Construction of solutions of nonlinear irregular singular differential equations by Borel summable functions

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Abstract

A system of nonlinear differential equations $x^{1+\gamma}\frac{dY}{dx} = F_0(x) + A(x)Y + F(x,Y)$ $(\gamma \geq 1)$ is considered. The origin $x = 0$ is irregular singular. There exist pioneering works about them. We study more precisely than preceding works, the meaning of asymptotic expansion of transformations and solutions by using Borel summable functions in asymptotic analysis and construct exponential series solutions often called transseries. [2]

0 Introduction

The main purpose of the present paper is to construct solutions of the following system of nonlinear differential equations with irregular singularity at $x = 0$

$$\begin{cases} Y =^t(y_1, y_2, \dots, y_n), \\ x^{1+\gamma}\dfrac{dY}{dx} =F_0(x) + A(x)Y + F(x,Y) \quad \gamma \geq 1. \end{cases} \tag{0.1}$$

There are pioneering researches about (0.1) by Hukuhara [14], Iwano [15] [16], Malmquist [17], Trjitzinsky [22] and many other important ones. They constructed formal solutions and showed the existence of genuine solutions under some conditions.

Firstly we mention a classical important result due to Malmquist [17]. Let $\Lambda = \{\lambda_1, \dots, \lambda_n\}$ be the set of eigenvalues of $A(0)$ and assume they are distinct and not zero. Let $\Lambda' = \{\lambda_1, \dots, \lambda_{n'}\}$ and $\Lambda'' = \{\lambda_{n'+1}, \dots, \lambda_n\}$. Λ' and Λ'' are separated by a straight line through the origin in the complex plane. It is shown that there exists an n'-parameter family of solutions in some sector corresponding to Λ'. After [17] there are fundamental works due to Iwano [15] and [16] which are generalizations of Malmquist's result. The following holds for solutions of (0.1) under the condition of this paper.

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There exist polynomials $\{\lambda_i(x)\}_{1\leq i\leq n}$ with degree $\leq \gamma$ and $\lambda_i(0) = \lambda_i$. Let $h_i(x) = \int^x \lambda_i(x)/x^{1+\gamma}dx$ and $H(x) = (h_1(x), \dots, h_{n'}(x))$. Then there exists an exponential series solution $Y(x) = {}^t(y_1(x), \dots, y_n(x))$ of (0.1) in some sector such that

$$y_i(x) = \sum_{k\in\mathbb{N}^{n'}} c_{i,k}(x)C^k e^{k\cdot H(x)}, \quad k\cdot H(x) = \sum_{i=1}^{n'} k_i h_i(x), \tag{0.2}$$

which are convergent for small $C = (C_1, \dots, C_{n'}) \in \mathbb{C}^{n'}$, $C^k = \prod_{i=1}^{n'} C_i^{k_i}$, $\mathbb{N} = \{0, 1, 2, \dots\}$. The coefficients $c_{i,k}(x)$ of (0.2) are holomorphic in an appropriate sector.

After the pioneering researches, the theories of multisummable functions in asymptotic analysis and Écalle's resurgence functions have been developed (see Balser [1] and Écalle [12]). Borel summability is a special case of multisummability. These theories are powerful to study correspondence between formal solutions and genuine solutions. For nonlinear equations it is shown that formal power series solutions of ordinary differential equations are multisummable in Braaksma [2] and those of some class of partial differential equations are multisummable in Ōuchi [21]. There are many works concerning Borel summability of solutions of ordinary or partial differential equations.

Écalle's theory can treat a broad class of problems in asymptotic analysis. It is used for equation (0.1) with $\gamma = 1$ (rank 1) in Braaksma and Kuik [3], Costin,O [7], [9] and Costin,O, Costin,R.D [8], where exponential series (transseries) solutions are studied under the condition that eigenvalues are simple. Procedure is required to apply Écalle's theory to (0.1). Several preparations and profound analysis are given there. The method in the present paper is different from theirs. We remark that it is known that (0.1) is reduced to a system with rank=1 ($\gamma = 1$) (Canalis-Durand, Mozo-Fernades, Schäfke [6]). But its size is $n\gamma$ and its eigenvalues of the linear part are not simple and have multiplicity γ, hence we can not apply [7] and [8] to the present case $\gamma \geq 1$. Really we have the difference of the exponential factor $e^{h(x)}$

$$e^{h(x)} = \begin{cases} \exp(-\dfrac{\lambda}{x})x^{a_1} & \gamma = 1, \\ \exp\big(-\dfrac{\lambda}{\gamma x^\gamma} + \dfrac{1}{x^\gamma}\displaystyle\sum_{\ell=1}^{\gamma-1} a_\ell x^\ell\big)x^{a_\gamma} & \gamma \geq 2. \end{cases} \tag{0.3}$$

It is the main purpose that we try to have another look at (0.1) for $\gamma \geq 1$, by applying theory of Borel summable functions. We use only the introductory properties of Borel summable functions, that is, they are represented by Laplace integral. Instead of formal calculus, we construct analytically transformations and solutions more precisely and clearly, by solving some singular differential equations in function classes with some Gevrey type estimates. Function spaces $\mathscr{O}_{\{1/\gamma\}}(S_0 \times \Omega)$ and $\mathscr{O}_{\{1/\gamma\}}(S_0)$ are available for our calculation. S_0 is a sectorial region with opening angle $> \pi/\gamma$ and $\Omega = \{Y \in \mathbb{C}^n; |Y| < R\}$ (see Definition 1.1). We make full use of these function spaces. In section 1 we define $\mathscr{O}(S_0(I) \times \Omega)$, Borel transform and Laplace transform. We sum up shortly what we need about Borel summable functions. It is known that the linear part $A(x)$ can be transformed formally to a diagonal matrix by a linear transformation with coefficients in $\mathbb{C}[[x]]$ (Wasow [23]). We transform $A(x)$ analytically to $diag.(\lambda_i(x))$ such that $\{\lambda_i(x)\}_{i=1}^m$ are polynomials with degree $\leq \gamma$ and $\lambda_i(0) = \lambda_i$, by using linear transformation with coefficients in $\mathscr{O}_{\{1/\gamma\}}(S_0)$. This process is called diagonalization of system (Balser [1]). We start to study this system. First we study the case $F_0(x) \equiv 0$

$$x^{1+\gamma}\frac{dY}{dx} = A(x)Y + F(x, Y). \tag{0.4}$$

Let $Z(x) = (C_1 e^{h_1(x)}, \ldots, C_{n'} e^{h_{n'}(x)})$ be a solution of

$$x^{1+\gamma}\frac{dz_i}{dx} = \lambda_i(x) z_i \quad 1 \leq i \leq n'. \tag{0.5}$$

By using $Z(x)$, we construct exponential series solutions. The process is as follows. We try to find $\Phi(x, Z) = (\phi_1(x, Z), \ldots, \phi_m(x, Z))$, $\phi_i(x, Z) \in \mathscr{O}_{\{1/\gamma\}}(S_0 \times \Omega')$, Ω' being a domain in $\mathbb{C}^{n'}$, such that $Y(x) = \Phi(x, Z)|_{Z=Z(x)}$ is a solution of (0.4). For this purpose we introduce some system of singular nonlinear partial equations that determines $\phi_i(x, Z)$. We give one of main results (Theorem 2.4), construction of exponential solutions. Its proof is given in section 3. The proof is to find a solution $\Phi(x, Z)$ of the introduced system of partial equations. The coefficients $\{c_{i,k}(x)\}$ of the exponential series in (0.2) are determined by $\Phi(x, Z)$, that is, by the system of partial differential equations. Hence it gives more information about $\{c_{i,k}(x)\}$ than classical algorithmic determination. We change the differential equations to convolution equations and solve them. In section 4 we study the case $F_0(x) \not\equiv$

0. By eliminating $F_0(x)$, we reduce to the former case. By this process (0.1) changes. The reduced system has a similar form as the original one and eigenvalues of its linear part are invariant. Hence we can apply Theorem 2.4. In section 5 we consider a special 2×2 system and study Painlevé IV as an example. It has a singular point at $t = x^{-1} = \infty$. We remark that Painlevé I-VI equations are important equations and investigated extensively. In section 6 for the readers we give a proof of the diagonalization of linear systems under the present condition. As for more details of block diagonalization (splitting) theorem of linear systems, we refer to [1]. We also refer the fundamental solution of systems of equations to [4]. The main results in this paper are obtained under the condition that the eigenvalues of matrix $A(0)$ are distinct. It will be studied in another paper for the case multiple eigenvalues of matrix $A(0)$ appear.

1 Borel summable functions with holomorphic parameters

We introduce some notations and definitions. Let $I = (\alpha, \beta)$ be an open interval and $\widetilde{\mathbb{C}}_{\{0\}}$ be the universal covering space of $\mathbb{C} - \{0\}$. $S(I) = S(\alpha, \beta) = \{x \in \widetilde{\mathbb{C}}_{\{0\}}; \arg x \in I\}$. $S_0(I) = S_0(\alpha, \beta) = \{x \in S(I); 0 < |x| < \rho(\arg x)\}$, where $\rho(t)$ is some positive continuous function on I. The same notation $S_0(\cdot)$ is used for various $\rho(\cdot)$. For arbitrary small $\epsilon > 0$, $I_\epsilon = (\alpha + \epsilon, \beta - \epsilon) \subset I$. $\mathscr{O}(U)$ is the set of holomorphic functions on a domain U. $\mathbb{C}[[x]]$ is the set of formal power series of x. $\mathbb{N}$ is the set of nonnegative integers and $\mathbb{Z}$ is the set of integers. Let $k = (k_1, \dots, k_n) \in \mathbb{N}^n$ and $Y = (y_1, \dots, y_n) \in \mathbb{C}^n$. Then we use notations $k! = k_1! k_2! \dots k_n!$, $|k| = \sum_{i=1}^n k_i$, $Y^k = y_1^{k_1} \dots y_n^{k_n}$, $|Y| = \max_{1 \le i \le n} |y_i|$ and $(\frac{\partial}{\partial Y})^k = \prod_{i=1}^n (\frac{\partial}{\partial y_i})^{k_i}$.

Definition 1.1. *Let $\kappa > 0$, $I = (\alpha, \beta)$ with $\beta - \alpha > \pi/\kappa$ and $\Omega = \{Y \in \mathbb{C}^n; |Y| < R\}$. A function $f(x, Y) \in \mathscr{O}(S_0(I) \times \Omega)$ is said to be κ-Borel summable with respect to x, if there exist constants M, C and $\{a_n(y)\}_{n=0}^\infty \subset \mathscr{O}(\Omega)$ such that for any $N \ge 0$*

$$|f(x, Y) - \sum_{n=0}^{N-1} a_n(Y) x^n| \le M C^N |x|^N \Gamma(\frac{N}{\kappa} + 1), \quad (x, Y) \in S_0(I) \times \Omega \quad (1.1)$$

holds. The totality of κ-Borel summable functions with respect to x on $S_0(I) \times \Omega$ is denoted by $\mathscr{O}_{\{1/\kappa\}}(S_0(I) \times \Omega)$.

We say that $f(x, Y)$ is κ-Borel summable in a direction θ, if there exists $\delta > \pi/(2\kappa)$ such that $f(x, Y) \in \mathscr{O}_{\{1/\kappa\}}(S_0(\theta - \delta, \theta + \delta) \times \Omega)$.

The notion of Borel summability is originally used for formal power series.

Definition 1.2. *Let $\widetilde{f}(x) = \sum_{n=0}^{\infty} a_n x^n \in \mathbb{C}[[x]]$ and $I = (\theta - \delta, \theta + \delta)$ $\delta > \pi/2\kappa$. $\widetilde{f}(x)$ is said to be κ-Borel summable in a direction θ, if there exists $f(x) \in \mathscr{O}_{\{1/\kappa\}}(S_0(I))$ such that for any $N \geq 0$*

$$|f(x) - \sum_{n=0}^{N-1} a_n x^n| \leq MC^N |x|^N \Gamma(\frac{N}{\kappa} + 1) \quad x \in S_0(I) \tag{1.2}$$

holds.

If $f(x)$ exists, then it is unique. Hence $\widetilde{f}(x)$ and $f(x)$ are often identified. Let $I = (\alpha, \beta)$ with $\beta - \alpha > \pi/\kappa$, $\theta = (\alpha + \beta)/2$ and $\delta = (\beta - \alpha)/2$. Then $I = (\theta - \delta, \theta + \delta)$ and $\delta > \pi/(2\kappa)$. Let $\psi(x, Y) \in \mathscr{O}(S_0(I) \times \Omega)$ and $|\psi(x, Y)| \leq C|x|^c (c > 0)$. κ-Borel transform of $\psi(x, Y)$ is defined by

$$(\mathfrak{B}_\kappa \psi)(\xi, Y) = \frac{1}{2\pi i} \int_{\mathcal{C}} \exp(\frac{\xi}{x})^\kappa \psi(x, Y) dx^{-\kappa}, \tag{1.3}$$

where $\mathcal{C}$ is a contour in $S_0(I)$ that starts from $0e^{i(\theta+\delta')}$ to $r_0 e^{i(\theta+\delta')}$ on a segment and next on an arc $|t| = r_0$ to $r_0 e^{i(\theta-\delta'')}$ and finally on a segment ends at $0e^{i(\theta-\delta'')}$ $(\delta > \delta', \delta'' > \pi/(2\kappa))$.

