w_{(1,r_i)},\dots,w_{(n_i,r_i)}\rangle, \quad 1\le i\le m, && W_{M-1}=\langle\mu\rangle. \end{aligned} \tag{11}$$

If $[\alpha]\in\mathcal{E}(\mathbb{L}(W))$, then for every $1\le i\le m$ and $1\le j\le n_i$, let us write

$$\begin{aligned} \alpha(w_{(j,r_i)}) &= \sum_{s_i=1}^{n_i}\lambda_{(j,r_i),s_i}w_{(j,r_i)}+A_{(j,r_i)}, && A_{(j,r_i)}\in\mathbb{L}^{\geq 3}(W_{\leq r_{i-1}}),\\ \alpha(\mu) &= a\mu+A_\mu, && A_\mu\in\mathbb{L}^{\geq 3}(W_{\leq M-2}), \end{aligned} \tag{12}$$

where all the coefficients $\lambda_{(r_i,j),s_i}, a$ are rationals.

Set $\tilde{\alpha}(w_{(j,r_i)})=\sum_{s_i=1}^{n_i}\lambda_{(j,r_i),s_i}w_{(j,r_i)}$, then (12) becomes

$$\alpha(w_{(j,r_i)})=\tilde{\alpha}(w_{(j,r_i)})+A_{(j,r_i)}.$$

Note that $\tilde{\alpha}(w_{(j,r_i)})\in W_{r_i}$. Moreover, if $l(A_{(j,r_i)})$ denotes the bracket length of $A_{(j,r_i)}$, then $l(A_{(j,r_i)})\geq 3$ because $|A_{(j,r_i)}|$ is odd and $W=W_{\text{odd}}$.

Theorem 3.3. *Let X be an F_0-space and let $(\mathbb{L}(W),\delta)$ be its Quillen model. If $[\alpha]\in\mathcal{E}_\#(\mathbb{L}(W))$, then α is homotopic to the DGL-map $\tilde{\alpha}$. Here $\mathcal{E}_\#(\mathbb{L}(W))$ is defined in* (2.1).

Proof. Let $[\alpha]\in\mathcal{E}_\#(\mathbb{L}(W))$ and $\alpha_{r_k}:(\mathbb{L}(W_{\leq r_k}),\delta)\to(\mathbb{L}(W_{\leq r_k}),\delta)$, the restriction of α to $\mathbb{L}(W_{\leq r_k})$. Since $H_{r_1}(\alpha)=id_{H_{r_1}(\mathbb{L}(W_{\text{odd}}))}=id_{W_{r_1}}$, we deduce that $\alpha_{r_3}=id$ on W_{r_1}.

First, from the relation (12), we have

$$\alpha_{r_2}(w_{(j,r_2)})=\tilde{\alpha}_{r_2}(w_{(j,r_2)})+A_{(j,r_2)},\quad l(A_{(j,r_2)})\geq 3,\qquad \alpha_{r_2}=id\ ,\ \text{on } \mathbb{L}(W_{r_1}),$$

implying that

$$\delta\alpha_{r_2}(w_{(j,r_2)})=\delta(\tilde{\alpha}_{r_2}(w_{(j,r_2)}))+\delta(A_{(j,r_2)}).$$

Next, as $\delta(t_{(j,r_2)})\in\mathbb{L}(W_{r_1})$, we get

$$\alpha_{r_2}\delta(w_{(j,r_2)})=\delta(w_{(j,r_2)}).$$

Since $\delta\alpha_{r_2}=\alpha_{r_2}\delta$, $l(A_{(j,r_2)})\geq 3$ and δ is purely quadratic, it follows that $\delta(A_{(j,r_2)})=0$ for every $1\le j\le n_2$. As a result, the homology class $\{A_{(j,r_2)}\}$ belongs to Γ_{r_2} which is, by proposition 3.2, trivial as r_2 is odd, therefore $A_{(j,r_2)}$ is a boundary. Now applying lemma 2.2, it follows that α_{r_2} and $\tilde{\alpha}_{r_2}$ are homotopic on $\mathbb{L}(W_{\leq r_2})$.