We denote $(\mathfrak{B}_\kappa \psi)(\xi, Y)$ by $\widehat{\psi}(\xi, Y)$. Let $\widehat{\alpha}(\kappa) = \alpha + \pi/2\kappa$, $\widehat{\beta}(\kappa) = \beta - \pi/2\kappa$ and $\widehat{I}(k) = (\widehat{\alpha}(k), \widehat{\beta}(k))$. Then $\widehat{\psi}(\xi, Y)$ is holomorphic in an infinite sector $S(\widehat{I}(k))$ with respect ξ. We can construct $\psi(x, Y)$ by κ-Laplace transform $\mathfrak{L}_\kappa \widehat{\psi}$, that is, $\psi(x, Y) = (\mathfrak{L}_\kappa \widehat{\psi})(x, Y)$

$$(\mathcal{L}_\kappa \widehat{\psi})(x, Y) = \int_0^{\infty e^{\sqrt{-1}\theta}} e^{-(\frac{\xi}{x})^k} \widehat{\psi}(\xi, Y) d\xi^\kappa \quad \theta \in \widehat{I}(\kappa). \tag{1.4}$$

If $f(x, Y) \in \mathscr{O}_{\{1/k\}}(S_0(I) \times \Omega)$ with $a_0(Y) = 0$, then it is known that there exists $r > 0$ such that

$$\widehat{f}(\xi, Y) = \sum_{n=1}^{\infty} \frac{a_n(Y)}{\Gamma(n/\kappa)} \xi^{n-\kappa} \tag{1.5}$$

holds in $\{0 < |\xi| < r\} \times \Omega$. $\xi^{\kappa-1} \widehat{f}(\xi, Y)$ is holomorphic in $\{|\xi| < r\} \times \Omega$. It holds that for any small $\epsilon > 0$ there exist positive constants K_ϵ and c_ϵ such that

$$|\widehat{f}(\xi, Y)| \leq \frac{K_\epsilon |\xi|^{1-\kappa} e^{c_\epsilon |\xi|^\kappa}}{\Gamma(1/\kappa)}, \quad (\xi, Y) \in \big(\{0 < |\xi| < r\} \cup S_{\widehat{I}_\epsilon(\kappa)}\big) \times \Omega. \tag{1.6}$$

As for the details of Borel summable functions, Borel transform and Laplace transform we refer to Balser [1]. By expanding $\widehat{f}(\xi, Y)$ at $Y = 0 \in \mathbb{C}^n$,

$$\widehat{f}(\xi, Y) = \sum_{k \in \mathbb{N}^n} \widehat{f}_k(\xi) Y^k, \quad \widehat{f}_k(\xi) = \frac{1}{k!}(\frac{\partial}{\partial Y})^k \widehat{f}(\xi, 0). \tag{1.7}$$

Let $f(x, Y) \in \mathscr{O}_{\{1/\kappa\}}(S_0(I) \times \Omega)$ with asymptotic expansion (1.1) and $f(0, Y) = 0$ means $a_0(Y) = 0$. The following holds by Cauchy's inequality.

Lemma 1.3. *Let $f(x, Y) \in \mathscr{O}_{\{1/\kappa\}}(S_0(I) \times \Omega)$ with $f(0, Y) = 0$. Then there exist positive constants K_ϵ and c_ϵ such that*

$$|\widehat{f}_k(\xi)| \le \frac{K_\epsilon |\xi|^{1-\kappa} e^{c_\epsilon |\xi|^\kappa}}{R^{|k|}\Gamma(1/\kappa)}, \quad \xi \in \{0 < |\xi| < r\} \cup S_{\widehat{I}_\epsilon(\kappa)}. \tag{1.8}$$

Proposition 1.4. *Let $\{f_k(x)\}_{k \in \mathbb{N}^n} \subset \mathscr{O}_{\{1/\kappa\}}(S_0(I))$ with $f_k(0) = 0$. Suppose that for any small $\epsilon > 0$ there exist positive constants K_ϵ and c_ϵ such that their κ-Borel transforms $\{\widehat{f}_k(\xi)\}_{k \in \mathbb{N}^n}$ have bounds*

$$|\widehat{f}_k(\xi)| \le \frac{K_\epsilon |\xi|^{1-\kappa} e^{c_\epsilon |\xi|^\kappa}}{R^{|k|}\Gamma(1/\kappa)} \quad \xi \in \{0 < |\xi| < r\} \cup S_{\widehat{I}_\epsilon(\kappa)}. \tag{1.9}$$

Then $f(x, Y) = \sum_{k \in \mathbb{N}^n} f_k(x) Y^k \in \mathscr{O}_{\{1/\kappa\}}(S_0(I_\epsilon) \times \Omega_0)$, $\Omega_0 = \{Y \in \mathbb{C}^n; |Y| < R_0\}$ $(R_0 < R)$.

Proof. Since $g^*(\xi, Y) = \xi^{\kappa-1} \sum_{k \in \mathbb{N}^n} \widehat{f}_k(\xi) Y^k$ converges in $(\{|\xi| < r\} \cup S_{\widehat{I}_\epsilon(\kappa)}) \times \Omega_0$, $|g^*(\xi, Y)| \le M_\epsilon e^{c_\epsilon |\xi|^\kappa}$ and there exist $\{g_n^*(Y)\}_{n \ge 1} \subset \mathscr{O}(\Omega_0)$ such that $g^*(\xi, Y) = \sum_{n=1}^\infty g_n^*(Y) \xi^{n-1}$ in $\{|\xi| < r\}$. We have

$$\sum_{k \in \mathbb{N}^n} \widehat{f}_k(\xi) Y^k = \xi^{1-\kappa} g^*(\xi, Y) = \sum_{n=1}^\infty g_n^*(Y) \xi^{n-\kappa}.$$

in $\{0 < |\xi| < r\} \times \Omega_0$, Hence $\widehat{f}(\xi, Y) = \sum_{n=1}^\infty g_n^*(Y) \xi^{n-\kappa}$ and $f(x, Y) \in \mathscr{O}_{\{1/\kappa\}}(S_0(I_\epsilon) \times \Omega_0)$. □

If $f(x, Y) = \sum_{k \in \mathbb{N}^n} f_k(x) Y^k$ satisfies the assumptions of Proposition 1.4. then $f(x, Y)$ has an asymptotic expansion $\sum_{n=1}^\infty a_n(Y) t^n$ in $\mathscr{O}_{\{1/\kappa\}}(S_0(I_\epsilon) \times \Omega_0)$, $a_n(Y) = \Gamma(n/\kappa) g_n^*(Y)$.

Let $\phi_i^*(\xi, Y) \in \mathscr{O}(S_0(I^*) \times U)$ $(i = 1, 2)$. The κ-convolution is defined by

$$(\phi_1^* \underset{\kappa}{*} \phi_2^*)(\xi, Y) = \int_0^{\xi} \phi_1^*((\xi^\kappa - \eta^\kappa)^{1/\kappa}, Y)\phi_2^*(\eta, Y) d\eta^\kappa. \tag{1.10}$$

The following lemma is used for calculations and estimates of convolution equations later.

Lemma 1.5. *Suppose that* $\phi_i^*(\xi, Y) \in \mathscr{O}(S_0(I^*) \times U)$ $(i = 1, 2)$ *satisfy*

$$|\phi_i^*(\xi, Y)| \le \frac{C_i |\xi|^{s_i - \kappa} e^{c|\xi|^\kappa}}{\Gamma(s_i/\kappa)} \quad (s_i > 0) \quad \textit{for } (\xi, Y) \in S_0(I^*) \times U. \tag{1.11}$$

Then $(\phi_1^* \underset{\kappa}{*} \phi_2^*)(\xi, Y) \in \mathscr{O}(S_0(I^*), U)$ *and*

$$|(\phi_1^* \underset{\kappa}{*} \phi_2^*)(\xi, Y)| \le \frac{C_1 C_2 |\xi|^{s_1 + s_2 - \kappa} e^{c|\xi|^\kappa}}{\Gamma((s_1 + s_2)/\kappa)}. \tag{1.12}$$

Proof. Let $\arg \xi = \theta$. Then it holds that

$$\begin{aligned}(\phi_1^* \underset{\kappa}{*} \phi_2^*)(\xi, Y) &= \int_0^{|\xi| e^{i\theta}} \phi_1^*((\xi^\kappa - \eta^\kappa)^{1/\kappa}, Y)\phi_2^*(\eta, Y) d\eta^\kappa \\ &= \int_0^{|\xi|} \phi_1^*((|\xi|^\kappa - r^\kappa)^{1/\kappa} e^{i\theta}, Y)\phi_2^*(re^{i\theta}, Y) e^{i\kappa\theta} dr^\kappa\end{aligned}$$

and

$$\begin{aligned}&|\phi_1^*((|\xi|^\kappa - r^\kappa)^{1/\kappa} e^{i\theta}, Y)\phi_2^*(re^{i\theta}, Y) e^{i\kappa\theta}| \\ &\le \frac{C_1 C_2}{\Gamma(s_1/\kappa)\Gamma(s_2/\kappa)} (|\xi|^\kappa - r^\kappa)^{s_1/\kappa - 1} e^{c(|\xi|^\kappa - r^\kappa)} |r|^{s_2 - \kappa} e^{cr^\kappa}.\end{aligned}$$

We have (1.12) from

$$\int_0^{|\xi|} (|\xi|^\kappa - r^\kappa)^{s_1/\kappa - 1} |r|^{s_2 - \kappa} dr^\kappa = \frac{\Gamma(s_1/\kappa)\Gamma(s_2/\kappa)}{\Gamma((s_1 + s_2)/\kappa)} |\xi|^{s_1 + s_2 - \kappa}.$$

□

We note that Lemma 1.5 holds for an infinite $S(I^*)$. Let $\widehat{\phi}_i(\xi, Y)$ be κ-Borel transform of $\phi_i(x, Y)$ $(i = 1, 2)$. Then

$$\phi_1(x, Y)\phi_2(x, Y) = \mathcal{L}_\kappa(\widehat{\phi}_1 \underset{\kappa}{*} \widehat{\phi}_2) \tag{1.13}$$

holds. This means $\phi_1(x, Y)\phi_2(x, Y)$ is κ-Laplace transform of $(\widehat{\phi}_1 \underset{\kappa}{*} \widehat{\phi}_2)$.

2 Nonlinear equation with irregular singularity I

First we study the case $F_0(x) \equiv 0$. The case $F_0(x) \not\equiv 0$ is considered in section 4. Let

$$\begin{aligned} &x^{1+\gamma}\frac{dY}{dx} = A(x)Y + F(x,Y), \quad Y = {}^t(y_1.y_2,\dots,y_n), \\ &A(x) = (a_{i,j}(x))_{1\le i,j\le n},\ F(x,y) = {}^t(f_1(x,Y),\dots,f_n(x,Y)), \end{aligned} \tag{2.1}$$

where γ is a positive integer. $a_{i,j}(x)$ is holomorphic in $\{|x| < r\}$ and $f_i(x,Y)$ is holomorphic in $\{(x,Y) \in \mathbb{C}\times\mathbb{C}^n; |x| < r, |Y| < R\}$ with $f_i(x,Y) = O(|Y|^2)$. If $F(0,Y) \neq 0$, let $y_i = xz_i$. Then $x^{-1}F(x,xZ) = O(x)$ and

$$x^{1+\gamma}\frac{dZ}{dx} =(A(x) - x^\gamma I)Z + x^{-1}F(x,xZ).$$

Hence we replace $A(x)-x^\gamma I$ by $A(x)$ and $x^{-1}F(x,xZ)$ by $F(x,Z)$. We have $F(0,Z) = 0$. Thus we study, by denoting Z by Y again,

$$\begin{cases} x^{1+\gamma}\dfrac{dy_i}{dx} = \displaystyle\sum_{j=1}^n a_{i,j}(x)y_j + f_i(x,Y) \quad i = 1,2,\dots,n \\ A(x) = (a_{i,j}(x)),\ F(x,Y) = {}^t(f_1(x,Y),\dots,f_n(x.Y)) \end{cases} \tag{2.2}$$

with $f_i(0,Y) = 0$. Let $\{\lambda_i\}_{1\le i\le n}$ be eigenvalues of $A(0)$ and $\mathbf{\Lambda} = \{\lambda_i; i = 1,\dots,n\}$. We assume

Condition 0. Eigenvalues are distinct. $\lambda_i \neq \lambda_j$ for $i \neq j$.

Set $\omega_{i,j} = \arg(\lambda_i - \lambda_j)$ $(0 \le \omega_{i,j} < 2\pi)$ for $i \neq j$ and

$$\begin{cases} \mathbf{\Lambda}^\sharp = \{\lambda_i - \lambda_j; i,j = 1,2,\dots n, i\neq j\} \\ \theta_{i,j,\ell} = (\omega_{i,j} + 2\pi\ell)/\gamma,\ \ell\in\mathbb{Z} \\ \mathbf{\Theta}_1 = \{\theta_{i,j,\ell};\ i\neq j,\ \ell\in\mathbb{Z}\}. \end{cases} \tag{2.3}$$

Let $\theta_* \notin \mathbf{\Theta}_1$. Then there exists $\epsilon_* > 0$ such that $(\theta_* - \epsilon_*, \theta_* + \epsilon_*) \cap \mathbf{\Theta}_1 = \emptyset$. Let $\delta_* = \pi/2\gamma + \epsilon_*$ and $I = (\theta_* - \delta_*, \theta_* + \delta_*)$. Then the following Proposition holds under Condition 0.

Proposition 2.1. *Let $\theta_* \notin \mathbf{\Theta}_1$ and $a_{i,j}(x) \in \mathscr{O}_{\{1/\gamma\}}(S_0(I))$. Then there exists a matrix $P(x)$ with elements in $\mathscr{O}_{\{1/\gamma\}}(S_0(I_\epsilon))$ for any small $\epsilon > 0$ and $P(0) = Id$ such that $Y = P(x)Z$ transforms $x^{1+\gamma}\frac{dY}{dx} = A(x)Y$ to*

$$x^{1+\gamma}\frac{dZ}{dx} = \Lambda(x)Z, \tag{2.4}$$

where $\Lambda(x) = diag.(\lambda_1(x), \lambda_2(x), \ldots, \lambda_n(x))$ *is a diagonal matrix and* $\lambda_i(x)$ *is a polynomial with degree* $\leq \gamma$ *and* $\lambda_i(0) = \lambda_i$.

This proposition is a special case of the splitting (block diagonalization) of systems of equations ([1]). We give a proof of this proposition in section 6. We take $\theta_* \notin \mathbf{\Theta}_1$ later so that it satisfies other conditions.

Hence we study the following system of nonlinear differential equations:

$$\begin{cases} Y =^t(y_1.y_2, \ldots, y_n), \\ x^{1+\gamma}\dfrac{dY}{dx} =\Lambda(x)Y + F(x,Y), \\ \Lambda(x) =diag.(\lambda_1(x), \ldots, \lambda_n(x)), \\ F(x,Y) =^t(f_1(x,Y), \ldots, f_n(x,Y)), \end{cases} \tag{2.5}$$

where

$$\{f_i(x,Y)\}_{1\leq i\leq n} \subset \mathscr{O}_{\{1/\gamma\}}(S_0(I)\times\Omega) \tag{2.6}$$

with $F(x,Y) = O(|Y|^2)$ and $F(0,Y) = 0$, and $\{\lambda_i(x)\}_{1\leq i\leq n}$ are polynomials with degree $\leq \gamma$ and $\lambda_i(0) = \lambda_i$,

We proceed to give other conditions on the eigenvalues. Let $\emptyset \neq \mathbf{\Lambda}' \subset \mathbf{\Lambda}$ and $\mathbf{\Lambda}' = \{\lambda_i; i = 1, \ldots, n'\}$. We assume the following two conditions on $\mathbf{\Lambda}'$.

Condition 1. Partial Poincaré condition. There exist $0 \leq \theta_{\mathbf{\Lambda}'} < 2\pi$ *and* $0 < \delta_{\mathbf{\Lambda}'} < \pi/2$ *such that* $\mathbf{\Lambda}' \subset \Sigma = \{\eta \neq 0; |\arg\eta - \theta_{\mathbf{\Lambda}'}| < \delta_{\mathbf{\Lambda}'}\}$.

Condition 2. Partial non resonance.

$$\sum_{j=1}^{n'}\lambda_j m_j - \lambda_i \neq 0 \ \ \textit{for } m\in\mathbb{N}^{n'} \textit{ with } |m|\geq 2,\ 1\leq i\leq n. \tag{2.7}$$

Let

$$\mathfrak{L} =\bigcup_{i=1}^{n}\Big\{\sum_{j=1}^{n'}\lambda_j m_j - \lambda_i;\ |m|\geq 2\Big\} \not\ni 0. \tag{2.8}$$

Function $\gamma\eta + \sum_{j=1}^{n'}\lambda_j m_j - \lambda_i$ ($|m| \geq 2$) does not vanish for $\gamma\eta \notin -\mathfrak{L}$. Let $L(\theta) = \{r \geq 0; re^{i\theta}\}$ be a half line in a direction θ.

Lemma 2.2. *There exist an interval* $\widehat{J} = (\widehat{\theta}_0 - \widehat{\epsilon}_0, \widehat{\theta}_0 + \widehat{\epsilon}_0)$ $(\widehat{\epsilon}_0 > 0)$ *and constants* $r_0, C_{\widehat{J}} > 0$ *such that* $\big(S(\widehat{J}) \cup \{|\eta| < r_0\}\big) \cap (-\overline{\mathfrak{L}}) = \emptyset$ *and*

$$|\gamma\eta + \sum_{j=1}^{n'}\lambda_j m_j - \lambda_i| \geq C_{\widehat{J}}(|\eta| + |m|) \ \ |m|\geq 2 \ \ 1\leq i\leq n \tag{2.9}$$

for $\eta \in S(\widehat{J}) \cup \{|\eta| < r_0\}$.

Proof. Let $\varepsilon > 0$ be a small constant with $\delta_{\mathbf{\Lambda}'} + \varepsilon < \pi/2$ and $\Sigma_\varepsilon = \{\eta \neq 0; |\arg \eta - \theta_{\mathbf{\Lambda}'}| < \delta_{\mathbf{\Lambda}'} + \varepsilon\}$. Then $\mathfrak{L} \cap \Sigma_\varepsilon^c$ and $(-\mathfrak{L}) \cap (-\Sigma_\varepsilon^c)$ are finite. Hence there exists $\widehat{\theta}_0$ with $L(\widehat{\theta}_0) \subset (-\Sigma_\varepsilon^c)$ and $\widehat{\epsilon}_0 > 0$ such that $L(\theta) \cap (-\overline{\mathfrak{L}}) = \emptyset$ for $\theta \in \widehat{J} = (\widehat{\theta}_0 - \widehat{\epsilon}_0, \widehat{\theta}_0 + \widehat{\epsilon}_0)$ and

$$|\gamma\eta + \sum_{j=1}^{n'} \lambda_j m_j - \lambda_i| \geq C'_{\widehat{J}}(|\eta| + |m|) \quad \eta \in S(\widehat{J}).$$

holds for some constant $C'_{\widehat{J}} > 0$. Since there exist $r_0, c_0 > 0$ such that $|\gamma\eta + \sum_{j=1}^{n'} \lambda_j m_j - \lambda_i| > c_0(|m| + 1)$ in $\{|\eta| < r_0\}$, the assertion holds. □

Under Condition 0,1,2 there exists an interval $\widehat{I}$ satisfying

Condition 3. Let $\widehat{I} = (\theta_* - \epsilon_*, \theta_* + \epsilon_*)$ $(\epsilon_* > 0)$ *with* $L(\gamma\theta) \cap \mathbf{\Lambda}^\sharp = \emptyset$ *for* $\theta \in \widehat{I}$ *and there exist* $C_{\widehat{I}}, r > 0$ *such that*

$$\begin{aligned} &|\gamma\xi^\gamma + \sum_{j=1}^{n'} \lambda_j m_j - \lambda_i| \geq C_{\widehat{I}}(|\xi|^\gamma + |m|) \\ &\xi \in S(\widehat{I}) \cup \{|\xi| < r\}, \quad |m| \geq 2 \;\; 1 \leq i \leq n \end{aligned} \tag{2.10}$$

holds.

The interval $\widehat{I}$ appears in Laplace integral (3.10). We show the existence of $\widehat{I}$ satisfying Condition 3. Let $\widehat{J}$ be that in Lemma 2.2. Since $\mathbf{\Lambda}^\sharp$ is finite, we can take $\widehat{J}$ such that $S(\widehat{J}) \cap \mathbf{\Lambda}^\sharp = \emptyset$. Hence $S(\widehat{J}) \cap \big((-\overline{\mathfrak{L}}) \cup \mathbf{\Lambda}^\sharp\big) = \emptyset$. Let $\theta_* = \widehat{\theta}_0/\gamma$, $\epsilon_* = \widehat{\epsilon}_0/\gamma$ and $\widehat{I} = (\theta_* - \epsilon_*, \theta_* + \epsilon_*)$. Then $\gamma\widehat{I} = \widehat{J}$ and this $\widehat{I}$ satisfies the Condition 3.

Remark 2.3. *We can choose* $\theta_{\mathbf{\Lambda}'} = \widehat{\theta}_0$*, by changing* $\delta_{\mathbf{\Lambda}'}$ *if necessary.* $L(\gamma\theta) \cap \mathbf{\Lambda}^\sharp = \emptyset$ *is equivalent to* $\theta \notin \mathbf{\Theta_1}$.

Let us define an interval I for $\widehat{I} = (\theta_* - \epsilon_*, \theta_* + \epsilon_*)$ under Condition 3. Let $\delta_* = \pi/2\gamma + \epsilon_*$ and $I = (\theta_* - \delta_*, \theta_* + \delta_*)$. Then the angle of sectorial domain $S_0(I)$ in x-space is more than π/γ. We get one of the main theorems.

Theorem 2.4. *There exists* $\Phi(x, Z) = (\phi_1(x, Z), \ldots, \phi_n(x, Z))$, $Z = (z_1, \ldots, z_{n'}) \in \mathbb{C}^{n'}$*, such that for any small* $\epsilon > 0$ *there exists* $r_\epsilon > 0$, $\phi_i(x, Z) \in \mathscr{O}_{\{1/\gamma\}}(S_0(I_\epsilon) \times \{|Z| < r_\epsilon\})$ *and the followings hold.*

(1) $\phi_i(x,Z) = z_i + O(|Z|^2)$ *for* $1 \le i \le n'$ *and* $\phi_i(x,Z) = O(|Z|^2)$ *for* $i > n'$.

(2) *Let* $\mathcal{S}$ *be an open set in* $S_0(I_\epsilon)$ *and* $Z(x) = (z_1(x),\dots,z_{n'}(x))$ $(x \in \mathcal{S}, |Z(x)| < r_\epsilon)$ *be a solution of*

$$x^{1+\gamma}\frac{dz_i}{dx} = \lambda_i(x)z_i,\ i = 1,2,\dots,n'. \tag{2.11}$$

Then $Y(x) = \Phi(x,Z(x))$, $y_i(x) = \phi_i(x, z_1(x),\dots,z_{n'}(x))$ $(1 \le i \le n)$, *satisfies* (2.5) *in* $\mathcal{S}$.

Remark 2.5. (1) $\Phi(x,Z)$ *is determined by solving a system of partial differential equations* (3.2) *in section 3 and depends on choice of* θ_*.
(2) *Theorem 2.4 means that there exist solutions of* (2.5) *with exponential series, often called transseries* (see also Remarks 3.4),

$$\begin{cases} z_i(x) = A_i \exp\left(\int^x \frac{\lambda_i(\tau)}{\tau^{\gamma+1}}d\tau\right)\ \ (1 \le i \le n'), \\ y_i = z_i(x) + \sum_{|p|\ge 2} C_{i,p}(x)Z(x)^p\ \ (1 \le i \le n'), \\ y_i = \sum_{|p|\ge 2} C_{i,p}(x)Z(x)^p\ \ (i > n'), \quad C_{i,p}(x) \in \mathscr{O}_{\{1/\gamma\}}(S_0(I_\epsilon)). \end{cases} \tag{2.12}$$

(3) *If* $\Lambda' = \{\lambda_1\}, \lambda_1 \ne 0$, *then Condition 1 is obviously holds and Condition 2 is* $m_1\lambda_1 - \lambda_i \ne 0$ *for* $m_1 \ge 2$ *and* $i = 2,\dots,n$.
(4) *Some case of* (2.5) *is treated in [18],[19] for* $n = 1$ *and [5] for* $n \ge 2$ *for the purpose of the normalization of vector fields. The following system*

$$\begin{aligned} &x^{1+\gamma}\frac{dY}{dx} = (\Lambda + z^\gamma A)Y + z^\gamma F(x,Y) \quad Y = {}^t(y_1.\dots,y_n) \\ &\Lambda = diag.(\lambda_1,\dots,\lambda_n), A = diag.(\alpha_1,\dots,\alpha_n) \end{aligned} \tag{2.13}$$

is studied in [5] under some diophantine condition of $\{\lambda_i\}_{i=1}^n$.

3 Proof of Theorem 2.4

Our assumptions are

$$\begin{cases} \{f_i(x,Y)\}_{1\le i\le n} \subset \mathscr{O}_{\{1/\gamma\}}(S_0(I) \times \{|Y| < R\}), \\ I = (\theta_* - \delta_*, \theta_* + \delta_*), \widehat{I} = (\theta_* - \epsilon_*, \theta_* + \epsilon_*)\ \delta_* = \pi/2\gamma + \epsilon_*, \\ L(\gamma\theta) \cap \mathbf{\Lambda}^\sharp = \emptyset\ for\ \theta \in \widehat{I} \end{cases} \tag{3.1}$$

with $f_i(0,Y)=0$, $f_i(x,Y)=O(|Y|^2)$ and (2.10) holds.