Assume by induction that $\alpha_{r_{k-1}}$ and $\tilde{\alpha}_{r_{k-1}}$ are homotopic on $\mathbb{L}(W_{\leq r_{k-1}})$. Therefore using (12) we get

$$\begin{aligned} \alpha_{r_k}(w_{(j,r_k)}) &= \tilde{\alpha}_{r_k}(w_{(j,r_k)})+A_{(j,r_k)}, && l(A_{(j,r_k)})\geq 3,\\ \alpha_{r_{k-1}} &\simeq \tilde{\alpha}_{r_{k-1}}, && \text{on } \mathbb{L}(W_{\leq r_{k-1}}). \end{aligned}$$

Due to lemma 2.3, we deduce that there is a cycle $A'_{(j,r_k)}$ such that $l(A'_{(j,r_k)})\geq 3$ and α_{r_k} is homotopic to the DGL-map α'_{r_k} given by

$$\begin{aligned} \alpha'_{r_k}(w_{(j,r_k)}) &= \tilde{\alpha}_{r_k}(w_{(j,r_k)})+A'_{(j,r_k)}, && l(A_{(j,r_k)})\geq 3,\\ \alpha'_{r_{k-1}} &= \tilde{\alpha}_{r_{k-1}}, && \text{on } \mathbb{L}(W_{\leq r_{k-1}}). \end{aligned}$$

The cycle $A'_{(j,r_k)}$ defines a homology class $\{A'_{(j,r_k)}\}$ belonging to $\Gamma_{\rm odd}$ which is trivial by 3.2 because $|A'_{(j,r_k)}| = r_k = $ odd. Therefore, from lemma 2.2, we deduce that $\alpha'_k \simeq \tilde{\alpha}_{r_k}$ and so are α_{r_k} and $\tilde{\alpha}_{r_k}$. Hence, $\alpha \simeq \tilde{\alpha}$. □

As a consequence of Theorem 3.3, we deduce the following fact

Corollary 3.4. *Let X be an F_0-space and let $(\mathbb{L}(W),\delta)$ be its Quillen model. If $[\alpha] \in \mathcal{E}_{\#}(\mathbb{L}(W))$, then for every $1 \le s \le m$ we have $\alpha(W_{r_s}) = W_{r_s}$ and $\alpha(\mu) = a\mu$, where a is a non-zero rational.*

Proof. It follows from Theorem 3.3 and the relations (12). □

Corollary 3.5. *Let X be an F_0-space and let $(\mathbb{L}(W),\delta)$ be its Quillen model. If $[\alpha] \in \mathcal{E}_{\#}(\mathbb{L}(W))$, then for every indecomposable cycle $w_{(j,r_s)} \in W$, we have $\alpha(w_{(j,r_s)}) = w_{(j,r_s)}$.*

Proof. By virtue of (2), if $[\alpha] \in \mathcal{E}_{\#}(\mathbb{L}(W))$, then $H_*(\alpha) = id_{H_*(\mathbb{L}(W))}$. Therefore, since $w_{(j,r_s)}$ is a cycle we get

$$H_*(\alpha)(\{w_{(j,r_s)}\}) = \{w_{(j,r_s)}\},$$

implying $\alpha(w_{(j,r_s)}) - w_{(j,r_s)}$ is a boundary in $(\mathbb{L}(W),\delta)$. As δ is purely quadratic, it follows that $\alpha(w_{(j,r_s)}) = w_{(j,r_s)}$. □

4. Properties of the Quillen model of an F_0-space

Let X be an F_0-space of formal dimension M and let

$$(\Lambda V,\partial) = (\Lambda(x_1,\dots,x_n;y_1,\dots,y_n,\partial)\ ,\ \partial(x_i) = 0\ ,\ 1 \le i \le n,$$

be its Sullivan model and $(\mathbb{L}(W),\delta)$ its Quillen model. Assume that

$$|x_1| \le \cdots \le |x_n|.$$

Recall that a basis of W_{r_s} is given by (see (11))

$$W_{r_s} = \langle w_{(1,r_s)},\dots,w_{(n_s,r_s)}\rangle, \qquad 1 \le s \le m, \qquad W_{M-1} = \langle \mu \rangle.$$

To each generator $w_{(j,r_s)}$ corresponds a non-trivial cohomology class $\{x_1^{i_1}\dots x_n^{i_n}\}$ such that

$$r_s = i_1|x_1| + \cdots + i_n|x_n| - 1, \qquad i_1 \ge 0,\dots,i_n \ge 0. \tag{13}$$

The differential is given by

$$\delta(w_{(j,r_s)}) = \sum \lambda_{(i,t)}[w_{(i,r_p)},w_{(t,r_q)}], \qquad r_p \le r_q\ ,\ r_p + r_q = r_s - 1, \tag{14}$$

where $\lambda_{(i,t)} \in \mathbb{Q}$ and where the generators $w_{(i,r_p)} \in W_{r_p}$ and $w_{(t,r_q)} \in W_{r_q}$ correspond respectively to the non-trivial cohomology classes $\{x_1^{p_1}\dots x_n^{p_n}\}$ and $\{x_1^{l_1}\dots x_n^{l_n}\}$ such that

$$x_1^{i_1}\dots x_n^{i_n} = \big(x_1^{p_1}\dots x_n^{p_n}\big)\big(x_1^{l_1}\dots x_n^{l_n}\big), \quad r_p = \sum_i^n p_i|x_i| - 1, \quad r_q = \sum_i^n l_i|x_i| - 1$$

$$p_1 \ge 0,\dots,p_n \ge 0, \qquad l_1 \ge 0,\dots,l_n \ge 0.$$

It well-known that if M is the formal dimension of the F_0-space X, then, thanks to the Poincaré duality ([4], §38), we have an isomorphism of vector spaces

$$\phi : W_{r_s} \to W_{M-2-r_s}.$$

So if $\{w_{(i,r_s)}\}_{1\leq i\leq n_s}$ is a basis for W_{r_s}, then $\{\phi(w_{(i,r_s)}) = w^*_{(i,r_s)}\}_{1\leq i\leq n_s}$ is a basis for W_{M-2-r_s}, called the dual basis. Consequently, we can choose a basis for W on the form

$$\mathcal{B} = \Big\{w_{(1,r_s)}, \dots, w_{(n_s,r_s)}; w^*_{(1,r_s)}, \dots, w^*_{(r_s,r_s)}, \mu\Big\}_{r_1\leq r_s\leq \frac{M-2}{2}}, \tag{15}$$

where $W_{M-1} = \langle\mu\rangle$. Moreover, due to (Theorem 2, [10]), we have

$$\delta(\mu) = \frac{1}{2}\sum_{r_s,t}[w_{(t,r_s)}, w^*_{(t,r_s)}], \qquad 1\leq s\leq m, \qquad 1\leq t\leq n_{r_s}. \tag{16}$$

Note that the integer M is even (see Remark 2.5), and if $r_p < r_q$, then $|w^*_{t,r_q}| < |w^*_{t,r_p}|$.

The following result plays a crucial role afterwards.

Lemma 4.1. *Let $(\mathbb{L}(W),\delta)$ be the Quillen model of an F_0-space X of formal dimension M. For every $w^*_{(j,r_s)} \in \mathcal{B}$, there exists $w^*_{(k,r_\sigma)} \in \mathcal{B}$ such that*

$$\delta(w^*_{(k,r_\sigma)}) = \beta_{(k,r_\sigma)}[w_{(s_1,r_p)}, w^*_{(j,r_s)}] + \Theta_{(k,r_\sigma)}, \tag{17}$$

where $\Theta_{(k,r_\sigma)}$ is a linear combination of 2-brackets where $w_{(s_1,r_p)}$ and $w^*_{(j,r_s)}$ are not involved. Moreover, $w_{(s_1,r_p)}$ is a cycle.

Proof. First, recall that $|w_{(j,r_s)}| = r_s$ and $|w^*_{(j,r_s)}| = M-2-r_s$. Next, by (13) and (14) we know that to $w_{(j,r_s)}$ and $w^*_{(j,r_s)}$ correspond two non-trivial cohomology classes $\{x^{t_1}_{s_1}\dots x^{t_h}_{s_h}\}$ and $\{x^{i_1}_{j_1}\dots x^{i_k}_{j_k}\}$ in the Sullivan model $(\Lambda V,\partial)$, such that

$$|x^{t_1}_{s_1}\dots x^{t_h}_{s_h}| = |w_{(j,r_s)}| + 1 = r_s+1, \qquad |x_{s_1}|\leq\dots\leq|x_{s_h}|.$$
$$|x^{i_1}_{j_1}\dots x^{i_k}_{j_k}| = |w^*_{(j,r_s)}| + 1 = M-1-r_s, \qquad |x_{j_1}|\leq\dots\leq|x_{j_k}|.$$

Here we can assume $t_1\geq 1,\dots,t_h\geq 1$ and $i_1\geq 1,\dots,i_k\geq 1$. Note that if the generator $w_{(j,r_s)}$ is a cycle, then the corresponding element in $(\Lambda V,\partial)$ is the cohomology class $\{x_{s_1}\}$.