3.1 Construction of $\Phi(x,Z)$-I

In order to construct $\Phi(x,Z)$ in Theorem 2.4 we introduce an auxiliary system of nonlinear partial differential equations

$$\begin{cases} \Phi(x,Z)=(\phi_1(x,Z),\phi_2(x,Z),\dots,\phi_n(x,Z)), \\ x^{1+\gamma}\dfrac{\partial \phi_i}{\partial x}+\displaystyle\sum_{j=1}^{n'}\lambda_j(x)z_j\frac{\partial \phi_i}{\partial z_j}-\lambda_i(x)\phi_i=f_i(x,\Phi) \quad 1\le i\le n, \\ (x,Z)=(x,z_1,\dots,z_{n'})\in\mathbb{C}\times\mathbb{C}^{n'}. \end{cases} \tag{3.2}$$

A similar type equation appeared in [20]. Assume we find a nice solution $\Phi(x,Z)$ of (3.2). Let $Z(x)=(z_1(x),\dots,z_{n'}(x))$ be a solution of

$$x^{1+\gamma}\frac{dz_i}{dx}=\lambda_i(x)z_i \quad 1\le i\le n' \tag{3.3}$$

and $Y(x)=\Phi(x,Z(x))$ $(y_i(z)=\phi_i(x,z_1(x),\dots,z_{n'}(x)))$. Then we have

$$\begin{aligned} x^{1+\gamma}\frac{dy_i}{dx} &= x^{1+\gamma}\frac{\partial\phi_i}{\partial x}+x^{1+\gamma}\sum_{j=1}^{n'}\frac{\partial \phi_i}{\partial z_j}\frac{dz_j}{dx} \\ &= x^{1+\gamma}\frac{\partial\phi_i}{\partial x}+\sum_{j=1}^{n'}\lambda_j(x)z_j\frac{\partial\phi_i}{\partial z_j}=\lambda_i(x)\phi_i+f_i(x,\Phi) \\ &= \lambda_i(x)y_i+f_i(x,Y(x)) \end{aligned}$$

and $Y(x)$ will be a solution of (2.5).

We construct $\Phi(x,Z)$ as follows. Let $\Psi(Z)=(\psi_1(Z),\psi_2(Z),\dots,\psi_n(Z))=(z_1,z_2,\dots,z_{n'},0,\dots,0)$ and $\Phi(x,Z)=U(x,Z)+\Psi(Z)$. We change (3.2) to a system of equations of $U(x,Z)=(u_1(x,Z),\dots,u_n(x,Z))$. By $\big(\lambda_i(x)-\sum_{j=1}^{n'}\lambda_j(x)z_j\frac{\partial}{\partial z_j}\big)\psi_i(Z)=0$ we have

$$\Big(x^{1+\gamma}\frac{\partial}{\partial x}+\sum_{j=1}^{n'}\lambda_j(x)z_j\frac{\partial}{\partial z_j}-\lambda_i(x)\Big)u_i=f_i(x,U+\Psi(Z)). \tag{3.4}$$

Let $f_i(x,\Phi) = \sum_{m\in\mathbb{N}^n,|m|\geq 2} f_{i,m}(x)\Phi^m$. Then there exist $g_{i,k,\ell}(x) \in \mathscr{O}_{\{1/\gamma\}}(S_0(I))$ $(k,\ell)\in\mathbb{N}^{n'}\times\mathbb{N}^n$ such that

$$\begin{aligned} f_i(x,U+\Psi(Z)) &= \sum_{m\in\mathbb{N}^n,|m|\geq 2} f_{i,m}(x)(U+\Psi(Z))^m \\ &= \sum_{\substack{(k,\ell)\in\mathbb{N}^{n'}\times\mathbb{N}^n,\\ |k|+|\ell|\geq 2,\ell\neq 0}} g_{i,k,\ell}(x)Z^kU^\ell + f_i(x,\Psi(Z)). \end{aligned}$$

It follows from $f_i(0,Y)=0$ that $g_{i,k,\ell}(0)=0$ and $|f_i(x,\Psi(Z))| \leq M|x||Z|^2$. Let

$$\begin{aligned} &L = x^{1+\gamma}\frac{\partial}{\partial x} + \sum_{j=1}^{n'} \lambda_j z_j \frac{\partial}{\partial z_j} - \lambda_i \quad \lambda_i = \lambda_i(0), \\ &\lambda_i^*(x) = \lambda_i(x) - \lambda_i, \;\; h_i(x,Z) := f_i(x,\Psi(Z)) = \sum_{|p|\geq 2} h_{i,p}(x)Z^p. \end{aligned} \tag{3.5}$$

Then $\lambda_i^*(0)=0$ and (3.4) is

$$\begin{aligned} Lu_i =& -(\sum_{j=1}^{n'} \lambda_j^*(x) z_j \frac{\partial}{\partial z_j} - \lambda_i^*(x))u_i \\ &+ \sum_{\substack{(k,\ell)\in\mathbb{N}^{n'}\times\mathbb{N}^n,\\ |k|+|\ell|\geq 2,\ell\neq 0}} g_{i,k,\ell}(x)Z^kU^\ell + h_i(x,Z). \end{aligned} \tag{3.6}$$

We introduce an auxiliary parameter ε in order to show clearly successive process of construction of a solution.

$$\begin{aligned} Lu_i =& -\varepsilon(\sum_{j=1}^{n'} \lambda_j^*(x) z_j \frac{\partial}{\partial z_j} - \lambda_i^*(x))u_i \\ &+ \varepsilon \sum_{\substack{(k,\ell)\in\mathbb{N}^{n'}\times\mathbb{N}^n,\\ |k|+|\ell|\geq 2,\ell\neq 0}} g_{i,k,\ell}(x)Z^kU^\ell + \varepsilon h_i(x,Z). \end{aligned} \tag{3.7}$$

If $\varepsilon = 1$, (3.7) coincides with (3.4) and (3.6). Our process is to find a solution $U(x,Z,\varepsilon) = (u_1(x,Z,\varepsilon),\ldots,u_n(x,Z,\varepsilon))$ of (3.7) and to take $\varepsilon = 1$.

It is constructed as follows. Let

$$u_i(x,Z,\varepsilon)=\sum_{\substack{(p,q)\in\mathbb{N}^{n'}\times\mathbb{N}\\ q\geq 1}} C_{i,p,q}(x)Z^p\varepsilon^q \quad 1\leq i\leq n, \tag{3.8}$$

and note $U^\ell=\prod_{s=1}^n u_s^{\ell_s},\ \ell=(\ell_1,\dots,\ell_s,\dots,\ell_n)$ and

$$u_s^{\ell_s}=\prod_{j=1}^{\ell_s}\Big(\sum_{\substack{(p^{s,j},q^{s,j})\in\mathbb{N}^{n'}\times\mathbb{N}\\ q^{s,j}\geq 1}} C_{s,p^{s,j},q^{s,j}}(x)Z^{p^{s,j}}\varepsilon^{q^{s,j}}\Big).$$

By substituting $u_i(x,Z,\varepsilon)$ into (3.7) and comparing the coefficient of $Z^p\varepsilon^q$, we get

$$\begin{aligned}(x^{1+\gamma}\frac{d}{dx}&+\sum_{j=1}^{n'}p_j\lambda_j-\lambda_i)C_{i,p,q}(x)=-\big(\sum_{j=1}^{n'}p_j\lambda_j^*(x)-\lambda_i^*(x)\big)C_{i,p,q-1}(x)\\ &+\sum_{\substack{(k,\ell)\in\mathbb{N}^{n'}\times\mathbb{N}^n\\ |k|+|\ell|\geq 2,\ell\neq 0}} g_{i,k,\ell}(x)\Big(\sum_{\substack{\sum_{s=1}^n\left(\sum_{j=1}^{\ell_s}p^{s,j}\right)+k=p\\ \sum_{s=1}^n\left(\sum_{j=1}^{\ell_s}q^{s,j}\right)+1=q}}\prod_{j=1}^{\ell_1}C_{1,p^{1,j},q^{1,j}}(x)\\ &\times\prod_{j=1}^{\ell_2}C_{2,p^{2,j},q^{2,j}}(x)\cdot\dots\cdot\prod_{j=1}^{\ell_n}C_{n,p^{n,j},q^{n,j}}(x)\Big)+\delta_{q,1}h_{i,p}(x),\end{aligned} \tag{3.9}$$

where $k,p^{s,j},p\in\mathbb{N}^{n'}$, $q,q^{s,j}\in\mathbb{N}$ and $\delta_{i,j}$ is Kronecker's delta.

3.2 Construction of $\Phi(x,Z)$-II

We try to construct $C_{i,p,q}(x)$ by Laplace integral

$$C_{i,p,q}(x)=\int_0^{\infty e^{\sqrt{-1}\theta}} e^{-(\frac{\xi}{x})^\gamma}\widehat{C}_{i,p,q}(\xi)d\xi^\gamma \quad \theta\in\widehat{I}. \tag{3.10}$$

We use the following notation

$$W_1(\xi)\underset{\gamma}{*}W_2(\xi)\underset{\gamma}{*}\dots\underset{\gamma}{*}W_N(\xi)=\underbrace{\prod_{i=1}^N}_{*\gamma}W_i(\xi).$$

By using (1.13), we get the following convolution equations from (3.9)

$$
\begin{aligned}
(\gamma\xi^{\gamma} + \sum_{j=1}^{n'} p_j\lambda_j - \lambda_i)\widehat{C}_{i,p,q}(\xi) &= -\big(\sum_{j=1}^{n'} p_j \widehat{\Lambda_j^*}(\xi) - \widehat{\Lambda_i^*}(\xi)\big) \underset{\gamma}{*} \widehat{C}_{i,p,q-1}(\xi) \\
+ \sum_{\substack{(k,\ell)\in\mathbb{N}^{n'}\times\mathbb{N}^n \\ |k|+|\ell|\geq 2, \ell\neq 0}} \widehat{g}_{i,k,\ell}(\xi) \underset{\gamma}{*} \Big(& \sum_{\substack{\sum_{s=1}^n(\sum_{j=1}^{\ell_s} p^{s,j})+k=p \\ \sum_{s=1}^n(\sum_{j=1}^{\ell_s} q^{s,j})+1=q}} \\
\underbrace{\prod_{j=1}^{\ell_1} \widehat{C}_{1,p^{1,j},q^{1,j}}(\xi) \prod_{j=1}^{\ell_2} \widehat{C}_{2,p^{2,j},q^{2,j}}(\xi) \cdots \prod_{j=1}^{\ell_n} \widehat{C}_{n,p^{n,j},q^{n,j}}(\xi)}_{*\gamma} \Big) & + \delta_{q,1}\widehat{h}_{i,p}(\xi).
\end{aligned}
\tag{3.11}
$$

The main result of this subsection is Proposition 3.1 concerning existence and estimate of $\widehat{C}_{i,p,q}(\xi)$. We proceed to solve (3.11). There are 2 steps, to determine $\widehat{C}_{i,p,q}(\xi)$ and to estimate them.

(I) *Determination of* $\widehat{C}_{i,p,q}(\xi)$. We notice that there exists a constant $C > 0$ such that for $\xi \in S(\widehat{I}) \cup \{|\xi| < r\}$ and $|p| \geq 2$

$$
|\gamma\xi^{\gamma} + \sum_{j=1}^{n'} p_j\lambda_j - \lambda_i| \geq C(|\xi|^{\gamma} + |p|). \tag{3.12}
$$

Let $q = 1$. Then $\widehat{C}_{i,p,1} = 0$ for $|p| \leq 1$ and

$$
\widehat{C}_{i,p,1}(\xi) = \frac{\widehat{h}_{i,p}(\xi)}{(\gamma\xi^{\gamma} + \sum_{j=1}^{n'} p_j\lambda_j - \lambda_i)} \tag{3.13}
$$

for $|p| \geq 2$. $\xi^{\gamma-1}\widehat{C}_{i,p,1}(\xi)$ is holomorphic at $\xi = 0$. Assume $\{\widehat{C}_{j,r,s}(\xi)\}_{j=1}^n$ ($s < q$) are determined such that $\widehat{C}_{j,r,s}(\xi) = 0$ for $|r| \leq 1$. We denote the right hand side of (3.11) by

$$
\begin{aligned}
&\mathcal{F}_{i,p,q}\big(\xi, \{\widehat{C}_{j,r,s}\}_{j=1}^n, (r,s) \in \Sigma_{p,q}\big) \\
&\Sigma_{p,q} = \{(r,s); 2 \leq |r| \leq |p|, 1 \leq s \leq q-1\}.
\end{aligned}
\tag{3.14}
$$

Then $\{\widehat{C}_{i,p,q}(\xi)\}_{i=1}^n$ are determined by

$$
\widehat{C}_{i,p,q}(\xi) = \frac{\mathcal{F}_{i,p,q}(\xi, \{\widehat{C}_{j,r,s}\}_{j=1}^n, (r,s) \in \Sigma_{p,q})}{(\gamma\xi^{\gamma} + \sum_{j=1}^{n'} p_j\lambda_j - \lambda_i)} \tag{3.15}
$$

and $\widehat{C}_{i,p,q}(\xi) = 0$ for $|p| \leq 1$. Thus $\widehat{C}_{i,p,q}(\xi)$ ($|p| \geq 2, q \geq 1$) are successively determined and they are holomorphic in $(\{0 < |\xi| < r\} \cup S(\widehat{I})$. Moreover $\xi^{\gamma-1}\widehat{C}_{i,p,q}(\xi)$ is holomrphic at $\xi = 0$.