Next, Poincaré duality implies that the multiplication

$$H^{r_s+1}(\Lambda V)\times H^{M-1-r_s}(\Lambda V)\to H^M(\Lambda V),$$

sending $\Big(\{x^{t_1}_{s_1}\dots x^{t_h}_{s_h}\};\{x^{i_1}_{j_1}\dots x^{i_k}_{j_k}\}\Big)$ to $\{x^{t_1}_{s_1}\dots x^{t_h}_{s_h}.x^{i_1}_{j_1}\dots x^{i_k}_{j_k}\}$, is non-degenerate. It follows that $x_{s_i}(x^{i_1}_{j_1}\dots x^{i_k}_{j_k})$ is not a coboundary for every $1\leq i\leq h$. As a result, we must have a generator $w^*_{(k,r_\sigma)}$ corresponding to cohomology class $\{x_{s_i}(x^{i_1}_{j_1}\dots x^{i_k}_{j_k})\}$ such that $\delta(w^*_{(k,r_\sigma)})$ satisfies the following formula

$$\delta(w^*_{(k,r_\sigma)}) = \beta_{(k,r_\sigma)}[w_{(s_1,r_p)}, w^*_{(j,r_s)}] + \Theta_{(k,r_\sigma)},$$

where $w_{(s_1,r_p)}$ corresponds to x_{s_1} which implies that $w_{(s_1,r_p)}$ is a cycle.
Finally, from the formula (14), it is clear that $\Theta_{(k,r_\sigma)}$ is a linear combination of 2-brackets where $w_{(s_1,r_p)}$ and $w^*_{(j,r_s)}$ are not involved. □

Remark 4.2. In the cohomology class $\{x^{t_1}_{s_1}\dots x^{t_h}_{s_h}\}$ corresponding to $w_{(j,r_s)}$, we might have

$$|x_{s_1}| = \dots = |x_{s_\tau}|, \qquad 1\leq\tau\leq h.$$

In this case, the formula (17) can be written as follows

$$\delta(w^*_{(k,r_\sigma)}) = \beta_1[w_{(s_1,r_p)}, w^*_{(j,r_s)}] + \sum_{j'\neq j\,,\,i>1}^{h}\beta_i[w_{(s_i,r_p)}, w^*_{(j',r_s)}] + \Theta_{(k,r_\sigma)},$$

furthermore, we have the following facts.

(1) Since $|x_{s_1}| \leq \cdots \leq |x_{s_h}|$, we deduce that $\Theta_{(k,r_\sigma)}$ is a linear combination of 2-brackets of the form $[w_{(a,b)}, w_{(c,d)}]$ such that
$$r_p < |w_{(a,b)}| \leq |w_{(c,d)}| < M - 2 - r_s.$$
(2) All the generators $w^*_{(j,r_s)}$ and $w^*_{(j',r_s)}$, where $j' \neq j$, are distinct and have the same degree $M - 2 - r_s$.
(3) All the generators $w_{(s_i,r_p)}, 1 \leq i \leq h$, are distinct cycles with $|w_{(s_i,r_p)}| = r_p$.
(4) All the rationals β_i are not zero.

Remark 4.3. A special case of Lemma 4.1 is when $r_s = \frac{M-2}{2}$. In this case the lemma still valid for any generator $w_{(j,r_s)}$ such that $\delta(w^*_{(j,r_s)}) \neq 0$ because the dual of $w^*_{(j,r_s)}$, namely $(w^*_{(j,r_s)})^*$, is $w_{(j,r_s)}$.

5. Main result

In all this section, let X denote an F_0-space of formal dimension M, $(\Lambda V, \partial)$ its Sullivan model, $(\mathbb{L}(W), \delta)$ its Quillen model and $\mathcal{B}$ the basis of W given in (15). Recall that by Corollary (3.4) there exists a rational $a \neq 0$ such that $\alpha(\mu) = a\mu$, where $W_{M-1} = \langle \mu \rangle$.

Subsequently, we prove some important lemmas concerning the properties of $(\mathbb{L}(W), \delta)$ needed to establish the main result in this paper. Indeed, if $[\alpha] \in \mathcal{E}_\#(\mathbb{L}(W))$, then by considering the basis (15) and Remark 4.2, we can summarize the next steps as follows.