(II) *Estimate of* $\widehat{C}_{i,p,q}(\xi)$. Let $\epsilon > 0$ be a small constant. We obtain estimates of $\widehat{C}_{i,p,q}(\xi)$ in a subsector $S(\widehat{I}_\epsilon) \subset S(\widehat{I})$. We often apply Lemma 1.5 to estimate. We have

Proposition 3.1. *Let $\epsilon > 0$ be an arbitrary small constant. Then there exist positive constants $r, M_{i,p,q}$ and c depending on ϵ such that*

$$|\widehat{C}_{i,p,q}(\xi)| \leq \frac{M_{i,p,q}|\xi|^{q-\gamma}e^{c|\xi|^\gamma}}{\Gamma(q/\gamma)} \quad \xi \in \{0 < |\xi| < r\} \cup S(\widehat{I}_\epsilon) \tag{3.16}$$

and the series $\sum_{\substack{(p,q)\in\mathbb{N}^{n'}\times\mathbb{N} \\ |p|\geq 2, q\geq 1}} M_{i,p,q}Z^p s^q$ converges in a neighborhood of $(Z, s) = (0, 0) \in \mathbb{C}^{n'} \times \mathbb{C}$.

Before the proof we note inequality (3.12) and that there exist constants $G_{i,k,\ell}, H_{i,p}$ and c such that

$$\begin{cases} |\widehat{g}_{i,k,\ell}(\xi)| \leq \dfrac{G_{i,k,\ell}|\xi|^{1-\gamma}e^{c|\xi|^\gamma}}{\Gamma(1/\gamma)} \\ |\widehat{h}_{i,p}(\xi)| \leq \dfrac{H_{i,p}|\xi|^{1-\gamma}e^{c|\xi|^\gamma}}{\Gamma(1/\gamma)} \qquad \xi \in \{0 < |\xi| < r_0\} \cup S(\widehat{I}_\epsilon). \end{cases} \tag{3.17}$$

Here $\sum_{\substack{(k,\ell)\in\mathbb{N}^{n'}\times\mathbb{N}^n \\ |k|+|\ell|\geq 2, \ell\neq 0}} G_{i,k,\ell}Z^k U^\ell$ converges in a neighborhood of $(Z, U) = (0, 0) \in \mathbb{C}^{n'} \times \mathbb{C}^n$ and $\sum_{|p|\geq 2} H_{i,p}Z^p$ converges in a neighborhood of $Z = 0 \in \mathbb{C}^{n'}$ (see Lemma 1.3).

Proof of Proposition 3.1. The proof consists of 2 parts, (1) determination of $M_{i,p,q}$ and (2) convergence of the series.

(1) *Determination of* $M_{i,p,q}$. Let r_0 in (3.17) and r in (3.16) with $0 < r < r_0$. First we show how to determine $M_{i,p,q}$ ($p \in \mathbb{N}^{n'}$ $|p| \geq 2, q \geq 1$) and study their relations. For $q = 1$ and $|p| \geq 2$ there exist a constant $C > 0$ such that

$$|\widehat{C}_{i,p,1}(\xi)| = \frac{|\widehat{h}_{i,p}(\xi)|}{|\gamma\xi^\gamma + \sum_{j=1}^{n'} p_j\lambda_j - \lambda_i|} \leq CH_{i,p}\frac{|\xi|^{1-\gamma}e^{c|\xi|^\gamma}}{\Gamma(1/\gamma)}$$

and take $M_{i,p,1} = CH_{i,p}$. Assume $\{M_{i,p,q'}\}_{i=1}^n$ $(q' < q)$ are determined such that

$$|\widehat{C}_{i,p,q'}(\xi)| \leq \frac{M_{i,p,q'}|\xi|^{q'-\gamma}e^{c|\xi|^\gamma}}{\Gamma(q'/\gamma)} \quad \xi \in \{0 < |\xi| < r\} \cup S(\widehat{I}_\epsilon). \tag{3.18}$$

Let us notice relation (3.11). It follows from Lemma 1.5 and (3.12) that there exists a constant $A > 0$ such that

$$D_1 = \frac{\left|\left(\sum_{j=1}^{n'} p_j|\widehat{\Lambda_j^*}(\xi) - \widehat{\Lambda_i^*}(\xi)\right) \underset{\gamma}{*} \widehat{C}_{i,p,q-1}(\xi)\right|}{|\gamma\xi^\gamma + \sum_{j=1}^{n'} p_j\lambda_j - \lambda_i|} \leq \frac{AM_{i,p,q-1}|\xi|^{q-\gamma}e^{c|\xi|^\gamma}}{\Gamma(q/\gamma)}. \tag{3.19}$$

Let

$$D_2 = |\gamma\xi^\gamma + \sum_{j=1}^{n'} p_j\lambda_j - \lambda_i|^{-1}\Big| \sum_{\substack{(k,\ell)\in\mathbb{N}^{n'}\times\mathbb{N}^n \\ |k|+|\ell|\geq 2, \ell\neq 0}} \widehat{g}^*_{i,k,\ell}(\xi) \underset{\gamma}{*} \Big(\sum_{\substack{\sum_{s=1}^n \left(\sum_{j=1}^{\ell_s} p^{s,j}\right)+k=p \\ \sum_{s=1}^n \left(\sum_{j=1}^{\ell_s} q^{s,j}\right)+1=q}} \underbrace{\prod_{j=1}^{\ell_1} \widehat{C}_{1,p^{1,j},q^{1,j}}(\xi) \prod_{j=1}^{\ell_2} \widehat{C}_{2,p^{2,j},q^{2,j}}(\xi) \cdot \ldots \cdot \prod_{j=1}^{\ell_n} \widehat{C}_{n,p^{n,j},q^{n,j}}(\xi)}_{*\gamma} \Big)\Big|.$$

Since

$$\begin{aligned}
&\Big|\underbrace{\prod_{j=1}^{\ell_1}\widehat{C}_{1,p^{1,j},q^{1,j}}(\xi) \prod_{j=1}^{\ell_2} \widehat{C}_{2,p^{2,j},q^{2,j}}(\xi) \cdot \ldots \cdot \prod_{j=1}^{\ell_n} \widehat{C}_{n,p^{n,j},q^{n,j}}(\xi)}_{*\gamma}\Big| \\
\leq &\prod_{j=1}^{\ell_1} M_{1,p^{1,j},q^{1,j}} \prod_{j=1}^{\ell_2} M_{2,p^{2,j},q^{2,j}} \cdot \ldots \cdot \prod_{j=1}^{\ell_n} M_{n,p^{n,j},q^{n,j}} \\
&\times (\underbrace{\prod_{j=1}^{\ell_1} \frac{|\xi|^{q^{1,j}-\gamma}e^{c|\xi|^\gamma}}{\Gamma(q^{1,j}/\gamma)} \prod_{j=1}^{\ell_2} \frac{|\xi|^{q^{2,j}-\gamma}e^{c|\xi|^\gamma}}{\Gamma(q^{2,j}/\gamma)} \cdot \ldots \cdot \prod_{j=1}^{\ell_n} \frac{|\xi|^{q^{n,j}-\gamma}e^{c|\xi|^\gamma}}{\Gamma(q^{n,j}/\gamma)}}_{*\gamma}),
\end{aligned}$$

we have

$$
\begin{aligned}
D_2 \leq & C \sum_{\substack{(k,\ell)\in\mathbb{N}^{n'}\times\mathbb{N}^n \\ |k|+|\ell|\geq 2, \ell\neq 0}} G_{i,k,\ell}\Big(\sum_{\substack{\sum_{s=1}^{n}\left(\sum_{j=1}^{\ell_s} p^{s,j}\right)+k=p \\ \sum_{s=1}^{n}\left(\sum_{j=1}^{\ell_s} q^{s,j}\right)+1=q}} \prod_{j=1}^{\ell_1} M_{1,p^{1,j},q^{1,j}} \prod_{j=1}^{\ell_2} M_{2,p^{2,j},q^{2,j}} \\
& \cdots\cdots \prod_{j=1}^{\ell_n} M_{n,p^{n,j},q^{n,j}}\Big) \frac{|\xi|^{q-\gamma}}{\Gamma(q/\gamma)} e^{c|\xi|^{\gamma}}.
\end{aligned}
\tag{3.20}
$$

Hence we define for $q \geq 2$

$$
\begin{aligned}
M_{i,p,q} = & A M_{i,p,q-1} + C \sum_{\substack{(k,\ell)\in\mathbb{N}^{n'}\times\mathbb{N}^n \\ |k|+|\ell|\geq 2, \ell\neq 0}} G_{i,k,\ell}\big(\sum_{\substack{\sum_{s=1}^{n}\left(\sum_{j=1}^{\ell_s} p^{s,j}\right)+k=p \\ \sum_{s=1}^{n}\left(\sum_{j=1}^{\ell_s} q^{s,j}\right)+1=q}} \\
& \prod_{j=1}^{\ell_1} M_{1,p^{1,j},q^{1,j}} \prod_{j=1}^{\ell_2} M_{2,p^{2,j},q^{2,j}} \cdots \prod_{j=1}^{\ell_n} M_{n,p^{n,j},q^{n,j}}\big).
\end{aligned}
\tag{3.21}
$$

Thus we have

$$
|C_{i,p,q}(\xi)| \leq \sum_{i=1}^{2} D_i \leq \frac{M_{i,p,q}|\xi|^{q-\gamma}e^{c|\xi|^{\gamma}}}{\Gamma(q/\gamma)}. \tag{3.22}
$$

(2) *Convergence of* $\sum_{|p|\geq 2, q\geq 1} M_{i,p,q} Z^p s^q$. For this purpose we use the method of implicit functions used in [13] and others. We introduce holomorphic functions

$$
G_i(Z,U) = \sum_{\substack{(k,\ell)\in\mathbb{N}^{n'}\times\mathbb{N}^n, \\ |k|+|\ell|\geq 2, \ell\neq 0}} G_{i,k,\ell} Z^k U^{\ell}, \quad H_i(Z) = \sum_{|p|\geq 2} H_{i,p} Z^p
$$

at $(Z,U) = (0,0) \in \mathbb{C}^{n'} \times \mathbb{C}^n$, $Z = (z_1, \ldots, z_{n'})$, $U = (u_1, \ldots, u_n)$. Let

$$
F_i(Z,s,U) = sAu_i + sCG_i(Z,U) + sCH_i(Z) \tag{3.23}
$$

and consider a system of functional equations with unknown functions $U = (u_1, u_2, \ldots, u_n)$

$$
u_i = F_i(Z,s,U) \quad i = 1, \ldots, n. \tag{3.24}
$$

We have

Lemma 3.2. *There exists a unique solution* $U(Z,s) = (u_1(Z,s),\dots,u_n(Z,s))$ *of* (3.24) *such that it is holomorphic at* $(Z,s) = (0,0)$ *and* $U(0,0) = 0$ *with expansion*

$$u_i(Z,s) = \sum_{|p|\geq 2, q\geq 1} u_{i,p,q} Z^p s^q. \tag{3.25}$$

Moreover $M_{i,p,q} = u_{i,p,q}$.

Proposition 3.1 follows from Lemma 3.2.