- In Lemma 5.1, we show that $\alpha(w^*_{j,r_s}) = aw^*_{j,r_s}$ for all j and $r_s < \frac{M-2}{2}$.
- In Lemma 5.2, we show that $\alpha(w_{j,r_s}) = w_{j,r_s}$ for every j and $r_s < \frac{M-2}{2}$.
- Lemmas 5.3 and 5.4, show that $\alpha(w_{j,\xi}) = aw_{j,\xi}$ and $\alpha(w^*_{j,\xi}) = aw^*_{j,\xi}$ for every j, where $\xi = \frac{M-2}{2}$.
- In Proposition 5.5, we show that $a = 1$.

Lemma 5.1. *If $[\alpha] \in \mathcal{E}_\#(\mathbb{L}(W))$, then for $w^*_{j,r_s} \in \mathcal{B}$ such that $r_s < \frac{M-2}{2}$, we have $\alpha(w^*_{j,r_s}) = aw^*_{j,r_s}$.*

Proof. First, let us prove that for every j we have
$$\alpha(w^*_{(j,r_1)}) = aw^*_{(j,r_1)}. \tag{18}$$
Indeed, let $w^*_{(j,r_1)} \in W_{M-2-r_1} = \langle w^*_{(1,r_1)}, w^*_{(2,r_1)}, \dots \rangle$. Using Corollary 3.4, we can write
$$\alpha(w^*_{(j,r_1)}) = \sum_{i\geq 1} \lambda_i w^*_{(i,r_1)}, \qquad \lambda_i \in \mathbb{Q}. \tag{19}$$
The formula (16) can be written as
$$\delta(\mu) = \frac{1}{2}\sum_j [w_{(j,r_1)}, w^*_{(j,r_1)}] + \frac{1}{2}\sum_{r_1\neq r_s}[w_{(t,r_s)}, w^*_{(t,r_s)}]. \tag{20}$$
Next, $w_{(j,r_1)}$ is obviously a cycle as $w_{(j,r_1)} \in W_{r_1}$. By Corollary 3.5, it follows that
$$\alpha(w_{(j,r_1)}) = w_{(j,r_1)}. \tag{21}$$
On the one hand, using (19), (20) and (21), we get
$$\alpha(\delta(\mu)) = \frac{1}{2}\sum_j\sum_{i\geq 1}\lambda_i[w_{(j,r_1)}, w^*_{(i,r_1)}] + \frac{1}{2}\sum_{r_1\neq r_s}[\alpha(w_{(t,r_s)}), \alpha(w^*_{(t,r_s)})].$$

On the other hand, by the relation (20) and Corollary 3.4, we have

$$\delta(\alpha(\mu))=a\delta(\mu)=\frac{a}{2}\ \sum_{j}[w_{(j,r_1)},w^*_{(j,r_1)}]+\frac{a}{2}\sum_{r_1\neq r_s}[w_{(t,r_s)},w^*_{(t,r_s)}].$$

Since $\alpha(\delta(\mu))=\delta(\alpha(\mu))$, and taking into account that $r_1\neq r_s$, which means that the generator $w^*_{(j,r_1)}$ cannot appear in the expression $\sum_{r_1\neq r_s}[\alpha(w_{(t,r_s)}),\alpha(w^*_{(t,r_s)})]$, it follows that all the rationals λ_i in (19) are zero except $\lambda_1=a$ showing (18).

Next, assume by induction that

$$\alpha(w^*_{(j,r_q)})=aw^*_{(j,r_q)}, \tag{22}$$

for all the generators w^*_{j,r_q} such that $r_q<r_s$. Let us prove it for every generator

$$w^*_{(j,r_s)}\in W_{M-2-r_s}=\langle w^*_{(1,r_s)},w^*_{(2,r_s)},\dots\rangle.$$

For this purpose, write

$$\alpha(w^*_{(j,r_s)})=\sum_{\tau\geq 1}\lambda_\tau w^*_{(\tau,r_s)},\qquad\qquad \lambda_\tau\in\mathbb{Q}. \tag{23}$$