Proof of Lemma 3.2. It follows from $F_i(0,0,0) = 0$ and $(\frac{\partial F_i}{\partial u_j}) = 0$ at $(Z,s,U) = (0,0,0)$ that there exists a unique solution $U(Z,s) = (u_1(Z,s),\dots,u_n(Z,s))$ of (3.24) with $u_i(0,0) = 0$ for $1 \leq i \leq n$. Let $u_i(Z,s) = \sum_{|p|+q\geq 1} u_{i,p,q} Z^p s^q$. We have from (3.23) and (3.24)

$$\begin{aligned} u_{i,p,q} = {} & A u_{i,p,q-1} + \\ & + C \sum_{\substack{(k,\ell)\in\mathbb{N}^{n'}\times\mathbb{N}^n \\ |k|+|\ell|\geq 2, \ell\neq 0}} G_{i,k,\ell}\Big(\sum_{\substack{\sum_{s=1}^n (\sum_{j=1}^{\ell_s} p_{s,j}) + k = p \\ \sum_{s=1}^n (\sum_{j=1}^{\ell_s} q_{s,j}) + 1 = q}} \prod_{j=1}^{\ell_1} u_{1,p_{1,j},q_{1,j}} \prod_{j=1}^{\ell_2} u_{2,p_{2,j},q_{2,j}} \cdots\cdots \\ & \cdots \prod_{j=1}^{\ell_n} u_{n,p_{n,j},q_{n,j}} \Big) + \delta_{q,1} C H_{i,p}. \end{aligned}$$

Since $\delta_{q,1} H_{i,p} = 0$ for $q \neq 1$ and $u_{i,p,0} = 0$, we have $u_{i,p,1} = CH_p = M_{i,p,1}$ for $|p| \geq 2$ and $u_{i,p,1} = 0$ for $|p| \leq 1$. Assume $u_{i,p,q'} = M_{i,p,q'}$ for $q' < q$. Then by (3.21) $u_{i,p,q} = M_{i,p,q}$. □

3.3 Construction of $\Phi(x,z)$-III

Let us show $\sum_{|p|\geq 2, q\geq 1} \widehat{C}_{i,p,q}(\xi) Z^p \varepsilon^q$ is convergent. It follows from Proposition 3.1 that there exist A, B such that $M_{i,p,q} \leq A^{|p|} B^q$. We have

$$\begin{aligned} \sum_{q\geq 1} |\widehat{C}_{i,p,q}(\xi)\varepsilon^q| &\leq \sum_{q\geq 1} \frac{M_{i,p,q}|\xi|^{q-\gamma}\varepsilon^q|}{\Gamma(q/\gamma)} e^{c|\xi|^\gamma} \\ &\leq \sum_{q\geq 1} \frac{A^{|p|}B^q|\xi|^{q-\gamma}|\varepsilon|^q}{\Gamma(q/\gamma)} e^{c|\xi|^\gamma} \leq C A^{|p|} \frac{|\xi|^{1-\gamma} e^{(c+c'_\varepsilon)|\xi|^\gamma}}{\Gamma(1/\gamma)}. \end{aligned} \tag{3.26}$$

Hence $\sum_{|p|\geq 2, q\geq 1} |\widehat{C}_{i,p,q}(\xi) Z^p \varepsilon^q|$ converges and $\{\widehat{C}_{i,p,q}(\xi)\}$ $(1 \leq i \leq n)$ satisfy (3.11). Take $\varepsilon = 1$ and let $\widehat{C}_{i,p}(\xi) = \sum_{q\geq 1} \widehat{C}_{i,p,q}(\xi)$ and

$$C_{i,p}(x) = \int_{L(\theta)} e^{-(\frac{\xi}{x})^\gamma} \widehat{C}_{i,p}(\xi) d\xi^\gamma \quad \theta \in \widehat{I}_\epsilon. \tag{3.27}$$

Then it follows from Proposition 1.4 that $u_i(x, Z) = \sum_{|p|\geq 2} C_{i,p}(x) Z^p \in \mathscr{O}_{\{1/\gamma\}}(S_0(I) \times \{|Z| < r_\epsilon\}$ and $\{u_i(x, Z)\}_{i=1}^n$ satisfy (3.6). We get a solution $U(x, Z) = (u_1(x, Z), \ldots, u_n(x, Z))$ of (3.4). Consequently $\Phi(x, Z) = U(x, Z) + \Psi(Z) = (\phi_1(x, Z), \ldots, \phi_n(x, Z))$ is a solution of equation (3.2) such that $\phi_i(x, Z) \in \mathscr{O}_{\{1/\gamma\}}(S_0(I_\epsilon) \times \{|Z| < r_\epsilon\})$ with $\phi_i(x, Z) = z_i + O(|Z|^2)$ for $1 \leq i \leq n'$ and $\phi_i(x, z) = O(|Z|^2)$ for $i > n'$. Thus we get Theorem 2.4.

3.4 Reduction to $\theta_{\mathbf{\Lambda}'} = 0$

We can take $\theta_{\mathbf{\Lambda}'} = \widehat{\theta}_0$ and $\theta_* = \theta_{\mathbf{\Lambda}'}/\gamma$ (see Remark 3.2). By transformation $x = t\exp(i\theta_*)$, $\frac{d}{dx} = \exp(-i\theta_*)\frac{d}{dt}$ and $x^{1+\gamma}\frac{d}{dx} = e^{i\gamma\theta_*} t^{1+\gamma}\frac{d}{dt}$,

$$x^{1+\gamma}\frac{d}{dx} - \lambda_i = e^{i\theta_{\mathbf{\Lambda}'}}(t^{1+\gamma}\frac{d}{dt} - e^{-i\theta_{\mathbf{\Lambda}'}}\lambda_i).$$

We have $\arg(e^{-i\theta_{\mathbf{\Lambda}'}}\lambda_i) = \omega_i - \theta_{\mathbf{\Lambda}'}$ and $|\omega_i - \theta_{\mathbf{\Lambda}'}| < \delta < \pi/2$. Hence we may assume $\theta_{\mathbf{\Lambda}'} = 0$ and

$$\arg\lambda_i = \omega_i, \ |\omega_i| < \delta < \frac{\pi}{2}, \quad 1 \leq i \leq n'.$$

Then there exist $\delta_* > \pi/2\gamma$ and $r = r(\delta_*) > 0$ such that

$$\phi_i(x, Z) \in \mathscr{O}_{\{1/\gamma\}}\big(S_0(I) \times \{|Z| < r\}\big), \ I = (-\delta_*, \delta_*). \tag{3.28}$$

Let

$$z_i(x) = e^{h_i(x)} \quad h_i(x) = \int^x \frac{\lambda_i(t)}{t^{\gamma+1}} dt = -\frac{\lambda_i}{\gamma x^\gamma} + \ldots, \quad 1 \leq i \leq n' \tag{3.29}$$

and $\varepsilon > 0$ be a small constant. If $|\gamma \arg x - \omega_i| < (\pi/2 - \varepsilon)$, $z_i(x)$ exponetially decreases with order γ. Let $\mathcal{I}(\varepsilon) = \cap_{i=1}^{n'}\{\theta; |\gamma\theta - \omega_i| < (\pi/2 - \varepsilon)\}$. Then $\{\theta; |\gamma\theta| < (\pi/2 - \delta - \varepsilon)\} \subset \mathcal{I}(\varepsilon)$ and $I(\varepsilon) \neq \emptyset$. We have from Theorem 2.4

Corollary 3.3. *Let $x \in S_0(\mathcal{I}(\varepsilon))$ and $y_i(x) = \phi_i(x, C_1 z_1(x), \ldots, C_{n'} z_{n'}(x))$, where $|C_k z_k(x)| < r$. Then $Y(x) = {}^t(y_1(x), \ldots, y_n(x))$ is a solution of* (2.5).

4 Nonlinear equation with irregular singularity II

In this section we study

$$\begin{cases} Y = {}^t(y_1.y_2, \dots, y_n) \\ x^{1+\gamma}\dfrac{dY}{dx} = F_0(x) + A(x)Y + F(x,Y), \end{cases} \tag{4.1}$$

where $A(x) = (a_{i,j}(x))_{1\le i,j\le n}$, $F_0(x)$ and $a_{i,j}(x)$ are holomorphic in a neighborhood of $x = 0$ and $F_0(0) = 0$. $F(x,Y)$ is holomorphic in a neighborhood of $(x,Y) = (0,0)$ and $F(x,Y) = O(|Y|^2)$. Let $\{\lambda_i\}_{1\le i\le n}$ be eigenvalues of $A(0)$ and $\lambda_i \neq 0$ for all i. Let $\omega_i = \arg\lambda_i$ $(0 \le \omega_i < 2\pi)$ and

$$\mathbf{\Theta}_0 = \{(\omega_i + 2\pi\ell)/\gamma,\ 1 \le i \le n, \ell \in \mathbb{Z}\}. \tag{4.2}$$

There exists a unique formal solution $\widetilde{K}(x) \in \mathbb{C}[[x]]^n$ with $\widetilde{K}(0) = 0$ of (4.1). Its Borel summability follows from [2]. We have

Proposition 4.1. *Suppose $\theta_* \notin \mathbf{\Theta}_0$. Then there exists a solution $K(x)$ of* (4.1)*, which is γ-Borel summable in the direction θ_* with the asymptotic expansion $\widetilde{K}(x)$.*

Put $Y = xW + K(x)$. Then

$$\begin{aligned} x^{1+\gamma}\frac{dW}{dx} &= (A(x) - x^\gamma I)W + x^{-1}\big(F(x, xW + K(x)) - F(x, K(x))\big) \\ x^{1+\gamma}\frac{dw_i}{dx} &= \sum_{j=1}^n (a_{i,j}(x) - \delta_{i,j}x^\gamma)w_j + x^{-1}\big(f_i(x, xW + K(x)) - f_i(x, K(x))\big) \\ &= \sum_{j=1}^n (a_{i,j}(x) - \delta_{i,j}x^\gamma)w_j + \sum_{j=1}^n \frac{\partial}{\partial y_j} f_i(x, K(x))w_j + g_i(x, W). \end{aligned}$$

Put $A'(x) = \big(a_{i,j}(x) - \delta_{i,j}x^\gamma + \frac{\partial}{\partial y_j} f_i(x, K(x))\big)$. Then

$$\begin{cases} W = {}^t(w_1.w_2, \dots, w_n), \\ x^{1+\gamma}\dfrac{dW}{dx} = A'(x)W + G(x,W), \end{cases} \tag{4.3}$$

where $A'(0) = A(0)$ and $G(x,W) = {}^t(g_1(x,W), \dots, g_n(x,W))$ with $G(0,W) = 0$ and $G(x,W) = O(|W|^2)$. Assume $\{\lambda_i\}_{1\le i\le n}$ are distinct (Condition 0) and

$\theta_* \notin \mathbf{\Theta}_0 \cup \mathbf{\Theta}_1$. Then by an invertible linear transformation $W = P(x)U$ whose elements in $\mathscr{O}_{\{1/\gamma\}}(S_0(I))$ ($I = (\theta_* - \delta_*, \theta_* + \delta_*)$, $\delta_* > \pi/2\gamma$), we have

$$\begin{cases} U =& {}^t(u_1.u_2, \dots, u_n), \\ x^{1+\gamma}\dfrac{dU}{dx} =& B(x)U + H(x,U), \\ B(x) =& diag.\ (b_1(x), b_2(x), \dots, b_n(x)), \\ H(x,U) =& {}^t(h_1(x,U), h_2(x,U), \dots, h_n(x,U)), \end{cases} \tag{4.4}$$

where $b_i(x)$ is a polynomial with degree $\le \gamma$ and $b_i(0) = \lambda_i$ and $h_i(x,U) \in \mathscr{O}_{\{1/\gamma\}}(S_0(I) \times \Omega)$, $\Omega = \{U \in \mathbb{C}^n; |U| < R\}$, with $h_i(0,U) = 0$ and $h_i(x,U) = O(|U|^2)$. Consequently we get (4.4) from (4.1) by a transformation $Y = K(x) + xW = K(x) + xP(x)U$. We remark that $B(x)$ depends on $K(x)$.

Set $\Lambda' = \{\lambda_i; 1 \le i \le n'\}$ and assume Conditions 1 and 2. Take θ_* so that it satisfies the assumptions of Theorem 2.4. Consider an $n' \times n'$ system of linear equations

$$\begin{cases} Z =& {}^t(z_1.z_2, \dots, z_{n'}), \\ x^{1+\gamma}\dfrac{dz_i}{dx} =& b_i(x)z_i,\ 1 \le i \le n'. \end{cases} \tag{4.5}$$

By applying Theorem 2.4, we have

Theorem 4.2. *There exists $\Phi(x,Z) = (\phi_1(x,Z), \dots, \phi_n(x,Z))$ such that for any small $\epsilon > 0$ there exists $r_\epsilon > 0$, $\phi_i(x,Z) \in \mathscr{O}_{\{1/\gamma\}}(S_0(I_\epsilon) \times \{Z \in \mathbb{C}^{n'}; |Z| < r_\epsilon\})$ and the followings hold.*

(1) *$\phi_i(x,Z) = z_i + O(|Z|^2)$ for $1 \le i \le n'$ and $\phi_i(x,Z) = O(|Z|^2)$ for $i > n'$.*

(2) *Let $\mathcal{S}$ be an open set in $S_0(I_\epsilon)$ and $Z(x) = (z_1(x), \dots, z_{n'}(x))$ ($x \in \mathcal{S}, |Z(x)| < r_\epsilon$) be a solution of* (4.5)*. Then $Y(x) = K(x) + xP(x)\Phi(x, Z(x))$ satisfies* (4.1) *in $\mathcal{S}$.*

5 Applications

5.1 A special 2×2 system

Let us consider a special 2×2 system as an example

$$x^{1+\gamma}\frac{dY}{dx} = F_0(x) + A(x)Y + F(x,Y) \quad Y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}. \tag{5.1}$$

The assumptions for (5.1) are the same as that for (4.1). Let λ_1, λ_2 be eigenvalues of $A(0)$. Further we assume

Condition 5-1

$\lambda_1\lambda_2 \neq 0$ and $\arg\lambda_1 = \omega$, $\arg\lambda_2 = \omega+\pi$.