By virtue of Lemma 4.1 and Remark 4.2, there exists $w^*_{(k,r_\sigma)}$ such that

$$\delta(w^*_{(k,r_\sigma)})=\beta_1[w_{(s_1,r_p)},w^*_{(j,r_s)}]+\sum_{j'\neq j\ ,\ i>1}^{h}\beta_i[w_{(s_i,r_p)},w^*_{(j',r_s)}]+\Theta_{(k,r_\sigma)}, \tag{24}$$

where $\beta_1\neq 0$. As a result, we obtain

$$\begin{aligned}\alpha(\delta(w^*_{(k,r_\sigma)}))\ &=\ \beta_1[\alpha(w_{(s_1,r_p)}),\alpha(w^*_{(j,r_s)})]+\sum_{j'\neq j\ ,\ i>1}^{h}\beta_i[\alpha(w_{(s_i,r_p)}),\alpha(w^*_{(j',r_s)})]+\alpha(\Theta_{(k,r_\sigma)})\\ &=\sum_{\tau\geq 1}\lambda_\tau\beta_1[w_{(s_1,r_p)},w^*_{(\tau,r_s)}]+\sum_{j'\neq j\ ,\ i>1}^{h}\beta_i[w_{(s_i,r_p)},\alpha(w^*_{(j',r_s)})]+\alpha(\Theta_{(k,r_\sigma)}).\end{aligned}$$

Note that, according to Remark 4.2, all the generators $w_{(s_i,r_p)}$ are cycles implying that $\alpha(w_{(s_i,r_p)})=w_{(s_i,r_p)}$ due to Corollary 3.5.
Next, as $|w^*_{(k,r_\sigma)}|>|w^*_{(j,r_s)}|$ which implies that $r_q<r_s$, using (22) and (24), we get

$$\delta(\alpha(w^*_{(k,r_\sigma)}))=a\delta(w^*_{(k,r_\sigma)})=a\beta_1[w_{(s_1,r_p)},w^*_{(j,r_s)}]+\sum_{j'\neq j\ ,\ i>1}^{h}a\beta_i[w_{(s_i,r_p)},w^*_{(j',r_s)}]+a\Theta_{(k,r_\sigma)}.$$

Since $\alpha(\delta(w^*_{(i,r_q)}))=\delta(\alpha(w^*_{(i,r_q)}))$ and taking into account that the bracket $[w_{(s_1,r_p)},w^*_{(j,r_s)}]$ does not appear in the expressions (see Remark 4.2)

$$\sum_{j'\neq j\ ,\ i>1}^{h}a\beta_i[w_{(s_i,r_p)},w^*_{(j',r_s)}]\qquad\text{and}\qquad\alpha(\Theta_{(k,r_\sigma)}),$$

we deduce that all the coefficients λ_τ in (23) are nil except $\lambda_1\beta_1=a\beta_1$ and because $\beta_1\neq 0$, we obtain $\alpha(w^*_{(j,r_s)})=aw^*_{(j,r_s)}$. □

Lemma 5.2. *If $[\alpha]\in\mathcal{E}_\#(\mathbb{L}(W))$, then for every $w_{j,r_s}\in\mathcal{B}$, where $r_s<\frac{M-2}{2}$, we have $\alpha(w_{j,r_s})=w_{j,r_s}$.*

Proof. First, we know from Corollary 3.5 that if $\delta(w_{j,r_s}) = 0$, then $\alpha(w_{j,r_s}) = w_{j,r_s}$, therefore we can suppose that $\delta(w_{j,r_s}) \neq 0$. Secondly, recall that from the formula (16) we can write

$$\delta(\mu) = \frac{1}{2}[w_{(j,r_s)}, w^*_{(j,r_s)}] + \frac{1}{2}\sum_{t\neq j}[w_{(t,r_s)}, w^*_{(t,r_s)}].$$

As a result, we get

$$\alpha(\delta(\mu)) = \frac{1}{2}[\alpha(w_{(j,r_s)}), \alpha(w^*_{(j,r_s)})] + \frac{1}{2}\sum_{t\neq j}[\alpha(w_{(t,r_s)}), \alpha(w^*_{(t,r_s)})],$$

and because $r_s < \frac{M-2}{2}$, Lemma 5.1 implies that

$$\alpha(w^*_{(j,r_s)}) = aw^*_{(j,r_s)}, \quad \alpha(w^*_{(t,r_s)}) = aw^*_{(t,r_s)}, \quad \forall t \neq j.$$

Next, by Corollary 3.4, we can write

$$\alpha(w_{(j,r_s)}) = \sum_i \rho_i w_{(i,r_s)}, \qquad \rho_i \in \mathbb{Q},$$

implying that

$$\alpha(\delta(\mu)) = \frac{a}{2}\rho_j[w_{(j,r_s)}, w^*_{(j,r_s)}] + \frac{a}{2}\sum_{i\neq j}\rho_i[w_{(i,r_s)}, w^*_{(j,r_s)}] + \frac{a}{2}\sum_{t\neq j}[\alpha(w_{(t,r_s)}), w^*_{(t,r_s)}]. \quad (25)$$

Finally, using Corollary 3.5, we obtain

$$\delta(\alpha(\mu)) = a\delta(\mu) = \frac{a}{2}[w_{(j,r_s)}, w^*_{(j,r_s)}] + \frac{a}{2}\sum_{t\neq j}[w_{(t,r_s)}, w^*_{(t,r_s)}]. \quad (26)$$

Since $\alpha(\delta(\mu)) = \delta(\alpha(\mu))$ and $w^*_{(j,r_s)} \neq w^*_{(t,r_s)}$, comparing (25) and (26), it follows that $\rho_i = 0$ for all $i \neq j$ and $\rho_j = 1$. Hence, $\alpha(w_{(j,r_s)}) = w_{(j,r_s)}$. □

Lemma 5.3. *If* $[\alpha] \in \mathcal{E}_\#(\mathbb{L}(W))$, *then* $\alpha(w^*_{(j,\xi)}) = aw^*_{(j,\xi)}$, *where* $\xi = \frac{M-2}{2}$.

Proof. By virtue of Lemma 4.1 and Remark 4.2, there exists $w^*_{(k,r_\sigma)}$ such that

$$\delta(w^*_{(k,r_\sigma)}) = \beta_1[w_{(s_1,r_p)}, w^*_{(j,\xi)}] + \sum_{j'\neq j\,,\, i>1}^{h}\beta_i[w_{(s_i,r_p)}, w^*_{(j',\xi)}] + \Theta_{(k,r_\sigma)}, \quad (27)$$

where the generators $w_{(s_i,r_p)}$ are cycles implying that $\alpha(w_{(s_i,r_p)}) = w_{(s_i,r_p)}$ for all $1 \leq i \leq h$. Next, since that a basis of W_ξ is formed by the generators $w_{(i,\xi)}$ and their duals $w^*_{(i,\xi)}$ because in this case we have $|w_{(i,\xi)}| = |w^*_{(i,\xi)}| = \xi = \frac{M-2}{2}$, by Corollary 3.5, we can write

$$\alpha(w^*_{(j,\xi)}) = \sum_{i\geq 1}\mu_i w^*_{(i,\xi)} + \sum_{\tau\geq 1}\gamma_\tau w_{(\tau,\xi)}, \quad (28)$$

As a result, we obtain

$$\begin{aligned}\alpha(\delta(w^*_{(k,r_\sigma)})) &= \sum_i \mu_i\beta_1[w_{(s_1,r_p)}, w^*_{(i,\xi)}] + \sum_\tau \gamma_\tau\beta_1[w_{(s_1,r_p)}, w_{(\tau,\xi)}] \\ &+ \sum_{j'\neq j\,,\, i>1}^{h}\beta_i[w_{(s_i,r_p)}, \alpha(w^*_{(j',\xi)})] + \alpha(\Theta_{(k,r_\sigma)}).\end{aligned}$$

Next, as $|w^*_{(k,r_\sigma)}| > |w^*_{(j,\xi)}| = \xi = \frac{M-2}{2}$, it follows that $r_\sigma < \xi$. Thus, using Lemma 5.1 and the relation (27), we get

$$\delta(\alpha(w^*_{(i,r_q)})) = a\delta(w^*_{(i,r_q)}) = a\beta_1[w_{(s_1,r_p)}, w^*_{(j,\xi)}] + \sum_{j'\neq j\,,\,i>1}^{h} a\beta_i[w_{(s_i,r_p)}, w^*_{(j',\xi)}] + a\Theta_{(k,r_\sigma)}.$$

Since $\alpha(\delta(w^*_{(i,r_q)})) = \delta(\alpha(w^*_{(i,r_q)}))$ and taking into account that the bracket $[w_{(s_1,r_p)}, w^*_{(j,\xi)}]$ does not appear in the expression $\alpha(\Theta_{(i,r_q)})$, according to Remark 4.2, we deduce that all the coefficients μ_i and γ_τ in (28) are nil except $\mu_j\beta_1 = a\beta_1$ implying that $\mu_j = a$ because $\beta_1 \neq 0$. Hence, $\alpha(w^*_{(j,\xi)}) = aw^*_{(j,\xi)}$ □