Then $\Theta_0 \cup \Theta_1 = \{(\omega+\ell\pi)/\gamma; \ell \in \mathbb{Z}\}$. If $\Lambda' = \{\lambda_1\}$ (resp. $\Lambda' = \{\lambda_2\}$), then $\mathfrak{L} \subset L(\omega)$ (resp. $\mathfrak{L} \subset L(\omega+\pi)$) (see (2.8)). Let $\theta_* \notin \Theta_0 \cup \Theta_1$ and $K(x) = {}^t(k_1(x), k_2(x))$ be a solution of (5.1) with γ-Borel summable in the direction θ_*. By eliminating $F_0(x)$ and a linear transformation of the unknowns $Y = K(x) + xP(x)U$, $U = {}^t(u_1, u_2)$, we can reduce (5.1) to

$$x^{1+\gamma}\frac{d}{dx}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} \lambda_1(x) & 0 \\ 0 & \lambda_2(x) \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + G(x,U),\ G(x,U) = \begin{bmatrix} g_1(x,U) \\ g_2(x,U) \end{bmatrix}. \tag{5.2}$$

$\lambda_i(x)$ $(i = 1,2)$ is a polynomial with degree $\leq \gamma$ and $\lambda_i(0) = \lambda_i$. $G(x,U)$ is γ-Borel summable in the direction θ_* with respect to x such that $G(x,U) = O(|U|^2)$ and $G(0,U) = 0$.

Let $h_i(x) = \int^x \frac{\lambda_i(\tau)}{\tau^{\gamma+1}}d\tau$. Then $Z = {}^t(C_1e^{h_1(x)}, C_2e^{h_2(x)})$ is a solution of

$$x^{1+\gamma}\frac{d}{dx}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} \lambda_1(x) & 0 \\ 0 & \lambda_2(x) \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}. \tag{5.3}$$

Take $\theta_* \in (\frac{\omega}{\gamma}, \frac{\omega+\pi}{\gamma})$ and let $I = (\frac{\omega}{\gamma} - \frac{\pi}{2\gamma}, \frac{\omega}{\gamma} + \frac{3\pi}{2\gamma})$. Then the assumptions of Theorem 4.2 hold for both $\Lambda' = \{\lambda_1\}$ and $\Lambda' = \{\lambda_2\}$. Hence there exist $\Psi_1(x,z_1) = {}^t(\psi_{1,1}(x,z_1), \psi_{1,2}(x,z_1))$ and $\Psi_2(x,z_2) = {}^t(\psi_{2,1}(x,z_2), \psi_{2,2}(x,z_2))$ with $\psi_{i,j}(x,z_i) = \sum_{n=1}^\infty \psi_{i,j}^n(x)z_i^n \in \mathscr{O}_{\{1/\gamma\}}(S_0(I_\epsilon) \times \{|z_i| < r_\epsilon\})$ for any small $\epsilon > 0$ and $|\partial_{z_i}\psi_{i,1}(x,0)| + |\partial_{z_i}\psi_{i,2}(x,0)| \not\equiv 0$ such that they have the following properties.

Theorem 5.1. *Let $\epsilon > 0$ be an arbitrary small constant.*
(1) *Let $\mathcal{I}^{\frac{\omega}{\gamma}} = (\frac{\omega}{\gamma} - \frac{\pi}{2\gamma}, \frac{\omega}{\gamma} + \frac{\pi}{2\gamma})$. Then there exists $C(\epsilon) > 0$ such that $Y(x) = {}^t(y_1(x), y_2(x))$,*

$$y_j(x) = k_j(x) + \sum_{n=1}^\infty \psi_{1,j}^n(x)C_1^n e^{nh_1(x)} \quad j = 1,2, \tag{5.4}$$

with $|C_1| < C(\epsilon)$ is a solution of (5.1) *in $S_0(\mathcal{I}_\epsilon^{\frac{\omega}{\gamma}})$.*
(2) *Let $\mathcal{I}^{\frac{\omega+\pi}{\gamma}} = (\frac{\omega}{\gamma} + \frac{\pi}{2\gamma}, \frac{\omega}{\gamma} + \frac{3\pi}{2\gamma})$. Then there exists $C(\epsilon) > 0$ such that*

$Y(x) = {}^t(y_1(x), y_2(x))$,

$$y_j(x) = k_j(x) + \sum_{n=1}^{\infty} \psi_{2,j}^n(x) C_2^n e^{nh_2(x)} \quad j = 1, 2, \tag{5.5}$$

with $|C_2| < C(\epsilon)$ *is a solution of* (5.1) *in* $S_0(\mathcal{I}_\epsilon^{\frac{\omega+\pi}{\gamma}})$.

Proof. Since $e^{h_1(x)}$ ($e^{h_2(x)}$) decays exponentially in $S_0(\mathcal{I}_\epsilon^{\omega/\gamma})$ $(resp. S_0(\mathcal{I}_\epsilon^{(\omega+\pi)/\gamma}))$. $Y(x) = {}^t(y_1(x), y_2(x))$ is a solution of (5.1) in $S_0(\mathcal{I}_\epsilon^{\omega/\gamma})$ $(resp. S_0(\mathcal{I}_\epsilon^{(\omega+\pi)/\gamma}))$. □

5.2 Painlevé 4

We apply Theorem 5.1 to Painlevé 4 equation as an example. Other Painlevé equations at irregular singular points are also studied in the same way. Transseries solutions are studied for Painlevé 1, 2, 5 equations by resurgence methods in [8], [9], [10] and [11].
Painlevé-4 is

$$(P_4): \quad y'' = \frac{(y')^2}{2y} + \frac{3}{2}y^3 + 4ty^2 + 2(t^2 - \alpha)y + \frac{\beta}{y}. \tag{5.6}$$

Let $t = 1/x$. Then

$$x^2(x^2\frac{d}{dx})^2 y = \frac{1}{2y}(x^3\frac{dy}{dx})^2 + \frac{3}{2}x^2y^3 + 4xy^2 + 2(1 - \alpha x^2)y + \frac{\beta x^2}{y}. \tag{5.7}$$

(I) Futher by multiplying (5.7) by x, we have

$$x^3(x^2\frac{d}{dx})^2 y = \frac{x}{2y}(x^3\frac{dy}{dx})^2 + \frac{3}{2}x^3y^3 + 4x^2y^2 + 2(1 - \alpha x^2)xy + \frac{\beta}{y}x^3.$$

Let $3c^2 + 8c + 4 = 0$ $(c = -\frac{2}{3}, -2)$. By transformation $y = \frac{1}{x}(c + z)$, we obtain

$$\begin{aligned} &x^3(x^2\frac{d}{dx})^2 y = (x^3\frac{d}{dx})^2 z - 3x^5\frac{d}{dx}z \\ =&\frac{x^2}{2c}(1 + \frac{z}{c})^{-1}(x^3\frac{d}{dx}\frac{c+z}{x})^2 + \frac{3}{2}(c+z)^3 + 4(c+z)^2 \\ &+ 2(c+z) - 2\alpha x^2(c+z) + \frac{\beta x^4}{c}(1 + \frac{z}{c})^{-1}. \end{aligned}$$

By $(1+z/c)^{-1} = \sum_{n=0}^{\infty}(-\frac{z}{c})^n$ we get

$$(x^3\frac{d}{dx})^2 z = 2x^5\frac{d}{dx}z + a(x)z + b(x,z,x^3\frac{d}{dx}z) + h(x), \tag{5.8}$$

where

$$\begin{aligned} a(x) =& \frac{9}{2}c^2 + 8c + 2 - 2\alpha x^2 + O(x^4),\ b(x,z,p) = \sum_{i+j\geq 2} b_{i,j}(x)z^i p^j \\ h(x) =& \frac{3}{2}c^3 + 4c^2 + 2c - 2\alpha c x^2 + O(x^4) = -2\alpha c x^2 + O(x^4). \end{aligned}$$

We have $a(x) = -4(c+1) + O(x^2)$ $(-4(c+1) = -4/3$ or $4)$. Let $u = z, v = x^3 z'$. Then we have

$$\begin{aligned} x^3\frac{d}{dx}\begin{bmatrix} u \\ v \end{bmatrix} =& \left(\begin{bmatrix} 0 & 1 \\ -4(c+1) & 0 \end{bmatrix} + x^2\begin{bmatrix} 0 & 0 \\ -2\alpha & 2 \end{bmatrix} + O(x^4)\right)\begin{bmatrix} u \\ v \end{bmatrix} \\ &+ \begin{bmatrix} 0 \\ b(x,u,v)+h(x) \end{bmatrix}. \end{aligned} \tag{5.9}$$

and $\gamma = 2$. Hence we can apply Theorem 5.1.
. (II) Let $2c^2 + \beta = 0$ $(\beta \neq 0)$ and $y = x(c+z)$. Then from (5.7)

$$\begin{cases} x^2(x^2\frac{d}{dx})^2 y = x(x^3\frac{d}{dx})^2 z + x^6\frac{dz}{dx} + 2x^5 z + 2cx^5, \\ \frac{3}{2}x^2y^3 + 4xy^2 - 2\alpha x^2 y = x^3(b_0(x) + b_1(x)z + b_2(x)z^2 + b_3(x)z^3), \\ 2y + \frac{\beta x^2}{y} = 4xz + \frac{\beta x}{c}\sum_{n=2}^{\infty}(-z/c)^n, \end{cases}$$

$b_1(x) = \frac{9c^2}{2}x^2 + 8c - 2\alpha$ and

$$\begin{aligned} \frac{1}{2y}(x^3\frac{dy}{dx})^2 =& x^5(\frac{d(xz)}{dx} + c)^2(\frac{1}{2c}\sum_{n=0}^{\infty}(-z/c)^n) \\ =& \frac{cx^5}{2} + (O(x^5))z + x^6\frac{dz}{dx} + x\sum_{i+j\geq 2} b_{i,j}(x)z^i(x^3\frac{dz}{dx})^j. \end{aligned}$$

$$(x^3\frac{d}{dx})^2 z = x^2h(x) + (4 + (8c-2\alpha)x^2 + O(x^4))z + b(x,z,x^3z'), \tag{5.10}$$

where $h(x)$ is holomorphic at $x = 0$. Let $u = z, v = x^3 z'$. Then we have

$$x^3 \frac{d}{dx} \begin{bmatrix} u \\ v \end{bmatrix} = \left(\begin{bmatrix} 0 & 1 \\ 4 & 0 \end{bmatrix} + x^2 \begin{bmatrix} 0 & 0 \\ 8c - 2\alpha & 0 \end{bmatrix} + O(x^4) \right) \begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} 0 \\ b(x, u, v) + x^2 h(x) \end{bmatrix} \tag{5.11}$$

and we can apply Theorem 5.1 to this case.

6 Transformation of systems of linear ordinary equations with irregular singularity

Let us return to an $n \times n$ system of linear ordinary equations

$$x^{1+\gamma} \frac{dY}{dx} = A(x)Y, \ A(x) = (a_{i,j}(x)). \tag{6.1}$$

We give a proof of Proposition 2.1, which is a special case of diagonalization of systems ([1]). Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of $A(0)$, $\lambda_i \neq \lambda_j$ $(i \neq j)$ and $A(0)$ be diagonal. Let us recall the definitions $\omega_{i,k}$, $\theta_{i,k,\ell}$ and $\mathbf{\Theta}_1$ (see (2.3)). Let $\theta_* \notin \mathbf{\Theta}_1$ and $\widehat{I} = (\theta_* - \epsilon_*, \theta_* + \epsilon_*)$ $(\epsilon_* > 0)$ be an interval such that $\widehat{I} \cap \mathbf{\Theta}_1 = \emptyset$. We assume $\{a_{i,j}(x)\}_{1 \le i,j \le n}$ are γ-Borel summable in the direction θ_*. Hence there exists $I = (\theta_* - \delta_*, \theta_* + \delta_*)$ $(\delta_* > \pi/2\gamma)$ such that $a_{i,j}(x) \in \mathscr{O}_{\{1/\gamma\}}(S_0(I))$. Firstly we have, by a transformation with polynomial elements.

Lemma 6.1. *There is a matrix $P(x)$ $(P(0) = Id)$ with polynomial elements such that linear transformation $Y = P(x)Z$ transforms* (6.1) *to*

$$x^{1+\gamma} \frac{dz_i}{dx} = \lambda_i(x) z_i + \sum_{k=1}^{n} a^*_{i,k}(x) z_k \quad (1 \le i \le n), \tag{6.2}$$

where $\lambda_i(x)$ is a polynomial with degree $\le \gamma$, $\lambda_i(0) = \lambda_i$ and $a^*_{i,k}(x)(1 \le i, k \le n) \in \mathscr{O}_{\{1/\gamma\}}(S_0(I))$ with $a^*_{i,k}(x) = O(x^{1+\gamma})$.

It is known that we can take $P(x)$ with formal power series elements so that $A^*(x) = (a^*_{i,k}(x))$ is also diagonal ([23]). We only have to stop at finite steps.