Lemma 5.4. *If $[\alpha] \in \mathcal{E}_\#(\mathbb{L}(W))$, then for every $w_{(j,\xi)} \in \mathcal{B}$, we have $\alpha(w_{(j,\xi)}) = aw_{(j,\xi)}$.*

Proof. The proof is as in Lemma 5.3 after taking into consideration Remark 4.3. □

Proposition 5.5. *If $(\mathbb{L}(W), \delta)$ is the Quillen model of an F_0-space of formal dimension M, then the group $\mathcal{E}_\#(\mathbb{L}(W))$is trivial.*

Proof. It suffices to prove that the rational a given in Lemmas 5.1, 5.3 and 5.4 satisfies $a = 1$. Indeed, first the formula (16) can be written as

$$\delta(\mu) = \frac{1}{2}\sum_j [w_{(j,\xi)}, w^*_{(j,\xi)}] + \frac{1}{2}\sum_{r_p<\xi}\sum_t [w_{(t,r_p)}, w^*_{(t,r_p)}].$$

It follows that

$$\alpha(\delta(\mu)) = \frac{1}{2}\sum_j [\alpha(w_{(j,\xi)}), \alpha(w^*_{(j,\xi)})] + \frac{1}{2}\sum_{r_p<\xi}\sum_t [\alpha(w_{(t,r_p)}), \alpha(w^*_{(t,r_p)})].$$

Now, for all t and $r_p < \xi$, Lemmas 5.1 and 5.2 yield the following

$$\alpha(w_{(t,r_p)}) = w_{(t,r_p)}, \qquad \alpha(w^*_{(t,r_p)}) = aw^*_{(t,r_p)},$$

and for for all t, by lemmas 5.3 and Corollary 5.4, we have

$$\alpha(w_{(t,\xi)}) = aw_{(t,\xi)}, \qquad \alpha(w^*_{(t,\xi)}) = aw^*_{(t,\xi)},$$

Therefore, on the one hand, we have

$$\alpha(\delta(\mu)) = \frac{1}{2}\sum_j a^2[w_{(i,\xi)}, w^*_{(j,\xi)}] + \frac{a}{2}\sum_{r_p<\xi}\sum_t [w_{(t,r_p)}, w^*_{(t,r_p)}].$$

On the other hand, by the relation (5) and Corollary 3.4 we have

$$\delta(\alpha(\mu)) = \frac{a}{2}\sum_j [w_{(j,\xi)}, w^*_{(j,\xi)}] + \frac{a}{2}\sum_{r_p<\xi}\sum_t [w_{(t,r_p)}, w^*_{(t,r_p)}].$$

Since $\alpha(\delta(\mu)) = \delta(\alpha(\mu))$, it follows that $a^2 = a$ and as $a \neq 0$, it follows that $a = 1$. □

Now we are able to announce the main result in this paper.

Theorem 5.6. *If X is an F_0-space, the $\mathcal{E}_\#(X)$ is finite.*

Proof. It suffices to apply Proposition 5.5 and the identification (2). □

References

1. Arkowitz, M., *Problems on Self-homotopy equivalences*, Contemporary Math., 274, (2001), 309-315.
2. M. Benkhalifa, *The effect of cell-attachment on the group of self-equivalences of an elliptic space*, Michigan Mathematical Journal, Vol. 71(2), 611-617, 2022.
3. M. Benkhalifa, *On the group of self-homotopy equivalences of an elliptic space*, Proceedings of the American Mathematical Society. Vol.148 (6),2695-2706, 2020
4. Y. Félix, S. Halperin, and J.-C. Thomas, *Rational homotopy theory*, Graduate Texts in Mathematics, Vol. 205, Springer-Verlag, New York, 2001.
5. Halperin S., *Finiteness in the minimal models of Sullivan,* Trans. Amer. Math. Soc. 230, 269-331, 1977.
6. G. Lupton, *A note on the conjecture of S. Halperin*, Lectures Notes in Mathematics, vol. 1440, Springer-Verlag, 148-163, 1990.
7. K. Maruyama, *Localization of a certain subgroup of self-homotopy equivalences,* Pacific J. Math. 136, 293-301, 1989.
8. W. Meier, *Rational universal fibrations and flag manifolds.* Math. Ann., 258(3):329–340, 1981.
9. J. Neisendorfer and T. Miller, *Formal and coformal spaces,* Illinois J. Math. Vol. 205 (4), 565-580, 1978
10. J. Stasheff, *Rational Poincaré duality spaces* . Illinois J. Math. Vol. 27 (1), 104–109, 1983.

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