Proof. Let $Y = P(x)Z$. Then

$$x^{1+\gamma}\frac{dY}{dx} = x^{1+\gamma}(\frac{dP}{dx}Z + P(x)\frac{dZ}{dx}) = A(x)P(x)Z.$$

Suppose $x^{1+\gamma}\dfrac{dZ}{dx} = B(x)Z$. Then

$$x^{1+\gamma}P'(x) + P(x)B(x) = A(x)P(x).$$

Let $A(x) = \sum_{k=0}^{\infty} A_k x^k$, $B(x) = \sum_{k=0}^{\infty} B_k x^k$ and $P(x) = \sum_{k=0}^{\gamma} P_k x^k$. We have

$$\sum_{k+l=m} A_l P_k - \sum_{k+l=m} P_k B_l = 0 \quad m \leq \gamma,$$
$$\sum_{k+l=m} A_l P_k - \sum_{k+l=m} P_k B_l = (m-\gamma)P_{m-\gamma} \quad m \geq \gamma + 1.$$

Let $P_0 = Id$ and $B_0 = A_0$. For $1 \leq m \leq \gamma$

$$P_m B_0 - A_0 P_m = \sum_{k=1}^{m-1}(A_{m-k}P_k - P_k B_{m-k}) + A_m - B_m.$$

Let $P_m = (p_{m,i,j})_{1\leq i,j\leq n}$ and $B_m = (b_{m,i,j})_{1\leq i,j\leq n}$. It follows from $A_0 = B_0 = diag.(\lambda_1, \ldots, \lambda_n)$ with $\lambda_i \neq \lambda_j (i \neq j)$ that we can take $p_{m,i,j}$ $(i \neq j)$ so that $b_{m,i,j} = 0$ $(i \neq j,\ 0 \leq m \leq \gamma)$. $P_m = 0$ for $m \geq \gamma + 1$. □

Therefore we may study

$$x^{1+\gamma}\frac{dy_i}{dx} = \lambda_i(x)y_i + \sum_{j=1}^{n} a_{i,j}(x)y_j \quad (1 \leq i \leq n) \tag{6.3}$$

with $a_{i,j}(x) = O(x^{1+\gamma})$. Set $n \times n$ matrices $\Lambda(x) = diag.(\lambda_1(x), \ldots .\lambda_n(x))$ and $A(x) = (a_{i,j}(x))$. Our aim is to transform (6.3) to a simpler form, that is, to the following linear system of equations

$$x^{1+\gamma}\frac{dz_i}{dx} = \lambda_i(x)z_i \quad (1 \leq i \leq n), \tag{6.4}$$

by using γ-Borel summable functions.

Proof of Proposition 2.1. Let $Y = (Id + C(x))Z$, where $C(x) = (C_{i,k}(x))$ is an $n \times n$ matrix with $C(0) = 0$. Then

$$x^{1+\gamma}\frac{dY}{dx} = x^{1+\gamma}(Id + C(x))\frac{dZ}{dx} + x^{1+\gamma}\frac{dC}{dx}Z$$
$$= (Id + C(x))\Lambda(x)Z + x^{1+\gamma}\frac{dC}{dx}Z = (\Lambda(x) + A(x))(Id + C(x))Z.$$

The equation to solve is

$$(Id + C(x))\Lambda(x) + x^{1+\gamma}\frac{dC}{dx} = (\Lambda(x) + A(x))(Id + C(x)), \tag{6.5}$$

more precisely

$$x^{1+\gamma}C'_{i,k}(x) + \big(\delta_{i,k} + C_{i,k}(x)\big)\lambda_k(x)$$
$$= \lambda_i(x)(\delta_{i,k} + C_{i,k}(x)) + \sum_{j=1}^{n} a_{i,j}(x)(\delta_{j,k} + C_{j,k}(x)).$$

Thus we get a system of differential equations with n^2 unknown functions $\{C_{i,k}(x); 1 \le i, k \le n\}$

$$\begin{aligned} x^{1+\gamma}C'_{i,k}(x) =& (\lambda_i(x) - \lambda_k(x))C_{i,k}(x) \\ &+ \sum_{j=1}^{n} a_{i,j}(x)C_{j,k}(x) + a_{i,k}(x). \end{aligned} \tag{6.6}$$

The aim is to show that $\{C_{i,k}(x)\}_{1\le i\le n}$ exist in some sectorial region and are γ-Borel summable functions. We construct $C_{i,k}(x)$ by γ-Laplace integral

$$C_{i,k}(x) = \int_0^{\infty e^{i\theta}} e^{-(\frac{\xi}{x})^\gamma}\widehat{C}_{i,k}(\xi)d\xi^\gamma. \tag{6.7}$$

Set $\lambda_{i,k} = \lambda_i - \lambda_k$ and $\lambda^*_{i,k}(x) = \lambda_i(x) - \lambda_k(x) - \lambda_{i,k}$. We have from (6.6)

$$x^{1+\gamma}C'_{i,k}(x) - \lambda_{i,k}C_{i,k}(x) = \lambda^*_{i,k}(x)C_{i,k}(x) + \sum_{j=1}^{n} a_{i,j}(x)C_{j,k}(x) + a_{i,k}(x) \tag{6.8}$$

and the following system of convolution equations

$$(\gamma\xi^\gamma - \lambda_{i,k})\widehat{C}_{i,k}(\xi) = \widehat{\lambda}^*_{i,k}(\xi) \underset{\gamma}{*} \widehat{C}_{i,k}(\xi) + \sum_{j=1}^{n} \widehat{a}_{i,j}(\xi) \underset{\gamma}{*} \widehat{C}_{j,k}(\xi) + \widehat{a}_{i,k}(\xi). \tag{6.9}$$

Lemma 6.2. (1) *There exists* $R > 0$ *such that* $\xi^{\gamma-1}\widehat{\lambda}^*_{i,k}(\xi)$ *and* $\xi^{\gamma-1}\widehat{a}_{i,j}(\xi)$ *are holomorphic in* $\Xi = \{|\xi| < R\} \cup S(\widehat{I})$.

(2) *For arbitrary small* $\epsilon > 0$ *there exist constants* M_ϵ *and* c_ϵ *such that*

$$|\widehat{\lambda}^*_{i,k}(\xi)|, |\widehat{a}_{i,j}(\xi)| \leq \frac{M_\epsilon |\xi|^{1-\gamma} e^{c_\epsilon |\xi|^\gamma}}{\Gamma(1/\gamma)}, \ |\widehat{a}_{i,j}(\xi)| \leq \frac{M_\epsilon |\xi| e^{c_\epsilon |\xi|^\gamma}}{\Gamma((1+\gamma)/\gamma)} \quad (6.10)$$

in $\{0 < |\xi| < R\} \cup S(\widehat{I}_\epsilon)$.

Proof. The statement (1) is obvious and the estimate (6.10) follows from $\lambda^*_{i,k}(x) = O(x)$ and $a_{i,j}(x) = O(x^{1+\gamma})$. □

Let us construct $\widehat{C}_{i,k}(\xi) = \sum_{m=1}^\infty \widehat{C}^m_{i,k}(\xi)$ as follows

$$\begin{aligned}(\gamma\xi^\gamma - \lambda_{i,k})\widehat{C}^1_{i,k}(\xi) =& \widehat{a}_{i,k}(\xi),\\ (\gamma\xi^\gamma - \lambda_{i,k})\widehat{C}^m_{i,k}(\xi) =& \widehat{\lambda}^*_{i,k}(\xi) \underset{\gamma}{*} \widehat{C}^{m-1}_{i,k}(\xi) + \sum_{j=1}^n \widehat{a}_{i,j}(\xi) \underset{\gamma}{*} \widehat{C}^{m-1}_{j,k}(\xi) \ \ m \geq 2.\end{aligned} \quad (6.11)$$

Let $\Xi_{\widehat{I}} = \{0 < |\xi| < R\} \cup S(\widehat{I})$ $(|\gamma R^\gamma| < \min_{\{i\neq k\}} |\lambda_{i,k}|)$. If $i \neq k$ and $\xi \in \Xi_{\widehat{I}}$, then $\gamma\xi^\gamma - \lambda_{i,k} \neq 0$ and $\widehat{C}^1_{i,k}(\xi) = \widehat{a}_{i,k}(\xi)/(\gamma\xi^\gamma - \lambda_{i,k})$. If $i = k$, then $\widehat{\lambda}^*_{i,k}(\xi) = 0$ and $\widehat{C}^1_{k,k}(\xi) = \widehat{a}_{k,k}(\xi)/\gamma\xi^\gamma$. The following lemma holds.

Lemma 6.3. *There exist* $\widehat{C}^m_{i,k}(\xi) \in \mathscr{O}(S(\widehat{I}) \cup \{0 < |\xi| < R\})$ *such that the following holds.*

(1) *Let* $\epsilon > 0$ *be an arbitrary small constant. Then there exist constants* A_ϵ *and* c_ϵ *such that*

$$|\widehat{C}^m_{i,k}(\xi)| \leq \frac{A^m_\epsilon |\xi|^{m-\gamma}}{\Gamma(m/\gamma)} e^{c_\epsilon |\xi|^\gamma}. \ \xi \in \Xi_{\widehat{I}_\epsilon} = \{0 < |\xi| < R\} \cup S(\widehat{I}_\epsilon). \quad (6.12)$$

(2) $\xi^{\gamma-1}\widehat{C}^m_{i,k}(\xi) \in \mathscr{O}(\{|\xi| < R\})$.

Proof. We show (6.12) by induction. Let $m = 1$. For $i \neq k$ inequality (6.12) holds. Let $i = k$. (6.12) holds from the second estimate of (6.10), Assume (6.12) holds for $m - 1$. Then there exists a constants B_ϵ such that

$$|\widehat{\lambda}^*_{i,k}(\xi) \underset{\gamma}{*} \widehat{C}^{m-1}_{i,k}(\xi)|, |\widehat{a}_{i,j}(\xi) \underset{\gamma}{*} \widehat{C}^{m-1}_{j,k}(\xi)| \leq \frac{B_\epsilon A^{m-1}_\epsilon |\xi|^{m-\gamma} e^{c_\epsilon |\xi|^\gamma}}{\Gamma(m/\gamma)}.$$

If $i \neq k$, $\gamma\xi^\gamma - \lambda_{i,k} \neq 0$. Then estimate (6.12) for m holds. If $i = k$, it follows from $\widehat{\lambda}^*_{i,k}(\xi) = 0$ and the second estimate of (6.10) that

$$|\widehat{a}_{k,j}(\xi) \underset{\gamma}{*} \widehat{C}^{m-1}_{j,k}(\xi)| \leq \frac{B_\epsilon A_\epsilon^{m-1} |\xi|^m e^{c_\epsilon |\xi|^\gamma}}{\Gamma(m/\gamma)}$$

and the estimate (6.12) also holds for m. We have the statement (2) in the same way as above. □

Thus we get

Proposition 6.4. *There exist* $\{\widehat{C}_{i,k}(\xi)\}_{1 \leq i,k \leq n}$ *such that*

(1) $\widehat{C}_{i,k}(\xi) \in \mathscr{O}(\Xi_{\widehat{I}})$, $\xi^{\gamma-1}\widehat{C}_{i,k}(\xi) \in \mathscr{O}(\{|\xi| < R\})$ *and* $\{\widehat{C}_{i,k}(\xi)\}_{1 \leq i,k \leq n}$ *satisfy convolution equations* (6.9).

(2) *For any small* $\epsilon > 0$ *there are positive constants* M_ϵ *and* c'_ϵ *such that*

$$|\widehat{C}_{i,k}(\xi)| \leq \frac{M_\epsilon |\xi|^{1-\gamma}}{\Gamma(1/\gamma)} e^{c'_\epsilon |\xi|^\gamma}, \; \xi \in \Xi_{\widehat{I}_\epsilon} = \{\xi; 0 < |\xi| < R\} \cup S(\widehat{I}_\epsilon). \quad (6.13)$$

Proof. We have $\{\widehat{C}^m_{i,k}(\xi)\}_{m=1,2,\dots}$ with (6.12). Then there exist constants M_ϵ, c_ϵ and c'_ϵ such that

$$\sum_{m=1}^{\infty} |\widehat{C}^m_{i,k}(\xi)| \leq \sum_{m=1}^{\infty} \frac{A_\epsilon^m |\xi|^{m-\gamma}}{\Gamma(m/\gamma)} e^{c_\epsilon |\xi|^\gamma} \leq M_\epsilon |\xi|^{1-\gamma} e^{c'_\epsilon \xi|^\gamma}.$$

Hence $\widehat{C}_{i,k}(\xi) = \sum_{m=1}^\infty \widehat{C}^m_{i,k}(\xi)$ $(1 \leq i, k \leq n)$ converge and satisfy (6.9). □

Let us define for $\theta \in I_\epsilon = (\theta_* - \epsilon, \theta_* + \epsilon)$

$$C_{i,k}(x) = \int_0^{e^{\imath\theta}\infty} e^{-(\frac{\xi}{x})^\gamma} \widehat{C}_{i,k}(\xi) d\xi^\gamma. \quad (6.14)$$

Thus we obtain a linear transformation $Y = (Id + C(x))Z$ and Proposition 2.1 is shown.

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